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Extensions in Model-Based System Analysis

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy

in

Engineering Sciences (Mechanical Engineering)

by

Matthew R. Graham

Committee in charge:

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2007

The dissertation of Matthew R. Graham is approved, and it is acceptable in quality and form for publication on microfilm:

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2007

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M.R. Graham, M.C. de Oliveira, and R.A. de Callafon, "An alternative Kalman-Yakubovich-Popov lemma and some extensions." *IEEE Transactions on Automatic Control*, Submitted, February 2007. (Chapters 3, 4, and 5)

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M.R Graham and R.A. de Callafon. *"Identification and Low-Order Control of Hard Disk Drives."* IEEE Transactions on Magnetics: INTERMAG Special Edition, 2006.

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ABSTRACT OF THE DISSERTATION

Extensions in Model-Based System Analysis

by

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Doctor of Philosophy in Engineering Sciences (Mechanical Engineering)

University of California, San Diego, 2007

Professor Raymond de Callafon, Chair
Professor Mauricio de Oliveira, Co-Chair

Model-based system analysis techniques provide a means for determining desired system performance prior to actual implementation. In addition to specifying desired performance, model-based analysis techniques require mathematical descriptions that characterize relevant behavior of the system. The developments of this dissertation give extended formulations for control-relevant model estimation as well as model-based analysis conditions for performance requirements specified as frequency domain inequalities.

A model estimation algorithm is proposed on the basis of identifying approximately normalized coprime factorizations from closed-loop system input-output measurements. In the proposed method a particular model structure is chosen such that a linear regression in the measurement data is formed, thus sanctioning the use of numerically efficient algorithms in computing the parameter estimates. Furthermore, methods based on closed-loop experimental data support model estimates that are accurate in the frequency region relevant for control design.

Specifications for performance and robustness of dynamical systems are commonly expressed in terms of frequency domain inequalities, which due to infinite dimensionality are not directly tractable for analysis and design. A pair of linear matrix inequality conditions are proposed that relate to checking frequency

domain inequalities over finite frequency intervals. The proposed conditions introduce an alternative formulation of the Kalman-Yakubovich-Popov Lemma in that the proposed conditions encompasses the lemma for the case when the coefficient matrix of the frequency domain inequality does not depend on frequency. Furthermore, the coefficient matrix can be made affine on the frequency variable at no extra computational cost, which can be significant in reducing conservatism in applications such as robustness analysis.

Extensions for the proposed conditions are developed via transformations on the frequency variable. The extensions have many applications including positivity analysis for matrix polynomials, robustness analysis of discrete-time linear systems, as well as analysis conditions on finite frequency intervals that transform to infinite intervals. Application of the alternative conditions and extensions are illustrated with numerical examples in the analysis of robustness.

Chapter 1

Modeling and Analysis

1.1 Introduction

Addressing the problem of designing control algorithms generally requires a sequence of steps in order to obtain a desired closed-loop (controlled) performance level. The first fundamental step in this problem involves the construction of a mathematical model for the physical system based on the laws of physics or the model's ability to describe a set of input-output data. The second fundamental step is the control algorithm design. Common with both of the fundamental steps for model-based control design, is the analysis of model and controller for qualities that enable them achieve a desired level of performance.

Models derived from the laws of physical are termed a first-principles models, since they are based mostly on the conservation of mass, energy, and momentum. Models derived from input-output data are termed identified or estimated models, since parameters are specified within a dynamical characterization and identified or estimated by fitting to the data. Interestingly, the parameters in a dynamical model generally do not correspond to physical quantities of the system and are merely chosen so as to provide a description of the input-output data. Although there exist methods for deriving models with nonlinear elements, both from first-principles and identification techniques, the results developed in this thesis will primarily focus on the large class of systems that can be adequately characterized with linear models.

There are many challenges associated with designing feedback controllers that make the closed-loop system stable as well as achieve a specified level of performance. Performance is commonly specified in terms of minimizing the unavoidable presence of external noise and disturbances that get introduced into the system through sensors and interaction with the environment. The presence of uncertainty between the physical system and the mathematical model describing it, labeled here as model uncertainty, presents challenges for the model-based control design in actually achieving the designed performance on the actual system. Model uncertainty is associated to the formulation of the model and can include approximation errors brought about by limitations on the *model structure*, *parametric* errors or even confidence intervals associated to the parameter estimation algorithm, and parameter perturbations inherent to the actual system.

Specifying the model and controller qualities that enable performance is largely the result of engineering experience in specifying appropriate cost functions for the design process. However, once a choice for the cost function has been made, the analysis tools often introduce some degree of conservatism in determining performance. As uncertainties limit achievable performance of the model-based controller, conservative analysis tools used in the control design process report limits on performance that are cautious with respect to actual limits on performance.

Furthermore, analysis is often a first step toward synthesis of controllers that by construction satisfy a desired level of performance. Reducing conservatism in the analysis is crucial for designing systems and controllers that achieve limits of performance without intentionally adding caution into the process. The feedback control algorithm design issue is not directly treated in this thesis, however it does contribute to the motivation for reducing conservatism in the analysis of systems with the intension of using these results for design purposes.

1.2   Contributions of Dissertation

The goal of this dissertation is to develop numerically efficient algorithms for model-based analysis of system performance, stability and robustness with respect to uncertainty induced by modeling approximations and parameter intervals. The

developments mainly focuses on analysis tools for systems that can be represented with linear time-invariant models. Models and associated uncertainty characterizations are assumed given from identification or first-principle methods and control algorithms are also assumed given such that an initial performance level for the closed-loop system is satisfied. The contributions of this research include the following.

- Since the analysis tools are based on models, an algorithm is presented for estimating control-relevant models of linear (possibly unstable) plant dynamics. The system identification algorithm of Section 2.3 is based on the theory of coprime factorizations and includes a particular model structure given by Proposition 2.3 that makes the estimation numerically efficient. These models are important for many control applications, particularly those with open-loop unstable dynamics which can occur in servomechanical and aerospace systems.

- Tools for evaluating performance, stability, and robustness specified by frequency domain inequalities are developed. The analysis tools provide conditions which convert frequency domain inequality specifications to convex inequalities which can be evaluated numerically. The convex inequalities incorporate a particular class of frequency-dependent multipliers and can be limited to finite frequency intervals, features which can significantly reduce conservatism as compared to existing conditions with similar complexity. The main results are presented in Theorems 3.1, 3.2, 4.1, and 4.2. These developments are made general in order to accommodate the variety of linear models, model uncertainty characterizations, and performance criteria that can be expressed in terms of frequency domain inequalities.

- Specific extensions of the convex inequalities for system analysis are explored.Properties of the bilinear transformation are used to construct analysis conditions that hold for finite as well as infinite frequency intervals. Additionally these extensions provide analysis conditions for different systems characterizations including discrete-time and polynomial systems.

- Applications of the analysis conditions are used to illustrate the numerical properties of the algorithms in reducing conservatism. The results are used in computing upper bound for the structured singular in order to verify stability and performance robustness. Section 3.5 presents the first application of the finite frequency analysis results, mainly Theorem 3.2, for hard disk drive servomechanism system with structured two-block complex uncertainty. Chapter 5 presents numerical examples of the proposed extension Theorem 4.2, particularly results which hold for infinite frequency range, and applications of structured real-parametric uncertainty.

Before continuing with presenting the results of this dissertation, some motivation will be given to further place the important role that analysis plays in model-based control design.

1.3   Motivation for Performance Analysis Tools

This section is largely devoted to giving perspective and motivation for system analysis with respect to performance and robustness. First Section 1.3.2 presents a model-based control design algorithm, slightly modified from the now common iterative model-based robust control designs, which serves to further motivate performance analysis.   Section 1.4 provides some background on the performance analysis problem.

1.3.1   Model-Based Control Design

Addressing the problem of approximate identification and model-based control generally requires iterative schemes [78] composed of separate model estimation, control design, and analysis steps. Many early model-based iterative schemes focused on nominal $\mathcal{H}_\infty$ performance enhancement of a closed-loop transfer function, denoted by $T(G, K)$ indicating dependence on a plant model G and controller K, while subsequent degradation of achieved performance of the controller K applied to the plant G_0 is evaluated via the triangle inequality [85]. A requirement that performance degradation should be small places emphasis on a closed-loop relevant

identification error and implies the need for performance robustness in both the identification and control design [92]. Model-based iterative schemes have been developed such that evaluation of achieved performance is bounded by the worst possible performance of a controller K evaluated over a set of models $\mathcal{G}$. Performance robustness can then be monitored at each stage of the iteration process such that a monotonic decrease in worst-case performance is guaranteed [17]. This iterative process is illustrated in Fig. 1.1, the elements of which are discussed subsequently.
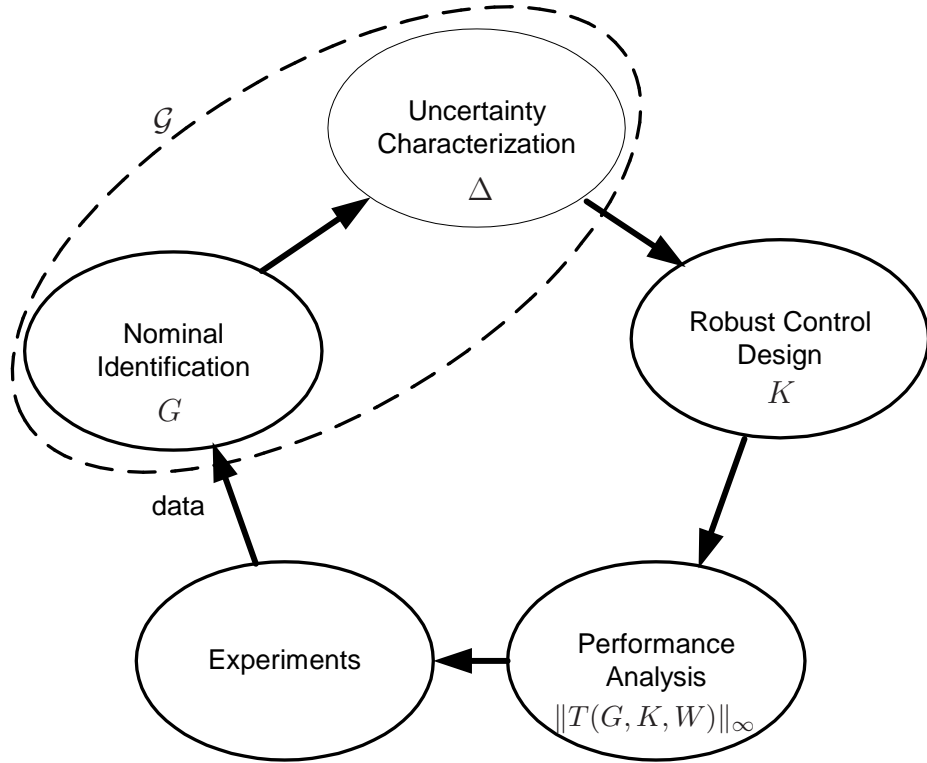
Figure 1.1. Model-based control design and analysis process diagram.

At iteration i consider a set of models $\mathcal{G}_i$ parametrized by a nominal model $\hat{G}_i$ along with an upper bound U_i on the modeling error

$$\mathcal{G}_i(\hat{G}_i, U_i) = \left\{ G \mid G = F(\hat{G}_i, \Delta), \text{ with } \|\Delta U_i\|_\infty < 1 \right\} \quad \text{s. t. } G_0 \in \mathcal{G}_i, \qquad (1.1)$$

where G_0 represents the unknown plant and F denotes a particular parametrization of the model set $\mathcal{G}_i$. Common choices for the parametrization of $\mathcal{G}_i$ with nominal model and norm bounded uncertainty include additive, multiplicative and coprime uncertainty descriptions [94].

The design of a controller K that satisfies performance specifications can be achieved through an explicit parametrization of the closed-loop system $T(G, K, W)$ as a function of W, where W reflects the general notion of a performance weighting function. Consider an initial controller K_i that internally stabilizes all $G \in \mathcal{G}_i$, then we can define a stable closed-loop transfer function $T(G, K_i, W)$ where W is a stable and stably invertible weighting filter such that $\|T(G, K_i, W)\|_\infty$ is bounded. For a given controller K_i and weighting function W the nominal performance and worst case performance are respectively defined by

$$\|T(G, K_i, W)\|_\infty,$$

and

$$\sup_{G \in \mathcal{G}_i} \|T(G, K_i, W)\|_\infty.$$

Consequently, a weighting filter W can be chosen such that

$$\sup_{G \in \mathcal{G}_i} \|T(G, K_i, W)\|_\infty \leq 1 \tag{1.2}$$

and constitutes a robust performance condition.

1.3.2 An Iterative Scheme for Cautious Control Design

Monitoring performance robustness only entails performance improvement if the closed-loop transfer function $T(G, K, W)$ is adjusted appropriately during iterations. One way to incorporate appropriate adjustments is through the explicit parametrization of $T(G, K, W)$ as a function of the performance weight W. Obviously selecting a fixed weighting function W allows a comparison between $\|T(G_0, K_{i+1}, W)\|_\infty$ and $\|T(G_0, K_i, W)\|_\infty$ as a measure of performance [17], whereas adjustment of W during subsequent identification and control design iterations would require a notion of performance improvement dependent on the choice of W [36].

For a given pair (K_i, W_i), the condition (1.1) and (1.2) constitute a modeling or identification problem in which a set of models $\mathcal{G}_i$ needs to be found that satisfies

$$G_0 \in \mathcal{G}_i \quad \text{and} \quad \sup_{G \in \mathcal{G}_i} \|T(G, K_i, W)\|_\infty \leq 1$$

referring to the estimation of an uncertainty set that satisfies a performance robustness condition. Once a set of models $\mathcal{G}_i$ is found, the pair (K_i, W_i) can be updated via a cautious control design that emphasizes performance robustness improvement. Since both K_i and W_i will change to (K_{i+1}, W_{i+1}) the value of the norm function $\|T(G_0, K_{i+1}, W_{i+1})\|_\infty$ does not suffice in characterizing performance robustness improvement. Instead the (normalized) evaluation of the performance robustness

$$\sup_{G \in \mathcal{G}_i} \|T(G, K_{i+1}, W_{i+1})\|_\infty \leq 1 \tag{1.3}$$

indicates that performance robustness has been satisfied for the pair (K_{i+1}, W_{i+1}) while the adjustment of the weighting function W_{i+1} (with respect to W_i) now provides an indication of the performance improvement.

Denote a function $J_{pw}(W)$ as a pseudometric that acts on stable transfer functions (on $\mathcal{RH}_\infty$) to produce a positive real number

$$\{J_{pw}(W): \ \mathcal{RH}_\infty \to [0, \infty)\},$$

where the second argument is the origin $J_{pw}(W) = \Gamma(W, 0)$ or some fixed desired performance weight $J_{pw}(W) = \Gamma(W, W_*)$. Since weighting functions are chosen to reflect design objectives, the evaluation criteria J_{pw} should support this choice and emphasize desired properties in the performance of the closed-loop system, for example bandwidth and disturbance rejection. Thus this framework, although general, still requires some level of engineering intuition to initialize the characterization of performance improvement in J_{pw} which allows the performance weight to be progressively tuned and monitored between iterations. For example

$$J_{pw}(W_{i+1}) \geq J_{pw}(W_i)$$

provides an ordering of the performance weighting functions in an iterative scheme in which both the controller K and the weighting function W are adjusted.

To address the trade-off between performance objectives and a tolerance to uncertainties while utilizing available tools for robust performance control design, the control objective function is restricted to being an $\mathcal{H}_\infty$-norm computation $\|T(G, K, W)\|_\infty$, although alternative objective functions are entirely possible [92]. A method for determining a controller that maximizes performance according to a weighted objective function but subject to robust performance constraints in an $\mathcal{H}_\infty$ framework is formulated below. The problem formulation follows the process diagram shown in Fig. 1.1 initialized with experiments for gathering data and terminated with the achievement of satisfactory system performance.

Problem 1.1. *Let a plant G_0 form a stable feedback connection with the currently implemented controller K_i. From data collected in closed-loop, estimate a set of models $\mathcal{G}_i$ where $G_0 \in \mathcal{G}_i$ and choose weighting function W_i such that*

$$\sup_{G \in \mathcal{G}_i} \|T(G, K_i, W_i)\|_\infty \leq 1 \tag{1.4}$$

Subsequently evaluate the following iterative procedure.

(a) *Given a set of models $\mathcal{G}_i$ design performance weight W_{i+1} and robust controller K_{i+1} to satisfy*

$$(K_{i+1}, W_{i+1}) = \max_{W} J_{pw}(W) \quad \text{such that} \sup_{G \in \mathcal{G}_i} \|T(G, K_{i+1}, W_{i+1})\|_\infty \leq 1, \tag{1.5}$$

where

$$K_{i+1} = \arg \min_{K} \sup_{G \in \mathcal{G}_i} \|T(G, K, W_{i+1})\|_\infty. \tag{1.6}$$

(b) *If the performance weight has improved, that is*

$$J_{pw}(W_{i+1}) > J_{pw}(W_i), \tag{1.7}$$

then implement the controller K_{i+1} and collect (new) data from a closed-loop experiment. Estimate a set of models $\mathcal{G}_{i+1}$ such that $G_0 \in \mathcal{G}_{i+1}$ and

$$\sup_{G \in \mathcal{G}_{i+1}} \|T(G, K_{i+1}, W_{i+1})\|_\infty < \sup_{G \in \mathcal{G}_i} \|T(G, K_{i+1}, W_{i+1})\|_\infty. \tag{1.8}$$

The iterations are terminated when the performance weight can not be improved via control design, that is when (1.7) is not satisfied, or when newly collected data does not provide information that enables reduction in model uncertainty (b).

In case the set of models $\mathcal{G}$ is characterized with a nominal model $\hat{G}$ and upper bound on the model uncertainty U such that $G_0 \in \mathcal{G}$, the formulation of Problem 1.1 generates an iterative sequence between simultaneous model-based control and performance weight synthesis (K, W) and model set identification $(\hat{G}, U)$. For the model-based control design of step (a), the condition $J_{pw}(W_{i+1}) \geq J_{pw}(W_i)$ in (1.5) enforces performance improvement imposed by the metric acting on the performance weighting function, while $\sup_{G \in \mathcal{G}_i} \|T(G, K_{i+1}, W_{i+1})\|_\infty \leq 1$ is a performance robustness condition that guarantees $\|T(G_0, K_{i+1}, W_{i+1})\|_\infty \leq 1$. Solving the argument for K_{i+1} in (1.6) is a standard $\mathcal{H}_\infty$ robust control design problem. For the model set identification of step (b), condition (1.8) considers closed-loop relevant identification of $\mathcal{G}_{i+1}$ such that $G_0 \in \mathcal{G}$ and where the information contained in the new model set improves the robust performance measure under the weighting W_{i+1} and new controller K_{i+1} and constitutes a robust identification problem.

The proposed iterative method in Problem 1.1 readily lends itself to the subset of objectives $\|T(G, K, W)\|_\infty$, uncertainty sets and performance weights that are numerically tractable for which standard tools are available [94]. The works in [16, 36] give examples that discusses the characterization of the model sets as well as simultaneous performance weight and control design which illustrates benefits in progressively adjusting performance weights while maintaining performance robustness. For this dissertation, the purpose of proposing Problem 1.1 is to point out the critical step that performance analysis, that is the evaluation of $\sup_{G \in \mathcal{G}_i} \|T(G, K, W)\|_\infty$, plays in model-based control design, particularly with respect to modeling uncertainties. For instance in Problem 1.1, performance analysis appears during initialization (1.4) as well as in the control design steps (1.5) and (1.6).

1.4 Preview of Performance Analysis Results

Classical control methods have been very effective in dealing with the design and analysis problems, at least when restricted to single input single output (SISO) systems. Even today, many control systems are designed and built following simple procedures that essentially reduce the analysis to dominant second order systems where time-domain specifications are easily translated to frequency-domain specifications, for instance cross-over frequency, gain and phase margins, and others illustrated in Fig. 1.2. Authors such as Bode, Nyquist, and Nichols are attributed to many of the early contributions that have led control theory to the simplicity and insight obtained from working in the frequency domain.
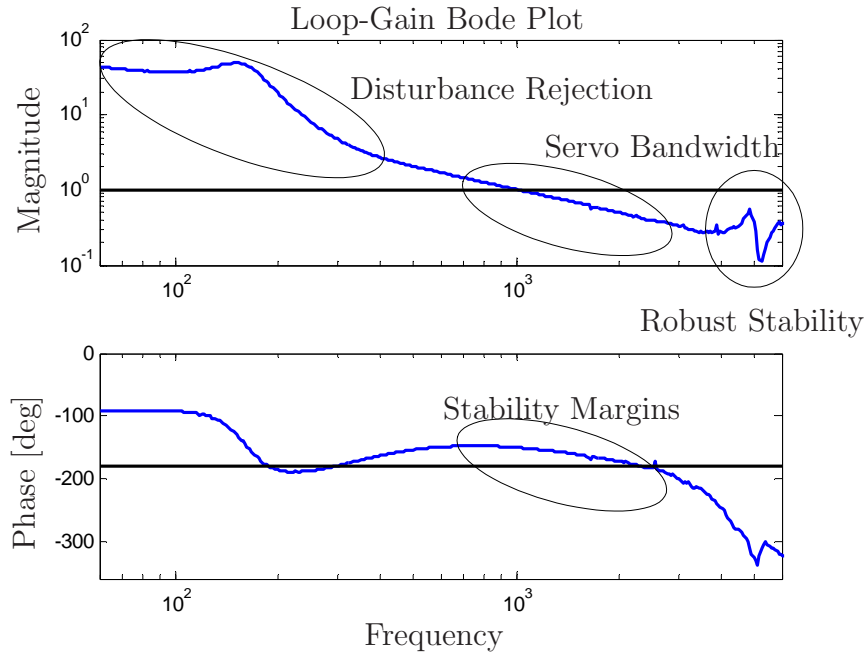
Figure 1.2. Specifications on Loop-Gain Bode Plot.

It is now common practice in systems and control to specify performance and robustness of dynamical systems using the frequency domain. Frequency domain plots, such as Bode plots [57, 28], Nyquist plots [28], or more recently singular value plots [94], convey lots of information about the system under consideration and are popular tools among the systems and control community [65]. Various important

properties in the systems and control community can be characterized by a set of inequality constraints in the frequency domain. For instance, specifications on the $\mathcal{H}_\infty$-norm such as $\|T(G, K, W)\|_\infty < 1$ as in (1.2) can be specified by FDI

$$\begin{bmatrix} T\left(G(j\omega), K(j\omega), W(j\omega)\right) \\ I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} T\left(G(j\omega), K(j\omega), W(j\omega)\right) \\ I \end{bmatrix} \prec 0,$$

for all frequencies $\omega \in \mathbb{R}$. The challenge in working with performance specifications in Frequency Domain Inequality (FDI) form is of course the infinite dimensionality of the problem, which must be specified for every frequency $\omega \in \mathbb{R}$. Indeed, various engineering design problems can be formulated as model-based specifications expressed by inequality constraints in the frequency domain. Conversion of such specifications into analytical conditions suitable to numerically tractable optimization problems remains an important area of research for dynamical systems analysis and design.

In the last decades, thanks mostly to the result known as Kalman-Yakubovich-Popov (KYP) Lemma (see [2, 72, 50] and [39] for a historical essay), FDIs became a major tool in the analysis of dynamic systems. The KYP Lemma, introduced formally in Section 3.2.3, establishes the equivalence between the FDI

$$\begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix} \prec 0, \qquad \text{for all } \omega \in \mathbb{R}, \qquad (1.9)$$

where matrices A, B and the Hermitian matrix Θ of appropriate finite dimensions are given, and the Linear Matrix Inequality (LMI)

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta \prec 0, \qquad (1.10)$$

which should hold for some Hermitian matrix P. The main role of the KYP Lemma is to convert the infinite dimensional inequality (1.9) into the finite dimensional inequality (1.10) where appropriate choices for Θ represent the analysis of various system properties, for example the $\mathcal{H}_\infty$-norm computations in Section 1.3.2. Being an LMI, the set of feasible solutions to inequality (1.10) is convex, and a suitable P can be computed (or proved that none exists) in polynomial time using interior-point methods [9].

Equivalent LMI conditions characterizing FDIs over the entire frequency range can be conservative in most practical applications where specifications of interest are considered only over finite frequency ranges. Including appropriate weighting functions in FDIs has demonstrated usefulness in practice for incorporating requirements over finite frequency intervals. However the process of selecting weights is non-trivial especially when considering the trade off between complexity and accuracy in capturing desired specifications [54], motivating the proposed iterative algorithm in Problem 1.1. Searching simultaneously for a frequency-dependent multiplier while satisfying the FDI over all frequencies has revealed to be a hard problem, see for example the challenges of robust analysis via the *structured singular value* (μ-analysis) [24, 68] and also later discussed in Section 3.4.1. An often used method is the reduction of the search domain by finite but sufficiently dense frequency grid. For some problems this technique is adequate, while for others it is unreliable, particularly in systems with narrow and high peaks in the frequency domain plot since it becomes possible to miss the critical frequency and thus under-evaluate the system response [25].

An approach that avoids both weighting functions and frequency griding is to generalize KYP lemma such that finite frequency intervals can be treated directly. An extension of the KYP Lemma, first proposed in [49], established the equivalence between the FDI

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0, \qquad \text{for all } \omega_1 \leq \omega \leq \omega_2, \qquad (1.11)$$

which is now evaluated only on a finite frequency interval, with the LMI

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1\omega_2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta \prec 0, \qquad (1.12)$$

where $\omega_c := (\omega_1 + \omega_2)/2$. The above LMI should hold for some Hermitian matrix P and some positive semidefinite matrix Q. The readers are referred to [46, 45, 48] for extensive discussions of the features and applications of this result. From a practical perspective, it allows to pose and check frequency domain specifications within a certain frequency range which might be the most relevant to a specific

application. Furthermore, by combining ranges one can pose frequency specifications in different ranges without augmenting the plant with frequency dependent scalings or weights.

The LMI formulation of the KYP Lemma (1.10) provides a constant multiplier relaxation for the FDI, see [75, 76] over the entire frequency axis $\omega \in \mathbb{R}$, that is Θ is constant over the frequency axis. The additional matrix Q that enables finite frequency intervals provides a conditions for constant multiplier relaxations only on that interval. By combining ranges one obtains piecewise constant multiplier relaxations, where Θ are constant coefficient matrices for each interval. For instance, a particular choice of structure for the coefficient matrix Θ in the generalized KYP lemma specifies a μ-analysis problem over finite frequency range using constant scaling matrices [49]. This particular result can also be shown through the losslessness of the scalings used in computing upper bounds for μ [67].

The KYP Lemma has been further generalized in [48], establishing the equivalence between the LMI

$$H^*(\Phi \otimes P + \Psi \otimes Q) H + \Theta \prec 0, \tag{1.13}$$

which should hold for some Hermitian matrix P and some positive semidefinite matrix Q, and the FDI

$$\left(\begin{bmatrix} I & -\xi I \end{bmatrix} H\right)_\perp^* \Theta \left(\begin{bmatrix} I & -\xi I \end{bmatrix} H\right)_\perp \prec 0, \qquad \text{for all } \xi \in \mathbf{\Lambda}(\Phi, \Psi),$$

where $H \in \mathbb{C}^{2n \times (n+m)}$ and

$$\mathbf{\Lambda}(\Phi, \Psi) := \{\xi \in \mathbb{C} : \quad \sigma(\xi, \Phi) = 0, \quad \sigma(\xi, \Psi) \geq 0\}, \quad \sigma(s, \Pi) := \begin{pmatrix} s \\ 1 \end{pmatrix}^* \Pi \begin{pmatrix} s \\ 1 \end{pmatrix}.$$

In the above, $\Phi, \Psi \in \mathbb{HC}^2$ are given matrices satisfying certain conditions (see [48] and Section 4.2) that generalize the LMI condition of (1.12). For instance, by setting

$$H = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}, \qquad \Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Psi = \begin{bmatrix} -1 & j\omega_c \\ -j\omega_c & -\omega_1\omega_2 \end{bmatrix},$$

one recovers the LMI (1.12).

The formulation (1.13) is a generalization of (1.12) in the sense that the frequency variable has been generalized to allow any curve that can be characterized by $\Lambda(\Phi, \Psi)$. However, the coefficient matrix Θ in both (1.12) and (1.13) provide similar constant multiplier relaxations over the finite frequency interval.

1.5   Overview of this thesis

The text of Chapter 2, is mostly a reprint of the material as it appears in [35]. A system identification algorithm is proposed based on a linear regression model parametrization for estimating approximately normalized coprime factorizations. The benefits for this algorithm include the possibility of estimating unstable open-loop plant models operating in stable closed-loop configurations, from which there is a link between modeling and control design objectives. Addition the numerical efficiency of linear regression form in estimating parameters is compared with existing methods. Alternative identification techniques are discussed along with a characterization of model uncertainty for use in model-based system analysis.

The text of Chapters 3 and 4, are mostly a reprint of the material as it appears in [38]. The proposed theory deals with system requirements that are specified in terms of frequency domain inequalities over possibly finite frequency regions. These chapters introduce an alternative formulation for the linear matrix inequality conditions of KYP Lemma, which relates an infinite dimensional frequency domain inequality with a pair of finite dimensional linear matrix inequalities. It is shown that this new formulation encompasses previous generalizations of the KYP Lemma for the case when the coefficient matrix of the frequency domain inequality does not depend on frequency. In addition, it allows the coefficient matrix of the frequency domain inequality to vary affinely with the frequency parameter.

The text of Chapter 5, in part is a reprint of the material as it appears in [37, 38] as well as some material that is not yet published. Here application of the results of Chapters 3 and 4 are discussed, such as stability analysis of continuous-time linear systems on finite as well as infinite frequency intervals, including a new way to allow for frequency-dependent scalings in computing upper bounds to the structured singular value. These applications demonstrate possible reduced conservatism in

the analysis, which is then illustrated with numerical examples.

1.6 Notation

The following notation will be used throughout this thesis. The scalar $j = \sqrt{-1}$. We denote by $\mathbb{C}^{m \times n}$ ($\mathbb{R}^{m \times n}$) the space of rectangular complex (real) matrices of dimension $m \times n$, and by $\mathbb{HC}^n$ ($\mathbb{S}^n$) the space of $\mathbb{C}^{n \times n}$ Hermitian ($\mathbb{R}^{n \times n}$ symmetric) matrices. For a matrix $X \in \mathbb{C}^{n \times n}$: X^{-1} is the inverse. For a matrix $X \in \mathbb{C}^{m \times n}$: $\overline{X}$, X^*, $X_\perp$ are, respectively, the complex-conjugate, the complex-conjugate transpose, and a basis for the null space of X, i.e., a full column rank matrix such that $XX_\perp = 0$ and $\begin{bmatrix} X^T & X_\perp \end{bmatrix}$ has also full column rank. $\text{He}\{X\}$ is short-hand notation for $X + X^*$. We use the symbol $X \succ 0$ ($X \succeq 0$) to denote that $X \in \mathbb{HC}^n$ is positive (semi)definite and $X \prec 0$ ($X \preceq 0$) to denote that $X \in \mathbb{HC}^n$ is negative (semi)definite. The notation $X \otimes Y$ indicates the Kronecker product of X and Y.

Chapter 2

Modeling via System Identification

2.1 Motivation for Control Relevant Modeling

The interconnection between system identification and model-based control design has motivated contributions in the area of "identification for control," whereby identification algorithms are tuned toward the intended purpose of the resulting model i.e. control design. The link between modeling and control is established by considering the difference between the designed performance $\|T(G, K, W)\|_\infty$ and the achieved performance of the controller implemented on the real system [85]. Typically models useful for control design are of low-order, capturing essential closed-loop dynamic behavior. In addition to safety and production requirements common in industrial applications, closed-loop experimental data supports the identification of models accurate in the frequency region relevant for control design [43].

Difficulties in closed-loop identification, mainly due to the correlation between disturbances and control signals in the feedback loop, have inspired numerous methods which can be classified into direct, indirect and joint input-output approaches [63]. Particular to control relevant identification are two-stage, dual-Youla and coprime factor methods where model quality depends upon the compensator used during the experiment. This suggests that control design on the

basis of identified models requires an iterative procedure [82, 78], for which several schemes have been studied, see [17, 53, 16] and [42] for an overview.

