GENERALIZED RECURSIVE ATOM ORDERING AND EQUIVALENCE TO CL-SHELLABILITY

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Submitted: Feb 7, 2023; Accepted: Mar 24, 2024; Published: Jun 30, 2024

Abstract. Björner and Wachs introduced CL-shellability as a technique for studying the topological structure of order complexes of partially ordered sets. They also introduced the notion of recursive atom ordering, and they proved that a finite bounded poset is CL-shellable if and only if it admits a recursive atom ordering.

In this paper, a generalization of the notion of recursive atom ordering is introduced. A finite bounded poset is proven to admit such a generalized recursive atom ordering if and only if it admits a traditional recursive atom ordering. This is also proven equivalent to admitting a CC-shelling (a type of shelling introduced by Kozlov) with a further property called self-consistency. Thus, CL-shellability is proven equivalent to self-consistent CC-shellability. As an application, the uncrossing posets, namely the face posets for stratified spaces of planar electrical networks, are proven to be dual CL-shellable.

Keywords. Poset topology, lexicographic shellability, EC-shellability, recursive atom ordering, uncrossing order

Mathematics Subject Classifications. 05E45, 06A07

1. Introduction

This paper introduces a new tool for studying the topological structure of order complexes of finite partially ordered sets (posets). This tool, called generalized recursive atom ordering, is a relaxation of the fundamental and widely used recursive atom ordering technique introduced by Björner and Wachs in [BW83]. Any recursive atom ordering (a notion that is reviewed in Section 2) of a finite bounded poset gives rise to a lexicographic shelling for the poset, thereby

*Supported by NSF grant DMS-1953931.

yielding the result that the order complex of the poset is either homotopy equivalent to a wedge of spheres or contractible.

We establish a number of fundamental properties of these generalized recursive atom orderings (GRAOs), including the property that any generalized recursive atom ordering may be transformed into a traditional recursive atom ordering (RAO) by a process we introduce called the atom reordering process. Since GRAOs are easier to construct than RAOs, this may give a useful new pathway to proving a poset is CL-shellable. These generalized recursive atom orderings further allow us to prove that several different forms of lexicographic shellability are all equivalent to each other, by which we mean that a finite bounded poset admits any one of these types of lexicographic shelling if and only if it admits each of the others. One might expect this to imply the stronger statement that any instance of any one of these types of lexicographic shelling is also an instance of any other of these types of lexicographic shelling, but this is not always true. For instance, one may deduce that not every "self consistent CC-shelling" is a CL-shelling from the fact that not every generalized recursive atom ordering is a recursive atom ordering; this latter fact is stated more precisely in Remark 3.3, and an example demonstrating this latter fact appears in Figure 1.2. These equivalence results clarify the hierarchy of different techniques for proving that a finite bounded poset is lexicographically shellable. Figure 1.1 gives a schematic of many of the implications proven in this paper, and it points readers (by way of the labels on the implication arrows) to where each result is proven in this paper or elsewhere in the literature.

The importance of the notion of recursive atom ordering stems from the fact that a finite bounded poset admits a recursive atom ordering if and only if it is CL-shellable. EL-shellability and CL-shellability are the original and predominant techniques for proving that posets are shellable. EL-shellability was first introduced by Björner in [Bjö80]. It was generalized to the notion of CL-shellability by Björner and Wachs in [BW83] when they observed that their idea for a way to shell Bruhat order did not meet the requirements of an EL-shelling but nonetheless gave rise to a shelling of a very similar flavor to an EL-shelling. Recursive atom orderings were introduced in [BW83] as an alternative approach to proving CL-shellability. For families of posets with inherent recursive structure, such as the partition lattice, recursive atom ordering can often be the easiest and most natural way to prove these posets are shellable. Both CL-shellability and the related notion of recursive atom ordering were extended to the non-graded case in [BW96] and [BW97]. In [Koz97], EL-shellability and CL-shellability were generalized to the notions of EC-shellability and CC-shellability, which allowed more flexibility in constructing labelings that could be used to establish lexicographic shellability. Some important examples of posets that have been proven to be lexicographically shellable (or dual lexicographically shellable), in some cases by way of a recursive atom ordering, include Bruhat order [BW82], posets with exponential structures [Sag86], supersolvable lattices [Bjö80], geometric lattices [Bjö80], geometric semilattices [WW85], various posets from finite group theory [Sha01, Woo07] and combinatorial commutative algebra [PRS98], intersection posets of k-equal subspace arrangements [BW96], and face posets of shellable d-complexes [Bjö84].

In Section 2, we review background, including some key ideas of Björner and Wachs that we will build upon later in the paper. We establish some fundamental properties of GRAOs and of the atom reordering process in Sections 3 and 4, respectively. In particular, Section 4 presents one of the main results of the paper, Theorem 4.10; this result establishes that a finite bounded

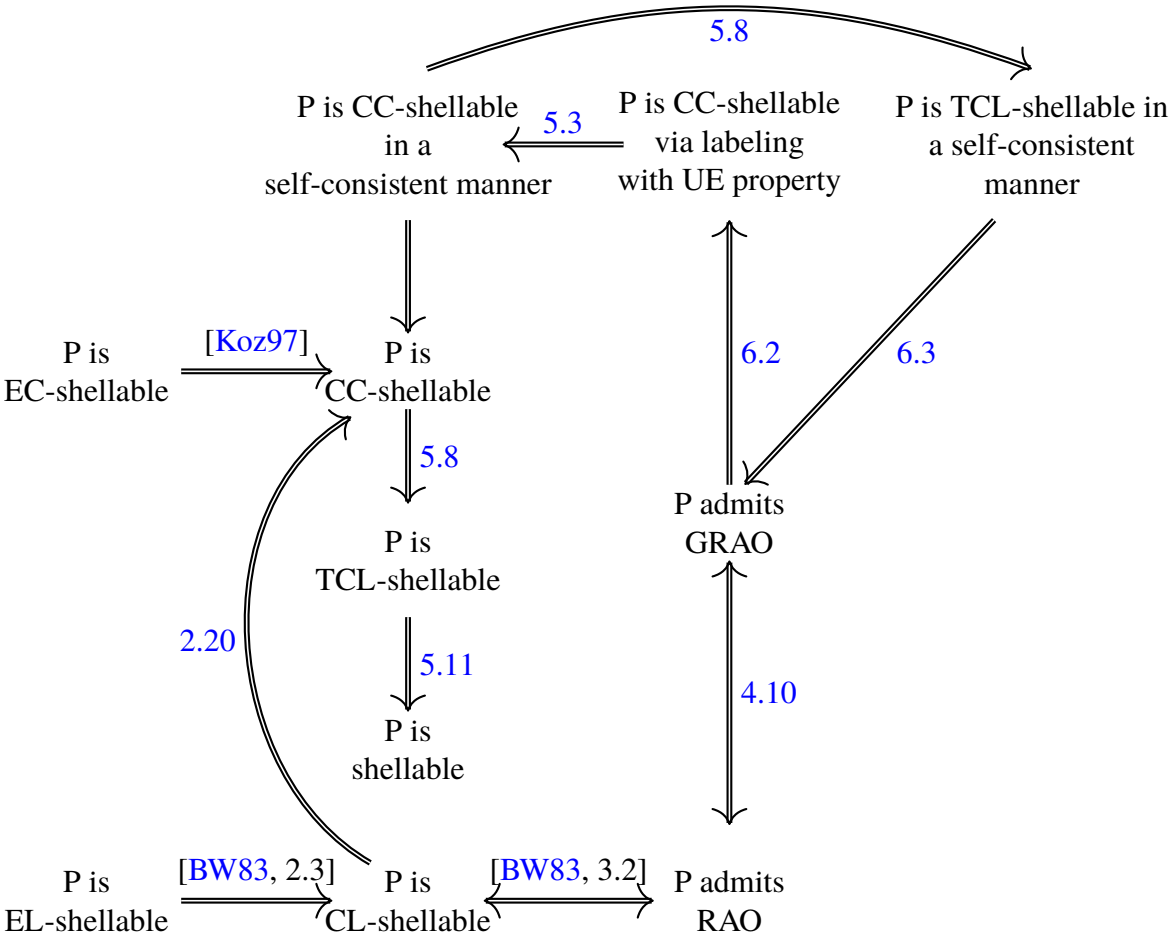

Figure 1.1: Implications and where they are proven.

poset admits a recursive atom ordering (RAO) if and only if it admits a generalized recursive atom ordering (GRAO). Figure 1.2 gives an example highlighting the subtlety of Theorem 4.10. It shows a small example of a poset P endowed with a GRAO that is not an RAO (on the left in the figure) and the same poset endowed with an RAO (on the right in the figure). A key ingredient to the proof of Theorem 4.10 is an algorithm that transforms any GRAO into an RAO on the same poset P, a procedure we call the "atom reordering process" (see Algorithm 4.1). This procedure transforms the GRAO shown on the left in Figure 1.2 into the RAO shown on the right in Figure 1.2.

Sections 5 and 6 establish a link that is not necessarily an equivalence between finite bounded posets that admit a GRAO and those that are CC-shellable. This link led us to introduce a variation on the notion of CC-shellability in Section 5.1, namely TCL-shellability (see Definition 5.7).

Many of the results proven throughout the paper are tied together in Theorem 6.4, a result which shows that several different notions of lexicographic shellability are equivalent to each

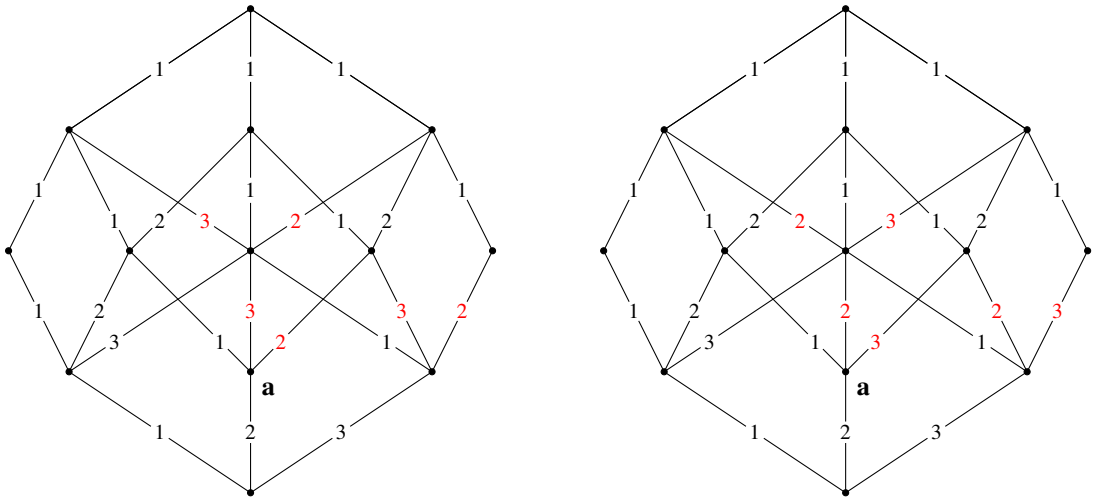

Figure 1.2: Poset with GRAO that is not RAO (left) and with RAO (right), with edge labels indicating the atom ordering.

other. In this result, we specify eight different versions of lexicographic shellability and show that a finite bounded poset admits any one of these types of lexicographic shelling if and only if it admits all of the others.

In Section 7, we apply our results to deduce that a class of posets previously shown to be CC-shellable in [HK21] is in fact CL-shellable. That is, we prove that the dual posets to the uncrossing orders (conjectured to be lexicographically shellable by Lam in [Lam15]) are CL-shellable. These uncrossing orders arise naturally as face posets of stratified spaces of planar electrical networks (see e.g. [Ken12, Lam15], and references therein). The fact that they are shellable posets combines with Lam's result from [Lam15] that they are Eulerian posets to imply that they are CW posets, i.e. face posets of regular CW complexes with finitely many cells. Thus, the shellability of uncrossing orders provides an important step in understanding the topological structure of these spaces of planar electrical networks.

The paper concludes with further results, observations, and open questions in Section 8.

2. Background

Let P denote a partially ordered set (poset). All posets throughout this paper are assumed to be finite and bounded but are not assumed to be graded. For background on posets, poset topology, and shellability beyond what appears below, we refer readers to [Sta12, Wac07, Bjö95, Sta96, Zie95].

A **cover relation** $u \lessdot v$ in a poset P is an order relation $u < v$ with the further requirement that there does not exist any $z \in P$ with $u < z < v$. In this case, we say that v **covers** u.

A poset P is **bounded** if it has both a unique least element (often denoted $\hat{0}$) and a unique greatest element (often denoted $\hat{1}$). A **closed interval**, denoted $[u, v]$, in a poset P is the subposet consisting of all elements $z \in P$ such that $u \leqslant z \leqslant v$.

The **atoms** of a bounded poset P (resp. a closed interval $[u, v]$) are those elements a in P (resp. $[u, v]$) that cover $\hat{0}$ (resp. u). Likewise the **coatoms** of a bounded poset P (resp. a closed interval $[u, v]$) are the elements in P (resp. $[u, v]$) covered by $\hat{1}$ (resp. v).

A **chain** in a poset P is a totally ordered subset $u_1 < \cdots < u_r$ of P. The **length** of a chain $u_1 < \cdots < u_r$ is the number $r-1$ of order relations in the chain. The **length** of a poset P, is the length of the longest chain in P. A chain is **maximal** in P if no additional elements of P may be inserted in it. A chain $u_1 < u_2 < \cdots < u_k$ is **saturated** in P if it is a maximal chain of $[u_1, u_k]$. We will also make the convention of sometimes referring to the maximal chains of a closed interval $[u_1, u_k]$ as the saturated chains of $[u_1, u_k]$. If all maximal chains in a poset P are of the same length, the poset is said to be **graded**.

Definition 2.1. The **order complex** of a poset P, denoted $\Delta(P)$, is the abstract simplicial complex whose $k-$faces are the chains of length k of P. In particular, this means the vertices are the chains consisting of single elements of P. Note that a face σ in $\Delta(P)$ is contained in another face τ in $\Delta(P)$ if and only if the chain corresponding to σ is contained in the chain corresponding to τ.

When we say that a poset P has a topological property (such as shellability or homotopy equivalence to a wedge of spheres), we mean that $\Delta(P)$ has this topological property.

The **dual** of a poset P, denoted P^*, has the same elements as P with $u \leqslant v$ in P^* if and only if $v \leqslant u$ in P.

Remark 2.2. Virtually everything in this paper has a dual version for the simple reason that a poset and its dual have the same order complex. This fact enables any poset theoretic technique to be applied to the dual poset to derive the same consequence regarding the order complex. There are indeed posets where this is a helpful thing to do (e.g. the uncrossing orders as discussed in Section 7). We leave it to the interested reader to fill in the dual versions of our results.

Definition 2.3. Given any simplicial complex K, its **face poset** $P(K)$ consists of the faces of K with order relation $\sigma \leqslant \tau$ if and only if the set of vertices in σ is a subset of the set of vertices in τ. The **closure** of a face τ, denoted $\overline{\tau}$, is the set of faces σ such that $\sigma \leqslant \tau$ in $P(K)$.

The **augmented face poset** $\hat{P}(K)$ is $P(K)$ with a maximal element $\hat{1}$ adjoined if $P(K)$ does not already have a unique maximal element, and $\hat{P}(K) = P(K)$ otherwise.

Definition 2.4. A simplicial complex is **shellable** if there is a total order $F_1, \ldots, F_k$ on its maximal faces (known as **facets**) such that $\overline{F_j} \cap (\cup_{i<j} \overline{F_i})$ is a pure, codimension one subcomplex of $\overline{F_j}$ for each $j \geqslant 2$. Such a facet ordering is known as a **shelling**.

Definition 2.5. A poset P is said to be **shellable** if its order complex $\Delta(P)$ is shellable.

Next we review various types of edge and chain-edge labelings of a finite poset P. These labelings will induce lexicographic shellings for $\Delta(P)$, namely shellings obtained by taking the facets of $\Delta(P)$ in order according to the lexicographic (i.e. dictionary) order of the label sequences of the corresponding maximal chains of P, breaking ties in any manner. Let $E(P)$ be the set of edges in the Hasse diagram of a finite poset P, that is, the pairs $x, y \in P$ such that $x \lessdot y$. An **edge labeling** of P is a map $\lambda : E(P) \to Q$ for Q a poset. Quite often, Q is the integers with their usual order.

We say that $x \lessdot y \lessdot z$ is an **ascent** with respect to the edge labeling λ if $\lambda(x,y) <_Q \lambda(y,z)$. Any $x \lessdot y \lessdot z$ that is not an ascent with respect to λ is said to be a **descent**. A maximal chain in a finite poset P (or more generally in a closed interval $[u,v]$ in P) is an **ascending chain** if it consists entirely of ascents.

Definition 2.6 ([Björ80, BW97])**.** An edge labeling of a finite, bounded poset P is called an **EL-labeling** (for edge lexicographical labeling) if for every interval $[x,y] \in P$ the following conditions are both met:

(i) There is a unique ascending maximal chain c in $[x,y]$.

(ii) The label sequence associated to c lexicographically precedes the label sequences associated to every other maximal chain in $[x,y]$.

A finite, bounded poset admitting such a labeling is **EL-shellable**.

Remark 2.7. In this paper, when we say that we take the lexicographic order on maximal chains of P, we mean that we take the maximal chains in order according to the lexicographic order on their label sequences.

Björner first introduced EL-labelings and EL-shellability in the graded case in [Björ80]. This was generalized by Björner and Wachs to the not necessarily graded case when they introduced the notion on nonpure shellability in [BW96] and [BW97]. The following fundamental result from [Björ80] explains the usage of the term EL-shellablility.

Theorem 2.8 ([Björ80], Theorem 2.3)**.** *If P is a finite, bounded, graded poset with an EL-labeling, then the lexicographic order of the maximal chains of P is a shelling order for the corresponding facets of $\Delta(P)$.*

Now we turn to a generalization of EL-shellability due to Björner and Wachs (see [BW83]) in which edge labelings are replaced by more general chain-edge labelings (defined next). In this context, we replace $E(P)$ by the set $E^*(P)$ defined as follows:

$$E^*(P) = \{(c,x,y) : c \text{ is a maximal chain}; x,y \in c; x \lessdot y\}.$$

Definition 2.9. Let P be a finite bounded poset and let Q be any poset. A **chain-edge labeling** (or **CE-labeling**) of P is a map $\lambda : E^*(P) \to Q$ that satisfies the following condition: If two maximal chains coincide along their first d edges, then they have the same labels as each other on these edges. In other words, if c is a maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot \ldots \lessdot x_n = \hat{1}$ and c' is a maximal chain $\hat{0} = x'_0 \lessdot x'_1 \lessdot \ldots \lessdot x'_n = \hat{1}$ where $x_i = x'_i$ for $i = 0,1,2,\ldots,d$, then $\lambda(c, x_{i-1}, x_i) = \lambda(c', x'_{i-1}, x'_i)$ for $i = 1,2,\ldots,d$.

