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**THE GENERALIZED RIEMANN PROBLEM
IN COMPRESSIBLE DUCT FLOW¹**

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Consider the Euler equations that model the time-dependent flow of an inviscid, compressible fluid through a duct of smoothly varying cross-section. Denoting by $A(r)$ the area of the cross-section at r , these equations are,

$$A \frac{\partial}{\partial t} U + \frac{\partial}{\partial r} [AF(U)] + A \frac{\partial}{\partial r} G(U) = 0, \quad (1)$$

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}, \quad F(U) = \begin{pmatrix} \rho u \\ \rho u^2 \\ (\rho E + p)u \end{pmatrix}, \quad G(U) = \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix},$$

where ρ, p, u are, respectively, density, pressure, and velocity, $E = e + \frac{1}{2}u^2$ is the total specific energy (with e being the internal specific energy), and an equation-of-state of the form $p = p(e, \rho)$ is assumed. Observe that the system (1) covers all cases of flows with spherical or cylindrical symmetry.

The numerical scheme proposed by Godunov [2] for the solution of (1) is well-known. To give a brief review, suppose that we use equally spaced grid-points $r_i = i \Delta r$ along the r -axis and uniform time intervals of size Δt . By "cell i " we shall refer to the interval extending between the "cell boundaries" $r_{i \pm \frac{1}{2}} = (i \pm \frac{1}{2}) \Delta r$. We let Q_i^n denote the average value of the quantity Q over cell i at time $n \Delta t$. Similarly, we designate by $Q_{i \pm \frac{1}{2}}^{n+\frac{1}{2}}$ the value of Q at the cell boundary $r_{i \pm \frac{1}{2}}$, averaged over the time interval $(n \Delta t, (n+1) \Delta t)$. Generally speaking, a "Godunov-type" difference scheme for (1) is given by

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta V_i} \left[A(r_{i+\frac{1}{2}}) F(U_{i+\frac{1}{2}}^{n+\frac{1}{2}}) - A(r_{i-\frac{1}{2}}) F(U_{i-\frac{1}{2}}^{n+\frac{1}{2}}) \right] - \frac{\Delta t}{\Delta r} \left[G(U_{i+\frac{1}{2}}^{n+\frac{1}{2}}) - G(U_{i-\frac{1}{2}}^{n+\frac{1}{2}}) \right], \quad (2)$$

where $\Delta V_i = \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} A(r) dr$. For the first order Godunov scheme one takes

$U_{i+\frac{1}{2}}^{n+\frac{1}{2}} = U_{i+\frac{1}{2}}^n$, where $U_{i+\frac{1}{2}}^n$ is the solution along $r = 0$ to the initial-value problem,

$$\frac{\partial}{\partial t}U + \frac{\partial}{\partial r}(F(U) + G(U)) = 0, \quad U(r,0) = \begin{cases} U_i^n, & r < 0, \\ U_{i+1}^n, & r > 0. \end{cases} \quad (3)$$

The notation in (3) is identical to that of (1). Problem (3) is known as a "Riemann Problem" and the fact that its solutions are "self-similar", i.e., depend on r/t only, is well known.

The transition to a second-order generalization of the Godunov scheme is accomplished by taking the following steps.

- (a) Assume that $U^n(r)$ is *linearly* distributed in the computational cells, with possible jumps at cell boundaries. Set $U_{i+\frac{1}{2},\pm}^n =$ limiting values of $U^n(r)$ as $r \rightarrow r_{i+\frac{1}{2}}\pm$.
- (b) Find the value $U_{i+\frac{1}{2}}^n$ by solving (3) as before, replacing U_i^n, U_{i+1}^n by $U_{i+\frac{1}{2},-}^n, U_{i+\frac{1}{2},+}^n$ respectively.
- (c) Let $(\Delta U)_i^n$ be the variation of $U^n(r)$ in cell i and let $U(r,t)$ be the solution of (1) subject to the initial conditions

$$U(r,0) = \begin{cases} U_{i+\frac{1}{2},-}^n + \frac{(\Delta U)_i^n}{\Delta r}(r - r_{i+\frac{1}{2}}), & r < r_{i+\frac{1}{2}}, \\ U_{i+\frac{1}{2},+}^n + \frac{(\Delta U)_{i+1}^n}{\Delta r}(r - r_{i+\frac{1}{2}}), & r > r_{i+\frac{1}{2}}. \end{cases} \quad (4)$$