Since the focus of this dissertation is on model-based analysis techniques, it seems fitting to include a discussion on the modeling of systems. A formulation of the problem discussed in this chapter is as follows.

Problem 2.1. *Given input-output measurement data obtained from a (possibly unstable) system operating under stable closed-loop (feedback controlled) conditions, estimate models that approximate the plant dynamics. In addition, the model parameters should be determined using computationally efficient model parametrization and identification algorithms.*

The aim of this chapter is to provide tools for the identification step in iterative schemes, such as Problem 1.1 for instance, which utilize the coprime factor representation for systems modeling and control design. Additionally under certain circumstances, for example in highly complex systems, employing simple linear (regression) identification algorithms may be computationally attractive. Here a control relevant coprime factor identification algorithm is presented that relies on a linear regression form, whereby the resulting coprime factors are restricted to be normalized. The algorithm used for estimating approximately normalized coprime factors was introduced in [86]. The proposed linear (regression) identification algorithm is an extension of that work.

2.2 Preliminaries

2.2.1 Access to System Measurements

The closed-loop system considered in this study is shown in Fig. 2.1 where C is a feedback controller that stabilizes the (possibly unstable) Linear Time Invariant (LTI) plant G_0, u is the plant input, y is the plant output, v is an additive disturbance represented at white noise e_0 filtered through a stable, monic and stably invertible H_0, r_1 and r_2 are the possible reference signals available. For convenience

define the general reference signal

$$r(t) := r_1(t) + C(q)r_0(t)$$

as the overall reference to the system, where q denotes the shift operator defined as $qr(t) := r(t+1)$. Note that the signals r_1 and r_0 can also be considered as unmeasurable disturbances if one assumes measurability of u and y and knowledge of the controller C since

$$r(t) = u(t) + C(q)y(t). \tag{2.1}$$

From Fig. 2.1, the system equations can be written as

$$y(t) = G_0(q)S_0(q)r(t) + W_0(q)H_0(q)e_0(t) \tag{2.2}$$

$$u(t) = S_0(q)r(t) - C(q)W_0(q)H_0(q)e_0(t) \tag{2.3}$$

where $S_0 = [1 + CG_0]^{-1}$ and $W_0 = [1 + G_0C]^{-1}$. For brevity the dependency on the delay operator q will be dropped whenever it is clear.
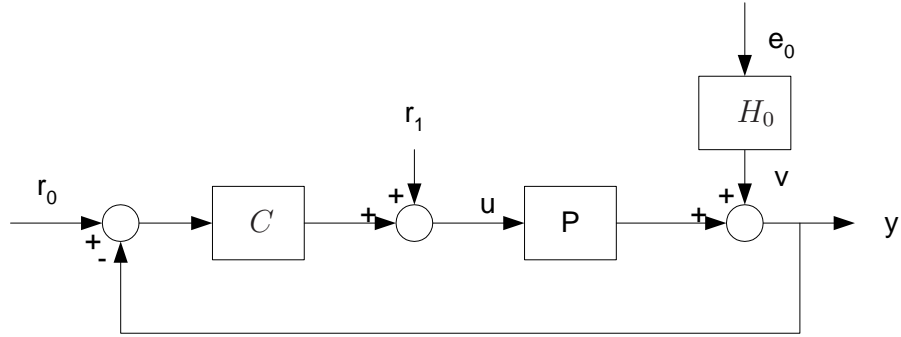
Figure 2.1. General feedback configuration for system identification.

Let the map from inputs $col(r0, r1)$ to outputs $col(y, u)$ be defined as

$$\mathcal{T}(G_0, C) := \begin{bmatrix} G_0 \\ I \end{bmatrix} (I + CG_0)^{-1} \begin{bmatrix} C & I \end{bmatrix},$$

such that $\mathcal{T}(G_0, C)$ describes the feedback system given by (2.2), (2.3) in connection with (2.1). The feedback connection $\mathcal{T}(G_0, C)$ is stable if and only if $\mathcal{T}(G_0, C) \in \mathcal{RH}_\infty$, where $\mathcal{RH}_\infty$ indicates the space of all proper, real rational and stable transfer functions [27].

2.2.2 Access to Coprime Factors from Measurements

Any system G has a *right coprime factorization* (r.c.f) (N, D) over $\mathcal{RH}_\infty$ if there exist $X, Y, N, D \in \mathcal{RH}_\infty$ such that [88]

$$G(z) = N(z)D^{-1}(z); \quad XN + YD = I \tag{2.4}$$

Dual definitions exist for left coprime factorizations and are denoted by $(\tilde{N}, \tilde{D})$. Normalized coprime factors are defined such that

$$N^T(z^{-1})N(z) + D^T(z^{-1})D(z) = I. \tag{2.5}$$

Numerically efficient algorithms for computing continuous-time and discrete-time normalized coprime plant factors can be found in [87], but the problem basically involves solving an appropriate Riccati equation.

The general framework for identification of coprime factors from closed-loop data is well established [79] and allows the flexibility of consistently estimating models from possibly unstable and/or non-minimum phase plants. The method has developed from the dual-Youla system formulation, parametrizing all plants stabilized by a given controller. Consider a stable filter F that generates an auxiliary signal

$$x(t) = F(q)r(t) = F(q)[u(t) + C(q)y(t)]. \tag{2.6}$$

According to Fig. 2.2 then (2.2),(2.3) can be written as

$$y(t) = N_0(q)x(t) + W_0(q)H_0(q)e_0(t) \tag{2.7}$$

$$u(t) = D_0(q)x(t) - C(q)W_0(q)H_0(q)e_0(t) \tag{2.8}$$

where $N_0 = G_0 S_0 F^{-1}$ and $D_0 = S_0 F^{-1}$.

The signal x is uncorrelated with the noise e_0 provided that r_1, r_2 are uncorrelated with e_0 thus the identification from x to $(y, u)^T$ is an open-loop identification of the factors (N_0, D_0) where the plant is constructed as $G_0 = N_0 D_0^{-1}$. For stable (N_0, D_0) and bounded x, the limited freedom in choosing F is summarized by the following proposition.

Proposition 2.1 ([86])**.** *Consider the filter*

$$F = (D_x + CN_x)^{-1} \tag{2.9}$$
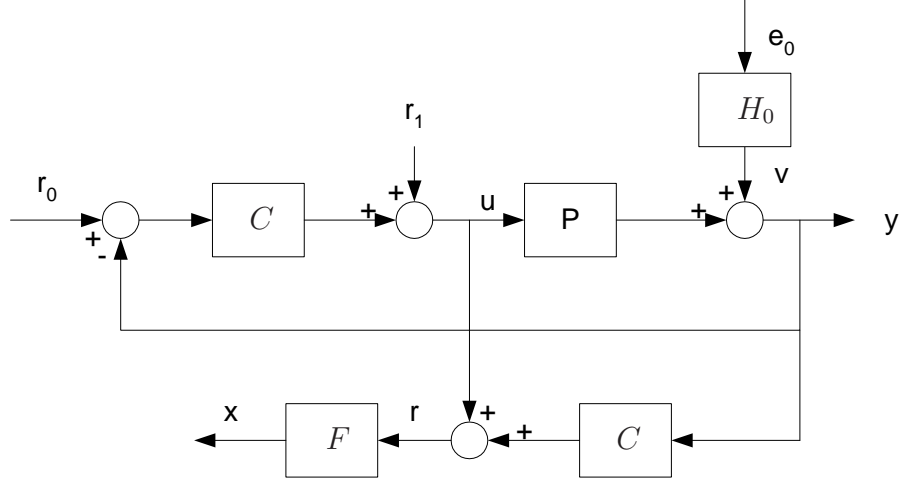
Figure 2.2. Construction of auxiliary signal x from closed-loop data for coprime factor identification.

where (N_x, D_x) are r.c.f. of an auxiliary system G_x, then F provides a stable mapping $(y, u)^T \rightarrow x$ and $x \rightarrow (y, u)^T$ if and only if the auxiliary system G_x is stabilized by C. For all such F the plant factors induced from closed-loop data satisfy

$$\begin{bmatrix} N_0 \\ D_0 \end{bmatrix} = \begin{bmatrix} G_0 \left(I + CG_0\right)^{-1} \left(I + CG_x\right) D_x \\ \left(I + CG_0\right)^{-1} \left(I + CG_x\right) D_x \end{bmatrix} \tag{2.10}$$

where $G_0 = N_0 D_0^{-1}$ is also a right coprime factorization.

The above proposition shows an obvious connection to the well-known dual-Youla parametrization. Since the feedback connection of the auxiliary model G_x with r.c.f. (N_x, D_x) and a controller C with r.c.f. (N_c, D_c) is stable then a system G_0 with r.c.f. (N_0, D_0) that is stabilized in feedback with C can be described by [79]

$$\begin{bmatrix} N_0 \\ D_0 \end{bmatrix} = \begin{bmatrix} N_x + D_c R_0 \\ D_x - N_c R_0 \end{bmatrix} \tag{2.11}$$

if and only if there exists a stable transfer matrix R_0. Additionally the R_0 that satisfies (2.11) is uniquely determined by

$$R_0 = D_c^{-1}(I + CG_0)^{-1}(G_0 - G_x)D_x. \tag{2.12}$$

In fact the dual-Youla parametrization provides all auxiliary models G_x stabilized by C. The freedom in choosing the filter F outlined in Proposition 2.1 suggests that access to the dual-Youla parameter may also be viewed as a natural method for tuning the auxiliary model G_x such that desired coprime factor representations in (2.10) are attained. Before further discussing how this may be used to tune the estimation of coprime factors through exploiting the freedom in choosing F, a brief overview of dual-Youla parameter identification is provided.

Proposition 2.2 ([85])**.** *Consider the data generating plant G_0 with r.c.f. (N_0, D_0) and an auxiliary model G_x with r.c.f. (N_x, D_x) such that both are internally stabilized by controller C with r.c.f. (N_c, D_c). Define the intermediate signal x as given by (2.6),(2.9) and the dual-Youla signal ξ as*

$$\xi(t) = (D_c(q) + G_x(q)N_c(q))^{-1}[I \quad - G_x(q)] \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \tag{2.13}$$

then the identification of the dual-Youla parameter R_0 is given by

$$\xi(t) = R_0(q)x(t) + \bar{H}(q)e(t) \tag{2.14}$$

where the signal x is uncorrelated with e since r in (2.6) is assumed uncorrelated with e and the transfer matrix R_0 is given by (2.12).

Thus the estimation of the dual-Youla parameter is an open-loop identification from the signals x and ξ, which can be constructed from known data filters and solved via standard identification techniques. The disadvantage in applying the dual-Youla parametrization, however, lies in the inability to directly control the order of the resulting model computed via (2.12) [3]. A way to circumvent this problem is a direct estimation of the coprime factors.

2.2.3 Prediction Error Identification

Consider, for the moment, the data generating system depicted in Fig. 2.1 operating in an open-loop configuration, that is the controller $C = 0$, reducing the system equations to

$$y(t) = G_0(q)u(t) + H_0(q)e_0(t) \tag{2.15}$$

where $u(t) = r_1(t)$ simply. Note that in this configuration the plant G_0 is required stable for boundedness of $y(t)$. Consider a model $G(q, \theta)$ for $G_0(q)$ and $H(q, \theta)$ for $H_0(q)$, then the one-step-ahead prediction of the output is given as [63]

$$y(t|t-1, \theta) = H^{-1}(q, \theta)G(q, \theta)u(t) + H^{-1}(q, \theta)[H(q, \theta) - 1]y(t),$$

which is used to obtain the one-step-ahead prediction error as

$$\varepsilon(t, \theta) := y(t) - y(t|t-1, \theta) = H^{-1}(q, \theta)\left[y(t) - G(q, \theta)u(t)\right].$$

Substituting (2.15) into the above yields the prediction error equation specified in terms of plant and noise model mismatch.

$$\varepsilon(t, \theta) = H^{-1}(q, \theta)\left[(G_0(q) - G(q, \theta))u(t) + (H_0(q) - H(q, \theta))e(t)\right] + e(t).$$

Given data up to time $t = N$, the least squares prediction error estimate is obtained by minimizing the square of the filtered prediction error

$$\hat{\theta}_N = \arg\min_\theta \frac{1}{N} \sum_{t=1}^N \varepsilon_F^T(t, \theta)\varepsilon_F(t, \theta),$$

where $\varepsilon_F(t, \theta) = L(q)\varepsilon(t, \theta)$ is a stable and monic filter. In subsequent sections motivation for filtered prediction errors will be discussed further in application, but for now $L(q) = 1$ is assumed for simplicity. Parseval's relation [63], gives the following frequency domain expression

$$\hat{\theta}_N = \arg\min_\theta \frac{1}{2\pi} \int_{-\pi}^\pi \Phi_{\varepsilon,\varepsilon}(\omega, \theta)d\omega, \tag{2.16}$$

where $\Phi_{\varepsilon,\varepsilon}(\omega, \theta)$ denotes the (auto) spectrum of the prediction error defined as

$$\Phi_{\varepsilon,\varepsilon}(\omega, \theta) := \sum_{\tau=-\infty}^\infty E\left\{\varepsilon(t, \theta)\varepsilon(t-\tau, \theta)\right\} e^{-j\tau\omega}.$$

Note that for a simplified system $y(t) = G(q)u(t)$ the output spectrum can be written in terms of the input spectrum [63]

$$\Phi_{y,y} = |G(e^{j\omega})|^2\Phi_{u,u}.$$

According to this, the spectrum of the prediction error can be written in terms of the input $u(t)$ and the white noise $e(t)$ as

$$\Phi_{\varepsilon,\varepsilon}(\omega,\theta) = |H^{-1}(e^{j\omega},\theta)|^2|G_0(e^{j\omega}) - G(e^{j\omega},\theta) + B_\theta(e^{j\omega})|^2\Phi_{u,u}+$$

$$|H^{-1}(e^{j\omega},\theta)|^2|H_0(e^{j\omega}) - H(e^{j\omega},\theta)|^2\left(\lambda - \frac{\Phi_{u,e}(e^{j\omega})}{\Phi_{u,u}(e^{j\omega})}\right),$$

where λ is the variance of the white noise $e(t)$ and

$$B_\theta(e^{j\omega}) = \left(H_0(e^{j\omega}) - H(e^{j\omega},\theta)\right)\frac{\Phi_{u,e}(e^{j\omega})}{\Phi_{u,u}(e^{j\omega})}.$$

Note that in case the identification is performed with data generated in open-loop, the above simplifies greatly since the input is not correlated with the noise, that is $\Phi_{u,e} = 0$. In this case, the parameter estimate can be represented as

$$\hat{\theta} = \arg\min_\theta \int_{-\pi}^\pi |H^{-1}(e^{j\omega},\theta)|^2|G_0(e^{j\omega}) - G(e^{j\omega},\theta)|^2\Phi_{u,u}(e^{j\omega})+$$

$$|H^{-1}(e^{j\omega},\theta)|^2|H_0(e^{j\omega}) - H(e^{j\omega},\theta)|^2\lambda. \quad (2.17)$$

2.3   Linear Regression Algorithm

The estimation of approximately normalized coprime factors in [86] came from the observation that the factors available in (2.10) can be shaped according to the choice of auxiliary model G_x. Stability restrictions on the auxiliary model G_x suggest a dual-Youla parametrization, however instead the estimate R_0 can be used as an initialization step in a direct coprime factor estimation, providing control over the order of the model being estimated. For example, servomechanisms typically have double integrator which would naturally be incorporated into an initial auxiliary model. This leads to the following algorithm similar to [86] with the main difference of using a structured linear regression model to allow for an affine optimization during the estimation of the coprime factors.

(1) *Initialization:*

   (a) Start with an auxiliary model G_x stabilized by C with (normalized) co-prime factors (N_x, D_x). Simulate auxiliary input x (2.6) with data filter

(2.9) and dual-Youla signal ξ according to (2.13) then accurately iden-
tify (possibly high order) dual-Youla parameter $\hat{R}_0$ (2.14) using linear
regression methods.

(b) Update estimated (high order) coprime factors $(\hat{N}_0, \hat{D}_0)$ (2.11) and let
$\hat{G} = \hat{N}_0 \hat{D}_0^{-1}$. Then compute a normalized coprime factorization of the
high-order plant (N_x, D_x) such that $\hat{G} = N_x D_x^{-1}$ and re-simulate auxil-
iary signal x (2.6) with the updated filter (2.9).

(2) *Identification:*

(a) Use signals $[y, u]^T$ and x in least squares multi-variable identification
minimizing the prediction error

$$\varepsilon(t, \theta) = A(q, \theta) \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} - B(q, \theta) x(t) \tag{2.18}$$

where
$$\begin{aligned} A(q, \theta) &= I + A_1 q^{-1} + ... + A_{n_a} q^{-n_a} \\ B(q, \theta) &= B_0 + B_1 q^{-1} + ... + B_{n_b} q^{-n_b} \end{aligned} \tag{2.19}$$

are *matrix polynomials* in q^{-1} and $B(q, \theta)$ can be written with the par-
tition $B = \begin{bmatrix} B_N^T, B_D^T \end{bmatrix}^T$.

(b) Compute the coprime factor estimate via

$$\begin{bmatrix} N(\theta) \\ D(\theta) \end{bmatrix} = A^{-1}(q, \theta) \begin{bmatrix} B_N(q, \theta) \\ B_D(q, \theta) \end{bmatrix}. \tag{2.20}$$

Obviously a general matrix polynomial $A(q, \theta)$ in an ARX model structure
does not provide a common left divisor in the coprime factorization, thus the
McMillan degree of the constructed model $G(\theta) = N(\theta)D^{-1}(\theta)$ is not the same as
the McMillan degree of $B_N(q, \theta)$ or $B_D(q, \theta)$. Additional structure on the matrix
polynomial may be imposed such that $A(q, \theta)$ is a common left divisor preserving
the McMillan degree of the individual coprime factors in the constructed model.

2.3.1 Structured Linear Regression Parametrization for Coprime Factor Identification

As a special case of prediction-error identification methods (PEM), the well-known least squares minimization criteria is a standard choice for its convenience in both computation and analysis [63]. Consider the multi-variable ARX model structure (2.18). Then the system description is given by

$$\begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = G(q,\theta)x(t) + H(q,\theta)e(t) \tag{2.21}$$

with

$$G(q,\theta) = A^{-1}(q,\theta)B(q,\theta), \quad H(q,\theta) = A^{-1}(q,\theta).$$

Recall that (2.18) can be parametrized by a linear regression with prediction error given by

$$\varepsilon(t,\theta) = \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} - \varphi^T(t)\theta \tag{2.22}$$

where additional structure may be imposed on the parametrization such that a d-dimensional column vector θ and a corresponding $n_y + n_u \times d$ matrix $\varphi^T(t)$ containing past input, output and auxiliary signals are used in the least squares minimization. Employing the linear regression prediction error (2.22), the least squares criterion is given by

$$V_N(\theta, Z^N) = \frac{1}{N}\sum_{t=1}^{N}\varepsilon_f^T(t,\theta)\varepsilon_f(t,\theta) \tag{2.23}$$

where $\varepsilon_f(t,\theta) = L(q)\varepsilon(t,\theta)$ with $L = diag(L_y, L_u)$ and $L \in \mathcal{RH}_\infty$. Filtering the prediction error can be made equivalent to filtering the identification input-output data, however in the multivariable case all signals must be subject to the same filter, i.e. L_y and L_u must be multiples of the identity matrix [63]. Different prefilters L_y and L_u account for the difference between the noise models made available from data (2.7), (2.8) where knowledge of the controller can be included into L_u. Prefiltering the input-output data, (2.23) remains quadratic in θ and can be minimized analytically giving

$$\hat{\theta}_N^{LS} = \left[\frac{1}{N}\sum_{t=1}^{N}\varphi(t)\varphi^T(t)\right]^{-1}\frac{1}{N}\sum_{t=1}^{N}\varphi(t)\begin{bmatrix} y(t) \\ u(t) \end{bmatrix}. \tag{2.24}$$

Additional structure imposed on the parametrization of the matrix polynomial $A(q, \theta)$ provides a common left divisor and preserves the McMillan degree of the constructed coprime factors in the constructed model (2.18)-(2.20).

Proposition 2.3. *Consider minimizing the least squares identification criterion (2.23) with the prediction error $\varepsilon(t, \theta)$ in (2.18). Let the matrix polynomial $A(q, \theta)$ be parametrized by*

$$A_i = a_i I_{n_y + n_u} \tag{2.25}$$

for $i = 1, ..., na$. Then the prediction error can be written into linear regression form (2.22) with

$$\theta^T = \begin{bmatrix} a_1...a_{n_a} & col(B_1)^T...col(B_{n_b})^T \end{bmatrix} \tag{2.26}$$

$$\varphi^T(t) = \begin{bmatrix} -y(t-1)... - y(t - n_a) \\ -u(t-1)... - u(t - n_a) \\ x^T(t-1) \otimes I_{n_y}...x^T(t - n_b) \otimes I_{n_y} \end{bmatrix}, \tag{2.27}$$

where the col operator stacks the columns of a matrix.

The parameter estimate $\hat{\theta}_N^{LS}$ from (2.24) with (2.28) can be computed using numerically efficient algorithms for solving equations of the form $z = Y x$, for instance recursive QR-algorithms [34]. Furthermore, increasing the order of $A(q, \theta)$ does not sacrifice the order of the model being estimated. Thus estimating a high order $A(q, \theta)$ which incorporates the noise filter into the estimation and can improve the fit of the coprime factors (see for instance [63] for a discussion on identifying the noise filter). As a result the least squares solution (2.24) provides an estimate of the coprime factorization (2.20) such that $A(q, \theta)$ preserves the McMillan degree of the coprime factors in the constructed model $G(q, \theta) = N(\theta)D(\theta)^{-1}$. For SISO systems the above proposition results in a parametrization

$$\begin{aligned} N(\theta) &= a^{-1}(q)b_N(q) \\ D(\theta) &= a^{-1}(q)b_D(q) \end{aligned} \tag{2.28}$$

where a, b_N and b_D are (matrix) polynomials of specified degree with coefficients collected in the parameter vector θ. For MIMO the diagonal form of $A(q, \theta)$ is equivalent to a common denominator parametrization. By imposing the structure

(2.25), the factorization $(N(\theta), D(\theta))$ has a common denominator which guarantees the McMillan degree of the coprime factorization is the same for the constructed model. Similar results are obtained when performing the least squares identification using an output error model structure [86], however one looses the computational benefits and unique analytic solution (2.24) provided by the linear regression.

2.4 Servomechanical Example

The development of high performance controllers of industrial servomechanical systems with significant product variations may require accurate modeling of each product on the basis of its own closed-loop experiment. Consider the frequency responses of several of the same servomechanism product presented in Fig. 2.3. The product variation between the servomechanisms would require too conservative a controller satisfying stability over all plants. The results presented in this section illustrate the proposed identification method applied to one test bench, see Fig. 2.4, with the intension of using the model in future work to design a high performance model based controller. Typically servomechanisms contains a double integrator, which makes it open-loop unstable. The current feedback loop is stabilized via a proportional integral derivative (PID) controller however the large resonance modes of the plant limit the bandwidth of the closed-loop system.

As a case study to illustrate the proposed identification method consider the frequency response presented in Fig. 2.4. Time series data $u(t)$ and $y(t)$ were obtained via simulation of a 20th order model fitted to the frequency response with input signals r_0 and r_1 chosen as zero mean white noise each with a variance of 1. The measurement noise that enters the system $v = H_0 e_0$ was modeled as zero mean white noise with variance 0.1.

The identification algorithm outlined in Section 2.3 is used to estimate normalized coprime plant factors. Results from the last step are presented in Fig. 2.5, where approximately normalized coprime factor estimates $(N(\theta), D(\theta))$ are obtained from a constrained ARX least squares linear regression identification. Because of the implicit high frequency weighting which results from using an ARX
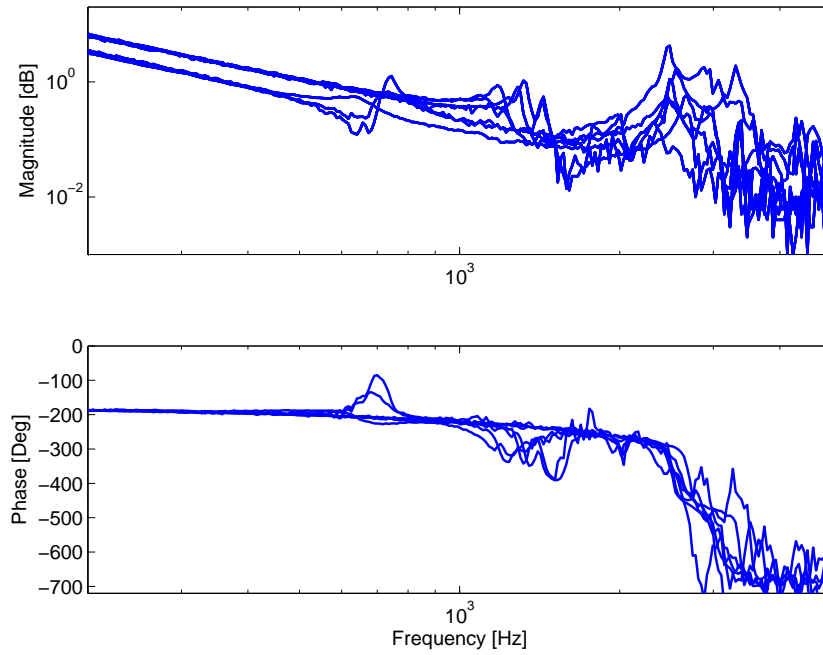
Figure 2.3. Frequency response of 16 of the same servomechanism product.
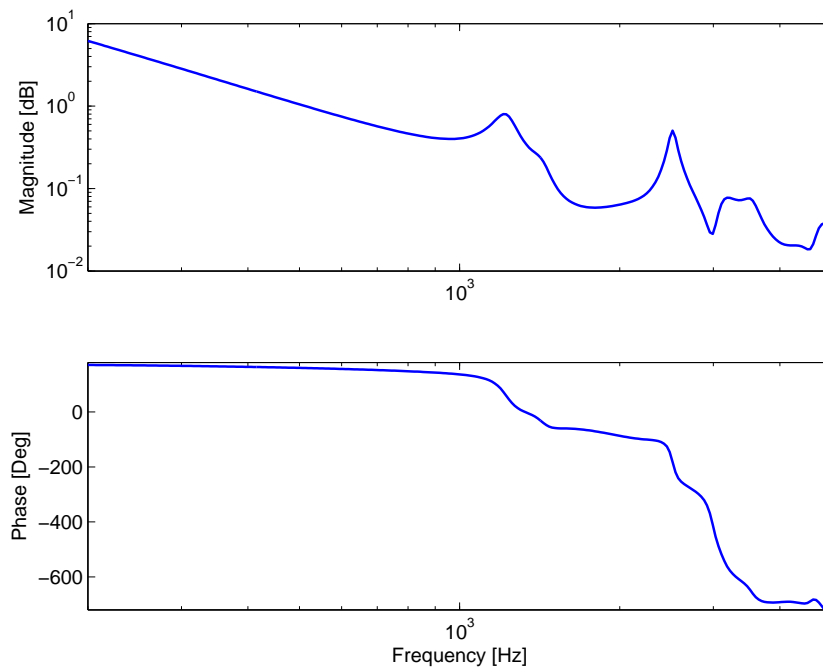
Figure 2.4. Frequency response of a 20th order case study model (solid-line).

model structure, the prediction error prefilter $L(q)$ is initially chosen as a 4th order low-pass butterworth filter with a cut-off at around half the sampling frequency. To improve the quality of the estimated factors the linear regression identification is performed a second time with prefilter chosen according to the Steiglitz-McBride method, $L(q) = A(q, \theta)$, where the prefilter is applied to the original data set. The resulting model constructed from the approximately normalized coprime factors $G(\theta) = N(\theta)D^{-1}(\theta)$ is presented in Fig. 2.6. The computational complexity involved in linear filtering and computing the linear regression identification is less than a non-linear optimization identification of the same order, yet these results show that for this experimental set-up and for sufficiently chosen order the resulting model is comparable. Additional benefits of linear regression identification come from the vast body of research available, i.e. so-called "fast algorithms" and the ability to compute estimates of multiple (high) orders [63].

2.5   Description of the Limit Model

Observe the filtered prediction error under the constrained ARX model structure

$$\varepsilon_f(t, \theta) = L(q)A(q, \theta) \begin{bmatrix} y(t) - N(q, \theta)x(t) \\ u(t) - D(q, \theta)x(t) \end{bmatrix}. \tag{2.29}$$

With fixed noise model given in (2.3.1), the asymptotic parameter estimate defined as $\theta^* := \lim_{N \to \infty} \hat{\theta}_N^{LS}$ is characterized by

$$\theta^* = \arg \min_\theta \int_{-\pi}^{\pi} |L(e^{j\omega})|^2 |A(e^{j\omega})|^2 \times \left\{ |N_0(e^{j\omega}) - N(e^{j\omega}, \theta)|^2 \right.$$

$$\left. + |D_0(e^{j\omega}) - D(e^{j\omega}, \theta)|^2 \right\} \Phi_x(\omega)d\omega,$$

where $\Phi_x(\omega)$ is the frequency spectrum of the auxiliary input signal (2.6). As a result of the ARX model structure chosen the asymptotic parameter estimate includes an implicit high-frequency weighting by $|A(e^{j\omega})|^2$. To enhance the model fit in the desired frequency range the prediction error is filtered through a low-pass filter $L(q)$. If a reasonable assumption is that the measurement errors are white this may also suggest the Steiglitz-McBride method for improving the estimate with a θ-dependent prefilter [63].
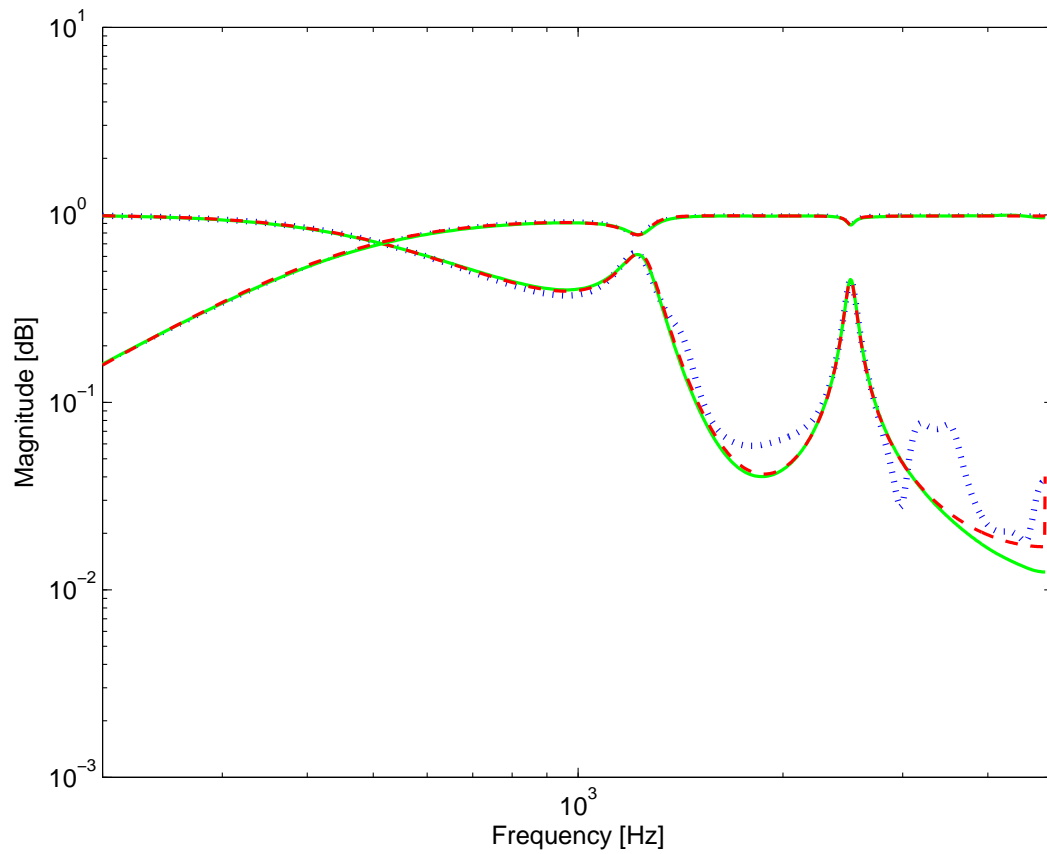
Figure 2.5. Frequency response of coprime factors computed from case-study model (dotted-line) and from 7th order: constrained ARX (solid-line), Prediction Error minimization with OE structure (dashed-line).
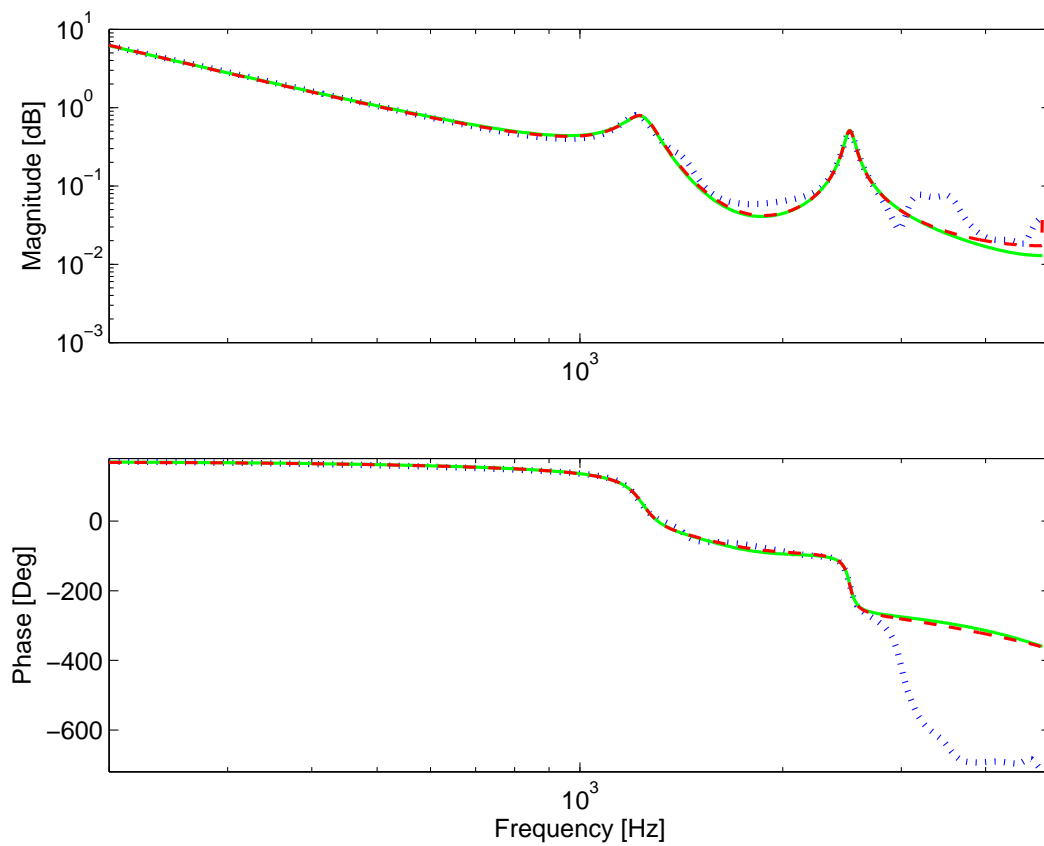
Figure 2.6. Frequency response of case study model (dotted-line) and 7th order constructed plant: $G(\theta) = N(\theta)D^{-1}(\theta)$ (solid-line) and Prediction Error minimization with OE structure (dashed-line).