To see a naturally arising example of a chain-edge labeling that is not an edge labeling, we refer readers to the chain-edge labeling due to Björner and Wachs for the dual poset to Bruhat order. This labeling is reviewed just prior to Proposition 8.4. Björner and Wachs proved in [BW83] that it is a CL-labeling, a notion defined shortly.

Definition 2.10. If $[x, y]$ is an interval in P and r is a saturated chain from $\hat{0}$ to x, then the pair $([x, y], r)$ is called a **rooted interval** with r as the **root** of this rooted interval. This is denoted by $[x, y]_r$.

Given a chain-edge labeling λ of a finite bounded poset P, λ associates to each maximal chain of P a label sequence as follows. If $m = (\hat{0} = x_0 \lessdot x_1 \lessdot \ldots \lessdot x_n = \hat{1})$, then the associated label sequence is

$$\lambda(m) := (\lambda(m, x_0, x_1), \lambda(m, x_1, x_2), \ldots \lambda(m, x_{n-1}, x_n)).$$

By definition of a chain-edge labeling, any two maximal chains both containing the same root r from $\hat{0}$ to x and the same saturated chain c in the interval $[x, y]$ will both have the same label sequence assigned to c. We will denote this label sequence by $\lambda_r(c)$ and the individual labels comprising it as $\lambda_r(x_i, x_{i+1})$ for $i = 0, \ldots, n-1$. Sometimes we will write $\lambda(c)$ for the label sequence and $\lambda(x_i, x_{i+1})$ for the label on the edge $x_i \lessdot x_{i+1}$ if the choice of root r is clear from context.

Given a label sequence $(\lambda_1, \ldots, \lambda_r)$, we say that a pair of consecutive labels λ_i, λ_{i+1} comprises an **ascent** if and only if $\lambda_i < \lambda_{i+1}$. The pair λ_i, λ_{i+1} comprises a **descent** otherwise.

Definition 2.11. A maximal chain c in a rooted interval $[x, y]_r$ is **ascending** with respect to a chain-edge labeling λ if the label sequence $\lambda_r(c) = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ has $\lambda_i < \lambda_{i+1}$ for $i = 1, 2, \ldots, r-1$.

Definition 2.12 ([BW83])**.** A CE-labeling λ of a finite bounded poset P is called a **CL-labeling** (for chain-lexicographical labeling) if for every rooted interval $[x, y]_r$ in P,

(i) there is a unique ascending chain c in $[x, y]_r$ and

(ii) the label sequence $\lambda_r(c)$ lexicographically precedes the label sequence for every other maximal chain in $[x, y]_r$.

If a finite bounded poset P admits a CL-labeling, then P is said to be **CL-shellable**.

Björner and Wachs proved in [BW82] (resp. [BW96]) that whenever a finite bounded poset P that is graded (resp. is not necessarily graded) admits a CL-labeling, then any linear extension of the lexicographic order on its maximal chains given by the CL-labeling is a shelling order on the corresponding facets of $\Delta(P)$.

One of the primary techniques for proving that a finite bounded poset is CL-shellable is to construct a recursive atom ordering (see Definition 2.14). The notion of recursive atom ordering (RAO) was introduced by Björner and Wachs in [BW83]. They extended it to posets that are not necessarily graded in [BW97]. Before defining recursive atom ordering, we lay the groundwork with a notion we call chain-atom ordering that will encompass all recursive atom orderings and all generalized recursive atom orderings (defined later) as special cases.

Definition 2.13. A **chain-atom ordering** Ω of a finite bounded poset P is a choice of ordering on the atoms of each rooted interval $[u, \hat{1}]_r$ of P. For the rooted interval $[u, \hat{1}]_r$, we will denote this ordering of atoms as $\Omega([u, \hat{1}]_r)$. On the other hand, we denote the restriction of the full

chain-atom ordering Ω to a rooted interval $[u,\hat{1}]_r$ as $\Omega|_{[u,\hat{1}]_r}$, and more generally we denote the restriction of Ω to $[u,v]_r$ as $\Omega|_{[u,v]_r}$. We sometimes call $\Omega|_{[u,v]_r}$ the chain-atom ordering on $[u,v]_r$ induced by Ω.

Whenever a finite bounded poset has a recursive atom ordering, defined next, this by definition guarantees the existence of an especially well-behaved type of chain-atom ordering. We will often refer to these especially nice chain-atom orderings themselves as recursive atom orderings. A key place where we will do this is when we introduce a relaxation of the notion of recursive atom ordering in Section 3.

Definition 2.14. A finite bounded poset P admits a **recursive atom ordering** if P has length 1 or if the atoms of P can be ordered $a_1, a_2, \ldots a_t$ such that:

(i) (a) For all $j = 1, \ldots, t$, $[a_j, \hat{1}]$ admits a recursive atom ordering.

(b) For $j \neq 1$, the atoms that come first in this recursive atom ordering for $[a_j, \hat{1}]$ are those that are greater than some atom a_k of P for $k < j$.

(ii) For all $i < j$ and $y \in P$ satisfying $y > a_i$ and $y > a_j$, there exists $k < j$ and $z \in P$ such that $a_j \lessdot z$ and $a_k < z \leqslant y$.

The following theorem of Björner and Wachs from [BW83] (which they extended to the not necessarily graded case in [BW97]) established a very useful relationship between the existence of a recursive atom ordering and of a CL-labeling for any finite bounded poset P.

Theorem 2.15 ([BW83], Theorem 3.2; [BW97], Theorem 5.11)**.** *A finite bounded poset P admits a recursive atom ordering if and only if P is CL-shellable.*

We sketch one direction of the proof of Theorem 2.15 shortly, since many of the ideas in this argument will be used in other proofs later in the paper. A key ingredient is the pair of sets $F_r(u)$ and $G_r(u)$ defined next. Björner and Wachs introduced these sets $F_r(u)$ and $G_r(u)$ for recursive atom orderings, but we find it convenient to define them more generally.

Definition 2.16. Consider any rooted interval $[u,\hat{1}]_r$ in a finite bounded poset P. Let u^- be the element of r covered by u and let r^- be the root for $[u^-,\hat{1}]$ obtained by omitting u from r and otherwise preserving r. Let Λ be either a total order on the atoms of $[u^-,\hat{1}]_{r^-}$ or any richer structure, such as a chain-atom ordering, which specifies such a total order.

Define $F_r^\Lambda(u)$ to be the set of atoms a of $[u,\hat{1}]_r$ such that $a >_P a'$ for some atom a' in $[u^-,\hat{1}]_{r^-}$ that comes earlier than u in Λ. Define $G_r^\Lambda(u)$ to be the set of all atoms of $[u,\hat{1}]_r$ that are not contained in $F_r^\Lambda(u)$.

Given any $v \in P$ satisfying $u < v$, define $F_r^\Lambda(u,v)$ to be the set of atoms a of $[u,v]_r$ such that $a >_P a'$ for some a' that covers u^- and comes earlier than u in $\Lambda|_{[u^-,v]_{r^-}}$. Define $G_r^\Lambda(u,v)$ to be the set of atoms of $[u,v]_r$ that are not contained in $F_r^\Lambda(u,v)$.

Sometimes we will denote these sets simply by $F_r(u,v)$, $G_r(u,v)$, $F_r(u)$ and $G_r(u)$ when the choice of Λ is clear from context.

Remark 2.17. By definition, we have $F_r^\Lambda(u) = F_r^\Lambda(u,\hat{1})$ and $G_r^\Lambda(u) = G_r^\Lambda(u,\hat{1})$.

Proof sketch of how an RAO yields a CL-labeling. Given an RAO $a_1, \ldots, a_n$ for P, start by labeling each cover relation of the form $\hat{0} \lessdot a_i$ with the integer i. By induction on the length of the longest saturated chain in P, one may assume that each rooted interval $[a_i, \hat{1}]$ has its own RAO that induces a CL-labeling for $[a_i, \hat{1}]$. The plan is to modify this CL-labeling for $[a_i, \hat{1}]$ for each i so that the modified labels may be taken together with the labels $\lambda(\hat{0}, a_i) = i$ for $i = 1, 2, \ldots, n$ to give a CL-labeling λ for all of P. It will suffice to describe how to modify the labels on the cover relations upward from a_i to the atoms of $[a_i, \hat{1}]$, then to apply this same label modification process inductively to handle the rooted intervals higher in the poset.

Letting $x_1, \ldots, x_u$ be our RAO for $[a_i, \hat{1}]$ which is guaranteed to exist by the definition of an RAO for P, Björner and Wachs made the key observation that there always exists some $j \in \{1, \ldots, u\}$ such that $x_1, \ldots, x_j \in F_{\hat{0}<a_i}(a_i)$ while $x_{j+1}, \ldots, x_u \in G_{\hat{0}<a_i}(a_i)$. For example, one may label each cover relation $a_i \lessdot x_{i'}$ for $i' \leqslant j$ with the label $\lambda(a_i, x_{i'}) = \lambda(\hat{0}, a_i) - (j - i') - 1$, and one may label each cover relation $a_i \lessdot x_{i'}$ for $i' \geqslant j+1$ with the label $\lambda(a_i, x_{i'}) = \lambda(\hat{0}, a_i) + (i' - j)$. The resulting labeling has the following two properties:

1. $\lambda(\hat{0}, a_i) \geqslant \lambda(a_i, x_{i'})$ if and only if $x_{i'} \in F_{\hat{0}<a_i}(a_i)$
2. $\lambda(a_i, x_{i'}) < \lambda(a_i, x_{i''})$ if and only if $i' < i''$.

In other words, the shifting of label values preserves the relative order of the labels on cover relations upward from a fixed element with the same fixed choice of root below, but at the same time it creates ascents and descents exactly where they are needed in order to have a CL-labeling. See Theorem 3.2 in [BW83] for further details of this proof. □

We next review the notions of CC-shellability and EC-shellability from [Koz97], doing so using the language of topological ascents and descents (defined next) that was introduced in [Her03a]. Let λ be an edge labeling on the cover relations of a poset P by elements of some poset Q. A **topological ascent** occurs in P at $u \lessdot v \lessdot w$ whenever the ordered pair of labels $(\lambda(u, v), \lambda(v, w))$ is lexicographically earlier than all other ordered sequences of labels on all other saturated chains from u to w. If $(\lambda(u, v), \lambda(v, w))$ is not a topological ascent, then it is a **topological descent**.

For λ a chain-edge labeling, we define topological ascents and descents in the same way, but now with respect to a choice of root. That is, we have a **topological ascent** at $u \lessdot v \lessdot w$ with respect to root r if the ordered pair $(\lambda(u, v), \lambda(v, w))$ is lexicographically smaller than all other label sequences on saturated chains from u to w with this same root r. We have a **topological descent** at $u \lessdot v \lessdot w$ with respect to root r otherwise.

If c is a saturated chain from an element u to an element v consisting entirely of topological ascents (in either an edge labeling or a chain-edge labeling), then we say that c is a **topologically ascending** chain from u to v. If c is a saturated chain from u to v consisting entirely of topological descents, then it is said to be **topologically descending**.

The next definition is rephrased from how it appears in [Koz97], using the language of topological ascents and descents, but is entirely equivalent to his definition.

Definition 2.18 ([Koz97])**.** An **EC-labeling** of a finite bounded poset P is an edge labeling $\lambda : E(P) \to Q$ on the cover relations of P with labels belonging to a poset Q, subject to

the requirements that (1) every interval $[x, y]$ has a unique saturated chain consisting entirely of topological ascents, and (2) the label sequences for the saturated chains of P (and hence of each interval $[x, y]$) are all distinct with no label sequence being a prefix of any other label sequence.

If P admits an EC-labeling, then P is said to be **EC-shellable**.

Just as EL-shellability was generalized to CL-shellability, the notion of EC-shellability was likewise generalized by Kozlov from edge labelings to chain-edge labelings. His definition from [Koz97] may be rephrased as follows.

Definition 2.19 ([Koz97])**.** A **CC-labeling** of a finite bounded poset P is a chain-edge labeling $\lambda : E^*(P) \to Q$ with labels belonging to a poset Q, subject to the requirements that (1) every rooted interval $[u, v]_r$ has a unique saturated chain consisting entirely of topological ascents, and (2) the label sequences for the saturated chains of $[u, v]_r$ are all distinct with no label sequence being a prefix of another label sequence.

If a finite bounded poset P admits a CC-labeling, then P is said to be **CC-shellable**.

Kozlov proved in Theorem 3.8 of [Koz97] that the order complex of any finite bounded poset P admitting a CC-labeling is shellable, doing so by taking any linear extension of the lexicographic order on the label sequences for the maximal chains of P as the shelling order for the corresponding facets of $\Delta(P)$.

Kozlov noted in [Koz97] that any finite bounded poset that is CL-shellable is also CC-shellable, leaving the proof to the reader. We include a proof of this result as Proposition 2.20. Our proof below closely follows parts of the proof of Björner and Wachs that any recursive atom ordering induces a CL-labeling (see Theorem 3.2 in [BW83]). This proof is included because it introduces several additional important ideas that will be used in various ways in other proofs later in the paper.

Proposition 2.20. *Any recursive atom ordering $a_1, \ldots, a_t$ of a finite bounded poset P gives rise to a CC-labeling of P. Thus, any finite bounded poset that is CL-shellable is CC-shellable.*

Proof. Given a recursive atom ordering Λ, we start by describing the desired CC-labeling λ derived from Λ. Letting $a_1, \ldots, a_t$ denote the atom ordering of P given by Λ, we assign the label $\lambda(\hat{0}, a_i) = i$ to each cover relation of the form $\hat{0} \lessdot a_i$. Now, for any $u \in P$ and any root r leading up to u, consider any recursive atom ordering $a'_1, \ldots, a'_{t'}$ for $[u, \hat{1}]_r$ of the type guaranteed to exist recursively in the definition of RAO. Assign the labels $\lambda_r(u, a'_i) = i$ for $i = 1, 2, \ldots, t'$.

By construction, this labeling has two of the requisite properties of a CC-labeling, namely the requirements that 1) no two saturated chains of $[u, \hat{1}]_r$ have the same label sequence and 2) no two saturated chains of $[u, \hat{1}]_r$ have the property that the label sequence of one is a prefix of the label sequence of the other. Now we verify that each rooted interval $[u, v]_r$ in P has a unique topologically ascending maximal chain and that this chain is lexicographically first. We do so by induction on the length of the longest chain in $[u, v]_r$. This hypothesis clearly holds for intervals of length two, the base case. Let c which is given by $u \lessdot u_1 \lessdot \ldots \lessdot u_k = v$ be the lexicographically first maximal chain in $[u, v]_r$. Because it is lexicographically first, this forces c to be topologically ascending.

Suppose that there is another topologically ascending maximal chain c' in $[u, v]_r$. Let c' be given by $u \lessdot u'_1 \lessdot u'_2 \lessdot \ldots \lessdot u'_{k'} = v$. As $u \lessdot u'_1 \lessdot u'_2$ must then be a topological ascent, it is the

lexicographically earliest maximal chain from u to u'_2. This implies $u'_2 \in G^{\Lambda}_{r\cup u'_1}(u'_1)$. As c is lexicographically earlier than c', we may conclude that $\lambda(u,u_1) \leqslant \lambda(u,u'_1)$. But our choice of chain-edge labeling based on an RAO ensures that $\lambda(u,u_1) \neq \lambda(u,u'_1)$, implying $\lambda(u,u_1) < \lambda(u,u'_1)$ from which we deduce that u_1 comes before u'_1 in the given RAO of $[u,\hat{1}]_r$. By (ii) of Definition 2.14, there exists an atom a of $[u,v]_r$ and an element $z \in P$ such that a comes before u'_1 in the RAO, $u'_1 \lessdot z$ and $a < z \leqslant v$. If $z = u'_2$, then $u'_2 \in F^{\Lambda}_{r\cup u'_1}(u'_1)$, contradicting our earlier claim that $u'_2 \in G^{\Lambda}_{r\cup u'_1}(u'_1)$. If $z \neq u'_2$, then $u'_1 \lessdot u'_2 \lessdot \ldots u'_{k'} = v$ is not lexicographically first in $[u'_1,v]_{r\cup u'_1}$ and, by the induction hypothesis, not topologically ascending. This gives a contradiction to the claim that c' consists entirely of topological ascents. Thus, we confirm that the chain-edge labeling λ has a unique topologically ascending chain in $[u,v]_r$, completing our proof that this is a CC-labeling.

Combining the above argument with the result of [BW83] that a finite bounded poset is CL-shellable if and only if it admits a recursive atom ordering shows that every CL-shellable poset is CC-shellable. □

3. Generalized recursive atom ordering

In this section we introduce a generalization of the notion of recursive atom ordering and prove several fundamental properties of these generalized recursive atom orderings.

Remark 3.1. The generalized recursive atom orderings (GRAOs) we are about to introduce will be defined as orderings on the atoms of a finite poset P that are extendible to chain-atom orderings of P meeting certain requirements. It will often be convenient to think of a generalized recursive atom ordering as a chain-atom ordering of this type.

Now we are ready to give the main new definition of the paper:

Definition 3.2. A finite bounded poset P admits a **generalized recursive atom ordering** (GRAO) if the length of P is 1 or if the length of P is greater than 1 and there is an ordering $a_1, a_2, \ldots a_t$ on the atoms of P satisfying:

(i) (a) For $1 \leqslant j \leqslant t$, $[a_j,\hat{1}]$ admits a GRAO.

(b) For any atom a_j and any $x, w \in P$ satisfying $a_j \lessdot x \lessdot w$, the following property holds when the chain-atom ordering given by the GRAO from (i)(a) is restricted to $[a_j,w]$: either the first atom of $[a_j,w]$ is above an atom a_i with $i < j$, or no atom of $[a_j,w]$ is above any atom a_i with $i < j$.

(ii) For any $y \in P$ and any atoms a_i, a_j satisfying $a_i < y$ and $a_j < y$ with $i < j$, there exists an element $z \in P$ with $z \leqslant y$ and an atom a_k with $k < j$ such that $a_j \lessdot z$ and $a_k < z$.

Remark 3.3. When we say that not all recursive atom orderings are generalized recursive atom orderings and when we say that GRAOs are strictly more general than RAOs, we are regarding RAOs and GRAOs as being types of chain-atom orderings.

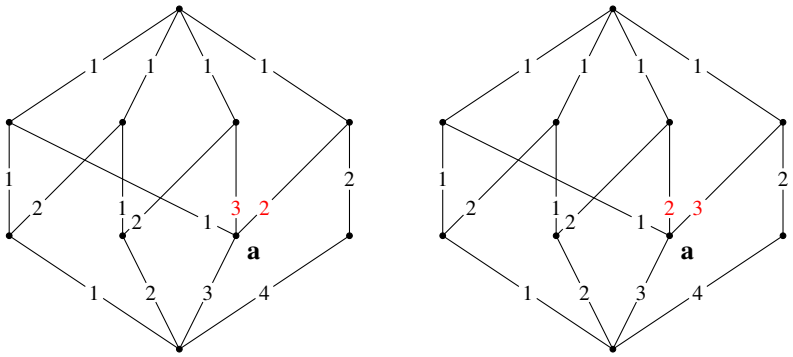

Figure 3.1: GRAO that is not RAO (left) and RAO for same poset (right).

