Title
Essays on Competition and Conflict

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Publication Date
2015

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA,
IRVINE

Essays on Competition and Conflict

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Economics

by

Blake A. Allison

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2015

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ACKNOWLEDGMENTS

I would like to thank my mentors Jason Lepore and Stergios Skaperdas, whose support and advice have been invaluable in the development of my potential as a researcher and without whom my accomplishments would not have been possible. I would also like to thank Michael McBride for his advice and for providing me with research opportunities which helped to enhance the breadth of my studies.

Additionally, I would like to thank the UCI department of economics and the Air Force Office of Scientific Research Award No. FA9550-10-1-0569 for their financial support.

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ABSTRACT OF THE DISSERTATION

Essays on Competition and Conflict

By

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Doctor of Philosophy in Economics

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These works contribute to the current understanding of conflict resolution and consumer behavior, as well as to our ability to formally analyze these and other economic models. The first chapter demonstrates that the use of negotiation in place of arbitration to settle disputes may lead to an increase in surplus by reducing the incentives of the involved parties to invest in their ability to engage in conflict. The second chapter derives the demand that faces competing firms when consumers shop strategically and the prices and capacities are fixed and common knowledge. The equilibrium motivates and justifies the implementation of the well-known proportional rationing rule in studying price competition. The final chapter, which is joint with Professor Jason Lepore, develops a verifiable sufficient condition for the existence of a mixed strategy equilibrium in discontinuous games. This condition is then used to demonstrate existence of equilibrium in a large class of games of price competition for which existence of equilibrium had not yet been proven.

Chapter 1

Do Players Prefer to Bargain Noncooperatively in the Shadow of Conflict?

1.1 Introduction

In this paper, we study a bargaining problem in which the disagreement point or outside option is endogenously determined by investments of the players. We investigate the impact of the bargaining process on the investments made when disagreement results in costly conflict. This allows for a Pareto ranking of bargaining solutions according to the efficiency of the equilibrium that they induce.

Conflict is a source of numerous inefficiencies that have plagued economies and societies throughout history. Even in times of peace, litigation and persuasion serve as a prominent manifestation of non-violent conflict. There are two primary inefficient components of conflict: destruction, which involves a reduction of potential surplus due to conflict, and wasteful investment, which results from a diversion of productive resources toward conflict. The destruction is clear with physical conflict, while generally there are additional costs associated with the utilization of the tools of conflict, be they weapons or influence, in excess of the investment in their capacity. This destructive component provides incentive for individuals to avoid conflict, requiring that all involved parties agree to a settlement through some bargaining process. Even if parties settle, however, they still incur the cost of their investments in the tools of conflict, as they are sunk prior to bargaining. Though unproductive, these investments may be unavoidable when conflict does not occur since they influence and enforce the settlements that are made. As such, it is common to observe an "armed peace" or "cold war" whereby all parties exert efforts that would increase their likelihood of winning in the event of conflict, though these efforts remain unused after an agreement is reached. These expenditures can be substantial, McCloskey and Klamer (1995) estimate that approximately one quarter of US GDP is persuasion, while Stiglitz and Bilmes (2012) estimates a conservative lower bound of three trillion dollars for the cost of the wars in Iraq and Afghanistan. Given the large costs that this behavior imposes, it is natural to understand the factors that contribute to the investment in conflict as well as search for methods to reduce such spending. To this end, this paper focuses on whether the bargaining process can be structured to incentivize players to reduce these inefficient expenditures.

We model the outcome of conflict as a winner take all contest in which the outcome is probabilistically determined by a contest function.[1] Players participate in a three stage game in which they choose efforts which determine the probability that either player wins in the event of conflict and then engage in a bargaining process which may be used to avoid conflict. Agreements made in the bargaining stage are non-binding; in the third stage of the game, players choose whether to engage in conflict. The ability to impose conflict after an

[1] This approach, pioneered by Tullock (1980), has become standard in the literature on conflict.

agreement is reached necessitates some amount of effort in order to enforce the settlement that is agreed upon. While it serves this instrumental purpose, we largely ignore the third stage as its equilibrium is trivial.

In the bargaining stage of the game, we examine two bargaining schemes in particular: the Nash (1950,1953) bargaining solution will represent the category of cooperative bargaining solutions and a Rubinstein-like alternating offers process in which conflict is an outside option will represent the category of noncooperative solutions.[2,3] Our main result is that the efforts induced in equilibrium by the cooperative solution are greater than those induced by the noncooperative solution. Consequently, implementation of the noncooperative solution results in greater efficiency. Interestingly, the noncooperative scheme does not typically admit a pure strategy equilibrium in effort choices even when the cooperative solution does.[4] While cooperative solutions tend to be easier to analyze and have desirable properties, our results suggest that their application may overestimate the equilibrium efforts and underestimate the equilibrium surplus gained through bargaining.

Before continuing, we shall describe the noncooperative bargaining process on which we focus. We adapt Binmore's (1987) variant of Rubinstein's (1982) alternating offers game to allow for outside options. The process consists of a countable number of bargaining rounds in which one player is selected at random to propose a division of the prize and the other responds to that proposal. A proposal specifies a utility that each player will receive upon agreement, while the possible responses are to accept the offer, reject the offer, or impose

[2] In the setting examined in this paper, utility is transferable. As a consequence, all well known cooperative solutions coincide with the Nash bargaining solution, including the Kalai Smorodinsky (1975), egalitarian (Kalai (1977), Roth (1979)), and equal sacrifice (O'Neill (1982), Aumann and Maschler (1985)) solutions.

[3] We focus on this scheme as it bears the greatest resemblence to realistic bargaining practices. We do not consider the continuous time model which Abreu and Gul (2000) and others have used to study reputation based bargaining as this model relies on asymmetric information, while ours is one of complete information.

[4] The only possible pure strategy equilibrium with the noncooperative scheme involves neither player investing any effort in the contest. This occurs when players and very patient and and engaging in conflict is very costly.

conflict. If the offer is accepted, each player receives the utility specified by the proposal provided that neither player imposes conflict in the third stage. If the offer is rejected, then the bargain process advances to the next round and repeats. If a player imposes conflict, then bargaining ceases and conflict occurs immediately. While players will never impose conflict on the equilibrium path, they may prefer immediate conflict to a delayed agreement as a result of the discounting of future payoffs. This environment admits a unique subgame perfect equilibrium. The equilibrium agreement is independent of the effort decisions of the players provided that the payoff to each player from conflict is sufficiently low. The reason is that neither would find it worthwhile to impose conflict, and so the proposer may ignore that option as an incredible threat.

The fact that conflict may not serve as a credible threat point to the noncooperative bargaining process is the driving force behind our results. When players make similar effort choices, they each expect a relatively small payoff in the event of conflict. Thus, in this case, the equilibrium agreement would be independent of those choices. This prohibits the existence of a symmetric pure strategy equilibrium. To see why, consider any symmetric pure strategy profile. Then since the equilibrium agreement is independent of the efforts chosen, either player may reduce their effort, maintaining the same agreement but reducing their expenditure. After such a deviation, the other player possesses a similar profitable deviation. By iterating this logic, we derive an upper bound on equilibrium strategies that is strictly lower than the unique pure strategy equilibrium effort that is induced by the cooperative solution. Moreover, we are able to show that this upper bound strictly dominates all higher effort choices. When conflict is very destructive and players are sufficiently patient, this upper bound may actually be zero, thereby inducing a completely efficient outcome which is impossible to achieve via the well known cooperative solutions.

Our result is not limited to the class of contests that possess pure strategy equilibria. Building upon the results of Ewerhart (2013) and Siegel (2009), we extend our results to a large class of contests which do not possess pure strategy equilibria. In these settings, there is not a universal incentive to reduce contest efforts, as a small increase in effort may drastically increase a player's probability of winning the conflict, thereby sharply increasing his bargaining power. In order to obtain our results, we appeal to Ewerhart and Siegel's results that rents are completely dissipated in all equilibria of such contests. Our results then extend easily since each player can guarantee himself a larger payoff by choosing zero effort when the noncooperative scheme is employed than the fully dissipated equilibrium rent when the cooperative solution is used. Thus, cooperative solutions induce equilibria in which players only earn the minimal payoff that they may secure with a choice of zero, while the analogous payoff serves as an lower bound for equilibrium payoffs induced by the noncooperative scheme.

To further investigate the robustness of our results, we consider a variant of the alternating offers game where, in addition to the discounting of future payoffs, there is an exogenous probability that the process breaks down after each rejected offer, the result of which is conflict. Unlike the prior formulation, the equilibrium division always depends on the efforts of the players since any response other than acceptance entails a positive probability of conflict. Regardless, our results extend to this formulation of the model provided that the probability of exogenous breakdown is less than one. The intuition remains unchanged. With a probability of breakdown that is less than one, only a fraction of the disagreement payoffs are considered in the equilibrium division. Given the reduced attention paid to the conflict outcome, the only candidate for a pure strategy equilibrium involves a low choice of effort for each player. If breakdown is sufficiently unlikely, then this candidate is near zero, which allows for a player to choose a slightly larger effort and secure a large share of the prize, ruling out pure strategy equilibrium. Otherwise, if breakdown is very likely, then there is a unique, symmetric pure strategy equilibrium in which players exert an effort that is lower than that induced by the cooperative solution. In either case, equilibrium efforts remain strictly below those induced by the cooperative solution.

Our results may seem surprising given the result of Binmore, Rubinstein, and Wolinsky (1986). They assume that there is a fixed probability that the bargaining process breaks down in every period and that there is no discounting of future payoffs. In this setting, they show that the equilibrium agreement converges to the Nash bargaining solution as the probability of breakdown tends to zero.[5] In our setting, one would expect from this result that the equilibrium efforts would converge to those induced by the Nash bargaining solution. Such convergence does not occur, however, when players possess an outside option of sufficiently high value, as has been observed by Binmore, Shaked, and Sutton (1989), Osborne and Rubinstein (1990), Chiu (1998), and Chiu and Yang (1999).The reason that conflict could have a greater value than rejection in our setting is due to a subtle distinction between discounting future payoffs and a probability of breakdown. In the former case, when a player rejects an offer and the process proceeds to the next round, both the utility possibility set and the value of the outside option shrink due to discounting. In the latter case, no shrinking occurs. More technically, when players reject an offer with exogenous breakdown they receive either their continuation value or engage in conflict. With discounting, rejection yields the discounted continuation value, which may be realized as a "probability" of receiving their continuation value plus the remaining "probability" of receiving nothing. As a result, when players discount future payoffs, they may prefer to engage in conflict immediately rather than wait for a better settlement in the future.

There has been some work examining the effect that the bargaining solution has on the ef-

[5] This result has since been generalized significantly. Recently, Britz, Herings, and Predtetchinski (2010) have allowed for a general stochastic process to determine the order in which players make proposals and shown convergence to the asymmetric Nash bargaining solution with parameters corresponding to the limiting properties of the stochastic process.

forts that players exert in equilibrium, though only using cooperative bargaining solutions.[6] Anbarci, Skaperdas, and Syropoulos (2002) provide a Pareto ranking of three well known cooperative solutions when expenditure on contest efforts diverts resources from production, showing that the equal sacrifice rule induces lower efforts than the Kalai-Smorodinsky and egalitarian rules. Skaperdas (2006) examines incentives for and against settlement in various settings and compares equilibrium efforts given various protocols within a class of cooperative solutions that includes Nash bargaining. These two papers both demonstrate that the solutions that are less reliant on the threat point (payoff from conflict) induce less effort and more efficiency in equilibrium. Consistent with this intuition, the noncooperative solution on which we focus may be independent of the players' efforts, thus the outside option may have minimal influence on the solution.

The remainder of the paper is organized as follows. In section 2 we construct the model. In section 3 we detail the noncooperative bargaining scheme and solve for its equilibrium. In section 4 we characterize the efforts induced by the solution to the bargaining process of section 3. In section 5 we characterize the efforts under cooperative bargaining schemes and prove the main result that players are better off with the noncooperative bargaining scheme of section 3. In Section 6 we extend the main result of Section 5 to a class of contests which do not possess a mixed strategy equilibrium. In Section 7, we consider an alternative bargaining scheme in which the process has an exogenous probability of breakdown. We demonstrate that the main results hold as long as players discount future payoffs. Finally, we conclude in Section 8, with some proofs relegated to the Appendix.

[6]Powell (1996) employs Rubinstein's bargaining framework in which conflict serves as an outside option with the purpose of understanding the role of asymmetric information in the bargaining process. The probability of winning in the event of conflict is exogenous in his model.

1.2 The Model

Two players contest a perfectly divisible prize with common value V. A typical player will be referred to as i, and we will use j to refer to the player other than i. The prize can be acquired in one of two ways: either the players can agree to a division of the prize or they can impose some sort of conflict to determine its allocation. The conflict may or may not be physical in nature; while war is a natural application, other prominent examples include litigation, contract negotiation and disputes, as well as lobbying.

There are three stages to the game. In the first stage, the players simultaneously choose effort levels $e_i \in \mathbb{R}_+$. These efforts determine the probability that each player wins the prize in the event of conflict. These efforts may represent military investment or the contracting of lawyers to prepare a legal case. In the second stage, players observe these efforts and then engage in a bargaining process which we will describe shortly. Finally, in the third stage, players decide whether to engage in conflict. In the case that an agreement is reached in the second stage, that agreement is realized when neither player chooses to impose conflict. Otherwise, conflict occurs if either player chooses to impose it or if no agreement was reached.

We make the following assumption regarding the cost of effort.

Assumption 1. *The cost of effort $c(\cdot)$ is continuous everywhere and twice continuously differentiable, with $c'(e) > 0$ and $c''(e) \geq 0$ for all $e > 0$. We normalize $c(0) = 0$.*[7]

This assumption implies the existence of an $E > 0$ such that for all $e > E$, $c(e) > V$. Thus, all $e > E$ are strictly dominated by $e' = 0$. We may therefore restrict attention to the compact strategy set $[0, E]$.

Conflict is inefficient, and the value of the prize is reduced to θV if conflict occurs, where

[7] The continuity of c allows for this normalization. We do not consider the possibility that participation ($e_i > 0$) incurs a fixed cost.

$\theta < 1$. This inefficiency may be the result of an additional cost of utilizing military forces, lawyers fees that must be paid during lengthy negotiations or court cases, as well as physical destruction or court fees that are incurred only if conflict occurs.[8] This inefficiency provides incentive for players to settle, as each player can receive a strictly higher payoff by reaching agreement. The case of $\theta = 1$ is theoretically and practically uninteresting. Engaging in conflict is almost always more costly than settling, and if conflict is efficient, then players cannot collectively (or individually) gain from settlement. A player could simply impose conflict to obtain his share of the prize, so the only bargaining solution would be for the players to receive their conflict payoff.

When conflict occurs, the players engage in a winner take all contest where the probability that each player i wins is determined by a contest function $p_i : \mathbb{R}_+ \times \mathbb{R}_+ \to [0,1]$.[9] We use a particular form for p_i,

$$p_i(e_i, e_j) = \begin{cases} \frac{f(e_i)}{f(e_1)+f(e_2)} & \text{if} \quad e_1 + e_2 > 0 \\ \frac{1}{2} & \text{otherwise} \end{cases}.$$

This functional form, axiomatized by Skaperdas (1996) and derived stochastically by Jia (2008), is commonly employed in the conflict literature.[10] As Skaperdas shows, this form is implied by some innocuous continuity and monotonicity assumptions along with an independence of irrelevant alternatives assumption.[11] The specification of an equal probability of winning when $e_1 = e_2 = 0$ is by a convention of symmetry, though other specifications would not alter the results of this paper. Note that if $f(0) = 0$, then there is a discontinuity

[8]It would be possible to instead model the inefficiency of conflict as resulting from a fixed cost F incurred by each player if conflict occurs, or dependent on the players' investments. This would not cause any qualitative changes to the equilibrium.

[9]Due to the risk neutrality of preferences, an equivalent interpretation is that p_i is the fraction of the prize that player i secures in the event of conflict.

[10]For further discussion on contest functions, see Jia and Skaperdas (2012).

[11]The I.I.A. assumption applies when there are $K > 2$ players. It states that the probability that a player wins the contest is independent of the efforts selected by those not participating in the contest.

at $e_1 = e_2 = 0$.

Assumption 2. *The function $f(\cdot)$ is twice continuously differentiable with $f(e) \geq 0$ for all $e \geq 0$ and $f'(e) > 0$ for all $e > 0$.*

Assumption 3. *The payoffs in the event of conflict $p_i(e)\theta V - c(e_i)$ are strictly concave in e_i on $[0, E] \times [0, E] \setminus \{(0, 0)\}$.*

This assumption of concavity is not particularly restrictive. It still allows for a discontinuity at zero efforts (when $f(0) = 0$), and does not require that f or p_i be concave, just that they are not too convex. If f is chosen to be concave, then assumption 3 is trivially satisfied. We will discuss the implications of relaxing Assumption 3 in section 6.

Assumption 4. *If $\theta > 1/2$, there exists an $e > 0$ such that $c(e) < \left(p_i(e, 0)\theta - \frac{1}{2}\right)V$.*

Assumption 4 is necessary to prevent the existence of an equilibrium in which both players choose zero efforts. While such a scenario is possible, there would be no need for any bargaining. The unique equilibrium efforts would be zero regardless of the bargaining scheme employed, and so comparing bargaining schemes would have no meaning. Note that if $f(0) = 0$, then Assumption 4 is trivially satisfied.

Remark 1. *Let us briefly discuss the contest functions that fit these assumptions. As mentioned, any concave f fits, so the most commonly used Tullock contest function $f(x) = \alpha x^m + \beta$ is accommodated. Another commonly used function is $f(x) = e^{kx}$ for some $k > 0$. This contest function does not generally admit a pure strategy equilibrium when applied to contest games with linear cost of effort, though it still has some desirable properties. Our model accommodates the exponential function provided that the cost function is sufficiently convex. For example, a cost function $c(x) = \alpha e^{\beta x}$ where $\beta = k$ and $\alpha \in \left(\left(5 + 3\sqrt{3}\right)\theta V / \left(3 + \sqrt{3}\right)^3, \theta V/4\right)$ allows $f(x) = e^{kx}$ to fit our assumptions. In general, many convex functions can be accommodated by selecting the cost to be of a similar form.*

Before we proceed, let us briefly discuss the equilibrium of the third stage of the game. If an agreement is reached, there are two equilibria, one in which neither player imposes conflict, and another in which both players impose conflict. The latter can be an equilibrium because a bilateral deviation is required to reach an outcome in which conflict does not occur. We will adopt as a convention that the players will choose to not impose conflict whenever the an agreement has been reached that is Pareto superior to the outcome of conflict. It follows that conflict will occur as an equilibrium outcome of the third stage exactly when either one of the players receives a higher payoff from conflict than from the agreement or if an agreement is not reached. In what follows, we will largely take these outcomes as given and will not present the third stage strategies in the characterization of the equilibrium of the game.

1.2.1 The Bargaining Process

We adopt the probabilistic version of Rubinstein's alternating offers bargaining model as first adapted by Binmore (1987). Bargaining takes place in a sequence of rounds. In each round, one player i is chosen at random to be the proposer. The proposer may either offer a division of the prize defined by the share $s_i \in [0,1]$ that he would receive, or he may choose to withdraw from the bargaining process and impose conflict. We will henceforth ignore the proposer's option to withdraw, as we will show that in the unique equilibrium of this process, the proposer obtains a share that is strictly larger than what he would obtain through conflict. As a convention of symmetry, we assume that each player is chosen to be the proposer with equal probability.

Once the proposer makes an offer, the responder either accepts the offer or rejects it. If he accepts the offer, then bargaining ends and the game proceeds to the third stage. If he rejects the offer, then he may choose to withdraw from the bargaining process and impose conflict, or he may delay until the next round. If he delays, bargaining proceeds to the next round and the process repeats. Players discount future payoffs, so rejecting an offer reduces the present value of the final division by a factor $\delta < 1$. If the process repeats indefinitely, then the prize is never allocated and each player obtains a payoff of zero less his cost of effort, which is sunk in the first stage. Note that accepting an offer and withdrawing from the bargaining process may lead to the same outcome if conflict yields a higher payoff for one of the players than he would receive in the agreement. We assume as a convention that if this is the case, then the responder will opt to withdraw rather than accepting the offer. This has no effect on the equilibrium outcomes of the game, it serves only to simplify the exposition, as reaching agreement becomes equivalent to allocating the prize according to the agreement.

It is worth discussing the interpretation of δ. In models without outside options, δ may be viewed as either a discount factor or an exogenous probability that bargaining continues. In our model, breakdown is endogenous. Players choose whether or not to end the bargaining process, and so it may not be fitting to have δ represent the probability of continuation. Furthermore, there is a mathematical distinction between the interpretations. To see this, let V_i denote the value to the current responder i in the next round if bargaining does not break down. Then with the exogenous breakdown interpretation, the expected utility of rejection is $\delta V_i + (1 - \delta) p_i \theta V - c(e_i)$. With the time discounting interpretation, the expected utility of rejection is $\delta V_i - c(e_i)$. This distinction has an important qualitative implication for the setup of the bargaining game: the threat point, the payoffs earned in the event of disagreement, cannot be normalized to zero. The reason is that consumption cannot occur until after the prize is allocated, which requires either conflict or agreement. As such, delay in the bargaining process delays all consumption, thereby reducing the present value of the consumption of the prize. This affects the value of the prize whether it is obtained via agreement or conflict. While seemingly subtle, this distinction is responsible for the nonconvergence of the equilibrium division to the Nash bargaining solution.[12]

Let t be the round in which agreement is reached or a players withdraws. If no agreement is reached and no player withdraws, then let $t = \infty$. Let s_i denote the share that i receives from agreement. Then we may write the expected utility of player i as

$$u_i = \begin{cases} \delta^{t-1} s_i V - c(e_i) & \text{if conflict does not occur,} \\ \delta^{t-1} p_i \theta V - c(e_i) & \text{if conflict occurs} \end{cases} .$$

We now analyze the equilibrium play of the game. We find a subgame perfect Nash equilibrium by first finding the solution to the bargaining game and then solving for the equilibrium efforts. Note that the solution of the bargaining scheme must be independent of assumptions 1-4, as the efforts are fixed prior to bargaining.

1.3 The Bargaining Subgame

In the second stage of the game, players bargain in the infinite game described above. The solution concept we use is that of a subgame perfect Nash equilibrium (SPNE). As we will show, there is a unique SPNE of this bargaining process. This equilibrium exhibits stationarity in the sense that each player i proposes the same proposal $s_i^* \in [0, 1]$ every time that player i is the proposer, and adheres to the same response rule R_i^* in every round that player i is the responder. The response rule R_i^* is a partition of the space of potential proposals $[0, 1]$ into three subsets, A_i^*,D_i^* and W_i^*. Formally, if $s_j \in A_i^*$ then player i accepts the proposal, if $s_j \in D_i^*$ then player i rejects the proposal and delays until the next round,

[12]This does not contradict the seminal results of Binmore, Rubinstein, and Wolinsky (1986) or other generalizations in the literature that show convergence of the noncooperative solution to the Nash bargaining solution. The noncooperative solutions in the literature that converge to Nash's solution are not invariant to affine transformations.

and if $s_j \in W_i^*$ then player i withdraws from the bargaining process and imposes conflict. As a convention, we assume that players accept rather than reject proposals when they are indifferent, and that they delay rather than withdraw when indifferent.[13] Note that we are free to ignore the cost of effort as it is sunk for both players after the first stage and does not influence the utility received from the prize per the bargaining agreement.

Before we present SPNE of the bargaining game, it is worthwhile to demonstrate the method by which the equilibrium is computed. Beginning with an assumption of stationarity (which can be shown to hold), we may define a value function $V_i(K)$ which specifies the expected payoff that of player i conditional on the state in which player $K \in \{i, j\}$ is the proposer. The value $V_i(I)$ is the solution to the proposer's optimization problem, which is as follows:

$$V_i(I) = \max_{s_i} s_i V$$

$$\text{subject to}$$

$$(1 - s_i) V \geq V_j(I).$$

As mentioned earlier, a responder may take actions to cause an immediate conflict. Thus the value of responding $V_i(J)$ must be at least as high as the i's payoff in the event of conflict. If a responder delays, he is equally likely to be the proposer and responder in the next round, so his expected continuation payoff is $\delta(V_i(I) + V_i(J))/2$. We rewrite the optimization

[13] This assumption is necessary for guaranteeing existence of equilibrium. If players were to reject when indifferent, then the proposer would have to offer the responder slightly more than his reservation value, but regardless of the offer, he would always have incentive to offer less.

problem with these two constraints as

$$V_i(I) = \max_{s_i} s_i V$$

subject to

$$(1 - s_i) V \geq p_j \theta V, \text{ and}$$

$$(1 - s_i) V \geq \frac{\delta}{2} (V_j(I) + V_j(J)).$$

Since the objective function is strictly increasing in s_i, it follows that at least one of these constraints must bind. The latter constraint is a function of the responder's value of proposing, and so both optimization problems must be solved simultaneously, meaning that two of the four constraints (one for each player) bind in any equilibrium. Therefore, there are four cases that must be checked corresponding to which constraints are binding. Solving the constraints in each case will yield a candidate for the bargaining solution, while the nonbinding constraint may be used to check the set of effort choices for which the candidate is indeed an equilibrium.

Which of the constraints that bind in equilibrium depends on the efforts employed by each player and their degree of patience as captured by δ. Intuitively, players with a lower probability of winning in the underlying contest would be less likely to withdraw, and would be expected to delay the bargaining process if offered an unsatisfactory share. Similarly, impatient players would find waiting less appealing, and thus would be expected to withdraw when faced with an unsatisfactory offer.

We now proceed to formalize the solution by first partitioning the strategy space of the first

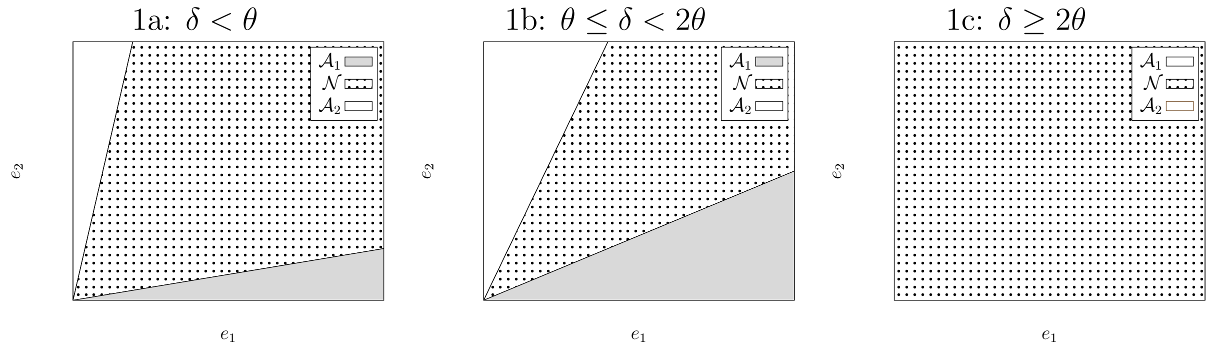


Figure 1.1: The partition of the strategy space.

stage $[0, E] \times [0, E]$ into three regions of interest:

$$\mathcal{A}_i(\delta) = \left\{ \begin{array}{ll} e : p_j < \frac{\delta}{2\theta}\frac{1-\theta}{1-\delta} & \text{if} \quad \delta < \theta \\ e : p_i > \frac{\delta}{2\theta} & \text{if} \quad \delta \geq \theta \end{array} \right\}, \ i = 1, 2, \text{ and}$$

$$\mathcal{N}(\delta) = \left\{ \begin{array}{ll} e : \min\{p_1, p_2\} \geq \frac{\delta}{2\theta}\frac{1-\theta}{1-\delta} & \text{if} \quad \delta < \theta \\ e : \max\{p_1, p_2\} \leq \frac{\delta}{2\theta} & \text{if} \quad \delta \geq \theta \end{array} \right\}.$$

The set $\mathcal{A}_i(\delta)$ corresponds to the region in which player i has a significant advantage in the underlying contest, while $\mathcal{N}(\delta)$ corresponds to the region in which neither player has a significant advantage. These regions are important as they will uniquely determine each player's threat of disagreement, i.e., which constraints bind, and determine the bargaining solution. Figure 1.1 illustrates an example of a partition of the effort space into these regions, with 1a corresponding to the case where $\delta < \theta$, 1b corresponding to the case of $\theta < \delta \leq 2\theta$, and 1c corresponding to $\delta \geq 2\theta$.