Note that the identification presented in Section 2.3 assumes linear regression models parametrized in discrete-time. Indeed most identification algorithms using time-domain measurement data have this characteristic [63], although some literature exists for directly estimating continuous-time models [11, 81]. Additionally the algorithm is limited by the imposed structure of the matrix polynomial $A(q)$ in (2.25). These limitations can be overcome through identification directly on frequency domain data, where model parametrizations $G(\theta) = B(q, \theta)A^{-1}(q, \theta)$ are possible in the least squares framework. Indeed frequency domain identification techniques provide a flexible framework for computing model estimates directly from frequency response (spectral analysis) data.

2.6 Frequency Domain Identification

A link between modeling and control can be established by considering the difference between the designed performance $\|T(G, K, W)\|_\infty$ and the achieved performance of the controller implemented on the real system. In order to design an enhanced performing robust controller, it is preferable to estimate a set of models $\mathcal{G}$ for which this difference is minimized with respect to the nominal model G in the cost equation

$$\|T(G_0, K, W) - T(G, K, W)\|_\infty, \tag{2.30}$$

where it is assumed that appropriate signals from the closed-loop system $T(G_0, K)$ are available for measurement. For most weighted closed-loop system characterizations $T(G, K, W)$, the identification of a model G from the control-relevant criteria (2.30) can be written into a weighted additive difference between the actual dynamics G_0 and the nominal model G via

$$\|(G_0 - G)\tilde{W}\|_2 \tag{2.31}$$

motivated by the fact that $\mathcal{L}_2$ approximation tends to $\mathcal{L}_\infty$ approximation [10]. The identification weight $\tilde{W}$ includes the weighting function W as well as closed-loop dynamics from $T(G, K)$.

The weighted additive difference between G_0 and the nominal model G in (2.31) can be written in terms of frequency domain criterion. Frequency domain data (measurements) of various transfer functions can be obtained via spectral analysis [63]. When using experimental frequency domain data $G_0(\omega_k)$, where ω_k, $k = 1, 2, \ldots, N$ refers to a frequency domain grid, the two-norm criterion in (2.31) can be approximated by

$$\sum_{k=1}^{N} |(G_0(\omega_k) - G(\omega_k))\tilde{W}(\omega_k)|^2. \qquad (2.32)$$

For example, consider a control-relevant identification criteria for disturbance rejection, that is in terms of a weighted sensitivity objective function,

$$\|W_S S_0 - W_S S\|_\infty = \|W_S(1 + G_0 K)^{-1} - W_S(1 + GK)^{-1}\|_\infty$$

such that the achieved closed-loop performance of the plant G_0 with the controller K is close to the designed closed-loop performance of the model G with controller K. Identification of a model G from the control-relevant criteria above can be written into a weighted additive difference between G_0 and G via

$$\|W_S(1 + G_0 K)^{-1}(G_0 - G)K(1 + GK)^{-1}\|_2.$$

Note that the above identification criteria can be used in frequency domain identification algorithms with (2.32) where the frequency weighting $W(\omega_k)$ given by

$$\tilde{W}(\omega_k) = \frac{W_S(\omega_k)}{1 + K(\omega_k)G_0(\omega_k)} \cdot \frac{K(\omega_k)}{1 + K(\omega_k)G(\omega_k)}.$$

Observed that the frequency weighting $\tilde{W}(\omega_k)$ requires information of the actual sensitivity function $(1 + KG_0)^{-1}$ and the sensitivity function $(1 + KG)^{-1}$ based on the model to be estimated. Data of the actual sensitivity $(1 + KG_0)^{-1}$ is easily constructed from closed-loop experiments using the controller K or can be computed from the open-loop frequency response data $G_0(\omega_k)$ and the controller frequency response $K(\omega_k)$. The weighting $(1 + KG)^{-1}$ can only be computed once the frequency response of a model $G(\omega_k)$ is available. However, an iterative update of the frequency weighting can be used to update the weighting $(1 + KG_i)^{-1}$ on the basis of a model G_i.

A discrete-time model $G(q, \theta)$ can be found by through the least squares minimization [18]

$$\hat{\theta}_i = \min_{\theta} \sum_{k=1}^{N} E_i(\omega_k, \theta_i)^* E_{i-1}(\omega_k, \theta_i) \qquad (2.33)$$

in which the curve fit error

$$E_{i-1}(\omega_k, \theta_i) = (G_0(\omega_k) - G(\omega_k, \theta_i))\tilde{W}_{i-1}(\omega_k),$$

is being minimized. Computation of a finite order discrete-time model $\hat{G}(q) = G(q, \hat{\theta}_i)$ using the above weighting function iteration is done with the (non-linear) least squares optimization techniques available in [18]. Note that algebraic model realization techniques for parameter estimation are available through subspace identification algorithms [66], which have also included control-relevant weightings [29, 15].

2.7 Uncertainty Characterizations

Often models of system dynamics are provided with some measure of uncertainty associated to the difference between the behavior of the model and the real system. Representations of uncertainty reflect the physical mechanisms that cause these differences and are constructed to facilitate convenient manipulation [94]. Furthermore, models developed from measurement data and subsequent system identification methods lack accurate information in some frequency ranges, and in particular, ranges that are not relevant for control design one does not require accurate models [33].

Consider a LTI closed-loop system subject to parametric uncertainty and neglected dynamics given by the interconnected structure of Fig. 2.7, where the transfer matrix $M(s)$ contains the nominal closed-loop dynamics (i.e. without uncertainty) as well as the locations that the model perturbations Δ enter the system. By choosing the number, size, and dynamic nature of the blocks of Δ, a variety of uncertainty structures can be translated into this standard form (see for instance [94]). The interconnection structure depicted in Fig. 2.7 forms a Linear

Fractional Transformation (LFT) feedback connection of the nominal model M and the uncertainty description Δ. The transfer function from w to z is denoted by

$$\mathcal{F}_u(M, \Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}$$

where M has been partitioned appropriately.
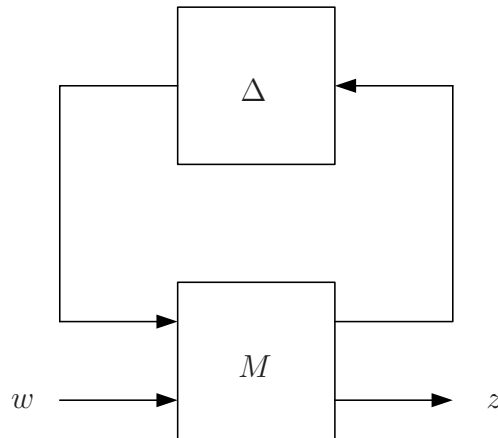
Figure 2.7. Standard uncertain system connection.

The actual construction of model sets for general uncertainty structure can be quite difficult and in general model sets are constructed to allow for plant dynamics that are not explicitly parametrized in the model structure. For example, a useful measure of uncertainty is to provide frequency domain bounds on the power spectrum of signal deviation from the nominal response to describe a model set. This gives a less structured representation of an uncertain transfer matrix $\Delta(s)$ of the neglected dynamics where it is only known to satisfy the $\mathcal{H}_\infty$ inequality

$$\|\Delta(s)\|_\infty = \overline{\sigma}(\Delta(j\omega)) \leq \eta, \qquad \text{for all } \omega \in \mathbb{R}, \qquad (2.34)$$

where $\overline{\sigma}$ indicates the maximum singular value [44]. The above norm-bounded uncertainty can be normalized ($\eta = 1$) by including stable and stably invertible weighting functions $W(s)$ so as to incorporate knowledge of the uncertainty in plant dynamics. For example consider for the moment a simple single block of neglected dynamics $\Delta(s)$ for taking into account the model uncertainty in the

additive model set representation $\mathcal{G} = \{G \mid G = G_0 + W\Delta, \quad \|\Delta\|_\infty \leq 1\}$. Then a weighting function can be chosen to satisfy the frequency domain inequality

$$W(j\omega) \geq \overline{\sigma}(G_0(j\omega) - G(j\omega)).$$

Spectral overbounding algorithms can be used to construct the above weighting function $W(s)$ that overbound the frequency dependent uncertainty [5, 6], which can also be made control-relevant [51].

Assuming that the nominal closed-loop $(\Delta(s) = 0)$ is stable, the core issue is to determine the maximal amount of uncertainty for which the closed-loop remains stable. A simple solution is provided by the small gain theorem, which gives a necessary and sufficient condition for stability of the closed-loop

$$\|\Delta(s)\|_\infty < \|M(s)\|_\infty^{-1}, \qquad \overline{\sigma}(\Delta(j\omega)) < \overline{\sigma}(M(j\omega))^{-1}.$$

As an alternative to determining weighting functions W that overbound uncertainty, one can simply evaluate the limits on allowable uncertainty (2.34) using the small gain theorem and $\overline{\sigma}(M(j\omega)) = \eta^{-1}$ for all frequencies $\omega \in \mathbb{R}$. Furthermore, levels of allowable uncertainty can be characterized over finite frequency intervals $\omega_1 \leq \omega \leq \omega_2$, as depicted in Fig. 2.8, which generally allows for reduced conservatism in structured uncertainty representations Δ.
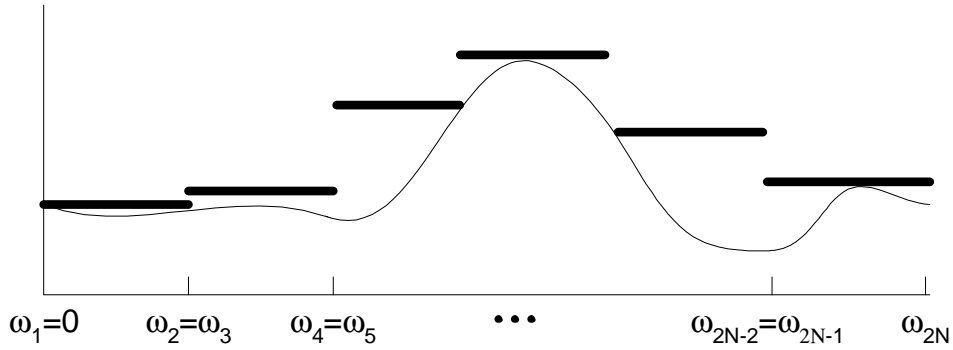
Figure 2.8. Allowable uncertainty overbound levels.

Note that determining allowable uncertainty levels can be posed as a FDI of the form (1.11) over finite frequency interval. With respect to Problem 1.1 this

corresponds to piece-wise performance weighting functions W_i as well as model uncertainty levels U_i that form the model set $\mathcal{G}_i$. Also note that although rational weighting functions U and W are required for standard $\mathcal{H}_\infty$ controller synthesis algorithms, the analysis problem of computing U and W such that $\sup_{G \in \mathcal{G}_i} \|T(G, K, W)\|_\infty < 1$ does not. The developments of subsequent chapters are devoted toward developing tools for model based analysis posed as FDI problems over finite frequency intervals. Evidence of reduced conservatism is illustrated through numerical examples.

2.8　Acknowledgements

Some text of this chapter includes the reprints of the following paper.

M.R. Graham and R.A. de Callafon, "Linear regression method for estimating approximate normalized coprime plant factors." In *Proceedings of IFAC Symposium on System Identification*, Newcastle, Australia, March 2006.

The dissertation author was the primary researcher and author in these works and the co-author listed in these publications directed and supervised the research.

Chapter 3

Extending the KYP Lemma

3.1 Motivation

With identification algorithms tuned toward the intended purpose of the resulting model, as discussed in the previous chapters, it seems logical that any model-based analysis should also be tailored for this purpose. That is, the model-based analysis conditions should be least conservative on the frequency range that is most important for the intended application. Keeping in mind that models contain limited information regarding system dynamics, the following problems formulate an outline for developing model-based analysis techniques.

Problem 3.1. *Given a model and performance specifications expressed as FDIs formulate conditions suitable to numerically tractable optimization for checking feasibility of the FDI over finite frequency range. Propose alternative performance specifications that allow for the coefficient matrix of the FDI to depend linearly on the frequency variable.*

This chapter will be devoted to the development of such analysis conditions that can be specified over finite frequency ranges, which will also be shown through an example to have significant impact on reducing conservatism in analysis of robustness.

3.2 Preliminaries

3.2.1 Various Linear Matrix Inequality (LMI) Results

A Linear Matrix Inequality (LMI) has the form

$$F(x) := F_0 + \sum_{i=1}^{m} x_i F_i \prec 0, \tag{3.1}$$

where $F = F^T \in \mathbb{R}^{n \times n}$ and $i = 1, \ldots, m$ are given. Considering strict LMIs in the subsequent sections is nonrestrictive since feasibility of nonstrict inequalities can be transformed to feasibility of strict inequalities [9]. It is easy to show that LMIs are convex [8], in fact they are affine functions composed of linear functions and a constant, therefor operations that preserve convexity can be exploited to pose new problems that remain convex which have other desirable properties. Perhaps the simplest example of this, which also plays a central role in the main results of this dissertation, is the convex combination of convex functions remains convex. That is, suppose a pair of LMIs $F(x)$ and $G(x)$ of the form (3.1) are given then the convex combination

$$\lambda F(x) + (1 - \lambda)G(x) \prec 0,$$

is also convex. Often, nonlinear convex inequalities can be converted to LMI form using Schur complements, which provides conditions for the positive definiteness of a partitioned matrix in term of its submatrices. That is the nonlinear inequality

$$R(x) \succ 0, \qquad\qquad Q(x) - S(x)R^{-1}(x)S^*(x) \succ 0,$$

is equivalent to the LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S^*(x) & R(x) \end{bmatrix} \succ 0, \qquad \text{with} \quad Q(x) = Q^*(x), \quad R(x) = R^*(x).$$

The advantage of working with the above higher dimensional form is the fact that the matrices Q, S, and R appear linearly, which is not true for the former inequality.

Often times, working with a higher dimensional form requires the introduction of extra variables in the search space. While this may appear disadvantageous, extending the search space for linear systems analysis often provides mathematically

more tractable problems. One such result plays a central role in the results of this dissertation, which is also used commonly in systems and control theory but first attributed to [26], and is known as Finsler's Lemma.

Lemma 3.1 (Finsler's Lemma). *Let $x \in \mathbb{C}^n$, $\mathcal{Q} \in \mathbb{HC}^n$ and $\mathcal{B} \in \mathbb{C}^{m \times n}$ such that* $\mathrm{rank}(\mathcal{B}) < n$ *be given. The following statements are equivalent:*

(i) $x^* \mathcal{Q} x < 0$, *for all* $\mathcal{B} x = 0$, $x \neq 0$.

(ii) $\mathcal{B}_\perp^* \mathcal{Q} \mathcal{B}_\perp \prec 0$.

(iii) *There exists a* $\mu \in \mathbb{C}$: $\mathcal{Q} - \mu \mathcal{B}^* \mathcal{B} \prec 0$.

(iv) *There exists an* $\mathcal{F} \in \mathbb{C}^{n \times m}$: $\mathcal{Q} + \mathcal{F} \mathcal{B} + \mathcal{B}^* \mathcal{F}^* \prec 0$.

Many additional sources for the proof are available, see [83, 22, 8] for instance. Since Finsler's Lemma plays an important role in the results of this dissertation a proof is provided here for completeness.

Proof: Note that any x such that $\mathcal{B} x = 0$ can be written as $x = \mathcal{B}_\perp y$ for some y. Then (i) can be written as $y^* \mathcal{B}_\perp^* \mathcal{Q} \mathcal{B}_\perp y \prec 0$, for all $y \neq 0$, which implies (ii). Assuming that (ii) holds, multiply on the right hand side by $y \neq 0$ and the left hand side by its conjugate transpose, which implies (i) after defining $x = \mathcal{B}_\perp y$.

Assume (iii) or (iv) holds, multiply on the right hand side by $\mathcal{B}_\perp$ and on the left hand side by the conjugate transpose to get (ii). Now assume (ii) holds and partition $\mathcal{B}$ into full rank factors $\mathcal{B} = \mathcal{B}_l \mathcal{B}_r$. Define $\mathcal{D} := \mathcal{B}_r^* (\mathcal{B}_r \mathcal{B}_r^*)^{-1} (\mathcal{B}_l^* \mathcal{B}_l)^{1/2}$ and apply the congruent transformation

$$
\begin{bmatrix} \mathcal{D} \\ \mathcal{B}_\perp^* \end{bmatrix} (\mathcal{Q} - \mu \mathcal{B}^* \mathcal{B}) \begin{bmatrix} \mathcal{D} & \mathcal{B}_\perp^* \end{bmatrix} = \begin{bmatrix} \mathcal{D}^* \mathcal{Q} \mathcal{D} & \mathcal{D}^* \mathcal{Q} \mathcal{B}_\perp \\ \mathcal{B}_\perp^* \mathcal{Q} \mathcal{D} & \mathcal{B}_\perp^* \mathcal{Q} \mathcal{B}_\perp^* \end{bmatrix} \prec 0.
$$

By assumption the second diagonal block is negative definite, therefor there exists a sufficiently large μ that makes the entire matrix negative definite, which implies (iii). Now choose $\mathcal{F} = -(\mu/2)\mathcal{B}^*$ to make the final implication (iv). $\square$

Statement (i) is a constrained quadratic form, where a vector $x \in \mathbb{C}^n$ is confined to lie in the null-space of $\mathcal{B}$. Statement (ii) is an unconstrained quadratic form

that results from parametrizing the vector x as $x = \mathcal{B}_\perp y$ for $y \neq 0$. Statements (iii) and (iv) give equivalent unconstrained quadratic forms where the constraint is accounted for by introducing an extra multiplier variable, μ in statement (iii) and the matrix $\mathcal{F}$ in statement (iv). Statement (iii) has the additional interpretation of a *Lagrange multiplier* (μ) for including constraints into optimization [40]. The methods of this dissertation will focus primarily on generating statement (iv) from unconstrained quadratic form (ii).

Finsler's Lemma has also been directly used to eliminate variables in matrix inequalities that appear commonly in systems and control literature, see for example [70, 7] where the result is referred to as the Elimination Lemma. In this context, applications move from statement (iv) to statement (ii) eliminating the matrix variable $\mathcal{F}$.

3.2.2   State-Space System Representation

The previous section presented various LMI results for constant matrices. Perhaps more interesting are results can be given for linear time invariant (LTI) dynamical systems represented in state-space form

$$\delta x(t) = Ax(t) + Bu(t), \qquad\qquad y(t) = Cx(t) + Du(t) \qquad (3.2)$$

where δ indicates the differential operator $\delta x = dx/dt$ in continuous-time or the forward step operator $\delta x(t) = x(t+1)$ in discrete-time. The above state-space representation forms a general class of models, which contains for example the linear models identified using the techniques in Chapter 2, and can be used in model based analysis for determining system properties such as robustness and performance. The LTI system of the form (3.2), gives a transfer matrix representation

$$M(\xi) = C(\xi I - A)^{-1}B + D,$$

for $\xi \in \mathbb{C}$. Note that the transfer matrix above can also be written as

$$M(\xi) = \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} (\xi I - A)^{-1}B \\ I \end{bmatrix}. \qquad (3.3)$$

Since matrices C and D do not depend on the frequency variable ξ, the results presented in future sections generally do not include these matrices explicitly.

3.2.3 The Kalman-Yakubovich-Popov (KYP) Lemma

Specifications for performance and robustness of dynamical systems are commonly described in terms of frequency domain inequalities (FDIs) on transfer matrices of the form (3.3), which due to infinite dimensionality of the frequency variable ξ are not directly tractable in analysis and design. As one of the fundamental results in dynamic systems analysis [39], the Kalman-Yakubovich-Popov (KYP) lemma establishes equivalence between FDIs for a transfer matrices and an LMI for its state space realization [2, 90, 72].

Lemma 3.2 (Kalman-Yakubovich-Popov Lemma). *Let $A \in \mathbb{C}^{n \times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{C}^{n \times m}$ and $\Theta \in \mathbb{HC}^{n+m}$ be given. The following statements are equivalent:*

(i) The FDI holds

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0, \qquad \text{for all } \omega \in \mathbb{R}, \qquad (3.4)$$

(ii) There exists a matrix $P \in \mathbb{HC}^n$ such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta \prec 0. \qquad (3.5)$$

Proof: See for instance [72]. An outline for the this proof will be omitted here since generalizations of this result will be provided in the next sections.

Many problems in systems and control theory can be posed in the form (3.4) where appropriate choices for Θ represent the analysis of various system properties. For example, positive realness of the transfer matrix $M(j\omega) = (j\omega I - A)^{-1}B$, i.e.

$$M(j\omega) + M^*(j\omega) \succ 0, \qquad \text{for all } \omega > 0,$$

can be specified in the LMI (3.5) with

$$\Theta = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}. \qquad (3.6)$$

For single input single output (SISO) systems, this condition can be checked graphically, by plotting the curve given by the real and imaginary parts of $M(j\omega)$ for all $\omega \in \mathbb{R}$ (called the Nyquist contour). Graphical methods became very popular in the systems and control work of the 1960's and 1970's, which is why Lemma 3.2 with (3.6) is sometimes referred to the Positive Real Lemma. In another common form, bounded realness of the transfer matrix $M(j\omega)$, i.e.

$$M(j\omega)^* M(j\omega) \prec I, \qquad \text{for all } \omega > 0,$$

can be specified in the LMI (3.5) with

$$\Theta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

Additionally, since the matrix Θ appears affinely in the LMI (3.5), it can also contain matrix variables that are included in the search for feasible solutions. For example, consider the matrix Θ given by

$$\Theta = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix},$$

where $X \in \mathbb{HC}^n$, $Y \in \mathbb{C}^{n \times m}$, and $Z \in \mathbb{HC}^m$ are matrix variables. At this point, it is not obvious why (3.5) with the above form for Θ is important, but in Section 3.4.1 it lends itself to the analysis of robustness via the search for scalings X, Y, and Z that provide upper bounds for the structured singular value.

3.3 The KYP Lemma on Frequency Intervals

In order to gain an understanding of the methods used in this dissertation for generalizing the KYP Lemma, this section presents an extension on the finite frequency KYP Lemma [49, 48]. A finite frequency KYP Lemma, first introduced in a continuous-time framework [49], was developed by establishing conditions for which the classic S-procedure is nonconservative, i.e. necessary and sufficient. The S-procedure [91] replaces inequalities of the form

$$x^* \Theta x < 0, \qquad \text{for all } x \in \mathcal{X}, \tag{3.7}$$

where

$$\mathcal{X} := \{x \in \mathbb{C}^p : x^* S x \geq 0, \quad x \neq 0, \quad \text{for all } S \in \mathcal{S}\}$$

$$\mathcal{S} := \left\{ \sum_{i=1}^{k} \tau_i S_i : \tau_i \geq 0, \ \forall \ i = 1, ..., k \right\} \quad (3.8)$$

and Θ, $S_i \in \mathbb{HC}^p$ are given by a single inequality given by the existence of $S \in \mathcal{S}$ such that $\Theta + S \prec 0$. The results in [49] give conditions on the set of matrices S under which the S-procedure is nonconservative, that is when the sufficient condition shown above also becomes necessary. Such sets S are termed *lossless*.

Definition 3.3.1 (Losslessness)**.** *A subset $\mathcal{S}$ is lossless if it has the following properties*

(i) *$\mathcal{S}$ is convex.*

(ii) *$S \in \mathcal{S}$ implies $\tau S \in \mathcal{S}$ for all $\tau \geq 0$.*

(iii) *For positive semidefinite matrix $X \in \mathbb{HC}^p$ such that*

$$\mathrm{tr}(SX) \geq 0, \quad \text{for all } S \in \mathcal{S},$$

there exist $x_i \in \mathbb{C}^p$ such that

$$X = \sum_{i=1}^{r} xx^*, \quad x^* S x \geq 0, \quad \text{for all } S \in \mathcal{S},$$

where r is the rank of X.

The nonconservative S-procedure is summarized in the following lemma.

Lemma 3.3 (Generalized S-Procedure)**.** *Let $\Theta \in \mathbb{HC}^p$ and lossless set $\mathcal{S}$ of Hermitian matrices be given. The following are equivalent.*

(i) *$x^* \Theta x < 0$ for all $x \in \mathcal{X}$ where $\mathcal{X}$ is defined in (3.8).*

(ii) *There exists $S \in \mathcal{S}$ such that $\Theta + S \prec 0$.*

Proof: See also [49]. The direction (ii) to (i) comes from Definition 3.3.1 property (ii) with $\tau = 0$. To show (i) implies (ii), suppose that (ii) does not hold. That is there exists no $S \in \mathcal{S}$ such that $\Theta + S \prec 0$, then from separating hyper-plane theorem arguments [64, 67] there exists a nonzero matrix $K \in \mathbb{HC}$, with $K \succeq 0$ such that

$$\operatorname{tr}\{(\Theta + S)K\} \geq 0, \qquad\qquad \text{for all } S \in \mathcal{S}.$$

Since $\mathcal{S}$ is lossless, from Definition 3.3.1 property (ii) the implication holds

$$\operatorname{tr}\{SK\} \geq 0, \quad \text{for all } S \in \mathcal{S} \qquad \Rightarrow \qquad \operatorname{tr}\{\Theta K\} \geq 0.$$

From Definition 3.3.1 property (iii), the first condition also implies the existence of vectors x_i such that the second condition becomes

$$\operatorname{tr}\{\Theta K\} = \sum_{i=1}^{r} x_i^* \Theta x_i \geq 0.$$

Then there exists an $x_i \in \mathcal{X}$ such that $x_i^* \Theta x_i \geq 0$, which contradicts the original assumption. $\qquad\square$

The following finite frequency KYP Lemma is developed from the nonconservative S-procedure above. It gives necessary and sufficient conditions for replacing the infinite dimensional FDI constraints (3.4) over finite frequency interval $\omega_1 \leq \omega \leq \omega_2$ with a single inequality by incorporating a multiplier matrix that is lossless in characterizing finite frequency intervals.

Lemma 3.4 (Finite Frequency KYP Lemma [49]). *Let matrices $A \in \mathbb{C}^{n \times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{C}^{n \times m}$ and a matrix $\Theta \in \mathbb{HC}^{n+m}$ be given. Let scalars $\omega_1 \leq \omega_2$, then the following statements are equivalent.*

(i) The finite FDI

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0, \tag{3.9}$$

holds for all $\omega_1 \leq \omega \leq \omega_2$.

(ii) There exist matrices $P, Q \in \mathbb{HC}^n$ such that $Q \succeq 0$ and

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1 \omega_2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta \prec 0, \quad \omega_c := (\omega_1 + \omega_2)/2.$$

(3.10)

Proof (Outline): From Finsler's Lemma (Lemma 3.1), the FDI (3.9) holds if and only if

$$x^* \Theta x < 0, \quad \text{for all } x \in \mathcal{X},$$

where

$$\mathcal{X} := \left\{ x \in \mathbb{C}^{n+m} : \begin{bmatrix} I & -j\omega I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} x = 0, \ x \neq 0, \ \omega_1 \leq \omega \leq \omega_2 \right\}.$$

Let $y_1 = Ax_1 + Bx_2$, $y_2 = x_1$ where x is partitioned appropriately $x_1 \in \mathbb{C}^n$, $x_2 \in \mathbb{C}^m$. Note that from the set $\mathcal{X}$, $y_1 = j\omega y_2$ holds for all $\omega_1 \leq \omega \leq \omega_2$ and that

$$y^* \begin{bmatrix} -Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1 \omega_2 Q \end{bmatrix} y = [-\omega^2 + 2\omega_c \omega - \omega_1 \omega_2] y_2^* Q y_2$$

$$= [(\omega - \omega_1)(\omega_2 - \omega)] y_2^* Q y_2.$$

From the last equality, since $Q \succeq 0$ and $(\omega - \omega_1)(\omega_2 - \omega) \geq 0$ for all $\omega_1 \leq \omega \leq \omega$ the following holds

$$y^* \begin{bmatrix} -Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1 \omega_2 Q \end{bmatrix} y \geq 0, \qquad \text{for all } P, Q \in \mathbb{HC}^n, \quad Q \succeq 0,$$

so that the set $\mathcal{X}$ can be replaced by

$$\mathcal{X} = \left\{ x \in \mathbb{C}^{n+m} : x^* S x \geq 0, \quad x \neq 0, \quad \text{for all } S \in \mathcal{S} \right\}$$

where $\mathcal{S}$ is given as

$$\mathcal{S} = \left\{ S = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1 \omega_2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} : \right.$$

$$\left. \text{for all } P, Q \in \mathbb{HC}^n, \quad Q \succeq 0, \quad \omega_1 < \omega_2 \right\}.$$

It is easy to verify that $\mathcal{S}$ satisfies properties (i) and (ii) in Definition 3.3.1. Property (iii) has been shown in [49] to establish losslessness, leading to the nonconservative S-procedure Lemma 3.3. $\square$

The previous lemma established the equivalence between the finite FDI (3.9) and the single LMI (3.10), hence it has been named the finite frequency KYP Lemma. From a practical perspective, it allows to pose and check frequency domain specifications within a certain frequency range which might be the most relevant to a specific application. Furthermore, by combining ranges one can pose frequency specifications in different ranges without augmenting the plant with frequency dependent scalings or weights. The next theorem establishes the equivalence between the finite frequency KYP Lemma 3.4, and an *extended* condition based on the of the work [22]. The *extended* condition is posed as a pair of LMI containing extra variables, which can bring about significant advantages and lead to mathematically more tractable problems, see [22] and as well as [21, 32, 74].