Remark 3.4. Conditions (i)(a) and (ii) in the definition of GRAO are exactly equivalent to corresponding statements within the definition of recursive atom ordering. However, condition (i)(b) above is considerably less restrictive than (i)(b) of the definition of RAO.

One of the main points of this relaxation of the notion of recursive atom ordering is that it gives considerably more flexibility than a recursive atom ordering in how we may order those atoms in a rooted interval $[u, \hat{1}]_r$ that are not the first atom of that interval. For this reason, the following rephrasing of condition (ii) also seems quite useful to note:

Remark 3.5. Condition (ii) in Definition 3.2 is logically equivalent to the following statement: if $a_i < y$ and $a_j < y$ for $i < j$, then there exists $z \in P$ such that $a_j \lessdot z \leqslant y$ where a_j is not the earliest atom in $[\hat{0}, z]$.

Example 3.6. The chain-atom ordering given on the left in Figure 3.1 is a generalized recursive atom ordering but is not a (traditional) recursive atom ordering. In particular, the atom of $[a, \hat{1}]$ labeled 3 in the poset on the left causes condition (i)(b) of Definition 2.14 to fail. The chain-atom ordering given on the right is a recursive atom ordering. We highlight in color the place where the two chain-atom orderings differ.

Remark 3.7. Implicit in the definition of GRAO is that a GRAO for a finite bounded poset P induces a GRAO for $[u, \hat{1}]_r$ for each $u \in P$ and each root r. Lemma 3.8 will show that it also induces a GRAO for each $[u, v]_r$. This result will allow us henceforth to refer to the restriction of any GRAO for a finite bounded poset to the rooted interval $[u, v]_r$ as the GRAO for $[u, v]_r$ induced by the GRAO for P.

Lemma 3.8. *Let P be a finite bounded poset with Γ a GRAO for P. Then for any $u < v$ and any root r for $[u, v]$, restricting the GRAO for $[u, \hat{1}]_r$ induced by Γ to $[u, v]_r$ yields a GRAO for $[u, v]_r$.*

Proof. Our proof will be by induction on the length of the longest saturated chain from u to $\hat{1}$. We can use maximum length 2 for our base case, obtaining this case by using the fact that every possible chain-atom ordering on a finite bounded poset with maximum chain length of 1 or 2 is a GRAO. We then obtain condition (i)(a) by induction as follows. Given any a_i satisfying $u \lessdot a_i \leqslant v$, the length of the longest saturated chain from u to $\hat{1}$ is strictly larger than the

length of the longest saturated chain from a_i to $\hat{1}$. Thus, our inductive hypothesis gives us that the GRAO for $[a_i, \hat{1}]_{r \cup a_i}$ will restrict to a GRAO for $[a_i, v]_{r \cup a_i}$. But it is immediate from the definition of GRAO that the properties of a GRAO given in conditions (i)(b) and (ii) from the definition of GRAO will restrict from $[u, \hat{1}]_r$ to $[u, v]_r$. Thus, the GRAO for P restricts just as desired. □

Next we show how the statement about cover relations in condition (i)(b) in the definition of GRAO can be strengthened to a corresponding statement about all order relations.

Lemma 3.9. *Let P be a finite bounded poset, and let Λ be a GRAO for P with atom ordering $a_1, a_2, \ldots a_t$. For each $\hat{0} \lessdot a_j < v$, restricting $\Lambda|_{[a_j, \hat{1}]}$ to $[a_j, v]$ yields a GRAO, denoted $\Lambda|_{[a_j, v]}$, for $[a_j, v]$ with the following property: either (a) the first atom of $[a_j, v]$ is greater than some atom a_i satisfying $i < j$ or (b) no atom of $[a_j, v]$ is greater than any atom a_i satisfying $i < j$.*

Proof. By Lemma 3.8, $\Lambda|_{[a_j, v]}$ is guaranteed to be a GRAO. Let x be the first atom in $[a_j, v]$ with respect to Λ. Suppose that some other atom y of $[a_j, v]$ is greater than some atom a_i of P with $i < j$. Since x and y are both less than v with x coming before y in $\Lambda|_{[a_j, \hat{1}]}$, Definition 3.2 (ii) guarantees the existence of elements w_1 and y_1, where y_1 comes before y in $\Lambda|_{[a_j, \hat{1}]}$ and where $y \lessdot w_1 \leqslant v$ and $y_1 < w_1$. Since y covers a_i with $i < j$, Definition 3.2 (i)(b) ensures that the first atom in $[a_j, w_1]$ must be greater than some atom $a_{i'}$ satisfying $i' < j$. We may take y_1 to be the first atom in $[a_j, w_1]$, i.e., we may replace y_1 by this first atom if it is not already this atom. If y_1 also equals x, then x is greater than some atom that comes before a_j in Λ, namely $a_{i'}$, as desired.

If, on the other hand, $y_1 \neq x$, then we repeat this process as many times as necessary to get the desired result that x is greater than some atom that comes before a_j in Λ, doing so as follows. Since x and y_1 are both less than v, we may use Definition 3.2 (ii) to deduce the existence of y_2 and w_2 where y_2 comes before y_1 in $\Lambda|_{[a_j, \hat{1}]}$ and where $y_2 < w_2$ and $y_1 \lessdot w_2 \leqslant v$. We may take y_2 to be the first atom in $[a_j, w_2]$. But then Definition 3.2 (i)(b) guarantees that y_2 must be greater than some atom $a_{i''}$ where $i'' < j$. If $y_2 = x$, this yields the desired result. If $y_2 \neq x$, then we repeat this process again obtaining elements y_3 and w_3 in place of y_2 and w_2.

Continuing in this manner, we obtain a sequence $y_1, y_2, y_3, \ldots$ of atoms of $[a_j, v]$ where for each $i \geqslant 2$ we have that:

1. y_i comes earlier in $\Lambda|_{[a_j, v]}$ than y_{i-1}, and
2. y_i is greater than an atom of P that comes before a_j in Λ.

Note that (1) implies each y_i is distinct, allowing us to deduce from finiteness of $[a_j, v]$ that the sequence $y_1, y_2, y_3, \ldots$ must terminate at some y_n. This terminal element y_n must satisfy $y_n = x$ since we have shown that otherwise there would be an atom y_{n+1} of $[a_j, v]$ coming still earlier than y_n. Thus we obtain the desired result that x is greater than some atom that comes before a_j in Λ. □

The sets $F_r(u, v)$ and $G_r(u, v)$ appearing in the next lemma will play a key role later in the paper in allowing us to transform any generalized recursive atom ordering into a recursive atom ordering. Our justification that this atom reordering process converts a GRAO to an RAO

will rely heavily on the property of the sets $F_r(u,v)$ and $G_r(u,v)$ that we justify next. See Definition 2.16 for the definitions of the sets $F_r(u,v)$ and $G_r(u,v)$.

Lemma 3.10. *Let Γ be a chain-atom ordering for a finite bounded poset P. Consider any two atoms a_i, a_{i+1} of a rooted interval $[u,\hat{1}]_r$ that are consecutive atoms of $[u,\hat{1}]_r$ with respect to Γ. Suppose the pair of atoms a_i, a_{i+1} has the further property for every rooted interval $[u,v]_r$ containing both a_i and a_{i+1} that neither a_i nor a_{i+1} is the first atom of $[u,v]_r$ with respect to $\Gamma|_{[u,v]_r}$. Let Λ be the chain-atom ordering on P obtained from Γ by switching the order of a_i and a_{i+1} in $[u,\hat{1}]_r$. Then $F^{\Gamma}_{r'}(u',v') = F^{\Lambda}_{r'}(u',v')$ and $G^{\Gamma}_{r'}(u',v') = G^{\Lambda}_{r'}(u',v')$ for each $u' < v'$ in P and each choice of root r' for $[u',v']$.*

Proof. Throughout this proof, let r be the chain $\hat{0} = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_k \lessdot u$, and let r^- be the chain $\hat{0} = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_k$ obtained from r by deleting u. We will first show how to handle all cases which do not satisfy both of the conditions $u \lessdot u'$ and $r' = r \cup \{u'\}$. Let $\hat{0} \lessdot u'_1 \lessdot \cdots \lessdot u'_l \lessdot u'$ be the root r'. Observe that if we do not have $u = u'_l$ and also have r equalling $\hat{0} \lessdot u'_1 \lessdot \cdots \lessdot u'_l$, then swapping the order of the atoms a_i and a_{i+1} in $[u,\hat{1}]_r$ has no impact on the ordering of the atoms of $[u'_l, v']_{\hat{0} \lessdot u'_1 \lessdot \cdots \lessdot u'_l}$. Thus, the sets $F^{\Gamma}_{r'}(u',v')$ and $G^{\Gamma}_{r'}(u',v')$ cannot be impacted by the swap of a_i and a_{i+1} unless $u \lessdot u'$ and $r' = r \cup \{u'\}$.

We subdivide the task of handling those situations with $u \lessdot u'$ and $r' = r \cup \{u'\}$ into three cases, namely: (a) $u' = a_i$, (b) $u' = a_{i+1}$, and (c) $u' \notin \{a_i, a_{i+1}\}$. We start with the easiest of these cases, namely (c). Our choice of a_i, a_{i+1} as consecutive atoms of $[u,\hat{1}]_r$ with respect to Γ implies that a_i, a_{i+1} are consecutive atoms of $[u,v']_r$ with respect to $\Gamma|_{[u,v']_r}$ for each $v' \in P$ having $a_i, a_{i+1} \in [u,v']_r$. This ensures for such $[u,v']_r$ we must either have both a_i and a_{i+1} before the atom u' or both a_i and a_{i+1} after u' with respect to $\Gamma|_{[u,v']_r}$. In either case, swapping the order of a_i, a_{i+1} does not impact which atoms of $[u,v']_r$ come earlier than u'. Thus, this swap does not impact which atoms of $[u',v']_{r'}$ are in $F_{r'}(u',v')$. If, on the other hand, we do not have both a_i and a_{i+1} in $[u,v']_r$, then swapping the order of a_i and a_{i+1} in $[u,\hat{1}]_r$ leaves the ordering of the atoms of $[u,v']_r$ unchanged. In particular, the swap preserves which atoms of $[u,v']_r$ come earlier than u' and which come later than u'. Again this implies that the swap does not impact which atoms of $[u',v']_{r'}$ are in $F_{r'}(u',v')$. This shows that $F^{\Gamma}_{r'}(u',v') = F^{\Lambda}_{r'}(u',v')$ and $G^{\Gamma}_{r'}(u',v') = G^{\Lambda}_{r'}(u',v')$ in case (c).

Now we turn to the cases (a) and (b), the cases in which $u' = a_i$ and $u' = a_{i+1}$, respectively. We handle these two cases simultaneously because the latter parts of the arguments for these two cases are intrinsically intertwined with each other. If an interval $[u,v']_r$ does not contain both of the atoms a_i and a_{i+1}, then swapping a_i and a_{i+1} has no impact on the ordering of the atoms of $[u,v']_r$, yielding the desired equalities in this case. For the remainder of cases (a) and (b), we therefore may assume the rooted interval $[u,v']_r$ under consideration contains both a_i and a_{i+1}. Since swapping a_i and a_{i+1} moves a_i to after a_{i+1} while otherwise preserving the chain-atom ordering, every atom of $[u,v']_r$ that came earlier than a_i in $[u,v']_r$ before the swap still comes earlier than a_i after the swap. In other words, we deduce the set containment $F^{\Gamma}_{r\cup a_i}(a_i,v') \subseteq F^{\Lambda}_{r\cup a_i}(a_i,v')$. Next we show $F^{\Gamma}_{r\cup a_{i+1}}(a_{i+1},v') \subseteq F^{\Lambda}_{r\cup a_{i+1}}(a_{i+1},v')$ by proving that $a' \in F^{\Gamma}_{r\cup a_{i+1}}(a_{i+1},v')$ implies $a' \in F^{\Lambda}_{r\cup a_{i+1}}(a_{i+1},v')$. Suppose there exists some $a' \in F^{\Gamma}_{r\cup a_{i+1}}(a_{i+1},v')$. Then by definition of $F^{\Lambda}_{r\cup a_{i+1}}(a_{i+1},v')$ the rooted interval $[u,a']_r$ has an earlier atom than a_{i+1} with respect to $\Gamma|_{[u,a']_r}$. Denote by a_1 the earliest atom of $[u,a']_r$ with

respect to Γ. If $a_i \in [u, a']_r$, then both a_i and a_{i+1} are atoms of $[u, a']_r$; since neither a_i nor a_{i+1} is allowed to be the first atom of any rooted interval $[u, w]_r$ containing both of them, $a_1 \notin \{a_i, a_{i+1}\}$ in this case. On the other hand, for $a_i \notin [u, a']_r$ we must have $a_1 \neq a_i$ since $a_1 \in [u, a']_r$; we may also deduce $a_1 \neq a_{i+1}$ in this case from our assumption that $a' \in F^{\Gamma}_{r \cup a_{i+1}}(a_{i+1}, v')$. Thus, regardless of whether $a_i \in [u, a']_r$ or not, swapping the order of a_i and a_{i+1} in $[u, \hat{1}]_r$ does not change the fact that a_1 is the earliest atom of $[u, a']_r$. This means that after the swap a' is still above the atom a_1 which still comes earlier than a_{i+1}, implying $a' \in F^{\Lambda}_{r \cup a_{i+1}}(a_{i+1}, v')$. This shows $F^{\Gamma}_{r \cup a_{i+1}}(a_{i+1}, v') \subseteq F^{\Lambda}_{r \cup a_{i+1}}(a_{i+1}, v')$.

If we swap a_i with a_{i+1} twice in succession, the second swap takes the chain-atom ordering Λ and produces from it the original chain-atom ordering Γ. The atom a_{i+1} comes earlier than a_i in $[u, \hat{1}]_r$ with respect to Λ, but a_{i+1} and a_i are still consecutive atoms of $[u, \hat{1}]_r$ with respect to Λ; also observe that a_{i+1} and a_i still satisfy our requirement with respect to Λ that any rooted interval $[u, v]_r$ containing both a_{i+1} and a_i must have neither a_{i+1} nor a_i as its first atom. Thus, we may apply the argument from the prior paragraph to the chain-atom ordering Λ using the swap of a_{i+1} and a_i in the rooted interval $[u, \hat{1}]_r$ which outputs the chain-atom ordering Γ. This yields the set containments $F^{\Lambda}_{r \cup a_{i+1}}(a_{i+1}, v') \subseteq F^{\Gamma}_{r \cup a_{i+1}}(a_{i+1}, v')$ and $F^{\Lambda}_{r \cup a_i}(a_i, v') \subseteq F^{\Gamma}_{r \cup a_i}(a_i, v')$. We may deduce $F^{\Gamma}_{r \cup a_i}(a_i, v') = F^{\Lambda}_{r \cup a_i}(a_i, v')$ from the series of set containments

$$F^{\Lambda}_{r \cup a_i}(a_i, v') \subseteq F^{\Gamma}_{r \cup a_i}(a_i, v') \subseteq F^{\Lambda}_{r \cup a_i}(a_i, v').$$

Likewise we deduce $F^{\Gamma}_{r \cup a_{i+1}}(a_{i+1}, v') = F^{\Lambda}_{r \cup a_{i+1}}(a_{i+1}, v')$ from the series of set containments $F^{\Lambda}_{r \cup a_{i+1}}(a_{i+1}, v') \subseteq F^{\Gamma}_{r \cup a_{i+1}}(a_{i+1}, v') \subseteq F^{\Lambda}_{r \cup a_{i+1}}(a_{i+1}, v')$. These set equalities imply the desired set equalities $G^{\Gamma}_{r \cup a_i}(a_i, v') = G^{\Lambda}_{r \cup a_i}(a_i, v')$ and $G^{\Gamma}_{r \cup a_{i+1}}(a_{i+1}, v') = G^{\Lambda}_{r \cup a_{i+1}}(a_{i+1}, v')$ of the complementary sets. □

Before proving the main result of the remainder of this section, Theorem 3.13, we prove a pair of lemmas that will provide two of the main pieces of the proof of Theorem 3.13.

Lemma 3.11. *Let Γ be a GRAO for a finite bounded poset P. Suppose a pair of consecutive atoms a_i, a_{i+1} in $[u, \hat{1}]_r$ with chain-atom ordering $\Gamma|_{[u,\hat{1}]_r}$ has the property for each rooted interval $[u, w]_r$ containing both a_i and a_{i+1} that neither a_i nor a_{i+1} is the earliest atom of $[u, w]_r$ with respect to $\Gamma|_{[u,w]_r}$. Then the chain-atom ordering Λ on P obtained from Γ by swapping the order of a_i and a_{i+1} in $[u, \hat{1}]_r$ satisfies condition (i)(b) of a GRAO.*

Proof. We break the proof into the following three cases based on the nature of the element $u \in P$ that is covered by a_i and a_{i+1}:

1. $u = \hat{0}$
2. u is an atom of P
3. u is some other element of P.

To show that Λ satisfies condition (i)(b), it suffices to prove for each $u \in P$ the following property for every atom $a_j \in P$ and every $v \in P$ such that there exists an $x \in P$ with $a_j \lessdot x \lessdot v$:

either the first atom of $[a_j, v]_{\hat{0}<a_j}$ with respect to Λ is above an earlier atom of P than a_j or no atom of $[a_j, v]_{\hat{0}<a_j}$ is above an earlier atom than a_j. We sometimes speak below of checking "condition (i)(b) for the rooted interval $[a_j, v]_{\hat{0}<a_j}$", by which we mean that we are checking the condition above just for that fixed choice of a_j and v. In the remainder of the proof, we suppress the notation indicating our choice of root for intervals $[a_j, v]$ where a_j is an atom so as to simplify notation, since it is clear from context for such intervals that the root must be $\hat{0} \lessdot a_j$.

We start with the easiest case, namely case (3) where we have $u \neq \hat{0}$ and u also not an atom of P. In this case, swapping the order of the atoms a_i and a_{i+1} in the rooted interval $[u, \hat{1}]_r$ cannot impact the ordering of the atoms of P. The swap also cannot impact the ordering on the atoms of $[a_j, v]$ for any atom a_j of P and any $v \in P$. Thus, swapping the order of a_i, a_{i+1} in $[u, \hat{1}]_r$ while otherwise leaving a chain-atom ordering unchanged cannot cannot whether (i)(b) is satisfied. Since P satisfies condition (i)(b) with respect to Γ, P therefore also satisfies condition (i)(b) with respect to Λ. This completes case (3).