Evaluate *exactly* $\left(\frac{\partial U}{\partial t}\right)_{i+\frac{1}{2}}^n = \frac{\partial U(r,t)}{\partial t} \Big|_{r=r_{i+\frac{1}{2}}, t=0}$.

- (d) Substitute in (2),

$$U_{i+\frac{1}{2}}^{n+\frac{1}{2}} = U_{i+\frac{1}{2}}^n + \frac{\Delta t}{2} \cdot \left(\frac{\partial U}{\partial t} \right)_{i+\frac{1}{2}}^n.$$

We refer to the problem presented in (c) as a *Generalized Riemann Problem (GRP)*.

The most difficult part in the treatment of the GRP is the resolution of a *Centered Rarefaction Wave (CRW)* in this case. Once such a resolution has been accomplished, it is fairly easy to complete the numerical scheme outlined above. A detailed account of this procedure along with some numerical examples may be found in [1].

In the remainder of this paper we shall concentrate on the analytic treatment of the CRW. Even though this is done in the context of compressible fluid flow, it can be extended to more general hyperbolic systems. In particular, the case of reactive flows can be handled similarly.

To fix the ideas, suppose that we are given a GRP with a jump at $r = 0$. It is convenient to change to a Lagrangian coordinate

$$d\xi = A \rho dr, \quad \xi(0) = 0. \quad (5)$$

The system (1) now takes the form,

$$\begin{aligned} \frac{\partial}{\partial t} V + \frac{\partial}{\partial \xi} [A \Phi(V)] + A \frac{\partial}{\partial \xi} \Psi(V) &= 0, \\ V = V(\xi, t) = \begin{pmatrix} \tau \\ u \\ E \end{pmatrix}, \quad \Phi(V) = \begin{pmatrix} -u \\ 0 \\ pu \end{pmatrix}, \quad \Psi(V) = \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix}, \quad \tau = 1/\rho. \end{aligned} \quad (6)$$

We assume that initially

$$V(\xi, 0) = \begin{cases} V_r + \left(\frac{\partial V}{\partial \xi} \right)_r \cdot \xi, & \xi > 0, \\ V_l + \left(\frac{\partial V}{\partial \xi} \right)_l \cdot \xi, & \xi < 0. \end{cases} \quad (7)$$

Corresponding to (6)-(7) we define the *Associated Riemann Problem* by,

$$\frac{\partial}{\partial t} V + \frac{\partial}{\partial \xi} [\Phi(V) + \Psi(V)] = 0, \quad V(\xi, 0) = \begin{cases} V_r, & \xi > 0, \\ V_l, & \xi < 0. \end{cases} \quad (8)$$

Again, the solution of (8) depends only on $\frac{\xi}{t}$. In particular, we set $V^* = V(0, t)$, $t > 0$. These are the values along the contact discontinuity $\xi = 0$. For variables that may have a jump across this discontinuity we add the indices l, r . Thus, ρ_l^* denotes the density value "immediately to the left" of the contact discontinuity.

Without loss of generality suppose that a CRW propagates to the left as shown in

Fig. 1.

Fig. 1

We denote by $\Gamma^\pm$ the characteristic curves (of (6)) with slopes $\pm gA$, where here and in what follows $g = \rho c$ is the "Lagrangian speed of sound". Throughout the wave we impose characteristic coordinates (α, β) as follows.

$\beta =$ Normalized slope of Γ^- at the origin, $\beta = 1$ at the head characteristic. (9)

$\alpha =$ Value of ξ at intersection point of Γ^+ with the curve $\beta = 1$.