Theorem 3.1. *Let matrices $A \in \mathbb{C}^{n \times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{C}^{n \times m}$ and $\Theta \in \mathbb{HC}^{n+m}$ be given. The following statements are equivalent.*

(i) The finite FDI

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0, \qquad (3.11)$$

holds for all $\omega_1 \leq \omega \leq \omega_2$.

(ii) There exist matrices $F \in \mathbb{C}^{n \times n}$ and $G \in \mathbb{C}^{m \times n}$ such that the pair of LMI holds

$$\mathrm{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\omega_i I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} + \Theta \prec 0, \qquad i = \{1, 2\}. \qquad (3.12)$$

Proof: To show that (ii) implies (i), assume the pair of inequalities (3.12) have feasible solutions. The sum of (3.12) for $i = 1$ multiplied by $\lambda(\omega) := (\omega_2 - \omega)/(\omega_2 - \omega_1) \in [0,1]$ and of (3.12) for $i = 2$ multiplied by $(1 - \lambda(\omega))$ implies that

$$\mathrm{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\omega I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} + \Theta \prec 0$$

is feasible for all $\omega_1 \leq \omega \leq \omega_2$. Set

$$N(\omega) := \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \tag{3.13}$$

which is a well defined matrix for all $\omega \in \mathbb{R}$ due to the assumption that A has no eigenvalue on the imaginary axis. Multiply the above inequality by $N(\omega)$ on the right and by its transpose conjugate on the left to obtain

$$N^*(\omega)\,\Theta\,N(\omega) = \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0,$$

which is the FDI in (i).

To show that (i) implies (ii), assume that (3.11) holds for all $\omega_1 \leq \omega \leq \omega_2$. From Lemma 3.4 there exist matrices $P, Q \in \mathbb{HC}^n$, $Q \succeq 0$ such that (3.10) holds. Define the matrix

$$X(\omega) := \begin{bmatrix} 1 & -j\omega \\ j\omega & \hat{\omega}^2 + 2\omega\,\omega_c - \omega_c^2 \end{bmatrix} \otimes Q, \tag{3.14}$$

where $\hat{\omega} = (\omega_2 - \omega_1)/2$ and $\omega_c = (\omega_1 + \omega_2)/2$. Because $Q \succeq 0$ one can establish the equivalence

$$\begin{bmatrix} 1 & -j\omega \\ j\omega & \hat{\omega}^2 + 2\omega\,\omega_c - \omega_c^2 \end{bmatrix} \otimes Q \succeq 0 \qquad \Leftrightarrow \qquad \hat{\omega}^2 - (\omega - \omega_c)^2 \geq 0,$$

where the last expression comes from using Schur complement. Note that

$$\hat{\omega}^2 - (\omega - \omega_c)^2 = (\omega - \omega_1)(\omega_2 - \omega) \geq 0,$$

which establishes the positive semidefiniteness of $X(\omega)$ for all $\omega_1 \leq \omega \leq \omega_2$. Now add the matrix

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* X(\omega) \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \succeq 0, \qquad \omega_1 \leq \omega \leq \omega_2,$$

to the right hand side of (3.10) so that

$$\Theta \prec \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \Upsilon \begin{bmatrix} A & B \\ I & 0 \end{bmatrix},$$

where

$$\Upsilon := \begin{bmatrix} Q & -P - j\omega_c Q \\ -P + j\omega_c Q & \omega_1 \omega_2 Q \end{bmatrix} + \begin{bmatrix} Q & -j\omega Q \\ j\omega Q & (\hat{\omega}^2 + 2\omega\,\omega_c - \omega_c^2)Q \end{bmatrix}.$$

Note that

$$\Upsilon = \begin{bmatrix} 2Q & -P - j(\omega + \omega_c)Q \\ -P + j(\omega + \omega_c)Q & 2\omega\,\omega_c Q \end{bmatrix}$$

$$= \begin{bmatrix} Q \\ -P + j\omega_c Q \end{bmatrix} \begin{bmatrix} I & -j\omega I \end{bmatrix} + \begin{bmatrix} I \\ j\omega I \end{bmatrix} \begin{bmatrix} Q & -P - j\omega_c Q \end{bmatrix},$$

which implies that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \left(\begin{bmatrix} Q \\ -P + j\omega_c Q \end{bmatrix} \begin{bmatrix} I & -j\omega I \end{bmatrix} + \right.$$

$$\left. \begin{bmatrix} I \\ j\omega I \end{bmatrix} \begin{bmatrix} Q & -P - j\omega_c Q \end{bmatrix} \right) \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta \prec 0.$$

Now choose

$$\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q \\ P - j\omega_c Q \end{bmatrix} \tag{3.15}$$

so that

$$\mathrm{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\omega I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} + \Theta \prec 0$$

for any $\omega_1 \leq \omega \leq \omega_2$, in particular for $\omega = \omega_1$ and $\omega = \omega_2$, which imply that the pair of inequalities (3.12) are feasible, therefore that (ii) should hold. $\square$

Remark 3.3.1. *The assumption that A has no eigenvalue on the imaginary axis can be relaxed to the absence of eigenvalues of A within the segment of the imaginary axis $j[\omega_1, \omega_2]$. Assumptions made in the following sections regarding the eigenvalues of A can also be relaxed in a similar manner.*

Remark 3.3.2. *The importance of the above result is in reducing the infinite number of inequalities to be checked in the FDI (3.9) to one, as in (3.10), or two, as in (3.12), finite dimensional inequalities in two variables, P and Q, or F and G, respectively. Furthermore, these finite dimensional inequalities are LMIs that can be efficiently solved to global optimality using Convex Programming [9].*

Remark 3.3.3. *The inequality (3.10) has $2n^2$ optimization variables in the matrices $P, Q \in \mathbb{HC}^n$. In contrast, the inequality (3.12) has $2n(n+m)$ optimization variables in the matrices $F \in \mathbb{C}^{n \times n}$ and $G \in \mathbb{C}^{m \times n}$.*

The above theorem presents the case where Θ is constant. This case can be extended to incorporate a particular class of frequency-dependent Θ without incurring extra computational cost in solving the LMI (3.12). One can recognize that in proving the theorem above, the forming of the parameter $\lambda(\omega) \in [0, 1]$ for all $\omega_1 \leq \omega \leq \omega_2$ introduces concepts from the analysis of polytopic systems [22] treating the frequency an real uncertain parameter. Indeed, it is possible to use form (ii) of the above theorem to handle the particular form of a frequency-dependent matrix Θ

$$\Theta(\omega) = \lambda(\omega)\Theta_1 + (1 - \lambda(\omega))\Theta_2, \tag{3.16}$$

where $\Theta_1, \Theta_2 \in \mathbb{HC}^{n+m}$ and

$$\lambda(\omega) := \frac{\omega_2 - \omega}{\omega_2 - \omega_1} \in [0, 1], \qquad \omega_1 \leq \omega \leq \omega_2. \tag{3.17}$$

Note that $\Theta(\omega)$ is not a proper rational function of ω

$$\Theta(\omega) = \frac{\omega_2 - \omega}{\omega_2 - \omega_1}\Theta_1 + \frac{\omega - \omega_1}{\omega_2 - \omega_1}\Theta_2, \qquad \omega_1 \leq \omega \leq \omega_2,$$

which means that $\Theta(\omega)$ cannot be realized as a proper rational transfer function of ω. The affine function $\Theta(\omega)$ considered in this section is restricted to a single interval only for simplicity of the exposition and will be extended in Section 4.5 to more general piecewise affine functions. The following theorem is obtained by constraining $\Theta(\omega)$ as in (3.16).

Theorem 3.2. *Let matrices $A \in \mathbb{C}^{n \times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{C}^{n \times m}$ and $\Theta_1, \Theta_2 \in \mathbb{HC}^{n+m}$ be given. If there exist matrices $F \in \mathbb{C}^{n \times n}$ and $G \in \mathbb{C}^{m \times n}$ such that*

$$\text{He}\left\{\begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\omega_i I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}\right\} + \Theta_i \prec 0, \quad i = \{1, 2\} \tag{3.18}$$

then the FDI

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta(\omega) \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0 \tag{3.19}$$

holds for all $\omega_1 \leq \omega \leq \omega_2$ with $\Theta(\omega)$ as given in (3.16).

Proof: Repeat the initial steps of proof of the implication from (ii) to (i) in the previous theorem replacing (3.12) by (3.18) and using (3.16) to conclude that

$$\text{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\omega I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} + \Theta(\omega) \prec 0 \tag{3.20}$$

is feasible for all $\omega_1 \leq \omega \leq \omega_2$ where $\Theta(\omega)$ is given in (3.16). Multiplication by the same matrix $N(\omega)$ on the right and by its transpose conjugate on the left produce

$$N^*(\omega)\, \Theta(\omega)\, N(\omega) = \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta(\omega) \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0.$$

which is the FDI (3.19). □

Theorems 3.1 and 3.2 reduce the infinite number of FDIs (3.11) to be checked to only two. Furthermore, Theorem 3.1 is a generalization of Lemma 3.4 in the sense that the pair of inequalities (3.12) are feasible whenever the inequality (3.10) has feasible solution. The significance of Theorem 3.2 is indicated by the sufficiency direction of the result. Allowing for frequency dependent coefficient matrices Θ can significantly reduce conservatism in applications that include searching over variables in Θ. Formal proof for the necessity of the conditions in Theorem 3.2 will not be included in the contents of this dissertation, however this topic will be further discussed in future work Section 6.2. The next sections will make the connection between Theorems 3.1 and 3.2 and the motivation in Chapters 1 and 2 through a presentation of results for robustness analysis via the structured singular value along with an application in performance analysis of disk drive servomechanisms.

3.4   Robust Stability Analysis

The structured singular value, often referred as μ and introduced in [23], has been a popular analytical tool for analyzing stability and performance robustness of linear systems with parametric and dynamic uncertainties. Despite the fact that computing μ, or even finding tight bounds for μ, has proved an extremely

hard problem, the use of the μ terminology and methodology in robust control is widespread.

A number of algorithms have been developed to compute the maximum of the structured singular value over a specific frequency interval. Many popular methods make use of the concept of frequency-dependent multipliers [68]. Searching simultaneously for a frequency-dependent multiplier and the maximum μ over all frequencies has also revealed to be a hard problem. An often used device is the reduction of the search domain by using finite yet sufficiently dense frequency grid. Search algorithms have been proposed for determining the maximum μ over frequency interval [58, 25] avoiding unnecessarily dense grids. Other approaches have considered the frequency itself as uncertain parameter augmented to the original system [80] eliminating difficulties associated with griding methods. For improved accuracy finite frequency intervals can be considered via linear fractional transformation mapping of the real frequency parameter [41].

The KYP Lemma (Lemma 3.2) establishes equivalence between frequency domain inequalities on the system transfer function (3.4) and the LMI (3.5). Generalization of the KYP Lemma allows for treatment of finite frequency ranges [48]. This result is closely related with standard μ-analysis [49], and can be proved from results on the losslessness of scalings for mixed-μ [67]. Recent results show the connection of these methods and propose generalizations in the context of powerful relaxation techniques [75, 77] for which robust analysis tests that approach exactness can be derived in a systematic way at the expense of increasing the large size of the problem to be solved and number of optimization variables.

The results presented in this chapter provide new robust stability conditions with a complexity that is comparable to the generalized KYP lemma [48]. The tests are expressed as a pair of LMI that, if solvable, provide an upper bound to μ over some specified and possibly finite frequency interval along with a particular frequency-dependent multiplier that is used to prove robust stability. The ideas are borrowed from robust analysis of uncertain polytopic systems [22] in the treatment of frequency as a real uncertain parameter to formulate the results in this paper.

3.4.1 μ-Analysis

Consider the standard LFT setup for robustness and performance μ analysis as depicted in Fig. 2.7. The nominal map $M(s)$ is assumed to be a rational function of the complex variable s, being a proper and square matrix that is analytic in the closed right-half plane. The unknown uncertainty is assumed to have the following structure

$$\boldsymbol{\Delta} := \{\mathrm{diag}[\phi_1 I_{s_1}, \cdots, \phi_r I_{s_r}, \delta_1 I_{s_1}, \cdots, \delta_c I_{s_c}, \Delta_1, \cdots, \Delta_F] :$$
$$\phi_i \in \mathbb{R}, \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j}\}. \quad (3.21)$$

Let $\mathcal{T}(\Delta)$ denote the set of all block diagonal and stable rational transfer function matrices that have block structures such as $\boldsymbol{\Delta}$

$$\mathcal{T}(\Delta) := \{\Delta(\cdot) \in \mathcal{RH}_\infty : \ \Delta(s_0) \in \boldsymbol{\Delta} \ \forall \ s_0 \in \mathbb{C}_+\} \quad (3.22)$$

The feedback connection of (M, Δ) is well-posed and internally stable for all $\Delta \in \mathcal{T}(\Delta)$ with $\|\Delta\|_\infty < \beta^{-1}$ if and only if [94]

$$\sup_{\omega \in \mathbb{R}} \mu_{\boldsymbol{\Delta}}(M(j\omega)) \leq \beta, \quad (3.23)$$

where $\mu_{\boldsymbol{\Delta}}$ denotes the structured singular value of a matrix, which is defined as

$$\mu_{\boldsymbol{\Delta}}(M) := \left(\inf_{\Delta \in \boldsymbol{\Delta}} \{\|\Delta\| : \det(I - M\Delta) = 0\} \right)^{-1}.$$

In case no $\Delta \in \boldsymbol{\Delta}$ makes $(I - M\Delta)$ singular $\mu_{\boldsymbol{\Delta}}(M) := 0$.

Note that one can consider robust performance of the feedback connection of (M, Δ) in Fig. 2.7 by considering an augmented uncertainty structure

$$\boldsymbol{\Delta}_P = \left\{ \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_f \end{bmatrix} : \Delta \in \boldsymbol{\Delta}, \quad \Delta_f \in \mathbb{C}^{q \times p} \right\},$$

where Δ_f is a fictitious uncertainty block of the same dimension as the input w and output z, respectively q and p. Such a test indicates the worst-case level of performance degradation associated with a given level of perturbations. That is, for all $\Delta \in \mathcal{T}(\Delta)$ with $\|\Delta\|_\infty < \beta^{-1}$, the feedback connection (M, Δ) is well-posed, internally stable, and $\|\mathcal{F}_u(M, \Delta)\|_\infty \leq \beta$ if and only if

$$\sup_{\omega \in \mathbb{R}} \mu_{\boldsymbol{\Delta}_P}(M(j\omega)) \leq \beta.$$

Checking for performance robustness with structured uncertainty $\mathbf{\Delta}_P$ is precisely the analysis problem specified in Problem 1.1.

In general, the structured singular value $\mu_{\mathbf{\Delta}}$ cannot be computed in polynomial time, being a problem for which no algorithm exists (NP-hard) [84]. In practice, the introduction of appropriate *scalings* or *multipliers* using duality theory is commonly used to provide computable upper bounds for $\mu_{\mathbf{\Delta}}$.

For instance, define the set of scaling matrices

$$\mathbf{Z} := \{ \mathrm{diag}[Z_1, \cdots, Z_{s_r+s_c}, z_1 I_{m_1}, \cdots, z_F I_{m_F}] :$$
$$Z_i \in \mathbb{C}^{s_i \times s_i}, Z_i = Z_i^* > 0, z_j \in \mathbb{R}, z_j > 0 \}, \quad (3.24)$$

and

$$\mathbf{Y} := \{ \mathrm{diag}[Y_1, \cdots, Y_{s_r}, 0, \cdots, 0] : \ Y_i = Y_i^* \in \mathbb{C}^{s_i \times s_i} \}. \quad (3.25)$$

Note that $\mathbf{Z}$ and $\mathbf{Y}$ commute with the matrices in $\mathbf{\Delta}$. Now define the matrix valued function

$$\Gamma_\beta(M, Z, Y) := M^* Z M - j \left(M^* Y - Y M \right) - \beta^2 Z, \quad (3.26)$$

and the optimization problem

$$\rho_{\mathbf{\Delta}}(M) := \inf_{\beta \in \mathbb{R}, Z \in \mathbf{Z}, Y \in \mathbf{Y}} \sup_{\omega \in \Omega} \{ \beta : \ \Gamma_\beta(M(j\omega), Z(\omega), Y(\omega)) \prec 0 \}. \quad (3.27)$$

It follows from duality theory [24] that

$$\sup_{\omega \in \Omega} \mu_{\mathbf{\Delta}}(M(j\omega)) \leq \rho_{\mathbf{\Delta}}(M). \quad (3.28)$$

The problem on the right hand side of the above inequality is, in some sense, simpler than the original problem (3.23). Yet it cannot be easily solved as well. The following are commonly found strategies for approaching this problem:

(i) **Constant multipliers on $\Omega = \mathbb{R}$:** When $\Omega = \mathbb{R}$ and Z and Y are assumed to be constant, i.e., $Z(\omega) = Z$ and $Y(\omega) = Y \ \forall \omega \in \mathbb{R}$, then problem (3.27) can be converted into an LMI using the KYP Lemma. The particular case $Z = I, Y = 0$ reduces to the well known BRL (Bounded-Real Lemma). This approach produces upper bounds for $\rho_{\mathbf{\Delta}}$.

(ii) **Constant multipliers on $\Omega \subset \mathbb{R}$:** When $\Omega = [\omega_1, \omega_2] \subset \mathbb{R}$ and Z and Y are assumed to be constant, i.e., $Z(\omega) = Z$ and $Y(\omega) = Y \, \forall \, |\omega| \in [\omega_1, \omega_2]$, then problem (3.27) can be converted into an LMI using the Generalized KYP results of [48]. This approach produces upper bounds for ρ_Δ with tight upper bounds obtained by splitting Ω in N segments $\Omega_i = [\omega_i, \omega_{i+1}]$, $i = 1, \ldots, N$ such that $\Omega = \cup_i \Omega_i$.

(iii) **Constant multipliers on $\Omega = \{w_1\}$:** For a single frequency ω_1, i.e., $\Omega = \{\omega_1\}$ problem (3.27) is an LMI. Lower bounds for ρ_Δ can be obtained by solving this LMI on a finite grid $\Omega = \cup_i \{\omega_i\}$. In most cases, to achieve a reasonable approximation for ρ_Δ a very dense grid must be used.

Theorem 3.2 can be used to produce upper bounds to ρ_Δ which has as its main advantage the fact that scaling matrices Z and Y are allowed to be affine functions of ω. The following corollary was presented by the authors in [37].

Corollary 3.1. *Let $A \in \mathbb{R}^{n \times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, and $D \in \mathbb{R}^{m \times m}$ be given. If there exist matrices $Z_1, Z_2 \in \mathbf{Z}$, $Y_1, Y_2 \in \mathbf{Y}$, $F \in \mathbb{C}^{n \times n}$, and $G \in \mathbb{C}^{m \times n}$ such that*

$$\text{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\omega_i I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} Z_i & -jY_i \\ jY_i & -\beta^2 Z_i \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \prec 0, \quad i = \{1, 2\} \tag{3.29}$$

has feasible solutions then $\rho_\Delta(C(j\omega I - A)^{-1}B + D, \Omega) \leq \beta$ for all $|\omega| \in \Omega = [\omega_1, \omega_2]$.

Proof: Follows from Corollary 4.2 noting that

$$\Theta_i = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} Z_i & -jY_i \\ jY_i & -\beta^2 Z_i \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}, \quad i = \{1, 2\}, \tag{3.30}$$

and that the matrix variables Z_i, Y_i appear linearly in (3.29). Feasibility of (3.29) directly implies that $\rho_\Delta(C(j\omega I - A)^{-1}B + D, \Omega) \leq \beta$ for all $\omega_1 \leq |\omega| \leq \omega_2$. $\square$

The proposed method for computing upper bounds on the structured singular value over finite frequency grid is demonstrated in the following section with a case study on hard disk drive servo performance analysis.

3.5   Case Study I: Hard Drive Servo Analysis

3.5.1   Motivation

As storage capacity of hard disk drives (HDDs) increases, the requirements for position control of the read/write head becomes more challenging to satisfy [13], especially for control firmware that is required to operate in the presence of product variability. Performance considerations under product variability are well suited under the worst-case $\mathcal{H}_\infty$-norm based optimal control which allows robustness issues to be incorporated into the servo design [19] in the form of uncertainty models [73]. Uncertainty models, as well as restrictions on the complexity of the controller to be used, make the design of a feedback control law for disk drive servo systems a challenging task.

This case study incorporates product variability based on model uncertainty in the performance analysis of low-order controllers designed using control-relevant model estimates. The controller is assumed given and the methods of Chapter 2 motivate control-relevant identification for determining the model used in the analysis. The method requires an upper bound on the characterization of model uncertainty, for instance the damping and frequency of flexible modes of the suspension, which can be achieved with uncertain but norm-bounded transfer matrix. Performance specifications can be characterized similarly, utilizing appropriate weighting functions that bound relevant closed-loop transfer functions, and can be combined with model uncertainty in an augmented structured uncertain transfer matrix. Subsequently satisfactory performance is determined via structured singular value analysis.

Alternatively, the performance analysis given in this section determines levels of possible performance taking into consideration model uncertainty. This problem is formulated similarly with performance robustness problems, however without the appropriate weighting functions on the performance channel for normalizing the acceptable bound to unity. Rather piece-wise performance levels are determined that overbound the structured singular values that indicate robust stability and an unweighted closed-loop transfer functions that are directly related to relevant

performance channels. The approach is demonstrated via implementation on a production disk drive servo system similar to that illustrated in Fig. 3.1. Specifications for the drive are given in Table 3.1.
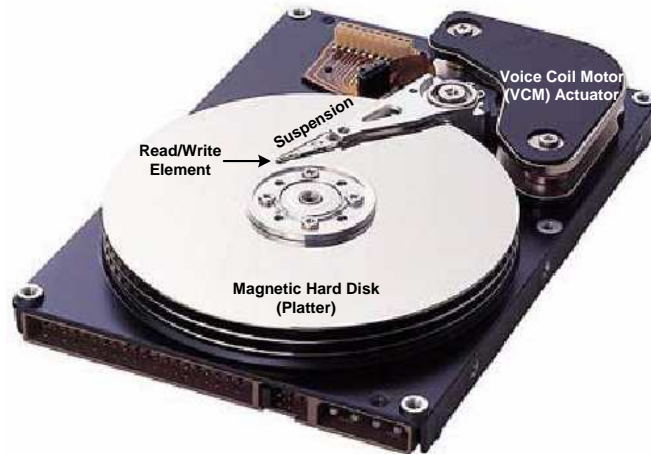
Figure 3.1. Hard disk drive cut-away picture.

Table 3.1. Experimental Disk Drive Specifications

Spindle motor speed	$7200rpm(120Hz)$
Number of data sectors, N	128
Servo sampling frequency, f_s	$15.36KHz$

3.5.2   HDD Model Set and Controller Formulation

For improved track following performance the bandwidth of the servo system should be increased, however this is limited by sampling frequency and mechanical resonances of the head/disk assembly. Bandwidth and disturbance rejection are characterized by the sensitivity function

$$S = (1 + GK)^{-1}$$

where G denotes a dynamical model of the voice coil motor (VCM) with flexibilities of the E-block and suspension and K denotes the VCM servo controller. The

feedback control law for the experimental drive is given by the discrete-time PID controller

$$u(t) = k_p e(t) + k_i \sum_{i=0}^{t} e(i) + k_d [e(t) - e(t-1)].$$

which was tuned to give a servo bandwidth of $1KHz$ as shown in Fig. 1.2.

For disk drive actuator dynamics, models G are commonly obtained from an intuitive modeling of a flexible structure as

$$G(s) = K_{VCM} \frac{\omega_l^2}{s^2 + \zeta_l \omega_l + \omega_l^2} \cdot \frac{\omega_h^2}{s^2 + \zeta_h \omega_h + \omega_h^2}$$

where low-frequency parameters, $(K_{VCM}, \zeta_l, \omega_l)$ modeling effects of the VCM motor, and high-frequency parameters, (ζ_h, ω_h) modeling flexibility in the suspension, are tuned to provide reasonable representation of the HDD dynamics. Motivated by the discussion in Section 2.6 the closed-loop performance analysis should incorporate a set of models $\mathcal{G}$ for which the difference between designed performance and achieved performance is minimized. This link between modeling and control is made explicit through the estimation criteria

$$\|S_0 - S\|_2 = \|(1 + G_0 K)^{-1} - (1 + GK)^{-1}\|_2,$$

which can also be written as

$$\|(G_0 - G)K(1 + G_0 K)^{-1}(1 + GK)^{-1}\|_2,$$

where $(1 + G_0 K)^{-1}$ comes directly from closed-loop frequency response measurements of the controller implemented on the real system. A 2nd order model estimate obtained through least squares minimization (2.33) is presented in Fig. 3.2.

An upper bound on model uncertainty can account for low-frequency effects such as friction as well as neglected high-frequency dynamics within product variation in flexible HDD suspensions. Consider a set of models $\mathcal{G}$ consisting of the nominal model G along with an upper bound allowable multiplicative model perturbation W_M

$$\mathcal{G} = \left\{ G \mid (I + \Delta W_M)G, \ \|\Delta\|_\infty \leq \beta^{-1} \right\}, \tag{3.31}$$

such that the real system, represented by G_0, is contained in the model set $G_0 \in \mathcal{G}$. A 3rd order uncertainty overbound is presented in Fig. 3.3. An upperbound on the

allowable perturbation level β can be normalized to unity [94], however to simplify the problem setup β will be determined during the analysis.

Performance analysis of the disk drive servo system characterized by the model set (3.31) is achieved through the construction of the generalized plant

$$\begin{pmatrix} z \\ y \end{pmatrix} = M \begin{pmatrix} w \\ v \end{pmatrix}, \qquad M = \begin{bmatrix} W_M GK(I+GK)^{-1} & W_M K(I+GK)^{-1} \\ G(I+GK)^{-1} & (I+GK)^{-1} \end{bmatrix},$$

with respect to the structured uncertainty

$$\Delta_T = \mathrm{diag}[\Delta, \Delta_P], \qquad \|\Delta\|_\infty < \beta^{-1}, \qquad \|\Delta_P\|_\infty \le \beta^{-1}.$$

Subsequently, structured singular value μ analysis is performed with respect to the structured uncertainty Δ_T.

A line search for the minimum β in $\Omega = [10, 7^3]$Hz for which M is robustly stable using four different methods. The results are presented in Fig. 3.4. The four methods shown in Fig. 3.4 are: a) solving problem (3.27) on a dense but finite frequency grid with 100 logarithmically spaced frequencies (solid line with data markers) b) solving the pair of LMIs in Theorem 3.2 (solid lines) with Θ_i given in (3.30), c) solving the Lemma 3.4 (dashed lines) with constant Θ given by

$$\Theta = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} Z_i & -jY_i \\ jY_i & -\beta^2 Z_i \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix},$$

and d) minimizing β using Lemma 3.2 (solid line) with constant Θ given above. Note that for this example the uncertainty structure is such that $\mu_\Delta = \rho_\Delta$ at each frequency ω. The extra freedom provided by the frequency-dependent multipliers in (3.18) allow for a much less conservative upper bound ρ_Δ when compared to Lemma 3.2.

The process is now repeated by further subdividing $\Omega = \cup_{i=1}^3 \Omega_i$ into three frequency ranges $\Omega_1 = [10, 10^2]$, $\Omega_2 = [10^2, 10^3]$, $\Omega_3 = [10^3, 7 \cdot 10^3]$. The problem is solved on each range using Lemma 3.4 and by solving the pair of LMIs in Theorem 3.2. The results are shown in Fig. 3.5. Note that for frequency ranges below $\omega_1 < 1KHz$ Lemma 3.4 provides an upper bound similar to the method of Theorem 3.2, however for $\omega_1 > 1KHz$ Lemma 3.4 provides a conservative upper bound. Note that smaller frequency intervals in general reduce the overall amount of conservatism for both methods.
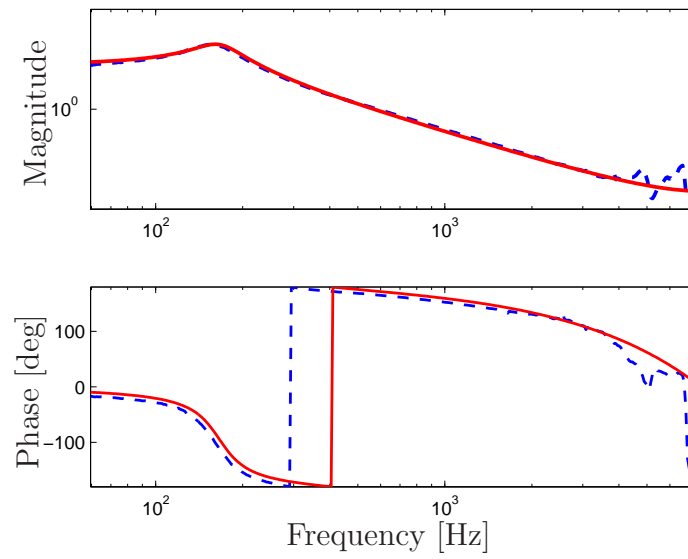
Figure 3.2. Control-relevant hard disk drive model (solid-line) estimated via frequency data (dashed-line) curve fitting.
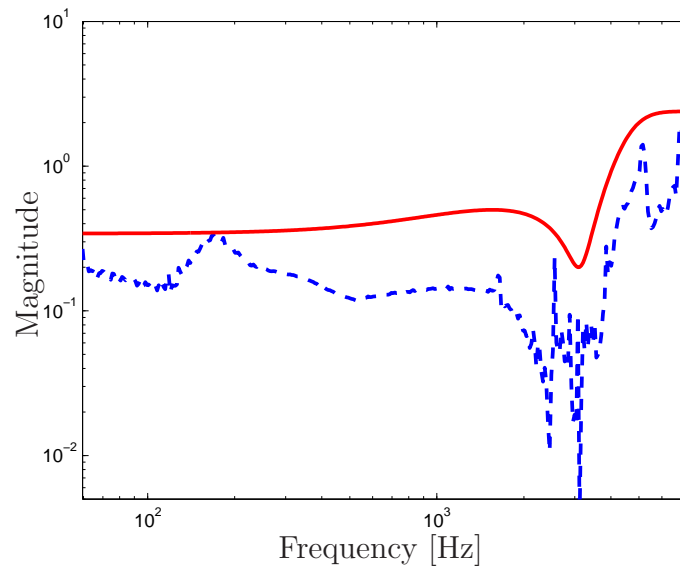
Figure 3.3. Hard disk drive model multiplicative uncertainty (solid line) overbounding frequency domain uncertainty (dashed-line).
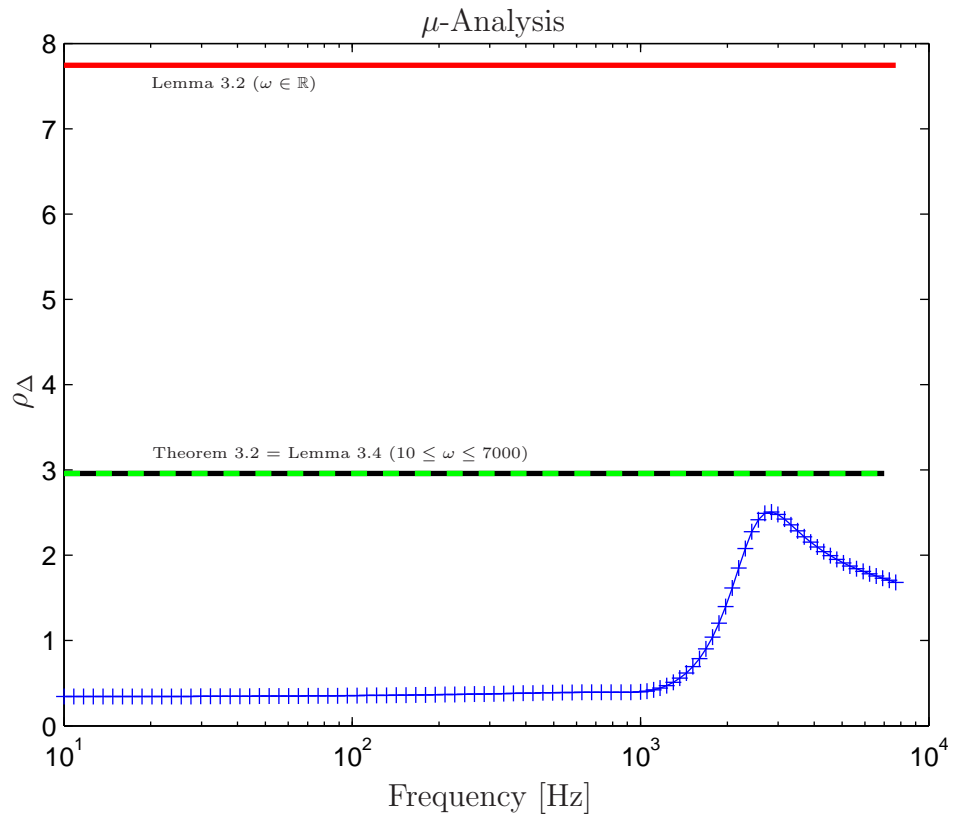
Figure 3.4. Structured singular value performance analysis of the hard disk servo system for a single frequency interval $\Omega = [10, 7000]$Hz. Labels are assigned according to the method used in computing the overbound.
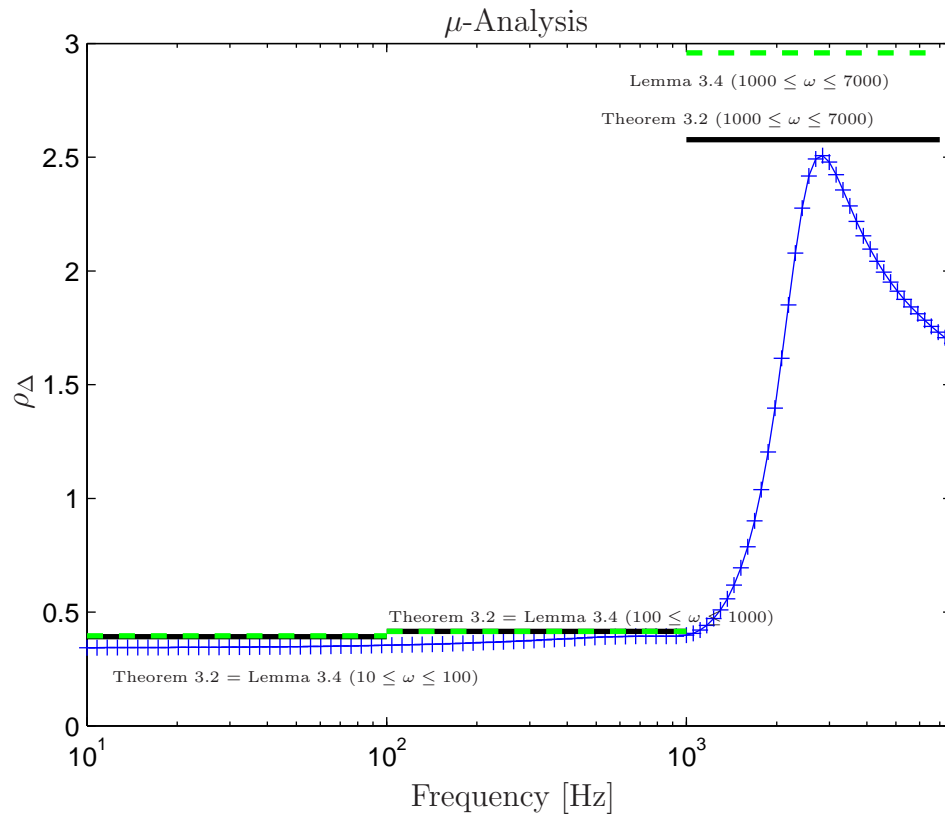
Figure 3.5. Structured singular value performance analysis of the hard disk servo system for three frequency intervals $\Omega = [10, 100]$, $[100, 1000]$, and $[1000, 7000]$Hz. Labels are assigned according to the method used in computing the overbound.