Define the following proposal and response rules:

$$
\begin{aligned}
s_i^* &= \begin{cases} \frac{2(1-\delta)}{2-\delta} + \frac{\delta}{2-\delta} p_i \theta & \text{if} \quad e \in \mathcal{A}_i(\delta) \\ (1-\theta) + p_i \theta & \text{if} \quad e \in \mathcal{A}_j(\delta) \text{ or } e \in \mathcal{N}(\delta) \text{ and } \delta < \theta \\ \frac{2-\delta}{2} & \text{if} \quad e \in \mathcal{N}(\delta) \text{ and } \delta \geq \theta \end{cases}, \\
A_i^* &= \left[p_j \theta, s_j^*\right], \\
D_i^* &= \begin{cases} \emptyset & \text{if} \quad e \in \mathcal{A}_i(\delta) \text{ or } e \in \mathcal{N}(\delta) \text{ and } \delta < \theta \\ [0,1] \diagdown A_i^* & \text{if} \quad e \in \mathcal{A}_j(\delta) \text{ or } e \in \mathcal{N}(\delta) \text{ and } \delta \geq \theta \end{cases}, \\
W_i^* &= \begin{cases} [0,1] \diagdown A_i^* & \text{if} \quad e \in \mathcal{A}_i(\delta) \text{ or } e \in \mathcal{N}(\delta) \text{ and } \delta < \theta \\ \emptyset & \text{if} \quad e \in \mathcal{A}_j(\delta) \text{ or } e \in \mathcal{N}(\delta) \text{ and } \delta \geq \theta \end{cases}.
\end{aligned}
$$

Lemma 1. *Suppose that each player employs the following strategy. In each round that player i is the proposer, he proposes s_i^*. In each round that player i is the responder, he accepts proposals $s_j \in A_i^*$, rejects proposals $s_j \in D_i^*$, and withdraws given proposals $s_j \in W_i^*$, where s_i^*, A_i^*, D_i^*, and W_i^* are as specified above. Then the expected share that each player i receives is*

$$
\sigma_i^* = \begin{cases} \frac{1-\delta}{2-\delta} + \frac{1}{2-\delta} p_i \theta & \textit{if} \quad e \in \mathcal{A}_i(\delta) \\ \frac{1-\theta}{2-\delta} + \frac{1}{2-\delta} p_j \theta & \textit{if} \quad e \in \mathcal{A}_j(\delta) \\ \frac{1}{2} & \textit{if} \quad e \in \mathcal{N}(\delta) \textit{ and } \delta \geq \theta \\ \frac{1-\theta}{2} + p_i \theta & \textit{if} \quad e \in \mathcal{N}(\delta) \textit{ and } \delta < \theta \end{cases}.
$$

The proof of Lemma 1 is located in the appendix.

We now proceed to characterize the equilibrium of the bargaining subgame.

Proposition 1. *The unique subgame perfect equilibrium of the bargaining subgame is as* *follows. In each round that player i is the proposer, he proposes s_i^*. In each round that player i is the responder, he accepts proposals $s_j \in A_i^*$, rejects proposals $s_j \in D_i^*$, and withdraws given proposals $s_j \in W_i^*$, where s_i^*, A_i^*, D_i^*, and W_i^* are as specified above.*

The following lemma will be useful in the proof of Proposition 1. The proof of this lemma consists of a simple computation and is located in a supplemental appendix.

Lemma 2. *The following two statements are true.*

(1) Suppose that $\delta < \theta$ and $p_j < \delta (1 - \theta) / (2\theta (1 - \delta))$. Then $p_i \geq \delta/2\theta$.

(2) Suppose that $\delta \geq \theta$ and $p_j \geq \delta/2\theta$. Then $\delta (1 - \theta) / (2\theta (1 - \delta))$.

In the proof that follows, we verify that the proposed solution is the unique equilibrium.

Proof of Proposition 1. The proof is broken into two parts. In the first part we verify that the proposed solution is an equilibrium, and then we verify that there are no equilibria in the second part.

The first part of the proof consists of two steps. Step 1: we show that the proposal strategy is optimal. Step 2: we show that the responder's strategy is optimal. In each step, it is sufficient to consider a single period deviation.

Step 1:

Note that the strategies dictate that the responder j accept a proposal s_i if and only if $s_i \in A_j^* = [p_i\theta, s_i^*]$. Since the proposer i always prefers a greater share of the prize, s_i^* is the maximum he can possibly propose and still have player j accept. The proposer would never find it optimal to make an offer that the responder will reject since he would get a lesser share of the prize if the roles of the proposer and responder are reversed. Therefore, s_i^* is the optimal proposal in each round.

Step 2:

To show that the responder's strategy is optimal, we will examine two cases corresponding to the player's effort choices: Case 1: $e \in \mathcal{A}_i(\delta)$ or $e \in \mathcal{N}(\delta)$ and $\delta < \theta$, and Case 2: $e \in \mathcal{A}_j(\delta)$ or $e \in \mathcal{N}(\delta)$ and $\delta \geq \theta$. In Case 1, we show that the responder i prefers withdrawal to delay given any proposal s_j. We then show that acceptance is preferred to withdrawal if $s_j \in A_i^*$ and vice versa if $s_j \notin A_i^*$. In Case 2, we show that the responder i prefers delay to withdrawal given any proposal s_j. We then show that acceptance is preferred to delay if $s_j \in A_i^*$ and vice versa if $s_j \notin A_i^*$.

Case 1: $e \in \mathcal{A}_i(\delta)$ or $e \in \mathcal{N}(\delta)$ and $\delta < \theta$.

Note that delay yields a payoff of $\delta\sigma_i^* V$, while withdrawal yields a payoff of $p_i\theta V$. We demonstrate that withdrawal is preferred to delay by manipulating the inequality

$$(1.1)\ \ \delta\sigma_i^* \leq p_i\theta$$

to show that it is equivalent to the conditions of Case 1. Since the structure of the payoffs varies based on the effort choices, we consider two subcases: Subcase 1.1: $e \in \mathcal{A}_i(\delta)$ and Subcase 1.2: $e \in \mathcal{N}(\delta)$ and $\delta < \theta$.

Subcase 1.1: $e \in \mathcal{A}_i(\delta)$.

In this case, (1) becomes

$$\begin{aligned}
\frac{\delta(1-\delta)}{2-\delta} + \frac{\delta}{2-\delta}p_i\theta &\leq p_i\theta \\
\frac{\delta(1-\delta)}{2-\delta} &\leq \frac{2(1-\delta)}{2-\delta}p_i\theta \\
\frac{\delta}{2\theta} &\leq p_i.
\end{aligned}$$

This holds based on the definition of $\mathcal{A}_i(\delta)$ and Lemma 2.

Subcase 1.2: $e \in \mathcal{N}(\delta)$ and $\delta < \theta$.

In this case, (1) becomes

$$\begin{aligned} \frac{\delta(1-\theta)}{2} + \delta p_i\theta &\leq p_i\theta \\ \frac{\delta(1-\theta)}{2} &\leq (1-\delta)\,p_i\theta \\ \frac{\delta}{2\theta}\frac{1-\theta}{1-\delta} &\leq p_i. \end{aligned}$$

This holds based on the definition of $\mathcal{N}(\delta)$.

Thus, the responder i prefers withdrawal to delay.

To show that acceptance is preferred to withdrawal if $s_j \in A_i^*$ and vice versa if $s_j \notin A_i^*$, note that in this case, $A_i^* = [p_j\theta, 1 - p_i\theta]$. Clearly, player i prefers agreement for any $s_j < 1 - p_i\theta$ and prefers conflict otherwise. If player i accepts an offer $s_j < p_j\theta$, then player j will impose conflict after agreement is reached, and so player i would be equally well off withdrawing given such an offer. Therefore, player i's response strategy is optimal in this case.

Case 2: $e \in \mathcal{A}_j(\delta)$ or $e \in \mathcal{N}(\delta)$ and $\delta \geq \theta$.

Similar to the previous case, we demonstrate that delay is preferred to withdrawal by manipulating the inequality

(1.2) $\delta\sigma_i^* \geq p_i\theta$

to show that it is equivalent to the conditions of Case 2. As before, the structure of the payoffs varies based on the effort choices, we consider two subcases: Subcase 1.1: $e \in \mathcal{A}_j(\delta)$

and Subcase 1.2: $e \in \mathcal{N}(\delta)$ and $\delta \geq \theta$.

Subcase 2.1: $e \in \mathcal{A}_j(\delta)$.

In this case, (2) becomes

$$\begin{aligned}
\frac{\delta(1-\theta)}{2-\delta} + \frac{\delta}{2-\delta} p_i\theta &\geq p_i\theta \\
\frac{\delta(1-\theta)}{2-\delta} &\geq \frac{2(1-\delta)}{2-\delta} p_i\theta \\
\frac{\delta}{2\theta}\frac{1-\theta}{1-\delta} &\geq p_i.
\end{aligned}$$

This holds based on the definition of $\mathcal{A}_j(\delta)$ and Lemma 2.

Subcase 2.2: $e \in \mathcal{N}(\delta)$ and $\delta \geq \theta$.

In this case, (2) becomes

$$\begin{aligned}
\frac{\delta}{2} &\geq p_i\theta \\
\frac{\delta}{2\theta} &\geq p_i.
\end{aligned}$$

This holds based on the definition of $\mathcal{N}(\delta)$.

Thus, the responder i prefers delay to withdrawal.

To show that acceptance is preferred to delay if $s_j \in A_i^*$ and vice versa if $s_j \notin A_i^*$, we will again consider Subcases 2.1 and 2.2. In each subcase, we verify that the proposal s_j^* is such that $1 - s_j^* = \delta\sigma_i^*$, so that delay is preferred to agreement if $s_j < s_j^*$ and vice versa if $s_j > s_j^*$.

Subcase 2.1: $e \in \mathcal{A}_j(\delta)$.

In this case, player i's share $1-s_j^*$ given the optimal proposal is

$$\begin{aligned} 1-\left(\frac{2(1-\delta)}{2-\delta}+\frac{\delta}{2-\delta}p_j\theta\right) &= 1-\frac{\delta}{2-\delta}\theta-\frac{2(1-\delta)}{2-\delta}+\frac{\delta}{2-\delta}p_i\theta \\ &= \frac{\delta(1-\theta)}{2-\delta}+\frac{\delta}{2-\delta}p_i\theta \\ &= \delta\sigma_i^*. \end{aligned}$$

Subcase 2.2: $e \in \mathcal{N}(\delta)$ and $\delta \geq \theta$.

In this case, player i's share $1-s_j^*$ given the optimal proposal is

$$\begin{aligned} 1-\left(\frac{2-\delta}{2}\right) &= \frac{\delta}{2} \\ &= \delta\sigma_i^*. \end{aligned}$$

Lastly, as argued in Case 1, if player i is indifferent between accepting and withdrawing if the offer s_j is such that $s_j < p_j\theta$. Thus, since delay is preferred to withdrawal, delay is also preferred to acceptance. Therefore, player i's response strategy is optimal in this case.

We conclude that the proposed solution is an equilibrium.

Next we show that this equilibrium is the unique subgame perfect equilibrium of the bargaining subgame. The proof is similar to that found in Appendix 1 of Binmore, Shaked, and Sutton (1989). In their paper, they prove a similar result given that the order in which players make proposals is nonrandom and only one player possesses an outside option. In our framework, the order in which players make proposals is random and each player possesses an outside option. This does not change the method of proof, though it does create for additional cases which must be exhausted.[14]

Given the existence of equilibrium as verified in Part 1 of this proof, there exists a minimum share m_i and maximum share M_i of the prize that player i receives in any subgame perfect equilibrium conditional on player i being the current proposer. Given these shares, the smallest share that the responder j can obtain by rejecting an offer is $\delta\left(m_j + (1 - M_i)\right)/2$, which is the discounted expectation given that j receives his minimum share when he is the proposer and i receives his maximum share when he is the proposer. Similarly, the largest share that the responder j can obtain by rejecting an offer is $\delta\left(M_j + (1 - m_i)\right)/2$, which is the discounted expectation given that j receives his maximum share when he is the proposer and i receives his minimum share when he is the proposer. It follows that the responder j should accept any offer $s_i \geq p_i\theta$ such that $1 - s_i > \max\left\{\delta\left(M_j + (1 - m_i)\right)/2, p_j\theta\right\}$ and should not accept any offer s_i such that $1 - s_i < \max\left\{\delta\left(m_j + (1 - M_i)\right)/2, p_j\theta\right\}$. These place a lower and upper bound on accepted proposals and thus on m_i and M_i, as follows:

$$\begin{aligned} m_i &\geq 1 - \max\left\{\delta\left(M_j + (1 - m_i)\right)/2, p_j\theta\right\} \\ 1 - M_i &\geq \max\left\{\delta\left(m_j + (1 - M_i)\right)/2, p_j\theta\right\}. \end{aligned}$$

The system of inequalities above actually consists of four inequalities since $i \in \{1, 2\}$. This gives rise to eight cases to check for uniqueness corresponding to which term is larger in each of the maximum operators. Due to the symmetry of the problem, it is sufficient to check the following six distinct cases: (i) $\delta\left(m_i + (1 - M_j)\right)/2 \geq p_i\theta$ for each player i, (ii) $\delta\left(m_j + (1 - M_i)\right)/2 < p_j\theta < \delta\left(M_j + (1 - m_i)\right)/2$ and $\delta\left(m_i + (1 - M_j)\right)/2 \geq p_i\theta$, (iii) $p_j\theta \geq \delta\left(M_j + (1 - m_i)\right)/2$ and $\delta\left(m_i + (1 - M_j)\right)/2 \geq p_i\theta$, (iv) $\delta\left(m_i + (1 - M_j)\right)/2 <$

[14] The proof found in Binmore, Shaked, and Sutton (1989) requires the exhaustion of three possible cases, while our proof requires six cases.

$p_i\theta < \delta\left(M_i + (1 - m_j)\right)/2$ for each player i, (v) $p_j\theta \geq \delta\left(M_j + (1 - m_i)\right)/2$ and $\delta\left(m_i + (1 - M_j)\right)/2 < p_i\theta < \delta\left(M_i + (1 - m_j)\right)/2$, and (vi) $p_i\theta \geq \delta\left(M_i + (1 - m_j)\right)/2$ for each player i. In each of these cases we show that either $m_i = M_i$ for each player i, or $m_i > M_i$ for some player i. If the former is true, then the subgame perfect equilibrium share is unique, whereas the latter is a contradiction, implying that the case cannot occur.

Case (i): $\delta\left(m_i + (1 - M_j)\right)/2 \geq p_i\theta$ for each player i.

In this case, the system of inequalities is

$$\begin{aligned}
m_i &\geq 1 - \delta\left(M_j + (1 - m_i)\right)/2 \\
1 - M_i &\geq \delta\left(m_j + (1 - M_i)\right)/2 \\
m_j &\geq 1 - \delta\left(M_i + (1 - m_j)\right)/2 \\
1 - M_j &\geq \delta\left(m_i + (1 - M_j)\right)/2.
\end{aligned}$$

This reduces to

$$\begin{aligned}
\frac{2-\delta}{\delta}\left(1 - m_i\right) &\leq M_j \\
1 - M_i &\geq \frac{\delta}{2-\delta} m_j \\
\frac{2-\delta}{\delta}\left(1 - m_j\right) &\leq M_i \\
1 - M_j &\geq \frac{\delta}{2-\delta} m_i.
\end{aligned}$$

Combining the first and fourth inequalities yields

$$\begin{aligned}\frac{2-\delta}{\delta}\left(1-m_i\right) &\leq 1-\frac{\delta}{2-\delta}m_i \\ \frac{2\left(1-\delta\right)}{\delta} &\leq \frac{4\left(1-\delta\right)}{\delta\left(2-\delta\right)}m_i \\ \frac{2-\delta}{2} &\leq m_i.\end{aligned}$$

By symmetry we obtain $m_j \geq \left(2-\delta\right)/2$, which when substituted into the second inequality yields

$$\begin{aligned}M_i &\leq 1-\frac{\delta}{2-\delta}\frac{2-\delta}{2} \\ M_i &\leq \frac{2-\delta}{2}.\end{aligned}$$

Thus, we have $\left(2-\delta\right)/2 \leq m_i \leq M_i \leq \left(2-\delta\right)/2$, and so $m_i = M_i$ for each player i.

Case (ii): $\delta\left(m_j+\left(1-M_i\right)\right)/2 < p_j\theta < \delta\left(M_j+\left(1-m_i\right)\right)/2$ and $\delta\left(m_i+\left(1-M_j\right)\right)/2 \geq p_i\theta$.

In this case, the system of inequalities is

$$\begin{aligned}m_i &\geq 1-\delta\left(M_j+\left(1-m_i\right)\right)/2 \\ 1-M_i &> \delta\left(m_j+\left(1-M_i\right)\right)/2 \\ m_j &\geq 1-\delta\left(M_i+\left(1-m_j\right)\right)/2 \\ 1-M_j &\geq \delta\left(m_i+\left(1-M_j\right)\right)/2,\end{aligned}$$

which reduces to

$$\begin{aligned}
\frac{2-\delta}{\delta}(1-m_i) &\leq M_j \\
1-M_i &> \frac{\delta}{2-\delta}m_j \\
\frac{2-\delta}{\delta}(1-m_j) &\leq M_i \\
1-M_j &\geq \frac{\delta}{2-\delta}m_i.
\end{aligned}$$

A repetition of the computations in Case (i) yields $(2-\delta)/2 \leq m_i \leq M_i < (2-\delta)/2$, a contradiction. Therefore, this case cannot occur.

Case (iii): $p_j\theta \geq \delta(M_j + (1-m_i))/2$ and $\delta(m_i + (1-M_j))/2 \geq p_i\theta$.

In this case, the system of inequalities is

$$\begin{aligned}
m_i &\geq 1-p_j\theta \\
1-M_i &\geq p_j\theta \\
m_j &\geq 1-\delta(M_i + (1-m_j))/2 \\
1-M_j &\geq \delta(m_i + (1-M_j))/2,
\end{aligned}$$

which reduces to

$$\begin{aligned}
m_i &\geq 1-p_j\theta \\
1-p_j\theta &\geq M_i \\
\frac{2-\delta}{\delta}(1-m_j) &\leq M_i \\
1-M_j &\geq \frac{\delta}{2-\delta}m_i.
\end{aligned}$$

From the first two inequalities we immediately obtain $m_i \leq M_i \leq m_i$, so that $m_i = M_i$. Substituting this equality into the third and fourth inequalities yields

$$\begin{aligned} \frac{2-\delta}{\delta}\left(1-M_j\right) &\geq \frac{2-\delta}{\delta}\left(1-m_j\right) \\ m_j &\geq M_j. \end{aligned}$$

Thus, $m_j = M_j$.

Case (iv): $\delta\left(m_i + \left(1-M_j\right)\right)/2 < p_i\theta < \delta\left(M_i + \left(1-m_j\right)\right)/2$ for each player i.

In this case, the system of inequalities is

$$\begin{aligned} m_i &\geq 1-\delta\left(M_j + \left(1-m_i\right)\right)/2 \\ 1-M_i &> \delta\left(m_j + \left(1-M_i\right)\right)/2 \\ m_j &\geq 1-\delta\left(M_i + \left(1-m_j\right)\right)/2 \\ 1-M_j &> \delta\left(m_i + \left(1-M_j\right)\right)/2, \end{aligned}$$

which reduces to

$$\begin{aligned} \frac{2-\delta}{\delta}\left(1-m_i\right) &\leq M_j \\ 1-M_i &> \frac{\delta}{2-\delta}m_j \\ \frac{2-\delta}{\delta}\left(1-m_j\right) &\leq M_i \\ 1-M_j &> \frac{\delta}{2-\delta}m_i. \end{aligned}$$

This may be solved as in Case (ii) to obtain a similar contradiction $\left(2-\delta\right)/2 < m_i \leq M_i <$

$(2-\delta)/2$. Therefore, this case cannot occur.

Case (v) $p_j\theta \geq \delta(M_j + (1-m_i))/2$ and $\delta(m_i + (1-M_j))/2 < p_i\theta < \delta(M_i + (1-m_j))/2$.

In this case, the system of inequalities is

$$\begin{aligned} m_i &\geq 1 - p_j\theta \\ 1 - M_i &\geq p_j\theta \\ m_j &\geq 1 - \delta(M_i + (1-m_j))/2 \\ 1 - M_j &> \delta(m_i + (1-M_j))/2 \end{aligned}$$

which reduces to

$$\begin{aligned} m_i &\geq 1 - p_j\theta \\ 1 - p_j\theta &\geq M_i \\ \frac{2-\delta}{\delta}(1-m_j) &\leq M_i \\ 1 - M_j &> \frac{\delta}{2-\delta} m_i. \end{aligned}$$

As in Case (iii), we immediately obtain $m_i = M_i$. Substituting this into the third and fourth inequalities yields

$$\begin{aligned} \frac{2-\delta}{\delta}(1-M_j) &> \frac{2-\delta}{\delta}(1-m_j) \\ m_j &> M_j. \end{aligned}$$

This is a contradiction, so this case cannot occur.

Case (vi): $p_i\theta \geq \delta\left(M_i + (1-m_j)\right)/2$ for each player i.

In this case, the system of inequalities is

$$\begin{aligned} m_i &\geq 1 - p_j\theta \\ 1 - M_i &\geq p_j\theta. \\ m_j &\geq 1 - p_i\theta \\ 1 - M_j &\geq p_i\theta. \end{aligned}$$

This immediately yields $m_i \leq M_i \leq m_i$ for each i, so $m_i = M_i$. Therefore, the subgame perfect equilibrium described in the proposition is the unique subgame perfect equilibrium of the bargaining subgame. ■

Note that in equilibrium, if $e \in \mathcal{A}_i(\delta)$, then player i's threat of disagreement is to impose conflict, while player j's threat is to delay bargaining. This is because of i's advantage in the underlying contest and so the conflict option is more desirable. Moreover, this results in i receiving a larger share of the prize in equilibrium, thereby increasing the cost of delay due to discounting. When $e \in \mathcal{N}(\delta)$, both players have the same threat point. Either they both find it better to impose conflict (if they are impatient) or they both prefer to delay (if they are patient) when the other makes an unsatisfactory offer as the proposer. This is not surprising; the cost due to discounting is higher for impatient players, and so their value of delaying is lower.

It is worth pointing out that if $\delta \geq 2\theta$, which requires that $\theta < 1/2$, then $\mathcal{N}(\delta) = [0, E] \times [0, E]$, the entire strategy space. The interpretation is that if players are very patient and conflict is very inefficient, then players always find it better to delay rather than impose conflict. This further implies that the share each player receives in equilibrium is independent of the players' effort choices.

Regardless of effort choices, the proposer always makes an offer so that agreement is reached in the first round of bargaining, as is typically the case with this alternating offers bargaining setup with complete information. Upon inspection, it becomes apparent that when $e \in \mathcal{A}_i(\delta)$ or $\delta < \theta$, the equilibrium strategies dictate that at least one player withdraws from bargaining and imposes conflict when they are offered extremely large shares of the prize by the proposer. Such a reaction may seem unreasonable, but recall that we adopted the convention that players choose to withdraw rather than accept when the one of the players will impose conflict in the third stage anyway. Intuitively, the responder realizes that the proposer would receive a lower payoff than under conflict, and so he anticipates that the proposer will impose conflict and preempts this action. We find that there is little to be gained in specifying all possible equilibria that lead to the same realized outcome in all subgames. Regardless of how the behavior is accommodated, such actions will always lie off the equilibrium path, and thus would never be observed in actual play.

Let us briefly discuss the patience of the players, measured by $\delta \geq \theta$ and $\delta < \theta$. We find the former case to be far more interesting, as the bargaining is meant to take place rapidly, and the latter would typically require that players be absurdly impatient. Further, we conduct analysis as $\delta \to 1$, necessitating that $\delta > \theta$. We thus focus primarily on the former case, though we analyze the latter for completeness.

1.4 Effort Decisions

Next we analyze the first stage of the game to find the equilibrium effort choices. Before we characterize the efforts induced by the bargaining scheme of Section 3, let us first characterize the equilibrium of the underlying contest.

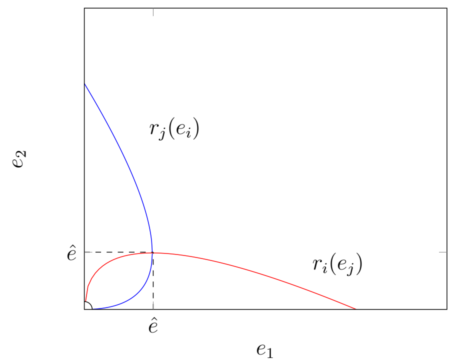


Figure 1.2: Effort best response functions for conflict

Proposition 2. *Consider a contest with payoffs of the form* $p_i \theta V \frac{1}{2-\delta} - c(e_i)$ *for some* $\delta \leq 1$. *Then if Assumptions 1-4 hold, there exists a unique pure strategy equilibrium. This equilibrium is symmetric with* $e_1 = e_2 = \widehat{e}$.

The proof of Proposition 2 requires the following technical definitions. These concepts were introduced by Reny (1999). Let $G = (X, u)$ denote an n player game with strategy space X and payoff vector u.

Definition 1. *A game* G *is* better reply secure *if whenever* x^* *is not an equilibrium and* (x^*, u^*) *is in the closure of the graph of* G, *there exists for some player* i *a strategy* x_i *and a neighborhood* V *of* x^*_{-i} *such that for all* $x_{-i} \in V$, $u_i(x_i, x_{-i}) > u^*_i$.

Definition 2. *A game* G *satisfies* payoff security *if for all* $\varepsilon > 0$ *and all* $x \in X$, *there exists for each player* i *a deviation* $x'_i \in X_i$ *and a neighborhood* $\mathcal{N}(x_{-i})$ *of* x_{-i} *such that* $u_i(x'_i, z) > u_i(x) - \varepsilon$ *for all* $z \in \mathcal{N}(x_{-i})$.

Definition 3. *A game* G *is* reciprocal upper semicontinuous *if for all* (x^*, u^*) *in the closure of the graph of* G, *if* $u_i(x^*) \leq u^*_i$ *for all players* i, *then* $u_i(x^*) = u^*_i$ *for all* i.

