SURJECTIVE ISOMETRIES OF REAL C^*-ALGEBRAS

CHO-HO CHU, TRUONG DANG, BERNARD RUSSO
AND BELISARIO VENTURA

In contrast to the situation for JB^*-algebras (and to some extent for C^*-algebras), Jordan triple systems over the reals have played no role in the analytic theory of JB^*-triples. This is due to the history of the area: JB^*-triples were born of an investigation into certain aspects of several complex variables [**14**]. However, a theory of real Jordan triples and real bounded symmetric domains in finite dimensions was developed by Loos [**17**]. This, together with the observation that many of the more recent techniques in Jordan theory [**8**, **13**, **1**] rely on functional analysis and algebra rather than holomorphy, suggests that it may be possible to develop a real theory and to explore its relationship with the complex theory.

This paper arose from a desire to study infinite dimensional real JB^*-triples via functional analysis. Our first attempt to formulate a definition came from a consideration of the range of a contractive projection on a real C^*-algebra. Although this can be analysed easily in the commutative case, see Section 7 below, the general case poses serious obstacles, and it remains open as to whether this range is isomorphic to a norm closed subspace of another real C^*-algebra stable for the triple product in that C^*-algebra (see [**9**] for the case of a complex C^*-algebra).

Upmeier [**25**, §20] has proposed a definition of a real JB^*-triple. His spaces include real C^*-algebras, JB^*-triples considered as vector spaces over the reals, the bounded operators between real Hilbert spaces, and the bounded operators between quaternionic Hilbert spaces. They also have the property that their open unit balls are real bounded symmetric domains. Since a real C^*-algebra is a real JB^*-triple, and hence essentially a geometric object, a natural test for its structure theory is whether the surjective linear isometries preserve the triple product. This is the main problem considered in this paper.

Our main result is the analog, for real C^*-algebras, of Kadison's celebrated theorem [**12**], and is based, in outline, on the recent affine geometric proof of that theorem [**4**]. Accordingly, the tools needed for that proof, which are standard results in the theory of (complex) C^*-algebras, need to be found for real C^*-algebras. In our initial search of the literature, we were warned that some of these results were not true (see [**6**]), and that others were true (see [**15**]), but we found that the published proof was sketchy at best. We therefore decided to develop the theory of real C^*-algebras and prove all the results that we needed for our main theorem. Although some of these results were expected or could be predicted, some of the proofs contain new ideas.

This paper is organized as follows. In §1 it is shown that the bidual of a real C^*-algebra is a real C^*-algebra. In §2 we give a definition of a real W^*-algebra. The main

Received 9 January 1991.

1991 *Mathematics Subject Classification* 46L05.

The research of the third author was supported in part by NSF Grant No. DMS8805256.

J. London Math. Soc. (2) 47 (1993) 97–118

result of §2, which has many consequences, is that the complexification of a real W^*-algebra is a W^*-algebra. In §3 the standard spectral theoretic type results are formulated for a real W^*-algebra. Using the results of §3, it is a simple matter to establish the fundamental relation between partial isometries and norm exposed faces which connects the algebraic structure of a real W^*-algebra with the geometric structure of the unit ball of its predual. This is done in §4 where it is used to prove that an isometry preserves orthogonality and 'cubes', and sends partial isometries to partial isometries. In §5 we prove the special case of our main result in which the two real C^*-algebras are W^*-factors of type I, that is, of the form $B(H)$ for some real, complex, or quaternionic Hilbert space H. Because of the lack of a polarization formula, the preservation of cubes does not automatically imply the preservation of the triple product, as it does in the complex linear case. Instead, we use the fact that $B(H)$ is generated by certain families of partial isometries, called grids, which occur in the general theory of Jordan triple systems.

The main result, that an isometry preserves the triple product is proved in §6 by a reduction to the special case worked out in §5. In the final section, §7, the structure of an arbitrary contractive projection on a commutative real C^*-algebra is given, complementing the known result in the commutative complex case [7].

If X is a real normed space, we denote its conjugate space by X', whereas if X is a complex normed space, its conjugate space will be denoted by X^*. A similar remark applies to the adjoints of operators on Banach spaces. We trust this will not cause any confusion with the notation for the adjoint operation in the involutive algebras which occur throughout the paper. Also, if X is a complex normed space, we denote its real restriction by X_r. The map $f \mapsto \Re f$ is a real linear isometry of $(X^*)_r$ onto $(X_r)'$, where $\Re f$ denotes the real part of $f \in X^*$. For any normed space X, real or complex, X_* will denote a normed space (when it exists) whose dual is X. We shall use the symbols $\mathbb{R}$ and $\mathbb{C}$ to denote the real and complex fields, and $\mathbb{H}$ to denote the division algebra of quaternions.

Part of this work was carried out during the first and fourth-named authors' visits to Irvine during the academic year 1988–89.

The authors wish to thank L. J. Bunce for pointing out the necessity of including the quaternionic case in Theorem 5.1.

1. *The bidual of a real C^*-algebra*

A *real C^*-algebra* is a real Banach *-algebra A such that $\|a^*a\| = \|a\|^2$ and $1+a^*a$ is invertible in A if A has a unit. If A is not unital we require that $1+a^*a$ be invertible for all a in the unit extension $\tilde{A}$ of A.

We note that, by [22, 4.1.13], if A is a non-unital real C^*-algebra, then the unit extension $\tilde{A} = A \oplus \mathbb{R}$ is a real C^*-algebra under the norm

$$\|(x, \lambda)\| = \sup\{\|xu+\lambda u\| : u \in A, \|u\| = 1\}.$$

The following lemma summarizes some equivalent definitions of real C^*-algebras. Let $A_h = \{a \in A : a = a^*\}$.

LEMMA 1.1. *For a real Banach *-algebra A, the following are equivalent:*

(i) *A is a real C^*-algebra;*

(ii) $\|a\|^2 \leqslant \|a^*a+b^*b\|$ *for all* $a, b \in A$;

(iii) *A is isometrically *-isomorphic to a norm-closed self-adjoint algebra of bounded operators on a real Hilbert space.*

Proof. The equivalence of (ii) and (iii) is given in [**19**, Theorem 1]. The equivalence of (i) and (iii) is Ingelstam's Theorem, given in [**10**, 8.2 and 15.3].

COROLLARY 1.2. *A closed *-subalgebra of a real C^*-algebra is a real C^*-algebra.*

COROLLARY 1.3. *Let A be a real C^*-algebra. Then $\|a^2\| \leqslant \|a^2+b^2\|$ for all $a, b \in A_h$. Hence $(A_h, \circ)$ is a JB-algebra, where $a \circ b = \frac{1}{2}(ab+ba)$. (Note that this Jordan algebra cannot be exceptional by the Gelfand–Naimark Theorem for real C^*-algebras* [**10**, 15.3].)

Proof. The second statement follows by definition [**11**, 3.1.4]; the rest is clear.

A real Banach algebra A is *Arens regular* if the two Arens products on the second dual A'' coincide. If A is a real Banach *-algebra which is Arens regular, then the involution * on A extends naturally to A'', and A'' becomes a real Banach *-algebra. Moreover, the extended involution is $\sigma(A'', A')$-$\sigma(A'', A')$-continuous.

LEMMA 1.4. *For a real Banach algebra A, the following are equivalent:*

(i) *A is Arens regular;*

(ii) *multiplication (with either Arens product) in A'' is separately $\sigma(A'', A')$-continuous;*

(iii) *for any pair of bounded sequences $\{a_n\}$ and $\{b_n\}$ in A and $f \in A'$,*

$$\lim_n \lim_m f(a_n b_m) = \lim_m \lim_n f(a_n b_m),$$

provided both limits exist.

Proof. As [**5**, p. 312].

If A is a real C^*-algebra, then its complexification $\mathscr{A} = A + iA$ can be given a norm so that it becomes a complex C^*-algebra, and A embeds isometrically as a real C^*-subalgebra of $\mathscr{A}$ [**10**, 15.4].

COROLLARY 1.5. *If B is a closed subalgebra of an Arens regular Banach algebra, then B is Arens regular (and hence multiplication in B'' is separately $\sigma(B'', B)$-continuous). In particular, a real C^*-algebra is Arens regular.*

Proof. By (iii) and the Hahn–Banach theorem, the first statement follows. If A is a real C^*-algebra, the real restriction $\mathscr{A}_r$ of $\mathscr{A}$ is clearly Arens regular (since $\mathscr{A}$ is, see [**5**]), so by the first statement, A, as a subalgebra of $\mathscr{A}_r$, is Arens regular.

By this corollary, there exists a natural $\sigma(A'', A')$-$\sigma(A'', A')$-continuous involution $\star$ on A'' which extends the involution * on A: for $x \in A''$, $x^\star(f) := \langle x, f^\star \rangle$ where $f^\star \in A'$ is defined by $f^\star(a) := f(a^*)$ for $a \in A$.

THEOREM 1.6. *Let A be a real C^*-algebra. Then its second dual A'', equipped with the Arens product and natural involution, is a real C^*-algebra.*

Proof. Let $\mathscr{A}$ be the complexification of A and let $\pi: A \to \mathscr{A}$ be the canonical real isometric $*$-isomorphism into. Let $\mathscr{A}^*, \mathscr{A}^{**}$ denote the complex dual spaces and let $\mathscr{A}_r, (\mathscr{A}^*)_r$ denote the real restrictions. The second dual map $\pi'': A'' \to (\mathscr{A}_r)''$ is a real linear isometry which is $\sigma(A'', A')$-$\sigma((\mathscr{A}_r)'', (\mathscr{A}_r)')$-continuous. By Arens regularity of A and $\mathscr{A}_r$, the multiplications in A'' and in $(\mathscr{A}_r)''$ are separately weak*-continuous. It follows that π'' is a $*$-isomorphism into.

As real Banach spaces we have

$$(\mathscr{A}_r)'' = ((\mathscr{A}_r)')' \cong ((\mathscr{A}^*)_r)' \cong ((\mathscr{A})^*)^*)_r$$

so we have a real linear isometry $\nu: (\mathscr{A}_r)'' \to \mathscr{A}^{**}$. It remains to show that ν is a $*$-homomorphism with respect to the Arens products on $(\mathscr{A}_r)''$ and $\mathscr{A}^{**}$.

We can write $\nu = \sigma \circ \tau'$, where $\sigma: ((\mathscr{A}^*)_r)' \to \mathscr{A}^{**}$ and $\tau: (\mathscr{A}^*)_r \to (\mathscr{A}_r)'$ are defined by

$$\sigma(F) = F(\cdot) - iF(i\cdot) \text{ for } F \in ((\mathscr{A}^*)_r)' \quad \text{and} \quad \tau(f) = \Re f \text{ for } f \in (\mathscr{A}^*)_r.$$

Using these formulas and the definition of the Arens multiplication and involution, a straightforward but tedious calculation shows that ν is a $*$-homomorphism.