3.6 Acknowledgements

Some text of this chapter includes the reprints of the following papers.

Chapter 4

Generalizations of the KYP Lemma

4.1 Motivation

Most system identification algorithms estimate models in a discrete-time framework, while the analysis conditions proposed in Chapter 3 are based on continuous-time system descriptions. Although discrete-time models are easily converted to continuous-time (and vise versa) through the well-known bilinear transformation, the quality of the resulting models for analysis depend upon the approximation method, i.e. zero-order-hold, Tustin or pole-matched algorithms. It is preferable to develop analysis conditions directly in the various systems descriptions so as to not introduce unnecessary model uncertainty.

Problem 4.1. *Investigate generalizations and specific extensions for the analysis conditions proposed in Chapter 3, such as stability analysis of continuous-time, discrete-time and polynomial systems.*

The technical developments of this chapter are devoted to constructing generalized versions of Theorem 3.1 and Theorem 3.2.

4.2 Frequency Characterization

One of the key developments in generalizing the original formulation of the KYP Lemma, that is Lemma 3.2, is the possibility of characterizing finite frequency ranges under a unified framework. A frequency range can be visualized as a curve (or curves) on the complex plane. For instance, a finite frequency interval in continuous-time is a segment of the imaginary axis while in discrete-time it is an arc on the unit circle. This section is focused on presenting the general framework for representing (possibly finite) frequency ranges as developed in [46, 47, 48]. The frequency ranges considered can all be characterized from a certain class of curves on the complex plane.

Let $s \in \mathbb{C}$ and $\Sigma \in \mathbb{HC}^2$ be given and define the function

$$\sigma(s, \Sigma) := \begin{pmatrix} s \\ 1 \end{pmatrix}^* \Sigma \begin{pmatrix} s \\ 1 \end{pmatrix}.$$

Consider the class of curves on the complex plane that can be parametrized by sets of the form

$$\Lambda(\Phi, \Psi) := \{s \in \mathbb{C} : \quad \sigma(s, \Phi) = 0, \quad \sigma(s, \Psi) \geq 0\}. \tag{4.1}$$

where $\Phi, \Psi \in \mathbb{HC}^2$ are given. In [48], conditions on Φ and Ψ have been presented for which the above set represents a curve. Some of these results will be presented here, since they play a central role in providing extensions of the original KYP Lemma. First a necessary and sufficient condition for $\Lambda(\Phi, \Psi)$ to characterize curves is presented through a common congruence transformation of the matrices Φ and Ψ. The result, is proved in [47, 48] and also shown here for completeness, is based on simultaneous matrix factorization.

Lemma 4.1. *(Simultaneous Matrix Factorization [47]) Let $\Phi, \Psi \in \mathbb{HC}^2$ be given. Suppose $\det(\Phi) < 0$. Then there exists a common congruence transformation $T \in \mathbb{C}^{2 \times 2}$ such that $\Phi = T^* \Phi_0 T$, $\Psi = T^* \Psi_0 T$ where*

$$\Phi_0 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad\qquad \Psi_0 := \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}, \tag{4.2}$$

and $\alpha, \beta, \gamma \in \mathbb{R}$ and $\alpha \leq \gamma$.

Proof: Since $\det(\Phi) < 0$, there exists a nonsingular matrix K such that

$$\Phi = K^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} K.$$

Define the matrix $\Upsilon := K^{-*}\Psi K^{-1} \in \mathbb{HC}^2$ and let the entries of Υ be defined as

$$\Upsilon = \begin{bmatrix} x & \beta + jy \\ \beta - jy & z \end{bmatrix}, \qquad \beta, x, y, z \in \mathbb{R},$$

then

$$\Upsilon_0 := \begin{bmatrix} x & y \\ y & z \end{bmatrix} = J\left(\Upsilon - \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}\right) J^*, \qquad J := \begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix} \tag{4.3}$$

One can compute the spectral factorization of $\Upsilon_0 \in \mathbb{S}^2$ which gives

$$\Upsilon_0 = U^T \begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix} U, \tag{4.4}$$

where the columns of U^T are the eigenvectors and α and γ are the eigenvalues. Since $\Upsilon_0 \in \mathbb{S}^2$, $\alpha, \gamma \in \mathbb{R}$ and $U \in \mathbb{R}^{2\times 2}$ can be chosen to satisfy $\det(U) = 1$. Define the set of matrices

$$\mathbf{L} := \left\{ J^*UJ : \quad U \in \mathbb{R}^{2\times 2}, \det(U) = 1 \right\},$$

and set (4.3) equal to (4.4) to get

$$\Upsilon = L^* \begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix} L + \beta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad L = J^*UJ.$$

Notice that for any $L \in \mathbf{L}$

$$L^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

so that plugging into the previous expression results in the factorization

$$\Upsilon = L^* \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} L,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. Finally defining $T := LK$ yields the desired simultaneous factorization $\Phi = T^*\Phi_0 T$ and $\Psi = T^*\Psi_0 T$. $\qquad\square$

The above lemma states that for any $\Phi, \Psi \in \mathbb{HC}^2$ with $\det(\Phi) < 0$ there exists a common congruence transformation that relates them back to Φ_0, Ψ_0. In terms of frequency range characterization, this translates to the statement that the set $\mathbf{\Lambda}(\Phi, \Psi)$ characterizes curves on the complex plane if the set $\mathbf{\Lambda}(\Phi_0, \Psi_0)$ does. This is stated formally in the following proposition that characterizes all curves parametrized by $\mathbf{\Lambda}(\Phi, \Psi)$.

Proposition 4.1. *Let* $\Phi, \Psi \in \mathbb{HC}^2$ *be given and define the set* $\mathbf{\Lambda}(\Phi, \Psi)$ *by* (4.1). *Then* $\mathbf{\Lambda}(\Phi, \Psi)$ *represents curves on the complex plane if and only if* $\det(\Phi) < 0$ *and either* $0 \leq \alpha \leq \gamma$ *or* $\alpha < 0 < \gamma$, *where* α *and* γ *are defined by the factorization* (4.2).

Proof (Outline): A formal proof can be found in [47], however only general arguments shall be presented here to convince the reader of this proposition. Note that $\sigma(s, \Phi_0) = s^* + s = 0$ which implies that $s = j\omega$ for any $\omega \in \mathbb{R}$, and as a result $\sigma(j\omega, \Psi_0) = \alpha\omega^2 + \gamma \geq 0$. In case $0 \leq \alpha < \gamma$, the inequality $\sigma(j\omega, \Psi) \geq 0$ holds for all $\omega \in \mathbb{R}$, thus parametrizing the entire imaginary axis. In case $\alpha < 0 < \gamma$, the inequality $\sigma(j\omega, \Psi) \geq 0$ holds for $-|\alpha/\gamma|^{1/2} \leq \omega \leq |\alpha/\gamma|^{1/2}$, thus parametrizing a symmetric segment of the imaginary axis. Now the use of Lemma 4.1 states that all curves parametrized by (4.1) are equivalent to the curve

$$\mathbf{\Lambda}(\Phi_0, \Psi_0) := \{s = j\omega, \quad \omega \in \mathbb{R} : \quad \alpha\omega^2 + \gamma \geq 0\}, \tag{4.5}$$

through a congruent transformation. $\qquad\qquad\square$

The previous proposition establishes a relationship between the curves specified by $\mathbf{\Lambda}(\Phi_0, \Psi_0)$ and $\mathbf{\Lambda}(\Phi, \Psi)$ through their common congruent transformation. This relationship can be made more precise through the *linear fractional transformation*[1], which provides a one-to-one mapping of (possibly infinite) curves on the complex plane to other curves on the complex plain. The following lemma was presented in [46, 48] and the proof will be shown here as well for completeness.

---

[1]Also known as the *bilinear transformation* in standard complex analysis textbooks [14].

Lemma 4.2. *(Bilinear Transformation) Let* Φ_0, Ψ_0 *in* (4.2) *and a nonsingular matrix*

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{C}^{2 \times 2} \tag{4.6}$$

be given. Define the linear-fractional transformation

$$\psi : \mathbb{C} \to \mathbb{C}, \quad \psi(s) = \frac{b - d\,s}{c\,s - a}. \tag{4.7}$$

Then the following is true

$$\{\xi \in \mathbb{C} : \quad \xi \in \Lambda(T^*\Phi_0 T, T^*\Psi_0 T), \quad c\xi + d \neq 0\} =$$
$$\{\psi(s) \in \mathbb{C} : \quad s \in \Lambda(\Phi_0, \Psi_0), \quad c\,s \neq a\}. \tag{4.8}$$

Proof: To show the direction $\Rightarrow$, let $\xi \in \Lambda(T^*\Phi_0 T, T^*\Psi_0 T)$ with $c\xi + d \neq 0$ so that $\sigma(\xi, T^*\Phi_0 T) = 0$ and $\sigma(\xi, T^*\Psi_0 T) \geq 0$ and let $\xi = \psi(s)$. Note that the inverse of the linear-fractional transformation gives

$$s = \frac{a\xi + b}{c\xi + d},$$

and the following identity holds

$$\sigma(\psi(s), T^*\Sigma T) = \begin{pmatrix} \frac{b-ds}{cs-a} \\ 1 \end{pmatrix}^* \begin{bmatrix} a & b \\ c & d \end{bmatrix}^* \Sigma \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} \frac{b-ds}{cs-a} \\ 1 \end{pmatrix},$$

$$= \begin{pmatrix} \frac{s^*(c^*b^* - d^*a^*)}{s^*c^* - a^*} & \frac{c^*b^* - d^*a^*}{s^*c^* - a^*} \end{pmatrix} \Sigma \begin{pmatrix} \frac{(bc-ad)s}{cs-a} \\ \frac{bc-ad}{cs-a} \end{pmatrix},$$

$$= \left| \frac{bc - ad}{cs - a} \right|^2 \sigma(s, \Sigma).$$

It follows that $\sigma(s, \Phi_0) = 0$, $\sigma(s, \Psi_0) \geq 0$ and from the nonsingularity of T

$$cs - a = \frac{bc - ad}{c\xi + d} \neq 0.$$

The direction $\Leftarrow$ immediately follows from the identity above, where letting $s \in \Lambda(\Phi_0, \Psi_0)$ with $cs \neq a$ implies that $\sigma(\xi, T^*\Phi_0 T) = 0$, $\sigma(\xi, T^*\Psi_0 T) \geq 0$ and

$$c\xi + d = \frac{bc - ad}{cs - a} \neq 0.$$

$\square$

In the context of future developments, a slightly modified version of Lemma 4.2, based on the inverse linear transformation introduced in its proof, is also enlightening. It is given as the next corollary, whose proof is omitted for brevity.

Corollary 4.1. *Let Φ_0, Ψ_0 in (4.2) and a nonsingular matrix T as in (4.6) be given. Define the inverse linear-fractional transformation*

$$\psi^{-1} : \mathbb{C} \to \mathbb{C}, \quad \psi^{-1}(\xi) = \frac{a\,\xi + b}{c\,\xi + d}. \tag{4.9}$$

The following is true

$$\{\omega \in \mathbb{R}: \quad j\omega \in \mathbf{\Lambda}(\Phi_0, \Psi_0), \quad jc\omega \neq a\} =$$
$$\{\psi^{-1}(\xi) \in j\mathbb{R}: \quad \xi \in \mathbf{\Lambda}(T^*\Phi_0 T, T^*\Psi_0 T), \quad c\xi + d \neq 0\}. \tag{4.10}$$

Lemma 4.2 and Corollary 4.1 are completely equivalent, however, (4.10) highlights the fact that any curve given by sets $\mathbf{\Lambda}(\Phi, \Psi)$ can be indeed parametrized by a transformation of a segment of or the entire imaginary axis. Notice that nonsingularity of T, i.e. $ad \neq bc$, excludes the possibility of the image of the mapping ψ be reduced to a single point in $\{-d/c, -jb/a, \infty, -\infty\}$, depending on whether the letters a, b, c and d are not zero. Conversely, nonsingularity of T excludes the possibility of the image of the mapping ψ^{-1} be reduced to a single point in $\{-jb/d, -ja/c, j\infty, -j\infty\}$.

4.3 Generalized KYP Lemma

The following generalized KYP Lemma is developed from the nonconservative S-procedure that was discussed Section 3.3. It gives necessary and sufficient conditions for replacing the infinite dimensional FDI constraints, where the frequency variable is specified by general curves on the complex plain, with a single inequality. Indicated by the discussion in the previous section, the bilinear transformation is used to map the general frequency curve to a finite (or semi-infinite) segment imaginary axis where the methods of Section 3.3 can be used to convert the inequality conditions. The following lemma gives the general KYP Lemma. A formal proof can be found in [48], which includes very many technical details and includes

derivation of generalized multiplier matrices that are lossless in characterizing generalized finite (semi-infinite) frequency intervals. Avoiding such repetition, only a general outline for the proof will be presented here.

Lemma 4.3 (Generalized KYP Lemma [48]). *Let $a, b, c, d \in \mathbb{C}$ with $ad \neq bc$, $T \in \mathbb{C}^{2 \times 2}$ as in (4.6), the inverse linear-fractional mapping $\psi^{-1} : \mathbb{C} \to \mathbb{C}$ be given as in (4.9), and Φ_0, Ψ_0 be given by (4.2) and define $\Phi := T^*\Phi_0 T$ and $\Psi := T^*\Psi_0 T$. Let matrices $H \in \mathbb{C}^{2n \times (n+m)}$ and $\Theta \in \mathbb{HC}^{n+m}$ be also given. Assume that $\alpha < 0 < \gamma < \infty$ and that $jc\tilde{\omega} \neq a$ for all $\tilde{\omega}_1 \leq \tilde{\omega} \leq \tilde{\omega}_2$. The following statements are equivalent.*

(i) The FDI

$$\left(\begin{bmatrix} I & -\xi I \end{bmatrix} H\right)^*_\perp \Theta \left(\begin{bmatrix} I & -\xi I \end{bmatrix} H\right)_\perp \prec 0, \tag{4.11}$$

holds for all $\xi \in \Lambda(\Phi, \Psi)$.

(ii) There exist matrices $P, Q \in \mathbb{HC}^n$ such that $Q \succeq 0$ and

$$H^* (\Phi \otimes P + \Psi \otimes Q) H + \Theta \prec 0. \tag{4.12}$$

Proof (Outline): From Finsler's Lemma (Lemma 3.1), the FDI (4.11) holds if and only if

$$x^*\Theta x < 0, \quad \text{for all } x \in \mathcal{X},$$

where

$$\mathcal{X} := \left\{ x \in \mathbb{C}^{n+m} : \begin{bmatrix} I & -\xi I \end{bmatrix} Hx = 0, \ x \neq 0, \ \xi \in \Lambda(\Phi, \Psi) \right\}.$$

Note that for any $s \in \mathbb{C}$ such that $cs \neq a$ we have

$$\begin{bmatrix} I & -sI \end{bmatrix} (T \otimes I)H = (a - cs) \begin{bmatrix} I & -\dfrac{b - ds}{cs - a}I \end{bmatrix} H = (a - cs) \begin{bmatrix} I & -\psi(s)I \end{bmatrix} H, \tag{4.13}$$

so that the set $\mathcal{X}$ is equivalent to

$$\mathcal{X} = \left\{ x \in \mathbb{C}^{n+m} : \begin{bmatrix} I & -sI \end{bmatrix} (T \otimes I)Hx = 0, \ x \neq 0, \ s \in \Lambda(\Phi_0, \Psi_0) \right\}.$$

Let $y := (T \otimes I)Hx \in \mathbb{C}^{2n}$, where y is partitioned as $y^* = [y_1^* \; y_2^*]$, $y_i \in \mathbb{C}^n$. Note that from the set $\mathcal{X}$, $y_1 = j\omega y_2$ holds for some $\omega \in \mathbb{R}$ such that $\alpha\omega^2 + \gamma \geq 0$ and

$$y^* \left(\Phi_0 \otimes P + \Psi_0 \otimes Q\right) y = \left[\alpha\omega^2 + \gamma\right] y_2^* Q \, y_2.$$

Since, $Q \succeq 0$ for all $\omega \in \mathbb{R}$ such that $\alpha\omega^2 + \gamma \geq 0$ the following holds (see Lemma 3.4 or [49], Lemma 2)

$$y^* \left(\Phi_0 \otimes P + \Psi_0 \otimes Q\right) y \geq 0, \qquad \text{for all } P, Q \in \mathbb{H}\mathbb{C}^n, \; Q \succeq 0,$$

so that the set $\mathcal{X}$ can equivalently be written as

$$\mathcal{X} = \{x \in \mathbb{C}^n : x^* S x \geq 0, \; x \neq 0, \; S \in \mathcal{S}\}, \tag{4.14}$$

where

$$\mathcal{S} = \{S = H^* \left(\Phi \otimes P + \Psi \otimes Q\right) H : P, Q \in \mathbb{H}\mathbb{C}^n, \; Q \succeq 0\}.$$

It is easy to verify that $\mathcal{S}$ satisfies properties (i) and (ii) in Definition 3.3.1. The property (iii) has been shown in [48] which leads to a nonconservative version of the S-procedure Lemma 3.3. $\qquad\square$

Note that the only case included in the Generalized KYP Lemma of [48] that is not presented in Lemma 4.3 is the case when Ψ_0 is such that $0 \leq \alpha \leq \gamma$. However, as noticed in [48], this implies $\Psi_0 \succeq 0$ which means that Q can be set to zero in (4.12), reducing the Generalized KYP Lemma to a standard frequency indepen-dent KYP Lemma. Indeed, for any choice of $0 \leq \alpha \leq \gamma$, the curve associated to $\Lambda(\Phi_0, \Psi_0)$ is the entire imaginary axis. For this reason, there is no need to treat such a case separately.

The previous lemma generalized the finite frequency KYP Lemma 3.4 for curves that can be characterized by the set $\Lambda(\Phi, \Psi)$. From a practical perspective, it al-lows to pose and check frequency domain specifications other than on the imaginary axis, which is relevant for specific applications and will be explored in Chapter 4. Working similar to the developments in Section 3.3, the next theorem establishes the equivalence between the previous lemma, that is with the general FDI (4.11) and LMI (4.12), and an *extended* condition based on the work [22].

Theorem 4.1. *Let $a, b, c, d \in \mathbb{C}$ with $ad \neq bc$, $T \in \mathbb{C}^{2\times 2}$ as in (4.6), the inverse linear-fractional mapping $\psi^{-1} : \mathbb{C} \to \mathbb{C}$ be given as in (4.9), and Φ_0, Ψ_0 be given by (4.2) and define $\Phi := T^*\Phi_0 T$ and $\Psi := T^*\Psi_0 T$. Let matrices $H \in \mathbb{C}^{2n\times(n+m)}$ and $\Theta \in \mathbb{HC}^{n+m}$ be also given. Assume that $\alpha < 0 < \gamma < \infty$ and define*

$$\tilde{\omega}_1 = -|\gamma/\alpha|^{1/2}, \qquad\qquad \tilde{\omega}_2 = |\gamma/\alpha|^{1/2}. \qquad (4.15)$$

Assume that $jc\tilde{\omega} \neq a$ for all $\tilde{\omega}_1 \leq \tilde{\omega} \leq \tilde{\omega}_2$. The following statements are equivalent.

(i) The FDI

$$\left(\begin{bmatrix} I & -\xi I \end{bmatrix} H\right)^*_{\perp} \Theta \left(\begin{bmatrix} I & -\xi I \end{bmatrix} H\right)_{\perp} \prec 0,$$

holds for all $\xi \in \Lambda(\Phi, \Psi)$.

(ii) There exist matrices $F \in \mathbb{C}^{n\times n}$, $G \in \mathbb{C}^{m\times n}$ such that the pair of LMI

$$\mathrm{He}\left\{\begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\tilde{\omega}_i I \end{bmatrix} (T \otimes I)H\right\} + \Theta \prec 0, \qquad i = \{1, 2\} \qquad (4.16)$$

Proof: To show sufficiency of (ii), i.e. the direction (ii) $\Rightarrow$ (i), assume that the pair of inequalities (4.16) have feasible solutions. The sum of (4.16) for $i = 1$ multiplied by $\lambda(\tilde{\omega}) = (\tilde{\omega}_2 - \tilde{\omega})/(\tilde{\omega}_2 - \tilde{\omega}_1) \in [0, 1]$ and of (4.16) for $i = 2$ multiplied by $(1 - \lambda(\tilde{\omega}))$ produces

$$\mathrm{He}\left\{\begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\tilde{\omega} I \end{bmatrix} (T \otimes I)H\right\} + \Theta \prec 0, \quad \text{for all } \tilde{\omega}_1 \leq \tilde{\omega} \leq \tilde{\omega}_2.$$

Recall that for any $s \in \mathbb{C}$ with $cs \neq a$ the relation (4.13) holds. Since $a \neq jc\tilde{\omega}$ for all $\tilde{\omega}_1 \leq \tilde{\omega} \leq \tilde{\omega}_2$, multiply the inequality above on the right by

$$\tilde{N}(\tilde{\omega}) := \left(\begin{bmatrix} I & -j\tilde{\omega} I \end{bmatrix} (T \otimes I)H\right)_{\perp} = \left(\begin{bmatrix} I & -\psi(j\tilde{\omega})I \end{bmatrix} H\right)_{\perp} \qquad (4.17)$$

and on the left by its transpose conjugate to obtain the frequency domain inequality

$$\left(\begin{bmatrix} I & -\psi(j\tilde{\omega})I \end{bmatrix} H\right)^*_{\perp} \Theta \left(\begin{bmatrix} I & -\psi(j\tilde{\omega})I \end{bmatrix} H\right)_{\perp} \prec 0,$$

which should hold for all $\tilde{\omega}_1 \leq \tilde{\omega} \leq \tilde{\omega}_2$. Since $\alpha < 0 < \gamma < \infty$ the set $\{s = j\tilde{\omega}, \ \tilde{\omega} \in \mathbb{R} : \ \tilde{\omega}_1 \leq \tilde{\omega} \leq \tilde{\omega}_2\}$ with (4.15) is equivalent to $\Lambda(\Phi_0, \Psi_0)$ as given in (4.5). Therefore, we can use Lemma 4.2 to establish (4.11) for all $\xi \in \Lambda(\Phi, \Psi)$.

To show necessity of item (ii), i.e. the direction (i) $\Rightarrow$ (ii), assume the FDI (4.11) holds for all $\xi \in \mathbf{\Lambda}(\Phi, \Psi)$. Then from Lemma 4.3 there exists some $P, Q \succeq 0$ satisfying (4.12). Define the matrix

$$\tilde{X}(\tilde{\omega}) := \begin{bmatrix} -\alpha Q & j\tilde{\omega}\alpha Q \\ -j\tilde{\omega}\alpha Q & \gamma Q \end{bmatrix} = \begin{bmatrix} -\alpha & j\tilde{\omega}\alpha \\ -j\tilde{\omega}\alpha & \gamma \end{bmatrix} \otimes Q.$$

Note that since $\alpha < 0$, then $\tilde{X}(\tilde{\omega}) \succeq 0$ for all $s = j\tilde{\omega} \in \mathbf{\Lambda}(\Phi_0, \Psi_0)$ because using Schur complement

$$Q \succeq 0, \quad \alpha\tilde{\omega}^2 + \gamma \geq 0, \quad \alpha < 0, \qquad \Longrightarrow \qquad \tilde{X}(\tilde{\omega}) \succeq 0$$

Now add the matrix

$$H^*(T \otimes I)^* \tilde{X}(\tilde{\omega})(T \otimes I)H \succeq 0, \qquad \text{for all } s = j\tilde{\omega} \in \mathbf{\Lambda}(\Phi_0, \Psi_0)$$

to the right hand side of (4.12) so that for all $s = j\tilde{\omega} \in \mathbf{\Lambda}(\Phi_0, \Psi_0)$ we have

$$\Theta \prec H^*(T \otimes I)^* \left[-\Phi_0 \otimes P - \Psi_0 \otimes Q + \tilde{X}(\tilde{\omega}) \right] (T \otimes I) H,$$

$$= H^*(T \otimes I)^* \left(\begin{bmatrix} -2\alpha Q & -(P + \beta Q) + j\tilde{\omega}\alpha Q \\ -(P + \beta Q) - j\tilde{\omega}\alpha Q & 0 \end{bmatrix} \right) (T \otimes I) H,$$

which can then be written as

$$H^*(T \otimes I)^* \left(\begin{bmatrix} \alpha Q \\ P + \beta Q \end{bmatrix} \begin{bmatrix} I & -j\tilde{\omega}I \end{bmatrix} + \right.$$

$$\left. \begin{bmatrix} I \\ j\tilde{\omega}I \end{bmatrix} \begin{bmatrix} \alpha Q & P + \beta Q \end{bmatrix} \right) (T \otimes I) H + \Theta \prec 0.$$

Now choose

$$\begin{bmatrix} F \\ G \end{bmatrix} = H^*(T \otimes I)^* \begin{bmatrix} \alpha Q \\ P + \beta Q \end{bmatrix} \tag{4.18}$$

so that

$$\mathrm{He} \left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\tilde{\omega}I \end{bmatrix} (T \otimes I) H \right\} + \Theta \prec 0$$

for any $\tilde{\omega}_1 \leq \tilde{\omega} \leq \tilde{\omega}_2$, where $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are given by (4.15), in particular, for $\tilde{\omega} = \tilde{\omega}_1$ and $\tilde{\omega} = \tilde{\omega}_2$ which imply that the pair of inequalities (4.16) are feasible. $\qquad \square$

Similar to the developments of Theorem 3.2, the case of constant Θ can be extended to incorporate a particular class of frequency-dependent Θ without incurring extra computational cost in solving the LMI (4.16). In the current context of the generalized KYP Lemma, consider the same class of functions (3.16) affine on the transformed frequency variable $\tilde{\omega}$, that is

$$\Theta(\tilde{\omega}) = \frac{\tilde{\omega}_2 - \tilde{\omega}}{\tilde{\omega}_2 - \tilde{\omega}_1}\Theta_1 + \frac{\tilde{\omega} - \tilde{\omega}_1}{\tilde{\omega}_2 - \tilde{\omega}_1}\Theta_2, \qquad -|\gamma/\alpha|^{1/2} \leq \tilde{\omega} \leq |\gamma/\alpha|^{1/2}. \qquad (4.19)$$

The following theorem is a version of Theorem 3.2 in the context of the generalized FDI (4.11), allowing Θ to depend affinely on frequency.

Theorem 4.2. *Let $a, b, c, d \in \mathbb{C}$ with $ad \neq bc$, $T \in \mathbb{C}^{2\times 2}$ as in (4.6), the inverse linear-fractional mapping $\psi^{-1} : \mathbb{C} \to \mathbb{C}$ be given as in (4.9), and Φ_0, Ψ_0 be given by (4.2). Let matrices $H \in \mathbb{C}^{2n \times (n+m)}$ and Θ_1, $\Theta_2 \in \mathbb{HC}^{n+m}$ be also given. Assume that $\alpha < 0 < \gamma < \infty$ and define*

$$\tilde{\omega}_1 = -|\gamma/\alpha|^{1/2}, \qquad\qquad \tilde{\omega}_2 = |\gamma/\alpha|^{1/2}.$$

Assume that $jc\tilde{\omega} \neq a$ for all $\tilde{\omega}_1 \leq \tilde{\omega} \leq \tilde{\omega}_2$. If there exist matrices $F \in \mathbb{C}^{n\times n}$, $G \in \mathbb{C}^{m\times n}$ such that the pair of LMI

$$\text{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\tilde{\omega}_i I \end{bmatrix} (T \otimes I)H \right\} + \Theta_i \prec 0, \qquad i = \{1, 2\} \qquad (4.20)$$

have feasible solution then the FDI

$$\left(\begin{bmatrix} I & -\xi I \end{bmatrix} H \right)_{\perp}^{*} \Theta(-j\psi^{-1}(\xi)) \left(\begin{bmatrix} I & -\xi I \end{bmatrix} H \right)_{\perp} \prec 0, \qquad (4.21)$$

holds for all $\xi \in \mathbf{\Lambda}(T^*\Phi_0 T, T^*\Psi_0 T)$ with $\Theta(\tilde{\omega})$ given by (4.19).

