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M. B. Halpern

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UNAMBIGUOUS FIELD-STRENGTH FORMULATION AND DUAL POTENTIALS

FOR GAUGE THEORIES

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ABSTRACT

Using completely-fixed axial-like gauges, I construct the unique inversion $A(F)$ for the potential A in terms of the field strength F . A change of variable to F results in an unambiguous field-strength formulation for any gauge theory; F is constrained to satisfy the Bianchi identities $\partial \tilde{F} - eA(F)\tilde{F} = 0$. Dual potentials $\tilde{A}$ may also be introduced as functional Fourier conjugate to the Bianchi forms. For quantum electrodynamics in four dimensions, duality $[F \leftrightarrow \tilde{F}, A \leftrightarrow \tilde{A}]$ is a perfect symmetry. Residuals of this symmetry persist in all theories however; e. g., $\tilde{A}$ is always canonically conjugate to the magnetic fields.

Let A be a generic potential for a generic gauge theory, and $F(A)$ its field strength. It is generally believed that the inversion $A(F)$ must be non-unique: For Abelian theories there is the certainty of a gauge ambiguity, and for non-Abelian theories there is the problem of field-strength copies.¹ However, in a recent publication,² I noted that field-strength copy is not a

gauge-invariant concept, and I proved that there are no such copies in completely fixed axial-like gauges.^{3,4} In these gauges, $A(F)$ is unique for all gauge theories. The purpose of this note is to report the explicit construction of $A(F)$ for all gauge theories, and the resulting immediate applications: unambiguous reformulation in terms of field-strengths and/or dual potentials. A more detailed paper is also in progress.⁵

The Inversion $A(F)$

The simplest example is the case of quantum chromodynamics⁶ in one space and one time dimension (QCD_2). In axial gauge $A_3 = 0$ (suppressing obvious color indices), all⁷ configurations have field-strength copies of the form $A_0(tz) \rightarrow A_0(tz) + \Delta(t)$, where $\Delta(t)$ is essentially arbitrary. However, there is a residual gauge freedom, which I use to choose the completely-fixed⁸ axial gauge,

$$A_3(tz) = A_0(tz_0) = 0 \quad (1)$$

where z_0 is a particular point in z . Then it is trivial to see that

$$A_0(tz) = -\int_{z_0}^z dz' F_{03}(tz') \quad (2)$$

is the unique inversion $A(F)$. In such gauges, all field-strength copies have become local-action copies.² This inversion is also correct for the Abelian theory (QED_2) in this gauge.

For theories in higher dimension, the algebra is more complicated, but the idea remains the same. Here I will simply state some results.

For QCD₃, in the gauge $A_2(txy) = A_1(txy_0) = A_0(tx_0y_0) = 0$, I obtain $A(F)$:

$$\begin{aligned} A_1(txy) &= - \int_{y_0}^y dy' F_{12}(txy') \\ A_0(txy) &= - \int_{y_0}^y dy' F_{02}(txy') - \int_{x_0}^x dx' F_{01}(tx'y_0). \end{aligned} \quad (3)$$

However, for all cases with dimension ≥ 3 , there is a consistency condition $F\{A(F)\} = F$, which turns out⁵ to be the Bianchi identities.

In Euclidean QCD₃, these are

$$\begin{aligned} I^a_\mu &= \partial_\mu F^a_\mu - e \epsilon^{abc} A^b_\mu (F) F^c_\mu = 0 \\ F^a_\mu &= \frac{1}{2} \epsilon_{\mu\nu\rho} F^a_{\nu\rho} \end{aligned} \quad (4)$$

with $\epsilon_{\mu\nu\rho}$ completely antisymmetric and $\epsilon_{012} = 1$. $A(F)$ is the form given in Eq. (3). Eq. (3) is also the correct inversion for QED₃, with the corresponding simplification in the Bianchi identities.

For QCD₄, in the fixed gauge $A_3(txyz) = A_1(txy z_0) = A_2(tx_0yz_0) = A_0(tx_0y_0z_0) = 0$, I obtain $A(F)$:

$$\begin{aligned} A_1 &= \int_{z_0}^z dz' F_{31}(txyz') \\ A_2 &= - \int_{z_0}^z dz' F_{23}(txyz') + \int_{x_0}^x dx' F_{12}(tx'y_0z_0) \\ A_0 &= - \int_{z_0}^z dz' F_{03}(txyz') - \int_{x_0}^x dx' F_{01}(tx'y_0z_0) \\ &\quad - \int_{y_0}^y dy' F_{02}(tx_0y'z_0). \end{aligned} \quad (5)$$