2. *Real W^*-algebras*

DEFINITION 2.1. Let A be a real C^*-algebra. We call A a *real W^*-algebra* if A is linearly isometric to the dual space E' of a real Banach space E such that multiplication in A is separately $\sigma(A, E)$-continuous.

We may and shall assume that $E \subset A'$. Then $E = \{f \in A' : f \text{ is } \sigma(A, E)\text{-continuous}\}$.

We now consider the complexification of a real W^*-algebra.

For any real linear space V, we let $M_n(V)$ be the real linear space of n by n matrices over V $(n = 1, 2, \ldots)$. If A is a real C^*-algebra, there is a unique norm on $M_n(A)$ making it a real C^*-algebra with the usual matrix multiplication as product and the involution $*$ defined by $[a_{ij}]^* = [a_{ji}^*]$ [**10**, 15.5]. We identify $M_n(A')$ with $M_n(A)'$ as real linear spaces by the mapping

$$[f_{ij}] \in M_n(A') \longmapsto \phi[f_{ij}] \in M_n(A)', \tag{1}$$

where $\phi[f_{ij}]([a_{ij}]) = \sum_{i,j=1}^n f_{ij}(a_{ij})$. It is easy to see that ϕ is a real linear isomorphism onto. We now equip $M_n(A')$ with the norm of the dual space $M_n(A)'$ of $M_n(A)$, thereby making $M_n(A')$ a Banach space.

If $\mathscr{A}$ is the complexification of a real C^*-algebra, then its real restriction $\mathscr{A}_r$ is isometrically (real) $*$-isomorphic to

$$\left\{\begin{pmatrix} x & y \\ -y & x \end{pmatrix} : x, y \in A\right\} \tag{2}$$

which is a real $*$-subalgebra of $M_2(A)$ (to establish the isometry, use [**10**, 8.2]). Of course, A identifies with the $*$-subalgebra

$$\left\{\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \in A\right\}. \tag{3}$$

PROPOSITION 2.2. *Let A be a real W^*-algebra with a predual E. Then there is a norm on $M_n(E)$ for which $M_n(E)' \cong M_n(A)$.*

Proof. Since $E \subset A'$, we have $M_n(E) \subset M_n(A') \cong M_n(A)'$, so that we can give $M_n(E)$ the norm it inherits from $M_n(A)'$. We have

$$M_n(E)' \cong M_n(A)''/M_n(E)^\circ,$$

where

$$M_n(E)^\circ = \{\xi \in M_n(A)'' : \langle \xi, f\rangle = 0 \,\forall f \in M_n(E)\}.$$

Let $q: M_n(A) \to M_n(A)''/M_n(E)^\circ \cong M_n(E)'$ be the restriction to $M_n(A) \subset M_n(A)''$ of the quotient map.

We first show that $M_n(E)^\circ$ is a two-sided ideal in $M_n(A)''$. Let $\eta \in M_n(E)^\circ$ and $\xi \in M_n(A)''$. We shall show that $\xi\eta \in M_n(E)^\circ$. With $\xi = \lim_\alpha a_\alpha$ in $(\sigma(M_n(A)'', M_n(A)')$, where $a_\alpha \in M_n(A)$, we have, by Arens regularity of $M_n(A), \xi\eta = \lim_\alpha a_\alpha \eta$. Letting $f \in M_n(E)$, say $f = [f_{ij}]$ with $f_{ij} \in E$, we shall prove that $\langle \xi\eta, f\rangle = 0$.

Now $\langle a_\alpha \eta, f\rangle = \langle \eta, fa_\alpha\rangle$, where $fa_\alpha \in M_n(A)'$ is defined by $\langle fa_\alpha, b\rangle = f(a_\alpha b)$ for $b \in M_n(A)$. By (1), $fa_\alpha = [g_{ij}]$ for some $g_{ij} \in A'$. From $[f_{ij}][a_{ij}] = [g_{ij}]$, where $a_\alpha = [a_{ij}] \in M_n(A)$, and the separate weak*-continuity of multiplication in A, it follows that $g_{ij} \in E$, so that $fa_\alpha \in M_n(E)$. Hence, $\langle a_\alpha \eta, f\rangle = \langle \eta, fa_\alpha\rangle = 0$ and $\langle \xi\eta, f\rangle = \lim_\alpha \langle a_\alpha \eta, f\rangle = 0$, proving that $\xi\eta \in M_n(E)^\circ$, and that $M_n(E)^\circ$ is a left ideal. Similarly, $M_n(E)^\circ$ is a right ideal, and thus $M_n(A)''/M_n(E)^\circ$ is a real C^*-algebra [**10**, Exercise 15C] and q is *-homomorphism.

We next show that q is a bijection. If $q([a_{ij}]) = 0$ let $\tilde{f} \in M_n(E)$, for $f \in E$, denote the matrix with f in the (i,j) entry and zeros elsewhere. Then $0 = \langle [a_{ij}], \tilde{f}\rangle = \langle a_{ij}, f\rangle$, proving that $[a_{ij}] = 0$ and q is injective. Now let $\xi + M_n(E)^\circ \in M_n(A)''/M_n(E)^\circ$. For fixed i,j, define $a_{ij} \in A$ as follows: for $f \in E$, let $\tilde{f}$ be as above and set $\langle a_{ij}, f\rangle = \langle \xi, \tilde{f}\rangle$. Since for any matrix $b = [b_{ij}] \in M_n(A)$, $\max_{i,j} \|b_{ij}\| \leqslant \|b\|$, we have $\|\tilde{f}\| = \|f\|$, implying that $a_{ij} \in E' = A$. We now have $q([a_{ij}]) = [a_{ij}] + M_n(E)^\circ = \xi + M_n(E)^\circ$ so that q is onto.

Since *-isomorphisms between real C^*-algebras are isometric (consider the complexifications), $M_n(A) \cong M_n(A)''/M_n(E)^\circ \cong M_n(E)'$.

COROLLARY 2.3. *Let A be a real W^*-algebra. Then $M_n(A)$ is a real W^*-algebra.*

Proof. We have $A = E'$, multiplication in A is separately $\sigma(A,E)$-continuous, and $M_n(E)' \cong M_n(A)$. Let $a^\alpha = [a^\alpha_{ij}] \in M_n(A)$ and suppose $a^\alpha \to 0$ in $\sigma(M_n(A), M_n(E))$. Then, for $f \in E$, again letting $\tilde{f}$ denote the matrix whose i,j entry is f and all other entries are zero, we have $\langle a^\alpha_{ij}, f\rangle \to 0$ for all i,j, that is, $a^\alpha_{ij} \to 0$ in $\sigma(A,E)$.

Now, let $b = [b_{ij}] \in M_n(A)$. We shall show that $a^\alpha b \to 0$ in $\sigma(M_n(A), M_n(E))$. Let $f = [f_{ij}] \in M_n(E)$. Then

$$\langle a^\alpha b, f\rangle = \langle [a^\alpha_{ij}][b_{ij}], [f_{ij}]\rangle = \left\langle \left[\sum_{k=1}^{n} a^\alpha_{ik} b_{kj}\right], [f_{ij}]\right\rangle = \sum_{i,j=1}^{n} \left\langle \sum_{k=1}^{n} a^\alpha_{ik} b_{kj}, f_{ij}\right\rangle.$$

For each $k, a^\alpha_{ik} b_{kj} \to 0$ in $\sigma(A,E)$, so $\langle a^\alpha b, f\rangle \to 0$.

THEOREM 2.4. *Let A be a real W^*-algebra. Then its complexification $\mathscr{A}$ is a W^*-algebra. Moreover, A is $\sigma(\mathscr{A}, \mathscr{A}_*)$-closed in $\mathscr{A}$, and for $a, a_\alpha \in A$,*

$$\sigma(A,E)\text{-}\lim_\alpha a_\alpha = a \quad \Leftrightarrow \quad \sigma(\mathscr{A}, \mathscr{A}_*)\text{-}\lim_\alpha a_\alpha = a.$$

Proof. Fix E such that $A = E'$ and multiplication in A is separately $\sigma(A, E)$-continuous. Let $\mathscr{A}$ be the complexification of A and consider the identifications (2) and (3). Let σ denote the $\sigma(M_2(A), M_2(E))$-topology on $M_2(A)$. Recall that

$$\langle [a_{ij}], [f_{ij}] \rangle = \sum_{i,j=1}^{2} f_{ij}(a_{ij}) \quad \text{for } [a_{ij}] \in M_2(A), [f_{ij}] \in M_2(E).$$

Thus

$$z_\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \longrightarrow z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{in } \sigma$$

if and only if

$$a_\alpha \longrightarrow a, \quad b_\alpha \longrightarrow b, \quad c_\alpha \longrightarrow c, \quad d_\alpha \longrightarrow d \quad \text{in } \sigma(A, E).$$

From this we see immediately that $\mathscr{A}_r$ and A are σ-closed in $M_2(A)$. Hence $\mathscr{A}_r$ has a predual $F = M_2(E)/\mathscr{A}_r^\circ$ and the topology $\sigma(\mathscr{A}_r, F)$ on $\mathscr{A}_r$ is the same as σ on $\mathscr{A}_r$. Also, A is $\sigma(\mathscr{A}_r, F)$-closed in $\mathscr{A}_r$ since it is σ-closed in $M_2(A)$.

We also note that

$$a_\alpha \longrightarrow a \text{ in } A(\sigma(A, E)) \Leftrightarrow \begin{pmatrix} a_\alpha & 0 \\ 0 & a_\alpha \end{pmatrix} \longrightarrow \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} (\sigma) \Leftrightarrow a_\alpha \longrightarrow a \text{ in } \mathscr{A}_r(\sigma(\mathscr{A}_r, F)).$$

Similarly,

$$z_\alpha = \begin{pmatrix} x_\alpha & y_\alpha \\ -y_\alpha & x_\alpha \end{pmatrix} \longrightarrow z = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \quad \text{in } (\sigma(\mathscr{A}_r, F))$$

if and only if

$$iz_\alpha = \begin{pmatrix} -y_\alpha & x_\alpha \\ -x_\alpha & -y_\alpha \end{pmatrix} \longrightarrow \begin{pmatrix} -y & x \\ -x & -y \end{pmatrix} = iz \quad \text{in } (\sigma(\mathscr{A}_r, F)),$$

that is, multiplication by i on $\mathscr{A}_r$ is $\sigma(\mathscr{A}_r, F)$-continuous.