Proof: Following the initial steps in the proof of Theorem 4.1, the sum of (4.20) for $i = 1$ multiplied by $\lambda(\tilde{\omega}) = (\tilde{\omega}_2 - \tilde{\omega})/(\tilde{\omega}_2 - \tilde{\omega}_1) \in [0, 1]$ and of (4.20) for $i = 2$ multiplied by $(1 - \lambda(\tilde{\omega}))$ produces

$$\text{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\tilde{\omega} I \end{bmatrix} (T \otimes I)H \right\} + \Theta(\tilde{\omega}) \prec 0, \quad \text{for all } \tilde{\omega}_1 \leq \tilde{\omega} \leq \tilde{\omega}_2 \qquad (4.22)$$

where $\Theta(\tilde{\omega})$ is given in (4.19). Multiply the inequality above on the right by (4.17) and on the left by its transpose conjugate to obtain the frequency domain inequality

$$\left(\begin{bmatrix} I & -\psi(j\tilde{\omega})I \end{bmatrix} H \right)_{\perp}^{*} \Theta(\tilde{\omega}) \left(\begin{bmatrix} I & -\psi(j\tilde{\omega})I \end{bmatrix} H \right)_{\perp} \prec 0,$$

which should hold for all $\tilde{\omega}_1 \leq \tilde{\omega} \leq \tilde{\omega}_2$. Since $\alpha < 0 < \gamma < \infty$ the finite frequency set $\{s = j\tilde{\omega}, \quad \tilde{\omega} \in \mathbb{R} : \quad \tilde{\omega}_1 \leq \tilde{\omega} \leq \tilde{\omega}_2\}$ with (4.15) is equivalent to $\boldsymbol{\Lambda}(\Phi_0, \Psi_0)$ as given in (4.5). Therefore, we can use Lemma 4.2, Corollary 4.1 and the correspondences

$$\xi = \psi(j\tilde{\omega}), \qquad\qquad \tilde{\omega} = -j\psi^{-1}(\xi),$$

which hold for all $\xi \in \boldsymbol{\Lambda}(T^*\Phi_0 T, T^*\Psi_0 T)$ to establish (4.21). $\qquad\square$

As with Lemma 4.3, the case when Ψ_0 is such that $0 \leq \alpha \leq \gamma$ is not explicitly treated in Theorem 4.2 since this implies that $\Psi_0 \succeq 0$ and the curve associated to $\boldsymbol{\Lambda}(\Phi_0, \Psi_0)$ is the entire imaginary axis. However we note that by exploring properties of linear-fractional mappings we are able to derive conditions that hold for the entire imaginary axis in which the associated matrix Ψ_0 is not positive semidefinite. As will be shown in Section 4.6, such conditions do not reduce to the standard KYP Lemma.

4.4 Revisiting the KYP Lemma on Frequency Intervals

Theorems 3.1 and 3.2 will be revisited in this section, where we illustrate how the results of Section 4.3 can be applied in the particular context of FDI analysis within finite frequency intervals. First note that the segment of the imaginary axis $j[\omega_1, \omega_2]$ can be described by the set $\boldsymbol{\Lambda}(\Phi, \Psi)$ with the choices (see [46, 48])

$$\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad\qquad \Psi = \begin{bmatrix} -1 & j\omega_c \\ -j\omega_c & -\omega_1\omega_2 \end{bmatrix}.$$

Recall that $\omega_c := (\omega_1 + \omega_2)/2$. From Lemma 4.1, there should exists a nonsingular transformation matrix $T \in \mathbb{C}^{2\times 2}$ such that this curve can be represented by

$\Lambda(T^*\Phi_0 T, T^*\Psi_0 T)$, where Φ_0 and Ψ_0 are given in the form (4.2). Note that the set $\Lambda(\Phi_0, \Psi_0)$ is (a segment of) the imaginary axis, in which case results for constant Θ exist [72, 4, 48]. One can verify that such transformation and the corresponding matrices Φ_0 and Ψ_0 are

$$T = \begin{bmatrix} 1 & -j\omega_c \\ 0 & 1 \end{bmatrix}, \qquad \Phi_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Psi_0 = \begin{bmatrix} -1 & 0 \\ 0 & \hat{\omega}^2 \end{bmatrix}. \qquad (4.23)$$

where $\hat{\omega} := (\omega_2 - \omega_1)/2$. Note that $-1 = \alpha < 0 < \gamma = \hat{\omega}^2$ for any $\omega_1 \neq \omega_2$ so that the proof of Theorems 4.1 and 4.2 applied with

$$H = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}, \quad \tilde{\omega}_1 = -|\hat{\omega}| = (\omega_1 - \omega_2)/2, \quad \tilde{\omega}_2 = |\hat{\omega}| = (\omega_2 - \omega_1)/2, \qquad (4.24)$$

implies feasibility of the FDIs (4.11) and (4.21) respectively. Furthermore, the case of Theorem 4.1 with (4.23) and (4.24) implies the existence of a solution F and G as in (4.18)

$$\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \left(\begin{bmatrix} 1 & -j\omega_c \\ 0 & 1 \end{bmatrix}^* \otimes I \right) \begin{bmatrix} -Q \\ P \end{bmatrix} = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q \\ P - j\omega_c Q \end{bmatrix},$$

which proves feasibility of the pair of inequalities (4.16) in the case Θ is constant. This is precisely the choice (3.15).

Finally, from both Theorems 4.1 and 4.2, note that

$$\begin{bmatrix} I & -j\tilde{\omega}_i I \end{bmatrix} (T \otimes I) = \begin{bmatrix} I & -j\omega_i I \end{bmatrix}, \qquad X(\omega) = (T \otimes I)^* \tilde{X}(\tilde{\omega})(T \otimes I),$$

after performing the change-of-variables

$$\omega := \tilde{\omega} + \omega_c, \qquad \omega_1 = \tilde{\omega}_1 + \omega_c \leq \omega \leq \tilde{\omega}_2 + \omega_c = \omega_2,$$

and where $X(\omega)$ was given in (3.14).

Another way to look at the change-of-variables introduced above is to note that the matrix T in (4.23) is associated with the transformation

$$\psi(s) = s + j\omega_c,$$

which could have been used to define the transformed frequency variable

$$jw = j(\tilde{\omega} + \omega_c).$$

In revisiting the results of Section 3.3 and from the context of the generalized Theorems 4.1 and 4.2, it should be clear that transformations of the frequency variable are what is at the root of the discussion in Section 4.3.

4.5 Piece-Wise Linear Coefficient Matrix

The results presented in previous sections can be extended to cope with piece-wise affine frequency-dependent matrices $\Theta(\omega)$ of the form

$$\Theta(\omega) := \begin{cases} \dfrac{\omega_2 - \omega}{\omega_2 - \omega_1}\,\Theta_1 + \dfrac{\omega - \omega_1}{\omega_2 - \omega_1}\,\Theta_2, & \omega_1 \le \omega \le \omega_2, \\[2ex] \dfrac{\omega_4 - \omega}{\omega_4 - \omega_3}\,\Theta_3 + \dfrac{\omega - \omega_3}{\omega_4 - \omega_3}\,\Theta_4, & \omega_3 \le \omega \le \omega_4, \\[2ex] \vdots \\[2ex] \dfrac{\omega_{2N} - \omega}{\omega_{2N} - \omega_{2N-1}}\,\Theta_{2N-1} + \dfrac{\omega - \omega_{2N-1}}{\omega_{2N} - \omega_{2N-1}}\,\Theta_{2N}, & \omega_{2N-1} \le \omega \le \omega_{2N}, \end{cases} \tag{4.25}$$

where N is any integer and $\Theta_i \in \mathbb{HC}^{n+m}$ for $i = 1, \ldots, 2N$, by simply solving the pair of inequalities (3.18) or (4.20) for each sub-interval $\omega_{2\ell-1} \le \omega \le \omega_{2\ell}$, $\ell = 1, \ldots, N$. The resulting LMI expressions from Theorem 3.2 are given by

$$\mathrm{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\omega_i I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} + \Theta_i \prec 0, \qquad i = 1, 2,$$

$$\mathrm{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\omega_i I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} + \Theta_i \prec 0, \qquad i = 3, 4,$$

$$\vdots \qquad\qquad \vdots$$

$$\mathrm{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\omega_i I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} + \Theta_i \prec 0, \qquad i = 2N-1, 2N,$$

Recall that $\Theta(\omega)$ is not a proper rational function of ω, and note that the piece-wise affine $\Theta(\omega)$ might not even be a continuous function of ω, or be defined on a contiguous interval. When the sub-intervals are contiguous, that is $\omega_{2\ell} = \omega_{2\ell+1}$, continuity of $\Theta(\omega)$ can be achieved by imposing $\Theta_{2\ell} = \Theta_{2\ell+1}$ for some or all $1 \le \ell < N$. Nevertheless, rational or other types of bounded functions of ω can be approximated by piecewise affine functions of ω, specially on finite frequency intervals. Fig. 4.1 illustrates a possible use of piece-wise affine $\Theta(\omega)$. Similar results

can be obtained for piece-wise $\Theta(\tilde{\omega})$ for intervals on the transformed frequency variable.
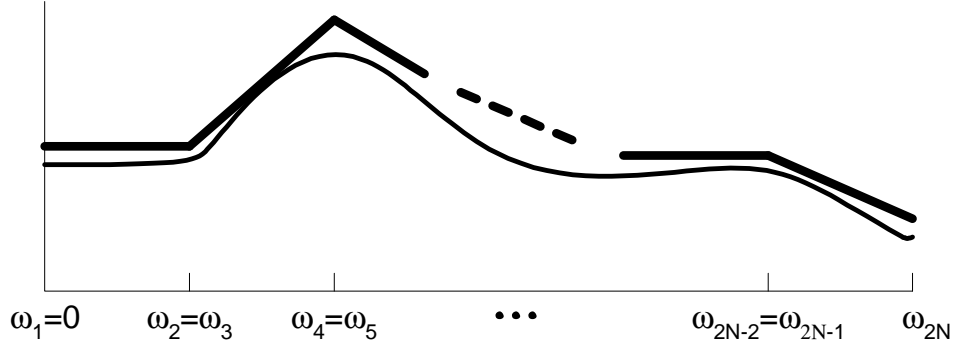
Figure 4.1. Illustrating piece-wise affine $\Theta(\omega)$.

One interesting application of piecewise affine matrices $\Theta(\omega)$ is to handle real valued matrices A and B in Theorem 3.2, or real valued H in Theorem 4.2. In such cases, as will be shown by the next lemma in the context of Theorem 4.2, there exists a piecewise affine matrix $\Theta(\omega)$ so that feasibility of the proposed test on the interval $\omega_1 \leq \omega \leq \omega_2$ also implies feasibility of the frequency domain inequality on the symmetric interval $-\omega_2 \leq \omega \leq -\omega_1$.

Theorem 4.3. *Let $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$, $T \in \mathbb{C}^{2\times 2}$ as in (4.6), the inverse linear-fractional mapping $\psi^{-1} : \mathbb{C} \to \mathbb{C}$ be given as in (4.9), and Φ_0, Ψ_0 be given by (4.2). Let matrices $H \in \mathbb{R}^{2n \times (n+m)}$ and Θ_1, $\Theta_2 \in \mathbb{HC}^{n+m}$ be also given. Assume that $\alpha < 0 < \gamma < \infty$ and define $\tilde{\omega}_1, \tilde{\omega}_2$ as in (4.15). If there exist matrices $F \in \mathbb{C}^{n \times n}$, $G \in \mathbb{C}^{m \times n}$ such that the pair of LMI (4.20) have feasible solutions then the FDI (4.21) holds for all $|\xi| \in \Lambda(T^*\Phi_0 T, T^*\Psi_0 T)$ with*

$$
\Theta(-j\psi(\xi)) := \begin{cases} \dfrac{\tilde{\omega}_2 + \tilde{\omega}}{\tilde{\omega}_2 - \tilde{\omega}_1} \overline{\Theta}_1 - \dfrac{\tilde{\omega}_1 + \tilde{\omega}}{\tilde{\omega}_2 - \tilde{\omega}_1} \overline{\Theta}_2, & -\tilde{\omega}_2 \leq \tilde{\omega} \leq -\tilde{\omega}_1, \\[2ex] \dfrac{\tilde{\omega}_2 - \tilde{\omega}}{\tilde{\omega}_2 - \tilde{\omega}_1} \Theta_1 + \dfrac{\tilde{\omega} - \tilde{\omega}_1}{\tilde{\omega}_2 - \tilde{\omega}_1} \Theta_2, & \tilde{\omega}_1 \leq \tilde{\omega} \leq \tilde{\omega}_2, \end{cases} \tag{4.26}
$$

where $\tilde{\omega} = -j\psi^{-1}(\xi)$.

Proof: Following the proof of Theorem 4.2 one can conclude on the feasibility

of (4.22) for all $\tilde{\omega}_1 \leq \tilde{\omega} \leq \tilde{\omega}_2$. Now take the complex conjugate of (3.20), i.e.

$$\text{He}\left\{\begin{bmatrix}\overline{F}\\\overline{G}\end{bmatrix}\begin{bmatrix}I & -j\tilde{\omega}I\end{bmatrix}(\overline{T}\otimes I)\,H\right\} + \Theta(\tilde{\omega}) \prec 0, \quad \text{for all } -\tilde{\omega}_2 \leq \tilde{\omega} \leq -\tilde{\omega}_1,$$

which should also hold with $\Theta(-j\psi^{-1}(\xi)) = \Theta(\tilde{\omega})$ as given in (4.26). Note that $\tilde{\omega}_i$, $i = \{1, 2\}$ in (4.15) form a symmetric interval so that $\tilde{\omega}_1 = -\tilde{\omega}_2$ and from (4.13) the relationship

$$\begin{bmatrix}I & j\tilde{\omega}\end{bmatrix}(\overline{T}\otimes I) = (\overline{a}+j\overline{c}\tilde{\omega})\begin{bmatrix}I & \left(\dfrac{\overline{b}+j\overline{d}\tilde{\omega}}{j\overline{c}\tilde{\omega}+\overline{a}}\right)I\end{bmatrix} = (\overline{a}+j\overline{c}\tilde{\omega})\begin{bmatrix}I & -\overline{\psi}(j\tilde{\omega})I\end{bmatrix}$$

holds for all $\tilde{\omega}_1 \leq \tilde{\omega} \leq \tilde{\omega}_2$. Multiply the inequality above on the right by

$$\left(\begin{bmatrix}I & -\overline{\psi}(j\tilde{\omega})\end{bmatrix}H\right)_\perp$$

and on the left by its transpose conjugate to conclude feasibility of (4.21) for all $|\xi| \in \Lambda(T^*\Phi_0 T, T^*\Psi_0 T)$. $\qquad\square$

Interestingly the transformation matrix T is not required to be real. To make this more clear, consider the above lemma in the context of Theorem 3.2, that is for $s = j\omega$ where $\omega_1 \leq \omega \leq \omega_2$.

Corollary 4.2. *Let matrices $A \in \mathbb{R}^{n\times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{R}^{n\times m}$ and $\Theta_1, \Theta_2 \in \mathbb{HC}^{n+m}$ be given. If there exist matrices $F \in \mathbb{C}^{n\times n}$, $G \in \mathbb{C}^{m\times n}$ such that the pair of LMI (3.18) have feasible solutions then the frequency domain inequality (3.19) holds for all $\omega_1 \leq |\omega| \leq \omega_2$ with*

$$\Theta(\omega) := \begin{cases} \dfrac{\omega_2+\omega}{\omega_2-\omega_1}\,\overline{\Theta}_1 - \dfrac{\omega_1+\omega}{\omega_2-\omega_1}\,\overline{\Theta}_2, & -\omega_2 \leq \omega \leq -\omega_1, \\[2ex] \dfrac{\omega_2-\omega}{\omega_2-\omega_1}\Theta_1 + \dfrac{\omega-\omega_1}{\omega_2-\omega_1}\Theta_2, & \omega_1 \leq \omega \leq \omega_2, \end{cases} \tag{4.27}$$

being piecewise affine.

Proof: Follows from application of Theorem 4.3 with (4.23) and (4.24).

Alternatively, one can prove the result directly noting that from the proof of Theorem 3.2 one can conclude on the feasibility of (3.20) for all $\omega_1 \leq \omega \leq \omega_2$. Now take the complex conjugate of (3.20), i.e.

$$\text{He}\left\{\begin{bmatrix}\overline{F}\\\overline{G}\end{bmatrix}\begin{bmatrix}I & -j\omega\end{bmatrix}\begin{bmatrix}A & B\\I & 0\end{bmatrix}\right\} + \Theta(\omega) \prec 0, \quad \text{for all } -\omega_2 \leq \omega \leq -\omega_1,$$

which should also hold with $\Theta(\omega)$ as given in (4.27). Feasibility of (3.19) can be concluded for all $\omega_1 \leq |\omega| \leq \omega_2$. $\qquad\square$

The previous corollary highlights the case when $\omega_1 = 0$. Note that $\Theta(\omega)$ defined in (4.27) may be discontinuous at $\omega = 0$. However, the above proof implies that the FDI (3.19) is satisfied at $\omega = 0$ for both $\Theta(0) = \Theta_1$ or $\Theta(0) = \overline{\Theta}_1$. The arguments used above do not require continuity of $\Theta(\omega)$ at $\omega = 0$ in order to conclude feasibility.

4.6   An Alternative KYP Lemma

Theorem 3.2 has difficulties in handling unbounded frequency ranges. For instance, in the case $\omega_2 \to \infty$, one could conceptually search for limits on the problem variables as ω_2 increases by solving a sequence of pairs of inequalities (3.18). A more elegant solution is to transform the frequency variable and solve a modified problem on the transformed frequency that now has a finite limit.

Consider the high-frequency condition $|\omega| \geq |z|$. This inequality can be described by the curve $\mathbf{\Lambda}(\Phi, \Psi)$ with

$$\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad\qquad \Psi = \begin{bmatrix} 1 & 0 \\ 0 & -z^2 \end{bmatrix}.$$

One can verify that this curve is associated with

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Phi_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Psi_0 = \begin{bmatrix} -z^2 & 0 \\ 0 & 1 \end{bmatrix},$$

with which application of Theorem 4.2 yields the pair of inequalities

$$\text{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} -j\tilde{\omega}_i I & I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} + \Theta_i \prec 0, \qquad i = \{1, 2\},$$

where

$$\tilde{\omega}_1 = -|1/z|, \qquad\qquad \tilde{\omega}_2 = |1/z|. \qquad\qquad (4.28)$$

The above approach, which corresponds to the linear-fractional transformation $\psi(s) = s^{-1}$, handles the limit $\omega \to \infty$ at the expense of a creating a singularity at $\omega = 0$. In this way, it cannot be used to construct a generalization of the KYP Lemma, that is, a condition that holds for all frequencies, since the extreme points (4.28) tend to infinity as $z \to 0$. This problem can be overcome by alternatively considering the linear-fractional mapping

$$\psi : \mathbb{C} \to \mathbb{C}, \quad \psi(s) = jz + \frac{jy\,(s - jx)}{(y - x)(jy - s)}, \qquad x, y, z \in \mathbb{R}, \quad x < y, \quad y > 0,$$

(4.29)

which maps the finite segment of the imaginary axis $s \in j[x, y)$ onto the infinite segment of the imaginary axis $\xi \in j[z, \infty)$, see illustration provided in Fig. 4.2. Note that the inverse transformation is given by
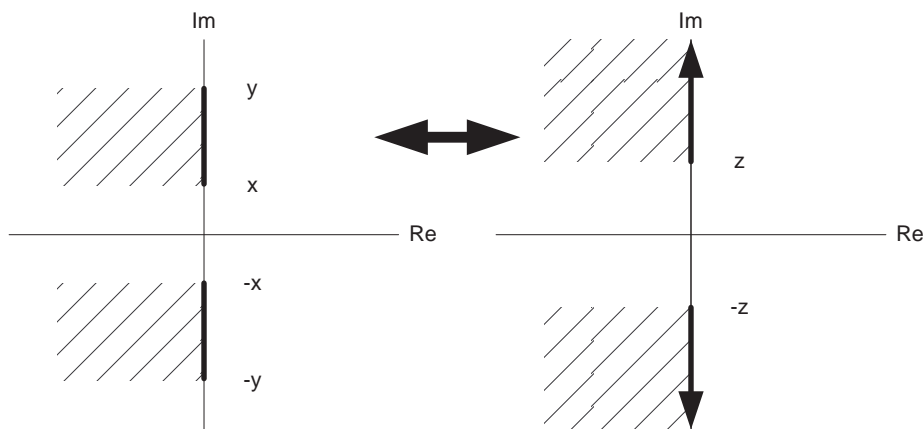
Figure 4.2. Bilinear transformation from finite segments to infinite segments of the frequency variable on the imaginary axis.

$$\psi^{-1} : \mathbb{C} \to \mathbb{C}, \quad \psi^{-1}(\xi) = \frac{y(y - x)\xi + jy[x + z(x - y)]}{j(x - y)\xi + z(x - y) + y},$$

which maps the infinite segment of the imaginary axis $s \in j[z, \infty)$ onto the finite segment of the imaginary axis $\xi \in j[x, y]$. The mapping (4.29) can be used in Theorem 4.2 by constructing the transformation matrix

$$T = \begin{bmatrix} 1 & j(x - y)/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y(y - x) & jy[x + z(x - y)] \\ j(x - y) & z(x - y) + y \end{bmatrix}.$$

(4.30)

We will not proceed with this general form, but rather focus on a particular choice in order to simplify the exposition. The next corollary is an alternative version of Theorem 3.2 that can handle limits of the extreme frequencies at both ends. It is obtained by specializing the above transformation ψ to cover the segment of the imaginary axis $s \in j[0, 1]$.

Corollary 4.3. *Let matrices $A \in \mathbb{C}^{n \times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{C}^{n \times m}$, and $\Theta_1, \Theta_2 \in \mathbb{HC}^{m+n}$ be given. Let $z \in \mathbb{R}$ be also given. If there exists matrices $F \in \mathbb{C}^{n \times n}$, $G \in \mathbb{C}^{m \times n}$ such that*

$$\mathrm{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} (1 - \omega_i)I & -j[z + \omega_i(1 - z)]I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} + \Theta_i \prec 0, \quad i = \{1, 2\}$$

(4.31)

where

$$\omega_1 = 0, \qquad\qquad\qquad \omega_2 = 1,$$

then the following frequency domain inequality holds

$$\begin{bmatrix} (j\eta I - A)^{-1}B \\ I \end{bmatrix}^* \Theta\left(\frac{\eta - z}{1 - z + \eta} \right) \begin{bmatrix} (j\eta I - A)^{-1}B \\ I \end{bmatrix} \prec 0$$

(4.32)

for all $z \leq \eta < \infty$.

Proof: Setting $x = 0$ and $y = 1$ in (4.30) we obtain

$$T = \begin{bmatrix} 1 & -j/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -jz \\ -j & 1 - z \end{bmatrix}, \qquad \Phi_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Psi_0 = \begin{bmatrix} -1 & 0 \\ 0 & 1/4 \end{bmatrix}.$$

Invoking Theorem 4.2 with (4.24) we obtain the pair of inequalities

$$\mathrm{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\tilde{\omega}_i I \end{bmatrix} \left(\begin{bmatrix} 1 & -j/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -jz \\ -j & 1 - z \end{bmatrix} \otimes I \right) \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} + \Theta_i \prec 0,$$

$$i = \{1, 2\}.$$

Inequalities (4.31) come from noticing that

$$\begin{bmatrix} I & -j\tilde{\omega}_i I \end{bmatrix} \left(\begin{bmatrix} 1 & -j/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -jz \\ -j & 1-z \end{bmatrix} \otimes I \right)$$

$$= \begin{bmatrix} I & -j(\tilde{\omega}+1/2)I \end{bmatrix} \left(\begin{bmatrix} 1 & -jz \\ -j & 1-z \end{bmatrix} \otimes I \right),$$

$$= \begin{bmatrix} (1-\omega)I & -j[z+\omega(1-z)]I \end{bmatrix},$$

where we have performed the change of variables $\omega = \tilde{\omega} + 1/2$ as discussed in previous sections.

Note that

$$\psi^{-1} : \mathbb{C} \to \mathbb{C}, \quad \psi^{-1}(\xi) = \frac{\xi - jz}{1 - z - j\xi},$$

so that for $\xi = j\eta$ we have the relationship

$$\omega = -j\psi^{-1}(\xi) = \frac{\eta - z}{1 - z + \eta},$$

which appear in the frequency domain inequality (4.32).

The above corollary handles the case $\eta < \infty$ by noting that (4.30) attains a finite limit as $\omega_2 \to y = 1$, so that the resulting pair of inequalities can be solved without facing any numerical complications. In fact, inequalities (4.31) involve only finite coefficients as long as z is finite, including zero. Combining Corollary 4.2 with Corollary 4.3 one obtains an interesting extension of the KYP Lemma, that is, an LMI condition that establishes an FDI for all $\eta \in \mathbb{R}$, while still allowing for a frequency dependent $\Theta(\omega)$. The particular pair of inequalities (4.31) associated with this case $z = 0$ are given by

$$\text{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} (1+\omega_i)I & -j\omega_i I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} + \Theta_i \prec 0, \quad \omega_1 = 0, \quad \omega_2 = 1. \quad (4.33)$$

The conditions of Corollary 4.3, in the general case z is arbitrary, are associated with the curve $\mathbf{\Lambda}(\Phi, \Psi)$ where

$$\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \qquad \Psi = \begin{bmatrix} 0 & j/2 \\ -j/2 & -z \end{bmatrix},$$

which is simply $\{s \in \mathbb{C} : \quad s = j\omega, \quad j(s^* - s)/2 \geq z\} = \{\omega \in \mathbb{R} : \quad \omega \geq z\}$. It is interesting to note that in the particular case $z = 0$ the matrix Ψ is not positive semidefinite, which implies that one cannot set $Q = 0$, $F = P$, $G = 0$, in order to reduce these conditions to the original KYP Lemma for constant Θ. Application of (4.33) compared to the standard KYP Lemma will be illustrated in the numerical examples.

In the case $z = 0$, the transformation (4.29) with $x = 0$ and $y = 1$ implies that

$$\omega = -j\psi^{-1}(j\eta) = \frac{\eta}{1 + \eta}$$

which, evaluated at the limits of the interval $\eta \in [0, \infty)$, yields the approximation

$$\omega \approx \eta, \quad \text{for } \eta \approx 0, \qquad\qquad \omega \to 1, \quad \text{for } \eta \to \infty.$$

This means that the frequency dependent scaling Θ should behave as

$$\Theta(\omega) \approx \Theta(\eta), \quad \text{for } \eta \approx 0, \qquad\qquad \Theta(\omega) \to \Theta(1), \quad \text{for } \eta \to \infty.$$

That is, Θ is a linear function of η near the origin and it approaches a constant as $\eta \to \infty$. One could have arrived at the opposite scenario by choosing different constants on the linear-fractional mapping (4.29).

Finally note that Theorem 3.2 and Corollary 4.3 are not equivalent and, in fact, they may produce different results for the very same frequency range. As seen above, the multiplier Θ is affine on $\omega = -j\psi^{-1}(j\eta)$, according to (3.16), but it is nonlinear on η.

4.7 A KYP Lemma for Discrete-Time Systems

A particular case of the Generalized KYP Lemma which might have some interest on its own is a version of the KYP lemma on finite frequency ranges for discrete-time systems. We start by examining the particular case of the linear-fractional mapping

$$\psi : \mathbb{C} \to \mathbb{C}, \quad \psi(s) = \frac{1 + s}{1 - s}. \tag{4.34}$$

The above transformation maps the imaginary axis $(-j\infty, j\infty)$ onto the unit disk $e^{j\theta} : \theta \in (-\pi, \pi)$, see illustration provided in Fig. 4.3.
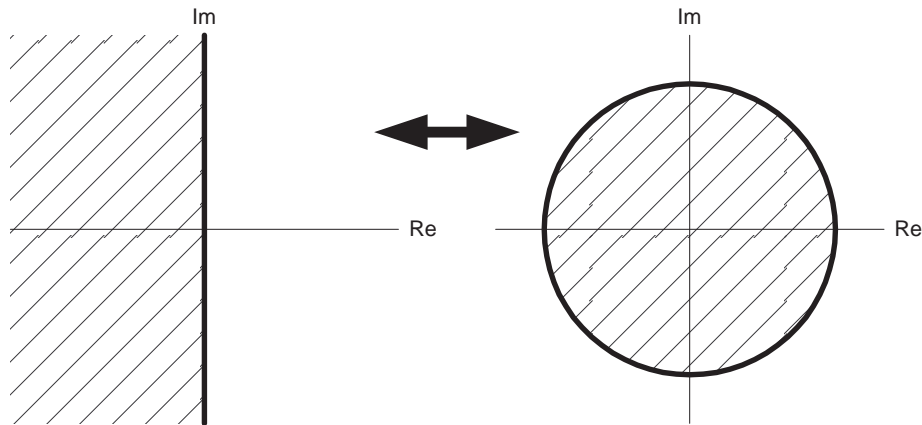
Figure 4.3. Bilinear transformation from continuous-time to discrete-time frequency variable.

Indeed, for any $s = j\omega \in j\mathbb{R}$ there exists one $\theta \in (-\pi, \pi)$ such that

$$e^{j\theta} = \frac{1 + j\omega}{1 - j\omega}, \tag{4.35}$$

namely $\theta = 2\arctan(\omega)$. Conversely $\omega = \tan(\theta/2)$. The next corollary is obtained after applying the above transformation ψ on the segment of the imaginary axis $j[\omega_1, \omega_2]$.

Corollary 4.4. *Let matrices $A \in \mathbb{C}^{n \times n}$ with no eigenvalues on the unit circle, $B \in \mathbb{C}^{n \times m}$, and $\Theta_1, \Theta_2 \in \mathbb{HC}^{m+n}$ be given. If there exists multiplier matrices $F \in \mathbb{C}^{n \times n}$, $G \in \mathbb{C}^{m \times n}$ such that*

$$\text{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\omega_i I \end{bmatrix} \begin{bmatrix} A - I & B \\ A + I & B \end{bmatrix} \right\} + \Theta_i \prec 0, \qquad i = \{1, 2\} \tag{4.36}$$

where

$$\omega_1 = \tan(\theta_1/2), \qquad\qquad \omega_2 = \tan(\theta_2/2),$$

then the following frequency domain inequality holds

$$\begin{bmatrix} (e^{j\theta}I - A)^{-1}B \\ I \end{bmatrix}^* \Theta(\tan(\theta/2)) \begin{bmatrix} (e^{j\theta}I - A)^{-1}B \\ I \end{bmatrix} \prec 0, \tag{4.37}$$

for all $-\pi \le \theta_1 \le \theta \le \theta_2 \le \pi$.

Proof: Composing the transformation (4.34) with (4.23) discussed in the previous section we obtain

$$
T = \begin{bmatrix} 1 & -j\omega_c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad \Phi_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Psi_0 = \begin{bmatrix} -1 & 0 \\ 0 & \hat{\omega}^2 \end{bmatrix},
$$

where $\omega_c = (\omega_1 + \omega_2)/2$, and $\hat{\omega} = (\omega_2 - \omega_1)/2$, so that invoking Theorem 4.2 for the above choices of matrices and (4.24) one obtains the pair of inequalities

$$
\mathrm{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\tilde{\omega}_i I \end{bmatrix} \left(\begin{bmatrix} 1 & -j\omega_c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \otimes I \right) \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} + \Theta_i \prec 0,
$$

for $i = \{1, 2\}$. Inequalities (4.36) come after noticing that

$$
\begin{bmatrix} I & -j\tilde{\omega}_i I \end{bmatrix} \left(\begin{bmatrix} 1 & -j\omega_c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \otimes I \right) \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} =
$$

$$
\begin{bmatrix} I & -j(\tilde{\omega}_i + \omega_c)I \end{bmatrix} \begin{bmatrix} A - I & B \\ A + I & B \end{bmatrix},
$$

and performing the change of variables $\omega = \tilde{\omega} + \omega_c$ as discussed in the previous section. The FDI (4.37) comes after defining θ as in (4.35) and from noticing that $\omega = -j\psi^{-1}(e^{j\theta}) = \tan(\theta/2)$.