There is a surprise in four dimensions however, as the consistency conditions are

$$\begin{aligned} I^a_i(txyz) &= \partial_\mu F^a_{\mu i} - e \epsilon^{abc} A^b_\mu (F) F^c_{\mu i} = 0 \quad (i = 0, 1, 2) \\ I^a_3(txyz_0) &= \left\{ \partial_\mu F^a_{\mu 3} - e \epsilon^{abc} A^b_\mu (F) F^c_{\mu 3} \right\}_{z=z_0} = 0 \\ F^a_{\mu\nu} &= \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^a_{\rho\sigma}, \quad \epsilon_{0123} = 1. \end{aligned} \quad (6)$$

I will refer to this fact as the phenomenon of "3.1 Bianchi identities." It can be shown however⁵ that the identities Eq.(6) imply $\partial_3 I^a_3(txyz) = 0$. This is the other ".9". This remark, and the form $A(F)$, is also correct for QED₄ in this gauge.

The pattern holds for all gauge theories in any fixed axial-like gauge: (1) $A(F)$ is linear in F . (2) The consistency conditions are the Bianchi identities. (3) The Bianchi identities are at most quadratic in F .

Field-Strength Formulation

I begin with a generic gauge theory in the form

$$Z \equiv \int \mathcal{D}A \delta[\text{CGF}] \exp \left[- \frac{1}{4} F^2(A) \right] \quad (7)$$

where $\delta[\text{CGF}]$ are products of functional δ -functions for the complete gauge-fixing. The unique inversion $A(F)$ makes it possible to change variables to the field-strengths themselves. The crucial identity⁵ is (up to multiplicative constants)

$$\int \mathcal{D}A \delta[\text{CGF}] \delta[G - F(A)] = \delta[I(G)], \quad (8)$$

Here $I(G) = \partial \tilde{G} - e A(G) \tilde{G}$ are the Bianchi identities as a function of G (which "is" F). In four dimensions, I have chosen, for the

sake of symmetry, to include the last ".9" Bianchi identity by hand.

(Formally, it is an extra constant factor $\delta[0]$, by virtue of the first 3.1 Bianchi identities). Starting with Z , inserting unity in the form $1 = \int \mathcal{D}G \delta[G - F(A)]$, and using Eq. (8), we find immediately

$$Z = \int \mathcal{D}G \delta[I(G)] \exp \left\{ -\frac{1}{4} \int G^2 \right\} \quad (9)$$

The form Eq. (9) provides an unambiguous field-strength formulation for any gauge theory: The field strengths are integrated over the Bianchi identities. There is no further operator measure. The Abelian theories are quite local; the non-Abelian theories are local in time (except in fixed A_0 gauges). I will continue to use G instead of F when it is used as a fundamental variable.

Any function of A in the original formulation translates immediately to that function of $A(F) \rightarrow A(G)$ in the field-strength formulation. In particular, for Abelian theories

$$W[C] \equiv \exp \left\{ -ie \oint_C A \cdot dx \right\} \rightarrow \exp \left\{ -\frac{ie}{2} \iint_{S(C)} dS_{\mu\nu} \tilde{G}_{\mu\nu} \right\}. \quad (10)$$

In fact, similar forms result for the non-Abelian case as well. I mention also that violation of the Bianchi identities $\delta[I(G)] \rightarrow \delta[I(G) + J]$ provides, at least for Abelian theories, an elegant approach to monopoles and monopole quantization.⁵

Dual Potentials

Having the field-strength formulation, a path to dual potentials is clear. Write

$$\delta[I(G)] = \int \mathcal{D}\tilde{A} \exp \left\{ -i \int \tilde{A} I(G) \right\}. \quad (11)$$

The variables $\tilde{A}$ have the indices of the Bianchi forms [so $\tilde{A}_\mu^a$ for QCD_4 , $\tilde{\phi}^a$ (magnetic scalar potential) for QCD_3 and so on],

and are identifiable as dual potentials. E. g. in QED_4 , the saddle point equations include⁹

$$\tilde{G}_{\mu\nu} = i F_{\mu\nu}(\tilde{A}) \equiv i(\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu) \quad (12)$$

This is precisely what is expected of a dual potential.

Note also the interpretation of the (Abelian) dual potentials in terms of monopoles: Inserting a factor $\exp\{i\oint \tilde{\phi}(\underline{y})\}$ in the Z for QED_3 is identical to taking $J(\underline{x}) = -g \delta^{(3)}(\underline{x} - \underline{y})$ in the violation of the Bianchi identities. Thus, summing $\tilde{\phi}$ emissions at $\underline{y}$ with strength g is equivalent to having a monopole g at $\underline{y}$. Similarly, a factor

$$W[C] = \exp \left\{ -ig \oint_C \tilde{A}_\mu dx_\mu \right\} \quad (13)$$

in QED_4 corresponds to a monopole-anti-monopole annihilation around C .