Reny shows that payoff security together with reciprocal upper semicontinuity imply better reply security. We will use the following theorem due to Reny (1999) to prove existence of equilibrium,

Fact 1 (Reny 1999)**.** *A compact, quasiconcave game that is better reply secure possesses a pure strategy Nash equilibrium.*

This result can be used to guarantee the existence of a mixed strategy Nash equilibrium by applying these conditions to the mixed extension of the game. The following fact is useful for proving that the mixed extension of a game is reciprocally upper semicontinuous.

Fact 2 (Reny 1999)**.** *If the sum of the payoffs* $\sum_i u_i$ *is upper semi continuous, then the mixed extension of the game is reciprocally upper semicontinuous.*

Proof of Proposition 2. First we prove existence. We show that the game is quasiconcave first. By assumption 3 we have for all e such that $e_1 + e_2 > 0$

$$\begin{aligned} \frac{d^2}{de_i^2}\left(p_i \theta V - c\left(e_i\right)\right) &< 0 \\ p'' \theta V &< c''. \end{aligned}$$

Since $2 - \delta \geq 1$, then for any e such that $p'' > 0$, we have $p''\theta V / \left(2 - \delta\right) < p''\theta V < c''$. If $p'' \leq 0$, then $p''\theta V / \left(2 - \delta\right) \leq 0 < c''$. Therefore the payoffs are strictly concave when $e_i + e_j > 0$. If $f\left(0\right) > 0$, then it follows that the game is strictly concave. Otherwise, if $f\left(0\right) = 0$, then we need to check that the game is quasiconcave for $e_j = 0$. In this case, utility is strictly decreasing in e_i for all $e_i > 0$, so we conclude that the game is strictly quasiconcave.

Next we show that the game is reciprocally upper semicontinuous. The condition is satisfied trivially for all points of continuity, so we need only check the point $e^* = (0, 0)$ if $f\left(0\right) = 0$. Note that any point in the closure of the graph $\left(e^*, u^*\right)$ is such that $u_1^* + u_2^* = \theta V / \left(2 - \delta\right)$. Since $u_i\left(e^*\right) = \theta V / 2\left(2 - \delta\right)$, then if $u_i\left(e^*\right) \leq u_i^*$ for each i, then $u_i\left(e^*\right) = u_i^*$ for each i.

Similarly, payoff security is trivially satisfied at all points of continuity. Note that

$$\lim_{e \to 0} u_i^C(e, 0) \geq u_i^C(0, 0),$$

so for all $\varepsilon > 0$, for some sufficiently small e_i', for all e_j in the neighborhood $[0, e_i')$, we have $u_i^C(e_i', e_j) > u_i^C(0, 0) - \varepsilon$. Therefore the game is payoff secure. We conclude that the game is quasiconcave and better reply secure, and so the game possesses a pure strategy Nash equilibrium.

Note that the game is strictly quasiconcave, so given $e_j > 0$, there exists a unique maximizer $r_i(e_j)$, and from the theorem of the maximum, the function r_i must be continuous. Due to a potential discontinuity at $(0, 0)$, $r_i(0)$ may not exist.

We now prove that the equilibrium must be unique and symmetric. Note that the first order condition for each player i when $e_1 + e_2 > 0$ is

$$\frac{f'(e_i) f(e_j)}{(f(e_i) + f(e_j))^2} \frac{\theta V}{2 - \delta} - c'(e_i) = 0.$$

We characterize the reaction function via the implicit function theorem:

$$\frac{\partial r_i(e_j)}{\partial e_j} = -\frac{f'(r_i(e_j)) f'(e_j) \frac{f(r_i(e_j)) - f(e_j)}{(f(r_i(e_j)) + f(e_j))^3} \frac{\theta V}{2-\delta}}{\frac{d^2}{de_i^2}\left(p_i \frac{\theta V}{2-\delta} - c\right)}.$$

Note that the numerator is positive if and only if $r_i(e_j) > e_j$. The denominator is negative as we just showed, so we conclude that the reaction function $r_i(e_j)$ is strictly increasing if and only if $r_i(e_j) > e_j$, and strictly decreasing if and only if $0 < r_i(e_j) < e_j$. If $r_i(e_j) = 0 < e_j$, then $r_i\left(e_j'\right) = 0$ for all $e_j' > e_j$, and so is weakly decreasing.

At any interior symmetric equilibrium (e, e), the first order conditions must be satisfied, which yield

$$\frac{f'(e)}{4f(e)}\frac{\theta V}{2-\delta}-c'(e)=0.$$

If we differentiate this expression with respect to e we get

$$\begin{aligned}\frac{d}{de}\left(\frac{f'(e)}{4f(e)}\theta V-c'(e)\right) &= \frac{f''(e)f(e)-(f'(e))^2}{4(f(e))^2}\frac{\theta V}{2-\delta}-c''(e)\\ &= \frac{d^2}{de_i^2}\left(p_i\frac{\theta V}{2-\delta}-c\right)\bigg|_{e_1=e_2=e}<0.\end{aligned}$$

Therefore, the first order condition may only be satisfied at a single interior symmetric strategy profile. Since Assumption 4 implies that $(0,0)$ is not an equilibrium, then there is a unique symmetric equilibrium if one exists.

Suppose that $r_i(x)\neq x$ for all x, then from above, either $r_i(x)>x$ for all x, implying the contradiction that $r_i(E)>E$, or $r_i(x)<x$ for all x, in which case $r_i(r_j(x))<x$ for all x, so that no equilibrium exists, arriving at another contradiction. Thus, there exists a $\widehat{e}$ such that $r_i(\widehat{e})=\widehat{e}$. It follows that $(\widehat{e},\widehat{e})$ is the unique symmetric Nash equilibrium.

Next, we show that there are no other asymmetric equilibria.

From above, $r_i(x)=x$ if and only if $x=\widehat{e}$. If $r_i(x)>x$ for some $x>\widehat{e}$, then since $r_i(E)\leq E$, continuity implies that $r_i(y)=y$ for some $y>x$, which would imply the existence of an additional symmetric equilibrium. Therefore, $r_i(x)<x$ for all $x>\widehat{e}$. If $r_i(x)<x$ for some $x<\widehat{e}$, then r_i would be decreasing for all $x<\widehat{e}$, implying that $r_i(\widehat{e})<\widehat{e}$, a contradiction. Therefore, $r_i(x)>x$ for all $x<\widehat{e}$. It follows that r_i achieves its maximum at $e_j=\widehat{e}$ and that it is quasiconcave.

Suppose that there is an asymmetric equilibrium (x,y) with $x<y$. If $x>\widehat{e}$, then $r_j(x)<x$,

so y cannot be a best response to x. Similarly, if $y < \widehat{e}$, then $r_i(y) > y$, so x cannot be a best response. Thus, it must be that $x < \widehat{e} < y$. Since r_j is maximized when $e_i = \widehat{e}$, achieving its unique maximum at $r_j(\widehat{e}) = \widehat{e}$, it follows that $y > r_i(x)$ for all x, and so cannot be a best response. We conclude that no asymmetric equilibria exist. ■

Let $r_i^\delta(\cdot)$ denote the best response function for player i in the contest with payoffs given as in Proposition 2.

Corollary 1. *The best response function $r_i^\delta(e_j)$ is quasiconcave and maximized at $e_j = \widehat{e}$.*

This corollary is shown in the proof of Proposition 2. Figure 1.2 displays an example of best response functions for such a contest, with the unique interior solution at the peak of the best response function. The functional form used for Figure 1.2 is a simple Tullock lottery contest, and so the best response is undefined at zero. When the response function is defined at zero, the best response must be positive, a consequence of Assumption 4. Inclusion of the term term $1/(2-\delta)$ will be useful for application in later proofs, as this term appears in the payoffs resulting from the bargaining process. Note that $\delta = 1$ is allowed in the statement of Proposition 2, so this result guarantees the existence of a unique, symmetric pure strategy equilibrium in the underlying contest of our game. The corollary will be used in the proof of our main result.

It will be helpful to summarize the expected utilities induced by the noncooperative bargaining solution.

$$u_i(e_1, e_2) = \begin{cases} \frac{1-\theta}{2-\delta}V + p_i \frac{\theta}{2-\delta}V - c(e_i) & \text{if} \quad e \in \mathcal{A}_j(\delta) \\ \frac{1-\theta}{2}V + p_i \theta V - c(e_i) & \text{if} \quad e \in \mathcal{N}(\delta) \text{ and } \delta < \theta \\ \frac{V}{2} - c(e_i) & \text{if} \quad e \in \mathcal{N}(\delta) \text{ and } \delta \geq \theta \\ \frac{1-\delta}{2-\delta}V + p_i \frac{\theta}{2-\delta}V - c(e_i) & \text{if} \quad e \in \mathcal{A}_i(\delta) \end{cases}.$$

It is easily shown that for some values of e_j, these payoff functions are not quasiconcave in each player i's own strategy. As a result, this game may not admit a pure strategy Nash equilibrium for $\theta \leq \delta < 2\theta$. Indeed the only potential pure strategy equilibrium involves $e_1 = e_2 = 0$, which requires that conflict is very destructive ($\theta < 1/2$) and either that players are very patient ($\delta \geq 2\theta$) or that players have a sufficiently large probability of winning the conflict when choosing zero effort ($f(0)$ sufficiently large).

Figure 1.3 depicts the best response functions for efforts for these utilities when $\theta \leq \delta < 2\theta$. It can be seen clearly that there is no pure strategy equilibrium. These response functions can be obtained by plotting Figures 1.1b and 1.2 in the same graph, then replacing segments of the best response functions with the boundaries of $\mathcal{N}(\delta)$. The reason is that when the graph of a conflict best response function falls within $\mathcal{N}(\delta)$, payoffs are strictly decreasing in efforts, and so the true best response would be to the boundary of $\mathcal{N}(\delta)$. When the graph of the conflict best response function falls within $\mathcal{A}_1(\delta) \cup \mathcal{A}_2(\delta)$, the marginal benefit of effort is of the same functional form as that of the underlying conflict, and so there is no local profitable deviation. Except for a small region, the best response functions with and without the noncooperative bargaining solution coincide whenever the graph is in $\mathcal{A}_1(\delta) \cup \mathcal{A}_2(\delta)$. The following proposition formalizes the nonexistence result when players are patient and conflict is not too inefficient.

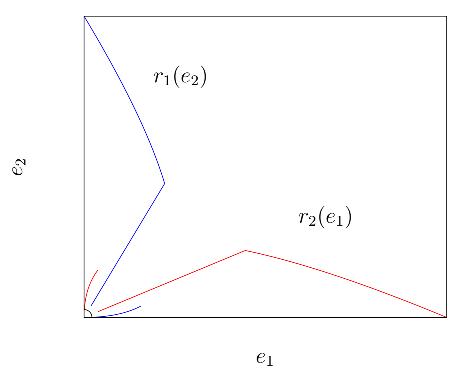


Figure 1.3: Effort best responses under noncooperative bargaining.

Proposition 3. *Suppose that Assumptions 1-4 hold. Then for all* $1/2 < \theta \leq \delta < 1$ *no pure strategy equilibrium exists.*

Proof of Proposition 3. Suppose that $\delta \geq \theta$. Note that there is a potential boundary solution with $e_i = 0$ for some player i. We first show that no interior solution exists, then we check the boundary.

Clearly no interior solution exists with $e \in \mathcal{N}(\delta)$ since payoffs are strictly decreasing in efforts in that region, so we check the interior of $\mathcal{A}_1(\delta) \cup \mathcal{A}_2(\delta)$. Proposition 2 implies that an equilibrium in this region must be symmetric, implying that $p_i^* = p_j^* = 1/2 < \delta/2\theta$. Therefore, this solution is in $\mathcal{N}(\delta)$, a contradiction.

Lastly, we must consider $e_j = 0$. If no best response function $r_i(0)$ exists at 0, then there is no equilibrium with a player choosing zero. Suppose that $r_i^\delta(0)$ exists. If $r_i^\delta(0) > 0$ and $r_j^\delta\left(r_i^\delta(0)\right) = 0$, then by symmetry both $\left(0, r_j^\delta(0)\right)$ and $\left(r_i^\delta(0), 0\right)$ must be equilibria of the underlying contest. But Proposition 2 proves that there is a unique equilibrium, so it must be that $r_i(0) = 0$.

If $\theta > \frac{1}{2}$, by Assumption 4, we have for some $e > 0$

$$p(e,0)\,\theta V - c(e) > \frac{V}{2},$$

and further since $c(e) > 0$, we have that

$$\begin{aligned} p(e,0)\,\theta &> \frac{1}{2}, \\ p(e,0) &> \frac{1}{2\theta} \\ &> \frac{\delta}{2\theta}. \end{aligned}$$

Therefore, $u_i(e,0) > u_i(0,0)$, contradicting 0 as a best response. We conclude that there is no pure strategy equilibrium. ■

The next proposition characterizes the first stage equilibrium for $\theta \leq 1/2$.

Proposition 4. *Suppose that Assumptions 1-4 hold. If $\delta \geq \theta$ and $\theta \leq 1/2$, then any pure strategy equilibrium must have $e_1 = e_2 = 0$. If $f(0) > 0$ then $\widehat{e}_1 = \widehat{e}_2 = 0$ may be a Nash equilibrium. If $f(0) = 0$ and $\delta < 2\theta$, then $e_i = e_j = 0$ is not a Nash equilibrium. For $\delta \geq 2\theta$, $\widehat{e}_1 = \widehat{e}_2 = 0$ is a dominant strategy equilibrium.*

Proof of Proposition 4. The first statement follows from the proof of the previous proposition. The last statement follows from the utility being strictly decreasing in efforts when $\delta \geq 2\theta$.

We provide a parametric example to show that when $f(0) > 0$, both players choosing zero effort may or may not be an equilibrium. Consider $f(e) = e+\alpha$, where $\alpha > 0$, and $c(e) = \beta e$, $\beta > 0$, and $V = 1$. It follows that Assumptions $1-3$ are satisfied.

Consider $\theta \leq 1/2$. We solve for the best response function conditional on $p_i > \delta/2\theta$.

$$\begin{aligned} \frac{e_j+\alpha}{(e_i+e_j+2\alpha)^2}\frac{\theta}{2-\delta} &= \beta \\ r_i^{\delta}(e_j) &= \sqrt{\frac{\theta(e_j+\alpha)}{\beta(2-\delta)}} - e_j - 2\alpha. \end{aligned}$$

Evaluating at zero yields

$$r_i^{\delta}(0) = \sqrt{\frac{\alpha\theta}{\beta(2-\delta)}} - 2\alpha.$$

Note that for $\beta' = \frac{\theta}{4\alpha(2-\delta)}$, $r_i^{\delta}(0) = 0$. It follows that for $\beta = \beta'$, the best response is zero, and so $e_i = e_j = 0$ is an equilibrium. Note further that $\lim_{\beta\to 0} r_i^\delta(0) = \infty$, and so for some sufficiently small β, $p_i\left(r_i^\delta(0), 0\right) > \frac{\delta}{2\theta}$. For β such that $p_i\left(r_i^\delta(0), 0\right) > \frac{\delta}{2\theta}$, the payoff at the best response to zero is

$$\begin{aligned} u_i\left(r_i(0), 0\right) &= \frac{1-\delta}{2-\delta} + \left(1 - \frac{\alpha}{\sqrt{\frac{\alpha\theta}{\beta(2-\delta)}}}\right)\frac{\theta}{2-\delta} - \beta\left(\sqrt{\frac{\alpha\theta}{\beta(2-\delta)}} - 2\alpha\right) \\ &= \frac{1+\theta-\delta}{2-\delta} - 2\sqrt{\frac{\alpha\beta\theta}{2-\delta}} - 2\alpha\beta. \end{aligned}$$

Taking the limit as $\beta \to 0$ we find

$$\lim_{\beta\to 0} \frac{1+\theta-\delta}{2-\delta} - 2\sqrt{\frac{\alpha\beta\theta}{2-\delta}} - 2\alpha\beta = \frac{1+\theta-\delta}{2-\delta}.$$

The following are equivalent.

$$\begin{aligned} \frac{1+\theta-\delta}{2-\delta} &> \frac{1}{2} \\ 2+2\theta-2\delta &> 2-\delta \\ 2\theta &> \delta. \end{aligned}$$

Therefore, if $\delta < 2\theta$, we may choose a sufficiently small β such that $r_i^\delta(0) > 0$ is a profitable deviation from 0, so that $e_i = e_j = 0$ is not an equilibrium.

Next we show that when $f(0) = 0$, then $e_i = e_j = 0$ is not a Nash equilibrium. Suppose that $f(0) = 0$. Note that for any $e > 0$, $p(e, 0) = 1 > \delta/2\theta$, so

$$\begin{aligned} u_i(e, 0) &= \frac{1-\delta}{2-\delta}V + \frac{\theta}{2-\delta}V - c(e) \\ &> \frac{V}{2} - c(e). \end{aligned}$$

Since the limit of the right hand side as $e \to 0$ is $V/2$, it follows that there exists an $e > 0$ such $u_i(e, 0) > u_i(0, 0)$, so $e_i = e_j = 0$ is not an equilibrium. ∎

The following proposition verifies that an equilibrium always exists.

Proposition 5. *Suppose that Assumptions 1-4 hold. Then a mixed strategy equilibrium exists for the effort decisions in the first stage when bargaining is conducted with the noncooperative scheme.*

The proof of Proposition 12 mimics the proof of the main theorem in Allison and Lepore (2014).

Proof of Proposition 12. Note that the sum of the payoffs $V - c(e_i) - c(e_j)$ is continuous, so the aforementioned fact implies that the mixed extension of the game is reciprocally upper semicontinuous. The reminder of the proof is devoted to showing that the mixed extension is payoff secure.

Denote by $\mathcal{M}$ the compact set of regular probability measures on $[0, E] \times [0, E]$. Let $\varepsilon 0$, and suppose that $\mu \in \mathcal{M}$. Note that for each player i there exists some strategy $\overline{e}_i$ in the support of μ_i such that

$$\int u_i(\overline{e}_i, e_j)\, d\mu_j \geq \int u_i(e)\, d\mu.$$

If $\overline{e}_i > 0$, continuity of u_i at $\overline{e}_i$ implies that $\lim_n \int u_i(\overline{e}_i, e_j)\, d\mu_j^n = \int u_i(\overline{e}_i, e_j)\, d\mu_j$ for any sequence of probability measures $\mu_j^n \to \mu_j$, and so payoff security is satisfied if $\overline{e}_i > 0$.

Suppose that $\overline{e}_i = 0$. Note that $\lim_{e_i \to 0} u_i(e_i, e_j) \geq u_i(0, e_j)$ for all e_j. Thus, there exists an $e_i' > 0$ and compact subset $K \subset [0, E]$ such that $u_i(e_i', e_j) > u_i(0, e_j) - \varepsilon$ for all $e_j \in K$, where $\mu_j(K) > 1 - \varepsilon$. Define $X = [0, E]$. Choose such an e_i'. Note that

$$\int_K u_i(e_i', e_j)\, d\mu_j > \int_K u_i(\overline{e}_i, e_j)\, d\mu_j - \varepsilon, \text{ and } \mu_j(X \setminus K) < \varepsilon.$$

Defining $M \equiv \sup |u_i| < \infty$ due to the fact that u_i is bounded on X, we get

$$\begin{aligned} \int u_i(e_i', e_j)\, d\mu_j &> \int_K u_i(\overline{e}_i, e_j)\, d\mu_j - \varepsilon + \int_{X \setminus K} u_i(e_i', e_j)\, d\mu_j \\ \int u_i(e_i', e_j)\, d\mu_j - \int u_i(\overline{e}_i, e_j)\, d\mu_j &> \int_{X \setminus K} u_i(e_i', e_j)\, d\mu_j - \int_{X \setminus K} u_i(\overline{e}_j, e_j)\, d\mu_j \\ &> -2M\mu_j(X \setminus K) \\ &> -2M\varepsilon. \end{aligned}$$

Therefore, we have

$$\int u_i(e_i', e_j)\, d\mu_{-i} > \int u_i(\overline{e}_i, e_j)\, d\mu_j - (1 + 2M)\,\varepsilon.$$

Since $u_i(e_i', e_j)$ is continuous in e_j, it follows that

$$\int u_i(e_i', e_j)\, d\mu_j$$

is continuous in μ_j. Therefore, there exists a neighborhood $\mathcal{N}(\mu_j)$ such that for all $\lambda \in$

$\mathcal{N}(\mu_j)$,

$$\int u_i(e_i', e_j)\, d\lambda > \int u_i(e_i', e_j)\, d\mu_j - \varepsilon.$$

Combining this with the previous inequality yields

$$\int u_i(e_i', e_j)\, d\lambda > \int u_i(\bar{e}_i, e_j)\, d\mu_j - 2(1+M)\varepsilon$$

for all $\lambda \in \mathcal{N}(\mu_j)$. Therefore, since $2(1+M)\varepsilon \to 0$ as $\varepsilon \to 0$, we conclude that the mixed extension of the game is payoff secure.

Therefore, Reny's theorem guarantees' that the game possesses a mixed strategy Nash equilibrium. ■

1.5 Comparison with Cooperative Bargaining

Before we may make comparisons, we need to briefly summarize the results of cooperative bargaining. As is well known, when utility is transferrable, all of the well known cooperative solutions (Nash, Kalai-Smorodinsky, egalitarian, split sacrifice, etc.) prescribe the same division of the prize, dividing the surplus, the aggregate gain from settlement, equally between the parties.[15]

These solutions are illustrated in Figure 1.4, with threat point D and utopia point U. The utopia point is the maximum utility each player could receive while the other receives at least their threat payoff. The Nash bargaining solution maximizes the product of the normalized

[15] If the boundary of the utility possibility set is strictly convex, then the cooperative bargaining solutions do not all prescribe the same division.

payoffs, and so we plot the maximal indifference curve for that product. The solution is the point of tangency between the indifference curve and the utility possibility frontier. By definition, the egalitarian solution divides the surplus over the disagreement equally between players, realized as the intersection of the 45 degree line emanating from the disagreement point. The equal sacrifice solution divides the loss relative to the utopia point equally between the players. Thus, the equal sacrifice is obtained by intersecting the 45 degree line emanating inward from the utopia point with the frontier. Lastly, the Kalai-Smorodinsky solution intersects the line connecting the threat and utopia points with the utility possibility frontier. As can be seen, with transferable utility, all these points coincide.

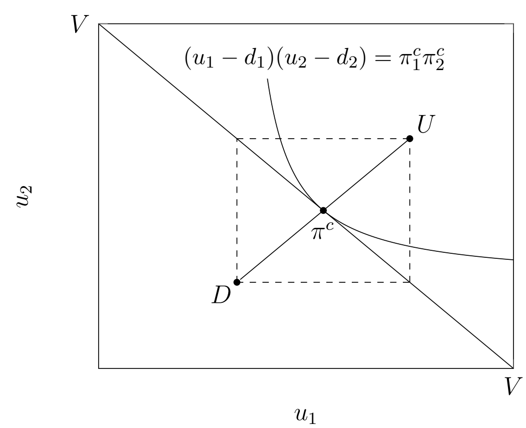


Figure 1.4: Cooperative bargaining solutions.

In this framework, the surplus from agreement is the fraction of the prize that is wasted by conflict, $(1 - \theta)$. Thus, cooperative bargaining solutions give each player their payoff under conflict plus half of this surplus, yielding a share for each player i of

$$s_i^C = \frac{1-\theta}{2} + p_i\theta.$$

Alternatively, it is possible to set a threat point of zero to corresponds to the infinite disagreement of the noncooperative bargaining game, and use the payoffs from conflict instead as constraints on the utility possibility set. Such a specification would be inappropriate in this setting, as the threat point should correspond to the payoffs that players receive in the event of disagreement. In this model, when a cooperative bargaining solution is used, players have a binary choice to either accept the specified division or not. If both players accept, then the prize is divided, and otherwise conflict occurs. We ignore the trivial equilibrium in which both players choose to reject the specified division, as accepting is weakly dominant.

The expected utility induced by cooperative bargaining is therefore

$$u_i^C(e_1, e_2) = \frac{1-\theta}{2}V + p_i \theta V - c(e_i).$$

Note that for $e \in \mathcal{A}_i(\delta)$ for either i, the variable portion of the payoffs under noncooperative bargaining are a constant fraction of those under cooperative bargaining, and so the marginal benefit of effort is strictly less with the former scheme. As players become more patient ($\delta \to 1$) the marginal benefit of effort in these regions converges to the marginal benefit earned under cooperative bargaining. For $e \in \mathcal{N}(\delta)$ and $\delta \geq \theta$, the same holds for the marginal benefit of effort since the payoffs are strictly decreasing under the noncooperative scheme, though the difference is not reconciled as $\delta \to 1$. Only when $e \in \mathcal{N}(\delta)$ and $\delta < \theta$ do the marginal incentives induced by cooperative and noncooperative schemes truly coincide. The following proposition characterizes the equilibrium efforts under cooperative bargaining.

Proposition 6. *Suppose that Assumptions 1-4 hold. Then when players employ a cooperative bargaining solution, there exists a unique pure strategy equilibrium in which each player chooses efforts* $e^C > 0$.

Proof of Proposition 6. From Proposition 2, we know that a unique pure strategy equilibrium exists and must be symmetric. We need only check that $e_1 = e_2 = 0$ is not an equilibrium.

Suppose that $e_j = 0$. First, consider the case that $\theta > 1/2$.

From Assumption 4, there exists an $e > 0$ such that

$$
\begin{aligned}
c(e) &< \left(p_i(e,0)\,\theta - \frac{1}{2}\right)V \\
\frac{V}{2} &< p_i(e,0)\,\theta V - c(e) \\
u_i^C(0,0) &< u_i^C(e,0) - \frac{1-\theta}{2}V - c(e).
\end{aligned}
$$

Thus, we conclude that e is a profitable deviation from 0, so $e_i = e_j = 0$ is not an equilibrium when $\theta > 1/2$.

We now consider the case that $\theta \leq 1/2$.

For $\theta' = 1/2 + \varepsilon$ where $0 < \varepsilon < \theta/2$, Assumption 4 gives us

$$
\begin{aligned}
c(e) &< \left(p_i(e,0)\,\theta' - \frac{1}{2}\right)V \\
\frac{1}{2}V &< p_i(e,0)\,\theta' V - c(e) \\
\frac{1}{2}V &< p(e,0)(\theta' - \theta)V + p(e,0)\,\theta V - c(e) \\
u_i^C(0,0) &< u_i^C(e,0) + p_i(e,0)(\theta' - \theta)V - \frac{1-\theta}{2}V - c(g).
\end{aligned}
$$

In order to show that this $e > 0$ is a profitable deviation from zero, it is sufficient to show that

$$
\begin{aligned}
p(e_i,0)(\theta' - \theta)V - \frac{1-\theta}{2}V &< 0 \\
p_i(e,0)\left(\frac{1}{2} + \varepsilon - \theta\right) &< \frac{1}{2} - \frac{\theta}{2}.
\end{aligned}
$$

The last line is obtained by substituting the definition of θ' and moving the second term to the right hand side. Note that $p_i \leq 1$ and $1/2 + \varepsilon - \theta < 1/2 - \theta/2$ by the assumption on ε. Therefore $e_i = e_j = 0$ is not an equilibrium when $\theta \leq 1/2$. We conclude that there exists a $e^C > 0$ such that $e_1 = e_2 = e^C$ is the unique Nash equilibrium. ■

We next show that the noncooperative solution does not converge to the Nash bargaining solution as the players become infinitely patient. Let σ_i^δ denote the equilibrium expected division from the noncooperative solution for a given δ.