We now find a *complex* predual for $\mathscr{A}$. Let $\Phi\colon (\mathscr{A}_r)' \to (\mathscr{A}^*)_r$ be the usual identification: $\Phi(f) = f(\cdot) - if(i\cdot)$. We have $F \subset F'' = (\mathscr{A}_r)'$. Set $\mathscr{F} = \Phi(F) \subset \mathscr{A}^*$. We shall show that $\mathscr{F}^* = \mathscr{A}$.

First of all, since multiplication by i is $\sigma(\mathscr{A}_r, F)$-continuous, for $f \in F$ we have $f(i\cdot) \in F$. Therefore $\Phi(f) \in \mathscr{F}$ implies $i\Phi(f) = \Phi(f(i\cdot)) \in \mathscr{F}$, showing that $\mathscr{F}$ is a complex subspace of $\mathscr{A}^*$. Now define $\pi\colon \mathscr{A} \to \mathscr{F}^*$ by

$$\langle \pi(a), \Phi(f) \rangle = \Phi(f)(a) = \langle a, f \rangle - i\langle ia, f \rangle \quad (a \in \mathscr{A}, f \in F).$$

Then, since $f(a) = \Re\Phi(f)(a)$,

$$\begin{aligned} \|\pi(a)\| &= \sup\{|\Phi(f)(a)| : \|\Phi(f)\| \leqslant 1, f \in F\} \leqslant \|a\| \\ &= \sup\{|f(a)| : f \in F, \|f\| \leqslant 1\} \quad (\text{consider } a \in \mathscr{A}_r) \\ &\leqslant \sup\{|\Phi(f)(a)| : f \in F, \|f\| \leqslant 1\} = \|\pi(a)\|. \end{aligned}$$

Thus π is an isometry.

Let $\delta \in \mathscr{F}^*$. Then $(\Re\delta)\circ\Phi \in F'$ so there exists $a \in \mathscr{A}_r$ such that $\langle(\Re\delta)\circ\Phi, f\rangle = \langle a, f\rangle$ for $f \in F$. Since

$$\langle \Re\delta(i\cdot), \Phi(f)\rangle = \Re(\delta(i\Phi(f))) = \Re(\delta(\Phi(f(i\cdot)))) = \langle \Re\delta, \Phi(f(i\cdot))\rangle$$
$$= \langle(\Re\delta)\circ\Phi, f(i\cdot)\rangle = \langle a, f(i\cdot)\rangle$$

and by definition $\langle a, f(i\cdot)\rangle = \langle ia, f\rangle$, we have

$$\langle \delta, \Phi(f)\rangle = \langle \Re\delta(\cdot) - i\Re\delta(i\cdot), \Phi(f)\rangle = \langle a, f\rangle - i\langle a, f(i\cdot)\rangle = \langle \pi(a), \Phi(f)\rangle.$$

Thus π is onto and $\mathscr{A}$ is a complex W^*-algebra.

To complete the proof in the theorem it suffices to observe that $x_\alpha \to x$ in $\mathscr{A}(\sigma(\mathscr{A}, \mathscr{F}))$ if and only if $x_\alpha \to x$ in $\mathscr{A}_r(\sigma(\mathscr{A}_r, F))$. To prove this, assume $x_\alpha \to x$ in $\mathscr{A}(\sigma(\mathscr{A}, \mathscr{F}))$ and let $f \in F$. Then

$$f(x_\alpha) - if(ix_\alpha) = \langle x_\alpha, \Phi(f)\rangle \longrightarrow \langle x, \Phi(f)\rangle = f(x) - if(ix)$$

which implies that $f(x_\alpha) \to f(x)$, that is, $x_\alpha \to x$ in $\mathscr{A}_r(\sigma(\mathscr{A}_r, F))$. Conversely, if $x_\alpha \to x$ in $\mathscr{A}_r\ (\sigma(\mathscr{A}_r, F))$, then $ix_\alpha \to ix$ in $\sigma(\mathscr{A}_r, F)$, so for any $\Phi(f) \in \mathscr{F}$ (with $f \in F$),

$$\langle x_\alpha, \Phi(f)\rangle = f(x_\alpha) - if(ix_\alpha) \longrightarrow f(x) - if(ix),$$

so $x_\alpha \to x$ in $\mathscr{A}(\sigma(\mathscr{A}, \mathscr{F}))$.

COROLLARY 2.5. *Let A be a real C^*-algebra. Then A is a real W^*-algebra if and only if A can be faithfully represented as a weak-operator closed real $*$-subalgebra of $B(H)$, for some complex Hilbert space H.*

Proof. If A is a real W^*-algebra, its complexification $\mathscr{A}$ can be represented as a weak*-closed $*$-subalgebra of $B(H)$, H complex, so that $\sigma(\mathscr{A}, \mathscr{A}_*) = \sigma(B(H), B(H)_*)$ on $\mathscr{A}$.

Conversely, if A can be faithfully represented as a weak-operator closed real $*$-subalgebra B of $B(H)$, for some complex Hilbert space H, then multiplication is separately $\sigma(B, B_*)$-continuous.

COROLLARY 2.6. *Let A be a real W^*-algebra. Then $(A_h, \circ)$ is a JBW-algebra. More precisely, if $A = E'$, then $A_h \cong (E/A_h^\circ)'$.*

Proof. By Corollary 1.3 and [**11**, Theorem 4.4.16], we need only to show that A_h is a dual space. This will follow if it is shown that A_h is $\sigma(A, E)$-closed, for then it is known that $(E/A_h^\circ) \cong (A_h)_*$ via the map $f + A_h^\circ \mapsto f|A_h$. Suppose $a_\alpha \to a\ (\sigma(A, E))$ and $a_\alpha^* = a_\alpha$. Then $a_\alpha \to a\ (\sigma(\mathscr{A}, \mathscr{A}_*))$ so that $a_\alpha^* \to a^*\ (\sigma(\mathscr{A}, \mathscr{A}_*))$. Therefore $a = a^*$.

COROLLARY 2.7. *Let A be a real W^*-algebra with preduals E_1 and E_2. Then $E_1 \cong E_2$.*

Proof. Since the topologies $\sigma(A, E_1)$ and $\sigma(A, E_2)$ both agree with $\sigma(\mathscr{A}, \mathscr{A}_*)$, we have

$$E_1 \cong \{f \in A' : f \text{ is } \sigma(A, E_1)\text{-continuous}\} = \{f \in A' : f \text{ is } \sigma(A, E_2)\text{-continuous}\} \cong E_2.$$

COROLLARY 2.8. *Every real W^*-algebra A has an identity.*

Proof. The complexification $\mathscr{A}$ of A has an identity $e = x+iy$ with $x, y \in A$. Since $e^* = e$, $x^* = x$ and $y^* = -y$, so that $x+iy = e = e^2 = x^2+iyx+ixy-y^2$. But $x = xe = x^2+iyx$, so $x+iy = x-y^2+ixy$ implying $y^2 = 0$, $y^*y = -y^2 = 0$, and $y = 0$.

COROLLARY 2.9. *Every weak*-closed C^*-subalgebra B of a real W^*-algebra A is a real W^*-algebra.*

Proof. With $A = E'$, we have $B = (E/B^\circ)'$ and

$$x_\alpha \to x\ (\sigma(B, E/B^\circ)) \quad \Leftrightarrow \quad x_\alpha \to x\ (\sigma(A, E)).$$

PROPOSITION 2.10. *Let L be a weak*-closed left ideal in a real W^*-algebra A. Then there is a (unique) projection $p \in A$ such that $L = Ap$. If L is a two sided weak*-closed ideal, then p is a central projection.*

Proof. Let $N = L \cap L^*$, where $L^* = \{x^*: x \in L\}$. Then N is a real W^*-algebra by Corollary 2.9. Let p be the identity element of N. Then p is a projection in A and $L = Ap$.

If $Ap = Aq$, then $p = aq$, $p = p^*p = qa^*aq$ implies $pq = p$ and $p \leqslant q$.

3. *Spectral, polar and Jordan decompositions*

For a W^*-algebra $\mathscr{A}$, $s(\mathscr{A}, \mathscr{A}_*)$ denotes the ultra-strong topology, that is, the topology defined by the family of semi-norms $x \mapsto \phi(x^*x)^{1/2}$, as ϕ varies over the positive $\sigma(\mathscr{A}, \mathscr{A}_*)$-continuous linear functionals on $\mathscr{A}$.

In this section and the next, we need to use the fact that the set of projections in a real W^*-algebra forms a complete lattice. This follows from Corollary 2.6 and [**11**, Lemma 4.2.8]. Alternatively, avoiding Jordan algebras, by Theorem 2.4 the extremum of a family of projections from A, calculated in the complexification $\mathscr{A}$, lies in A.

Given an element x in a real W^*-algebra A, the smallest projection e with $ex = x$ is called the *range projection* or the *left support* of x, and is denoted by $s_l(x)$. Similarly, the *right support* $s_r(x)$ is the smallest projection q with $xq = x$.

If A is a real C^*-algebra and $a \in A$, then by [**10**; 13.3, 13.4], $|a| = (a^*a)^{1/2} \in A$.

PROPOSITION 3.1. *Let A be a real W^*-algebra, and let $a \in A$. There is a unique partial isometry $u \in A$ with the property $a = u|a|$ where $|a| = (a^*a)^{1/2}$ and uu^* is the range projection of a.*

Proof. Let $\mathscr{A}$ be the complexification of A and consider the polar decomposition of a in $\mathscr{A}$. According to [**18**], $a = u|a|$ where $u = s(\mathscr{A}, \mathscr{A}_*)\text{-}\lim_{\varepsilon \to 0^+} a(|a|+\varepsilon)^{-1}$, uu^* is the range projection of a and u^*u is the right support of x. By [**23**, 1.8.9], $u = \sigma(\mathscr{A}, \mathscr{A}_*)\text{-}\lim_{\varepsilon \to 0^+} a(|a|+\varepsilon)^{-1}$, so by Theorem 2.4, $u \in A$. For the uniqueness, see [**20**, 2.2.9].

PROPOSITION 3.2. *Let A be a real W^*-algebra. Then*

(i) *for each $a \in A_h$ and $\varepsilon > 0$, there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and orthogonal projections $e_1, \ldots, e_n$ such that $\|a - \sum_{j=1}^n \lambda_j e_j\| < \varepsilon$;*

(ii) *for each $a \in A$ and $\varepsilon > 0$, there exist $\lambda_1, \ldots, \lambda_n > 0$ and orthogonal partial isometries $u_1, \ldots, u_n$ such that $\|a - \sum_{j=1}^n \lambda_j u_j\| < \varepsilon$.*

Proof. (i) Since $(A_h, \circ)$ is a JBW-algebra, the result follows from **[11**, 4.2.3].

(ii) With $a = u|a|$, we have, by (i),

$$\left\| |a| - \sum_{j=1}^{n} \lambda_j e_j \right\| < \varepsilon,$$

so that

$$\left\| a - \sum_{j=1}^{n} \lambda_j u e_j \right\| < \varepsilon \text{ and } ue_j \in A.$$

Moreover $u_j := ue_j$ are orthogonal partial isometries in A.