Because the Bianchi identities are at most quadratic in G , one can if desired, integrate out the field-strengths, to formulate the theories entirely in terms of the dual potentials. For QED_4 , the resulting action density is particularly elegant

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}(\tilde{A}) F_{\mu\nu}(\tilde{A}) \quad (14)$$

In this sense, the theory is perfectly dual.

For QCD_4 , the equations corresponding to Eqs. (12) and (14) are

$$\begin{aligned} [1 + ie \int \tilde{A} G]_{\mu\nu}^a &= i F_{\mu\nu}^{(0)a}(\tilde{A}) \\ \mathcal{L} &= -\frac{1}{4} F^{(0)}(\tilde{A}) [1 + ie \int \tilde{A} G^{-1} F^{(0)}(\tilde{A})] \end{aligned} \quad (15)$$

where $F_{\mu\nu}^{(0)a}(\tilde{A}) \equiv \partial_\mu \tilde{A}_\nu^a - \partial_\nu \tilde{A}_\mu^a$, and $[1 + ie \int \tilde{A} G]$ is an easily-derived integral operator, linear in $\tilde{A}_\mu^a$. There is also a measure

$\{\det[1 + ie \hat{A}]\}^{-\frac{1}{2}}$. Although for small coupling, one still verifies $G_{\mu\nu}^a \sim iF_{\mu\nu}^{(0)a}(\hat{A})$, attention is drawn to the zeroes of the operator $[1 + ie \int \hat{A}]$. The inverse operator is not singular for real $\hat{A}_\mu^a$, so the Euclidean theory is well defined.¹⁰ However, the inverse may be singular at certain saddle points, and will tend toward singularity at large coupling. In that regime, these singularities may be enhanced, and the matter deserves further investigation.

Canonical Formulation

I can pass quickly to a canonical form, by integrating out $E_i = G_{0i}$ in Minkowski space, leaving $\hat{A}$ and B , the magnetic fields, as variables. The resulting Hamiltonia are all of the form

$$H = \frac{1}{2} \int \{B^2 + E^2(\hat{A}, B)\} \quad (16)$$

and $B, \hat{A}$ are canonically conjugate. Here I list some results.

$$\text{QED}_4: \quad \underline{E} = \nabla \times \underline{\hat{A}}, \quad [B_i(\underline{x}), \hat{A}_j(\underline{y})] = i\delta_{ij}\delta^{(3)}(\underline{x} - \underline{y}). \quad (17)$$

This Hamiltonian must be taken with the constraint $I_0(G) = I_0(B) = \nabla \cdot \underline{B} = 0$ (I chose not to introduce $\hat{A}_0$, leaving the fourth component of the Bianchi identity as a constraint). Note that $I_0 = \nabla \cdot \underline{B}$ is the generator of gauge transformations for $\hat{A}$, so duality

$$\underline{B} \leftrightarrow \underline{E}, \quad \underline{A} \leftrightarrow \underline{\hat{A}}, \quad \nabla \cdot \underline{E} \leftrightarrow \nabla \cdot \underline{B} \quad (18)$$

is quite perfect. $\hat{A}$ gauges may be fixed in the usual ways.

Because $\hat{A}_i$ is canonical to B_i , it is not hard to verify that

$$\begin{aligned} \hat{W}[C]W[C] &= W[C]\hat{W}[C']\exp\{ie\int N\} \\ N &= \iint_{S(C)} d\underline{S}(\underline{x}) \cdot \oint_{C'} d\underline{x}' \delta^{(3)}(\underline{x} - \underline{x}') \end{aligned} \quad (19)$$

where $W, \hat{W}$ are defined in Eqs. (10) and (13). The contours C, C' are spatial. $\hat{W}[C']$ is thus the Abelian analogue of the local operator 't Hooft defines implicitly¹¹

$$\begin{aligned} \text{QCD}_3: \quad E_1^a &= \partial_2 \hat{\phi}^a - e\delta(y-y_0)\{\theta(x-x_0)\int_x^\infty dx' - \theta(x_0-x)\int_{-\infty}^x dx'\} \int_{-\infty}^{+\infty} dy' g^a(x'y') \\ E_2^a &= -(\partial_1 \hat{\phi}^a - e\epsilon^{abc}A_1^b(B)\hat{\phi}^c) - e\{\theta(y-y_0)\int_y^\infty dy' - \theta(y_0-y)\int_{-\infty}^y dy'\} g^a(xy') \\ [B^a(\underline{x}), \hat{\phi}^b(\underline{y})] &= i\delta^{ab}\delta^{(2)}(\underline{x} - \underline{y}). \end{aligned} \quad (20)$$