Lemma 3. *The division σ_i^δ does not converge to s_i^C as $\delta \to 1$. In particular, $\sigma_i^\delta(e) \to s_i^C(e)$ if and only if $e_1 = e_2$.*

Proof of Lemma 3. First consider $\theta < 1/2$. Then for $\delta > 2\theta$, $\sigma_i^\delta = 1/2$. Further, $s_i^C = 1/2$ if and only if $e_1 = e_2$.

Next consider $\theta \geq 1/2$. Recall that the expected divisions for $\delta > \theta$ are

$$\sigma_i^\delta = \begin{cases} \frac{1-\theta}{2-\delta} + p_i \frac{\theta}{2-\delta} & \text{if} \quad e \in \mathcal{A}_j(\delta) \\ \frac{V}{2} & \text{if} \quad e \in \mathcal{N}(\delta) \\ \frac{1-\delta}{2-\delta} + p_i \frac{\theta}{2-\delta} & \text{if} \quad e \in \mathcal{A}_i(\delta) \end{cases}$$

$$\to \begin{cases} 1-\theta+p_i\theta & \text{if} \quad e \in \mathcal{A}_j(1) \\ \frac{V}{2} & \text{if} \quad e \in \mathcal{N}(1) \\ p_i\theta & \text{if} \quad e \in \mathcal{A}_i(1) \end{cases}.$$

Note that $\mathcal{N}(1) = \{e : e_1 = e_2\}$, so this piecewise solution agrees with the Nash bargaining solution if and only if $e_1 = e_2$. ■

Note that this nonconvergence result implies that the Nash bargaining solution may be an inaccurate approximation of the noncooperative bargaining solution as the solutions disagree almost everywhere. In particular when δ is close to 1 and players choose slightly different efforts, the noncooperative scheme results in one player receiving nearly the entire surplus, while cooperative solutions prescribe an equal division of the surplus. Recall that equilibrium efforts are in mixed strategies. When $\delta \geq \theta$, in any equilibrium, efforts are chosen in $\mathcal{A}_1(\delta) \cup \mathcal{A}_2(\delta)$ with positive probability, else the players would always receive a share of 1/2 and would have incentive to reduce their effort choices. Thus, in any realization of an equilibrium, there is a positive probability that the cooperative and noncooperative schemes prescribe drastically different divisions of the surplus.

This lack of convergence allows for the main result that both players prefer the noncooperative bargaining scheme due to the Pareto superior efforts it induces in equilibrium. Denote by u_i^C the equilibrium expected payoffs under a cooperative bargaining scheme. Let u_i^δ denote the equilibrium expected payoffs under the noncooperative bargaining scheme for a given $\delta < 1$. Let $\overline{e}^\delta$ denote the maximum effort chosen in equilibrium, that is, $\overline{e}^\delta$ is the supremum of the support of a player's equilibrium mixed strategy.

Theorem 1. *Suppose that Assumptions 1-4 hold. Then for all $\delta < 1$ such that $\theta \leq \delta < 2\theta$, $u_i^\delta > u_i^C$ and $\overline{e}^\delta < e^C$. Moreover, $\lim_{\delta \to 1} u_i^\delta > u_i^C$ and $\lim_{\delta \to 1} \overline{e}^\delta < e^C$. If $\delta \geq 2\theta$, then $\overline{e}^\delta = 0$ and the equilibrium is fully efficient.*

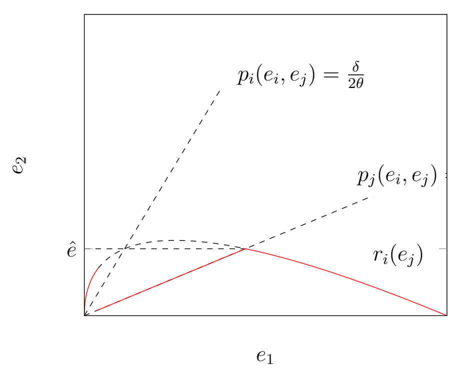


Figure 1.5: Bounding the undominated strategies.

Figure 1.5 plots the best response in a simple Tullock lottery contest along with the partition of the strategy space into $\mathcal{A}_1(\delta) \cup \mathcal{A}_2(\delta) \cup \mathcal{N}(\delta)$. This will allow us to clearly visualize the result, as we will describe below. Note that Corollary 1 implies that e^C is the maximum of the the best response function in the absence of bargaining.

We provide an outline before we conduct the formal proof of this result. Define $\overline{e} < e^C$ to be argument where the graph of the best response function intersects the boundary of $\mathcal{N}(\delta)$, as in Figure 1.5. Our objective is to show that $\overline{e}$ strictly dominates all effort choices $e > \overline{e}$. In order to do this, we compare $\overline{e}$ to an arbitrary $e > \overline{e}$ in three cases corresponding to the region in which $(\overline{e}, e_j)$ lies, $\mathcal{A}_1(\delta)$, $\mathcal{A}_2(\delta)$, or $\mathcal{N}(\delta)$. The argument is as follows. Within any of these three regions, the payoff function is strictly quasiconcave in e_i. Thus, given any $e > e' \geq r(e_j)$ with (e, e_j) and (e', e_j) in the same region $\mathcal{A}_1(\delta)$ or $\mathcal{A}_2(\delta)$, e' yields a strictly higher payoff than does e. Given $e > e'$ with (e, e_j) and (e', e_j) in $\mathcal{N}(\delta)$, e' yields a strictly higher payoff since the utility of each player is strictly decreasing in their own strategy in this region. It therefore suffices to show that $\overline{e} \geq r(e_j)$ for all e_j such that $(\overline{e}, e_j) \in \mathcal{A}_1(\delta) \cup \mathcal{A}_2(\delta)$. In order to deal with the case in which (e, e_j) and $(\overline{e}, e_j)$ lie in different regions, we simply find an intermediate effort with $e > e' > \overline{e}$ such that (e', e_j) lies on the boundary of the two regions and compare e to e' and e' to $\overline{e}$. After establishing that $\overline{e}$ strictly dominates all effort choices $e > \overline{e}$, we need only show that a player can guarantee himself a payoff strictly higher than u^C by choosing $\overline{e}$.

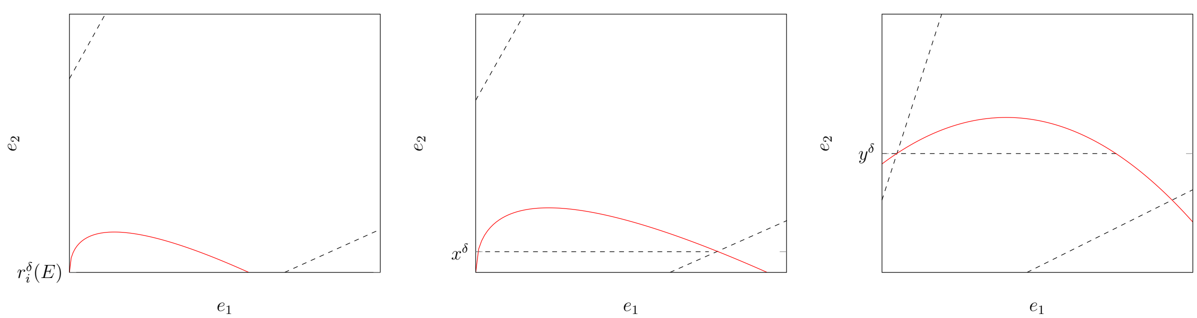
Generally, the best response function $r_i(\cdot)$ may intersect the boundary of $\mathcal{N}(\delta)$ at multiple distinct effort levels. In this case we simply consider $\overline{e}$ to be the supremum of these intersection points. Note that this upper bound on the support of the mixed strategy equilibrium is larger than necessary. A tighter bound could be computed via iterated deletion of strictly dominated strategies, however, such a process would not add value, as we make no attempt to characterize the mixed strategy equilibrium of this game.

Proof of Theorem 1. Fix $\theta \leq \delta < 2\theta$. Let $r_i^\delta(e_j)$ denote player i's best response function in the contest with payoffs given by

$$p_i \frac{\theta}{2-\delta} V - c(g_i).$$

Note that r_i^δ gives the best response for payoffs defined for $e \in \mathcal{A}_1 \cup \mathcal{A}_2$. It is possible that $r_i^\delta(0)$ does not exist, which we consider separately as needed. We use $\widehat{e}^\delta$ to denote the symmetric pure strategy equilibrium effort in this contest, that is, $r_i^\delta\left(\widehat{e}^\delta\right) = \widehat{e}^\delta$. Recall that the expected payoffs are

$$u_i(e_i, e_j) = \begin{cases} \frac{1-\theta}{2-\delta} V + p_i \frac{\theta}{2-\delta} V - c(e_i) & \text{if} \quad e \in \mathcal{A}_j(\delta) \\ \frac{V}{2} - c(e_i) & \text{if} \quad e \in \mathcal{N}(\delta) \text{ and } \delta \geq \theta \\ \frac{1-\delta}{2-\delta} V + p_i \frac{\theta}{2-\delta} V - c(e_i) & \text{if} \quad e \in \mathcal{A}_i(\delta) \end{cases}.$$



(a) $\bar{e}^\delta = r_i^\delta(E)$ (b) $\bar{e}^\delta = r_i^\delta(E)$ (c) $\bar{e}^\delta = x^\delta$

Figure 1.6: Possible bounds for the undominates strategies.

Define x^δ and y^δ as in Figure 1.6. Formally,

$$x^\delta = \sup\left\{r_i^\delta(e_j) : e_j < \widehat{e}^\delta \text{ and } p_i\left(r_i^\delta(e_j), e_j\right) = \delta/2\theta\right\}$$

and $y^\delta = \sup\left\{r_i^\delta(e_j) : e_j > \widehat{e}^\delta \text{ and } p_j\left(r_i^\delta(e_j), e_j\right) = \delta/2\theta\right\}$. The supremum operator is necessary as these sets may be empty. Note that if $\max\left\{x^\delta, y^\delta\right\} = \widehat{e}^\delta$, then since r_i^δ only obtains its maximum $\widehat{e}^\delta$ only at $e_j = \widehat{e}^\delta$, this implies that $p_i\left(\widehat{e}^\delta, \widehat{e}^\delta\right) = \delta/2\theta > 1/2$, a contradiction.

Therefore, $\max\left\{x^\delta, y^\delta\right\} < \widehat{e}^\delta$. Next, define $\overline{e}^\delta = \max\left\{x^\delta, y^\delta, r_i^\delta(E)\right\}$, noting that $\overline{e}^\delta < \widehat{e}^\delta$. We show that $\overline{e}^\delta$ strictly dominates all efforts $e > \overline{e}^\delta$ and so $\overline{e}^\delta$ is an upper bound for the efforts employed in any Nash equilibrium.

Consider any $e_i > \overline{e}^\delta$. We consider three cases corresponding to whether $(e_i, e_j) \in \mathcal{A}_i(\delta)$, $(e_i, e_j) \in \mathcal{A}_j(\delta)$, or $(e_i, e_j) \in \mathcal{N}(\delta)$.

Case (i): $\left(\overline{e}^\delta, e_j\right) \in \mathcal{A}_i(\delta)$.

It follows that $(e_i, e_j) \in \mathcal{A}_i(\delta)$ as well, so given the quasiconcavity of the payoffs in this region it is sufficient to show that $r_i^\delta(e_j) \leq \overline{e}^\delta$. Suppose to the contrary that $r_i^\delta(e_j) > \overline{e}^\delta$. It follows that $(r_i(e_j), e_j) \in \mathcal{A}_i(\delta)$. Since $\overline{e}^\delta < \widehat{e}^\delta$, it must be that $e_j < \widehat{e}^\delta$, so Corollary 1 implies that r_i^δ is increasing in e_j. Since $p_i\left(r_i^\delta(e_j), e_j\right) > \delta/2\theta$ and $p_i\left(r_i^\delta\left(\widehat{e}^\delta\right), \widehat{e}^\delta\right) < \delta/2\theta$, then there must be some $\widetilde{e} \in \left(e_j, \widehat{e}^\delta\right)$ such that $p_i\left(r_i^\delta(e_j), e_j\right) = \delta/2\theta$. Thus, $r_i^\delta(e_j) \leq r_i^\delta(\widetilde{e}) \leq x^\delta \leq \overline{e}^\delta$. This is a contradiction, so we conclude that $r_i^\delta(e_j) \leq \overline{e}^\delta$ and thus $u_i^\delta(e_i, e_j) < u_i^\delta\left(\overline{e}^\delta, e_j\right)$.

Case (ii): $\left(\overline{e}^\delta, e_j\right) \in \mathcal{N}(\delta)$.

If $(e_i, e_j) \in \mathcal{N}(\delta)$ then since the payoffs are strictly decreasing in efforts in that region, it must be that $u_i^\delta(e_i, e_j) < u_i^\delta\left(\overline{e}^\delta, e_j\right)$. Suppose that $(e_i, e_j) \in \mathcal{A}_1(\delta)$. Define $\widetilde{e} \in \left(\overline{e}^\delta, e_i\right]$ to be such that $(\widetilde{e}, e_j) \in \mathcal{A}_1(\delta) \cap \mathcal{N}(\delta)$. Then we have that $u_i^\delta(\widetilde{e}, e_j) < u_i^\delta\left(\overline{e}^\delta, e_j\right)$. A repetition of the argument in case (i) will yield that $r_i^\delta(e_j) \leq \widetilde{e}$, and so the quasiconcavity of payoffs in $\mathcal{A}_1(\delta)$ implies that $u_i^\delta(e_i, e_j) \leq u_i^\delta(\widetilde{e}, e_j)$.

Case (iii): $\left(\overline{e}^\delta, e_j\right) \in \mathcal{A}_j(\delta) \backslash \mathcal{N}(\delta)$.

We first show that $r_i^\delta(e_j) \leq \overline{e}^\delta$. Suppose first that $y^\delta = -\infty$. then either $r_i^\delta(e_j) = 0$ or $\left(r_i^\delta(E), e_j\right) \in \mathcal{N}(\delta)$. If $r_i^\delta(e_j) = 0$, then we have $r_i^\delta(e_j) \leq \overline{e}^\delta$ as desired. Otherwise, if $\left(r_i^\delta(E), e_j\right) \in \mathcal{N}(\delta)$, then it must be that $\left(\overline{e}^\delta, e_j\right) \in \mathcal{N}(\delta)$, a contradiction. Next suppose that $y^\delta > -\infty$, and let $\widetilde{e} > e^\delta$ such that $p_j\left(r_i^\delta(\widetilde{e}), \widetilde{e}\right) = \delta/2\theta$. It follows that $e_j > \widetilde{e}$, and so from Corollary 1 we conclude that $r_i^\delta(e_j) < r_i^\delta(\widetilde{e}) \leq \overline{e}^\delta$.

If $(e_i, e_j) \in \mathcal{A}_j(\delta)$, then the quasiconcavity of the payoffs in this region ensure that $u_i^\delta(e_i, e_j) < u_i^\delta\left(\overline{e}^\delta, e_j\right)$. If $(e_i, e_j) \in \mathcal{N}(\delta)$, then consider e_i' such that $(e_i', e_j) \in \mathcal{A}_j(\delta) \cap \mathcal{N}(\delta)$. Then it must be that $u_i^\delta(e_i', e_j) < u_i^\delta\left(\overline{e}^\delta, e_j\right)$. Moreover, it follows from Case (ii) that $u_i^\delta(e_i, e_j) \leq u_i^\delta(e_i', e_j)$.

We conclude that $\overline{e}^\delta$ strictly dominates e_i. Thus in equilibrium, each player will only play strategies in $\left[0, \overline{e}^\delta\right]$. Given this restriction, equilibrium payoffs must be such that

$$\begin{aligned} u_i^\delta &\geq \min_{e_j \in [0, \overline{e}^\delta]} E\left[u_i\left(\overline{e}^\delta, e_j\right)\right] \\ &= \frac{V}{2} - c\left(\overline{e}^\delta\right) \\ &= u_i^C + c\left(e^C\right) - c\left(\overline{e}^\delta\right) \\ &> u_i^C. \end{aligned}$$

It remains to be shown that these results hold in the limit as $\delta \to 1$. Define $\overline{e}$ analogously to $\overline{e}^\delta$ but for the game whose payoffs are given by $u_i^\delta(e_i, e_j)$ with $\delta = 1$. Let r_i be the best response function for the contest with payoffs $p_i \theta V - c(e_i)$. Then note that $\overline{e}^\delta \to \overline{e}$ as $\delta \to 1$. From this we obtain

$$\begin{aligned} \lim_{\delta \to 1} u_i^\delta &\geq \lim_{\delta \to 1} \min_{e_j \in [0, \overline{e}^\delta]} E\left[u_i\left(\overline{e}^\delta, e_j\right)\right] \\ &= \lim_{\delta \to 1} \frac{V}{2} - c\left(\overline{e}^\delta\right) \\ &= \frac{V}{2} - c(\overline{e}) \\ &= u_i^C + c\left(e^C\right) - c(\overline{e}) \\ &> u_i^C. \end{aligned}$$

The final statement of the theorem has already been discussed. ■

Theorem 1 guarantees that the noncooperative bargaining scheme always induces lower effort choices relative to a cooperative solution. Since efforts are entirely wasteful, this means that the noncooperative bargaining scheme induces a more efficient outcome. The expected efforts induced by the noncooperative scheme are lower still than the upper bound in the Theorem since the equilibrium is in mixed strategies.

While the equilibrium strategies are not easily solved for, Theorem 1 demonstrates that it is relatively easy to find an upper bound on the equilibrium efforts. Moreover, that strategy dominates all larger effort choices, allowing a player to guarantee himself a payoff that is strictly larger with the noncooperative scheme than he would receive under a cooperative solution, even if he is unable to determine the equilibrium strategies.

1.5.1 Choosing a Bargaining Scheme

The implication of Theorem 1 is that players are strictly better off if bargaining is done noncooperatively with the scheme described in section 3. Thus, given the option, players would commit to this scheme prior to making effort decisions. Even if commitments could not be made to use a particular scheme, the implementation of a cooperative solution would require the consent of each player, and the following proposition shows that this would be difficult to acquire.

Proposition 7. *Suppose that $e_i < e_j$. Then for all $\delta > \theta$, $\sigma_i^\delta > s_i^C$ and* $\lim_{\delta \to 1} \sigma_i^\delta > s_i^C$.

Proof of Proposition 7. Given any effort choices such that $e_i < e_j$,

$$\sigma_i^\delta \in \left\{ \frac{1}{2}, \frac{1-\theta}{2-\delta} + p_i \frac{\theta}{2-\delta} \right\}.$$

If $\sigma_i^\delta = 1/2$, then $\sigma_i^\delta > s_i^C$ since $s_i^C = (1-\theta)/2 + p_i\theta < 1/2$. Otherwise, note that the following are equivalent.

$$\begin{aligned}
\frac{1-\theta}{2-\delta} + p_i \frac{\theta}{2-\delta} &> \frac{1-\theta}{2} + p_i\theta \\
\frac{2-2\theta}{2(2-\delta)} + \frac{2p_i}{2(2-\delta)}\theta &> \frac{2-\delta-2\theta+\delta\theta}{2(2-\delta)} + \frac{4p_i - 2\delta p_i}{2(2-\delta)}\theta \\
\frac{\delta(1-\theta)}{2(2-\delta)} &> 2p_i \frac{\theta(1-\delta)}{2(2-\delta)} \\
\delta(1-\theta) &> 2p_i\theta(1-\delta).
\end{aligned}$$

The final inequality holds since $\delta > \theta$ and $2p_i < 1$. Due to the continuity of σ_i^δ, the limiting statement follows since $\delta \to 1 > \theta$. ■

Proposition 7 implies that the player with the disadvantage in the underlying contest strictly prefers the noncooperative solution. This provides justification for the use of noncooperative over cooperative schemes, as noncooperative bargaining would be imposed by the player that chooses the least effort. Even if both players choose the same level of effort, they are indifferent between solutions. After choosing efforts, players are at least as well off bargaining than not, so there would be no advantage to choosing a different level of effort in the anticipation of imposing conflict. Moreover, choosing a level of effort in order to gain an advantage under a cooperative bargaining solution would not be beneficial, as gaining an advantage would induce the other player to elect for the noncooperative scheme.

1.6 Contests with Mixed Strategy Equilibria

In this section, we extend our analysis to consider contest functions which do not satisfy our previous assumptions. In particular, we examine contests which do not admit pure strategy equilibria. Our objective is to extend the previous results to a large majority of the remaind of contests studied in the conflict literature. In the class of contests considered here, the equilibrium involves total rent dissipation, whereas we show that there is less dissipation when the noncooperative bargaining scheme is employed. We first present a general sufficient condition for the extension of our main result.

Theorem 2. *Suppose that in every equilibrium of the contest (the game in which efforts are chosen and then conflict occurs), both players have an expected payoff of zero. Then equilibrium expected payoffs are greater under noncooperative bargaining,* $u_i^\delta > u_i^C$*. Moreover,* $\lim_{\delta \to \infty} u_i^\delta > u_i^C$*.*

Proof of Theorem 2. Since the cooperative solution shares equilibria with the underlying contest, we immediately obtain

$$u_i^C = \frac{1-\theta}{2} V.$$

Note that by choosing $e_i = 0$, player i is guaranteed to receive a payoff of at least $(1-\theta) V > u_i^C$. ■

We now present a large class of contests that satisfy the conditions of Theorem 2. Assume that each player i's utility function in the event of conflict is given by

$$u_i(e) = p_i(e)\,\theta V - c(e_i),$$

where p_i is symmetric.[16] In order to extend our results to a more general setting, we replace Assumptions 2-4 with the following elasticity assumption due to Ewerhart (2013).

Assumption 5. u_i *is differentiable for all* e *such that* $e_1 + e_2 > 0$*. For all* $e_i > 0$*, there is a*

[16] By symmetric, we mean that $p_1(e, e') = p_2(e', e)$ for all e, e'.

constant $\lambda(e_i) > 0$ *such that*

$$\frac{\partial u_i}{\partial e_i}(e) > \lambda u_i(e)$$

for any $e_j \geq e_i$.

As Ewerhart demonstrates, a large class of contest functions satisfy this assumption, including the Tullock function with exponent greater than 2 and the logistic specification mentioned in Remark 1 when the cost function is linear. He obtains the following result that rent is completely dissipated in all equilibria of contests satisfying Assumption 5.

Fact 3 (Ewerhart (2013))**.** *Suppose that Assumption 5 is satisfied. Then in any equilibrium, each player earns a payoff* $u_i^* = 0$.

Corollary 2. *Suppose that Assumption 5 is satisfied. Then equilibrium expected payoffs are greater under noncooperative bargaining,* $u_i^\delta > u_i^C$. *Moreover,* $\lim_{\delta\to\infty} u_i^\delta > u_i^C$.

Not that all we have shown is that there is less rent dissipation when the noncooperative scheme is used relative to the when the cooperative solution is employed. We are unable to verify whether rent is completely dissipated (save for the surplus that can guaranteed by choosing zero effort) in the equilibrium when the noncooperative bargaining scheme is employed. The reason is that the payoffs after bargaining no longer satisfy Assumption 5, and thus Ewerhart's analysis does not apply.

Another common model of conflict is as a deterministic contest in which the player that exerts the highest effort wins with certainty. In this case, we are able to completely characterize the equilibrium payoffs under each bargaining scheme.

Assumption 6. *The probability that player i wins the contest is given by*

$$p_i(e) = \begin{cases} 0 & \text{if} \quad e_i < e_j \\ \frac{1}{2} & \text{if} \quad e_i = e_j \\ 1 & \text{if} \quad e_i > e_j \end{cases}.$$

Contests satisfying Assumption 6 may be realized as the limit of a Tullock contest in which

$$p_i(e) = \frac{e_i^m}{e_1^m + e_2^m}$$

and $m \to \infty$. In this setting we show that rent is completely dissipated in all equilibria under any bargaining scheme.

Siegel (2009) characterizes the equilibrium payoffs of deterministic all-pay contests with complete information. His results may be interpreted in our model as follows.

Fact 4 (Siegel (2009))**.** *Suppose that Assumptions 1 and 6 are satisfied. Then in any equilibrium, each player earns a payoff $u_i^* = 0$.*[17]

Corollary 3. *Suppose that Assumptions 1 and 6 are satisfied. Then equilibrium expected payoffs are greater under noncooperative bargaining, $u_i^\delta > u_i^C$. Moreover, $\lim_{\delta \to \infty} u_i^\delta > u_i^C$.*

For completeness, we shall verify that the game in which Assumptions 1 and 6 are satisfied and the noncooperative scheme is employed fits the assumptions of Siegel's model. This implies that rent is completely dissipated in any equilibrium, and so players would receive only that which they could secure by choosing zero effort. The reason that the intuition from the previous section does not extend is that the region $\mathcal{N}(\delta)$ coincides with the diagonal, as can be seen in Figure 1.7. The incentives when players tie ($e_1 = e_2$) are thus reversed so

[17] Assumption 1 is stronger than necessary. It is sufficient that c be continuous and nondecreasing with $\lim_{e \to \infty} c(e) > V$.

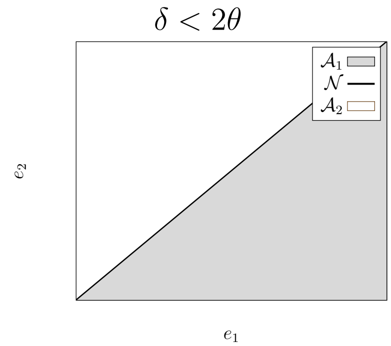


Figure 1.7: The partition of the strategy space with a deterministic contest.

that players ought to marginally increase their efforts from ties rather than decrease, as this will tip them into the advantaged region. Thus, it is not possible to remove strategies that would fully dissipate rent via deletion of dominated strategies.

Proposition 8. *Suppose that Assumption 6 is satisfied and that the noncooperative bargaining scheme is employed. If $\delta < 2\theta$, then the equilibrium expected payoff to each player i is $u_i^\delta = (1-\theta)/(2-\delta)$. If $\delta \geq 2\theta$, then the dominant strategy equilibrium is $\widehat{e}_1 = \widehat{e}_2 = 0$, and the expected payoff to each player i is $u_i^\delta = V/2$.*

Proof of Proposition 8. Note that the partition of the strategy space with the deterministic contest is

$$\mathcal{A}_i(\delta) = \left\{ \begin{array}{ccc} e : e_i > e_j & \text{if} & \delta < 2\theta \\ \emptyset & \text{if} & \delta \geq 2\theta \end{array} \right\} \text{ and}$$

$$\mathcal{N}(\delta) = \left\{ \begin{array}{ccc} e : e_1 = e_2 & \text{if} & \delta < 2\theta \\ [0, E] & \text{if} & \delta \geq 2\theta \end{array} \right\}.$$

This is illustrated in Figure 1.7. Thus, for $\delta < 2\theta$, payoffs can be normalized with the addition of the constant $-(1-\theta)/(2-\delta)$ to

$$u_i(e_i, e_j) = \begin{cases} -c(e_i) & \text{if} \quad e_i < e_j \\ \left(\frac{1}{2} - \frac{1-\theta}{2-\delta}\right) V - c(e_i) & \text{if} \quad e_1 = e_2 \\ \frac{2\theta-\delta}{2-\delta} V - c(e_i) & \text{if} \quad e_i > e_j \end{cases} .$$

The previous fact then implies that the payoff to each player in this game is zero. Reversing the normalization, we find that the true expected payoff to each player is

$$u_i^\delta = \frac{1-\theta}{2-\delta} V.$$

If $\delta \geq 2\theta$, then the result follows from the fact that $\mathcal{N}(\delta) = [0, E]$. ■

This proposition states that the total surplus is dissipated to $2(1-\theta)/(2-\delta)$ when $\delta < 2\theta$. As noted in the proof of Theorem 2, the total surplus under the cooperative scheme is dissipated to $(1-\theta)$. Thus for sufficiently patient players the noncooperative scheme yields approximately double the surplus of the cooperative solution. As in previous sections, when conflict is sufficiently destructive ($\delta \geq 2\theta$) the noncooperative scheme induces maximal efficiency, while the cooperative solution would otherwise yield minimal efficiency. While the analysis of this section does not guarantee that our result extend to all contests, it does demonstrate its robustness, applying to the vast majority of models employed in the literature on conflict.