Recall that for a real C^*-algebra A, $f \in A'$ is said to be *hermitian* if $f(a^*) = f(a)$ for all $a \in A$. Let $(A')_h$ denote the set of hermitian functionals on A. It is easy to see that $(A_h)' \cong (A')_h$ via $f \mapsto f \oplus 0$, where we are using the decomposition $A = A_h \oplus A_{sh}$, A_{sh} denoting the set of skew-hermitian elements of A. A functional $f \in A'$ is *positive* if it is hermitian and if $f(x^*x) \geqslant 0$ for every $x \in A$. We shall indicate this as usual by $f \geqslant 0$. Obviously, $f \geqslant 0$ if and only if $f \mid A_h$ is a positive functional on the JB-algebra A_h. From these remarks and the fact that $E \cap (A')_h \cong (A_h)_*$ via $f \mapsto f|_{A_h}$, we obtain the following proposition.

PROPOSITION 3.3. *Let A be a real W^*-algebra with predual E. For each $f \in E \cap (A')_h$, we have $f = f^+ - f^-$ where $f^\pm \geqslant 0$, $f^\pm \in E$, and $\|f\| = \|f^+\| + \|f^-\|$.*

Let $A = E'$ be a real W^*-algebra and let $f \in E$ with $f \geqslant 0$. Then

$$L := \{x \in A : f(x^*x) = 0\}$$

is a left ideal in A. There exists $\tilde{f} \in \mathscr{A}_{*+}$ such that $\Re \tilde{f}|_A = f$, where $\mathscr{A}$ is the complexification of A (see the proof of Theorem 2.4). Since $L = A \cap \{y \in \mathscr{A} : \tilde{f}(y^*y) = 0\}$, L is $s(\mathscr{A}, \mathscr{A}_*)$-closed and hence, by **[23**, 1.8.11], $\sigma(\mathscr{A}, \mathscr{A}_*)$-closed. By Theorem 2.4 L is $\sigma(A, E)$-closed. Hence, by Proposition 2.10, $L = Ap$ for some projection $p \in A$, and p is the greatest of all projections q with $f(q) = 0$. Define $s(f) = 1 - p$ to be the *support* of f. Then $f(x) = f(xs(f)) = f(s(f)x) = f(s(f)xs(f))$ for all $x \in A$.

PROPOSITION 3.4. *Let A be a real W^*-algebra with a predual E. For $f, g \in E$ with $f \geqslant 0, g \geqslant 0$, we have $\|f - g\| = \|f\| + \|g\|$ if and only if f and g have orthogonal support projections.*

Proof. As above, $(A_h, \circ)$ is a JBW-algebra with $(A_h)_* \cong E \cap (A')_h$. Note that the support projection of $f \geqslant 0$ in A is the same as that of $f|_{A_h}$ in A_h. Therefore

$$\begin{aligned} & s(f) \text{ and } s(g) \text{ are orthogonal in } A \\ \Leftrightarrow\ & s(f|_{A_h}) \text{ and } s(g|_{A_h}) \text{ are orthogonal in } A_h \\ \Leftrightarrow\ & \|f|_{A_h} - g|_{A_h}\| = \|f|_{A_h}\| + \|g|_{A_h}\| \\ \Leftrightarrow\ & \|f - g\| = \|f\| + \|g\|. \end{aligned}$$

Since A is a real W^*-algebra with a predual E, if $f \in E$ and if e is a projection in A, then $f.e$ denotes the functional $x \mapsto f(ex)$. The functional $f.e$ belongs to E by the separate $\sigma(A, E)$-continuity of multiplication.

PROPOSITION 3.5. *Let A be a real W^*-algebra with a predual E. Let $f \in E$ and let e be a projection in A. Then $\|f\| = \|f.e\|$ if and only if $f = f.e$.*

Proof. As [**24**, Lemma 4.1, p. 140].

LEMMA 3.6. *Let A be a unital real C^*-algebra, and let $f \in A'$. Suppose there is $a \in A$ with $0 \leqslant a \leqslant 1$ and $\|f\| = f(a)$. Then $f \geqslant 0$. (Note that we do not assume that f is hermitian.)*

Proof. Consider the real number $\lambda = f(1-a)$. If $\lambda \geqslant 0$, then

$$\|f\| = f(a) \leqslant f(a) + f(1-a) = f(1) \leqslant \|f\|$$

and thus $\lambda = 0$. If $\lambda < 0$, then

$$0 \leqslant a \leqslant 1 \Rightarrow -1 \leqslant 2a-1 \leqslant 1 \Rightarrow \|2a-1\| \leqslant 1$$

and

$$\|f\| = f(a) \leqslant f(a) - f(1-a) = f(2a-1) \leqslant \|f\|$$

so that again $\lambda = 0$.

Thus $f(1) = f(a) = \|f\|$ so that by [**10**, 14.4], $f/\|f\|$ is a real state, that is, $f \geqslant 0$.

PROPOSITION 3.7. *Let A be a real W^*-algebra with predual E. For $f \in E$ there is a unique partial isometry $u \in A$ and an element $\phi \in E$ with $\phi \geqslant 0$ such that $f = u\phi$, $u^*u = s(\phi)$ and $\|\phi\| = \|f\|$.*

Proof. As [**24**, Theorem 4.2, p. 140], using Lemma 3.6.

Let $A = E'$ be a real W^*-algebra and let $a \in A_h$. Then $L := \{x \in A : xa = 0\}$ is a $\sigma(A, E)$-closed left ideal in A and so $L = Ap$ for some projection $p \in A$. Let $s(a) = 1 - p$. Then $s(a)$ is the least of all projections with $qa = a = aq$. Call $s(a)$ the *support* of a.

LEMMA 3.8. *Let $W(a)$ be the real W^*-subalgebra generated by $a \in A_h$, where A is a real W^*-algebra. Then $s(a) \in W(a)$.*

Proof. As [**23**, Proposition 1.10.4].

Let A be a real W^*-algebra. By [**11**, 3.2.4], each $a \in A_h$ can be written $a = a^+ - a^-$ for unique positive elements $a^{\pm} \in A_h$ such that $a^+a^- = 0$. Now fix $a \in A_h$. For $\lambda \in \mathbb{R}$, define $e(\lambda) = s((\lambda - a)^+)$. Then $e(\lambda) \in W(a)$ and $e(\lambda) \leqslant e(\mu)$ if $\lambda \leqslant \mu$.

PROPOSITION 3.9. *For any self adjoint element a in a real W^*-algebra A, there exists a family of projections $\{e(\lambda) : \lambda \in \mathbb{R}\}$ such that*

1. $\lambda \leqslant \mu \Rightarrow e(\lambda) \leqslant e(\mu)$,
2. $\lambda_n \uparrow \lambda \Rightarrow e(\lambda_n) \to e(\lambda)$ *in the* $\sigma(A, E)$-*topology*,
3. $\lim_{\lambda \to \infty} e(\lambda) = 1$ *and* $\lim_{\lambda \to -\infty} e(\lambda) = 0$,
4. $a = \int_{-\infty}^{\infty} \lambda \, de(\lambda) = \int_{-|a|}^{|a|^+} \lambda \, de(\lambda)$, *where the integral converges in the* $\sigma(A, E)$-*topology*.

Proof. As [**23**, Theorem 1.11.3].

4. *Partial isometries and faces*

Let v be a partial isometry in a real C^*-algebra A. Setting $l = vv^*$ (= the left support projection of v) and $r = v^*v$ (= the right support projection of v), the contractive projections $P_j(v)$, $j = 0, 1, 2$ on A are defined by

$$P_2(v)x = lxr, \quad P_1(v)x = (1-l)xr + lx(1-r), \quad P_0(v)x = (1-l)x(1-r) \quad (x \in A).$$

Note that if w is a partial isometry belonging to $P_2(v)A$, then

$$P_2(w)A \subset P_2(v)A. \tag{4}$$

The decomposition $x = x_2 + x_1 + x_0$, where $x_j = P_j(v)x$, is called the *Peirce decomposition* of x relative to v. Note that $P_j(v)A$ is the j-eigenspace of the map $x \mapsto vv^*x + xv^*v$, $j = 0, 1, 2$. We have

$$\|P_2(v)x + P_0(v)x\| = \max(\|P_2(v)x\|, \|P_0(v)x\|) \quad (x \in A), \tag{5}$$

and

$$\|P_2(v)'g + P_0(v)'h\| = \|P_2(v)'g\| + \|P_0(v)'h\| \quad (g, h \in A'). \tag{6}$$

The following five lemmas can now be proved exactly as in [**4**, §2], using the results of Sections 2 and 3 on real C^*-algebras and real W^*-algebras. In Lemma 4.2 and Lemma 4.4, we need the uniqueness of the polar decomposition in Proposition 3.7; for Lemma 4.3 we need the integral form of the spectral theorem (that is, Proposition 3.9); in Lemma 4.4 we need the fact that the projections in a real W^*-algebra form a complete lattice.

LEMMA 4.1. *Let v be a partial isometry in a real C^*-algebra A.*

(a) *$A_v := v^*Ar$, with $r = v^*v$, is a real C^*-subalgebra of A with unit r. If A is a real W^*-algebra, so is A_v.*

(b) *The map $x \mapsto vx$ is a linear isometric bijection of A_v onto $P_2(v)A$ with inverse $a \mapsto v^*a$. Thus $P_2(v)A$ becomes a real C^*-algebra with unit v, multiplication $a \cdot b := av^*b$ and involution $a^\# := va^*v$.*

(c) *The map $f \mapsto f|_{P_2(v)A}$ is an affine isometry of $\{f \in A' : f(v) = \|f\|\}$ onto $(P_2(v)A)'_+$. If A is a real W^*-algebra with predual A_*, this map restricts to an affine isometry of $\{f \in A_* : f(v) = \|f\|\}$ onto $(P_2(v)A)_{*+}$.*

Proof. As [**4**, Lemma 1].

Partial isometries u and v are *orthogonal* if their left and right support projections are orthogonal, that is, $uu^*vv^* = u^*uv^*v = 0$. More generally, elements x, y in a real C^*-algebra are orthogonal if $xy^* = y^*x = 0$. As in the complex case, this is equivalent to $D(x, y) = 0$, where $D(x, y)$ is the operator $z \mapsto (xy^*z + zy^*x)/2$ on A. Note that if u is a partial isometry in A and $x \in A$, then x and u are orthogonal if and only if $x \in P_0(u)A$.