Thus, $\alpha = \text{const.}$ (resp. $\beta = \text{const.}$) represents a Γ^+ (resp. Γ^-) curve. Observe that the "curve" $\alpha = 0$ corresponds to the point of singularity. The section of the CRW near the origin is now projected onto the rectangular domain $\alpha^* \leq \alpha \leq 0$, $\beta^* \leq \beta \leq 1$, where $\alpha^* < 0$ is sufficiently small and $\beta^* = g_l^*/g_l$ is the normalized slope (at the origin) of the tail characteristic. All flow variables (including ξ, t) become now functions of α, β . In particular, the values $Q(0, \beta)$, for every variable Q , are those obtained from the associated Riemann problem (8), by the standard self-similar solution mentioned before. The following lemma gives simple expressions for ξ, t in terms of α, β near $\alpha = 0$.

Lemma. *The functions $\xi(\alpha, \beta)$, $t(\alpha, \beta)$ satisfy*

$$\begin{aligned}\xi(\alpha, \beta) &= \alpha\beta^{1/2} + \epsilon(\alpha, \beta)\alpha^2, \\ t(\alpha, \beta) &= -k\alpha\beta^{-1/2} + \eta(\alpha, \beta)\alpha^2, \quad k = (g_l A(0))^{-1},\end{aligned}\tag{10}$$

where $\epsilon(\alpha, \beta), \eta(\alpha, \beta)$ are smooth functions in $[\alpha^*, 0] \times [\beta^*, 1]$.

Proof. The definition of the characteristic coordinates implies the relations

$$\frac{\partial \xi}{\partial \alpha} = -gA \frac{\partial t}{\partial \alpha}, \quad \frac{\partial \xi}{\partial \beta} = gA \frac{\partial t}{\partial \beta},\tag{11}$$

where $A = A(r) = A(r(\xi, t))$. Differentiate the first relation with respect to β , the second with respect to α , and note that, at $\alpha = 0$, $\frac{\partial A}{\partial \beta} = \frac{\partial t}{\partial \beta} = 0$. We obtain,

$$2g(0, \beta) \frac{\partial}{\partial \beta} \left(\frac{\partial t}{\partial \alpha}(0, \beta) \right) + \frac{\partial g}{\partial \beta}(0, \beta) \cdot \frac{\partial t}{\partial \alpha}(0, \beta) = 0.$$

By definition $\xi(\alpha, 1) = \alpha, g(0, \beta) = g_l \beta$, so that the first equation in (11) implies

$\frac{\partial t}{\partial \alpha}(0,1) = -k$ and the above equation yields $\frac{\partial t}{\partial \alpha}(0,\beta) = -k \beta^{-1/2}$ and therefore $\frac{\partial \xi}{\partial \alpha}(0,\beta) = \beta^{1/2}$, from which (10) follows.

It turns out that in order to determine the time derivatives of variables at the singularity one needs to know the directional derivatives $\frac{\partial Q}{\partial \alpha}(0,\beta)$ (the time derivatives $\left(\frac{\partial V}{\partial t}\right)^*$ are then determined by using the chain rule and the directional derivatives at $\beta = \beta^*$). It can be shown that once $\frac{\partial u}{\partial \alpha}(0,\beta)$ is determined, all other derivatives follow easily. Our theorem gives a "propagation of singularities" result for the evaluation of this quantity.