1.7 Exogenous Breakdown

In this section we consider an alternative specification of the bargaining process to allow for an exogenous probability of breakdown as in Binmore, Rubinstein, and Wolinsky (1986).

As noted in the Introduction, if players do not discount future payoffs and players have no outside option, the equilibrium of this bargaining scheme converges to the Nash bargaining solution. Our goal here is to demonstrate that the results of Section 5 are robust to the possibility of breakdown provided that players discount future payoffs.

The framework of the bargaining problem is altered as follows. In each bargaining round, if the offer is rejected by the responder, there is a probability that the process breaks down, in which case conflict occurs. Let $1-\Delta>0$ denote the probability that bargaining breaks down each round, so that the probability of continuation is Δ. Then the expected value of delay to the responder j is given by

$$\Delta\delta\left(\frac{1}{2}V_j\left(I\right)+\frac{1}{2}V_j\left(J\right)\right)+\left(1-\Delta\right)p_j\theta V,$$

where $V_j\left(K\right)$ denotes the expected payoff that j receives when $K\in\{1,2\}$ is the proposer.

The key difference is that when both player's optimal threat is to delay rather than withdraw, the equilibrium division will depend on the effort choices since there is a probability that delay results in conflict. As before, we partition the strategy space of effort choices, though the partition now depends on Δ as well.

$$\begin{aligned}\mathcal{A}_i\left(\Delta,\delta\right) &= \left\{\begin{array}{cc} e: p_j<\frac{\delta}{2\theta}\frac{1-\theta}{1-\delta} & \text{if}\quad \delta<\theta \\ e: p_i>\frac{\delta}{2\theta}\frac{1-\Delta\delta-(1-\Delta)\theta}{1-\delta} & \text{if}\quad \delta\geq\theta \end{array}\right\} \text{ and} \\ \mathcal{N}\left(\Delta,\delta\right) &= \left\{\begin{array}{cc} e:\min\{p_1,p_2\}\geq\frac{\delta}{2\theta}\frac{1-\theta}{1-\delta} & \text{if}\quad \delta<\theta \\ e:\max\{p_1,p_2\}\leq\frac{\delta}{2\theta}\frac{1-\Delta\delta-(1-\Delta)\theta}{1-\delta} & \text{if}\quad \delta\geq\theta \end{array}\right\}.\end{aligned}$$

Define the following proposal and response rules:

$$s_i^{**} = \begin{cases} \frac{2(1-\Delta\delta-(1-\Delta)\theta)}{2-\Delta\delta} + \frac{\Delta\delta+2(1-\Delta)}{2-\Delta\delta} p_i\theta & \text{if} \quad e \in \mathcal{A}_i(\Delta,\delta) \\ (1-\theta)+p_i\theta & \text{if} \quad e \in \mathcal{A}_j(\Delta,\delta) \text{ or } e \in \mathcal{N}(\Delta,\delta) \text{ and } \delta < \theta \\ \frac{2-\Delta\delta}{2}\frac{1-\Delta\delta-(1-\Delta)\theta}{1-\Delta\delta} + \frac{1-\Delta}{1-\Delta\delta} p_i\theta & \text{if} \quad e \in \mathcal{N}(\Delta,\delta) \text{ and } \delta \geq \theta \end{cases},$$

$$A_i^{**} = \left[p_j\theta, s_j^*\right],$$

$$D_i^{**} = \begin{cases} \emptyset & \text{if} \quad e \in \mathcal{A}_i(\Delta,\delta) \text{ or } e \in \mathcal{N}(\Delta,\delta) \text{ and } \delta < \theta \\ [0,1] \diagdown A_i^{**} & \text{if} \quad e \in \mathcal{A}_j(\Delta,\delta) \text{ or } e \in \mathcal{N}(\Delta,\delta) \text{ and } \delta \geq \theta \end{cases},$$

$$W_i^{**} = \begin{cases} [0,1] \diagdown A_i^{**} & \text{if} \quad e \in \mathcal{A}_i(\Delta,\delta) \text{ or } e \in \mathcal{N}(\Delta,\delta) \text{ and } \delta < \theta \\ \emptyset & \text{if} \quad e \in \mathcal{A}_j(\Delta,\delta) \text{ or } e \in \mathcal{N}(\Delta,\delta) \text{ and } \delta \geq \theta \end{cases}.$$

Lemma 4. *Suppose that each player employs the following strategy. In each round that player i is the proposer, he proposes s_i^{**}. In each round that player i is the responder, he accepts proposals $s_j \in A_i^{**}$, rejects proposals $s_j \in D_i^{**}$, and withdraws given proposals $s_j \in W_i^{**}$, where s_i^{**}, A_i^{**}, D_i^{**}, and W_i^{**} are as specified above. Then the expected share that each player i receives is*

$$\sigma_i^{**} = \begin{cases} \frac{1-\Delta\delta-(1-\Delta)\theta}{2-\Delta\delta} + \frac{2-\Delta}{2-\Delta\delta} p_i\theta & \textit{if} \quad e \in \mathcal{A}_i(\Delta,\delta) \\ \frac{1}{2-\Delta\delta}(1-\theta) + \frac{2-\Delta}{2-\Delta\delta} p_i\theta & \textit{if} \quad e \in \mathcal{A}_j(\Delta,\delta) \\ \frac{1}{2}\frac{1-\Delta\delta-(1-\Delta)\theta}{1-\Delta\delta} + \frac{1-\Delta}{1-\delta\Delta} p_i\theta & \textit{if} \quad e \in \mathcal{N}(\Delta,\delta) \textit{ and } \delta \geq \theta \\ \frac{1-\theta}{2} + p_i\theta & \textit{if} \quad e \in \mathcal{N}(\Delta,\delta) \textit{ and } \delta < \theta \end{cases}.$$

The proof of Lemma 4 is identical to that of Lemma 1 and is located in the supplemental appendix.

The following propositions characterize the unique Markov perfect equilibrium of the bargaining subgame when there is an exogenous probability of breakdown between bargaining rounds.

Proposition 9. *Suppose that there is an exogenous probability of breakdown* $\Delta \in [0,1]$ *and* $\delta < 1$*. The unique subgame perfect equilibrium of the bargaining subgame is as follows. In each round that player* i *is the proposer, he proposes* s_i^{**}*. In each round that player* i *is the responder, he accepts proposals* $s_j \in A_i^{**}$*, rejects proposals* $s_j \in D_i^{**}$*, and withdraws given proposals* $s_j \in W_i^{**}$*, where* s_i^{**}*,* A_i^{**}*,* D_i^{**}*, and* W_i^{**} *are as specified above.*

The proof is identical to that of Proposition 1 and is located in the supplemental appendix. The following proposition demonstrates that so long as players discount future payoffs and are reasonably patient ($\delta \geq \theta$), then they will prefer the noncooperative bargaining scheme. Let $u_i^{\Delta,\delta}$ denote the payoff earned by player i in an equilibrium induced by this bargaining scheme, and let $\overline{e}^{\Delta,\delta}$ denote the supremum of the support of the equilibrium strategies.

Proposition 10. *Suppose that Assumptions 1-4 hold. Then for all* $\delta \in [\theta, 1)$*, for any* $\Delta \in (0,1]$*,* $u_i^{\Delta,\delta} > u_i^C$ *and* $\overline{e}^{\Delta,\delta} < e^C$*. Moreover,* $\lim_{(\Delta,\delta)\to(1,1)} u_i^{\Delta,\delta} > u_i^C$ *and* $\lim_{(\Delta,\delta)\to(1,1)} \overline{e}^{\Delta,\delta} < e^C$*.*

Proof of Proposition 10. Let $\widehat{e}^{\Delta,\delta}$ denote the symmetric pure strategy equilibrium effort for the game whose payoff to each player i is

$$\frac{2-\Delta}{1-\Delta\delta} p_i \theta.$$

The existence of this equilibrium is guaranteed by Proposition 2. Let $r_i^{\Delta,\delta}$ denote the best response function of that game.

Then analogous to the proof of Theorem 1, define

$$x^{\Delta,\delta} = \sup\left\{r_i^{\Delta,\delta}(e_j) : e_i < \widehat{e}^{\Delta,\delta} \text{ and } p_i\left(r_i^{\Delta,\delta}(e_j), e_j\right) = \frac{\delta}{2\theta}\frac{1-\Delta\delta-(1-\Delta)\theta}{1-\delta}\right\}$$

and

$$y^{\Delta,\delta} = \sup\left\{r_i^{\Delta,\delta}(e_j) : e_j > \widehat{e}^{\Delta,\delta} \text{ and } p_j\left(r_i^{\Delta,\delta}(e_j), e_j\right) = \frac{\delta}{2\theta}\frac{1-\Delta\delta-(1-\Delta)\theta}{1-\delta}\right\}.$$

Now define $z^{\Delta,\delta}$ to be the symmetric pure strategy equilibrium effort for the game whose payoff to each player is

$$\frac{1-\Delta}{1-\Delta\delta}p_i\theta,$$

which reflects the payoffs in $\mathcal{N}(\Delta,\delta)$. If we define the upper bound on equilibrium strategies to be $\overline{e}^{\Delta,\delta} = \max\left\{x^{\Delta,\delta}, y^{\Delta,\delta}, r_i^{\Delta,\delta}(E), z^{\Delta,\delta}\right\}$, then the remainder of the proof is identical to that of Theorem 1, where $\overline{e}^{\Delta,\delta}$ takes on the role of $\overline{e}^{\delta}$. ■

This result states that players are better off in the equilibrium induced by the noncooperative scheme even if there is an exogenous probability of breakdown as long as the have even the slightest time preferences.

Even though the payoffs in $\mathcal{N}(\Delta,\delta)$ depend on the effort choices of the players, the marginal benefit is still lower than under conflict. If the probability of breakdown is sufficiently high, then there is a unique pure strategy equilibrium in which both players choose symmetric, positive efforts, though these efforts are strictly below the cooperative effort level since the probability of conflict is less than one and in a sense, the conflict outcome is discounted. Otherwise, if the probability of breakdown is sufficiently low, then the pure strategy equilibrium candidate effort is not optimal since either player could slightly increase their efforts and obtain a sufficient advantage so as to be guaranteed a much larger share of the prize. In this case, a dominating strategy may be computed as before that players may choose and guarantee themselves a higher payoff than they would receive in the equilibrium induced by the cooperative solution.

As a final note, if breakdown is certain ($\Delta = 0$) or players do not discount future payoffs ($\delta = 1$), then the equilibrium efforts induced by the noncooperative scheme and the cooperative solution are identical, as are expected payoffs. When $\Delta = 0$, then the bargaining process reduces to the ultimatum game, in which the only equilibrium is for the proposer to take the entire surplus, leaving the responder with his conflict payoff. If $\delta = 1$, then not only does the equilibrium division converge to the Nash bargaining solution as shown by Binmore, Rubinstein, and Wolinsky (1986), but the equilibrium division is a function of Δ plus the conflict payoff. Thus, if either $\Delta = 0$ or $\delta = 1$, the expected payoff for each player coincides with that specified by the cooperative solution, inducing the same equilibrium.

1.8 Conclusions

We have provided a general characterization of concave, symmetric contests. Using this, we have shown that the noncooperative alternating offers bargaining scheme in which conflict serves as an outside option induces equilibrium effort choices in mixed strategies for the most interesting portion of the parameter space, and zero effort for the remainder. Payoffs are strictly higher for both players in the equilibrium induced by the noncooperative scheme relative to the equilibrium induced by any cooperative scheme. Moreover, the use of a cooperative bargaining solution always induces positive effort choices, while a noncooperative solution can induce an efficient, zero effort equilibrium. We have further shown that our results extend to symmetric contests which do not possess pure strategy equilibria.

While some situations such as war lend themselves naturally to the timing of actions used in this paper, it is at other times natural to only exert effort in the event that conflict occurs. The unique property of conflict is that it supersedes agreements, so that regardless of the bargaining scheme or solution that is used, the agreement is nonbinding. It follows that after efforts have been chosen, players would have an opportunity to renegotiate their agreement. Regardless of the bargaining scheme or solution that is employed prior to effort selection, if either player would receive a lower payoff after effort selection than he would under the noncooperative scheme considered here, then he would have incentive to renegotiate to earn that expected payoff. Thus, as long as players have the ability to renegotiate then the timing of actions will not change the results of this paper.

These results should extend predictably to games with multiple players and to the case where players are mildly asymmetric. If they are too asymmetric, however, then the pure strategy equilibrium efforts of the underlying contest will lie sufficiently outside of the symmetric region $\mathcal{N}$ and thus be a pure strategy equilibrium of the game with any bargaining scheme. It may be possible that the results hold in asymmetric contest games if the players have different discount factors or with a more general random process by which a proposer is selected. An interesting avenue for future research would be to investigate the extension of our results to compare a noncooperative scheme with these asymmetries to the corresponding asymmetric Nash bargaining solution.

An implication of our results is that the use of cooperative bargaining solutions in modeling may overestimate effort decisions and underestimate efficiency. In the process of proving our results, we have demonstrated that the marginal benefit of an increase in effort is zero for large subset of the strategy space. This has further implications for the motivation for investments such as government defense spending and firms hiring dedicated lawyers, possibly even lobbying. While an argument for increasing investment in these activities may be that this would increase the government or firm's bargaining power, we have shown that such increases may have no effect whatsoever on bargaining outcomes, and thus may be entirely wasteful.

Chapter 2

Directed Search and Consumer Rationing

2.1 Introduction

In this paper, we study a model in which heterogenous consumers engage in directed search to obtain a good. Distinct from other models of directed search, when consumers are rationed, they may search another firm in a later round. Using this model, we derive equilibrium consumer rationing rules and the demand facing each firm in the market, providing a better understand of the role that consumers play in firm competition.

Demand rationing, the process by which firms' goods are allocated to consumers, is a fundamental aspect of price competition that can have vast influence on firms' equilibrium behavior. Rationing has traditionally been considered as exogenous in the literature on Bertrand-Edgeworth price competition, which studies the pricing strategies of capacity constrained firms. In these models, consumers shop according to the surplus they would obtain from purchase. The rationing rules employed are equivalent to all consumers shopping at the firm from which they derive the greatest surplus, then moving to the next firm only after their preferred firm exhausts its capacity. Such a convention is satisfactory with Bertrand competition, where a single firm is able to satiate the entire market, though it may fail to reflect strategic consumer behavior when firms are capacity constrained. As a simple example, consider two firms selling homogenous goods, with one firm setting a price higher than the other by an arbitrarily small amount. If both firms have sufficiently low capacity, then there will be consumers that are unable to purchase a good. If all consumers shop first at the firm with the lower price, it stands to reason that a consumer that would receive a large surplus from either firm would be better off shopping at the higher priced firm initially. This would guarantee that consumer a purchase, while the potential surplus loss is arbitrarily small. Given that traditional rationing rules may not reflect strategic behavior on the part of consumers, how might strategic shopping influence the behavior of firms?

We show that given any capacities and prices set by the firms, there always exists an equilibrium in which the quantity sold by each firm coincides with that specified by the proportional rationing rule, one of the two most prominently employed traditional rationing rules. Moreover, if there exists an equilibrium in which a firm sells a different quantity, the "proportional equilibrium" maximizes the expected payoffs of all consumers whose strategies differ across equilibria, suggesting a coordination on this equilibrium. This result suggests that the use of traditional rationing rules accurately represents market demand from the firms' perspective, thereby justifying its application in the literature.

The model we use is as follows. There are two firms selling substitute goods with exogenously fixed capacities and prices. There are a continuum of consumers each demanding a single unit of either good, for which they possess possibly asymmetric valuations. Consumers take part in a two stage game in which they choose where to shop in each period. In a given stage, all consumers that shop at a particular firm arrive in a uniformly random order at that firm.

The equilibrium of this game is characterized by a critical ratio of surplus at firm 1 versus firm 2 such that all consumers whose surplus ratio is larger will shop at firm 2 in the first round, while the remaining consumers shop at firm 1. This rule is most easily interpreted when products are homogenous. In this case, individuals with the highest valuation of the good shop first at the firm with the higher price, which exhibits the same logic that we used to argue that traditional rationing rules do not portray strategic behavior. In addition to the clearer interpretation of equilibrium strategies, when goods are homogenous, we are able to show that concavity of demand is a sufficient condition for *all* equilibrium quantities to coincide with the proportional rule.

The literature on directed search is rooted in the study of the labor market, though has been applied to markets for consumer goods as well.[1] The standard assumption in this literature is that rationed consumers are unable to search again. That is, if consumers do not receive a good at the firm they initially select, then they go unsatiated. While this may the case for some markets, it is not true in general. A prominent result of these models is to explain market frictions, inefficiencies due to failures of the market to match willing buyers and sellers. Such an outcome occurs because consumers are unable to coordinate their decisions and thus some firms face excess demand while others hold excess capacity. As we consider markets in which additional search is possible, such frictions are not present. Instead, we focus on the quantity that each firm is able to sell, as this corresponds most closely with the Bertrand-Edgeworth literature.[2]

[1] See for example, Montgomery (1991), Moen (1997), Burdett, Shi, and Wright (2001), Lester (2010), and Geromichalos (2012).

[2] Another literature which studies the strategic actions of consumers is that of undirected search. These models focus on the role of asymmetric information regarding prices. See for example Diamond (1971) and Stahl (1989).

Outside of the literature on consumer search, the literature on Bertrand-Edgeworth price competition has modeled the consumer side of the market via demand rationing rules. There are two prominent rationing rules that are employed in the this literature: the efficient rule and the proportional rule. In characterizing the equilibria of these games, these two rules have been used exclusively.[3] The efficient rule, so named because it maximizes consumer surplus, is the first of the rules described in our example above, specifying that the highest value consumers receive the lowest price goods.[4] This rule was first introduced by Levitan and Shubik (1972), which motivates the rule with the fact that it is generated by the demand of a representative consumer. A single consumer would be best off purchasing as much as possible (or desired) from the low price firm before shopping at the high price firm. Alternatively, Dixon (1987) notes that the efficient rule can be realized when each consumer receives an equal share of the low price good.[5] The alternative proportional rule was formally introduced by Shubik (1959). The proportional rationing rule formalizes the following statement by Shubik, "if a percentage of consumers is satisfied by the low price firm, then the same percentage of consumers at any price level will have been satisfied." This story is equivalent to a uniformly random arrival of consumers at the low price firm. Both of these rules are widely used in the literature, though the former is much more prominent.

While Shubik is the first to formalize the proportional rule, its origin can be traced back to the first use of a rationing rule by Edgeworth (1925). The pricing story he describes is dynamic in nature and informal, with firms each monopolizing a distinct half of the market, then sequentially lowering prices to steal customers until reaching some threshold, at which

[3]Maskin (1986), Bagh (2010), and Allison and Lepore (2014) prove existence of equilibria with more general rationing rules.

[4]The efficient rule is difficult to describe when products are differentiated, and depends critically on how the consumer side of the market is specified.

[5]Such behavior may arise from policies such as "limit x units per customer," however, this requires the individuals have downward sloping demand. If individuals each demand one unit of a good, then a limiting policy would not induce efficient rationing.

point they raise their prices to the aforementioned monopoly price.[6] The terminology he uses is that firms serve individual customers, with an indication that those served first are the first to arrive. Informally, Edgeworth was describing the proportional rationing rule. Since each customer is satiated before the next makes a purchase, even after the low price firm sells its capacity, the high price firm faces the full demand of the remaining customers. Given the symmetry of consumers, this means that the residual demand is proportional to the total demand, with the proportion being the fraction of consumers not served by the low price firm.

Dixon (1987) derives a different rationing rule using a traditional Marshallian demand approach which he calls the true contingent demand. In his specification there is a representative household and one of the two relevant firms is not capacity constrained. The rule he derives is similar to the efficient rule (since there is a representative consumer) except that there are endogenous income effects due to purchasing goods at different prices. Due to the computational complexity of the income effects, this rule has not been used in the studying price competition.

Little attention has been paid to rationing rules when products are differentiated, owing largely to the continuous demand structure that is commonly employed to avoid capacity constraints and enable the existence of a pure strategy equilibrium of the pricing game. At a glance, it seems that there might not be rationing issue without demand discontinuities, but in fact, only the sharing rule is irrelevant in this case.[7] When capacity constraints have been introduced into a differentiated product framework, the efficient rationing rule has been used exclusively and without explicit reference. In this paper, we introduce a more general definition of proportional rationing that applies to both homogenous and differentiated products

[6]This story is formalized as an equilibrium of the dynamic pricing game by Maskin and Tirole (1988).

[7]A sharing rule determines how utility is split at points of discontinuity. In a pricing game, a sharing rule would determine the fraction of consumers which purchase from each firm when prices are equal. Thus, a rationing rule includes a sharing rule, but not vice versa.

and is independent of the structure of the differentiation.

A segment of the literature that has most notably put focus on the choice of rationing rule is that which examines the equivalence between the Cournot equilibrium and the equilibrium of the two stage Bertrand-Edgeworth in which firms first choose capacities and then choose prices. Kreps and Scheinkman (1983) showed that the equilibrium outcomes of these two models coincide when the efficient rationing rule is used. This result has since been extended to different settings including less restrictive demand [Osborne and Pitchik (1986)], non-binding capacity constraints [Boccard and Wauthy (2000)], differentiated products [Maggi (1996)], asymmetric information regarding costs [Lepore 2008], proportional rationing [Lepore (2009)], and demand uncertainty [Lepore (2012)]. One such finding is that the Cournot outcome is not necessarily an equilibrium under all specifications, and in particular, with all rationing rules. Davidson and Deneckere (1986) find a counterexample using the proportional rationing rule, while Lepore (2009) demonstrates that the Cournot outcome is an equilibrium if and only if the costs of capacity are sufficiently high. All of this research gives rise to the question of whether this equivalence would hold for any rationing rule. Our results suggest that answering this question is not a priority, as there is always an equilibrium in which the proportional rationing rule specifies the quantity sold, and there is reason for consumers to coordinate on such an equilibrium.

The discussion of rationing rules is not limited to the justification of the Cournot equilibrium. Indeed any study of Bertrand Edgeworth competition requires the use of a rationing rule.[8] This necessitates both the selection of a rule and the justification of that selection. Our results may be used to fill both of these roles, which have previously been filled by informal

[8] Some examples include studies of collusion [Brock and Scheinkman (1985), Benoit and Krishna (1987), and Davidson and Deneckere (1990)], large markets and the competitive equilibrium [Allen and Hellwig (1986a,1986b)], sequential pricing [Deneckere and Kovenock (1992)], triopoly [Hirata (2009) and De Francesco and Salvadori (2010)], competition with input and output prices [Loertscher (2008)], and price discriminating monopoly [Dana (2001)].

arguments.

The remainder of the paper is organized as follows. In the following section, we describe the formal model. In Section 3 we characterize the consumer behavior, with the homogenous goods case characterized in Section 4. Finally, we conclude in Section 5.

2.2 The Model

Consider a duopoly in which firms sell substitute goods. We use the subscript i to refer to a typical firm and j to refer to the firm other than i. The price $p_i \geq 0$ and capacity $x_i > 0$ of each firm i are exogenously fixed and common knowledge.[9] There are a continuum of consumers in the market with possibly heterogenous valuations for each good. Each consumer demands exactly one unit of a good and may be satiated by either firm.[10] The distribution of consumer valuations $v = (v_1, v_2)$ is described by the function $D : \mathbb{R}_+^2 \to \mathbb{R}_+$, where $D(p)$ is the mass of consumers with value $v > p$.[11] We refer to $D(p)$ as the market demand. Let Δ represent the measure induced by D. Often more relevant than prices or values is the surplus that a consumer receives from purchase. Since we frequently refer to this, we use $s_i(p_i)$ to denote the surplus that a consumer with value v_i receives when purchasing a good from firm i, that is, $s_i(p_i) = v_i - p_i$. For notational convenience, we will suppress the argument of $s_i(p_i)$ in proofs and refer to the surplus as simply s_i.

Assumption 7. *$D(p)$ is nonincreasing for all p. For all $p \neq (0,0)$, $D(p) < \infty$.*

We need only assume that the law of demand holds and that demand is finite whenever prices

[9] If one firm's capacity is zero, then the market is a monopoly with trivial outcome.

[10] The assumption that each consumer demands a single good is not necessary. We will discuss how the model extends to the case in which consumers have individual demand curves in the conclusion.

[11] The notation $v > p$ denotes $v_i > p_i$ for both i. To avoid confusion, we will always include subcripts when referring to an element of a vector and omit subscripts when referring to a vector. Thus, we use p_i and v_i to refer to the price and value of good i, whereas p and v refer to vectors of prices and values.

are not simultaneously zero. We place no restriction on the shape of the demand curve for now, nor do we make any assumptions about continuity. Our model thus accommodates any structure of product differentiation, demand functions which are convex, concave, or neither, as well as those for which $\lim_{p\to 0} D(p) = \infty$ or $\lim_{q\to 0} D^{-1}(q) = \infty$. Later, we will make additional assumptions in order to obtain more precise results for the case in which the two firms' goods are homogenous.

Consumers decide where to shop in a two round game. In the first round, each consumer chooses a firm at which to shop. Shopping is voluntary, so that individuals that would receive a negative surplus from purchase will not shop in our model. Consumers arrive in a uniformly random order at their selected firm, so all consumers who shop at a given firm receive a good with equal probability. In the second round, any consumers that did not receive a good (due to insufficient firm capacity) may shop at the other firm, provided that it has remaining capacity.[12]

Given that market demand is not assumed to be continuous, it is possible that a mass of consumers is indifferent between purchasing and not purchasing. As such, it is necessary to specify the actions of such consumers. As a convention, we assume that any individual with $s_i(p_i) = 0$ for some i will attempt to shop at firm i rather than not shop. This particular specification is purely for convenience; any specification for the fraction of indifferent consumers that decide to shop may be used. Different specifications will alter the shopping decisions of consumers, as having more indifferent consumers shopping will result in a decreased likelihood of any particular consumer receiving a good. If there are more of these indifferent consumers for one firm than the other, then some of the consumers who would otherwise have shopped at that firm will switch to the other firm to increase their likelihood of receiving a good. While consumer behavior may be nominally changed, the qualitative aspects of

[12] We explicitly prohibit the possibility of a secondary market for the good. This is natural in cases where the good must be consumed immediately, for example, when the good is service at a restaurant.

the equilibrium and the results regarding the quantity sold by the firms in equilibrium will remain unchanged if a different decision rule were used by the indifferent consumers.