Note that if w_1 and w_2 are orthogonal partial isometries with $w_1 + w_2 \in P_2(u)A$ for some other partial isometry u, then by (4),

$$w_1 \in P_2(w_1)A \subset P_2(w_1 + w_2)A \subset P_2(u)A. \tag{7}$$

LEMMA 4.2. *Let f and g be normal functionals on a real W^*-algebra A, that is, $f, g \in E$, where $A \cong E'$, and let u and v be the partial isometries occurring in their polar decompositions respectively. Then u and v are orthogonal if and only if*

$$\|f+g\| = \|f-g\| = \|f\| + \|g\|. \tag{8}$$

Proof. As [**4**, Lemma 2].

A *norm exposed face* of the unit ball $(W)_1$ of a real Banach space W is a non-empty subset F_x of $(W)_1$ of the form

$$F_x = \{f \in W: \langle f, x\rangle = \|f\| = 1\} \text{ for some } x \in W' \text{ of norm } 1.$$

Note that, as in the complex case, if u is a non-zero partial isometry, then by Lemma 4.1(c), $F_u \neq \varnothing$. Note also that you cannot have $A_h = \{0\}$ in a real W^*-algebra A.

LEMMA 4.3. *For each x in a real W^*-algebra A with $\|x\| = 1$ and $F_x \neq \varnothing$, there is a partial isometry $w \in A$ such that $F_x = F_w$ and $x - w$ is orthogonal to w.*

Proof. As [**4**, Lemma 3].

Lemma 4.3 says that the map $u \mapsto F_u$ from the set of partial isometries in a real W^*-algebra to the set of norm exposed faces in the unit ball of the predual is onto. Unlike the complex case, this map is not in general one-to-one, this being due to the presence of skew-hermitian elements.

LEMMA 4.4. *Let u and v be partial isometries in a real W^*-algebra A. Then u and v are orthogonal if and only if* (8) *holds for every $(f, g) \in F_u \times F_v$.*

Proof. As [**4**, Lemma 4].

LEMMA 4.5. *Let x be an element of a real W^*-algebra A. Then x is a partial isometry if and only if $\|x\| = 1$, $F_x \neq \varnothing$, and $f(x) = 0$ for all f which satisfy* (8), *for all $g \in F_x$.*

Proof. As [**4**, Lemma 5].

PROPOSITION 4.6. *Let ϕ be a weak*-weak*-continuous surjective linear isometry of a real W^*-algebra A onto a real W^*-algebra B.*

(a) *If u is a partial isometry in A, then $\phi(u)$ is a partial isometry in B.*
(b) *If u and v are orthogonal partial isometries in A, then $\phi(u)$ and $\phi(v)$ are orthogonal partial isometries in B.*
(c) *If $x \in A$ then $\phi(xx^*x) = \phi(x)(\phi(x))^*\phi(x)$.*
(d) *If x and y are orthogonal elements of A, then $\phi(x)$ and $\phi(y)$ are orthogonal elements of B.*

Proof. The assertions (a) and (b) are proved as in [**4**, Proposition 1], and the assertion (c) is proved as in [**4**, (2.3)]. For (d) it suffices to observe that if $x = u|x|$ and $y = v|y|$ are the polar decompositions of x and y then u and v are orthogonal.

5. *Isometries of W^*-factors of Type I*

Our goal in this section is to prove the following.

THEOREM 5.1. *Let H and K be Hilbert spaces over the same set of scalars, which is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and let $\phi: B(H) \to B(K)$ be a weak*-weak*-continuous surjective real-linear isometry. Then ϕ preserves the triple product, that is, for $a, b, c \in B(H)$,*

$$\phi(ab^*c+cb^*a) = \phi(a)\,\phi(b)^*\phi(c)+\phi(c)\,\phi(b)^*\phi(a).$$

Note that $B(H)$ is necessarily a *real* C^*-algebra if H is real or quaternionic.

If we define the *triple product* in any associative *-algebra as $\{abc\} = (ab^*c+cb^*a)/2$, our conclusion can be rewritten more compactly as $\phi\{abc\} = \{\phi(a)\,\phi(b)\,\phi(c)\}$.

The Peirce projections $P_k(v), k = 0,1,2$ relative to a partial isometry v were defined in the previous section, as well as the notion of orthogonality: $P_0(u)\,v = v$ (or $P_0(v)\,u = u$), denoted by $u \perp v$. We say that two partial isometries u and v are *colinear* if $P_1(u)\,v = v$ and $P_1(v)\,u = u$. We indicate this relation by the notation $u \top v$. If the stronger conditions $P_2(u)\,B(H) \subset P_1(v)\,B(H)$ and $P_2(v)\,B(H) \subset P_1(u)\,B(H)$ are satisfied, we say that u and v are *strongly colinear*.

For complex Hilbert spaces, Theorem 5.1 is proved in **[2]**. We next prepare some tools for proving the theorem in the other cases. Unless otherwise stated, all Hilbert spaces are over one of the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$.

By the *rank* of a partial isometry $v \in B(H)$ is meant the common dimension of $v(H)$, lH and rH, where $l = vv^*, r = v^*v$. The partial isometry v is *primitive* if it cannot be written as a sum of two orthogonal non-zero partial isometries.

DEFINITION 5.2. Let $\{u_i\colon i = 1,2,3,4\}$ be four primitive partial isometries on a Hilbert space H. The quadruple (u_1, u_2, u_3, u_4) is said to form a *quadrangle* if

1. u_i and u_{i+1} are strongly colinear and $u_i \perp u_{i+2}$;
2. $\{u_k\, u_{k+1}\, u_{k+2}\} = \frac{1}{2}u_{k+3}$ for some k.

(The indices are computed modulo 4.)

LEMMA 5.3. *The triple products among the partial isometries belonging to a quadrangle satisfy*

1. $u_i^3 = u_i$ *(partial isometry property);*
2. $\{u_i, u_i, u_{i+1}\} = \frac{1}{2}u_{i+1}$ *and* $\{u_i, u_i, u_{i+3}\} = \frac{1}{2}u_{i+3}$ *(colinearity property);*
3. $\{u_i, u_{i+1}, u_{i+2}\} = \frac{1}{2}u_{i+3}$ *and* $\{u_i, u_{i+3}, u_{i+2}\} = \frac{1}{2}u_{i+1}$.
4. *All triple products among the partial isometries belonging to a quadrangle which are not of the form in* 2. *or* 3. *vanish.*

Proof. This follows from the computation rules

$$\{A_i(u)\,A_j(u)\,A_k(u)\} \subset A_{i-j+k}(u)$$

and

$$\{A_2(u)\,A_0(u)\,A\} = \{0\} = \{A_0(u)\,A_2(u)\,A\},$$

where $A_k(u) = P_k(u)\,B(H)$ if $k = 0,1,2$ and $A_k(u) = \{0\}$ otherwise, together with the identity

$$\{u, v, \{xyz\}\} = \{\{uvx\}, y, z\} - \{x, \{vuy\}, z\} + \{x, y, \{uvz\}\},$$

which is easily verified.

REMARK 5.4. Let v be a partial isometry on a Hilbert space H.

(i) The rank of v is one if and only if v is primitive;

(ii) If v is the sum of two orthogonal primitive partial isometries w_1 and w_3, then there are orthogonal primitive partial isometries w_2 and w_4 such that (w_1, w_2, w_3, w_4) form a quadrangle with

$$P_2(v)\,B(H) = \bigoplus_{j=1}^{4} P_2(w_j)\,B(H).$$

Indeed, if w_j is the operator $\eta_j \otimes \xi_j\colon \alpha \mapsto \langle \alpha \mid \xi_j \rangle \eta_j$, $j = 1, 3$, with ξ_1, ξ_3 an orthonormal set in H, then we can choose $w_2 = \eta_3 \otimes \xi_1$ and $w_4 = \eta_1 \otimes \xi_3$.

The reason for the terminology in the following is that this definition and the previous one make sense and are useful if the partial isometries map one Hilbert space into another (see [3]).

DEFINITION 5.5. Let I be some index set. A family $G = \{u_{ij}\colon i, j \in I\}$ of primitive partial isometries on a Hilbert space is called a *rectangular grid* if $(u_{ij}, u_{il}, u_{kl}, u_{kj})$ is a quadrangle for all choices of indices i, j, k, l with $j \neq l$ and $i \neq k$.

It is important to note that any two distinct elements of a rectangular grid are either colinear or orthogonal. Also, the triple product among any three elements of a rectangular grid vanishes, unless they all belong to some quadrangle. Thus, the triple product on the real span of a rectangular grid is determined by the quadrangles which are formed by elements of the grid.

Let $\{\xi_i\}$ be an orthonormal basis for the real Hilbert space H. Let e_{ij} be the primitive partial isometry $\xi_i \otimes \xi_j$ defined by $\eta \mapsto \langle \eta \mid \xi_j \rangle \xi_i$. The family $\{e_{ij}\}$ is a rectangular grid and will be referred to as a family of *elementary matrices* on H.

LEMMA 5.6. *The span of a family $\{e_{ij}\}$ of elementary matrices on a real Hilbert space H is weak*-dense in $B(H)$.*

Proof. This follows, just as in the complex case, from the fact that $B(H)_*$ is the trace class operator, denoted $T(H)$. This latter fact also follows exactly as in the complex case (see [**21**; VI.9, VI.10, VI.18, VI.19(a), VI.24], and [**19**; 3.5.2, 3.5.3, 3.5.4]).

REMARK 5.7. Let H be a Hilbert space over $\mathbb{C}$ ($\mathbb{H}$ respectively). If $\{\xi_j\}$ is an orthonormal basis for H, then with $H^{\mathbf{R}} :=$ the closed real span of $\{\xi_j\}$, $H^{\mathbf{R}}$ is a real Hilbert space, and we have

$$H = \mathbb{C} \otimes_{\mathbf{R}} H^{\mathbf{R}} \quad (H = \mathbb{H} \otimes_{\mathbf{R}} H^{\mathbf{R}} \text{ respectively})$$

and

$$B(H) = \mathbb{C} \otimes_{\mathbf{R}} B(H^{\mathbf{R}}) \quad (B(H) = \mathbb{H} \otimes_{\mathbf{R}} B(H^{\mathbf{R}}) \text{ respectively}).$$

In particular, if H is of finite dimension n over $\mathbb{H}$, we can identify the real algebras $B(H)$ and $M_n(\mathbb{H}^{\mathrm{op}})$ by associating $a\xi_i \otimes \xi_j \in B(H)$ with $a \otimes e_{ij}$, which is the matrix with a in the i,j-position and zeros elsewhere. In this case, $B(H)_2(e_{ij}) = \mathbb{H} \otimes_{\mathbf{R}} e_{ij}$.

The following is the analog of Lemma 5.6 for quaternionic Hilbert spaces.