Theorem. Let $a(\beta) = \frac{\partial u}{\partial \alpha}(0,\beta)$, $\beta^* \leq \beta \leq 1$. We have

$$a'(\beta) = H(\beta) + T(\beta) \frac{\partial t}{\partial \alpha}(0,\beta), \quad (12)$$

where $H(\beta), T(\beta)$ can be determined explicitly from the equation of state and the initial conditions **ahead** of the rarefaction (i.e., $V_1, \left(\frac{\partial V}{\partial \xi}\right)_1$). Furthermore, $H(\beta), T(\beta)$ reflect, respectively, the thermodynamic and geometrical non-uniformity ahead of the rarefaction. In particular,

$$H(\beta) \equiv 0 \quad \text{if} \quad \left(\frac{\partial S}{\partial \xi}\right)_1 = 0 \quad (S\text{-entropy}), \quad (13)$$

$$T(\beta) = \frac{\lambda}{2} \frac{\partial}{\partial \beta}(u(0,\beta) \cdot c(0,\beta)), \quad \lambda = A'(0)/A(0). \quad (14)$$

The initial condition for (12) is given by

$$a(1) = \left(\frac{\partial u}{\partial \xi} \right)_l + g_l^{-1} \left(\frac{\partial p}{\partial \xi} \right)_l. \quad (15)$$

Proof. The characteristic relations for (6) are

$$\begin{aligned} \text{(i)} \quad & \frac{\partial p}{\partial \alpha} - g \frac{\partial u}{\partial \alpha} + \frac{ucgA'}{A} \frac{\partial t}{\partial \alpha} = 0, \\ \text{(ii)} \quad & \frac{\partial p}{\partial \beta} + g \frac{\partial u}{\partial \beta} + \frac{ucgA'}{A} \frac{\partial t}{\partial \beta} = 0. \end{aligned} \quad (16)$$

Differentiating (i) with respect to β , (ii) with respect to α and eliminating p we get

$$\begin{aligned} 2g(0,\beta)a'(\beta) + \frac{\partial g}{\partial \alpha}(0,\beta) \cdot \frac{\partial u}{\partial \beta}(0,\beta) + a(\beta) \frac{\partial g}{\partial \beta}(0,\beta) \\ - \lambda \frac{\partial t}{\partial \alpha}(0,\beta) \cdot \frac{\partial}{\partial \beta}[u(0,\beta)c(0,\beta)g(0,\beta)] = 0. \end{aligned} \quad (17)$$

Since the CRW is non-isentropic we cannot evaluate g_α from p_α . So to eliminate g_α we proceed as follows. Given a state (g_0, p_0) we let $g = G(g_0, p_0, p)$ represent the isentropic curve through (g_0, p_0) in the (g, p) plane. With $\xi(\alpha, \beta)$ given by (10) we evaluate the initial conditions at $\xi(\alpha, \beta)$

$$p_0(\alpha, \beta) = p_l + \left(\frac{\partial p}{\partial \xi} \right)_l \cdot \xi(\alpha, \beta), \quad g_0(\alpha, \beta) = g_l + \left(\frac{\partial g}{\partial \xi} \right)_l \cdot \xi(\alpha, \beta).$$

Since the flow is isentropic along streamlines we conclude that

$$g(\alpha, \beta) = G(g_0(\alpha, \beta), p_0(\alpha, \beta), p(\alpha, \beta)),$$

and evaluating the derivative of this expression with respect to α at $(0, \beta)$ we get

$$\begin{aligned}
\frac{\partial g}{\partial \alpha}(0, \beta) &= \left[G_{g_0}(g_l, p_l, p(0, \beta)) \cdot \left(\frac{\partial g}{\partial \xi} \right)_l + G_{p_0}(g_l, p_l, p(0, \beta)) \left(\frac{\partial p}{\partial \xi} \right)_l \right] \cdot \\
&\quad \cdot \frac{\partial \xi}{\partial \alpha}(0, \beta) + G_p(g_l, p_l, p(0, \beta)) \cdot \frac{\partial p}{\partial \alpha}(0, \beta) \\
&= I(\beta) \cdot \beta^{\frac{1}{2}} + G_p(g_l, p_l, p(0, \beta)) \cdot \\
&\quad \cdot \left[g(0, \beta) a(\beta) - \lambda u(0, \beta) c(0, \beta) g(0, \beta) \cdot \frac{\partial t}{\partial \alpha}(0, \beta) \right],
\end{aligned}$$

where $I(\beta)$ is the expression in the square brackets and we have made use of (10) and