Next we handle case (2), i.e., the case where u is an atom of P. First we check condition (i)(b) for each interval $[a_j, v]$ with $a_j \neq u$ where a_j is an atom of P and v satisfies $a_j \lessdot x \lessdot v$ for some $x \in P$; then we will separately handle the intervals with $a_j = u$. Since swapping the order of a_i and a_{i+1} in $[u, \hat{1}]_{\hat{0}<u}$ has no impact on the order of the atoms of P or on the order of the atoms of $[a_j, v]$ for $a_j \neq u$, it follows from the fact that $[a_j, v]$ satisfied condition (i)(b) before the swap that it also satisfies condition (i)(b) after the swap. Next we prove condition (i)(b) for those intervals $[a_j, v]$ with $a_j = u$ such that there exists x with $u \lessdot x \lessdot v$, splitting this in two parts based on whether or not a_i and a_{i+1} are both elements of $[u, v]_{\hat{0}<u}$. First consider any such interval $[u, v]$ that contains at most one of a_i and a_{i+1}. Notice that $[\hat{0}, v]$ contains at most one of a_i, a_{i+1} by virtue of our having $u \lessdot a_i$ and $u \lessdot a_{i+1}$ with a_i, a_{i+1} not both in $[u, v]$. But then $\Gamma|_{[\hat{0},v]}$ has the property that swapping the order of a_i, a_{i+1} in $[u, \hat{1}]$ does not change the chain-atom ordering $\Gamma|_{[\hat{0},v]}$ which therefore is still a GRAO on $[\hat{0}, v]$ after the swap. Therefore, condition (i)(b) of a GRAO holds for $\Lambda|_{[\hat{0},v]}$, implying that $[u, v]$ with at most one of a_i, a_{i+1} in $[u, v]$ satisfies condition (i)(b) with respect to Λ. Next consider the intervals $[u, v]_{\hat{0}<u}$ containing both a_i and a_{i+1}. Neither a_i nor a_{i+1} may be the first atom with respect to Γ of any interval to which both belong, implying neither is the first atom in $[u, v]$ with respect to Γ. Therefore, swapping the order of a_i and a_{i+1} has no impact on whether the first atom of $[u, v]$ is greater than an earlier atom of P than u. This swap also has no impact on whether there exists an atom of $[u, v]$ that is greater than an earlier atom of P than u. Thus, we use that condition (i)(b) holds for each such interval $[u, v]$ with respect to Γ to deduce that it also holds for each such interval with respect to Λ.

Finally, we turn to case (1), the case in which $u = \hat{0}$. Here we subdivide the task of checking condition (i)(b) for the pertinent rooted intervals $[a_j, v]$ based on whether we have (a) $a_j \in \{a_i, a_{i+1}\}$ with $a_i, a_{i+1} \in [\hat{0}, v]$, (b) $a_j \in \{a_i, a_{i+1}\}$ with exactly one of a_i, a_{i+1} in $[\hat{0}, v]$, or (c) $a_j \notin \{a_i, a_{i+1}\}$. It is not possible to have $a_j \in \{a_i, a_{i+1}\}$ with neither a_i nor a_{i+1} in $[\hat{0}, v]$ since $a_j \in [\hat{0}, v]$.

First consider for $u = \hat{0}$ any rooted interval $[a_j, v]$ of type (a). Since neither a_i nor a_{i+1} is allowed to be the first atom with respect to Γ in any rooted interval containing both a_i and a_{i+1}, there exists an atom a of $[\hat{0}, v]$ that comes earlier than both a_i and a_{i+1} with respect to Γ. Since Γ is a GRAO, the ordered pairs of atoms (a, a_i) and (a, a_{i+1}) each satisfy condition (ii) of a GRAO when using the chain-atom ordering Γ, in the following sense: for each $[\hat{0}, y]$ having $a, a_i \in [\hat{0}, y]$

(resp. $a, a_{i+1} \in [\hat{0}, y]$) there exists $z \in [\hat{0}, y]$ and an atom $a_{l'}$ coming earlier than a_i (resp. a_{i+1}) in Γ with $a_{l'} < z$ and $a_i \lessdot z$ (resp. $a_{i+1} \lessdot z$). Therefore, there exists an atom z of $[a_i, v]$ (resp. $[a_{i+1}, v]$) that is above an earlier atom $a_{l'}$ of $[\hat{0}, v]$ than a_i (resp. a_{i+1}) with respect to Γ. This together with the fact that Γ is a GRAO allows us to deduce that the earliest atom of $[a_i, v]$ (resp. $[a_{i+1}, v]$) is above an earlier atom than a_i (resp. a_{i+1}) with respect to Γ. But then the earliest atom b' of $[a_i, v]$ is still above an earlier atom than a_i after swapping the order of a_i and a_{i+1} since every atom that came before a_i with respect to Γ still comes before a_i after the swap. This confirms condition (i)(b) for $[a_j, v]$ when $a_j = a_i$. Next consider the possibility that $a_j = a_{i+1}$. We have already shown above that the earliest atom b' of $[a_{i+1}, v]$ is above an earlier atom than a_{i+1} with respect to Γ. What remains is to rule out the possibility that a_i is the only earlier atom than a_{i+1} with respect to Γ that is below b'. But this would imply $a_i < b'$ and $a_{i+1} < b'$, which by our hypotheses implies that neither a_i nor a_{i+1} is the earliest atom of $[\hat{0}, b']$. This gives a contradiction to our assumption that a_i was the only earlier atom than a_{i+1} below b', completing our treatment of intervals of type (a).

Continuing the $u = \hat{0}$ case, next we handle the intervals of type (b). That is, we consider intervals $[a_j, v]$ with $a_j = a_i$ but $a_{i+1} \not\leqslant v$ (resp. $a_j = a_{i+1}$ but $a_i \not\leqslant v$) where $a_j \lessdot x \lessdot v$ for some $x \in P$. In each of these two cases, swapping the order of the atoms a_i, a_{i+1} of P leaves $\Gamma|_{[\hat{0},v]}$ unchanged. Thus, we may use the fact that condition (i)(b) holds for Γ to deduce condition (i)(b) for intervals of type (b) with respect to Λ.

We now complete the $u = \hat{0}$ case by handling the intervals of type (c). That is, consider the intervals $[a_j, v]$ where $a_j \notin \{a_i, a_{i+1}\}$ such that there exists $x \in P$ with $a_j \lessdot x \lessdot v$. Since a_i and a_{i+1} are consecutive atoms in Γ, a_j must either come before both a_i and a_{i+1} in Γ or come after both a_i and a_{i+1} in Γ. In either event, the following three facts may easily be checked: that Γ satisfies condition (i)(b) due to Γ being a GRAO, that swapping a_i and a_{i+1} preserves $\Gamma|_{[a_j,v]}$, and that the set of atoms coming earlier than a_j with respect to Γ equals the set of atoms coming earlier than a_j in the chain-atom ordering obtained from Γ by swapping the order of a_i and a_{i+1}. These facts combine to imply that $[a_j, v]$ satisfies condition (i)(b) after the swap, namely with respect to Λ. □

Lemma 3.12. *Let Γ be a GRAO for a finite bounded poset P. Suppose a pair of consecutive atoms a_i, a_{i+1} in the induced GRAO for $[u, \hat{1}]_r$ has the property that each rooted interval $[u, w]_r$ containing both a_i and a_{i+1} has neither a_i nor a_{i+1} as its earliest atom with respect to $\Gamma|_{[u,w]_r}$. Then the chain-atom ordering Λ obtained from Γ by swapping the order of a_i and a_{i+1} in $[u, \hat{1}]_r$ satisfies condition (ii) from the definition of GRAO.*

Proof. If $u \neq \hat{0}$, swapping the order of the atoms a_i and a_{i+1} of $[u, \hat{1}]_r$ does not impact the ordering of the atoms of P. Since condition (ii) held before the swap, condition (ii) therefore also holds after the swap. Thus, we may assume $u = \hat{0}$ for the remainder of the proof.

For each interval $[\hat{0}, y]$ in P, we need to show the following: for any pair of atoms a_l, a_m of $[\hat{0}, y]$ with a_l coming before a_m with respect to Λ, there exists an atom a_k coming earlier than a_m in Λ and an element $z \in [\hat{0}, y]$ such that $a_k < z$ and $a_m \lessdot z$. Within this proof we call what must be checked for any fixed choice of $y \in P$ "condition (ii-int) for the interval $[\hat{0}, y]$ within P". We do not call this condition (ii) because we wish to emphasize the fact which will be important to our proof that this condition on $[\hat{0}, y]$ is strictly easier to check than regarding $[\hat{0}, y]$

as a poset with a chain-atom ordering and checking condition (ii) from the definition of GRAO on that poset.

First we show for each $y \in P$ having $a_i, a_{i+1} \in [\hat{0}, y]$ that condition (ii-int) holds for $[\hat{0}, y]$ with respect to Λ. Consider any pair a_l, a_m of atoms of $[\hat{0}, y]$ where a_l comes earlier than a_m with respect to Λ. Since a_m is not the first atom of $[\hat{0}, y]$ with respect to Λ, we claim that it also cannot be the first atom of $[\hat{0}, y]$ with respect to Γ; we confirm this claim by noting that the swap of a_i, a_{i+1} converting Λ back to Γ does not impact which atom is first in $[\hat{0}, y]$ since neither a_i nor a_{i+1} is allowed to be the first atom of $[\hat{0}, y]$ with respect to Γ which implies they also cannot be first with respect to Λ. Since Γ is a GRAO, $[\hat{0}, y]$ satisfies condition (ii-int) with respect to Γ. Since a_m is not the first atom of $[\hat{0}, y]$ with respect to Γ, there exists an atom in $[\hat{0}, y]$ coming earlier than a_m with respect to Γ. But then condition (ii) of a GRAO implies there exists an atom $a_k \in [\hat{0}, y]$ and an element $z \in [\hat{0}, y]$ such that a_k comes earlier than a_m with respect to Γ, $a_k < z$ and $a_m \lessdot z$. We will show that we may use this same z and a_k after the swap of a_i and a_{i+1} to demonstrate the existence of both an atom $a'_k \in [\hat{0}, y]$ that comes earlier than a_m with respect to Λ and an element $z' \in [\hat{0}, y]$ that covers a_m and is greater than a'_k. What needs to be checked in order to justify letting $a'_k = a_k$ and $z' = z$ is that swapping a_i and a_{i+1} cannot cause a_k to come after a_m with respect to Λ. This could only potentially happen if $a_k = a_i$ and $a_m = a_{i+1}$. But then we could use the fact that $a_i, a_{i+1} \in [\hat{0}, z]$ (due to having $a_k, a_m \in [\hat{0}, z]$) to conclude that neither a_i nor a_{i+1} could be the first atom of $[\hat{0}, z]$. This would imply the existence of some atom $a \in [\hat{0}, z]$ coming earlier than both a_i and a_{i+1}. But then we could use this element a to serve as our desired atom a_k coming earlier than a_m, and we could use the same z as before the swap. This confirms condition (ii-int) for all intervals $[\hat{0}, y]$ having $a_i, a_{i+1} \in [\hat{0}, y]$.

Finally, we verify that condition (ii-int) holds with respect to Λ for all intervals $[\hat{0}, y]$ such that a_i, a_{i+1} are not both elements of $[\hat{0}, y]$. Again, consider any pair of atoms $a_l, a_m \in [\hat{0}, y]$ such that a_l comes earlier than a_m with respect to $\Gamma|_{[\hat{0},y]}$. Since Γ is a GRAO, condition (ii) of a GRAO ensures that there exists $z \in [\hat{0}, y]$ and an atom a_k of $[\hat{0}, y]$ which comes earlier than a_m with respect to Λ such that we also have $a_k < z$ and $a_m \lessdot z$. Since at most one of the elements a_i, a_{i+1} is an atom of $[\hat{0}, y]$ in this case, the swap of a_i, a_{i+1} cannot change the relative order of the atoms of $[\hat{0}, y]$. Thus, we may use this same a_k and z after the swap of a_i and a_{i+1} to demonstrate the existence of an element $z \in [\hat{0}, y]$ and an atom a_k coming earlier than a_m such that $a_k < z$ and $a_m \lessdot z$. This confirms condition (ii-int) with respect to Λ for all intervals $[\hat{0}, y]$ not having both a_i and a_{i+1} as elements of $[\hat{0}, y]$. □

While the statement of the next result may seem somewhat technical, it captures in a precise and seemingly useful way how certain types of localized moves are guaranteed to transform a GRAO into a new GRAO. Thus, this theorem pins down a certain type of flexibility one has in choosing a chain-atom ordering that will be a GRAO. We will use this result in our proof later in the paper that any GRAO may be transformed into an RAO.

Theorem 3.13. *Let Γ be a GRAO for a finite bounded poset P. Suppose a pair of consecutive atoms a_i, a_{i+1} in the GRAO for $[u, \hat{1}]_r$ induced by Γ has the property that each rooted interval $[u, w]_r$ containing both a_i and a_{i+1} has neither a_i nor a_{i+1} as its earliest atom with respect to Γ. Then the chain-atom ordering Λ obtained from Γ by swapping the order of a_i and a_{i+1} in $[u, \hat{1}]_r$ is itself a GRAO for P.*

Proof. Our proof is by induction on the length of the longest saturated chain in P. It will suffice to show that each of the requirements of a GRAO is preserved under swapping the order of two consecutive atoms a_i and a_{i+1} in a rooted interval $[u, \hat{1}]_r$, provided that neither a_i nor a_{i+1} is the earliest atom of any rooted interval $[u, w]_r$ that contains both a_i and a_{i+1}.

To show that condition (i)(a) in the definition of GRAO still holds after the swap, it suffices to show for each of the atoms a in P that $\Lambda|_{[a,\hat{1}]}$ is a GRAO. First consider the case with $u \in [a, \hat{1}]$. One may easily observe in this case that $\Lambda|_{[a,\hat{1}]}$ is the same chain-atom ordering one obtains by restricting Γ to $[a, \hat{1}]$ and then performing the swap on this restricted GRAO. Since $\Gamma|_{[a,\hat{1}]}$ is a GRAO, our inductive hypothesis allows us to deduce that $\Lambda|_{[a,\hat{1}]}$ is also a GRAO. Next consider the case with $u \notin [a, \hat{1}]$. Then the swap of a_i, a_{i+1} leaves $\Gamma|_{[a,\hat{1}]}$ unchanged; since this chain-atom ordering on $[a, \hat{1}]$ is a GRAO before the swap, it remains a GRAO after the swap. This proves condition (i)(a) for Λ.

Since Γ is a GRAO, we may apply Lemma 3.11 to deduce that Λ also satisfies condition (i)(b) from the definition for GRAO. Likewise we may use that Γ is a GRAO and apply Lemma 3.12 to deduce that condition (ii) from the definition of GRAO holds for Λ. □

4. Equivalence of admitting a generalized recursive atom ordering to admitting a recursive atom ordering

In this section we prove that a finite bounded poset admits a generalized recursive atom ordering (see Definition 3.2) if and only if it admits a recursive atom ordering. We accomplish the more challenging half of this result constructively by a procedure that transforms any generalized recursive atom ordering into a recursive atom ordering, a process we call "atom reordering." Along the way, we will develop several properties of such reorderings.

First we carry out the other much easier direction of the result.

Lemma 4.1. *Every recursive atom ordering is a generalized recursive atom ordering.*

Proof. We first verify that conditions (i)(a) and (i)(b) of Definition 3.2 hold for any given RAO, doing so by way of a proof by induction on the length l of the longest saturated chain of P. Any ordering of the atoms of a finite bounded poset whose longest saturated chain is of length 1 or 2 is a generalized recursive atom ordering, giving the base case. Let $a_1, a_2, \ldots, a_t$ be an RAO for a finite bounded poset P having $l > 2$. By induction, we may assume for each atom $a_j \in P$ that the RAO on $[a_j, \hat{1}]$ guaranteed to exist by condition (i)(a) of the definition of RAO is a GRAO, giving (i)(a) from the definition of GRAO.

Now we turn to condition (i)(b) of Definition 3.2. By the definition of RAO, each atom in $F_{\hat{0}<a_j}(a_j)$ (see Definition 2.16) must come earlier in our given RAO for $[a_j, \hat{1}]$ than every atom in $G_{\hat{0}<a_j}(a_j)$. This implies that if any atom of $[a_j, \hat{1}]$ is greater than a_i for some $i < j$, then the first atom of $[a_j, \hat{1}]$ is greater than $a_{i'}$ for some $i' < j$. Once we check that any RAO for $[a_j, \hat{1}]$ restricts to an RAO for $[a_j, w]$ for each $w > a_j$ in P, we will likewise be able to deduce the following implication: if any atom of $[a_j, w]$ is greater than a_i for some $i < j$, then the first atom of $[a_j, w]$ is greater than $a_{i'}$ for some $i' < j$. Now we verify the desired claim about restriction

of an RAO. Given an RAO, we first apply the construction of Björner and Wachs from [BW83] that produces a CL-labeling from any RAO. Then we note that this CL-labeling restricts to a CL-labeling for the interval $[a_j, w]$. Finally we apply the construction of Björner and Wachs from [BW83] which produces an RAO from any CL-labeling to get an RAO for $[a_j, w]$. It is easy to see that the chain-atom ordering obtained this way is exactly the restriction of our given RAO to $[a_j, w]$. Having verified this claim, we have completed the confirmation of condition (i)(b) of Definition 3.2.

Condition (ii) in the definition of GRAO is the same as condition (ii) in the definition of RAO, hence is guaranteed to hold for any RAO. □

Remark 4.2. It is not true that every generalized recursive atom ordering is a recursive atom ordering. See Figure 4.1 for an example illustrating this.

Next, in Algorithm 4.1, we describe the atom reordering process that will allow us to transform any generalized recursive atom ordering into a recursive atom ordering. This algorithm takes as its input any chain-atom ordering Λ, and it outputs a chain-atom ordering that we denote by Λ^{re}. One may think of this superscript re as shorthand for "reordered." The atom reordering process is designed to output a chain-atom ordering that will satisfy condition (i)(b) from the definition of recursive atom ordering. Moreover, it is set up to do so in such a way that when applied to a GRAO, it preserves useful structure that is present in a GRAO, including preserving condition (ii) from the definition of GRAO.

Broadly, the algorithm starts at the bottom of the poset P and works its way to the top, reordering the atoms of each rooted interval in a way that takes into account the reordering that has already occurred lower in the poset. Each of these reordering steps moves those atoms of a rooted interval $[u, \hat{1}]_r$ that are above an earlier atom than u in the rooted interval $[u^-, \hat{1}]_{r^-}$ ahead of those that are not, otherwise preserving the ordering on atoms. The algorithm progressively builds up a chain-atom ordering Λ^{pr} by defining $\Lambda^{pr}([u, \hat{1}]_r)$ for more and more choices of $u \in P$ and of root r for $[u, \hat{1}]$.The superscript pr in Λ^{pr} is shorthand for "partially reordered". Once the algorithm has made Λ^{pr} into an entire chain-atom ordering, it outputs this chain-atom ordering and calls it Λ^{re}. Readers may find it helpful to refer to Definition 2.16 for an explanation of the notations $F_r^{\Lambda^{pr}}(u, v)$ and $G_r^{\Lambda^{pr}}(u, v)$ as they read Algorithm 4.1.