The approach taken in [45, 48] may seem slightly different as the segment of the unit disk $e^{j\theta} : \theta \in (\theta_1, \theta_2)$ is alternatively parametrized by

$$
|\theta - \theta_c| \leq \hat{\theta} \qquad \Leftrightarrow \qquad \cos(\theta - \theta_c) \geq 2\cos\hat{\theta},
$$

$$
\Leftrightarrow \qquad \{\xi \in \mathbb{C} : \quad \xi^*\xi = 1, \quad \xi e^{-j\theta_c} + \xi^* e^{j\theta_c} \geq 2\cos\hat{\theta}\},
$$

where $\theta_c := (\theta_1 + \theta_2)/2$ and $\hat{\theta} := (\theta_2 - \theta_1)/2$. The above describes a curve $\mathbf{\Lambda}(\Phi, \Psi)$ with

$$
\Phi = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \Psi = \begin{bmatrix} 0 & e^{j\theta_c} \\ e^{-j\theta_c} & -2\cos\hat{\theta} \end{bmatrix}.
$$

One can verify that this curve is associated with

$$
T = \begin{bmatrix} 1 & -e^{j\theta_c} \\ 1 & e^{j\theta_c} \end{bmatrix}, \qquad \Phi_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Psi_0 = \begin{bmatrix} -1 - \cos\hat{\theta} & \cos\hat{\theta} \\ \cos\hat{\theta} & 1 - \cos\hat{\theta} \end{bmatrix},
$$

with which application of Theorem 4.2 yields the pair of inequalities

$$\text{He}\left\{\begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\tilde{\omega}_i I \end{bmatrix} \begin{bmatrix} A - e^{j\theta_c}I & B \\ A + e^{j\theta_c}I & B \end{bmatrix}\right\} + \Theta_i \prec 0, \qquad i = \{1, 2\}, \qquad (4.38)$$

where

$$\tilde{\omega}_1 = \tan(-\hat{\theta}/2), \qquad\qquad \tilde{\omega}_2 = \tan(\hat{\theta}/2).$$

The above extreme frequencies were obtained using the trigonometric identity $\tan^2(\hat{\theta}/2) = (1 - \cos\hat{\theta})/(1 + \cos\hat{\theta})$.

Note that in the particular case $\theta_c = 0$ inequalities (4.36) and (4.38) are identical. However, when $\theta_c \neq 0$ they are not the same. Indeed, the inverse linear-fractional transformation associated with the form used in [45, 48] is

$$\psi^{-1} : \mathbb{C} \to \mathbb{C}, \quad \psi^{-1}(\xi) = \frac{\xi - e^{j\theta_c}}{\xi + e^{j\theta_c}}, \quad \Rightarrow \quad \tilde{\omega} = -j\psi^{-1}(e^{j\theta}) = \tan[(\theta - \theta_c)/2].$$

Note that when $\theta_c \neq 0$, it is not possible to find a constant ω_c, i.e. independent of θ, such that

$$\omega_c = \tan(\theta/2) - \tan[(\theta - \theta_c)/2], \qquad\qquad \omega = \tan(\theta/2).$$

In fact, inequalities (4.36) and (4.38) are related through a rotation $e^{j\theta_c}$ of the system matrices (A, B). That is, if we apply inequalities (4.38) for the system matrices $(e^{j\theta_c}A, e^{j\theta_c}B)$

$$0 \succ \text{He}\left\{\begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} I & -j\tilde{\omega}_i I \end{bmatrix} \begin{bmatrix} e^{j\theta_c}A - e^{j\theta_c}I & e^{j\theta_c}B \\ e^{j\theta_c}A + e^{j\theta_c}I & e^{j\theta_c}B \end{bmatrix}\right\} + \Theta_i,$$

$$= \text{He}\left\{\begin{bmatrix} e^{j\theta_c}F \\ e^{j\theta_c}G \end{bmatrix} \begin{bmatrix} I & -j\tilde{\omega}_i I \end{bmatrix} \begin{bmatrix} A - I & B \\ A + I & B \end{bmatrix}\right\} + \Theta_i,$$

for $i = \{1, 2\}$, which are precisely inequalities (4.36) with shifted matrices F and G. From Corollary 4.4, feasibility of the above inequalities implies that

$$0 \succ \begin{bmatrix} (e^{j\theta}I - e^{j\theta_c}A)^{-1}e^{j\theta_c}B \\ I \end{bmatrix}^* \Theta(\tan(\theta/2)) \begin{bmatrix} (e^{j\theta}I - e^{j\theta_c}A)^{-1}e^{j\theta_c}B \\ I \end{bmatrix},$$

$$= \begin{bmatrix} (e^{j(\theta - \theta_c)}I - A)^{-1}B \\ I \end{bmatrix}^* \Theta(\tan(\theta/2)) \begin{bmatrix} (e^{j(\theta - \theta_c)}I - A)^{-1}B \\ I \end{bmatrix},$$

for all $|\theta - \theta_c| \leq \hat{\theta}$.

4.8 Regular Descriptor Systems

The frequency variable transformations discussed in the previous sections can be put into a regular descriptor system representation given by

$$\mathcal{M}(j\omega) = (j\omega\mathcal{E} - \mathcal{A})^{-1}(\mathcal{B} - j\omega\mathcal{H}). \tag{4.39}$$

Consider, for instance, the frequency variable transformation (4.29) with $x = 0$ and $y = 1$. Note that

$$
\begin{aligned}
(\xi I - A)^{-1}B &= \left(\left(\frac{j\omega}{1-\omega} + jz\right)I - A\right)^{-1}B, \\
&= (j\omega I + jzI - jz\omega I - A + \omega A)^{-1}(1-\omega)B, \\
&= (j\omega(I - zI - jA) - (A - jzI))^{-1}(B - j\omega(jB)),
\end{aligned}
$$

which is represented in the form (4.39) with

$$\mathcal{A} = A - jzI, \qquad \mathcal{B} = B, \qquad \mathcal{E} = I - zI - jA, \qquad \mathcal{H} = jB.$$

Similarly the discrete time transformation (4.34) can be put into regular descriptor form, where it can be verified that

$$\mathcal{A} = A - I, \qquad \mathcal{B} = \mathcal{H} = B, \qquad \mathcal{E} = A + I.$$

In addition to these specific transformations, the descriptor form is useful in describing non-rational systems with general $\mathcal{A}$, $\mathcal{B}$, $\mathcal{E}$ and $\mathcal{H}$ matrices. This motivates the presentation of a corollary of Theorem 4.2 for a class of regular descriptor systems.

Corollary 4.5. *Let matrix $\mathcal{A} \in \mathbb{C}^{n\times n}$ and $\mathcal{E} \in \mathbb{C}^{n\times n}$ be given such that the $\det(j\omega\mathcal{E} - \mathcal{A}) \neq 0$ for all $\omega \in \mathbb{R}$. Let matrices $\mathcal{B} \in \mathbb{C}^{n\times m}$, $\mathcal{H} \in \mathbb{C}^{n\times m}$ and $\Theta_1, \Theta_2 \in \mathbb{HC}^{n+m}$ and $\omega_1, \omega_2 \in \mathbb{R}$ be also given. If there exist matrices $F \in \mathbb{C}^{n\times n}$ and $G \in \mathbb{C}^{m\times n}$ such that*

$$\text{He}\left\{\begin{bmatrix} F \\ G \end{bmatrix}\begin{bmatrix} I & -j\omega_i I \end{bmatrix}\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{E} & \mathcal{H} \end{bmatrix}\right\} + \Theta_i \prec 0, \quad i = \{1,2\} \tag{4.40}$$

then the FDI

$$\begin{bmatrix} (j\omega\mathcal{E} - \mathcal{A})^{-1}(\mathcal{B} - j\omega\mathcal{H}) \\ I \end{bmatrix}^{*}\Theta(\psi)\begin{bmatrix} (j\omega\mathcal{E} - \mathcal{A})^{-1}(\mathcal{B} - j\omega\mathcal{H}) \\ I \end{bmatrix} \prec 0, \tag{4.41}$$

holds for all $\omega_1 \leq \omega \leq \omega_2$ with $\Theta(\cdot)$ as given in (3.16).

Proof: Follows from application of Theorem 4.2 with (4.23) and

$$H = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{E} & \mathcal{H} \end{bmatrix},$$

and noting that

$$\left(\begin{bmatrix} I & -j\omega I \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{E} & \mathcal{H} \end{bmatrix} \right)_\perp = \left(\begin{bmatrix} -(j\omega\mathcal{E} - \mathcal{A}) & \mathcal{B} - j\omega\mathcal{H} \end{bmatrix} \right)_\perp,$$

$$= \begin{bmatrix} (j\omega\mathcal{E} - \mathcal{A})^{-1}(\mathcal{B} - j\omega\mathcal{H}) \\ I \end{bmatrix}.$$

$\square$

Note that the assumption $\det(j\omega\mathcal{E} - \mathcal{A})$ can be relaxed to the existence of

$$\left(\begin{bmatrix} I & -j\omega I \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{E} & \mathcal{H} \end{bmatrix} \right)_\perp,$$

for all $\omega_1 \leq \omega \leq \omega_2$, where in such a case the FDI (4.41) becomes

$$\left(\begin{bmatrix} I & -j\omega I \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{E} & \mathcal{H} \end{bmatrix} \right)_\perp^* \Theta(\psi) \left(\begin{bmatrix} I & -j\omega I \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{E} & \mathcal{H} \end{bmatrix} \right)_\perp.$$

4.9   Polynomial Systems

As a final note in this chapter, it is important to discuss the role played by the low complexity of the conditions (3.18) and (4.20), since recently developed polynomial techniques can be used to produce robust stability tests with little or no conservatism (see [56, 69, 76, 75, 12]). Indeed, arbitrary polynomial dependence of Θ on ω could be obtained, of course, at the expense of an exponential growth in the number of variables and size of inequalities. A remarkable feature of our results is that there is absolutely no extra cost associated with solving the inequalities (3.18) as compared with (3.12) while still enlarging the class of matrices $\Theta(\omega)$ being tested from constant to affine in ω.

The previous sections give specific extensions that pertain to general state-space system descriptions. Indeed, using the results of [30], the results of this

chapter can provide an alternative characterization of positive pseudo-polynomial matrices on finite frequency intervals, a reformulation that might be useful on its own and will be explored in Section 4.9. Last but not least, being close in form and structure to the KYP Lemma may prove useful in developing control design methods for limited frequency ranges, a problem which seems to be hard in the context of polynomial techniques.

4.9.1 Pseudo-Polynomial Para-Hermitian Matrices

Pseudo-polynomial matrices have entries that depend polynomially on a single variable $s \in \mathbb{C}$ and admit a finite expansion in positive and negative powers of s. The work [31, 30] provides conditions under which certain pseudo-polynomial matrices are positive (nonnegative) on specific curves in the complex plane, namely the entire imaginary axis, the entire real axis and the unit circle. Using the conditions of this paper one can rewrite these conditions for any curve $s \in \mathbf{\Lambda}(\Phi, \Psi)$ as discussed in Section 4.3. First consider the case $\mathbf{\Lambda}(\Phi, \Psi) = \{s = j\omega,\, \omega_1 \leq \omega \leq \omega_2,\, \omega \in \mathbb{R}\}$, i.e. segments of the imaginary axis, through application of Theorem 3.1.

Any para-Hermitian pseudo-polynomial matrix can be written in the form

$$\Xi(s) = \sum_{k=0}^{2t} s^k \, \Xi_k, \qquad\qquad \Xi_k = (-1)^k \Xi_k^*, \qquad\qquad (4.42)$$

The task is to find whether

$$\Xi(s) \succ 0, \qquad \text{for all } s = j\omega, \quad \omega_1 \leq \omega \leq \omega_2.$$

In the sequel we assume $\omega_1 > 0$ to avoid technical complications. This assumption can be removed by transforming s as done in Section 4.6. Following [30], the above para-Hermitian pseudo-polynomial matrix $\Xi(s)$ can be rewritten in the form

$$\Xi(s) = \begin{bmatrix} (sI_{tn} - A)^{-1}B \\ I_n \end{bmatrix}^* \Theta \begin{bmatrix} (sI_{tn} - A)^{-1}B \\ I_n \end{bmatrix} \qquad (4.43)$$

where $A \in \mathbb{R}^{tn \times tn}$, $B \in \mathbb{R}^{tn \times n}$ are given by

$$A = \begin{bmatrix} 0 & I_n & & \\ & 0 & \ddots & \\ & & \ddots & I_n \\ & & & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_n \end{bmatrix}, \qquad (4.44)$$

and $\Theta \in \mathbb{HC}^{tn}$ is given by

$$\Theta = \begin{bmatrix} \Xi_0 & \frac{-j}{2}\Xi_1 & & 0 \\ \frac{j}{2}\Xi_1 & \Xi_2 & \ddots & \\ & \ddots & \ddots & \frac{-j}{2}\Xi_{2t-1} \\ 0 & & \frac{j}{2}\Xi_{2t-1} & \Xi_{2t} \end{bmatrix}.$$

Positivity of the $\Xi(s)$ in $\mathbf{\Lambda}(\Phi, \Psi) = \{s = j\omega, \ 0 < \omega_1 \leq \omega \leq \omega_2, \ \omega \in \mathbb{R}\}$ can be checked by applying Theorem 3.1 with the above matrices A, B and $-\Theta$.

Similar developments can be obtained for more general $\mathbf{\Lambda}(\Phi, \Psi)$ by applying Theorem 4.2 and the methods of [30]. For instance consider the para-Hermitian discrete-time transfer function given by the pseudo-polynomial expansion

$$\Xi(z) = \sum_{k=-t}^{t} z^k \Xi_k, \qquad \qquad \Xi_{-k} = \Xi_k^*,$$

again with the intent of finding whether

$$\Xi(z) \succ 0, \qquad \text{for all } z = e^{j\theta}, \quad \theta_1 \leq \theta \leq \theta_2,$$

with the unrestrictive assumption that $\theta > 0$. Following [30], the above para-Hermitian pseudo-polynomial matrix $\Xi(z)$ can be rewritten in the form

$$\Xi(z) = \begin{bmatrix} (zI_{tn} - A)^{-1}B \\ I_n \end{bmatrix}^* \Theta \begin{bmatrix} (zI_{tn} - A)^{-1}B \\ I_n \end{bmatrix}$$

where $A \in \mathbb{R}^{tn \times tn}$, $B \in \mathbb{R}^{tn \times n}$ are given in (4.44) and $\Theta \in \mathbb{HC}^{tn}$ is given by

$$\Theta = \begin{bmatrix} \Xi_0 & \Xi_1 & \cdots & \Xi_t \\ \Xi_{-1} & 0 & \vdots & 0 \\ \vdots & \vdots & & \vdots \\ \Xi_{-t} & 0 & \cdots & 0 \end{bmatrix}. \qquad (4.45)$$

Positivity of $\Xi(z)$ in $\mathbf{\Lambda}(\Phi, \Psi) = \{z = e^{j\theta}, \ 0 < \theta_1 \leq \theta \leq \theta_2, \ \theta \in \mathbb{R}\}$ can be checked by applying Theorem 4.2 with the matrices A, B given in (4.44) and $-\Theta$ given in (4.45).

4.9.2 General Polynomial System KYP Lemma

In the previous section, the coefficient matrix was specified in such a way as to give a polynomial matrix with respect to the desired variable. Consider a polynomial expansion in powers of a general frequency variable ξ parametrized by the curve $\mathbf{\Lambda}(\Phi, \Psi)$

$$\Xi(\xi) = \sum_{k=0}^{t} \xi^k \Xi_k,$$

where $\Xi_k \in \mathbb{C}^{p \times m}$. Rather than simply checking for positivity of the above polynomial expansion, one might be concerned with more general properties that can be posed in terms of the quadratic form

$$\begin{bmatrix} \Xi(\xi) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} \Xi(\xi) \\ I \end{bmatrix} \prec 0, \qquad \qquad \xi \in \mathbf{\Lambda}(\Phi, \Psi),$$

where $\Pi \in \mathbf{HC}^{p+m}$. Feasibility of the above FDI can be checked by applying Theorem 4.2 with the matrices A, B given in (4.44) and Θ as given by

$$\Theta = \begin{bmatrix} \Xi_0 & \Xi_1 & \cdots & \Xi_N \\ I & 0 & \cdots & 0 \end{bmatrix}^* \Pi \begin{bmatrix} \Xi_0 & \Xi_1 & \cdots & \Xi_N \\ I & 0 & \cdots & 0 \end{bmatrix}.$$

Various polynomial system properties can be checked with the coefficient matrix form above, which depends on the choice of Π. For instance, to check positivity of discrete time polynomial system let

$$\Pi = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix},$$

with $\Xi_k = (-1)^k \Xi_k$ and $\Pi \in \mathbb{HC}^n$, which gives (4.45). Although this matrix coefficient form seems general enough to cover the presentation in Section 4.9.1, this coefficient matrix form does admit realizations for checking some important system properties. For instance positivity of continuous-time polynomial systems can not be checked with any choice for $\Pi \in \mathbb{HC}^n$.

4.10 Acknowledgements

Some text of this chapter includes the reprints of the following paper.

M.R. Graham, M.C. de Oliveira, and R.A. de Callafon, "An alternative Kalman-Yakubovich-Popov lemma and some extensions." *IEEE Transactions on Automatic Control*, Submitted, February 2007.

The dissertation author was the primary researcher and author in these works and the co-author listed in these publications directed and supervised the research.

Chapter 5

Application of the Extended KYP Lemma for Analyzing Robustness

5.1 Motivation

Conceptually the work proposed here is similar to techniques used in μ-analysis for avoiding frequency griding by considering the frequency itself as a real parametric uncertainty [80, 41]. The method in this paper however borrows ideas from robust analysis of uncertain polytopic systems [22] in the treatment of frequency as a real uncertain parameter.

Problem 5.1. *Relate the extensions proposed in Chapter 4 to standard μ-analysis results, allowing for frequency-dependent scalings in computing upper bounds to the structured singular value.*

This chapter is dedicated to presenting numerical examples that demonstrate the theoretical developments of previous chapters to the application of robustness analysis.

5.2 Numerical Examples

In this section, Corollary 4.3 and Corollary 3.1 are used to illustrate the possible reduction in conservativeness when using the generalization proposed by Theo-

rem 3.2 in the context of μ-analysis. The examples explore the fact that Θ appears affinely in the LMIs to synthesize frequency dependent multipliers appearing in $\Theta(\omega)$.

5.2.1   Example 1

The first example is taken from [37, 55]. Consider the nominal plant model, uncertainty weight, and controller are respectively given by

$$P = \frac{10}{s\,(s+10)}, \qquad W = \frac{10\,(s+5)}{(s+100)}, \qquad C = 5 + \frac{1}{10\,s} + \frac{5\,s}{10+s}.$$

The dependence on s is omitted for simplicity. The feedback connection with the uncertainty is as in Fig. 2.7 where the generalized plant and the associated uncertainty structure are

$$M = \frac{1}{1+CP} \begin{bmatrix} WCP & WC \\ P & 1 \end{bmatrix}, \qquad \boldsymbol{\Delta} = \{\mathrm{diag}[\delta_1, \delta_2] : \delta_i \in \mathbb{C}, \quad i = \{1,2\}\}.$$

We first used the KYP Lemma (3.5) to compute an upper bound for $\rho_{\boldsymbol{\Delta}}$ over all frequencies using Θ as in (3.30) with the constant scalings

$$Z \in \mathbf{Z}, \qquad\qquad Y = 0 \qquad\qquad (5.1)$$

to account for the complex uncertainty. We searched for the smallest value of $\beta > 0$ for which the LMI (3.5) has a feasible solution. This same procedure was used in the methods we will describe in the sequel. We then used the Generalized KYP Lemma (3.10) with scalings of same structure as (5.1) within the frequency range

$$\omega_1 = 10^{-1}, \qquad\qquad \omega_2 = 10^3. \qquad\qquad (5.2)$$

Next we tested the inequalities in Corollary 4.3 with

$$x = z = 0, \qquad\qquad y = 1 \qquad\qquad (5.3)$$

and multipliers

$$Z_i \in \mathbf{Z}, \qquad\qquad Y_i = 0, \qquad\qquad i = \{1,2\}. \qquad\qquad (5.4)$$

Table 5.1. Example 1: Upper bounds for $\rho_{\mathbf{\Delta}}$ (complex uncertainty); $\rho_{\mathbf{\Delta}} \geq 1.821$ computed on a dense grid.

METHOD	UPPER BOUND $(\rho_{\mathbf{\Delta}})$	FREQUENCY RANGE
KYP Lemma 3.2	4.535	$0 \leq |\omega| \leq \infty$
Generalized KYP Lemma 3.4	4.524	$10^{-1} \leq |\omega| \leq 10^3$
Corollary 4.3	1.835	$0 \leq |\omega| \leq \infty$
Corollary 3.1	1.826	$10^{-1} \leq |\omega| \leq 10^3$

to find an upper bound for $\rho_{\mathbf{\Delta}}$ over all frequencies. Finally we tested the inequalities in Corollary 3.1 with the scalings (5.4) over the frequency range (5.2). The smallest upper bounds found by each method are listed in Table 5.1. These values should be compared against the greatest lower bound for $\rho_{\mathbf{\Delta}}$ of 1.821 found with a dense frequency grid.

All comparisons are shown in Fig. 5.1. For this example, the proposed alternative conditions for the KYP Lemma significantly reduce the conservatism in computing the μ upper bound when evaluated over the entire imaginary axis as well as over semi-infinite and finite frequency ranges.

5.2.2 Example 2

The second example is taken from [20, 41]. Consider the feedback connection depicted in Fig. 2.7 where the generalized plant is

$$
M = \left[
\begin{array}{cccc|ccc}
-4 & 0 & -800 & 6400 & 80 & -0.2 & 0 \\
1 & -6 & 0 & 0 & 0 & 0 & -0.3 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -10 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 8 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0
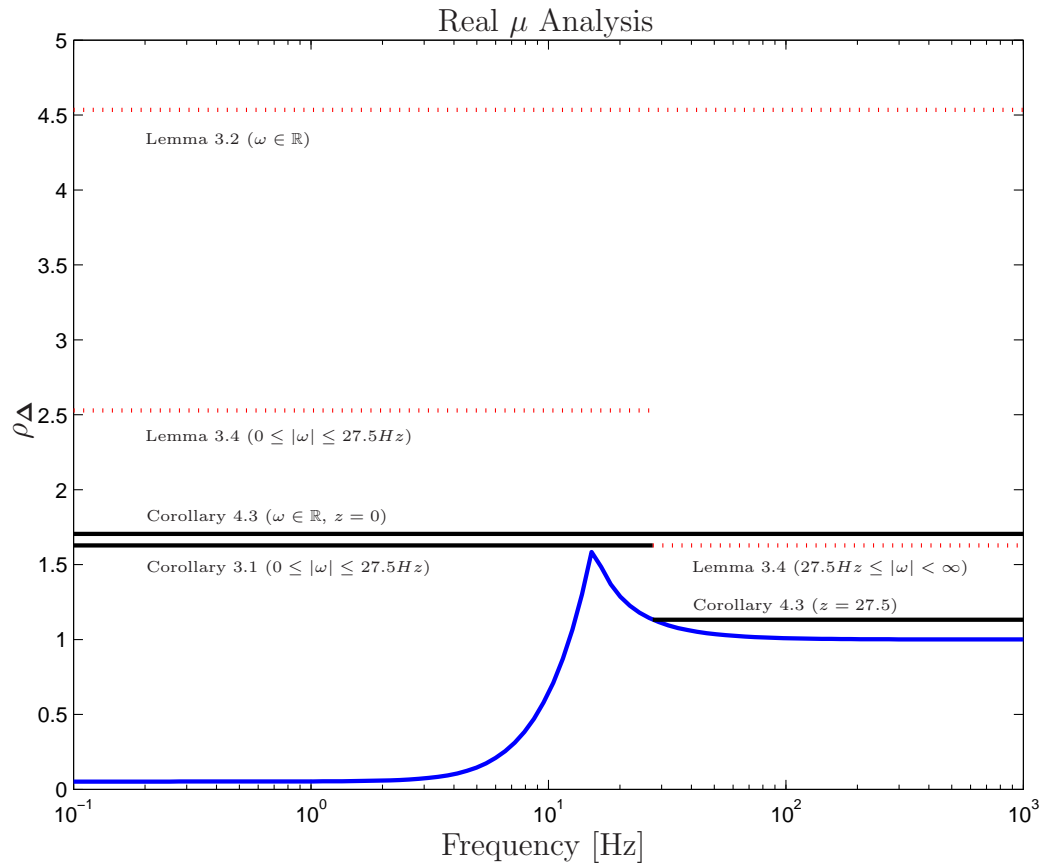\end{array}
\right]
$$

Figure 5.1. Computation of upper bounds for μ. The curved line is the greatest lower bound for $\rho_{\boldsymbol{\Delta}}$. Other lines are labeled according to the method and frequency range used to compute them.

Table 5.2. Example 2: Upper bounds for $\rho_{\boldsymbol{\Delta}}$ (real uncertainty); $\sup_{w \in \mathbb{R}} \mu_{\boldsymbol{\Delta}} = 0.291$ (from [20]).

METHOD	UPPER BOUND ($\rho_{\boldsymbol{\Delta}}$)	FREQUENCY RANGE
KYP Lemma 3.2	$\beta_\infty = 0.458$	$0 \leq |\omega| \leq \infty$
Generalized KYP Lemma 3.4	$\beta_0 = 0.293$	$10^{-1} \leq |\omega| \leq 10^2$
Corollary 4.3	$\eta_\infty = 0.293$	$0 \leq |\omega| \leq \infty$
Corollary 3.1	$\eta_0 = 0.293$	$10^{-1} \leq |\omega| \leq 10^2$

and the uncertainty structure is

$$\boldsymbol{\Delta} = \{\mathrm{diag}[\phi_1, \phi_2, \phi_3] : \phi_i \in \mathbb{R}, \quad i = \{1, 2, 3\}\}.$$

We have used the same methods as in Example 1 with the constant scalings

$$Z \in \mathbf{Z}, \qquad\qquad Y \in \mathbf{Y}$$

used for the KYP Lemma and the Generalized KYP Lemma. Note the presence of the Y associated with the real uncertainty. We used

$$Z_i \in \mathbf{Z}, \qquad\qquad Y_i \in \mathbf{Y}, \qquad\qquad i = \{1, 2\}$$

to search for frequency dependent scalings using Corollary 4.3 and Corollary 3.1. The smallest upper bounds found by each method are listed in Table 5.2. These values should be compared against the exact value for $\sup_{w \in \mathbb{R}} \mu_{\boldsymbol{\Delta}}$ with real uncertainty which in this case is known to be 0.291 (see [20]).

5.2.3 Example 3

Example 3 uses the same data as Example 2 [20, 41]. A rough comparison with the results of [41] is now attempted, where tighter bounds for $\rho_{\boldsymbol{\Delta}}$ are produced by successively bisecting the frequency range into smaller intervals. Each iteration produces one more real uncertainty parameter to which a new (constant) multiplier must be computed. In order to compare the results developed in this dissertation with this approach we take only two steps in the algorithm of [41], splitting the

Table 5.3. Example 3: Upper bounds for $\rho_{\boldsymbol{\Delta}}$ (real uncertainty); $\sup_{w\in\mathbb{R}}\mu_{\Delta} = 0.291$ (from [20]).

	UPPER BOUND $(\rho_{\boldsymbol{\Delta}})$	
METHOD	$0 \le |\omega| \le 1$	$1 \le |\omega| \le \infty$
Method of [41][1]	$\gamma_1 = 0.111$	$\gamma_2 = 0.509$
Generalized KYP Lemma [48]	$\beta_1 = 0.115$	$\beta_2 = 0.458$
Corollary 3.1 / Corollary 4.3	$\eta_1 = 0.102$	$\eta_2 = 0.293$

positive frequencies into two intervals $\{0 \le \omega \le 1\} \cup \{1 \le \omega \le \infty\}$. The results of [41] are given in the first row of Table 5.3. Note that the values shown are taken directly from [41], where they have been computed using the Matlab μ-toolbox.

Constant scalings Z and Y are used in the Generalized KYP Lemma (3.10) with $\omega_1 = -\omega_2 = 1$ to compute the first bound shown in the second column and second row of Table 5.3. In order to compute the "high-frequency" bound on the third column of the second row, the high-frequency version of the Generalized KYP Lemma from [48] is used with a search for constant scalings. The values on the third row of Table 5.3 have been computed using Corollary 3.1 (second column) and Corollary 4.3 (third column) with

$$x = 0, \qquad\qquad y = z = 1.$$

Additionally we solved the same problem limiting the lowest and highest frequencies to be 0.1 and 100, respectively. The upper bounds, shown in Table 5.4, have been computed using the Generalized KYP Lemma and Corollary 3.1 as before.

The upper bounds from Table 5.3 and Table 5.4 are compared with the greatest lower bound for $\rho_{\boldsymbol{\Delta}}$ obtained on a dense grid in Figures 5.2 to 5.4. It is worth noticing that the largest peak on the plot is very sharp, and that the max value of $\rho_{\boldsymbol{\Delta}}$ obtained with 100 logarithmically spaced frequency points between 10^{-1} and 10^2 was only 0.223. In Figures 5.2 and 5.4, the known critical frequency

---

[1] Computed using `mu` command from Matlab μ-toolbox

Table 5.4. Example 3: Upper bounds for $\rho_{\boldsymbol{\Delta}}$ (real uncertainty); $\sup_{w \in \mathbb{R}} \mu_{\Delta} = 0.291$ (from [20]).

	UPPER BOUND ($\rho_{\boldsymbol{\Delta}}$)	
METHOD	$0.1 \leq |\omega| \leq 1$	$1 \leq |\omega| \leq 10^2$
Generalized KYP Lemma 3.4	$\beta_3 = 0.104$	$\beta_4 = 0.293$
Corollary 3.1	$\eta_3 = 0.071$	$\eta_4 = 0.293$

$\omega = 8.22$ (see [20]) was added to this grid in order to obtain the exact value of $\rho_{\boldsymbol{\Delta}} = \mu_{\boldsymbol{\Delta}} = 0.291$. Also note the sharp variations occurring in the lower frequency range shown in Figure 5.2 and zoomed in Figure 5.3. Surprisingly, the upper bound η_{∞} computed for the range $\omega \in [0, \infty)$ using Corollary 4.3 matches the best bounds obtained for all other finite frequency results.

Figure 5.2. Robust analysis for real structured uncertainty in Examples 2 and 3, showing the bounds computed using the results of this paper. The labels have been defined in Tables 5.2 to 5.4. The curved solid line is the greatest lower bound for $\rho_{\mathbf{\Delta}}$.