Without capacity constraints, the minimal quantity demanded from a given firm i is the mass of consumers that both receive a positive surplus from shopping at firm i and receive a greater surplus at firm i than firm. Formally, the minimal demand for firm i in the absence of capacity constraints is given by the function

$$D_i(p) = \Delta\left(\{v : s_i(p_i) > s_j(p_j) \text{ and } s_i(p_i) \geq 0\}\right).$$

Note that the realized demand facing firm i may be larger than $D_i(p)$ if there is a mass $I(p)$ of consumers that is indifferent between purchasing from either firm, given by

$$I(p) = \Delta\left(\{v : s_1(p_1) = s_2(p_2) \geq 0\}\right).$$

We do not specify the decision rule of these consumers, as their decisions are determined in equilibrium.

Lastly, in order to completely specify the demand facing each firm, we will refer to the consumers that receive a nonnegative surplus at both firms, as these are the consumers that would be willing to shop in the second round conditional on not receiving a good in the first round. In particular, we define $d_i(p)$ to be the mass consumers that strictly prefer firm j to firm i at prices p, but prefers i to not shopping. Formally,

$$d_i(p) = \Delta\left(\{v : s_j(p_j) > s_i(p_i) \geq 0\}\right).$$

2.3 Equilibrium Rationing

The classic assumption is that consumers shop from the firm with the lowest price, but this may not reflect equilibrium behavior. To see why, suppose that both firms sell out their capacity. Then there is a positive probability that a given individual does not obtain a good if they shop at the low price firm first, whereas this shopping rule would guaranteed them a good if they were to shop at the high price firm first. If the firms' prices are close enough, then some of the consumers are better off guaranteeing themselves a reduced surplus rather than taking the risk of receiving no surplus at all.

In the general setting, the classic assumption corresponds to consumers shopping at the firm from which they receive the greatest surplus. Such behavior has most commonly been considered with two consumer rationing schemes: efficient and proportional. Under efficient rationing, the consumers with the highest surplus receive the goods first. As this requires that firms discriminate between perfectly between individuals, the efficient rationing rule cannot be reasonably implemented. Note that in our model, efficient rationing cannot occur since this would require a nonrandom order of arrival by consumers. More plausible is the proportional rule, which is realized within our model when consumers all shop first at the firm from which they receive the greatest surplus.[13] This rule is formalized below.

Definition 4. *Under* proportional rationing*, the residual demand facing firm i is*

$$D_i^P(p) = D_i(p) + \lambda_i(p) I(p) + \max\left\{0, \left(1 - \frac{x_j}{D_j(p) + \lambda_j(p) I(p)}\right)(d_i(p) + I(p))\right\},$$

where $\lambda_i(p) \in [0,1]$ *and* $\lambda_1(p) + \lambda_2(p) = 1$.[14]

[13] The proportional rationing rule does not rely on the specification provided here in which each consumer demands a single good. This rule may instead be derived from a continuum of identical consumers each with downward sloping demand.

[14] For application to a pricing game, one would require that each λ_i be measurable. As the prices are fixed in our model, this assumption is not required.

The λ_i's simply represent the fraction of indifferent consumers that shop at firm i first. We now characterize the consumers' equilibrium shopping rules.

Proposition 11. *Any equilibrium is characterized by a critical ratio* $r^* \in [0, \infty]$ *and a fraction* $\lambda^* \in [0, 1]$ *such that consumers shop according to the following rule:*

(i) any consumer with $s_i(p_i) \geq 0 > s_j(p_j)$ *shops at firm i first and does not shop in the second round,*

(ii) any consumer with $\min\{s_1(p_1), s_2(p_2)\} \geq 0$ *and* $s_2(p_2)/s_1(p_1) > r^*$ *shops at firm 2 in the first round,*

(iii) any consumer with $\min\{s_1(p_1), s_2(p_2)\} \geq 0$ *and* $s_2(p_2)/s_1(p_1) < r^*$ *shops at firm 1 in the first round, and*

(iv) λ^* *of the consumers with* $\min\{s_1(p_1), s_2(p_2)\} \geq 0$ *and* $s_2(p_2)/s_1(p_1) = r^*$ *shop at firm 1 first while the remaining* $1 - \lambda^*$ *of these consumers shop at firm 2 first.*

Proof of Proposition 11. Clearly, any consumer with $s_i \geq 0 > s_j$ will shop at firm i in the first round and not shop in the second round. Taking this behavior and any shopping rule by the consumers with $\min\{s_1, s_2\} \geq 0$, let α_{ij} denote the probability that a consumer with $\min\{s_1, s_2\} \geq 0$ receives a good from firm j when shopping at firm i first. Given these probabilities, a consumer with values $v \geq p$ will strictly prefer to shop at firm 1 in the first round if and only if

(2.1) $\alpha_{11}s_1 + \alpha_{12}s_2 > \alpha_{21}s_1 + \alpha_{22}s_2.$

Similarly, such a consumer strictly prefers to shop at firm 2 first if and only if the inequality in (1) is reversed. Note that we may rearrange (1) as

$$\frac{s_2}{s_1} < \frac{\alpha_{11} - \alpha_{21}}{\alpha_{22} - \alpha_{12}}.$$

Thus, defining

$$r^* = \begin{cases} \frac{\alpha_{11}-\alpha_{21}}{\alpha_{22}-\alpha_{12}} & \text{if} \quad \alpha_{12} < \alpha_{22} \\ \infty & \text{o.w.} \end{cases},$$

we can say that consumers with values $v \geq p$ strictly prefer to shop at firm 1 first if and only if $s_2/s_1 < r^*$. To complete the proofs of parts (ii) and (iii), it suffices to show that r^* is well defined. This is the case if either $\alpha_{11} - \alpha_{21} > 0$ or $\alpha_{22} - \alpha_{21} > 0$. Given the shopping decisions of the consumers, let N_{ij} denote the mass of consumers that shops at firm j in round i. Note that

$$\begin{aligned} \alpha_{ii} &= \min\left\{\frac{x_i}{N_{11}}, 1\right\} \text{ and} \\ \alpha_{ij} &= (1 - \alpha_{ii}) \min\left\{\max\left\{\frac{x_j - N_{1j}}{N_{2j}}, 0\right\}, 1\right\}. \end{aligned}$$

Thus, if $\alpha_{ii} < 1$, then $\alpha_{ji} = 0$ and $\alpha_{ii} - \alpha_{ji} > 0$, while if $\alpha_{ii} = 1$, then $\alpha_{ij} = 0$ and $\alpha_{jj} - \alpha_{ij} > 0$. It follows that r^* is well defined.

Lastly, note that any consumer with valus $v \geq p$ and $s_2/s_1 = r^*$ is indifferent between shopping at either firm first, and so any shopping order by these consumers is optimal. If the mass of these consumers is zero, then their shopping decisions will not influence the probabilities α_{ij}, and so any decisions by these consumers represents equilibrium behavior. Alternatively, if the mass of these consumers is positive, then the probabilities α_{ij} and critical surplus ratio r^* will depend on the fraction of these consumers who shop at firm 1 first. Let

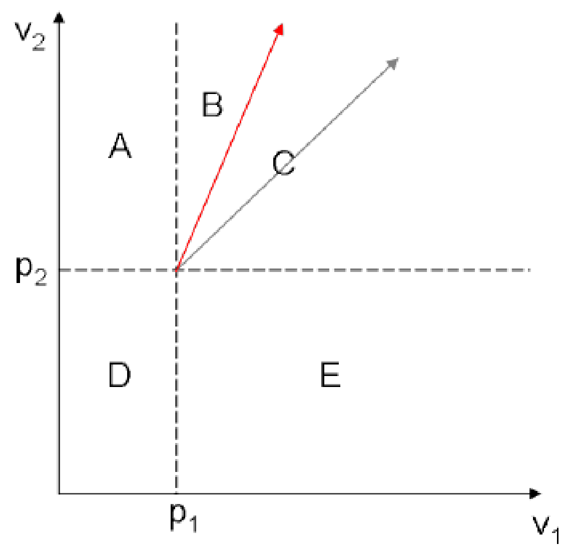


Figure 2.1: Equilibrium shopping rule.

$r^*(\lambda)$ represent the critical surplus ratio as a function of this fraction. We have shown that in equilibrium, there is a critical surplus ratio r^* such that (ii) and (iii) in the statement of the proposition hold. It follows that in any equilibrium, the fraction λ^* of consumers with $s_2/s_1 = r^*$ that shop at firm 1 first must be chosen such that $r^*(\lambda^*) = r^*$. ■

This equilibrium shopping rule is depicted in Figure 2.1. Individuals with values in regions A and E shop only at firm 2 and firm 1, respectively. Those in region B shop at firm 2 first, then at firm 1 if unsatiated. Similarly, those in region C shop at firm 1 first and then at firm 2. Lastly, individuals whose valuations are below the prices (region D), do not shop. Note that the region betweenn the red and grey lines corresponds to individuals who shop at firm 1 first even though they would receive the greatest surplus from firm 2. These individuals choose to shop at their least preferred firm in order to increase their probability of consumption.

Note that the equilibrium is necessarily nonunique when $0 < \lambda^* < 1$ as there are a mass of consumers with surplus ratio $s_2/s_1 = r^*$. In this case, any allocation of λ^* of the indifferent consumers shopping at firm 1 first and $1-\lambda^*$ at firm 2 first would be an equilibrium. If there is not a mass of indifferent consumers, then any $\lambda \in [0,1]$ would constitute an equilibrium, as it would not influence the probability of any consumer receiving a good. This is not the only source of multiplicity of equilibrium. As we will demonstrate shortly with Example 1, there may be multiple equilibria in which the quantity sold differs across equilibria.

Before continuing, we may express now the demand facing a firm i in an equilibrium. To do so, we must first define some notation. Given an equilibrium characterized by (r^*, λ^*), let $D_i(p, r^*)$ denote the mass of individuals who strictly prefer to shop at firm i in the first round, $d_i(p, r^*)$ the mass of individuals who strictly prefer to shop at firm j in the first round but would shop in the second round, and $I(p, r^*)$ the mass of individuals who are indifferent between shopping at either firm in the first round. Formally,

$$\begin{aligned}
D_1(p, r^*) &= \Delta\left(\{v : s_2(p_2)/s_1(p_1) < r^* \text{ or } s_1(p_1) \geq 0 > s_2(p_2)\}\right), \\
d_1(p, r^*) &= \Delta\left(\{v : s_2(p_2)/s_1(p_1) > r^* \text{ and } s_1(p_1) \geq 0\}\right), \\
I(p, r^*) &= \Delta\left(\{v : s_2(p_2)/s_1(p_1) = r^*\}\right),
\end{aligned}$$

where $D_2(p, r^*)$ and $d_2(p, r^*)$ are defined analogously. Given this notation, we may express the demand facing firm 1 as follows, with the demand facing firm 2 defined similarly.

$$\begin{aligned}
& D_1^*(p, r^*, \lambda^*) \\
= \; & D_1(p, r^*) + \lambda^* I(p, r^*) \\
& + \max\left\{0, \left(1 - \frac{x_2}{D_2(p, r^*) + (1-\lambda^*) I(p, r^*)}\right)\left(d_1(p, r^*) + (1-\lambda^*) I(p, r^*)\right)\right\}.
\end{aligned}$$

Note that the basic form of the demand is identical to the proportional rationing rule. In particular, $r^* = 1$ yields a proportional rationing rule.

The following proposition guarantees that a pure strategy equilibrium exists.

Proposition 12. *A pure strategy equilibrium exists.*

The following lemma will be useful in proving the existence of an equilibrium. Define N_{ij} to be the mass of consumers that shop at firm j in round i and $N = (N_{ij})_{i,j=1,2}$. Define $r(N)$ to be the set of critical surplus ratios in the sense of the equilibrium shopping rule of Proposition 11.

Lemma 5. *Suppose that $N_{ij}^n \to N_{ij}^*$ for $i, j = 1, 2$ and that $r^n \in r(N^n)$ for each n with $r^n \to r^*$. Then $r^* \in r(N^*)$.*

Proof of Lemma 5. Suppose that $r^* < \infty$. Then $r^n < \infty$ for sufficiently large n, and so

$$r^n = \frac{\alpha_{11}(N^n) - \alpha_{21}(N^n)}{\alpha_{22}(N^n) - \alpha_{12}(N^n)}.$$

It follows that

$$\frac{\alpha_{11}(N^n) - \alpha_{21}(N^n)}{\alpha_{22}(N^n) - \alpha_{12}(N^n)} \to \frac{\alpha_{11}(N^*) - \alpha_{21}(N^*)}{\alpha_{22}(N^*) - \alpha_{12}(N^*)},$$

so

$$r^* = \frac{\alpha_{11}(N^*) - \alpha_{21}(N^*)}{\alpha_{22}(N^*) - \alpha_{12}(N^*)} = r(N^*).$$

Note that we are free to redefine the critical ratio as $q = s_1/s_2$ via a relabelling of firms. If $r^* = \infty$, then $q^* = 0$, and the previous analysis applies. ■

The main idea behind the proof of Proposition 12 is to realize the critical surplus ratio as a rotation of an arbitrary line through p, which can parameterized by its angle $\theta \in \Theta = [0, \pi/2]$. We then use the best response correspondence for all consumers as an upper hemicontinuous correspondence and apply Kakutani's fixed point theorem to obtain an equilibrium.

Proof of Proposition 12. Note that for any $r \in [0, \infty]$, there is a unique angle $\theta \in [0, \pi/2]$ such that the all values with $s_2/s_1 = r$ lie on the line of angle θ through p. Let $\rho(\theta)$ denote the ratio r corresponding to the angle θ. Then the angle θ is defined by the law of cosines, and so $\rho(\theta)$ is such that

$$\begin{aligned} \theta^n &\to \theta \text{ if and only if } \rho(\theta^n) \to \rho(\theta) \text{ and} \\ r^n &\to r \text{ if and only if } \theta(r^n) \to \theta(r). \end{aligned}$$

Let $\iota(\theta) = \{v : r = \rho(\theta) \text{ and } s_1 > 0\}$. Define the correspondence $\Phi : [0, \pi/2] \times [0, 1] \to [0, \pi/2] \times [0, 1]$ as follows. For any $(\theta, \beta) \in [0, \pi/2] \times [0, 1]$, let $\psi(\theta, \beta)$ denote the set of all critical ratios (angles) $\rho^{-1}(r^*)$ when consumers shop according to the rule (r^*, λ) for some $\lambda \in [\beta\mu(\{p\}) / (\beta\mu(\{p\}) + \mu(\iota(\theta))), 1]$. Then define $\Phi(\theta, \beta) = (\psi(\theta, \beta), [0, 1])$.

By continuity of each N_{ij} in λ and the continuity of $r(N)$, the sets $\Phi(\theta, \beta)$ must be closed, convex, and nonempty. In order to apply Kakutani's fixed point theorem, we need only show that Φ has a closed graph.

Let $(\theta^n, \beta^n) \to (\theta^*, \beta^*)$ with $(\phi^n, \gamma^n) \in \Phi(\theta^n, \beta^n)$ and $(\phi^n, \gamma^n) \to (\phi^*, \gamma^*)$. Note that for all n, there exists a $\lambda^n \in [\beta^n\mu(\{p\}) / (\beta^n\mu(\{p\}) + \mu(\iota(\theta^n))), 1]$ such that ϕ^n is the angle of a critical ratio when consumers shop according to $(\rho(\theta^n), \lambda^n)$. This corresponds to a $\varphi^n \in [0, 1]$ such that φ^n of the $\iota(\theta^n)$ consumers shop at firm 1 first. Define the following sets.

$$\begin{aligned} A_i &= \{v : s_i \geq 0 > s_j\} \\ B(\theta) &= \Delta(\{v : s_2/s_1 < \rho(\theta),\ \min\{s_1, s_2\} > 0\}),\ \text{and} \\ C(\theta) &= \Delta(\{v : s_2/s_1 > \rho(\theta),\ \min\{s_1, s_2\} > 0\}). \end{aligned}$$

We will use these sets to show that $N_{ij}^n \to N_{ij}^*$. The previous lemma will thus imply that $(\phi^*, \gamma^*) \in \Phi(\theta^*, \beta^*)$.

Consider three cases: (i) $\theta^n < \theta^{n+1} < \theta$ for all n, (ii) $\theta^n = \theta$ for all n, (iii) $\theta^n > \theta^{n+1} > \theta$ for all n. Note that an arbitrary sequence (θ^n, β^n) may not satisfy (i), (ii), or (iii), but there must exist a subsequence that satisfies on of these cases. Since all subsequences converge to (θ^*, β^*), we are free to pick a subsequence which falls into one of the three cases, and without loss of generality, we may take that subsequence to be θ^n itself.

(i) Note that

$$\begin{aligned} B(\theta^n) \cup \iota(\theta^n) &\subset B\left(\theta^{n+1}\right) \\ C(\theta^n) &\supset C\left(\theta^{n+1}\right) \cup \iota\left(\theta^{n+1}\right). \end{aligned}$$

Then note that

$$\begin{aligned} &\lim_n B(\theta^n) \cup \iota(\theta^n) = B(\theta^*)\ \text{and} \\ &\lim_n C(\theta^n) = C(\theta^*) \cup \iota(\theta^*). \end{aligned}$$

The statement then follows from Theorems 9D and 9E of Halmos (1974), which states that if a sequence of sets is monotonic (by the order of containment) and at least one of the sets has finite measure, then the measure of the limit of the sets is equal to the limit of the measure of the sets. Thus

$$\begin{aligned}
\lim_n N_{11}^n(\varphi^n) &= \mu(A_1) + \lim_n \mu(B(\theta^n)) + \varphi^n \mu(\iota(\theta^n)) + \beta^n \mu(\{p\}) \\
&= \mu(A_1) + \mu(B(\theta^*)) + \beta^* \mu(\{p\}) \\
&= N_{11}^*(\varphi = 0), \\
\lim_n N_{12}^n(\varphi^n) &= \mu(A_2) + \lim_n \mu(C(\theta^n)) + (1-\varphi^n)\mu(\iota(\theta^n)) + (1-\beta^n)\mu(\{p\}) \\
&= \mu(A_2) + \mu(C(\theta^*)) + \mu(\iota(\theta^*)) + (1-\beta^*)\mu(\{p\}) \\
&= N_{12}^*(\varphi = 0), \\
\lim_n N_{21}^n(\varphi^n) &= \lim_n (1-\alpha_{11}^n)(N_{12}^n - \mu(A_2)) \\
&= (1-\alpha_{11}^*)(N_{22}^*(\varphi = 0) - \mu(A_2)) \\
&= N_{21}^*(\varphi = 0) \\
\lim_n N_{22}^n(\varphi^n) &= \lim_n (1-\alpha_{22}^n)(N_{11}^n - \mu(A_1)) \\
&= (1-\alpha_{22}^*)(N_{11}^*(\varphi = 0) - \mu(A_1)) \\
&= N_{22}^*(\varphi = 0).
\end{aligned}$$

(ii) Note that $B(\theta^n) = B(\theta^*)$, $C(\theta^n) = C(\theta)$, and $\iota(\theta^n) = \iota(\theta)$ for all n. Thus, the limits in the previous case all hold, except with $N^n \to N^*(\varphi = \lim \varphi^n)$.

(iii) This case is identical to case (i) except that the containments of the B's and C's are reversed. Thus, $N^n \to N^*(\varphi = 1)$.continuous in θ', and so Φ is an upper hemicontinuous correspondence.

We conclude by Lemma 1 that $(\phi^*, \gamma^*) \in \Phi(\theta^*, \beta^*)$. Thus, we may apply Kakutani's fixed point theorem to obtain a fixed point (θ^*, β^*). Note that there must be a corresponding $\lambda^* \in [\beta^* \mu(\{p\}) / (\beta^* \mu(\{p\}) + \mu(\iota(\theta^*))), 1]$ for which the shopping rule $(\rho(\theta^*), \lambda^*)$ is an equilibrium for consumers. ■

A natural follow up question is whether the equilibrium rationing rule is unique. The following example shows that this is not the case, even when demand is continuous and the revenue function $(p_i D(p_i))$ is concave.

Example 1. *Suppose that products are homogenous and that market demand is given by $D(p_i) = 2 - \sqrt{p_i}$. Since demand is continuous, then the fraction λ has no influence on the equilibrium, so we omit the λ. There exists a neighborhood $\mathcal{N}$ of $(0, .3, 1, .75)$ such that for all $(p_1, p_2, x_1, x_2) \in \mathcal{N}$, $r_1^* = 1$ is an equilibrium and some r_2^* in a neighborhood of 0.82 is an equilibrium. Moreover, the quantity sold by firm 2 under r_1^* is strictly less than under r_2^*.*

To see this, note that for $r^* = 1$, firm 2 does not sell its full capacity. It follows that $\alpha_{12} = 1 - \alpha_{11}$, so shopping at firm 1 is strictly better than shopping at firm 2 for all consumers. To find the other equilibrium, note that r^* corresponds to some $\omega > p_2$, for consumers have $v > \omega$ if and only if $s_2/s_1 > r^*$.[15] *Thus, this equilibrium is defined by*

$$\frac{x_1}{\sqrt{\omega} - \sqrt{p_1}}(\omega - p_1) + \frac{x_2 - (2 - \sqrt{\omega})}{\sqrt{\omega} - \sqrt{p_2}}(\omega - p_2) = (\omega - p_2),$$

which holds when the individual with value ω is indifferent provided that all consumers with $v > \omega$ shop at firm 2 first and the others shop at firm 1 first. This equation reduces to

$$x_1\left(\sqrt{\omega} + \sqrt{p_1}\right) = (2 - \sqrt{p_2} - x_2)\left(\sqrt{\omega} + \sqrt{p_2}\right)$$

which can be solved numerically.

[15] This transformation from r to ω is formalized in the following section.

Example 1 shows that not only is the equilibrium rationing rule not unique, but the quantity sold by firms is not unique either. In general, this may result in the existence of an equilibrium selection that varies discontinuously in the price and at some prices may actually be increasing in price. While this is problematic, the following proposition shows that there is an equilibrium selection that coincides with the proportional rationing rule, making it a natural selection for application. Let $q_i(r^*, \lambda^*)$ denote the quantity sold by firm i when consumers shop according to the rule (r^*, λ^*).

Proposition 13. *For any prices and capacities, there exists an equilibrium in which the quantity sold by each firm coincides with that specified by the proportional rationing rule. If* (r_1^*, λ_1^*) *and* (r_2^*, λ_2^*) *are both equilibria in which* $q_i(r_1^*, \lambda_1^*) < q_i(r_2^*, \lambda_2^*)$ *for some firm* i*, then* $r_1^* = 1$ *and* $q_i(r_1^*, \lambda_1^*) = D_i^P(p)$ *for each firm* i*.*

Proof of Proposition 13. For any equilibrium in which $r^* \neq 1$, both firms must sell their capacity. Otherwise, the shopping rule specifies that some individuals shop at their least preferred firm first. Since firms do not sell out, these individuals are guaranteed a good regardless of their shopping decision, and thus should shop for their most preferred good in the first round. It follows that the quantity sold by a firm in equilibrium may only differ from the proportional quantity if a firm does not sell its full capacity under the proportional rule, that is, $D_i^P(p) < x_i$ for some firm i.

Suppose that some firm i does not sell its capacity with proportional rationing. Without loss of generality, suppose that $i = 2$. Given the rule $r^* = 1$, $\alpha_{12} = 1 - \alpha_{11}$, $\alpha_{22} = 1$, and $\alpha_{21} = 0$. Clearly, individuals with $s_2 > s_1$ should shop at firm 2 first. For any individual with $s_1 > s_2$, the following are equivalent

$$\begin{aligned}\alpha_{11}s_1 + \alpha_{12}s_2 &> \alpha_{22}s_2 + \alpha_{21}s_1 \\ \alpha_{11}s_1 + (1-\alpha_{11})\,s_2 &> s_2 \\ \alpha_{11}(s_1 - s_2) &> 0.\end{aligned}$$

so these individuals should shop at firm 1 first. Therefore, $r^* = 1$ is an equilibrium. ■

It is worth highlighting the latter result of Proposition 2. The standard assumption in economics is that is that individuals shop for the lowest price or greatest surplus, which corresponds to the shopping rule $r^* = 1$. In addition to this being the intuitive equilibrium, it is also a surplus maximizing equilibria for the consumers whose equilibrium strategies differ across equilibria.

Proposition 14. *Suppose that* (r_1^*, λ_1^*) *and* (r_2^*, λ_2^*) *are equilibria with the property that* $q_i(r_1^*, \lambda_1^*) = D_i^P(p) < q_i(r_2^*, \lambda_2^*)$ *for some firm* i*. Then all consumers with* $s_2(p_2)/s_1(p_1) \in (r_1^*, r_2^*)$ *receive a greater expected utility under* (r_1^*, λ_1^*) *than under* (r_2^*, λ_2^*)*.*

Proof of Proposition 14. Note that all consumers with $s_2/s_1 \in (r_1^*, r_2^*)$ receive a good with certainty under (r_1^*, λ_1^*), while this is not necessarily the case under (r_2^*, λ_2^*). Further, under (r_1^*, λ_1^*), all such consumers have a higher probability of receiving a good from their most preferred firm than under (r_2^*, λ_2^*). ■

This proposition suggests that a subset of the consumers would have incentive to coordinate their strategies and select the equilibrium in which the proportional quantity is sold. Thus, both intuition and collective action justify the selection of the "proportional" equilibrium, while other equilibria require the expectation that consumers fail to coordinate optimally. The next section focuses on homogenous goods and strengthening the result of Proposition

2.

2.4 Homogenous Goods

In this section, we focus on the case of homogenous goods. This is most relevant to the literature, as the vast majority of studies of Bertrand-Edgeworth competition is conducted with homogenous goods. In this setting, we are able to find stronger support for the proportional rationing rule, as well as gain some additional insight into the equilibrium rationing rule.

For this section, we let $D(p_i)$ be the measure of consumers with values strictly higher than p_i. We maintain the subscript on the p_i so as to clarify that the price is a scalar. Henceforth, we use v to refer to a consumer's value of consumption. There should be no more ambiguity whether v is a vector since $v_1 = v_2 = v$. Define $A = \lim_{p_i \to 0} D(p_i)$, possibly infinite.

Assumption 8. *Market demand $D(\cdot)$ is continuous.*

The following lemma characterizes the equilibrium rationing rule in a more salient manner than Proposition 1.

Lemma 6. *Suppose that goods are homogenous, Assumption 2 holds, and $p_1 < p_2$. Then any equilibrium is characterized by a critical value $\omega > p_2$ such that*

(i) all consumers with value $v \in [p_1, \omega]$ shop at firm 1 in the first round,

(ii) all consumers with value $v > \omega$ shop at firm 2 in the first round.

If $p_1 = p_2$ and $x_1 + x_2 \leq D(p_i)$, then $x_i / (x_1 + x_2)$ of the consumers with value $v \geq p_i$ shop at firm i. If $p_1 = p_2$ and $x_1 + x_2 > D(p_i)$, then any division of consumers is an equilibrium.