It is sometimes necessary (e.g. in the proof of Lemma 4.8) to take a different viewpoint on this algorithm, keeping track of more data at the intermediate stages in the algorithm in a way that does not impact the output of the algorithm or the essentials of how the algorithm proceeds. In this enriched version of the algorithm, Λ^{pr} will denote an entire chain-atom ordering at each step of the atom reordering process. This is accomplished by initializing Λ^{pr} to equal Λ and then otherwise leaving the algorithm unchanged. The effect is that progressively more and more of the values $\Lambda^{pr}([u, \hat{1}]_r)$ are re-set from what they equal in Λ to what they will equal in Λ^{re}. This allows one to think of Λ as evolving into Λ^{re} over the course of the algorithm. Our upcoming proofs of various properties of the atom reordering process will all hold regardless of which of these two viewpoints one takes, namely regarding Λ^{pr} as growing or as evolving, as one may easily check by noting how these two versions of the algorithm really only differ in terms of notation, not in substance.

The reason we take the former viewpoint within Algorithm 4.1 itself is so that the

sets $F_r^{\Lambda^{pr}}(u,v)$ and $G_r^{\Lambda^{pr}}(u,v)$ will be defined in an unambiguous way when they are used later (e.g. within the proof of Lemma 4.6). When we speak of the atom reordering process transforming Λ into Λ^{re}, both in the introduction of the paper and in the proof of Lemma 4.8, we are taking the latter viewpoint.

Example 4.3. Figure 4.1 shows a poset P with a GRAO on the left and a RAO on the right. The RAO on the right is obtained by applying the atom reordering process described in Algorithm 4.1 to the GRAO on the left. In this case, the atom reordering process changes the order of the atoms above the element a, in particular swapping the 2nd and 3rd atoms above a (highlighted in red). Note that this example was chosen to have the further property in both the ordering on the left and the reordering on the right that the atom ordering for any interval $[u,\hat{1}]_r$ is independent of choice of root r. This makes the figures more understandable, but is a very special case.

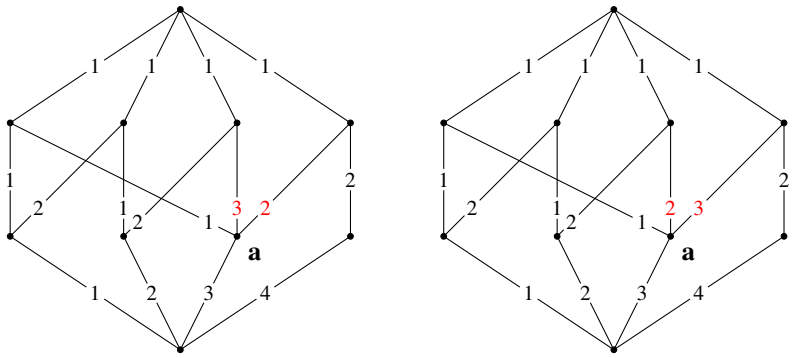

Figure 4.1: GRAO that is not RAO (left) and RAO for same poset (right).

We next deduce several fundamental properties of atom reorderings.

Proposition 4.4. *Let P be a finite bounded poset with a chain-atom ordering Λ. Let $\Lambda|_{[\hat{0},v]}$ (resp. $\Lambda^{re}|_{[\hat{0},v]}$) be the chain-atom ordering for $[\hat{0},v]$ obtained by restricting Λ (resp. Λ^{re}) to $[\hat{0},v]$. Then $\Lambda^{re}|_{[\hat{0},v]}$ equals the chain-atom ordering for $[\hat{0},v]$ obtained by applying the atom reordering process to $\Lambda|_{[\hat{0},v]}$.*

Proof. Let $r=\hat{0}\lessdot t_1\lessdot t_2\ldots\lessdot u$ be a root for the interval $[u,v]$ in P. Let r^- be the root obtained by eliminating u from r, and let u^- be the highest element of r^-. Recall from Definition 2.16 that $F_r^{\Lambda^{re}}(u,v)$ refers to the set of atoms of $[u,v]$ that cover an earlier atom of $[u^-,v]_{r^-}$ than u in $\Lambda^{re}|_{[u^-,v]_{r^-}}$. Similarly, $G_r^{\Lambda^{re}}(u,v)$ refers to the set of atoms of $[u,v]$ that are not in $F_r^{\Lambda^{re}}(u,v)$

Our main task will be to prove that $F_r^{\Lambda^{re}}(u,\hat{1})\cap[u,v]_r = F_r^{\Lambda^{re}}(u,v)$ and that $G_r^{\Lambda^{re}}(u,\hat{1})\cap[u,v]_r = G_r^{\Lambda^{re}}(u,v)$. We claim that $F_r^{\Lambda^{re}}(u,\hat{1})\cap[u,v]_r \subseteq F_r^{\Lambda^{re}}(u,v)$ and $G_r^{\Lambda^{re}}(u,\hat{1})\cap[u,v]_r \subseteq G_r^{\Lambda^{re}}(u,v)$, and we call this Claim (I). Notice that Claim (I) would imply that the union of sets $(F_r^{\Lambda^{re}}(u,\hat{1})\cap[u,v]_r)\cup(G_r^{\Lambda^{re}}(u,\hat{1})\cap[u,v]_r)$ is also contained in the union of sets $F_r^{\Lambda^{re}}(u,v)\cup G_r^{\Lambda^{re}}(u,v)$. But this last set containment would actually be an equality of sets by virtue of both of the sets in the containment equalling the set of all atoms in $[u,v]$. This set

Algorithm 4.1: The atom reordering process applied to Λ which outputs Λ^{re}.

input: A finite bounded poset P equipped with a chain-atom ordering Λ

output: A chain-atom ordering of P, denoted Λ^{re}, also called the **atom reordering of** P.

begin

1. Choose any linear extension $u_0, u_1, u_2, \ldots, u_n$ of P;
2. Set $\Lambda^{pr}([\hat{0}, \hat{1}]) := \Lambda([\hat{0}, \hat{1}])$, thereby determining $\Lambda^{pr}([u_0, \hat{1}])$;
3. **for** $i = 1$ *to* n **do**
   - (a) Choose an ordering of the roots $r_1, r_2 \ldots r_{t_i}$ for $[u_i, \hat{1}]$;
   - (b) **for** $j = 1$ *to* t_i **do**
     - i. Calculate $F_{r_j}^{\Lambda^{pr}}(u_i)$ and $G_{r_j}^{\Lambda^{pr}}(u_i)$;
     - ii. Order the elements of $F_{r_j}^{\Lambda^{pr}}(u_i)$ in the same relative order as in $\Lambda([u_i, \hat{1}]_{r_j})$;
     - iii. Order the elements of $G_{r_j}^{\Lambda^{pr}}(u_i)$ in the same relative order as in $\Lambda([u_i, \hat{1}]_{r_j})$;
     - iv. Determine $\Lambda^{pr}([u_i, \hat{1}]_{r_j})$ as follows: put all elements of $F_{r_j}^{\Lambda^{pr}}(u_i)$ in the order from step ii, followed by all elements of $G_{r_j}^{\Lambda^{pr}}(u_i)$ in the order from step iii;
     - v. Increase j by 1;
   - **end**
   - (c) Increase i by 1;

   end
4. Set Λ^{re} to be the chain-atom ordering that has $\Lambda^{re}([u_i, \hat{1}]_{r_j}) = \Lambda^{pr}([u_i, \hat{1}]_{r_j})$ for every $u_i \in P$ and every choice of root r_j for $[u_i, \hat{1}]$;

end

equality would imply that each of the component set containments would also be a set equality, which would complete the proof.

What remains is to prove Claim (I). We do so by induction on the length of the longest saturated chain from $\hat{0}$ to u. The point is that $a \in F_r^{\Lambda^{re}}(u, \hat{1})$ (resp. $a \in G_r^{\Lambda^{re}}(u, \hat{1})$) implies that there exists (resp. does not exist) an atom a' of $[u^-, \hat{1}]_{r^-}$ with $a' < a$ in P such that a' is earlier than u in $\Lambda^{re}|_{[u^-, \hat{1}]_{r^-}}$. But any such a' is also an atom of $[u^-, v]_{r^-}$ since $u^- \lessdot a' < a \leqslant v$ in P. Moreover, our inductive hypothesis ensures that a' is an earlier atom than u in $[u^-, v]_{r^-}$ regardless of whether we are reordering within P or within $[\hat{0}, v]$. This is exactly what is needed to show that $a \in F_r^{\Lambda^{re}}(u, v)$. On the other hand, when such a' does not exist within P, this implies that no such a' exists within $[\hat{0}, v]$ since $[\hat{0}, v]$ is a subset of P, yielding the desired claim that $G_r^{\Lambda^{re}}(u, \hat{1}) \cap [u, v]_r \subseteq G_r^{\Lambda^{re}}(u, v)$. □

In light of Proposition 4.4, we may henceforth speak interchangeably of the restriction to $[u, v]_r$ of the atom reordering Λ^{re} of a chain-atom ordering Λ of a finite bounded poset P and of the restriction to $[u, v]_r$ of the atom reordering of $\Lambda|_{[\hat{0}, v]}$ for any given $u < v$ in P and any

root r for $[u, v]$.

Next we give a variation on condition (i)(b) from the definition of GRAO that will be useful in upcoming inductive arguments.

Proposition 4.5. *Let P be a finite bounded poset with a GRAO. Consider $t, u, v \in P$ such that $t \lessdot u < v$ and consider a choice of root r for $[t, v]$. Then either the first atom of $[u, v]_{r\cup u}$ in the GRAO is above an earlier atom of $[t, v]_r$ than u or else u is the first atom of $[t, v]_r$.*

Proof. Lemma 3.9 immediately implies that either the first atom of $[u, v]_{r\cup u}$ is above an earlier atom of $[t, v]_r$ than u or no atom of $[u, v]_{r\cup u}$ is above an earlier atom of $[t, v]_r$ than u. It suffices to consider the case where no atom of $[u, v]_{r\cup u}$ is above an earlier atom than u in the GRAO of $[t, v]_r$, and to show in this case that u is the first atom in the GRAO of $[t, v]_r$. Suppose by way of contradiction that u' is the first atom in the GRAO of $[t, v]_r$ for some $u' \neq u$. Since both u' and u are below v, Definition 3.2, part (ii), implies there must exist some atom u'' of $[t, v]_r$ that comes before u in the GRAO and some element x' such that $u \lessdot x'$ and $u'' < x' < v$. The existence of such x' contradicts the fact that no atom of $[u, v]_{r\cup u}$ is above an atom earlier than u in the GRAO of $[t, v]_r$. This completes the proof. □

Next is a result that plays an important role in explaining (and in justifying) the fact that our atom reordering process transforms any GRAO into an RAO.

Lemma 4.6. *Let P be a finite, bounded poset with Λ a GRAO for P. Then for any $u < v$ in P and any root r for $[u, v]$, the first atom of $[u, v]_r$ in Λ is the first atom of $[u, v]_r$ in Λ^{re}, namely in the atom reordering of Λ.*

Proof. Proposition 3.8 ensures that Λ restricts to a GRAO for $[u, v]_r$. Proposition 4.4 allows us to speak interchangeably about the atom reordering of $[\hat{0}, v]$ restricted to $[u, v]_r$ and the atom reordering of P restricted to $[u, v]_r$. Let $r = \hat{0} \lessdot t_1 \lessdot t_2 \ldots \lessdot t_n \lessdot u$ be a root for the interval $[u, v]$ in P and let r^- denote the root for $[t_n, v]$ obtained from r by removing u. Recall from Definition 2.16 that $F_r^\Lambda(u, v)$ refers to the set of atoms of $[u, v]_r$ that cover an earlier atom of $[t_n, v]_{r^-}$ than u in $\Lambda([t_n, v]_{r^-})$. Also recall that $G_r^\Lambda(u, v)$ refers to the set of atoms of $[u, v]$ that are not in $F_r^\Lambda(u, v)$.

We will prove that the first atom of $\Lambda([u, v]_r)$ is the first atom of $\Lambda^{re}([u, v]_r)$ for all $u \in P$ and all choices of root r. We will do so by induction on the length l of the root r from $\hat{0}$ to u. For the base case, let u be an atom of $[\hat{0}, v]$. Let x be the first atom in $\Lambda([u, v]_{\hat{0}<u})$. If x is in $F_r^\Lambda(u, v)$, then x is in $F_r^{\Lambda^{re}}(u, v)$ since u is an atom of P and the atom reordering process does not change the order of the elements covering $\hat{0}$. Since we put the elements of $F_r^{\Lambda^{pr}}(u, v)$ before all other atoms of $[u, v]_r$ during the atom reordering of $\Lambda|_{[\hat{0},v]}$, x remains first among the atoms of $[u, v]_r$ in $\Lambda^{pr}([u, v]_r)$ in this $x \in F_r^\Lambda(u, v)$ case. Now suppose x is not in $F_r^\Lambda(u, v)$. By Proposition 4.5, u is the first atom in $\Lambda([\hat{0}, v])$. Since the atom reordering process does not change the order of the elements covering $\hat{0}$, u is also the first atom in $\Lambda^{re}([\hat{0}, v])$. The fact that u is first in $\Lambda^{re}([\hat{0}, v])$ implies that every atom in $[u, v]_r$ must be in $G_r^{\Lambda^{pr}}(u, v)$. Since the atom reordering process preserves the relative ordering of the elements of $G_r^{\Lambda^{pr}}(u, v)$ given by $\Lambda([u, v]_r)$, the atom reordering process preserves the ordering of all atoms of $[u, v]_r$. Thus, x remains first in $\Lambda^{re}([u, v]_r)$. This completes the base case.

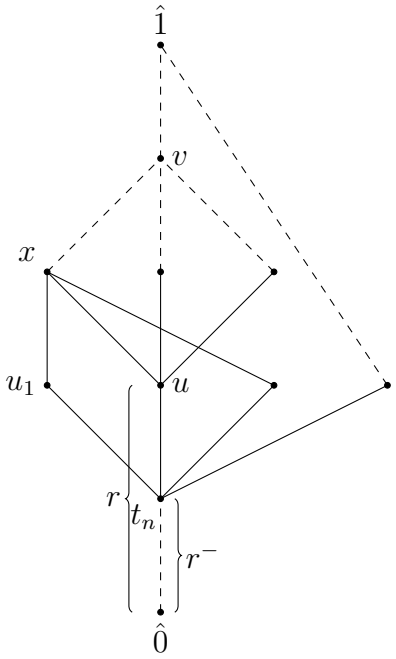

Figure 4.2: Case 1 in the proof of Lemma 4.6.

For the inductive step, assume that the first atom in $\Lambda([u,v]_r)$ is the first atom in $\Lambda^{re}([u,v]_r)$ for any $u < v$ and any root r of length l, where $l \leqslant n$. We will prove that the first atom in $\Lambda([u,v]_r)$ is also the first atom in $\Lambda^{re}([u,v]_r)$ for any $u < v$ and any root r of length $n+1$. Let x be the first atom in $\Lambda([u,v]_r)$ where r is a root of length $n+1$. Suppose by way of contradiction that x is not first in $\Lambda^{re}([u,v]_r)$. This implies x is moved to a later position by the atom reordering process applied to Λ, hence it implies $x \in G_r^{\Lambda^{pr}}(u,v)$. Since x is the first atom in $\Lambda([u,v]_r)$, Lemma 3.9 gives us that either (1) x is greater than an atom that comes before u in $\Lambda([t_n,v]_{r^-})$ or (2) no atom of $[u,v]_r$ is greater than an atom that comes before u in $\Lambda([t_n,v]_{r^-})$. We consider these two cases separately.

Case 1. This is the case in which x is above an earlier atom than u in $\Lambda([t_n,v]_{r^-})$. See Figure 4.2 for an illustration of this case. Let u_1 be the first atom in $\Lambda([t_n,x]_{r^-})$. By our assumption about x, note that $u_1 \neq u$; this implies that u_1 comes earlier than u in $\Lambda([t_n,v]_{r^-})$. Since we already showed $x \in G_r^{\Lambda^{pr}}(u,v)$, u_1 must come later than u in $\Lambda^{re}([t_n,v]_{r^-})$. Because of Proposition 4.4, u_1 must come later than u in $\Lambda^{re}([t_n,x]_{r^-})$ as well. In particular, u_1 is not first in $\Lambda^{re}([t_n,x]_{r^-})$. But the root r^- has length n, which contradicts our inductive hypothesis that requires u_1 to remain the earliest atom in $\Lambda^{re}([t_n,x]_{r^-})$. Thus, we rule out this case.

Case 2. This is the case in which no atom of $[u,v]_r$ is above an atom that is earlier than u in $\Lambda([t_n,v]_{r^-})$. See Figure 4.3 for an illustration of this case. By Proposition 4.5, u must then be first in $\Lambda([t_n,v]_{r^-})$. Since r^- has length n, our inductive hypothesis ensures that u must also be first in $\Lambda^{re}([t_n,v]_{r^-})$. This means that all atoms of $[u,v]_r$ are in $G_r^{\Lambda^{pr}}(u,v)$. This implies that

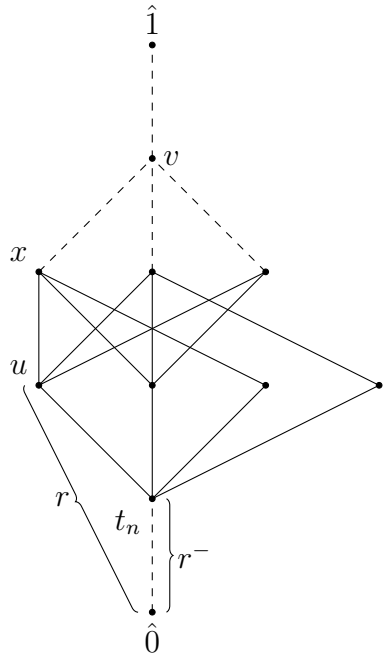

Figure 4.3: Case 2 in the proof of Lemma 4.6.

the ordering of all atoms of $[u, v]_r$ is preserved by the atom reordering process. In particular, x remains first among the atoms of $[u, v]_r$, contradicting our assumption that x is not the first atom of $[u, v]_r$ in $\Lambda^{re}([u, v]_r)$. This completes case 2. □

Next is a lemma that sheds light on what the reordering process does to each rooted interval $[u, \hat{1}]_r$ in a finite bounded poset P endowed with a chain-atom ordering. This will be helpful for proving that applying the atom reordering process to a GRAO yields a chain-atom ordering that is still a GRAO.