Proposition 5.8. *If H is a quaternionic Hilbert space, then the real span of $\{\mathbb{H} \otimes_{\mathbb{R}} e_{ij}: i, j \in I\}$ is weak*-dense in $B(H)$, where $\{e_{ij}\}$ is a set of elementary matrices on $H^{\mathbb{R}}$.*

Proof. We first observe that $(\mathbb{H} \otimes T(H^{\mathbb{R}}))'$ can be identified with

$$\mathbb{H} \otimes B(H^{\mathbb{R}}) = B(\mathbb{H} \otimes H^{\mathbb{R}}).$$

In fact, $\phi \in (\mathbb{H} \otimes T(H^{\mathbb{R}}))'$ is associated with the element

$$1 \otimes x_1 + i \otimes x_2 + j \otimes x_3 + k \otimes x_4,$$

where the $x_j \in B(H^{\mathbb{R}})$ are such that $\phi(1 \otimes t) = \operatorname{trace}(tx_1)$, $\phi(i \otimes t) = \operatorname{trace}(tx_2)$, and so on, for $t \in T(H^{\mathbb{R}})$, and the duality of $\mathbb{H} \otimes T(H^{\mathbb{R}})$ and $\mathbb{H} \otimes B(H^{\mathbb{R}})$ is given by

$$\langle \mu \otimes t, \lambda \otimes x \rangle = \left(\sum_{p=1}^{4} \mu_p \lambda_p \right) \operatorname{trace}(tx)$$

for $t \in T(H^{\mathbb{R}})$, $x \in B(H^{\mathbb{R}})$ and $\lambda, \mu = \mu_1 + \mu_2 i + \mu_3 j + \mu_4 k \in \mathbb{H}$.

To show the weak*-density, let $\psi \in \mathbb{H} \otimes T(H^{\mathbb{R}})$ and suppose that ψ vanishes on all $\mathbb{H} \otimes e_{ij}$. To show that $\psi = 0$, define $\psi^{(p)} \in (B(H^{\mathbb{R}}))'$, as above, by $\psi^{(1)}(a) = \psi(1 \otimes a)$, $\psi^{(2)}(a) = \psi(i \otimes a)$ and so on, for $a \in B(H^{\mathbb{R}})$. By the weak*-continuity of the map $a \mapsto \lambda \otimes a$, $\psi^{(p)} \in B(H^{\mathbb{R}})_*$. Thus, since $\psi^{(p)}(e_{ij}) = 0$ implies $\psi^{(p)} = 0$ for all p we have, for $\lambda \otimes a \in \mathbb{H} \otimes B(H^{\mathbb{R}})$,

$$\psi(\lambda \otimes a) = \lambda_1 \psi(1 \otimes a) + \lambda_2 \psi(i \otimes a) + \ldots = 0.$$

Now let H and K be real Hilbert spaces and let $\phi: B(H) \to B(K)$ be a weak*-weak*-continuous surjective linear isometry. To show that ϕ preserves the triple product, we only need to show that $\{\phi(e_{ij})\}$ is a rectangular grid in $B(K)$. For this purpose, it suffices to show that ϕ maps the quadrangles in $B(H)$ into quadrangles in $B(K)$. This is done in Proposition 5.11 below.

LEMMA 5.9. *Let H, K be two-dimensional real Hilbert spaces and let (w_1, w_2, w_3, w_4) be a quadrangle (of rank 1 partial isometries) in $B(H, K)$. Let $z = aw_1 + bw_2 + cw_3 + dw_4$ for $a, b, c, d \in \mathbb{R}$. Then z is a real multiple of a primitive partial isometry if and only if $ac - bd = 0$. Moreover, in this case, $\|z\|^2 = a^2 + b^2 + c^2 + d^2$.*

Proof. Choose orthonormal bases for H and K in which $w_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and so on. Then $z = \begin{bmatrix} a & b \\ d & c \end{bmatrix}$ and applying it to the vectors $(1, 0)$ and $(0, 1)$ shows that the vectors (a, d) and (b, c) are proportional. This proves the first statement. For the second statement, let η_1 be a unit vector in the range of z^*z. Then

$$\|z\|^2 = \|z^*z\| = \langle z^*z\eta_1 \mid \eta_1 \rangle = \operatorname{trace}(z^*z).$$

LEMMA 5.10. *Let H and K be a pair of Hilbert spaces over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and let $\phi: B(H) \to B(K)$ be a surjective real-linear isometry. Let u be any partial isometry in $B(H)$. Then $\phi[P_2(u) B(H)] = P_2(\phi(u)) B(K)$. In particular if u_1 and u_3 are orthogonal partial isometries of rank 1 in $B(H)$, then $\phi(u_1)$ and $\phi(u_3)$ are orthogonal partial isometries of rank 1 in $B(K)$, and ϕ restricts to an isometry of $P_2(u_1 + u_3) B(H)$ onto $P_2(\phi(u_1) + \phi(u_3)) B(K)$.*

Proof. Let w be a partial isometry such that $u+w$ is a maximal partial isometry. Then with $M = B(H)$ and $N = B(K)$, we have $M_2(u) = M_0(w) = \{w\}^{\perp}$, where $M_i(u) = P_i(u)M$. Since ϕ preserves orthogonality, $\phi(M_2(u)) \subset \{\phi(w)\}^{\perp} = N_0(\phi(w)) = N_2(\phi(u))$, where similarly $N_i(v) = P_i(v)N$. By considering the inverse of ϕ, we obtain equality.

PROPOSITION 5.11. *Let H and K be a pair of real Hilbert spaces, and let $\phi: B(H) \to B(K)$ be a surjective real-linear isometry. If (u_1, u_2, u_3, u_4) is a quadrangle of rank 1 partial isometries of $B(H)$, then $(\phi(u_1), \phi(u_2), \phi(u_3), \phi(u_4))$ is a quadrangle in $B(K)$.*

Proof. Let $w_j = \phi(u_j)$ for $j = 1, 3$, and choose, by Remark 5.4, w_2, w_4 to be primitive partial isometries such that (w_1, w_2, w_3, w_4) is a quadrangle spanning $P_2(w_1+w_3)B(K)$.

According to Lemma 5.10, we may write $\phi(u_2) = aw_1+bw_2+cw_3+dw_4$. Since $\phi(u_2)$ is a primitive partial isometry, we have by Lemma 5.9

$$a^2+b^2+c^2+d^2 = 1. \tag{9}$$

On the other hand, by the same lemma, $(u_1+u_2)/\sqrt{2}$ is a partial isometry of rank 1. Thus

$$\|\phi(u_1+u_2)\|^2 = (a+1)^2+b^2+c^2+d^2 = 2, \tag{10}$$

which together with (9) implies $a = 0$. This same argument when applied to u_2 and u_2+u_3 yields $c = 0$. Since the rank of $\phi(u_2)$ is one, we also have $d = 0$ or $b = 0$. In the case $d = 0$, $\phi(u_2) = \pm w_2$ and since $u_4 \perp u_2$, $\phi(u_2) \in N_0(w_2) = N_2(w_4)$, implying $\phi(u_4) = \pm w_4$ (here N denotes $B(K)$).

Since $(w_1, -w_2, -w_3, -w_4)$ is also a quadrangle, we may assume without loss of generality that $\phi(u_2) = w_2$. The proof will be completed by showing that $\phi(u_4) = w_4$. Suppose instead that $\phi(u_4) = -w_4$, and let $z = u_1+u_2+u_3+u_4$. Then $\phi(z) = w_1+w_2+w_3-w_4$, which contradicts Lemma 5.9.

The proof for the case $b = 0$ is the same with w_4 and w_2 interchanged.

This completes the proof of Theorem 5.1 in the case of real Hilbert spaces. We now complete the proof in the case of quaternionic Hilbert spaces, thereby completing the proof of Theorem 5.1. As in the case of real Hilbert spaces, there will be a reduction to the two by two matrix case. We formulate this case in the following proposition.

PROPOSITION 5.12. *Let $M = B(H,K), N = B(H',K')$, with two-dimensional Hilbert spaces H, K, H', K' over the quaternions $\mathbb{H}$. Let $\phi: M \to N$ be a surjective real-linear isometry. Then ϕ preserves the triple product.*

Proof. Let $\{\xi_i\}$ and $\{\eta_i\}$ be orthonormal bases for H and K respectively, and define the operators $e_{ij} = \eta_i \otimes \xi_j \in B(H,K)$. Let $e'_{ii} = \phi(e_{ii})$, so that e'_{11} and e'_{22} are orthogonal primitive partial isometries. Make a preliminary choice of unit vectors to satisfy $e'_{ii} := \eta'_i \otimes \xi'_i$, and define $w_{ij} := \eta'_i \otimes \xi'_j \in B(H',K')$. An argument similar to the one used to prove (10) shows that $\phi(e_{12}) \in [\mathbb{H} \otimes w_{12}] \cup [\mathbb{H} \otimes w_{21}]$. A modification of the bases results in orthonormal bases $\{\eta'_i\}$ and $\{\xi'_i\}$ such that either

$$\phi(e_{ii}) = w_{ii} (i = 1,2), \quad \phi(e_{12}) = w_{12} \quad \text{and} \quad \phi(e_{21}) \in N_0(w_{12}) = N_2(w_{21}) = \mathbb{H} \otimes w_{21},$$

or

$$\phi(e_{ii}) = w_{ii}\ (i = 1,2), \quad \phi(e_{12}) = w_{21} \quad \text{and} \quad \phi(e_{21}) \in \mathbb{H} \otimes w_{12}.$$

In what follows, we shall assume the second alternative above, that is, the case $\phi(e_{12}) = w_{21}$, the proof of the other case being similar. By Lemma 5.10, $\phi(M_2(e_{ij})) = N_2(w_{ji})$ $(1 \leqslant i,j \leqslant 2)$. Since $M_2(e_{ij}) = \mathbb{H} \otimes e_{ij}$ and $N_2(w_{ji}) = \mathbb{H} \otimes w_{ji}$ (see Remark 5.7), there exist maps $\rho_{ij}: \mathbb{H} \to \mathbb{H}$ satisfying

$$\phi(a \otimes e_{ij}) = \rho_{ij}(a) \otimes w_{ji}, \quad a \in \mathbb{H}.$$

We now assert that all the ρ_{ij} coincide with a map ρ which is a *-anti-isomorphism of $\mathbb{H}$ (in the case $\phi(e_{12}) = w_{12}$, the corresponding map is a *-isomorphism). With $z = a \otimes e_{11} + b \otimes e_{12} + c \otimes e_{21} + d \otimes e_{22}$, we have the implications $ab^{-1} = cd^{-1} \Rightarrow z$ has one-dimensional range $\Rightarrow z$ is a real multiple of a partial isometry of rank $1 \Rightarrow \phi(z)$ is a real multiple of a partial isometry of rank $1 \Rightarrow \rho_{11}(a)\rho_{21}(c)^{-1} = \rho_{12}(b)\rho_{22}(d)^{-1}$ (recall that $\phi(e_{12}) = w_{21}$). In particular, with $a = cd^{-1}b$ we obtain

$$\rho_{11}(cd^{-1}b) = \rho_{12}(b)\rho_{22}(d)^{-1}\rho_{21}(c). \tag{11}$$

By our choice of the bases, ρ_{11}, ρ_{12} and ρ_{22} are unital. Using $c = d = b = 1$ in (11) shows that ρ_{21} is unital, that is, $\phi(e_{21}) = w_{12}$. From this and (11), our assertion follows.