(16)(i). Inserting the expression for $\frac{\partial g}{\partial \alpha}(0, \beta)$ in (17) results in

$$\begin{aligned}
&2g(0, \beta) a'(\beta) + E(\beta) \cdot a(\beta) \\
&+ \frac{\partial u}{\partial \beta}(0, \beta) \cdot \left[I(\beta) \cdot \beta^{\frac{1}{2}} - \lambda G_p(g_l, p_l, p(0, \beta)) u(0, \beta) c(0, \beta) g(0, \beta) \frac{\partial t}{\partial \alpha}(0, \beta) \right] \quad (18) \\
&- \lambda \frac{\partial}{\partial \beta}(u(0, \beta) c(0, \beta) g(0, \beta)) \frac{\partial t}{\partial \alpha}(0, \beta) = 0,
\end{aligned}$$

where $E(\beta) = \frac{\partial g}{\partial \beta}(0, \beta) + G_p(g_l, p_l, p(0, \beta)) g(0, \beta) \frac{\partial u}{\partial \beta}(0, \beta)$. However, $E(\beta) \equiv 0$, by

(16)(ii) and the identity $g(0, \beta) = G(g_l, p_l, p(0, \beta))$. Also, using (16)(ii) once again,

$$\begin{aligned}
\frac{\partial u}{\partial \beta}(0, \beta) \cdot G_p(g_l, p_l, p(0, \beta)) \cdot u(0, \beta) c(0, \beta) g(0, \beta) &= - u(0, \beta) c(0, \beta) \frac{\partial g}{\partial \beta}(0, \beta) \\
&= - g_l u(0, \beta) c(0, \beta),
\end{aligned}$$

and equation (18) takes the form (12) with $T(\beta)$ as in (14) and $H(\beta) =$

$-\frac{1}{2} g_l^{-1} \beta^{-\frac{1}{2}} I(\beta) \frac{\partial u}{\partial \beta}(0, \beta)$. To prove (13) we note that if $\left(\frac{\partial S}{\partial \xi} \right)_l = 0$ then

$$G(g_0(\alpha, \beta), p_0(\alpha, \beta), p(0, \beta)) = g(0, \beta) + o(\alpha^2),$$

and differentiating with respect to α yields $I(\beta) \equiv 0$.

Finally, (15) follows from (6), (10) and the chain rule,

$$a(1) = \frac{\partial u}{\partial \alpha}(0,1) = \left(\frac{\partial u}{\partial \xi} \right)_l \frac{\partial \xi}{\partial \alpha}(0,1) + \left(\frac{\partial u}{\partial t} \right)_l \cdot \frac{\partial t}{\partial \alpha}(0,1),$$

when we use that $\left(\frac{\partial u}{\partial t} \right)_l + A(0) \left(\frac{\partial p}{\partial \xi} \right)_l = 0$.

Remark. For the γ -law equation of state,

$$p = (\gamma-1)\rho e, \quad \gamma > 1,$$

the expression for $a(\beta)$ is

$$\begin{aligned} a(\beta) = a(1) &+ \frac{2}{g_l(3\alpha-1)} \left[c_l \left(\frac{\partial g}{\partial \xi} \right)_l - \frac{\gamma+1}{2} \left(\frac{\partial p}{\partial \xi} \right)_l \right] \cdot \left(\beta^{\frac{3\gamma-1}{2(\gamma+1)}} - 1 \right) \\ &- \frac{\lambda}{\gamma-3} A(0)^{-1} \rho_l^{-1} [(\gamma-1)u_l + 2c_l] \cdot \left(\beta^{\frac{\gamma-3}{2(\gamma+1)}} - 1 \right) \\ &+ \frac{4\lambda c_l}{3\gamma-5} A(0)^{-1} \rho_l^{-1} \cdot \left(\beta^{\frac{3\gamma-5}{2(\gamma+1)}} - 1 \right), \end{aligned}$$

with some modification for the cases $\gamma = 5/3, 3$.

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