Here $g^a = \epsilon^{abc}B^b\gamma^c$ is a rotation operator for $\hat{\phi}^a$ and B^a , and $A_1^b(B) = A_1^b(F_{12})$ is that form in Eq. (3). Notice that the magnetic scalar potential $\hat{\phi}^a$ has a Higgs-like coupling to the magnetic field. But for the g^a terms, and with the proper stability conditions, one might then expect a Meissner effect. Notice also the term proportional to $\delta(y-y_0)$ in E_1^a . It is not clear whether this is a divergence (perhaps related to that of Chodos⁴) or the statement that $\hat{\phi}^a$ is always discontinuous at y_0 . (Certainly, for any regular $\underline{E}^a, B^a, \hat{\phi}^a$ is discontinuous at y_0). The Higgs structure and the singular terms can also be seen in the equations of motion.

QCD₄: I treat the theory as discussed above for QED₄, with the result

$$\begin{aligned} E_i^a &= A_i^a + J_i^a \\ A_i^a &= \epsilon_{ijk}\{\partial_j \hat{A}_k^a - e\epsilon^{abc}A_j^b(B)\hat{A}_k^c\} \\ J_1^a &= -e\delta(z-z_0)\{\theta(x-x_0)\int_x^\infty dx' - \theta(x_0-x)\int_{-\infty}^x dx'\} \int_{-\infty}^{+\infty} dz' g^a(x'yz') \\ J_2^a &= -e\delta(z-z_0)\delta(x-x_0)\{\theta(y-y_0)\int_y^\infty dy' - \theta(y_0-y)\int_{-\infty}^y dy'\} \\ &\quad \cdot \int_{-\infty}^{+\infty} dz' \int_{-\infty}^{+\infty} dz'' g^a(x'y'z'') \\ J_3^a &= -e\{\theta(z-z_0)\int_z^\infty dz' - \theta(z_0-z)\int_{-\infty}^z dz'\} g^a(xyz') \\ [B_i^a(\underline{x}), \hat{A}_j^b(\underline{y})] &= i\delta^{ab}\delta_{ij}\delta^{(3)}(\underline{x} - \underline{y}), \quad g^a = \epsilon^{abc}B_i^b\gamma_i^c. \end{aligned} \quad (21)$$

and $A_j^b(B)$ are those forms given in Eq. (5). As with QED₄, this Hamiltonian must be taken with the constraint $I_0^a = \partial_i B_i^a - e^{abc} A_i^b(B) B_i^c = 0$. Curiously, this fourth component of the Bianchi identities does (at least classically) generate Abelian symmetries of the Hamiltonian: $\frac{d}{dt} I_0^a|_{t=0} = 0$ if $I_0^a(t=0) = 0$. I_0^a acts entirely on A_i^a , and the symmetry is intimately related to the "3.1 Bianchi identity" phenomenon [more precisely, the fact that I chose to complete to all four Bianchi identities in Eq. (8).⁵]

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FOOTNOTES AND REFERENCES

1. T. W. Wu and C. N. Yang, Phys. Rev. D12, 3843 (1975).
2. M. B. Halpern, "Field Strength Copies and Local Action Copies in Quantum Chromodynamics". To be published in Nuclear Physics.
3. The field-copies have become local-action copies. See Ref. 2.
4. Fixed axial gauges have been studied independently and for different purposes by A. Chodos. Phys. Rev. D17, 2624, (1978).
5. M. B. Halpern. To be published.
6. My notation for the non-Abelian theories is $F_{\mu\nu}^a(A) \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - e\epsilon^{abc} A_\mu^b A_\nu^c$, with the obvious simplification in the Abelian cases. I work in Euclidean space until I discuss canonical quantization.
7. M. B. Halpern, Phys. Rev. D16, 1798 (1977).

8. Strictly speaking, I am leaving a global (constant) gauge freedom. This too can be fixed, but it is not relevant to my discussion.
9. The saddle points are thus at purely imaginary λ_μ . This is to be expected in Euclidean space. See, e. g., Ref. 7.
10. Not so the Minkowski theory: In an attempt to integrate out the field-strengths, one runs head-on into these singularities.
11. G. 't Hooft. "On the Phase Transition Towards Permanent Quark Confinement". To be published.

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