We now have $\phi(a \otimes e_{ij}) = \rho(a) \otimes w_{ji}$, $a \in \mathbb{H}$, with ρ a *-anti-isomorphism of $\mathbb{H}$, and from this it is easy to check that ϕ preserves the triple product. For example,

$$\{a \otimes e_{11}, b \otimes e_{12}, c \otimes e_{22}\} = \frac{a\bar{b}c}{2} \otimes e_{21}$$

and

$$\{\rho(a) \otimes w_{11}, \rho(b) \otimes w_{21}, \rho(c) \otimes w_{22}\} = \frac{\rho(c)\overline{\rho(b)}\rho(a)}{2} \otimes w_{12}$$

so that

$$\phi(\{a \otimes e_{11}, b \otimes e_{12}, c \otimes e_{22}\}) = \rho\left(\frac{a\bar{b}c}{2}\right) \otimes w_{12} = \frac{\rho(c)\overline{\rho(b)}\rho(a)}{2} \otimes w_{12}.$$

We can now complete the proof of Theorem 5.1. Suppose that the Hilbert spaces H, K in Theorem 5.1 are over $\mathbb{H}$. With $\{e_{ij}\}$ a rectangular grid formed by elementary matrices, by Proposition 5.8, $\oplus_{i,j} M_2(e_{ij})$ is weak *-dense in $M := B(H)$. Thus it suffices to prove that

$$\phi(\{xyz\}) = \{\phi(x)\,\phi(y)\,\phi(z)\} \tag{12}$$

holds for $x \in M_2(e_{ij})$, $y \in M_2(e_{kl})$, $z \in M_2(e_{mn})$. If $\{e_{ij}, e_{kl}, e_{mn}\}$ is not part of a quadrangle, then both sides of (12) are zero since ϕ preserves strong colinearity and orthogonality. Otherwise $x, y, z \in M_2(e_{pq} + e_{rs}) \cong B(\tilde{H}, \tilde{K})$ for some two dimensional Hilbert spaces $\tilde{H}, \tilde{K}$.

REMARK 5.13. Let H be a Hilbert space over $\mathbb{C}$ ($\mathbb{H}$ respectively) and let $H^{\mathbb{R}}$ be a Hilbert space over $\mathbb{R}$ such that

$$B(H) = \mathbb{C} \otimes_{\mathbb{R}} B(H^{\mathbb{R}}) \quad (B(H) = \mathbb{H} \otimes_{\mathbb{R}} B(H^{\mathbb{R}}) \text{ respectively}).$$

The proof of Proposition 5.12 showed that an isometry ϕ of $B(H)$ factors as $\phi = \rho \otimes \phi'$ where ϕ' is an isometry of $B(H^{\mathbb{R}})$ and ρ is a *-isomorphism or *-anti-isomorphism of $\mathbb{C}$ ($\mathbb{H}$ respectively). The isometry ϕ' is the restriction of ϕ to the closed real span of an appropriate grid. The proof also indicates that the complex and quaternionic cases follow in a unified way from the real case proved above.

6. *Isometries of real C^*-algebras*

For a projection p in a real W^*-algebra, $c(p)$ will denote the smallest central projection dominating p. The projection $c(p)$ is the unique central projection such that $Ac(p)$ is the intersection of all weak*-closed two sided ideals containing p. We call $c(p)$ the *central support* of p.

LEMMA 6.1. *In a real W^*-factor A, any two minimal projections are equivalent.*

Proof. Let p and q be two minimal projections in A. Then their central supports $c(p)$ and $c(q)$ are each equal to the identity.

Consider pAq. We claim that $pAq \neq \{0\}$. In fact, if $pAq = \{0\}$, then

$$I := \{x \in A : pAx = \{0\}\}$$

is a weak*-closed two sided ideal and we can find a central projection e such that $I = Ae$. Since $q \in I$, $q \leqslant e$, and hence $c(q) \leqslant e$. Thus $e = 1$ and $p = 0$, a contradiction.

If $x \neq 0$ and $x \in pAq$, we have that $x = pxq$, so that $s_l(x) \leqslant p$. Since p is minimal, $s_l(x) = p$ and similarly $s_r(x) = q$. The lemma now follows from Proposition 3.1.

Let $c(f)$ denote the central support of $s(f)$, the support of a state f belonging to the predual E of a real W^*-algebra A.

LEMMA 6.2. *Let $f \in \partial S_A$ be a pure state of a real C^*-algebra A, and let $c(f)$ be the central support of f in A''. Then $A''c(f) \cong B(H_f)$ for some Hilbert space H_f over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*

Proof. Since f is a pure state, it follows that $s(f)$ is a minimal projection, $c(f)$ is a minimal central projection, and $s(f) \leqslant c(f)$. Thus $A''c(f)$ is a real W^*-factor.

Let $\{e_i\}$ be a maximal family of mutually orthogonal minimal projections in $A''c(f)$. We claim that $c(f) = \sum_i e_i$. Suppose this is not the case. Then, as in the proof of Lemma 6.1, we have, for $p = e_{i_0}$ for some i_0 and $q = c(f) - \sum_i e_i$, that $pA''c(f)q \neq \{0\}$. Thus choosing $x \neq 0, x \in pA''c(f)q$, the argument of the proof of Lemma 6.1 shows that $s_l(x) = p$ and $s_r(x) \leqslant q$ are two orthogonal minimal equivalent projections. Thus, $\{e_i\} \cup \{s_r(x)\}$ is a family of mutually orthogonal minimal projections in $A''c(f)$, which is a contradiction.

Thus $c(f) = \sum_i e_i$, and choosing partial isometries $\{u_i\}$ in $A''c(f)$ such that $u_i^* u_i = e_1$, $u_i u_i^* = e_i$, we see that $\{w_{ij}\}$, where $w_{ij} = u_i u_j^*$, forms a set of matrix units. Since $e_i A''c(f) e_i$ is a real Banach division algebra, it is isomorphic to one of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ [**10**, Theorem 9.7]. It follows that $A''c(f) \cong B(H_f)$ for some real, complex, or quaternionic Hilbert space H_f.

Define the *atomic part* of A'' to be $A''z_A$, where $z_A := \bigvee_{f \in \partial S_A} c(f)$.

LEMMA 6.3. *Let ϕ be a surjective linear isometry from a real C^*-algebra A to a real C^*-algebra B. Then $\phi''(A''z_A) = B''z_B$.*

Proof. We remark first that if $f, g \in \partial S_A$, then either $c(f) = c(g)$ or $c(f)c(g) = 0$. Indeed, if $c(f)c(g) \neq 0$, then $c(f)A''c(g) \neq \{0\}$, and hence the proof of Lemma 6.1

would give a subprojection of $c(f)$ equivalent to a subprojection of $c(g)$. That is, choosing $x \neq 0$, $x \in c(f)A''c(g)$, we have that $s_l(x)$ is equivalent to $s_r(x)$ and $s_l(x) \leqslant c(f)$, $s_r(x) \leqslant c(g)$. Let p be a minimal subprojection of $s_l(x)$ and q the corresponding minimal subprojection of $c(g)$ which is equivalent to p. By the proof of Lemma 6.2, p (q respectively) is equivalent to $s(f)$ ($s(g)$). Thus $s(f)$ is equivalent to $s(g)$, and therefore $c(f) = c(g)$.

Thus, we obtain that $A''z_A \cong \bigoplus B(H_f)$, and $A''z_A$ is the weak*-closed span of its minimal projections. It remains to show that ϕ'' maps a minimal projection into $B''z_B$.

From Proposition 4.6, we know that ϕ'' carries partial isometries to partial isometries. In fact, it carries primitive partial isometries to primitive partial isometries. Thus, if p is a minimal subprojection of z_A, $\phi''(p)$ is a primitive partial isometry. Then $\phi''(p)^*\phi''(p)$ is a minimal projection so that $\phi''(p)^*\phi''(p)(1-z_B) = 0$. From $\phi''(p) = \phi''(p)z_B + \phi''(p)(1-z_B)$ we see that $\phi''(p) = \phi''(p)z_B$, so that $\phi''(p) \in B''z_B$.

We can now prove the main result of our paper.

THEOREM 6.4. *A surjective linear isometry ϕ between two real C^*-algebras preserves the triple product:* $\phi(ab^*c+cb^*a) = \phi(a)\phi(b)^*\phi(c)+\phi(c)\phi(b)^*\phi(a)$.

Proof. As $A \to A''z_A$ is an isomorphism, by the previous two lemmas, it is enough to check that ϕ'' preserves the triple product on $A''z_A$.

Also, from Lemma 6.2, $A''z_A \cong \bigoplus_{f \in \partial S_A} B(H_f)$ and $B''z_B \cong \bigoplus_{g \in \partial S_B} B(K_g)$, so we can view ϕ'' as a map from $\bigoplus_{f \in \partial S_A} B(H_f)$ to $\bigoplus_{g \in \partial S_B} B(K_g)$. In this case the result follows from Theorem 5.1 since for each f, there is a g such that K_g and H_f have the same scalars and $\phi''(B(H_f)) = B(K_g)$. To prove the last statement, it suffices, by symmetry to show that $\phi''(B(H_f))$ lies in some $B(K_g)$. If (u_1, u_2, u_3, u_4) is a quadrangle in $B(H_f)$, then $\phi''(u_1)$, being primitive, lies in a summand of $B''z_B$, and if $\phi''(u_1)$ and $\phi''(u_2)$ belonged to different summands, then $u_1 = \phi''^{-1}(\phi''(u_1))$ and u_2 would be orthogonal, which is a contradiction. Two more applications of this argument shows that ϕ'' maps the quadrangle into some summand of $B''z_B$. Moreover, any two quadrangles in a given rectangular grid are mapped into the same summand. Indeed, picking an element of each quadrangle, there is a third quadrangle in the grid which contains both elements.

7. *Contractive projections*

By the Arens–Kaplansky theorem [10, 12.5], every commutative real C^*-algebra is isomorphic to a norm closed real *-subalgebra of a commutative complex C^*-algebra $C(X, \mathbb{C})$. For each $\xi \in A'$ there is a complex Borel measure μ on X such that $\|\mu\| = \|\xi\|$ and

$$\xi(f) = \Re \int_X f\,d\mu \quad \text{for } f \in A.$$

We shall call μ a *representing measure* for ξ.