Proof of Lemma 6. The first fact follows immediately from the fact that the ratio s_2/s_1

is strictly increasing in v for $p_1 < p_2$. The second fact is necessary since consumers in equilibrium must have the same probability of receiving a good at each firm. The last result follows as the when $x_1 + x_2 > D(p_i)$ for $p_1 = p_2$, all consumers are guaranteed a good regardless of shopping decisions. ■

Thus, we find that those with the highest valuation for the good will pay a premium to guarantee themselves a good. Note that for a given ω, the equilibrium demand facing each firm is

$$D_i(p,\omega) = \begin{cases} D(p_i) - D(\omega) & \text{if} \quad p_i < p_j \\ \max\{\lambda_i(p) D(p_i), D(p_i) - x_j\} & \text{if} \quad p_i = p_j \\ D(\omega) + \max\left\{0, \left(1 - \frac{x_j}{D(p_j) - D(\omega)}\right)(D(p_i) - D(\omega))\right\} & \text{if} \quad p_i > p_j \end{cases}.$$

where $\lambda_i = x_i / (x_1 + x_2)$ if $x_1 + x_2 \leq D(p_i)$, $\lambda_i \in [0,1]$ otherwise, and $x_1 \leq D(p_i) - D(\omega)$ if $p_i < p_j$ and $\omega < A$.

It is worth highlighting that the residual demand facing the firm with the higher price is a function of the capacity of that firm through ω. While this is fairly intuitive, the traditional efficient and proportional rules lack this property.

We use the following example to demonstrate how one might go about deriving the equilibrium rationing rule, and demonstrate the different properties that it exhibits when compared with traditional rationing rules.

Example 2. *Suppose that demand is given by $D(p_i) = A - bp_i$. We begin by finding the equilibrium candidates. Any candidate equilibrium, characterized by ω, must satisfy the condition that a consumer with value ω is indifferent between shopping at either firm in the first round. Since the low price firm must sell out in the first round, then any such ω must* *be defined by either*

$$\text{(2.2)} \qquad \frac{x_1}{D(p_1) - D(\omega)}(\omega - p_1) = \frac{x_2}{D(\omega)}(\omega - p_2) \quad \textit{or}$$

$$\text{(2.3)} \qquad \frac{x_1}{D(p_1) - D(\omega)}(\omega - p_1) + \frac{x_2 - D(\omega)}{D(p_2) - D(\omega)}(\omega - p_2) = (\omega - p_2).$$

The equation in (1) corresponds to the case in which $x_2 \leq D(\omega)$*, so that firm 2 exhausts its capacity in the first round. Thus, the expected utility from shopping at either firm would be the probability of receiving a good at that firm multiplied by the surplus received from such a purchase. The equation in (2) corresponds to the case in which* $x_2 > D(\omega)$*, so that firm 2 does not exhaust its capacity in the first round. As such, the expected utility from shopping at firm 2 first is simply the surplus from purchase, while the expected utility from shopping at firm 1 is as in (1) except with an additional term corresponding to the probability of being rationed and receiving a good from firm 2 multiplied by the corresponding surplus.*

In the linear case, (1) reduces to

$$\frac{x_1}{b} = \frac{x_2}{A - b\omega}(\omega - p_2),$$

which may easily be solved to obtain

$$\text{(2.4)} \qquad \omega = \frac{x_1(A - bp_2)}{b(x_1 + x_2)} + p_2.$$

Alternatively, (2) reduces to

$$x_1 = D(p_2) - x_2.$$

Thus, any ω *is a candidate for an equilibrium if* $x_1 + x_2 = D(p_2)$*.*

Next, we check whether these candidates are equilibria. For the first candidate, ω was found by assuming that $x_2 \leq D(\omega)$. Further, as in the characterization, it must be that $\omega > p_2$. Thus, we need only verify the parameters for which these conditions are satisfied. Clearly, $\omega > p_2$ as defined in (3). It remains to check whether $x_2 \leq A - b\omega$. The following are equivalent.

$$\begin{aligned} x_2 &\leq A - bp_2 - \frac{x_1 (A - bp_2)}{x_1 + x_2} \\ x_1 x_2 + x_2^2 &\leq A x_2 - x_2 b p_2 \\ x_1 + x_2 &\leq D(p_2) . \end{aligned}$$

Thus, ω as defined in (3) is an equilibrium if $x_1 + x_2 \leq D(p_2)$. Any $\omega > p_2$ such that $x_2 \geq D(\omega)$ is an equilibrium if $x_1 + x_2 = D(p_2)$, and so $\omega = A$ is an equilibrium otherwise. Lastly, it is easy to verify that $\omega = A$ is not an equilibrium if $x_1 + x_2 < D(p_2)$.

In summary, the set of equilibria Ω is given by

$$\Omega = \begin{cases} \left\{ \frac{x_1(A-bp_2)}{b(x_1+x_2)} + p_2 \right\} & \text{if} \quad x_1 + x_2 < A - bp_2 \\ (p_2, A] & \text{if} \quad x_1 + x_2 = A - bp_2 \\ \{A\} & \text{if} \quad x_1 + x_2 > A - bp_2 \end{cases} .$$

This simple example allows us to easily compare the equilibrium rationing rule to the traditional proportional and efficient rules by examining the residual demand associated with each rule. Under efficient rationing, the residual demand is a parallel shift of the market demand curve.[16] Under proportional rationing, the residual demand is a pivot of the demand curve from the price at which demand is zero. Unlike these traditional rules, equilibrium residual

[16]The alternative name for the efficient rule, the parallel rule, is taken from this property.

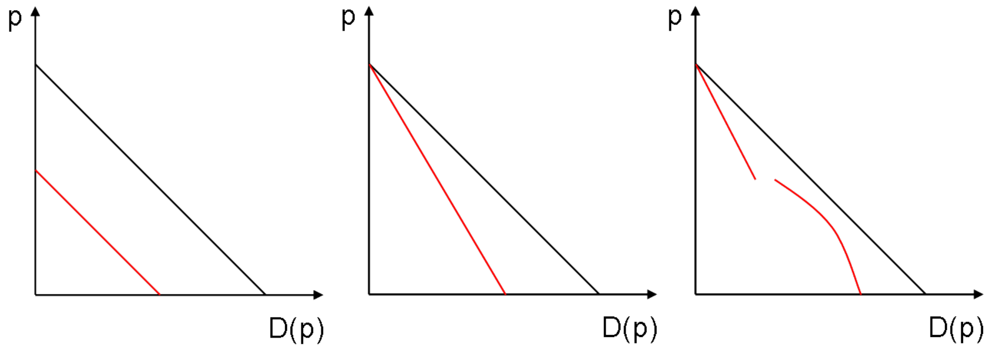


Figure 2.2: Residual demand with equilibrium and traditional rationing.

demand is not a rigid transformation of the market demand. Despite market demand being linear, the equilibrium residual demand is nonlinear and is discontinuous. The nonlinearity results from the fact that ω is a function of prices, while the discontinuity in this example is due to the fact that there are a continuum of equilibria when $x_1 + x_2 = D(p_2)$.

For clarity, it is worthwhile to examine the proportional rule when goods are homogenous and $I(p) = 0$. In this case, proportional rule is given by

$$D_i^P(p) = \begin{cases} D(p_i) & \text{if} \quad p_i < p_j \\ \max\{\lambda_i(p) D(p_i), D(p_i) - x_j\} & \text{if} \quad p_i = p_j \\ \max\left\{0, \left(1 - \frac{x_j}{D(p_j)}\right) D(p_i)\right\} & \text{if} \quad p_i > p_j \end{cases}.$$

Recall that in equilibrium, the low price firm must sell its capacity in the first round if $\omega < A$. Thus, it is immediately apparent that the quantity sold by each firm in equilibrium is at least as large as the proportional quantity. Formally, $\min\{x_i, D_i(p,\omega)\} \geq \min\left\{x_i, D_i^P(p)\right\}$ for each firm i. Example 1 shows that this inequality may be strict, even if the revenue function is concave. Naturally, we would like reasonable conditions under which no such "ill-behaved" equilibria exist.

Assumption 9. *Market demand $D(p)$ is continuously differentiable and concave.*

The following proposition shows that Assumption 3 is a sufficient condition for all equilibria to be quantity equivalent, and thus profit equivalent for the firms.

Proposition 15. *Suppose that goods are homogenous and Assumption 3 is satisfied. Then the quantity sold by each firm in any equilibrium coincides with the quantity specified by the proportional rationing rule. If* $D_i^P < x_i$ *for some firm* i*, then the unique equilibrium is* $\omega^* = A$ $(r^* = 1)$*.*

Proof of Proposition 15. The latter statement follows as a corollary from Proposition 13.

Since both firms sell their full capacity in any equilibrium that is not proportional, it suffices to consider prices and capacities such that $D_i^P(p) < x_i$ for some firm i. Without loss of generality, suppose that $p_1 < p_2$, so that $D_2^P(p) < x_2$.

Suppose that $x_1 + x_2 \leq D(p_2)$. Then note it must be that

$$\begin{aligned} \frac{D(p_2)}{D(p_1)} x_1 + x_2 &< D(p_2), \\ x_2 &< D(p_2)\left(1 - \frac{x_1}{D(p_1)}\right) = D_2^P. \end{aligned}$$

This is a contradiction, since we assumed that $D_2^P < x_2$. Therefore, $x_1 + x_2 > D(p_2)$.

Let $u_i(v, \omega)$ denote the expected utility of a consumer with value v when shopping at firm i first and all other consumers shop according to the rule ω. Note that in any equilibrium the individual with value ω must be be indifferent, so that $u_1(\omega, \omega) = u_2(\omega, \omega)$. Thus, ω must be defined by one of the two following equations:

$$\text{(2.5)} \qquad \frac{x_1}{D(p_1) - D(\omega)}(\omega - p_1) = \frac{x_2}{D(\omega)}(\omega - p_2) \text{ or}$$

$$\text{(2.6)} \qquad \frac{x_1}{D(p_1) - D(\omega)}(\omega - p_1) + \frac{x_2 - D(\omega)}{D(p_2) - D(\omega)} = (\omega - p_2).$$

We begin by showing that (5) may not define an equilibrium when $x_1 + x_2 > D(p_2)$. We rewrite (5) as

$$\frac{x_1}{D(p_1) - D(\omega)}(\omega - p_1) = \frac{D(p_2) - x_2}{D(p_2) - D(\omega)}(\omega - p_2).$$

Note that $x_1 > D(p_2) - x_2$, so the left hand side is such that

$$\frac{x_1}{D(p_1) - D(\omega)}(\omega - p_1) > \frac{D(p_2) - x_2}{D(p_1) - D(\omega)}(\omega - p_1).$$

The following are equivalent.

$$\begin{aligned}
\frac{D(p_2) - x_2}{D(p_1) - D(\omega)}(\omega - p_1) &\geq \frac{D(p_2) - x_2}{D(p_2) - D(\omega)}(\omega - p_2) \\
\frac{D(p_2) - D(\omega)}{\omega - p_2} &\geq \frac{D(p_1) - D(\omega)}{\omega - p_1} \\
-\frac{D(p_2) - D(\omega)}{p_2 - \omega} &\geq -\frac{D(p_1) - D(\omega)}{p_1 - \omega} \\
\frac{D(p_1) - D(\omega)}{p_1 - \omega} &\geq \frac{D(p_2) - D(\omega)}{p_2 - \omega}.
\end{aligned}$$

The final statement is true since D is concave and $p_1 < p_2$. Therefore, we conclude that

$$\frac{x_1}{D(p_1) - D(\omega)}(\omega - p_1) > \frac{D(p_2) - x_2}{D(p_2) - D(\omega)}(\omega - p_2),$$

and so (5) cannot hold.

We next use (4) to show that if $\omega^* = A$ is an equilibrium, then there is no largest equilibrium $\overline{\omega} < A$. Let Ω be the set of equilibria and consider $\overline{\omega} = \sup \Omega \backslash \{A\}$. Since Ω is closed, then $\overline{\omega} \in \Omega$. Note that

$$\lim_{v \to A} \frac{x_1}{D(p_1) - D(v)} (v - p_1) = \frac{x_1}{D(p_1)} (A - p_1),$$

while

$$\lim_{v \to A} \frac{x_2}{D(v)} (v - p_2) = \infty.$$

Thus, it must be that $\overline{\omega} < A$. We subtract the right hand side from both sides of (4) and differentiate with respect to ω to get

$$\begin{aligned}
& \frac{\partial}{\partial \omega} \left(\frac{x_1}{D(p_1) - D(\omega)} (\omega - p_1) - \frac{x_2}{D(\omega)} (\omega - p_2) \right) \\
= \; & \frac{x_1}{D(p_1) - D(\omega)} + \frac{x_1}{(D(p_1) - D(\omega))^2} D'(\omega)(\omega - p_1) \\
& - \frac{x_2}{D(\omega)} + \frac{x_2}{(D(\omega))^2} D'(\omega)(\omega - p_2). \qquad (2.7)
\end{aligned}$$

Evaluating at $\omega = \overline{\omega}$, we substitute the relation (4) into (6) to get

$$\begin{aligned}
& \frac{x_1}{D(p_1) - D(\omega)} + \frac{x_1}{(D(p_1) - D(\omega))^2} D'(\omega)(\omega - p_1) \\
& - \frac{x_1}{D(p_1) - D(\omega)} \frac{\omega - p_1}{\omega - p_2} + \frac{x_2}{(D(\omega))^2} D'(\omega)(\omega - p_2) \\
= \; & \frac{x_1}{D(p_1) - D(\omega)} \left(1 - \frac{\omega - p_1}{\omega - p_2} \right) \\
& + \frac{x_1}{(D(p_1) - D(\omega))^2} D'(\omega)(\omega - p_1) + \frac{x_2}{(D(\omega))^2} D'(\omega)(\omega - p_2).
\end{aligned}$$

Since $\omega - p_1 > \omega - p_2$, then the first term is negative, while the last two terms are nonpositive since $D' \leq 0$. Thus, we conclude that there is some $\varepsilon > 0$ such that

$$u_1 (\overline{\omega} + \varepsilon, \overline{\omega} + \varepsilon) < u_2 (\overline{\omega} + \varepsilon, \overline{\omega} + \varepsilon) .$$

Since $D_2^P < x_2$, there is some $M > 0$ such that for all $\omega > M$,

$$D(\omega) + \left(1 - \frac{x_1}{D(p_1) - D(\omega)}\right)(D(p_2) - D(\omega)) < x_2.$$

This further implies that

$$u_1 (\omega, \omega) > u_2 (\omega, \omega) .$$

Therefore, $\overline{\omega}+\varepsilon < M$. By continuity of u_1 and u_2, the intermediate value theorem guarantees the existence of an $\omega^* \in (\overline{\omega} + \varepsilon, M)$ such that $u_1 (\omega^*, \omega^*) = u_2 (\omega^*, \omega^*)$. But this implies that $\omega^* \in (\overline{\omega}, A)$ is an equilibrium, contradicting the definition of $\overline{\omega}$. The result follows. ■

This result is particularly relevant for the literature on the coincidence of the Cournot outcome and Bertrand Edgeworth equilibrium. Concavity of demand is a common assumption in this literature, as well as in many studies of BE competition, and thus the previous proposition suggests that the proportional rationing rule is the only rule that should be considered in such studies.

2.5 Conclusion

In this paper, we have provided a general model of direct search in which consumers endogenously determine the demand faced by firms. We have shown that the quantity specified by the proportional rationing rule is always an equilibrium quantity. This is not necessarily the only equilibrium quantity, though we have shown that it is obtained in a surplus maximizing equilibrium when other equilibrium quantities exist. Moreover, when goods are homogenous, concavity of demand is a sufficient condition for the proportional rationing quantity to be realized in every equilibrium. This provides a strong support for the use of the proportional rationing rule in studying Bertrand Edgeworth oligopoly, with or without product differentiation.

It should be cautioned that this does not imply that the equilibrium of this game is proportional rationing. Only the quantity sold by firms agrees with the proportional rule, not consumer decisions. In this sense, equilibrium rationing and proportional rationing are identical from the firms' perspective, though not the consumers. In particular, when goods are homogenous, we have shown that in equilibrium, consumers with the highest valuations of the good shop at the high price firm initially. This results in an allocative inefficiency that is not present with the proportional rationing rule. In general this will lead to a disparity between the consumer surplus realized in equilibrium and that computed using the proportional rule.

There are three directions in which the results of this paper could be generalized. First, one could allow for oligopoly rather than duopoly. The model would need to be appended to add enough rounds of shopping so that each consumer would have an opportunity to shop at each firm, and the equilibrium rules would be cumbersome to formally define, however, they ought to share the same basic properties as our analysis. Equilibria would be characterized by surplus ratio thresholds which would determine the order of shopping as in Proposition 11, and proportional rationing quantity will always be an equilibrium. Second, one could allow for individual consumers' demand to depend on the prices. This would be accommodated simply by modifying the measure that represents the distribution of consumers so that it reflects the quantity demanded by consumers at a given price rather than the mass or density of consumers with a given valuation. Lastly, the results of Section 4 may be trivially extended to a case of vertical differentiation in which the surplus ratio is nondecreasing in the consumers' underlying value.

Chapter 3

Verifying Payoff Security in the Mixed Extension of Discontinuous Games

3.1 Introduction

We provide a new sufficient condition for payoff security that generalizes the set of games in which existence of mixed strategy equilibrium can be readily verified. Reny (1999) introduced the class of better reply secure games and showed existence of pure strategy Nash equilibrium in such games.[1] In addition, Reny provided two conditions that together are sufficient for a game to be better reply secure: payoff security and reciprocal upper semicontinuity.[2] The results apply to both the normal form of a game and its mixed extension. In the context of mixed strategies, the sufficient conditions of Reny are difficult to verify since they are based

[1]A game is better reply secure if for every nonequilibrium strategy profile x^* and every limiting payoff vector u^* at x^*, there is a player i that has a strategy that gives payoff strictly higher than u_i^* even when other players deviate slightly from x^*.

[2]A game is payoff secure if at any strategy profile x, each player has a strategy that earns a payoff close to that of x against slight deviations from x by the other players. Roughly speaking, a game is reciprocally upper semicontinuous, if whenever some player's payoff jumps down, some other player's payoff jumps up.

on the mixed extension of a game.[3] Our new sufficient condition is relatively straightforward to verify for a large class of games in which other sufficient conditions have not been readily applicable.

We introduce the concept of *disjoint payoff matching*, which imposes minor structure on the discontinuities of the game instead of solely on the payoffs. A game satisfies disjoint payoff matching if, given any strategy of a player, that player possesses a sequence of deviations that are at least as good in the limit and whose discontinuity sets are sufficiently disjoint. Sufficiently disjoint in this context means that there is no strategy profile by the other players that constitutes a discontinuity for a subsequence of these deviations.[4] The advantage that disjoint payoff matching has over other sufficient conditions for existence of mixed strategy equilibrium is that it is easily verifiable, owing to the fact that other conditions appeal to arbitrary probability measures or neighborhoods of strategies. Disjoint payoff matching can replace payoff security of the mixed extension of the game in Reny's or Bagh and Jofre's (2006) theorems that additionally require (weak) reciprocal upper semicontinuity to guarantee better reply security and thus existence of equilibrium in mixed strategies.[5]

Most closely related to the concept of disjoint payoff matching is a sufficient condition introduced in Bagh (2010). He introduces the notion of variational convergence of finite approximations of games. A result of this analysis is a sufficient condition for existence that involves computation of limits of mixed strategies of the finite approximations. To alleviate the difficulty of this computation, he establishes a stronger sufficient condition on the set of all mixed strategies of the game. Disjoint payoff matching places less restriction on the discontinuity sets along with more restriction on the payoffs than does Bagh's condition,

[3]Recent papers by Tian (2010), Nessah and Tian (2010), and McLennan et al. (2011), have worked to generalize the work of Reny (1999) and made great pushes toward a better understanding of Nash equilibria.

[4]We require that the limit superior of the discontinuity sets of the deviations be empty.

[5]Bagh and Jofre (2006) show that reciprocal upper semicontinuity can be replaced with the weaker concept of weak reciprocal upper semicontinuity.

facilitating greater ease of verification.

Other sufficient conditions in the literature that have attempted to alleviate the burden of computations of payoffs in the mixed extensions of games are uniform payoff security due to Monteiro and Page (2007) and uniform diagonal security due to Prokopovych and Yannelis (2012). Uniform payoff security, a condition on the set of pure strategies, is a sufficient condition for payoff security of the mixed extension of a compact game. Uniform diagonal security is similarly a condition on the set of pure strategies which under certain conditions is a generalization of uniform payoff security, but with the advantage of being a sufficient condition for existence of equilibrium rather than just payoff security of the mixed extension.[6]

The rest of the paper is formatted as follows. In Section 2, we introduce the necessary preliminaries. Section 3 defines disjoint payoff matching and proves that it implies payoff security for the mixed extension of the game. In Section 4, we use our results to show existence of equilibrium for a Bertrand-Edgeworth price setting oligopoly and we use an example from Sion and Wolfe (1957) to demonstrate how an equilibrium may not exist when disjoint payoff matching does not hold.

3.2 Preliminaries

An N-player compact game is a $2N$-tuple $G = (X_i, u_i)_{i=1}^N$, where the strategy space of each player i is a compact Hausdorff space X_i and the payoff of each player $u_i : X_1 \times ... \times X_N \mapsto \mathbb{R}$ is bounded and measurable. The mixed extension of the game is $\overline{G} = (\mathcal{M}_i, U_i)_{i=1}^N$ where the strategy space of each player i is $\mathcal{M}_i$, the set of regular probability measures on X_i, which is compact and convex, and the payoff function of player i is $U_i = \int u_i d\mu$, $\mu \in \mathcal{M} = \prod_{i=1}^N \mathcal{M}_i$.

[6]Prokopovych and Yannelis (2012) also adapt the concept of hospitality from Duggan (2007) to the domain of nonzero sum games. This condition involves the verification of deviations to a specific subset of the set of mixed strategies.

Note that, as defined, G and $\overline{G}$ are also the graphs of the games. The closure of the graph is denoted cl$G$. Finally, the frontier of G, denoted FrG, is defined to be the elements of the closure that are not in the graph, that is FrG =cl$G \setminus G$. The closure and frontier of the mixed extension are defined analogously.

Our condition and proofs will make reference to the sets of discontinuities of each player. Specifically, we reference the points at which a player's payoff is discontinuous in the other players' strategies. These are given by the discontinuity map $D_i : X_i \mapsto \mathcal{P}(X_{-i})$, where $\mathcal{P}(X_{-i})$ is the power set of X_{-i}, defined for all $x_i \in X_i$ as

$$D_i(x_i) = \{x_{-i} \in X_{-i} : u_i \text{ is discontinuous in } x_{-i} \text{ at } (x_i, x_{-i})\}.$$

3.3 Disjoint payoff matching and payoff security

We now introduce disjoint payoff matching. The condition has two parts: the first is that any player can deviate from any strategy and remain almost as well off, while the second imposes that the discontinuity sets of each deviation have limited intersection. The name "disjoint payoff matching" for our condition follows from the existence of a sequence of strategies which "match" the payoff of the original strategy and for which the discontinuity sets are sufficiently disjoint.

Definition 5. *The game G satisfies* disjoint payoff matching *if for all $x_i \in X_i$, there exists a sequence of deviations $\{x_i^k\} \subset X_i$ such that the following holds:*

(i) $\liminf_k u_i\left(x_i^k, x_{-i}\right) \geq u_i(x_i, x_{-i})$ *for all $x_{-i} \in X_{-i}$,*

(ii) $\limsup_k D_i\left(x_i^k\right) = \emptyset$.[7]

[7]Given a sequence of sets E_n, the limit superior is $\limsup_n E_n = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n$. This is equivalently

Remark 2. *It follows that one need only check this condition for x_i such that $D_i(x_i)$ is nonempty. That is, if $u_i(x_i, x_{-i})$ is continuous in x_{-i} at x_i, then the conditions of disjoint payoff matching are trivially satisfied by the constant sequence $x_i^k = x_i$.*

The second condition is clearly satisfied when $D_i\left(x_i^k\right) \cap D_i\left(x_i^l\right) = \emptyset$ for all $k \neq l$. This stronger empty intersection condition holds for the prominent examples in the literature. Unlike security concepts, there is no reference here to neighborhoods of the opponents' strategies. Further, the payoffs at the discontinuity points of the deviations are irrelevant since they are completely avoided in the limit, making the condition easy to verify and unrestrictive. It is worth noting that games in which discontinuities do not satisfy part (ii) of DPM often share best responses with a game that does satisfy DPM. That is, if $u_i(x)$ is player i's utility function in the game of interest and does not satisfy DPM, then there is often some strategically equivalent game for which player i's utility is of the form $v_i(x) = u_i(x) + f(x_{-i})$, where v_i satisfies condition (ii) of DPM.

We need one more definition before we prove the main result. The following concept was introduced by Reny (1999).

Definition 6. *The game G satisfies* payoff security *if for all $\varepsilon > 0$ and all $x \in X$, there exists for each player i a deviation $x_i' \in X_i$ and a neighborhood $\mathcal{N}(x_{-i})$ of x_{-i} such that $u_i(x_i', z) \geq u_i(x) - \varepsilon$ for all $z \in \mathcal{N}(x_{-i})$.*

The definition is analogous for the mixed extension of the game. Reny (1999) showed that payoff security combined with another condition is sufficient to guarantee the existence of a pure strategy Nash equilibrium.[8]

Payoff security is easily verified in the set of pure strategies, but is particularly difficult to

all points $x \in X$ such that $x \in E_n$ for infinitely many n.

[8]Payoff security along with reciprocal upper semicontinuity together imply that a game is better reply secure, which in turn guarantees existence of equilibrium in a compact, quasiconcave game.

verify in the mixed extension of a game. Our main result shows that disjoint payoff matching implies that the mixed extension of the game is payoff secure.

Theorem 3. *Let G be a compact game. Suppose that G satisfies DPM, then $\overline{G}$ is payoff secure.*

The advantage of DPM is that it is straightforward to verify and still fairly general. The condition in the following lemma is easier to use in the proof of our main result, but more difficult to verify directly.

Lemma 7. *Suppose that the compact game G satisfies disjoint payoff matching. Then for all $\varepsilon > 0$, $x_i \in X_i$, and $\mu_{-i} \in \mathcal{M}_{-i}$ there exists a deviation $x_i' \in X_i$ and a compact set $K \subset X_{-i} \setminus D_i(x_i')$ such that the following holds:*

(i) $u_i(x_i', x_{-i}) > u_i(x_i, x_{-i}) - \varepsilon$ for all $x_{-i} \in K$,

(ii) $\mu_{-i}(X_{-i} \setminus K) < \varepsilon$.[9]

Proof of Lemma 7. Assume that G satisfies disjoint payoff matching and consider any player i, $\varepsilon > 0$, and $\mu_{-i} \in \mathcal{M}_{-i}$. Take $\{x_i^k\}$ to be a defection sequence from the definition of DPM. Define the collection of sets $E_k = \{x_{-i} \in X_{-i} : u_i(x_i^k, x_{-i}) > u_i(x) - \varepsilon\}$. Then notice that $\liminf_k E_k = X_{-i}$, so $\mu_{-i}(\liminf_k E_k) = 1$.[10] Further, $\limsup_k D_i(x_i^k) = \emptyset$, so $\mu_{-i}(\limsup_k D_i(x_i^k)) = 0$. By statement (5) in Section 9 of Halmos (1974), $\mu_{-i}(\liminf_k E_k) \leq \liminf_k \mu_{-i}(E_k)$ and $\mu_{-i}(\limsup_k D_i(x_i^k)) \geq \limsup_k \mu_{-i}(D_i(x_i^k))$, and so $\lim_k \mu_{-i}(E_k) = 1$ and $\lim_k \mu_{-i}(D_i(x_i^k)) = 0$. It follows that there exists a k such that $\mu_{-i}(E_k) > 1 - (\varepsilon/3)$ and $\mu_{-i}(D_i(x_i^k)) < \varepsilon/3$. Choose such a k and by regularity of μ_{-i}, we may choose a closed

[9] These conditions are equivalent to a slight weakening of disjoint payoff matching: for all players i and all $x_i \in X_i$ and $\mu_{-i} \in \mathcal{M}_{-i}$, there exists a sequence $\{x_i^k\} \subset X_i$ such that (i) $\liminf_k u_i(x_i^k, x_{-i}) \geq u_i(x_i, x_{-i})$ μ_{-i}-almost everywhere, and (ii) $\limsup_k D_i(x_i^k)$ is μ_{-i}-measure zero. This definition seems less useful due to its dependence on an arbitrary probability measure.