Lemma 4.7. *Let Σ be a chain-atom ordering on a finite bounded poset P. Given any $u \in P$ and any root r for $[u, \hat{1}]$, consider the chain-atom ordering $\Sigma^{re}|_{[u,\hat{1}]_r}$ on $[u, \hat{1}]_r$ that is obtained by applying the atom reordering process to Σ and then restricting the resulting chain-atom ordering for P to $[u, \hat{1}]_r$.*

1. *The chain-atom ordering $\Sigma^{re}|_{[u,\hat{1}]_r}$ may alternatively be obtained from $\Sigma|_{[u,\hat{1}]_r}$ by first permuting the atoms of $[u, \hat{1}]_r$ by a permutation π to obtain a chain-atom ordering on $[u, \hat{1}]_r$, and then applying the atom reordering process to $\pi(\Sigma|_{[u,\hat{1}]_r})$.*

2. *If Σ is a GRAO for P, then the permutation π from part (a) is a product of adjacent transpositions in which each of these adjacent transpositions fixes which atom of $[u, v]_r$ is first for every $v > u$ in P.*

3. *If Σ is a GRAO for P, then the chain-atom ordering for P obtained from Σ by applying the permutation π from (a) to permute the order of the atoms of $[u, \hat{1}]_r$ while otherwise leaving Σ unchanged is a GRAO for P.*

Proof. First we prove (a). Applying the reordering process to all of P, the point will be to observe what happens to $[u,\hat{1}]_r$. The earliest step in which the chain-atom ordering for $[u,\hat{1}]_r$ gets modified by the reordering process is the step where the atoms of $[u,\hat{1}]_r$ get permuted by moving the elements of $F_r^{\Sigma}(u,\hat{1})$ ahead of the elements of $G_r^{\Sigma}(u,\hat{1})$, preserving the order of the elements within $F_r^{\Sigma}(u,\hat{1})$ and within $G_r^{\Sigma}(u,\hat{1})$. This gives the desired permutation π on the ordering of the atoms of $[u,\hat{1}]_r$. Now one may observe that the subsequent reordering of the atoms of each rooted interval $[u',\hat{1}]_{r'}$ for $u' \in (u,\hat{1}]$ and each root r' containing r depends only on π and on the reordering of $[u,\hat{1}]_r$ that takes place prior to reaching u' and r'. More precisely, one may confirm by induction on the length of the longest saturated chain from u to u' that the reordering of the atoms of $[u',\hat{1}]_{r'}$ accomplished within the course of reordering P gives the same atom ordering for $[u',\hat{1}]_{r'}$ as we get by first restricting Σ to $[u,\hat{1}]_r$, then applying the permutation π to the atoms of $[u,\hat{1}]_r$, and finally applying the reordering process just to the resulting chain-atom ordering on $[u,\hat{1}]_r$ (i.e. not working our way up from $\hat{0}$). This completes the proof of (a)

Next we prove (b). Lemma 4.6 implies that the permutation π from (a) has the property for each $v > u$ and each root r that the earliest atom of $[u,v]_r$ before applying π is still the earliest atom of $[u,v]_r$ after applying π; equivalently, the permutation that π induces on the atoms of $[u,v]_r$ fixes the first atom of $[u,v]_r$. Let $s_1 \cdots s_k$ be any reduced expression for π, i.e. any expression for π as a product of adjacent transpositions $(i,i+1)$ with k as small as possible. The natural correspondence between reduced expressions for any fixed $\pi \in S_n$ and saturated chains from e to π in left weak order for S_n (which is discussed extensively e.g. in [BB05]) may be combined with Corollary 3.1.4 in [BB05] to give us the well-known fact that for any $1 \leqslant l < m \leqslant n$ satisfying $\pi(l) < \pi(m)$, we also must have $s_j \cdots s_k(l) < s_j \cdots s_k(m)$ for $1 \leqslant j \leqslant k$.

We will apply this property of reduced expressions to certain pairs of atoms $a, a' \in [u,v]_r$ such that a comes before a' both in the ordering $\mathcal{A}$ on the atoms of $[u,\hat{1}]_r$ given by Σ and in the ordering $\pi(\mathcal{A})$ on the atoms of $[u,\hat{1}]_r$ obtained by applying the permutation π to $\mathcal{A}$. For any two atoms a, a' of an interval $[u,v]_r$ such that a is the first atom of $[u,v]_r$ both in the ordering for the atoms of $[u,v]_r$ inherited from $\mathcal{A}$ and in the ordering for the atoms of $[u,v]_r$ inherited from $\pi(\mathcal{A})$, a comes before a' both in $\mathcal{A}$ and in $\pi(\mathcal{A})$. Thus, our above property of reduced expressions implies that a must come before a' in $s_j \cdots s_k(\mathcal{A})$ for each $j \leqslant k$. This implies that a must be the first atom in the ordering on the atoms of $[u,v]_r$ that is inherited from the ordering $s_j \cdots s_k(\mathcal{A})$ on the atoms of $[u,\hat{1}]_r$ for each $j \leqslant k$. Thus, each adjacent transposition in $s_1 \ldots s_k$ applied to $\mathcal{A}$ from right to left preserves which atom is first in every rooted interval of P.

Finally we prove (c). Part (b) tells us that each of the adjacent transpositions in $s_1 \cdots s_k$ has the property when it is applied in turn to $\mathcal{A}$ that it preserves which atom is first in each $[u,v]_r$. This enables us to use Theorem 3.13 repeatedly, once for each of the adjacent transpositions in $s_1 \cdots s_k$, to deduce that the chain-atom ordering obtained from Σ by reordering the atoms of $[u,\hat{1}]_r$ by π is a GRAO for P. □

We are now prepared to prove that applying the atom reordering process to a GRAO yields a GRAO and then to go on from there to prove that the resulting GRAO is in fact an RAO.

Lemma 4.8. *The atom reordering process applied to a GRAO for a finite bounded poset P yields a GRAO for P.*

Proof. Throughout this proof, we will work with the more enriched variation on the atom reordering process discussed just prior to Algorithm 4.1. That is, we regard Λ^{pr} at each step in the process as being a full chain-atom ordering, doing so as follows. We initialize Λ^{pr} to equal Λ, and then otherwise run the algorithm just as in Algorithm 4.1. In other words, at each of the steps of the algorithm in which $\Lambda^{pr}([u, \hat{1}]_r)$ has not yet been defined for a given $u \in P$ and a given choice of root r in the usual algorithm, we set $\Lambda^{pr}([u, \hat{1}]_r)$ equal to $\Lambda([u, \hat{1}]_r)$ in our modified atom reordering process, otherwise leaving the process entirely unchanged. Since this is merely an enrichment of the data contained in Λ^{pr} at the intermediate stages of the algorithm, one may easily see that this does not impact the output of the algorithm or the applicability of the proofs of Lemmas 4.4, 4.6 and 4.7 to this enriched version of the atom reordering process. With these conventions, Λ^{pr} progressively evolves from being the chain-atom ordering Λ at the start of the algorithm to equalling the chain-atom ordering Λ^{re} at the end of the algorithm, as will be necessary for the argument below to apply.

By Lemma 4.6, the atom reordering process for Λ takes $[u, \hat{1}]_r$ for each $u \in P$ and each root r for $[u, \hat{1}]$ and permutes the atoms $a_1, \ldots, a_m$ of $[u, \hat{1}]_r$ in a way that preserves which atom comes first in every rooted interval $[u, v]_r$ for $v > u$ with our fixed root r. Thus, we may apply Lemma 4.7, part (b), to deduce that this permutation on the atoms of $[u, \hat{1}]_r$ is expressible as a reduced expression comprised of a series of adjacent transpositions, each of which preserves which atom comes first in $[u, v]_r$ for each $v > u$. But this means that we may express this enriched version of the atom reordering process as a series of such steps by considering each u and each r in turn, applying such a series of adjacent transpositions as in Lemma 4.7, part (b), for each u and each r. If we start with a GRAO, then Lemma 3.13 tells us that after each of these steps, i.e. after each of these adjacent transpositions, we still have a GRAO. In particular, the end-result of this enriched version of the atom reordering process applied to a GRAO for P is a GRAO for P. Since by design this enriched version of the atom reordering process has the same output Λ^{re} as the atom reordering process given in Algorithm 4.1, this completes the proof. □

Before getting to our main result of this section, we mention a consequence of what we have just proven that could give useful insight into the atom reordering process.

Corollary 4.9. *The reordering process applied to a GRAO of a finite bounded poset P preserves the sets $F_r(u, v)$ and $G_r(u, v)$ for each $u < v$ in P and each root r for $[u, v]$.*

Proof. This follows from Lemma 3.10 and the proof of Lemma 4.8. □

Now we are ready to prove the main result of this section.

Theorem 4.10. *A finite bounded poset admits a generalized recursive atom ordering (GRAO) if and only if it admits a recursive atom ordering (RAO). Moreover, every RAO is a GRAO while every GRAO may be transformed into an RAO by the atom reordering process.*

Proof. By Lemma 4.1, any RAO is a GRAO. What remains is to prove that applying the reodering process from Algorithm 4.1 to any GRAO for a finite bounded poset P yields an RAO. We will do this by induction on the length l of the longest saturated chain in P. For $l = 2$, any ordering of the atoms of P is a recursive atom ordering, giving the base case for our proof by induction.

Now assuming the result for $l \leqslant n$, it will suffice to show that this implies the result for $l = n+1$. Let Λ be a GRAO for P, regarded as a chain-atom ordering. Let $a_1, a_2, \ldots, a_q$ be the ordering of the atoms of P with $l = n+1$ in Λ. In this case, for each atom $a_j \in P$, we have that $[a_j, \hat{1}]$ is an interval whose longest saturated chain is of length at most n. By condition (i)(a) in the definition of GRAO, $\Lambda|_{[a_j,\hat{1}]}$ is a GRAO for $[a_j, \hat{1}]$. By Lemma 4.7, parts (a) and (b), applying the atom reordering process to Λ and then restricting the resulting chain-atom ordering Λ^{re} to $[a_j, \hat{1}]$ yields the same chain-atom ordering for $[a_j, \hat{1}]$ that we get by instead permuting the atoms of $[a_j, \hat{1}]$ in a way that yields a new GRAO for $[a_j, \hat{1}]$ that is denoted by $\pi(\Lambda|_{[a_j,\hat{1}]})$ and then applying the atom reordering process directly to $\pi(\Lambda|_{[a_j,\hat{1}]})$. By our inductive hypothesis, the result of this latter series of operations is an RAO for $[a_j, \hat{1}]$. This implies that condition (i)(a) of Definition 2.14 holds for Λ^{re}. By definition of the atom reordering process, those atoms of $[a_j, \hat{1}]$ that are above an earlier atom of P than a_j all come before all of the other atoms of $[a_j, \hat{1}]$ in Λ^{re}, so condition (i)(b) of Definition 2.14 is also satisfied for Λ^{re}.

What remains is to prove that condition (ii) of Definition 2.14 holds for Λ^{re}. Here we may use Lemma 4.8 which tells us that applying the atom reordering process to Λ yields a GRAO for P. Therefore Λ^{re} satisfies condition (ii) in the definition of GRAO. But condition (ii) in the definition of GRAO is the same as condition (ii) in the definition of RAO, albeit phrased slightly differently, completing our proof. □

5. Self-consistency, the UE property and TCL-shellability

In this section, we introduce a condition that a CC-labeling may have called self-consistency. We also introduce a fairly natural and readily checkable property called the UE property that will imply self-consistency. We prove that all CL-labelings have the UE property.

These notions are introduced in preparation for a result later in the paper where we will prove that a finite bounded poset is CL-shellable if and only if it is CC-shellable by way of a self-consistent CC-labeling. To help us further clarify the relationship between the established notions of CC-shellability and CL-shellability, we also introduce a variation on CC-shellability that we call TCL-shellability.

Definition 5.1. Consider a chain-edge labeling λ such that each rooted interval has a unique lexicographically earliest saturated chain. We define such λ to be **self-consistent** if for any rooted interval $[u, v]_r$ we have the following condition: if a is the atom in the lexicographically first saturated chain of $[u, v]_r$ and $b \neq a$ is also an atom of $[u, v]_r$, then for any $[u, v']_r$ containing a

and b, all saturated chains of $[u, v']_r$ containing b come lexicographically later than all saturated chains of $[u, v']_r$ containing a. If a chain-edge labeling is not self-consistent, then it is said to be **self-inconsistent**.

Next we introduce a property that will imply self-consistency that is more readily checkable.

Definition 5.2. A chain-edge labeling λ of a finite bounded poset P has the **unique earliest (UE) property** if for each rooted interval $[u, v]_r$ in P, the smallest label occurring on any cover relation upward from u only occurs on one such cover relation.

Lemma 5.3. *If a CC-labeling has the UE property, then it is self-consistent.*

Proof. Let a be the unique atom of $[u, v]_r$ for which $\lambda(u, a)$ is smallest among all labels upward from u. Consider any other atom $b \in [u, v]_r$ and any v' satisfying $u < v'$ with $a, b \in [u, v']$. The label sequences on saturated chains of $[u, v']_r$ containing a must be lexicographically smaller than those containing b by virtue of $\lambda(u, a)$ being smaller than $\lambda(u, b)$ with respect to root r. □

Corollary 5.4. *Any recursive atom ordering for a finite bounded poset gives rise to a CC-labeling with the UE property, hence to a self-consistent CC-labeling.*

Proof. Simply observe that the CC-labeling constructed in the proof of Proposition 2.20 has the UE property, then apply Lemma 5.3. □

The next result gives some evidence that the UE property is not an unduly burdensome condition to impose.

Lemma 5.5. *Every CL-labeling has the UE property.*

Proof. Consider a pair of elements $u < v$ in a finite bounded poset P with a CL-labeling λ, and consider any root r of $[u, v]$ from $\hat{0}$ to u. Suppose that there are distinct atoms $a, a' \in [u, v]$ such that $\lambda_r(u, a) = \lambda_r(u, a')$. Further suppose $\lambda_r(u, a) \leqslant \lambda_r(u, a'')$ for all other atoms $a'' \in [u, v]_r$. We may choose a to belong to the lexicographically earliest saturated chain M from u to v in $[u, v]_r$ since this saturated chain will begin with the smallest possible first label. Let $u \lessdot a \lessdot x_2 \lessdot x_3 \lessdot \cdots \lessdot x_k \lessdot v$ be this lexicographically first saturated chain of $[u, v]_r$. Let $u \lessdot a' \lessdot y_2 \lessdot y_3 \lessdot \cdots \lessdot y_l \lessdot v$ be the lexicographically first saturated chain of $[u, v]_r$ that contains a'. Denote by M' this latter saturated chain from u to v. We must have $\lambda_r(u, a) < \lambda_{r\cup a}(a, x_2)$ since M is an ascending chain. But we also have $\lambda_{r\cup a}(a, x_2) \leqslant \lambda_{r\cup a'}(a', y_2)$ since the label sequences for M and M' both start with the same label and M has a lexicographically smaller label sequence than M'. Thus, we have

$$\lambda_r(u, a') = \lambda_r(u, a) < \lambda_{r\cup a}(a, x_2) \leqslant \lambda_{r\cup a'}(a', y_2)$$

implying that M' has an ascent at a'. But M' is also ascending from a' to v since M' is lexicographically first in $[a', v]_{r\cup a'}$. Thus, M and M' are both ascending chains on $[u, v]_r$, giving a contradiction. □

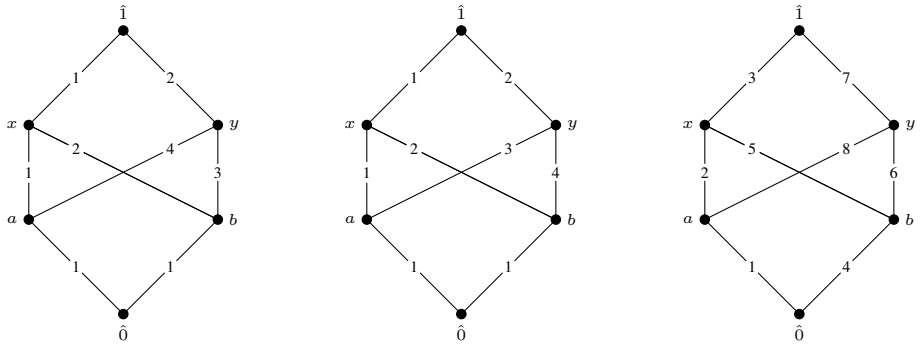

Figure 5.1: Self-inconsistent and self-consistent CC-labelings.

Example 5.6. Figure 5.1 shows a poset P and three different CC-labelings (all of which are in fact EC-labelings). The leftmost CC-labeling is self-inconsistent because both $[\hat{0}, x]$ and $[\hat{0}, y]$ contain a and b as atoms, but a is first in $[\hat{0}, x]$ while b is first in $[\hat{0}, y]$. Given that this labeling is self-inconsistent, it also cannot have the UE property. The CC-labeling in the middle of Figure 5.1 is self-consistent but does not have the UE property because both cover relations up from $\hat{0}$ have label 1. The CC-labeling on the right has the UE property and is self-consistent.

It seems plausible that many of the most interesting and the most natural examples of CC-labelings will have the UE property. Section 7 shows that an EC-labeling from [HK21] for the dual poset to the uncrossing order has the UE property, allowing us to invoke the upcoming Corollaries 6.5 and 6.6 to deduce dual CL-shellability for the uncrossing order. Section 8.1 shows how readily checkable the UE property is for numerous well-known EL-labelings and CL-labelings.

5.1. TCL-shellability: a variation on CC-shellability

One may replace Kozlov's requirements of a CC-labeling (a) that saturated chains have distinct label sequences and (b) that no label sequence be a prefix of any other label sequence by simply requiring each rooted interval to have a unique lexicographically earliest saturated chain. With just this requirement, the proof of Björner and Wachs that a CL-labeling yields a shelling of the order complex will carry over using topological descents in place of descents and using topological ascents in place of ascents. In particular, the UE property suffices in place of these two aforementioned conditions of Kozlov to guarantee we get a shelling.

This yields a class of poset chain-edge labelings inducing shellings that includes all CL-labelings, whether or not the labels upward from the lowest element u in any rooted interval $[u, v]_r$ are all distinct from each other. In justifying his assertion that CC-shellability is the most general possible version of lexicographic shellability, Kozlov notes in [Koz97] that any CL-labeling may be modified into a CC-labeling, implying that every CL-shellable poset is CC-shellable.

Definition 5.7. A **TCL-labeling** of a finite bounded poset P is a chain-edge labeling λ for P such that (i) each rooted interval $[u, v]_r$ has a unique saturated chain from u to v with lexicographically smallest label sequence, and (ii) every other saturated chain from u to v in $[u, v]_r$ has at least one topological descent. If a poset has a TCL-labeling, then it is **TCL-shellable**.

Theorem 5.8. *Every CC-labeling is a TCL-labeling. However, not every TCL-labeling is a CC-labeling.*

Proof. First notice that the requirement of a CC-labeling that the saturated chains in any rooted interval have distinct label sequences in particular implies the uniqueness of the lexicographically earliest label sequence on each rooted interval. Since the saturated chain from u to v with the lexicographically earliest label sequence on a rooted interval $[u, v]_r$ cannot have any topological descents, condition (1) in the definition of CC-labeling implies that every other saturated chain of $[u, v]_r$ must have a topological descent. This shows that every CC-labeling is a TCL-labeling.