Figure 5.3. Robust analysis for real structured uncertainty in Examples 2 and 3, comparing the bounds computed using the results of this paper with other results in the literature (low frequency). The labels have been defined in Tables 5.2 to 5.4. The curved solid line is the greatest lower bound for $\rho_{\boldsymbol{\Delta}}$.
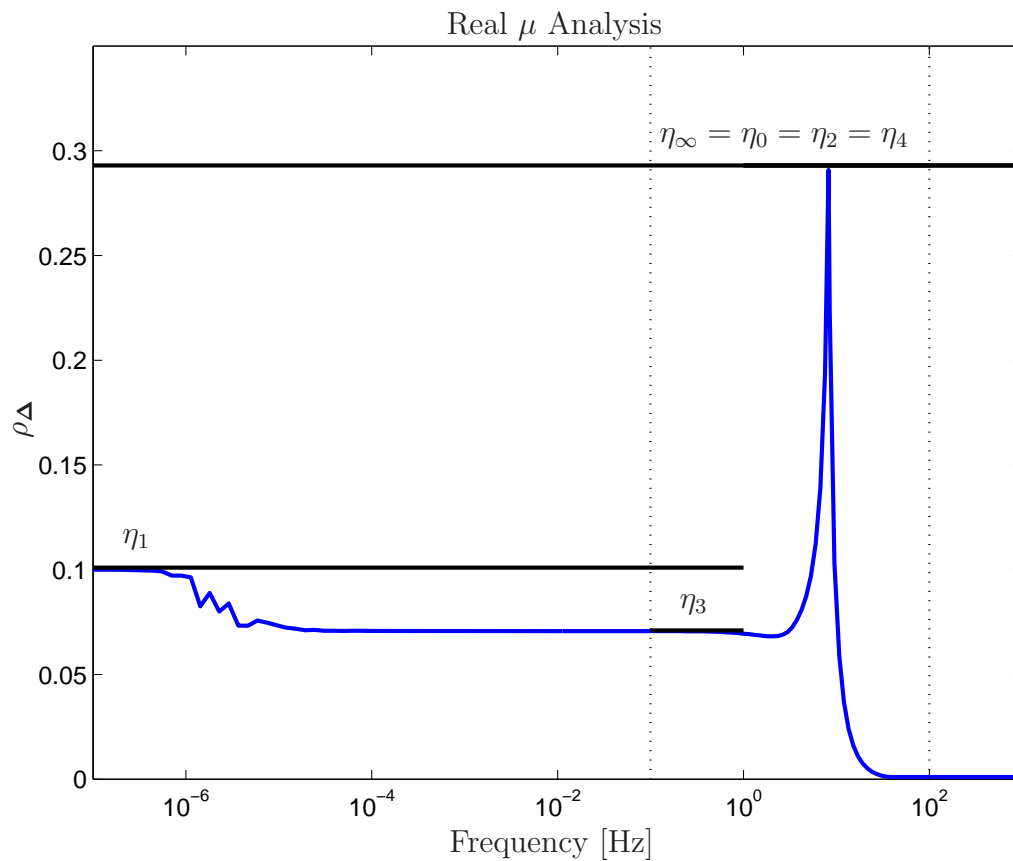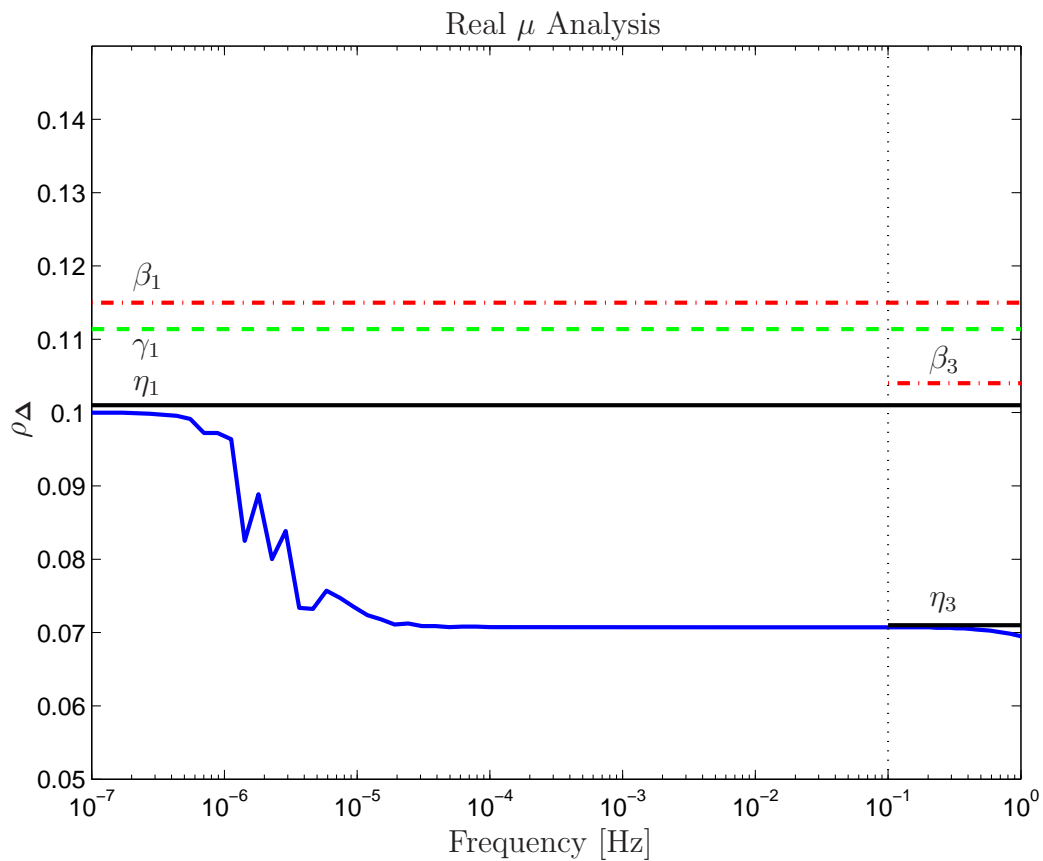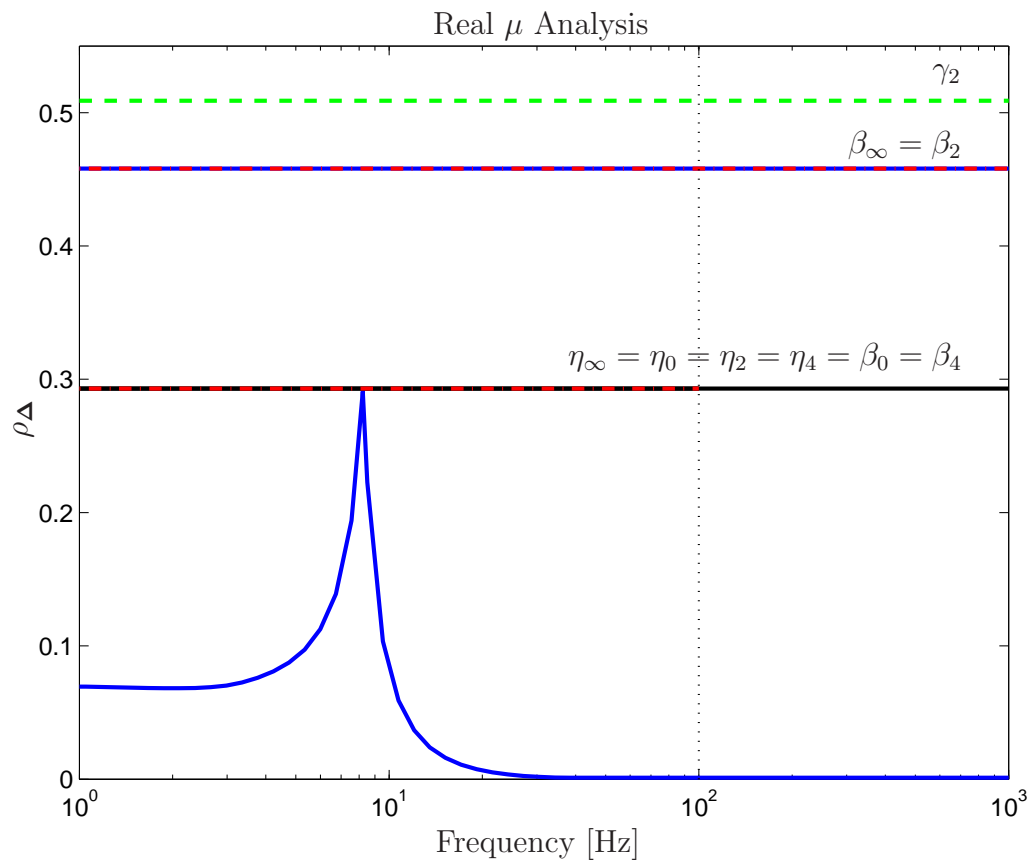
Figure 5.4. Robust analysis for real structured uncertainty in Examples 2 and 3, comparing the bounds computed using the results of this paper with other results in the literature (high frequency). The labels have been defined in Tables 5.2 to 5.4. The curved solid line is the greatest lower bound for $\rho_{\mathbf{\Delta}}$.

5.3 Case Study I: Aeroservoelastic (ASE) System Analysis

5.3.1 Motivation

Modern aerostructural design emphasize a progression toward flexible and multifunctional wing structures, which increase the likelihood of dynamic instabilities [89] and require extensive testing and analysis to safely explore the boundary of the flight envelope. In case aerodynamic instabilities are experienced during flight-testing active structural servo control strategies can be used to stabilize and expand the boundary of the flight envelope. Aeroservoelastic (ASE) models provide a means for analyzing stability and performance [61], possibly online monitoring of aerodynamic instabilities [60], as well as for designing active structural control systems.

Aeroservoelastic models can be formulated by considering a nominal (nonlinear) model with unknown but bounded perturbations. The nominal ASE model captures the inherent dynamic interaction between structural elasticity and aerodynamic loads and contains the effects of actuator bandwidth, major structural modes and sensor locations. The unknown but bounded model perturbation is used to account for uncertainties such as variations in flight conditions, unmodeled (high frequency) elastic modes and actuator nonlinearity. Nominal flight behavior and perturbations can be estimated by identification of possibly non-linear dynamics within a feedback control framework for an ASE system [62] or derived from first principles modeling [1].

Of concern in the context of this dissertation is the analysis of aeroservoelastic systems and reducing the conservatism with which stability and performance is guaranteed without significantly increasing the computational effort. The theoretical developments for robust aeroservoelastic stability analysis can be found in [61], while the computation method used in evaluating the analysis consists of the methods discussed in the previous chapters.

5.3.2 Pitch-Plunge ASE System Formulation

In order to demonstrate the model-based robust aeroservoelastic stability analysis procedure presented in the previous section, consider the low-order pitch plunge system illustrated in Fig. 5.5. The system is modeled as a rigid airfoil attached to a support structure composed of springs and cams that allow pitch and plunge motion. The airfoil also has a trailing edge control surface (flap) that adds an extra degree of freedom for the system.
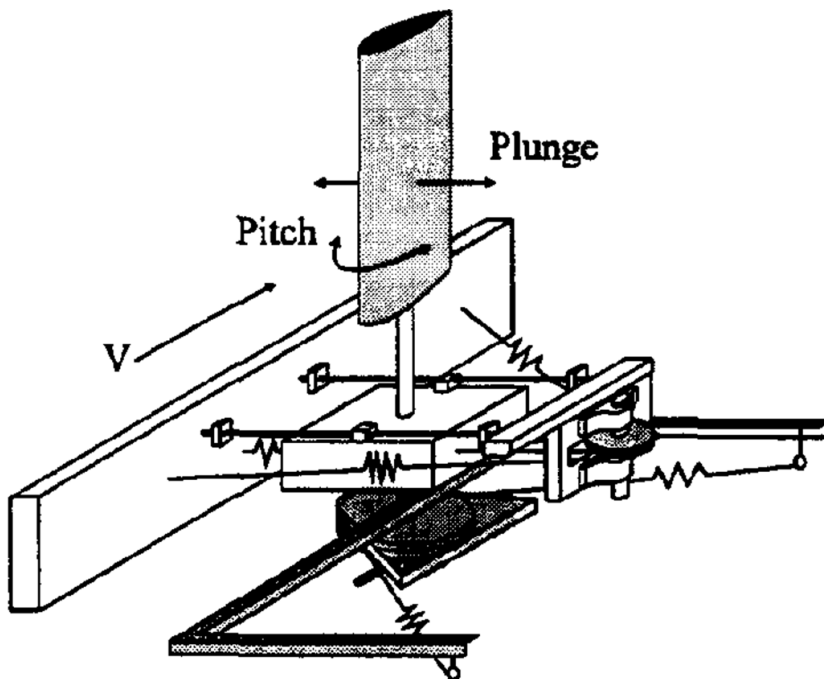
Figure 5.5. Isometric view of the pitch-plunge mechanism for studying aeroservoelastic systems.

The equations of motion describing the pitch and plunge motion during aeroelastic response are derived from force and moment equations [52, 93, 61] and written in matrix form,

$$
\begin{bmatrix} m & mx_\alpha b \\ mx_\alpha b & I_\alpha \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\alpha} \end{bmatrix} + \begin{bmatrix} c_y & 0 \\ 0 & c_\alpha \end{bmatrix} \begin{bmatrix} \dot{y} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} k_y & 0 \\ 0 & k_\alpha \end{bmatrix} \begin{bmatrix} y \\ \alpha \end{bmatrix} = \begin{bmatrix} -L \\ M \end{bmatrix}, \tag{5.5}
$$

where y is plunge deflection and α is pitch angle. The other variables include the airfoil mass m, distance to center of mass x_α, moment of inertia I_α, chord length

b, structural damping coefficients c_y, c_α, and spring constants k_y, k_α. The above system description can also be extended to generalized second order equations of motion relating structural dynamics and unsteady aerodynamics, however this discussion is beyond the intended scope of this chapter and the interested reader is referred to [61].

The aeroelastic system (5.5) becomes an ASE system by including active servo control via the control surface. The control surface deflection angle β affect the lift L and moment M through the relation

$$L = 2\tilde{q}bc_{l_\alpha}\left(\alpha + 1.1\frac{b}{U}\dot{\alpha} + \frac{1}{U}\dot{y}\right) + \tilde{q}b^2 c_{l_\beta}\beta, \tag{5.6}$$

$$M = 2\tilde{q}bc_{m_y}\left(\alpha + 1.1\frac{b}{U}\dot{\alpha} + \frac{1}{U}\dot{y}\right) + \tilde{q}b^2 c_{m_\beta}\beta, \tag{5.7}$$

where U is the nominal airspeed, c_{l_α}, c_{m_α} are lift and moment coefficients for pitch angle and c_{l_β}, c_{m_β} are lift and moment coefficients for control surface angle. The airspeed U directly corresponds to a dynamic pressure $\tilde{q} \in \mathbb{R}$, that represents a flight condition. The system equations relate rigid body and control surface displacement and velocities through rectangular matrices of the vibration and control modes.

The aeroservoelastic model for nominal stability (flutter) analysis in the μ-analysis framework is formulated using a model formulated at some nominal dynamic pressure and additional input and output signals to introduce perturbations to the dynamic pressure. Given a nominal flight condition $\tilde{q}_0$, consider perturbations to dynamic pressure $\delta_{\tilde{q}}$ through a feedback relationship as in Fig. 5.6(a). The state-space matrices for the general transfer matrix P are given by [61]

$$\begin{bmatrix} \dot{y} \\ \dot{\alpha} \\ \ddot{y} \\ \ddot{\alpha} \\ z_l \\ z_m \\ y \\ \alpha \end{bmatrix} = \left[\begin{array}{cccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_{3l} & b_{3m} & b_{3\beta} \\ a_{41} & a_{42} & a_{43} & a_{44} & 4_{3l} & b_{4m} & b_{4\beta} \\ \hline 0 & z_{l\alpha} & z_{l\dot{y}} & z_{l\dot{\alpha}} & 0 & 0 & z_{l_\beta} \\ 0 & z_{m\alpha} & z_{m\dot{y}} & z_{m\dot{\alpha}} & 0 & 0 & z_{m_\beta} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} y \\ \alpha \\ \dot{y} \\ \dot{\alpha} \\ w_l \\ w_m \\ \beta \end{bmatrix},$$
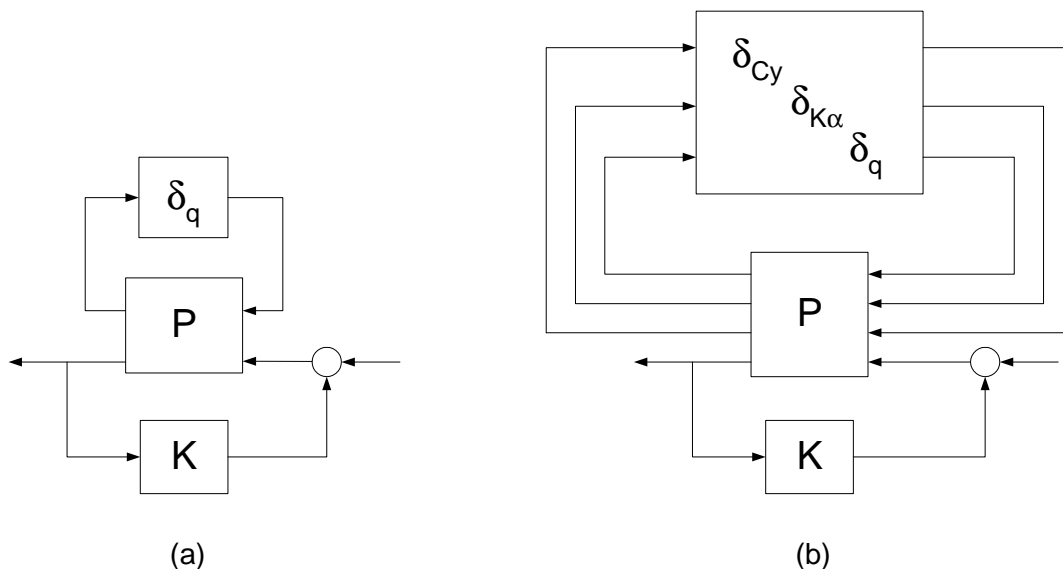
Figure 5.6. Pitch plunge uncertainty block diagram.

where z_l, z_m, w_l and w_m are the input-output signals of the feedback connection with the dynamic pressure perturbation and y, α correspond to pitch-plunge sensor measurements.

In the traditional aeroservoelastic stability analysis, the nominal stability (or flutter) margin is defined as the largest perturbation in the dynamic pressure such that the nominal aeroservoelastic model is stable. Nominal models however are subject to errors due to the accuracy of model parameters, neglected unmodeled dynamics, and nonlinear effects. A robust aeroservoelastic model can be generated by associating uncertainty operators Δ with the nominal model and include the parametrization along with a perturbation in dynamic pressure. The robust aeroservoelastic stability (robust flutter) margin is the largest perturbation in dynamic pressure such that all possible feedback connections, formulated as in Fig. 2.7 with nominal model and perturbation $\Delta \in \mathbf{\Delta}$, are stable. Modeling uncertainties, both parametric and dynamic in nature, can be incorporated into the aeroservoelastic stability analysis within the μ-analysis framework. The choice of uncertainty structure plays an important role in determining realistic stability (flutter) boundaries, however this discussion is beyond the scope of this dissertation. The aim here is to investigate the tools presented in the previous chapters for application in

aeroservoelastic stability analysis, and in particular, model based stability analysis methods that include model uncertainty.

Consider modeling uncertainties as perturbations to the structural spring constant k_α and damping coefficient c_y given by

$$k_\alpha = k_{\alpha_0} + W_{k_\alpha}\delta_{k_\alpha}, \qquad\qquad c_y = c_{y_0}(I + W_{c_y}\delta_{c_y}),$$

with $\|\delta_{k_\alpha}\|_\infty \le 1$ and $\|\delta_{c_y}\|_\infty \le 1$. Note the difference between the uncertainty descriptions corresponds to additive and multiplicative uncertainty structures for the spring constant and damping coefficient respectively. These model uncertainties are considered along with perturbations to dynamic pressure $\delta_{\bar{q}}$ through a feedback relationship as in Fig. 5.6(b). The control law is given by negative feedback of the velocity measurement,

$$K = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

The state-space matrices of the general transfer matrix for this uncertainty and feedback linear fractional transformation configuration is given by [61]

$$
\begin{bmatrix} \dot{y} \\ \dot{\alpha} \\ \ddot{y} \\ \ddot{\alpha} \\ z_l \\ z_m \\ z_y \\ z_\alpha \\ y \\ \alpha \end{bmatrix}
=
\left[\begin{array}{cccc|ccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
a_{31} & a_{32} & a_{33} & a_{34} & b_{3l} & b_{3m} & b_{3y} & b_{3\alpha} & b_{3\beta} \\
a_{41} & a_{42} & a_{43} & a_{44} & b_{4l} & b_{4m} & b_{4y} & b_{4\alpha} & b_{4\beta} \\
0 & z_{l\alpha} & z_{l_{\dot{y}}} & z_{l_{\dot{\alpha}}} & 0 & 0 & 0 & 0 & z_{l_\beta} \\
0 & z_{m\alpha} & z_{m_{\dot{y}}} & z_{m_{\dot{\alpha}}} & 0 & 0 & 0 & 0 & z_{m_\beta} \\
0 & 0 & W_{c_y} & 0 & 0 & 0 & 0 & 0 & 0 \\
W_{k_\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\begin{bmatrix} y \\ \alpha \\ \dot{y} \\ \dot{\alpha} \\ w_l \\ w_m \\ w_y \\ w_\alpha \\ y \\ \alpha \end{bmatrix},
\qquad (5.8)
$$

where numeric values are given for each parameter, and the generalized transfer

matrix P is given by

$$
P = \left[
\begin{array}{cccc|ccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-291.11 & -1.21 & -3.39 & -0.06 & -0.10 & -0.65 & -0.10 & 0.65 & -8.51 \\
1846.89 & -54.43 & -20.20 & -0.35 & 0.65 & 19.50 & 0.65 & -19.50 & 207.16 \\
\hline
0 & 1.70 & 0.28 & 0.04 & 0 & 0 & 0 & 0 & 0.91 \\
0 & -0.02 & -0.01 & -0.00 & 0 & 0 & 0 & 0 & 0.45 \\
0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.35 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\right].
$$

More complex model uncertainties can be considered by augmenting the outputs of the state-space system with feedback perturbations. Additional model perturbations could include parametric uncertainty in the aerodynamic models, structural modes and model properties, as well as dynamic uncertainty overbounding responses measured from flight data [71, 59].

5.3.3 Robust Analysis of the Pitch-Plunge System

Computation of robust aeroservoelastic stability (robust flutter) margins is considered in the μ-analysis framework of Section 3.4.1 through an augmented generalized plant (5.8) and uncertainty structure given by

$$
\tilde{\mathbf{\Delta}} = \left\{ \tilde{\Delta} : \tilde{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \delta_{\tilde{q}} I \end{bmatrix}, \ \Delta \in \mathrm{diag}(\delta_{c_y}, \delta_{k_\alpha}), \ \|\Delta\|_\infty \leq 1, \|\delta_{\tilde{q}}\|_\infty \leq 1 \right\}.
$$

The norm bound $\|\delta_{\tilde{q}}\|_\infty \leq 1$ restricts the search over dynamic pressure to perturbations of $\tilde{q} = \tilde{q}_0 \pm 1$ with respect to the nominal dynamic pressure $\tilde{q}_0$. Searching over larger range of dynamic pressure simply requires introducing an appropriate weighting $W_{\tilde{q}}$ to the additive perturbation of $\tilde{q}$,

$$
\tilde{q} = \tilde{q}_0 + W_{\tilde{q}} \delta_{\tilde{q}}. \tag{5.9}
$$

The dynamic pressure weighting is incorporated into the generalized plant to produce a scaled plant $\tilde{P}$

$$\tilde{P} = P \begin{bmatrix} W_{\tilde{q}} & 0 \\ 0 & I \end{bmatrix}. \tag{5.10}$$

Lemma 5.1. *Let nominal plant P, derived for nominal dynamic pressure $\tilde{q}_0$, be given and define the relationship for perturbations to dynamic pressure $\delta_{\tilde{q}}$ as in Fig. 5.6(a). Let scaling $W_{\tilde{q}}$ and uncertainty structure $\mathbf{\Delta}$ be also given such that $\|\Delta\|_\infty < 1$, for all $\Delta \in \mathbf{\Delta}$. Define the scaled plant $\tilde{P}$ as in (5.10). Then W_q is the robust aeroservoelastic stability (robust flutter) margin*

$$\Gamma_{rob} = W_{\tilde{q}}$$

if and only if $\mu_{\tilde{\mathbf{\Delta}}}(\tilde{P}(j\omega)) = 1$, where $\mu_{\mathbf{\Delta}}(\cdot)$ is defined in (3.23). Furthermore, $\tilde{q}_{rob} := \tilde{q}_0 + W_q$ is the dynamic pressure which is at the limit of stability.

Proof: See [61].

The robust aeroservoelastic stability (robust flutter) margin is determined by iterating over the scaling $W_{\tilde{q}}$ until the largest dynamic pressure $\tilde{q}$ given in (5.9) is found such that $\tilde{P}$ is robustly stable with respect to the set of uncertainties $\mathbf{\Delta}$. This idea is presented formally in the following proposition.

Proposition 5.1. *Let nominal plant P, derived for nominal dynamic pressure $\tilde{q}_0$, be given and define the relationship for perturbations to dynamic pressure $\delta_{\tilde{q}}$ as in Fig. 5.6(a). Let scaling $W_{\tilde{q}}$ and uncertainty structure $\mathbf{\Delta}$ be also given such that $\|\Delta\|_\infty < 1$, for all $\Delta \in \mathbf{\Delta}$. Evaluate the following iterative procedure.*

(a) Compute the scaled plant (5.10).

(b) Compute an upperbound for $\mu_{\tilde{\mathbf{\Delta}}}(\tilde{P}(j\omega)) \le \rho_{\tilde{\mathbf{\Delta}}}(M)$ as given in (3.27).

(c) For a given tolerance level ε, if $(\rho_{\tilde{\mathbf{\Delta}}}(M) < 1 - \varepsilon)$ or $(\rho_{\tilde{\mathbf{\Delta}}}(M) > 1 + \varepsilon)$ then

$$W_{\tilde{q}} = W_{\tilde{q}}/\rho_{\tilde{\mathbf{\Delta}}}(M)$$

and return to step (a). Otherwise let $\tilde{q}_{rob} = \tilde{q}_0 + W_q$ and $\Gamma_{rob} = W_{\tilde{q}}$ and exit.

The robust aeroservoelastic stability margin use μ-analysis tools to give worst-case stability parameters. Safe and efficient expansion of the flight envelope can be performed using on-line implementation of the above analysis algorithm. Since computing the upperbound is a convex problem, it does not introduce excessive computational burden. The predictive nature of model based analysis methods, such as the μ-method allow for so called flutterometer tools to be developed for tracking flutter margin during flight tests [60].

The generalized plant is scaled such that the peak structured singular value plot is approximately, but not over, unity. That is the system is robustly stable, with respect to the model uncertainty and scaled perturbation in dynamic pressure. For this feedback control configuration, the singular value plot approaches the robust stability limit at frequencies much lower than the predominant dynamics of the system. Had analysis been performed on a finite frequency range above $10^{-4}Hz$ one would have overestimated the stability margin. Conversely had the analysis been performed using conventional techniques that hold for all frequencies $\omega \in \mathbb{R}$, such as the bounded real lemma, the stability margin would have been greatly underestimated due to the conservativeness introduced by the numerical computations.

This simple example illustrates the usefulness of the results presented in previous chapters for practical problems in which performance and robustness are considered along with numerical difficulty.

5.4 Acknowledgements

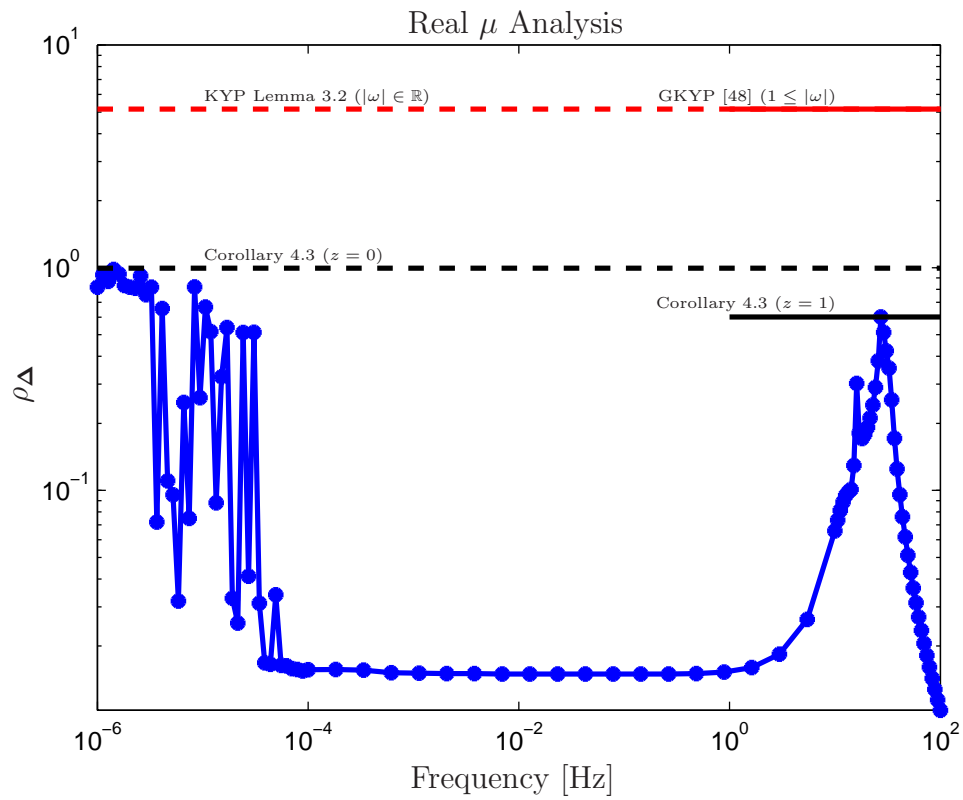Figure 5.7. Robust flutter analysis with velocity feedback controller. The curved line is the greatest lower bound for $\rho_{\boldsymbol{\Delta}}$ with dots located at the frequency grid. Other lines are labeled according to the method and frequency range used to compute them.

Chapter 6

Conclusions and Comments

6.1 Achieved Contributions

In model-based system analysis, the modeling of a dynamical system is the first step in obtaining information regarding the satisfactory compliance with system requirements. Studies on iterative identification and control design have lead to model estimation schemes that are tuned toward their intended purpose, for example control design suggests models estimated from closed-loop data. The resulting model is then used in the analysis of the closed-loop system. The modeling objectives summarized above form the basis for Problem 1. To this end, Chapter 2 provides an extension of the two-step method proposed in [86] for to identifying approximately normalized coprime factorization from closed-loop data. The proposed algorithm demonstrates the use of a constrained ARX model structure, maintaining a linear regression form for numerical efficiency in computing estimates. Furthermore, the model structure preserves the McMillan degree of a constructed model from its coprime factors while allowing the number of denominator of each coprime factor individually to grow. The servomechanical example illustrated that the proposed identification algorithm effectively estimates coprime plant factors from closed-loop data.

The discussion on iterative identification and model based control design provides motivation for studying model-based analysis techniques that incorporate different requirements in various frequency regions of interest. This allows the

control engineer to pose and analyze system requirements specified over a frequency regions, for example around the closed-loop bandwidth, without the use of weighting functions. These analysis objectives are formulated in Problem 2. Chapter 3 addresses these objectives by proposing an alternative formulation of the KYP Lemma, relating an infinite dimensional frequency domain inequality with a pair of finite dimensional linear matrix inequalities. The proposed formulation encompasses previous finite frequency KYP Lemmas for the case when the coefficient matrix of the frequency domain inequality does not depend on frequency. In addition, the proposed formulation allows the coefficient matrix of the frequency domain inequality to vary affinely with the frequency variable. The result has many applications including a new way to allow for frequency-dependent scalings in computing upper bounds to the structured singular value, which can be used to verify stability and performance robustness. The conditions, expressed as a pair of convex inequalities, are computationally efficient and can be limited to finite frequency intervals, features which can significantly reduce conservatism as compared to existing conditions with similar complexity. This effect was illustrated by a case study for hard disk drive servo example.

Noticing the benefits of the alternative formulation of the finite frequency KYP Lemma, Problem 3 asks if generalizations of this result are possible. Chapter 4 explores the same generalizations proposed in [48] via transformations of the frequency variable which leads to several useful extensions. These extensions many applications such as analysis of discrete-time linear systems and analysis of infinite frequency intervals specified with numerically tractable finite limits. The proposed extensions to infinite frequency intervals are particularly interesting since it provides a generalization of the original KYP Lemma, which holds over all frequencies. The particular extension for infinite frequency intervals is illustrated with numerical examples particularly in demonstrating reduced conservatism in the analysis.

6.2 Future Research

Future work on developing control-relevant identification algorithms will be to extend the proposed linear regression method of Chapter 2 to include multivariable

system identification. Also of interest will be to explore explicit control-relevancy implications directly from the limit-model expressions, most likely in terms of a gap-metric.

Future work on system analysis tools will be to show that the sufficient LMI test for frequency domain inequalities in finite frequency ranges presented in Chapters 3 and 4 are also necessary in the case the coefficient matrix Θ is frequency dependent. The methods for proving necessity involve introducing an augmented system followed by subsequent reduction of the analysis conditions through appropriate congruent transformations. Regarding the application of these tools to robust analysis, the question of characterizing the class of uncertainties that is represented by the proposed robust stability test remain open. Additionally, extensions for the conditions may be proposed to include even broader class of system descriptions particularly polynomial systems.

Finally, many of the control synthesis and filter design applications proposed in [48] can be posed under the extended conditions. Although through example these extended conditions show numerical benefits in system analysis it is not clear how these traits might carry over to synthesis.

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