[10] The limit inferior of a sequence of sets E_n is $\liminf_n E_n = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n$. This is equivalently the set of points are are in E_n for all but finitely many n.

(and thus compact) subset $K \subset E_k \setminus D_i\left(x_i^k\right)$ such that $\mu_{-i}(K) > \mu_{-i}\left(E_k \setminus D_i\left(x_i^k\right)\right) - (\varepsilon/3)$. It follows that $\mu_{-i}(X_{-i} \setminus K) < \varepsilon$. ■

Now we proceed to the proof of Theorem 1 which is based on showing that the condition in Lemma 1 implies the mixed extension is payoff secure.

Proof of Theorem 3. Let $\varepsilon > 0$ and suppose that $\mu \in \mathcal{M}$. Note that for each player i there exists some strategy $\overline{x}_i$ in the support of μ_i such that

$$\int u_i(\overline{x}_i, x_{-i})\, d\mu_{-i} \geq \int u_i(x)\, d\mu. \tag{3.1}$$

From disjoint payoff matching and Lemma 7, there exists a deviation x_i' and a set $K(\varepsilon) \subset X_{-i} \setminus D_i(x_i')$ such that

$$u_i(x_i', x_{-i}) > u_i(\overline{x}_i, x_{-i}) - \frac{\varepsilon}{6} \text{ for all } x_{-i} \in K\ (\varepsilon)$$

and

$$\mu_{-i}(X_{-i} \setminus K(\varepsilon)) < \frac{\varepsilon}{6M},$$

where $M \equiv \sup |u_i|$. It follows that

$$\int\limits_{K(\varepsilon)} u_i(x_i', x_{-i})\, d\mu_{-i} > \int\limits_{K(\varepsilon)} u_i(\overline{x}_i, x_{-i})\, d\mu_{-i} - \frac{\varepsilon}{6}. \tag{3.2}$$

Further, we have that

$$\begin{aligned}
& \int\limits_{X_{-i}\setminus K(\varepsilon)} u_i\left(x_i', x_{-i}\right) d\mu_{-i} - \int\limits_{X_{-i}\setminus K(\varepsilon)} u_i\left(\overline{x}_i, x_{-i}\right) d\mu_{-i} \\
> & -\int\limits_{X_{-i}\setminus K(\varepsilon)} \left(\left|u_i\left(\overline{x}_i, x_{-i}\right)\right| + \left|u_i\left(x_i', x_{-i}\right)\right|\right) d\mu_{-i} \\
> & -2\sup\left|u_i\right| \mu\left(X_{-i}\setminus K\left(\varepsilon\right)\right) \\
> & -\frac{2\varepsilon}{6}. \qquad (3.3)
\end{aligned}$$

Combining (2) and (3) yields

$$\int u_i\left(x_i', x_{-i}\right) d\mu_{-i} > \int u_i\left(\overline{x}_i, x_{-i}\right) d\mu_{-i} - \frac{\varepsilon}{2}. \qquad (3.4)$$

Define

$$\underline{u}_i\left(x_{-i}\right) = \sup_{V\ni x_{-i}} \inf_{x_{-i}'\in V} u_i\left(x_i', x_{-i}'\right),$$

where the supremum is taken over all neighborhoods V of x_{-i}. As noted by Reny (1999) in the proof of Theorem 3.1, $\underline{u}_i\left(x_{-i}\right)$ is lower semicontinuous. From Reny's proof of Proposition 5.1, it follows that $\int \underline{u}_i\left(x_{-i}\right) d\mu_{-i}$ is lower semicontinuous in μ_{-i}. This property implies the existence of a neighborhood $\mathcal{N}\left(\mu_{-i}\right)$ such that for all $\lambda \in \mathcal{N}\left(\mu_{-i}\right)$,

$$\int \underline{u}_i\left(x_{-i}\right) d\lambda > \int \underline{u}_i\left(x_{-i}\right) d\mu_{-i} - \frac{\varepsilon}{6}. \qquad (3.5)$$

Since M bounds $\underline{u}_i$ as well as u_i, we have that

$$\begin{aligned}
\int\limits_{X_{-i}\setminus K(\varepsilon)} \left(\underline{u}_i\left(x_{-i}\right) - u_i\left(x_i', x_{-i}\right)\right) d\mu_{-i} &\geq -\int\limits_{X_{-i}\setminus K(\varepsilon)} \left(\left|\underline{u}_i\left(x_{-i}\right)\right| + \left|u_i\left(x_i', x_{-i}\right)\right|\right) d\mu_{-i} \\
&> -2M\mu\left(X_{-i}\setminus K\left(\varepsilon\right)\right) \\
&= -\frac{2\varepsilon}{6}.
\end{aligned}$$

Further, since $u_i\left(x_i', x_{-i}\right)$ is continuous in x_{-i} at all $x_{-i} \in K\left(\varepsilon\right)$, then $\underline{u}_i\left(x_{-i}\right) = u_i\left(x_i', x_{-i}\right)$ on $K\left(\varepsilon\right)$.[11] Therefore,

$$\begin{aligned}
\int \underline{u}_i\left(x_{-i}\right) d\mu_{-i} &= \int u_i\left(x_i', x_{-i}\right) d\mu_{-i} + \int\limits_{X_{-i}\setminus K(\varepsilon)} \left(\underline{u}_i\left(x_{-i}\right) - u_i\left(x_i', x_{-i}\right)\right) d\mu_{-i} \\
&> \int u_i\left(x_i', x_{-i}\right) d\mu_{-i} - \frac{2\varepsilon}{6}. \qquad (3.6)
\end{aligned}$$

Using the fact that $u_i\left(x_i', x_{-i}\right) \geq \underline{u}_i\left(x_{-i}\right)$ and combining (5) and (6), we have that for all $\lambda \in \mathcal{N}\left(\mu_{-i}\right)$,

$$\begin{aligned}
\int u_i\left(x_i', x_{-i}\right) d\lambda &\geq \int \underline{u}_i\left(x_{-i}\right) d\lambda \\
&> \int \underline{u}_i\left(x_{-i}\right) d\mu_{-i} - \frac{\varepsilon}{6} \\
&> \int u_i\left(x_i', x_{-i}\right) d\mu_{-i} - \frac{\varepsilon}{2}. \qquad (3.7)
\end{aligned}$$

[11] The continuity here is with respect to the topology on X_{-i}, not to be confused with the subspace topology on $K\left(\varepsilon\right)$. Otherwise, if the function were only continuous with respect to the subspace topology, it might be that $u_i > \underline{u}_i$ on the boundary of $K\left(\varepsilon\right)$.

Lastly, we combine (1), (4), and (7) and find that for all $\lambda \in \mathcal{N}(\mu_{-i})$,

$$\begin{aligned} \int u_i(x_i', x_{-i})\, d\lambda &> \int u_i(x_i', x_{-i})\, d\mu_{-i} - \frac{\varepsilon}{2} \\ &> \int u_i(\overline{x}_i, x_{-i})\, d\mu_{-i} - \varepsilon \\ &\geq \int u_i d\mu - \varepsilon. \end{aligned}$$

Therefore, the mixed extension $\overline{G}$ is payoff secure. ■

3.4 Examples

In the first part of this section, we use Theorem 1 to prove existence of mixed strategy equilibrium for a Bertrand-Edgeworth price-setting oligopoly with general specifications of costs, residual demand rationing, and tie breaking rules. In the second part of this section, an example from Sion and Wolfe (1957) is used to demonstrate that equilibrium may not exist if disjoint payoff matching does not hold.

3.4.1 Bertrand-Edgeworth oligopoly

A Bertrand-Edgeworth (BE) price-setting oligopoly is a competition between producers of homogenous products where prices are the only strategic variables. We apply disjoint payoff matching to a BE oligopoly specification that subsumes much of the large literature and offers a basis to generalize the analysis in these games.[12] Existence of equilibrium in such

[12]The literature on BE games includes: Kreps and Scheinkman (1983), Davidson and Deneckere (1986), Osborne and Pitchick (1986), Deneckere and Kovenock (1992), Allen and Hellwig (1993), Deneckere and Kovenock (1996), Allen et al.(2000), Boccard and Wauthy (2000) and Lepore (2009)).

games has been examined by Dixon (1984), Allen and Hellwig (1986a), Dasgupta and Maskin (1986a&b), Maskin (1986), Deneckere and Kovenock (1996), and Bagh (2010). With the exception of Allen and Hellwig, which studies a symmetric oligopoly with constant marginal cost, these papers only demonstrate existence for a BE duopoly.[13] Most of these results rely upon Dasgupta and Maskin (1986a) to guarantee existence, while Deneckere and Kovenock construct an equilibrium and Bagh (2010) develops and applies the concept of variational convergence to show existence. In addition to extending existence results to an oligopoly setting, our formulation greatly generalizes the set of rationing rules which are permitted.[14]

Consider a homogeneous product industry with a set of firms N, with $|N| = n$. All firms simultaneously announce prices, then production decisions are made after demand is realized. Each firm i has a continuous, nondecreasing cost of production c_i with $c_i(0) = 0$.[15] The market demand $F : \mathbb{R} \mapsto \mathbb{R}$ is continuous and nonincreasing in x with $F(0) > 0$. Further, assume that there exists a $\overline{x} > 0$ such that $F(x) = 0$ for all $x \geq \overline{x}$. Note that any price $x > \overline{x}$ is weakly dominated by $x' = \overline{x}$, so we may restrict the strategy space to $X = [0, \overline{x}]^n$. We denote by p_i the price of any firm i and by p the vector of all firms' prices.

Each firm i has a capacity k_i, which serves as an upper bound on the quantity that it can produce. Thus, the production problem faced by the firm at a price p_i is

$$\max_{z \in [0, k_i]} \pi_i(p_i, z) = p_i z - c_i(z).$$

We refer to the solution to this problem as $s_i(p_i)$.[16] We assume that $s_i(p_i)$ is a continuous

[13]The result of Allen and Hellwig has been used to study BE oligopoly in other settings. Vives (1986) studies the an BE oligopoly as the number of firms gets large with efficient rationing and constant marginal cost up to capacity. Two recent articles Hirata (2009) and De Francesco and Salvadori (2010) characterize equilibria of a BE triopoly with efficient rationing and constant marginal cost up to capacity.

[14]Most of the literature focuses on either efficient or proportional rationing. Maskin (1986) and Bagh (2010) consider a larger class of rationing rules, although many reasonable rules are excluded from their frameworks.

[15]It is well known that equilibrium may not exist if c_i is discontinuous or $c_i(0) > 0$.

[16]The specification of $s_i(p_i)$ follows from Dixon (1984), Maskin (1986) and Bagh (2010).

nondecreasing function. Further, we assume that if $q < q' < s_i(p_i)$ and $\pi_i(p_i, s_i(p_i))$, then $\pi_i(p_i, q') > \pi_i(p_i, q)$. This is necessarily true if c_i is strictly convex.[17] The quantity $s_i(p_i)$ may be referred to as firm i's supply, the maximum quantity that it is willing to produce at any given price. Inherently, $s_i \leq k_i$, so the supply functions account for the capacity constraints.

For any price vector p, order the players so that $p_1 \leq p_2 \leq ... \leq p_n$. The demand served by firm 1 is $Q_1 = \min\{F(p_1), s_1(p_1)\}$. We make minimal assumptions as to which portion of demand is served by firm i, only that for all $j > i$ there is a continuous function $\lambda_{ij}(p)$ which denotes the share of i's quantity that satiates j's demand.[18] In the event that multiple firms choose the same price, there are multiple ways to order the players such that prices are nondecreasing. In this case, some tie breaking rule α is used to allocate the demand. Specifically, α serves to give each player some weighted average of the demand they would receive under each possible ordering of the prices. Most commonly in application α gives a uniform weight to each possible ordering, however, this is not necessary. Let O be the collection of possible orderings of the prices. For each $o \in O$, we let α_o denote the weight applied to the order o, $o(i)$ the position of player i in the ordering o, and Q_i^o the quantity served by firm i as if the ordering under o were a strictly increasing order of prices. That is,

$$Q_i^o = \min\left\{F(p_i) - \sum\nolimits_{j<i} Q_j^o \lambda_{ji}(p), s_i(p_i)\right\},$$

where $\lambda_{ji} = 1$ for all j such that $p_j = p_i$.[19] We require that $\sum_{o \in O} \alpha_o = 1$, so that demand is

[17]Notice that for symmetric constant marginal cost $c \geq 0$, we can restrict the strategy space to $X = [c, \overline{x}]^n$ and the supply functions satisfy our assumptions.

[18]A simple way to understand the purpose of λ_{ij} is to consider the case in which a continuum of consumers have unit demand. In this case, λ_{ij} specifies the fraction of consumers served by firm i that have willingness to pay of at least p_j.

[19]One way to interpret Q_i^o is that o represents a strict preference order for consumers, whereby p_i at firm i is strictly preferred to p_j at firm j for all $i < j$. Thus, the demand Q_i^o reflects the notion that this preference induces consumers to shop at firm i before firm $i + 1$.

always fully allocated, though α may be any measurable function.[20] Let I denote the set of players that charge p_i and J the set of players that charge a price strictly less than p_i. The actual demand served by firm i is then given by

$$Q_i = \min\left\{F\left(p_i\right) - \sum\nolimits_{j\in J} Q_j\lambda_{ji}\left(p\right) - \sum\nolimits_{o\in O}\alpha_o\sum\nolimits_{o(j)<o(i)} Q_j^o, s_i\left(p_i\right)\right\}.$$

Thus, each firm i serves the minimum of its capacity, supply, and the demand left by the firms with lower prices than i. The purpose for this formulation is to allow any possible rationing between tied firms. When multiple firms are tied, this allows any order of satiation of supply, be it simultaneous, partially sequentially, or fully sequentially.

This very general framework captures the notion that consumers shop first at firms with lower prices. Consider two choice for the functions λ_{ij} given by $\lambda_{ij}^e\left(p\right) = 1$ and λ_{ij}^p defined iteratively as

$$\lambda_{ij}^p = \min\left\{\frac{Q_i}{D\left(p_i\right) - \sum_{j<i} Q_j\lambda_{ji}^p\left(p\right)}, 1\right\}.$$

The rationing rule under λ_{ij}^e is the well known efficient, or parallel rule, whereas the rule under λ_{ij}^p is the proportional rationing rule.

The profit of each firm i can then be written as

$$u_i(p) = p_iQ_i\left(p\right) - c_i\left(Q_i\left(p\right)\right).$$

We now turn to establishing that this game satisfies DPM.

Proposition 16. *The BE oligopoly game satisfies disjoint payoff matching.*

[20]Both α and the λ functions may depend on the full vector of prices as well as the capacities. We suppress the capacity arguments for clarity. The quantities Q_i^o and Q_i depend on capacities only through the supply functions, α, and the λ functions.

Proof of Proposition 16. For any firm i, $u_i(0, p_{-i}) = 0$ for all p_{-i}. Consequently, $D_i(0) = \emptyset$. Let $p_i > 0$. Note that the set of discontinuities $D_i(p_i)$ is a subset of points where $p_j = p_i$ for some $i \neq j$. Thus, if $p_i \neq p_i'$, then $D_i(p_i) \cap D_i(p_i') = \emptyset$. It follows that for any sequence $p_i^l \to p_i$ with $p_i^l < p_i^{l+1} < p_i$ for all l, condition (ii) of DPM is satisfied. Note that $\lim_l Q_i\left(p_i^l, p_{-i}\right) \geq Q_i(p)$ for all p_{-i}. Since s_i is continuous, $\lim_l Q_i\left(p_i^l, p_{-i}\right) \leq s_i(p_i)$. By definition, u_i is increasing in $Q_i(p)$ for $Q_i(p) \leq s_i(p_i)$, and since u_i is continuous in $Q_i(p)$, it follows that

$$\begin{aligned}
& \lim_l \left(p_i Q_i\left(p_i^l, p_{-i}\right) - c_i\left(Q_i\left(p_i^l, p_{-i}\right)\right)\right) \\
= \;& p_i \lim_l Q_i\left(p_i^l, p_{-i}\right) - c_i\left(\lim_l Q_i\left(p_i^l, p_{-i}\right)\right) \\
\geq \;& p_i \lim_l Q_i(p) - c_i\left(Q_i(p)\right).
\end{aligned}$$

Therefore, the game satisfies DPM. ■

Since this game satisfies DPM, we know from Theorem 1 that the mixed extension is payoff secure. Now we establish that the game has a mixed strategy equilibrium by appealing to results from Reny (1999) and Bagh and Jofre (2006). The following definitions are necessary for our proof of existence.

Definition 7. *The game G is* weakly reciprocal upper semicontinuous (WRUSC) *if for all $(x^*, u^*) \in \mathrm{Fr}G$, there exists for some player i with a deviation $x_i \in X_i$ such that $u_i\left(x_i, x_{-i}^*\right) > u_i^*$.*

Definition 8. *The game G is* better reply secure *if whenever x^* is not an equilibrium and (x^*, u^*) is in the closure of the graph of G, there exists for some player i a strategy x_i and a neighborhood $\mathcal{N}(x_{-i}^*)$ of x_{-i}^* such that for all $x_{-i} \in \mathcal{N}(x_{-i}^*)$, $u_i(x_i, x_{-i}) > u_i^*$.*

The definitions are analogous for the mixed extension of the game. Reny (1999) showed that a compact game whose mixed extension is better reply secure possesses a Nash equilibrium. He also showed that payoff security together with reciprocal upper semicontinuity implies that a game is better reply secure. Bagh and Jofre (2006) showed that the latter condition can be replaced with WRUSC.

To prove that the game has a mixed strategy equilibrium we only need to show that the mixed extension of the game is WRUSC.

Proposition 17. *The BE oligopoly game has a mixed strategy equilibrium.*

Proof of Proposition 17. Since the strategy space X is compact and Hausdorff, we need only show that the game satisfies WRUSC. We begin by defining for each player i

$$\overline{u}_i(p) = \limsup_{x \to p} u_i(x)$$

and $\overline{u} = (\overline{u}_1, ..., \overline{u}_n)$, noting that each $\overline{u}_i$ is upper semicontinuous. Since $s_i(p)$ is continuous and $Q_i(p) \leq s_i(p)$ for all p, then $\limsup_{x \to p} Q_i(x) \leq s_i(p)$. Further, by assumption, for any Q such that $Q_i(p) < Q \leq s_i(p)$, $\pi_i(p_i, Q) > \pi_i(p_i, Q_i(p))$. Thus, it follows that

$$\overline{u}_i(p) = p_i \limsup_{x \to p} Q_i(x) - c_i\left(\limsup_{x \to p} Q_i(x)\right).$$

Note that

$$\begin{aligned} \lim_{x_i \to p_i^-} Q_i(x_i, p_{-i}) &= \limsup_{x \to p} Q_i(x) \\ &\equiv \overline{Q}_i(p). \end{aligned}$$

Thus, for any $p \in X$ with $p_i > 0$,

$$(3.8) \quad \lim_{x_i \to p_i^-} u_i(x_i, p_{-i}) = \overline{u}_i(p).$$

Let $(\mu^*, u^*) \in \mathrm{Fr}\overline{G}$ and let $\mu^l \to \mu^*$ be such that $\int u d\mu^l \to u^*$. Note that

$$\begin{aligned} u^* &= \lim_l \int u d\mu^l \\ &\leq \limsup_l \int \overline{u} d\mu^l. \end{aligned}$$

Since each $\overline{u}_i$ is upper semicontinuous, then as Reny (1999) shows in the proof of Proposition 5.1,

$$\limsup_l \int \overline{u} d\mu^l \leq \int \overline{u} d\mu^*.$$

Thus, we have that $u^* \leq \overline{u}(\mu^*)$.

Define $Y_i = \{p \in X : Q_i(p) < \overline{Q}_i(p)\}$ and $Y = \bigcup_i Y_i$. We consider two cases: (i) $\mu^*(Y) = 0$, and (ii) $\mu^*(Y) > 0$.

(i) In this case, $Q_i(p) = \overline{Q}_i(p)$ μ^*-almost everywhere for all players i. Thus, as noted above, $u_i = \overline{u}_i$ μ^*-almost everywhere, so we conclude that $u(\mu^*) = \overline{u}(\mu^*) \geq u^*$. Since $(\mu^*, u^*) \notin \overline{G}$, then it must be that $u_i(\mu^*) > u_i^*$ for some player i. It follows that $\mu_i = \mu_i^*$ satisfies the definition of WRUSC.

(ii) We begin by showing that $\overline{u}_i(\mu^*) > u_i^*$ for some player i. For any $p \in Y$, at least two firms must charge the same positive price, and at least one such firm i must have $Q_i(p) < \overline{Q}_i(p) \leq s_i(p)$. Let I be the set of firms j with $p_j = p_i$ and J the set of firms j

with $p_j < p_i$. Note that

$$(3.9)\quad \sum_{j\in I} Q_j(p) \leq F(p_i) - \sum_{j\in J} Q_j \lambda_{ji}(p)$$

for any choice of α. The inequality in (3.9) must hold with equality else there would be excess demand that firm i would be able to satiate.

Let $A(p) \equiv \sum_{j=1}^{n} u_j(p)$ and $\overline{A}(p) \equiv \limsup_{x\to p} A(x)$. We will show that

$$\overline{A}(p) < \sum_{j=1}^{n} \overline{u}_j(p).$$

Let $x^m \to p$ be such that $A(x^m) \to \overline{A}(p)$, and for each j let $\widetilde{Q}_j(p) \equiv \lim_m Q_j(x^m)$. Since

$$\sum_{j\in I}^{n} \widetilde{Q}_j(p) = F(p_i) - \sum_{j\in J}^{n} \widetilde{Q}_j \lambda_{ji}(p),$$

then there still exists at least one firm i' such that $\widetilde{Q}_{i'}(p) < \overline{Q}_{i'}(p)$. By our assumption, $\pi_{i'}(p_{i'}, \widetilde{Q}_{i'}(p)) < \pi_{i'}\left(p_{i'}, \overline{Q}_{i'}(p)\right)$, so

$$
\begin{aligned}
(3.10)\sum_{j=1}^{n} \overline{u}_j(p) - \overline{A}(p) &= \sum_{j=1}^{n} \pi_j\left(p_{i'}, \overline{Q}_{i'}(p)\right) - \pi_j(p_{i'}, \widetilde{Q}_{i'}(p)) \\
&\geq \pi_{i'}\left(p_{i'}, \overline{Q}_{i'}(p)\right) - \pi_{i'}(p_{i'}, \widetilde{Q}_{i'}(p)) \\
&> 0.
\end{aligned}
$$

The inequality in (3.10) holds based on the facts that: (i) $\overline{u}_j(p) \geq \lim_m u_j(x^m)$ for all players j, and (ii) based on (3.8), $\overline{u}_j(p) = \pi_{i'}\left(p_{i'}, \overline{Q}_{i'}(p)\right)$ and $\lim_m u_j(x^m) = \pi_{i'}(p_{i'}, \widetilde{Q}_{i'}(p))$. Therefore for all $p \in Y$,

$$\overline{A}(p) < \sum_{i=1}^{n} \overline{u}_i(p).$$

Since $\overline{A}$ is upper semicontinuous, then

$$\begin{aligned}\sum_{i=1}^{n} u_i^* &= \int A(p)\, d\mu^l \\ &\leq \int \overline{A}(p)\, d\mu^* \\ &< \int \sum_{i=1}^{n} \overline{u}_i(p)\,.\end{aligned}$$

The final line follows from the fact that $\mu^*(Y) > 0$.

Let i be the player with $u_i(\mu^*) < u_i^*$. Consider the deviation functions

$$f_m(p_i) = \left(1 - \frac{1}{m}\right) p_i.$$

We construct a sequence of measures that transfers any mass or density from each p_i to $f_m(p_i)$. For each m and every measurable set E, define $\mu_i^m(E) = \mu_i^*\left(f_m^{-1}(E)\right)$. By Theorem 39C in Halmos (1974),

$$\lim_m \int u_i(p_i, p_{-i})\, d\mu_i^m d\mu_{-i}^* = \int u_i(f_m(p_i), p_{-i})\, d\mu^*.$$

Since u_i is bounded, there is a constant, integrable function which bounds u_i, so the Lebesgue dominated convergence theorem states that

$$\lim_m \int u_i(f_m(p_i), p_{-i})\, d\mu^* = \int \lim_m u_i(f_m(p_i), p_{-i})\, d\mu^*.$$

As noted, for all p, $\lim_m Q_i(f_m(p_i), p_{-i}) = \limsup_{x \to p} Q_i(x)$, so $\lim_m u_i(f_m(p_i), p_{-i}) = \overline{u}_i$. It follows that

$$\lim_m \int u_i d\mu_i^m d\mu_{-i}^* = \int \overline{u}_i d\mu^*.$$

Therefore, for sufficiently large m, μ_i^m satisfies the definition of WRUSC. ■

3.4.2 Nonexistence

The following example is constructed by Sion and Wolfe (1957) as an example of a game without equilibrium.

There are two players with strategy spaces $X_1 = X_2 = [0, 1]$. The game is zero-sum, with

$$u_1(x_1, x_2) = \begin{cases} 1 & \text{if} \quad x_1 > x_2 \\ 0 & \text{if} \quad x_1 = x_2 \text{ or } x_1 + \frac{1}{2} = x_2 \\ -1 & \text{if} \quad x_1 < x_2 < x_1 + \frac{1}{2} \\ 1 & \text{if} \quad x_1 + \frac{1}{2} < x_2 \end{cases}.$$

It is easy to see that this game does not satisfy disjoint payoff matching. The discontinuities occur at ties of the form $x_1 = x_2$ and $x_1 + 1/2 = x_2$. For true ties ($x_1 = x_2$), player 1 benefits from deviating to points with $x_1' > x_1$, while at shifted ties ($x_1 + 1/2 = x_2$), player 1 benefits from deviations to points with $x_1' < x_1$ or $x_1' > x_1 + 1/2$. If we consider $x_1 = 1/2$, then if $x_2 = 1/2$, a improvement requires $x_1 > 1/2$, while if $x_2 = 1$, then an improvement requires $x_1' < 1/2$. This tension where one discontinuity demands deviations upward while another demands deviations downward to improve is what causes DPM to fail. Indeed, given any sequence of deviations from $x_1 = 1/2$, each individual deviation must make player 1 discretely worse off at either $x_2 = 1/2$ or $x_2 = 1$.

A lesson in general is that DPM tends to hold whenever players can always improve their payoffs at all discontinuities by deviations in a single direction, as is the case with the Bertrand-Edgeworth game where firms can always lower their prices any be at least as well off, or in any contests, where players can increase their bids or efforts and be at least as well off. In the current example, different discontinuities require conflicting deviations to improve, and so a single sequence of deviations cannot uniformly improve a player's positions.

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