For any complex measure $\mu \in C(X, \mathbb{C})^*, \mu = \phi \cdot |\mu|$ denotes the measure theoretic polar decomposition of μ.

Let $P\colon A \to A$ be a (real) linear contractive projection and let $P'\colon A' \to A'$ be its Banach space adjoint.

LEMMA 7.1. *Let ξ be an extreme point of the convex set $P'(A')_1$ and let μ be a representing measure for ξ. Then for every $f \in A$*

$$Pf = \langle \xi, f \rangle \bar{\phi}, \quad |\mu|\text{-a.e.}$$

Proof. Suppose that $(Pf)\phi$ is not constant $|\mu|$-a.e. Then there is a real α such that, with

$$E_1 = \{x \in X : \Re(Pf(x)\phi(x)) \geqslant \alpha\},$$

and

$$E_2 = \{x \in X : \Re(Pf(x)\phi(x)) < \alpha\},$$

we have $t = |\mu|(E_1) > 0$ and $1-t = |\mu|(E_2) > 0$. We have

$$\mu = t\frac{\mu|_{E_1}}{t} + (1-t)\frac{\mu|_{E_2}}{1-t},$$

and therefore, with $\xi_1, \xi_2 \in A'$ defined by

$$\xi_1(f) = \Re\frac{1}{t}\int_{E_1} f\,d\mu, \quad \xi_2(f) = \Re\frac{1}{1-t}\int_{E_2} f\,d\mu,$$

we have $\xi = P'\xi = tP'\xi_1 + (1-t)P'\xi_2$.

Since ξ is extreme, $P'\xi_1 = P'\xi_2$. But

$$\langle f, P'\xi_1 \rangle = \frac{1}{t}\Re\int_{E_1} Pf\,d\mu = \frac{1}{t}\Re\int_{E_1} (Pf)\phi\,d|\mu| = \frac{1}{t}\int_{E_1} \Re(Pf)\phi\,d|\mu| \geqslant \alpha.$$

Similarly, $\langle f, P'\xi_2 \rangle < \alpha$, contradiction.

Therefore, $(Pf)\phi = k$, $|\mu|$-a.e., and

$$\langle f, \xi \rangle = \langle Pf, \xi \rangle = \Re\langle Pf, \mu \rangle = \Re\langle Pf, \phi|\mu| \rangle = \Re\langle k\bar{\phi}, \phi|\mu| \rangle = k.$$

REMARK 7.2. Let T be a subset of A' and let $S = \{\xi_j\}_{j \in J}$ be a subset of T which is maximal with respect to the property: $\xi_i \neq \pm\xi_j$ *if* $i \neq j$. Then given any $\xi \in T$ either $\xi \in S$ or $-\xi \in S$.

THEOREM 7.3. *Let A be a commutative real C^*-algebra, say $A \subset C(X, \mathbb{C})$, and let P be a contractive projection on A. Then there exists a family of complex Borel measures $\{\mu_i\}_{i \in I}$ such that with $\mu_i = \phi_i \cdot |\mu_i|$ and $S = \bigcup_i \operatorname{supp}|\mu_i|$,*

1. $\|\mu_i\| = 1$ *for each* $i \in I$,
2. *there is a bounded linear transformation $Q\colon A \to C_b(S, \mathbb{C})$ such that for each $i \in I$,*

$$Qf|_{\operatorname{supp}|\mu_i|} = \Re\langle f, \mu_i \rangle \bar{\phi}_i, \quad |\mu_i|\text{-a.e.},$$

3. *there is an isometric simultaneous extension operator $E\colon Q(A) \to A$ such that $P = EQ$.*

Proof. Let $\{\xi_j\}_{j \in J}$ be a family of extreme points of $P'(A')_1$ which is maximal with respect to the property: $\xi_i \neq \pm\xi_j$ for all $i \neq j$ in J. Let μ_j be a representing measure for ξ_j, let $S_j = \operatorname{supp}|\mu_j|$, and $S = \bigcup_j S_j$. Define

$$Q\colon A \to C_b(S) \text{ and } E\colon Q(A) \to A$$

by

$$Qf = Pf|_S \text{ and } E(Qf) = Pf.$$

The theorem will follow as soon as it is proved that

$$\|Qf\|_{C_b(S)} = \|Pf\|_A,$$

and for this it suffices to prove

$$\|Pf\|_A = \sup_{x \in S} |Pf(x)|.$$

We have, by Krein–Milman,

$$\|Pf\|_A = \sup_{\xi \in P'(A')_1,\, \xi \text{ extreme}} |\langle \xi, Pf \rangle| = \sup_{j \in J} |\langle \xi_j, Pf \rangle|$$

$$= \sup_{j \in J} \left| \Re \int_{S_j} Pf\, d\mu_j \right| \leqslant \sup_{x \in S} |Pf(x)|.$$

The third-named author wishes to take this opportunity to make a correction in [7]. In that paper, Lemma 1.3 (and hence Lemma 1.4) is false. Therefore the condition (1.2) should be deleted as an assumption in Proposition 1.1 and as a conclusion in Theorem 1. This change does not affect any of the later results in [7].

The following result is a straightforward consequence of Theorem 7.3, as in [7].

PROPOSITION 7.4. *Let P be a contractive projection on a commutative real C^*-algebra A.*

(a) *For $f, g, h \in A$, we have*

$$Qf\overline{Qg}Qh = Q(Pf\overline{Pg}h) = Q(Pf\bar{g}Ph).$$

Therefore $Q(A)$ is a ternary subalgebra of $C_b(S, \mathbb{C})$, that is, it is closed under the triple product $f\bar{g}h$,

(b) *the range $P(A)$ is a real C^*-ternary algebra with the triple product*

$$[f, g, h] = P(f\bar{g}h) \text{ for } f, g, h \in P(A),$$

that is,

$$[[f_1, f_2, f_3], f_4, f_5] = [f_1, f_2, [f_3, f_4, f_5]] = [f_1, [f_2, f_3, f_4], f_5],$$

$$\|[f, g, h]\| \leqslant \|f\|\,\|g\|\,\|h\|,$$

and

$$\|[f, f, f]\| = \|f\|^3.$$

In view of [7, Theorem 5], it is natural to ask whether $P(A)$ is isometric to a real C_σ-space. This is false. The complex field $\mathbb{C}$, considered as a real C^*-algebra, is real isometric to a two-dimensional Hilbert space. If $\mathbb{C}$ were isometric to a real C_σ-space, then by [**16**, Corollary, p. 343], $\mathbb{C}$ would be real isometric to all continuous real functions on some compact Hausdorff space, so its (real) dual would be isometric to a real L^1-space, which gives a contradiction.

The following remains a challenging and important open problem in the study of real JB^*-triples.

PROBLEM 7.5. *Is the range of a contractive projection on a real C^*-algebra isometric to a linear subspace of some real C^*-algebra, closed for the natural triple product $ab^*c + cb^*a$?*

References

1. T. BARTON and R. TIMONEY, 'Weak*-continuity of Jordan triple products and applications', *Math. Scand.* 59 (1986) 177–191.
2. T. DANG, 'Real isometries between JB^*-triples', *Proc. Amer. Math. Soc.* 114 (1992) 971–980.
3. T. DANG and Y. FRIEDMAN, 'Classification of JBW^*-triple factors and applications', *Math. Scand.* 61 (1987) 292–330.
4. T. DANG, Y. FRIEDMAN and B. RUSSO, 'Affine geometric proofs of the Banach–Stone theorems of Kadison and Kaup', Proc. of the 1987 Great Plains Operator Theory Seminar, *Rocky Mountain J. Math.* 20 (1990) 409–428.
5. J. DUNCAN and S. A. R. HOSSEINIUM, 'The second dual of a Banach algebra', *Proc. Roy. Soc. Edinburgh* 84A (1979) 309–325.
6. K. EKMAN, 'Unitaries and partial isometries in a real W^*-algebra', *Proc. Amer. Math. Soc.* 54 (1976) 138–140.
7. Y. FRIEDMAN and B. RUSSO, 'Contractive projections on $C_0(K)$', *Trans. Amer. Math. Soc.* 273 (1982) 57–73.
8. Y. FRIEDMAN and B. RUSSO, 'Structure of the predual of a JBW^*-triple', *J. Reine Angew. Math.* 356 (1985) 67–89.
9. Y. FRIEDMAN and B. RUSSO, 'Solution of the contractive projection problem', *J. Functional Analysis* 60 (1985) 56–79.
10. K. R. GOODEARL, *Notes on real and complex C^*-algebras* (Shiva, England, 1982).
11. H. HANCHE-OLSEN and E. STØRMER, *Jordan operator algebras* (Pitman, London, 1984).
12. R. V. KADISON, 'Isometries of operator algebras', *Ann. of Math.* 54 (1951) 325–338.
13. W. KAUP, 'A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces', *Math. Z.* 138 (1983) 503–529.
14. M. KOECHER, *An elementary approach to bounded symmetric domains* Lecture Notes (Rice University, Houston, 1969).
15. BING REN LI, 'Real operator algebras', *Sci. Sinica* 22 (1979) 733–746.
16. J. LINDENSTRAUSS and D. E. WULBERT, 'On the classification of the Banach spaces whose duals are L_1 spaces', *J. Functional Analysis* 4 (1969) 332–349.
17. O. LOOS, *Bounded symmetric domains and Jordan pairs* Lecture Notes (University of California, Irvine, 1977).
18. E. NELSON, 'Notes on non-commutative integration', *J. Functional Analysis* 15 (1974) 103–116.
19. T. W. PALMER, 'Real C^*-algebras', *Pacific J. Math.* 35 (1970) 195–204.
20. G. K. PEDERSEN, *C^*-algebras and their automorphism groups* (Academic Press, New York, 1979).
21. M. REED and B. SIMON, *Methods of modern mathematical physics I: Functional analysis* (Academic Press, New York, 1972).
22. C. E. RICKART, *General theory of Banach algebras* (Van Nostrand, New York, 1960).
23. S. SAKAI, *C^*-algebras and W^*-algebras* (Springer-Verlag, New York, 1971).
24. M. TAKESAKI, *Theory of operator algebras* I (Springer-Verlag, New York, 1979).
25. H. UPMEIER, *Symmetric Banach manifolds and Jordan C^*-algebras* (North-Holland, Amsterdam, 1985).

(C-H.C.)
Goldsmiths' College
University of London
London SE14 6NW

(T.D. and B.R.)
Department of Mathematics
University of California
Irvine
California 92717

(B.V.)
Department of Mathematics
California State University
San Bernardino
California 92407