To see that the converse fails, simply consider the CL-labeling λ (which is also a TCL-labeling) for a graded bounded poset of rank 2 consisting of elements $\hat{0}, \hat{1}, x_1, x_2, x_3$ with $\hat{0} \lessdot x_i \lessdot \hat{1}$ for $i = 1, 2, 3$ having $\lambda(\hat{0}, x_1) = \lambda(x_2, \hat{1}) = \lambda(x_3, \hat{1}) = 1$ and $\lambda(\hat{0}, x_2) = \lambda(\hat{0}, x_3) = \lambda(x_1, \hat{1}) = 2$. This fails the requirement for CC-labelings that distinct saturated chains have distinct label sequences. □

Remark 5.9. It may very well be true that a finite bounded poset is CC-shellable if and only if it admits a TCL-labeling. The labeling given in the second half of the proof of Theorem 5.8 can easily be modified into a CC-labeling, and indeed it remains open whether these two notions of shellability are equivalent in terms of which posets admit such shellings. Likewise, as far as we know, it is open whether CC-shellability is equivalent to CL-shellability, though this seems to be quite a tricky question.

Theorem 5.10. *Any recursive atom ordering of a finite bounded poset P induces a TCL-labeling for P that is self-consistent.*

Proof. One may simply check that the CC-labeling constructed in the proof of Theorem 2.20 is in fact self-consistent (by virtue of having the UE property) and is a TCL-labeling. One may also invoke Theorem 5.8 for the latter claim. □

Theorem 5.11. *If a finite bounded poset P admits a TCL-labeling, then this induces one or more lexicographic shellings for $\Delta(P)$. Specifically, any linear order on the maximal chains of P that is a linear extension of the partial order obtained by ordering maximal chains lexicographically is a shelling order.*

Proof. The proof of Björner and Wachs in [BW82] that any CL-labeling of a finite bounded poset P induces a lexicographic shelling for $\Delta(P)$ (for each choice of linear extension of the lexicographic order on saturated chains) still applies unchanged for TCL-labelings when we simply replace ascents by topological ascents, descents by topological descents, ascending chains by topologically ascending chains, and descending chains by topologically descending chains throughout the proof. □

6. Equivalence of CL-shellability to self-consistent CC-shellability

In this section, we will prove that several different conditions on a finite bounded poset are equivalent to CL-shellability.

Theorem 6.1. *If a finite bounded poset P admits a generalized recursive atom ordering, then P admits a self-consistent CC-labeling.*

Proof. We proved in Theorem 4.10 that any finite bounded poset P with a generalized recursive atom ordering also admits a recursive atom ordering. We showed in Corollary 5.4 that any such P admits a self-consistent CC-labeling, completing the proof. □

Remark 6.2. One may alternatively prove Theorem 6.1 by using any generalized recursive atom ordering for a finite bounded poset P to construct a self-consistent CC-labeling for P that is compatible (in the sense of Definition 8.5) with the GRAO without needing to use the atom reordering process at all. The construction of such a CC-labeling is carried out as follows. For each $u \in P$ and each root r for $[u, \hat{1}]$, the GRAO gives an ordering $a_1, \ldots, a_{t(u,r)}$ of the atoms of the rooted interval $[u, \hat{1}]_r$. Assign the label i to the cover relation $u \lessdot a_i$ for this choice of root r. One may prove that this produces a UE (and thus self-consistent) CC-labeling for P. It is worth noting that this is typically a different self-consistent CC-labeling for P than the one produced as in the proof above of Theorem 6.1, as we do not apply the atom reordering process to P in this more direct approach. See the proof of Theorem 8.6 for all of the details of this approach.

Theorem 6.3. *If a finite bounded poset P admits a self-consistent TCL-labeling, then P admits a generalized recursive atom ordering.*

Proof. Let λ be a self-consistent TCL-labeling for P. By definition, λ restricts to a TCL-labeling for $[u, \hat{1}]_r$ for each $u \in P$ and each root r. Consider the following chain-atom ordering of P that is compatible with λ in the sense that each rooted interval $[u, v]_r$ will have its earliest atom in the chain-atom ordering belonging to the lexicographically earliest saturated chain of $[u, v]_r$ according to λ. For any $u \in P$ and any choice of root r, define the first atom of $[u, \hat{1}]_r$ in our chain-atom ordering to be the unique atom belonging to the lexicographically first saturated chain of $[u, \hat{1}]_r$ with respect to λ. Denote this atom as a_1. Now assuming we have already chosen atoms $a_1, a_2, \ldots, a_i$ of $[u, \hat{1}]_r$ for some $i \geqslant 1$, choose a_{i+1} as follows. Among the saturated chains of $[u, \hat{1}]_r$ that do not include any of the atoms $a_1, a_2, \ldots, a_i$, choose one that is as small as possible in lexicographic order. There may be more than one choice, but pick any such saturated chain. Let a_{i+1} be the atom of $[u, \hat{1}]_r$ belonging to this chosen saturated chain. Continuing in this manner, we specify a total order $\Omega = a_1, \ldots, a_t$ on the atoms of $[u, \hat{1}]_r$. By construction, Ω is compatible with λ by virtue of λ being self-consistent.

We will prove that Ω is a generalized recursive atom ordering (GRAO), doing so with a proof by induction on the length of the longest saturated chain of P. For P of length 2, our base case, any ordering on the atoms of P is a GRAO. Let P be a finite bounded poset whose longest saturated chain is of length $l \geqslant 3$. We assume by induction that the ordering Ω on the atoms of $[a_j, \hat{1}]$ is a GRAO for $[a_j, \hat{1}]$, directly yielding condition (i)(a) of Definition 3.2.

Next we prove that this GRAO on $[a_j, \hat{1}]$ satisfies condition (i)(b) of Definition 3.2. That is, we verify the following assertion for any w covering an element x satisfying $a_j \lessdot x$: if there exists an atom x' of $[a_j, w]$ such that x' is greater than a_i for some atom a_i with $i < j$, then the first atom x_1 of $[a_j, w]$ is greater than an atom $a_{i'}$ with $i' < j$. Suppose this assertion is false. By definition of Ω and by its compatibility with λ, x_1 must be in the saturated chain of $[a_j, \hat{1}]$ with the lexicographically smallest label sequence among those saturated chains of $[a_j, \hat{1}]$ that contain w. Since λ is self-consistent, x_1 is also in the saturated chain of $[a_j, w]$ having the lexicographically smallest label sequence. Let $a_j \lessdot x_1 \lessdot u_1 \lessdot u_2 \lessdot \ldots \lessdot u_n \lessdot w$ be this lexicographically first saturated chain in $[a_j, w]$ according to λ. Since λ is a TCL-labeling, all pairs of consecutive cover relations in $a_j \lessdot x_1 \lessdot u_1 \lessdot \ldots \lessdot u_n \lessdot w$ must be topological ascents. But our assumption that x_1 is not above an earlier atom than a_j implies that a_j is the first atom of $[\hat{0}, x_1]$. Since the GRAO denoted by Ω and the TCL-labeling λ are compatible, this implies that $\hat{0} \lessdot a_j \lessdot x_1$ is a topological ascent with respect to λ. Thus, we have shown that $M = \hat{0} \lessdot a_j \lessdot x_1 \lessdot x_2 \lessdot u_1 \lessdot u_2 \lessdot \cdots \lessdot u_n \lessdot w$ is a topologically ascending saturated chain from $\hat{0}$ to w with respect to the labeling λ.

We now show how our assumption that x_1 is not above an earlier atom than a_j while there is some atom x' for $[a_j, w]$ that is above an earlier atom will imply the existence of another topologically ascending chain for $[\hat{0}, w]$ besides M. Let a_w be the first atom of $[\hat{0}, w]$ with respect to the atom ordering Ω. Since $a_i < x'$ for some $i < j$, we have $a_i < w$ and hence have $a_w \neq a_j$. Since λ is a TCL-labeling compatible with Ω, $(\lambda(\hat{0}, a_w), \lambda(a_w, y))$ is a topological ascent for any y such that $a_w \lessdot y < w$. Let c be the unique topologically ascending chain in $[a_w, w]$. Then the saturated chain M' for $[\hat{0}, w]$ obtained by appending $\hat{0}$ to c is a topologically ascending chain in $[\hat{0}, w]$. Since $M' \neq M$, this contradicts M being the unique topologically ascending chain from $\hat{0}$ to w, completing our proof of condition (i)(b) of Definition 3.2.

Now let us check that condition (ii) of Definition 3.2 holds for Ω. Consider any atoms a_i, a_j with $i < j$ satisfying $a_i < y$ and $a_j < y$ for some element y. Let M be the lexicographically first saturated chain in $[a_j, y]$ and let $a_j \lessdot x$ be the lowest cover relation in M. Let a_y be the first atom of $[\hat{0}, y]$ with respect to the atom ordering Ω. Since a_i comes before a_j in Ω and $a_i < y$, we must have $a_j \neq a_y$. Let M' be the lexicographically first saturated chain in $[a_y, y]$ and let $a_y \lessdot z$ be the lowest cover relation in M'. Since λ is a TCL-labeling compatible with Ω, a_y and z are both in the lexicographically first saturated chain of $[\hat{0}, y]$. Thus, $(\lambda(\hat{0}, a_y), \lambda(a_y, z))$ is a topological ascent, implying M' is a topologically ascending chain from $\hat{0}$ to y. Since M is topologically ascending from a_j to y but is not the unique topologically ascending chain in $[\hat{0}, y]$, M must have a topological descent at $(\lambda(\hat{0}, a_j), \lambda(a_j, x))$. This means there must exist $\hat{0} \lessdot a_k \lessdot u$ such that $u \leqslant x$ and $\hat{0} \lessdot a_k \lessdot u$ belongs to the lexicographically first saturated chain of $[\hat{0}, x]$. Since λ is self-consistent, a_k is in the lexicographically first saturated chain of P that contains an atom of $[\hat{0}, x]$. As a_j and a_k are both atoms of $[\hat{0}, x]$, we conclude that a_k comes before a_j in Ω. This confirms that condition (ii) of Definition 3.2 holds for Ω. □

Theorem 6.4. *Let P be a finite, bounded poset. Then the following are equivalent:*

1. *P admits a recursive atom ordering*
2. *P admits a generalized recursive atom ordering*
3. *P admits a CL-labeling*

4. *P admits a CL-labeling with the UE property*
5. *P admits a self-consistent CC-labeling.*
6. *P admits a CC-labeling with the UE property*
7. *P admits a self-consistent TCL-labeling*
8. *P admits a TCL-labeling with the UE property*

Moreover, all of these implications are proven constructively. That is, for each implication, either a construction is given showing how to construct the latter type of object from the former or else the former type of object is proven also to be the latter type of object.

Proof. The results we cite below for the various implications are all proven constructively.

We proved the equivalence of (1) and (2) in Theorem 4.10. Björner and Wachs proved the equivalence of (1) and (3) in [BW83]. We proved in Lemma 5.5 that every CL-labeling has the UE property, giving the equivalence of (3) and (4).

Next we confirm the equivalence of (2) and (5). Theorem 5.8 followed by Theorem 6.3 gives one direction. The converse is given by Theorem 6.1.

Now we prove the equivalence of (2) and (6). If λ is a CC-labeling with the UE property, then Lemma 5.3 implies λ is a self-consistent CC-labeling. This gives (5) which was already shown to imply (2). For the converse, Remark 6.2 explains how to construct a CC-labeling with the UE property from a GRAO; the justification that this indeed yields a CC-labeling may be found within the proof of Theorem 8.6. It is self-evident from how it is constructed that this CC-labeling has the UE property.

Next we verify the equivalence of (2) and (7). Theorem 6.1 followed by Theorem 5.8 gives the forward direction. The reverse direction is proven in Theorem 6.3.

Finally we prove the equivalence of (7) and (8). To show that (8) implies (7), we use the result from Lemma 5.3 that the UE property implies self-consistency. For the other direction, first use the result already proven above that (7) implies (4). Then use the fact that every CL-labeling is a TCL-labeling (by virtue of how these are defined) to get that (4) implies (8). Combining these implications shows that (7) implies (8). □

We conclude this section with two simple consequences of the above results. They are included because they will both be used in Section 7 in our proof that the uncrossing order is dual CL-shellable.

Corollary 6.5. *Let P be a finite bounded poset with a self-consistent EC-labeling λ. Then λ may be transformed algorithmically into a CL-labeling for P.*

Proof. This is immediate from Theorem 6.4, using the fact that every EC-labeling is a CC-labeling. □

Corollary 6.6. *A finite bounded poset P admits a generalized recursive coatom ordering if and only if it admits a recursive coatom ordering.*

Proof. Simply apply Theorem 4.10 to P^*. □

7. Dual CL-shellability of the uncrossing order

As an application of our earlier results, in this section we prove that the uncrossing order P_n is dual CL-shellable. Interest in the family of uncrossing posets stems from their role as the face posets of naturally arising stratified spaces of planar electrical networks (see e.g. [Lam15]). In [Lam15], Lam proved P_n to be Eulerian and conjectured that the dual to the uncrossing order is lexicographically shellable. Indeed, the uncrossing order was proven dual EC-shellable in [HK21]. Our results clarifying the relationship between CL-shellability and CC-shellability (as well as EC-shellability) will allow us to deduce dual CL-shellability of P_n from the dual EC-shelling of [HK21] for P_n once we prove that the EC-labeling from [HK21] has the UE property.

Let us begin by recalling the definition of P_n. Figure 7.1 depicts two cover relations in P_3 while Figure 7.2 shows all of P_3. These examples may be helpful for understanding both the definition of P_n and how its cover relations are labeled. The poset P_n has a unique minimal element $\hat{0}$. The other elements of P_n are the various complete matchings on a set of $2n$ vertices (called boundary nodes) positioned around a circle and labeled clockwise $1, 2, \ldots, 2n$. By convention, we will typically mark the node labeled 1 (by making it larger than the other nodes) and leave the other nodes unlabeled in our examples. These complete matchings giving rise to the elements of $P_n \setminus \{\hat{0}\}$ may be represented by collections of n strands, with each strand connecting a matched pair of boundary nodes. P_n is graded with the rank of any element other than $\hat{0}$ being one more than the minimal number of strand crossings needed to represent its strand diagram in the plane. One of these strand diagrams u will be covered by another strand diagram v whenever the rank of u is one less than the rank of v and u is obtainable from v by uncrossing a single pair of strands. Thus, a strand diagram u will be less than a strand diagram v in P_n whenever u may be obtained from v by a series of steps, with each step uncrossing a pair of strands. The elements of P_n covering $\hat{0}$ are exactly those strand diagrams with no two strands crossing each other; thus, for C_n the n-th Catalan number, there are exactly C_n such elements covering $\hat{0}$.

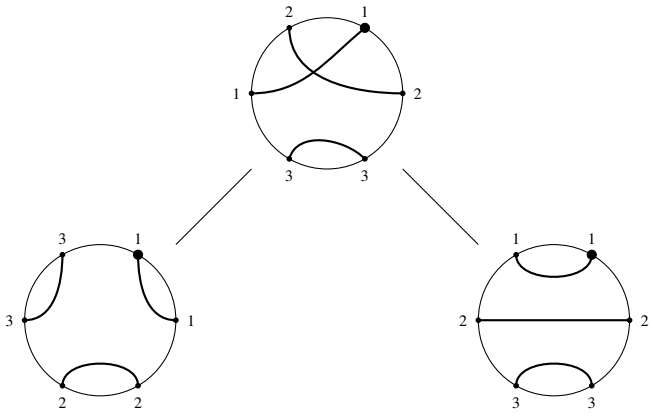

Figure 7.1: Cover relations uncrossing strands in two different ways in P_3.

Next we describe an encoding of the elements of $P_n \setminus \{\hat{0}\}$ that was introduced in [HK21]. This will be used to define the dual EC-labeling for P_n from [HK21]. Again, Figures 7.1 and 7.2 may help with parsing this discussion by illustrating the case with $n = 3$. Start by choosing one of the 2n boundary nodes to treat as a basepoint in each of the strand diagrams of P_n. In Figures 7.1 and 7.2, this base node is enlarged and labeled with a 1. We associate to each strand diagram a sequence of 2n numbers as follows. First we assign the number $1, 2, \ldots, n$ to the n strands by numbering the strands in the order they are first encountered as we proceed clockwise about the circle starting at our basepoint. Next, we label each strand endpoint with the number that has been assigned to its strand. Finally, we proceed clockwise about the circle from the basepoint reading off the sequence of numbers assigned to the $2n$ nodes. This procedure assigns to each strand diagram a sequence of length $2n$ comprised of two copies of each of the numbers from 1 to n. Among such sequences, the ones that arise this way are exactly those in which the first copy of i occurs earlier than the first copy of j for each $i < j$, and this map from $P_n \setminus \{\hat{0}\}$ to the set of such sequences (introduced in [HK21]) is a bijection.

As an example, Figure 7.1 shows the strand diagram giving rise to the sequence 123312 as well as the two strand diagrams it covers that are obtained by uncrossing the pair of strands in the two different possible ways. These two elements it covers give rise to the sequences 112233 and 123321, with the former shown on the left and the latter shown on the right in Figure 7.1. See Figure 7.2 for all of the poset P_3 with the strand diagrams each labeled numerically so that the associated sequence may be read off by proceeding clockwise from the basepoint.

Uncrossing the pair of strands labeled i and j for $i < j$ in a strand diagram will change a sequence having the subsequence $ijij$ into a sequence having either the subsequence $iijj$ or the subsequence $ijji$ (with some of these letters possibly appearing in new positions in the sequence), depending on which of the two possible ways the uncrossing is done. This is discussed in more depth in [HK21], but these further details are not critical to our upcoming proof that the dual EC-labeling of [HK21] (which is also a dual CC-labeling) may be transformed into a dual CL-labeling.

With this encoding in hand, we next describe the dual EC-labeling λ for P_n from [HK21]. Given a cover relation $u \lessdot v$ which replaces subsequence $ijij$ in v with $ijji$ in u, we assign the label $\lambda(u, v) = (i, j)$. Given a cover relation $u \lessdot v$ which replaces $ijij$ in v with $iijj$ in u, we assign the label $\lambda(u, v) = (j, i)$. Label each cover relation which proceeds downward from a strand diagram not having any crossings to the element $\hat{0}$ with the label L. Totally order the label set as follows:

- For $i < j$ we have $(i, j) < L < (j, i)$.
- We make the label (i_1, j_1) smaller than the label (i_2, j_2) for $i_1 < j_1$ and $i_2 < j_2$ if and only if we either have $i_1 < i_2$ or we have $i_1 = i_2$ and $j_1 < j_2$. In other words, we order these labels lexicographically.
- We make the label (j_1, i_1) smaller than the label (j_2, i_2) for $i_1 < j_1$ and $i_2 < j_2$ if and only if we either have $j_1 > j_2$ or we have $j_1 = j_2$ and $i_1 > i_2$. In other words, we order these labels in the reverse order to lexicographic order.

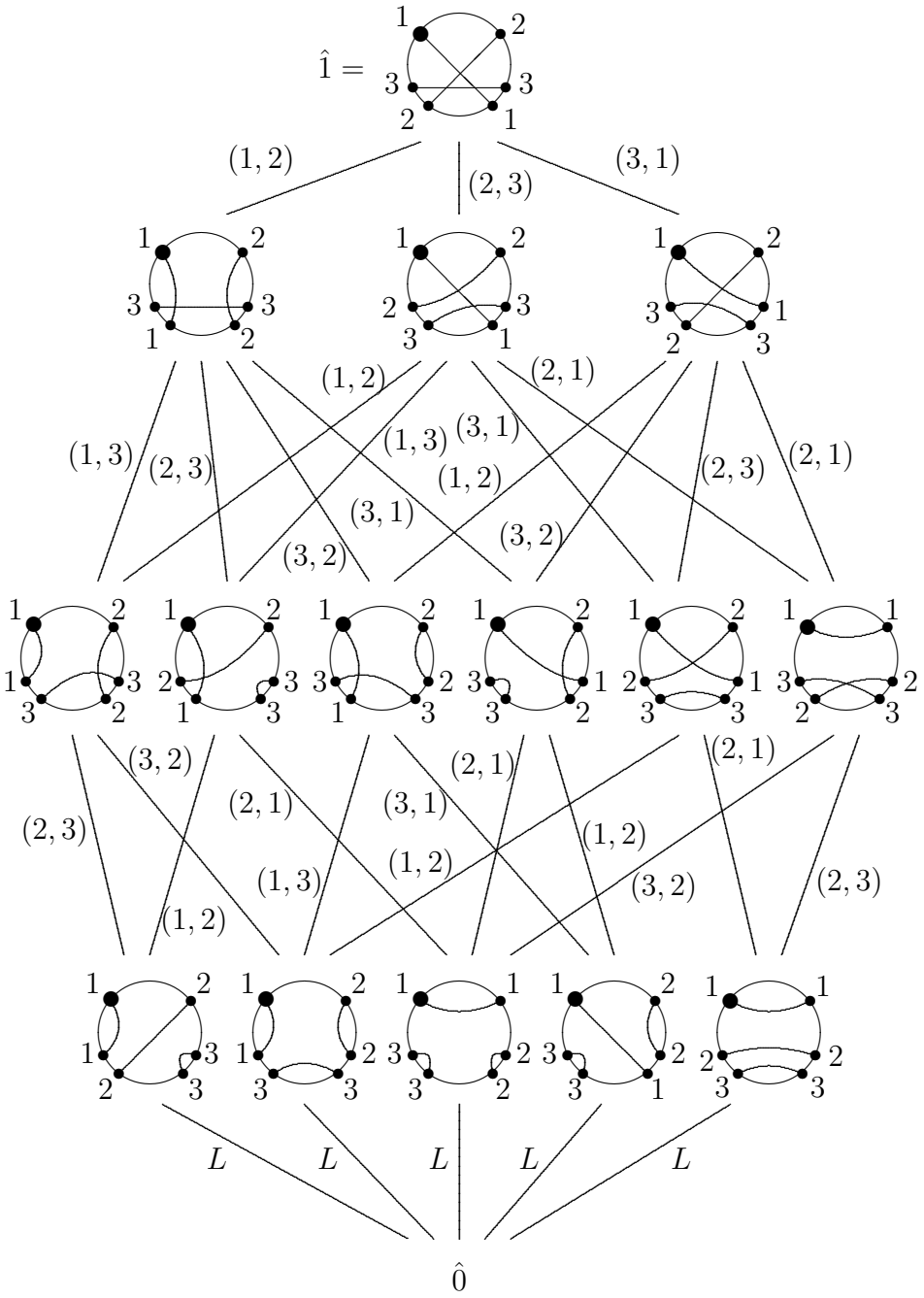

Figure 7.2: P_3 and its edge labeling inducing a shelling.

It is proven in [HK21] that this edge labeling λ is a dual EC-labeling for the uncrossing order P_n (i.e. with label sequences proceeding from top to bottom in P_n).

Theorem 7.1. *The uncrossing poset P_n is dual CL-shellable.*

Proof. By Corollary 6.5, it suffices to show that the dual EC-labeling λ for P_n given in [HK21] is self-consistent. To this end, we will show that the labels on the cover relations downward from a given element $v \in P_n$ are all distinct. This will immediately imply that the resulting edge labeling for P_n^* has the UE property. Lemma 5.3 shows that the UE property implies that the labeling is self-consistent.

Given any fixed element $v \neq \hat{0}$ in P_n and given a label K on an edge down from v, we may determine the unique u for which $\lambda(u, v) = K$. This holds because we can read from the label K the names of the strands being uncrossed as well as which way they are being uncrossed. As these two pieces of information are the only data used to determine u from v when $u \lessdot v$, this means the labels on the cover relations downward from v uniquely determine each u and thus must be distinct. This implies this dual EC-labeling for P_n has the UE property. □

8. Further results, remarks, and open questions

We conclude with some further connections between our work and results in the literature, some open questions, and assorted other remarks.

8.1. Classical EL-labelings and CL-labelings with the UE property

In light of the role of self-consistency in constructing a GRAO or a CL-labeling from a CC-labeling and also given the fact that self-consistency is implied by the UE property (see Lemma 5.3), it is natural to ask how readily checkable the UE property is. We already proved that every CL-labeling (and hence every EL-labeling) is self-consistent in Lemma 5.5. We now show how easy it is to verify the UE property directly for many of the classic examples of EL-labelings and CL-labelings. This together with the proof in Section 7 of the UE property for the EC-labeling for dual uncrossing orders from [HK21] may give readers some sense of how the UE property may well be readily verifiable for many of the sorts of CC-labelings and EC-labelings that people are likely to construct for posets of interest in the future.

Proposition 8.1. *Let L be a geometric lattice and let $a_1, \ldots, a_t$ be any total order on the atoms of L. For each $x \in L$, denote by $A(x)$ the set of atoms $a \in L$ satisfying $a \leqslant x$. Then the EL-labeling λ for geometric lattices which labels each cover relation $u \lessdot v$ with the smallest atom in $A(v) \setminus A(u)$ satisfies the UE property.*

Proof. Notice for each $u \in L$ and any cover relation $u \lessdot v$ with edge label a_i that we must have $v = u \vee a_i$. In particular, v may be recovered from u and $\lambda(u, v)$, implying that the labels on the cover relations upward from u are all distinct from each other. Thus, the smallest such label can only occur once, as needed. □

Proposition 8.2. *Let L be a finite, distributive lattice. Consider the EL-labeling obtained by first interpreting L as the poset $J(P)$ of order ideals in P ordered by containment and then labeling the cover relation $I \lessdot I \cup \{x\}$ with the label x, using any linear extension on P to order the set of edge labels. Then this EL-labeling λ satisfies the UE property.*

Proof. Again the proof is based on observing for the cover relations upward from u that any element $v \in L$ covering u may be determined from knowing u and $\lambda(u, v)$. This implies that the edge labels on cover relations upward from a fixed element u are all distinct. Since the smallest such label only occurs once, this implies the UE property. □

Next we consider an example in which some labels upward from a poset element u occur more than once, but where the smallest upward from u occurs only once:

Proposition 8.3. *Let Π_n be the partition lattice, namely let Π_n be the lattice of set partitions of $\{1, 2, \ldots, n\}$ ordered by refinement. That is, Π_n has $\pi \leqslant \tau$ if and only if π is a refinement of τ or, in other words, each block of π is contained in a block of τ. Consider the EL-labeling λ of Π_n given by*

$$\lambda(\pi, \tau) = \max(\min B_1, \min B_2)$$

where B_1 and B_2 are the two blocks of π being merged to obtain τ. Then λ has the UE property.

Proof. Order the blocks $B_1, \ldots, B_r$ of a set partition π in such a way that the smallest element of B_i is smaller than the smallest element of B_j for $i < j$. By definition of λ, the smallest edge label that is achievable on any cover relation upward from u is $\min B_2$. The only way this label arises is by merging blocks B_1 and B_2. In particular, this smallest label only occurs on one cover relation upward from π. This completes the proof of the UE property for this EL-labeling of Π_n. □

Next we will verify the UE property for the chain-edge labeling λ described next for the dual poset to the Bruhat order of any finite Coxeter group W. Björner and Wachs introduced this chain-edge labeling and proved it was a CL-labeling in [BW82]. In order to work with the dual poset to Bruhat order, we will speak of maximal chains proceeding from top to bottom in Bruhat order, allowing the label on a cover relation $u \lessdot v$ to depend on u, v and the choice of saturated chain downward from $\hat{1}$ to v. We call each saturated chain from $\hat{1}$ to v a co-root. We sometimes denote a cover relation $u \lessdot v$ by $v \to u$ below to highlight the fact that we are proceeding down each cover relation.

Choose a reduced expression $R_{w_0} = s_1 \cdots s_d$ for the longest element, denoted w_0, in W. Label each cover relation $w_0 \to v$ by the position j of the unique letter s_j that may be deleted from R_{w_0} to obtain a reduced expression for v. Let $R_{v,r} = s_1 \cdots s_{j-1} \hat{s}_j s_{j+1} \cdots s_d$ be this reduced expression for v, with r denoting the co-root downward from $\hat{1}$ to v. Continue proceeding down a saturated chain from $\hat{1}$ in Bruhat order, labeling each downward cover relation $y \to x$ that we encounter in turn as follows. Assume inductively that we have chosen a co-root r downward from $\hat{1}$ to $y \in W$, and assume that y and r together have been used to specify a subexpression $R_{y,r} = s_{i_1} \cdots s_{i_k}$ of R_{w_0} that is a reduced expression for y. The set $\{i_1, \ldots, i_k\}$ of indices

in $R_{y,r}$ indicates the positions of the letters in R_{w_0} that appear in $R_{y,r}$; the co-root r uniquely specifies this set of indices. Label the cover relation downward from y to x for our given choice of co-root r with the position i_l within R_{w_0} of the unique letter s_{i_l} that may be deleted from $R_{y,r}$ to obtain a reduced expression for x. Denote by $R_{x,r'}$ this reduced expression $s_{i_1} \cdots s_{i_{l-1}} \hat{s}_{i_l} s_{i_{l+1}} \cdots s_{i_k}$ for x that is determined by x and r' where r' is the co-root $r' := r \cup \{x\}$.

As an example, let W be the symmetric group S_3 generated by simple reflections $a = (1,2)$ and $b = (2,3)$. The cover relation $aba \to ba$ is labeled 1 while the cover relation $ba \to a$ proceeding further down this maximal chain is labeled 2; the latter label reflects the fact that b is the second letter in aba even though it is the first letter in ba. The cover relation $a \to e$ is either labeled 1 or 3, depending on the choice of co-root downward from aba to a.

Proposition 8.4. *Consider the dual CL-labeling for Bruhat order for any finite Coxeter group W that was given by Björner and Wachs in [BW82] and is recalled above. This CL-labeling for the dual poset to Bruhat order has the UE property.*

Proof. Let $s_1 \cdots s_d$ be our chosen reduced expression R_{w_0} for w_0. The label on the cover relation downward from an element v to an element u for a choice of co-root r is the position i_j within $s_1 \cdots s_d$ of the unique letter that may be deleted from $R_{v,r} = s_{i_1} \cdots s_{i_k}$ to obtain a reduced expression for u, as described above. Therefore the label i_j uniquely determines the reduced expression $R_{u,r'} = s_{i_1} \cdots s_{i_{j-1}} \hat{s}_{i_j} s_{i_{j+1}} \cdots s_{i_k}$ that we get by deleting s_{i_j} from $R_{v,r}$, letting $r' = r \cup \{u\}$. One may determine u from v, r, and i_j as the unique Coxeter group element for which $R_{u,r'}$ is an expression. This shows that u is uniquely determined by v, r and the label $\lambda_r(v,u)$. Thus, for any fixed v and r, the value of the label on a cover relation $v \to u$ downward from v uniquely determines this element u that is covered by v. Since all cover relations downward from v for fixed co-root r lead to distinct elements, the labels on these cover relations must therefore be distinct. This implies that this dual CL-labeling has the UE property. □

8.2. Obtaining a self-consistent CC-labeling directly from a GRAO

Our next result shows that one does not need to use the atom reordering process to convert a GRAO to an RAO in order to show that every finite bounded poset with a GRAO has a self-consistent CC-labeling. It shows that alternatively, one may obtain a self-consistent CC-labeling directly from any GRAO.

Definition 8.5. Any CC-labeling λ for a finite bounded poset P gives rise to a total order on the maximal chains of P by ordering the label sequences for the maximal chains lexicographically. On the other hand, any GRAO for this same poset P gives rise to a total order on the maximal chains of P as we describe shortly. Let Γ be such a GRAO. We say that Γ is **consistent** with λ if Γ and λ both give rise to the same total order on the maximal chains of P.

Now we describe the total order on maximal chains induced by Γ. Given any two maximal chains m_1, m_2 of P, let x (resp. x') be the lowest element in m_1 (resp. m_2) that is not in m_2 (resp. m_1). Let u be the element in both m_1 and m_2 that is covered by both x and x'. Denote by r the unique saturated chain from $\hat{0}$ to u with the property that r is contained in both m_1 and m_2. In this case, m_1 comes earlier than m_2 in the total order on maximal chains induced by Γ if and only if x is an earlier atom than x' in the GRAO for $[u, \hat{1}]_r$ induced by Γ.

Theorem 8.6. *Any generalized recursive atom ordering of a finite bounded poset P gives rise to a self-consistent CC-labeling which orders the maximal chains of P consistently with the GRAO.*

Proof. Given a generalized recursive atom ordering (GRAO) for the poset P, construct the following chain-edge labeling λ for P. In the rooted interval $[u, v]_r$, label the cover relation $u \lessdot a_i$ with the label i, where a_i is the ith atom in the GRAO of $[u, v]_r$ that is induced by the GRAO of P as justified in Lemma 3.8.

By construction, λ has the property that distinct cover relations upward from u within $[u, v]_r$ will have distinct labels on them, ensuring that this chain-edge labeling will have the UE property and hence be self-consistent. Also by construction, λ has the following pair of properties required for a CC-labeling (see Definition 2.19):

1. The saturated chains on a rooted interval $[u, v]_r$ have distinct label sequences.

2. No saturated chain in $[u, v]_r$ has as its label sequence a prefix of the label sequence of another saturated chain in $[u, v]_r$.

For any rooted interval $[u, v]_r$, consider any saturated chain M from u to v other than the one which comes earliest with respect to λ for the choice of root r. We will show that M must have a topological descent with respect to the labeling λ, thereby proving that λ is indeed a self-consistent CC-labeling.

Suppose there exists some $u < v$ in P and some root r with a saturated chain $M = u \lessdot u_1 \lessdot u_2 \lessdot \cdots \lessdot u_s \lessdot v$ in $[u, v]_r$ that is not the lexicographically earliest saturated chain of $[u, v]_r$ but where M nonetheless does not have any topological descents. Among all such triples u, v, r, choose one for which the length of the longest saturated chain from u to v is as small as possible. This choice ensures that $u \lessdot u_1$ does not belong to the lexicographically smallest saturated chain in $[u, v]_r$, since otherwise, M restricted to $[u_1, v]_{r \cup u}$ would have the same properties within an interval with strictly shorter longest saturated chain, contradicting the sort of minimality $[u, v]_r$ was chosen to have.

We may deduce that u_1 does not come first amongst the atoms of $[u, v]_r$ in the GRAO, as M is not the lexicographically earliest saturated chain in $[u, v]_r$. Consider first the case where u_2 is the first atom in the GRAO of $[u_1, v]_{r \cup u_1}$. Condition (ii) in Definition 3.2 then ensures that there exists an atom u_1' of $[u, v]_r$ where u_1' comes earlier than u_1 in our GRAO and that there exists some element z such that $u_1 \lessdot z < v$ and $u_1' < z < v$. Lemma 3.9 then guarantees that u_2 must satisfy $u_1'' < u_2$ for some atom u_1'' in $[u, v]_r$ with u_1'' coming earlier than u_1 in the GRAO. This implies that $(\lambda(u, u_1), \lambda(u_1, u_2))$ is a topological descent, contradicting our assumption that M does not have any topological descents.

Next consider the case where u_2 is not the smallest atom in the GRAO of $[u_1, v]_{r \cup u_1}$. Our minimality assumption in choosing $[u, v]_r$ above implies that M restricted to $[u_1, v]_{r \cup u_1}$ must have a topological descent. But this topological descent also gives a topological descent in M itself, again giving a contradiction. This completes the proof that λ is a self-consistent CC-labeling. □

Remark 8.7. The key difference between this construction and our use of a GRAO to construct an RAO (and thereby a CL-labeling and hence a self-consistent CC-labeling) is that this con-

struction we have just given goes directly from a GRAO to a self-consistent CC-labeling without needing to invoke the atom reordering process or construct an RAO as an intermediate step.

8.3. Dual CC-shellability of face lattices of d-complexes

Björner proved in [Bjö84] that shellability of a d-complex is equivalent to dual CL-shellability of its face lattice.

Recall that a **d-complex** is a polyhedral complex whose maximal faces are all of dimension d for a fixed d. The **face lattice** $L(\Delta)$ of a d-complex Δ is the partial order on its cells in which one cell is less than or equal to another if the former is in the closure of the latter, with a unique minimal element $\hat{0}$ representing the empty cell and with an additional maximal element denoted by $\hat{1}$ appended if there is not a unique maximal cell in Δ.

Proposition 8.8. *Shellability of a d-complex Δ is equivalent to $L(\Delta)$ admitting a self-consistent dual CC-labeling. This is also equivalent to $L(\Delta)$ admitting a self-consistent dual TCL-labeling.*

Proof. We use Björner's result from [Bjö84] that a d-complex Δ is shellable if and only if its face lattice $L(\Delta)$ is dual CL-shellable. We combine this with Theorem 6.4 applied to $L(\Delta)^*$, giving the equivalence of CL-shellability to self-consistent TCL-shellability and to self-consistent CC-shellability. □

As an application of Björner's aforementioned result from [Bjö84], we may deduce directly from Theorem 7.1 that the d-complexes having the uncrossing posets as their posets of closure relations are shellable complexes. The recursive nature of the definition of shelling for polyhedral complexes in particular thereby yields shellability of the boundary of the unique maximal cell in these d-complexes coming from uncrossing orders.

8.4. A possible further direction

The notions of CC-shellability and CL-shellability are generalized in [Her03a] from poset order complexes to more general balanced simplicial complexes (and balanced boolean cell complexes). The class of balanced quotient complexes $\Delta(B^{2n})/S_2 \wr S_n$ is proven to be CC-shellable in [Her03a] but is not proven to be CL-shellable.

Question 8.9. It remains open whether self-consistent CC-shellability is equivalent to CL-shellability in the more general setting of balanced complexes.

This could be an interesting avenue for further study, with potential applications to quotient cell complexes such as $\Delta(B^{2n})/S_2 \wr S_n$ (see [Her03b] for non-shellability of $\Delta(B^{kn})/S_k \wr S_n$ for $k > 2$ and see [Her03a] for CC-shellability when $k = 2$).

Acknowledgements

The authors thank the referee for their very helpful feedback.

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