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Spectral Gaps on Riemannian Manifolds

DISSERTATION

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ABSTRACT OF THE DISSERTATION

Spectral Gaps on Riemannian Manifolds

By

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In this text, we survey some basic results related to the New Weyl criterion for the essential spectrum of the Laplace-Beltrami operator on Riemannian manifolds. We then use the language of Gromov-Hausdorff convergence to prove a spectral gap theorem. Finally, we discuss the Hodge-Laplacian operator on differential k-forms, and generalize our main theorem to that setting.

Chapter 0

Geometry and Analysis Background

This first chapter is a collection of notation and basic theorems used in the entirety of this document. Each section should be independent of the other, but makes little sense by itself.

The goal is to make this document fairly self-contained for the benefit of a novice, while still readable and enjoyable to a specialist. As such, the recommendation is that this chapter is skipped altogether, and referred back to for clarification, when necessary, on the contents of the other chapters.

0.1 Definitions and Notation

A *topological manifold* is a topological space M, with the following properties:

(i) *M is a Hausdorff space*: For every pair of points $p, q \in M$, there are disjoint open sets $U, V \subset M$ such that $p \in U$ and $q \in V$.

(ii) *M is second countable*: There exist a countable basis for the topology of M.

(iii) *M is locally euclidean of dimension n*: Every point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. More specifically, for each $p \in M$ we can find

- An open set $U \subset M$ containing p,
- An open set $\tilde{U} \subset \mathbb{R}^n$, and
- A homeomorphism $\varphi : U \to \tilde{U}$.

The pair (U, φ) is called a *coordinate chart*, and the set of all such charts is called an *atlas*.

An atlas $\mathcal{A}$ is said to be *smooth* if the *transition maps*

$$\psi \circ \varphi^{-1} : \text{Image}(\varphi) \subset \mathbb{R}^n \to \text{Image}(\psi) \subset \mathbb{R}^n$$

have partial derivatives of all orders, for all coordinate charts $(\varphi, U), (\psi, V) \in \mathcal{A}$. A smooth atlas $\mathcal{A}$ is said to be maximal if it is not contained in any strictly larger smooth atlas. Finally, a *smooth manifold* is a pair $(M\mathcal{A})$ where M is a topological manifold and $\mathcal{A}$ is a maximal smooth atlas of M. The differentiability of functions on a smooth manifold M is defined in terms of their composition with coordinate charts. Recall:

$$C^k(M) := \{\ k\text{-times differentiable real-valued functions on } M\},$$

$$C^\infty(M) := \{\text{infinitely differentiable real-valued functions on } M\}.$$

A linear map $V : C^\infty(M) \to \mathbb{R}$ is called a derivation at $p \in M$ if

$$V(fh) = f(p)V(h) + h(p)V(f), \forall f, h \in C^\infty(M).$$

The set of all such derivations at p is a vector space called the *tangent space* to M at p, denoted by T_pM. Each element of T_pM is most commonly called a *tangent vector* to M at

p.

In coordinates, given a chart (φ, U) around a point $p \in M$, the tangent vectors

$$\begin{aligned} \frac{\partial}{\partial x^i} : C^\infty(M) &\longrightarrow \mathbb{R} \\ f : &\longmapsto \frac{\partial}{\partial x^i}\left(f \circ \varphi^{-1}\right)\Big|_{\varphi(p)} \end{aligned}$$

form a basis to T_pM. Alternatively, one writes $\partial_i = \frac{\partial}{\partial x^i}$ for simplicity. Note that, in fact, $\{\partial_1, \cdots, \partial_n\}$ defines a basis for $T_qM, \forall q \in U$. Under this perspective, we call $\{\partial_1, \cdots, \partial_n\}$ a local *coordinate frame* around p.

The *tangent bundle* of M is the disjoint union

$$TM := \bigsqcup_{p \in M} T_pM.$$

TM is a smooth manifold of dimension $2n$, and comes equipped with a natural projection $\pi(p, v) = p$. A *section* of TM is a map $\sigma : M \to TM$ satisfying $\pi \circ \sigma(p) = p, \forall p \in M$. A *vector field* is a smooth section of TM, and the set of all vector fields on M is denoted by $\mathfrak{X}(M)$.

A differential *k-form* on M is a smooth map ω that associates, to each point $p \in M$, an alternating multilinear map $w(p) : \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ times}} \to \mathbb{R}$. The map is alternating in the sense that, for each $p \in M$ and $v_1, \cdots, v_k \in T_pM$, we have

$$\omega(p)(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = \operatorname{sign}(\sigma)\omega(p)(v_1, \cdots, v_k),$$

where $\sigma \in S_k$ is a permutation of the indices $\sigma : \{1, \cdots, k\} \to \{1, \cdots, k\}$, and $\operatorname{sign}(\sigma)$ is the sign of the permutation, which equals 1 is it is an even permutation and -1 if it is odd. The differential form ω is smooth in the sense that, for each $X_1, \cdots, X_k \in \mathfrak{X}(M)$, the map

$$p \longmapsto \omega(p)(X_1, \cdots, X_k)$$

is smooth. The set of all such k-forms on a manifold M is denoted $\Lambda^k M$. From the tangent frame defined above, its dual frame $\{dx^1, \cdots, dx^n\}$ can be defined by taking $dx^i(\partial_j) = \delta_{ij}$, end extending to T_pM linearly, where δ_{ij} is the kronecker delta $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

If $\omega \in \Lambda^k M$ and $\eta \in \Lambda^l M$ are differential forms on M, then the *wedge product* is defined as

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!} \operatorname{Alt}(\omega \otimes \eta),$$

where the *alternation* map $\operatorname{Alt} : T^k M \to \Lambda^k M$ is defined as the $(k+l)$-form

$$\operatorname{Alt}(\alpha)(x)(v_1, \cdots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \alpha(x)(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

for any $\alpha \in T^k M$ and $\sigma \in S_k$ permutations. It is noteworthy that $\operatorname{Alt}(\alpha) \in \Lambda^k M, \forall \alpha T^k M$ and $\operatorname{Alt}(\alpha) = \alpha \iff \alpha \in \Lambda^k M$.

Because the dimension of each tangent space T_pM as a vector space is n, which is the same dimension of the manifold, and the fact that k-forms are multilinear and alternating, there are no nonzero k-forms for $k > n$, but for $k = n$, we can deduce the existence of a globally defined n-form dV defined as

$$dV = dx^1 \wedge \cdots \wedge dx^n.$$

Because every k-form can be written in coordinates as

$$\omega = a_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

for $a_{i_1 \cdots i_k} \in C^\infty(M)$, all n-forms on M are multiples of the volume form, that is,

$$\omega \in \Lambda^n M \Rightarrow \omega = \alpha dV, \quad \text{for some } \alpha \in C^\infty(M).$$

An important operation on differential forms is the *interior multiplication* by a vector field. Given $\omega \in \Lambda^k M$ and $X \in \mathfrak{X}(M)$, for every $p \in M$ and $v_1, \cdots, v_k \in T_pM$ we define the $(k-1)$-form $i_X\omega$ through

$$i_X\omega(p)(v_1, \cdots, v_k) = \omega(p)(X, v_1, \cdots, v_k).$$

A tensor T of type $\binom{k}{l}$ on M is a smooth multilinear map

$$T : \underbrace{\Lambda^1 M \times \cdots \times \Lambda^1 M}_{l \text{ times}} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k \text{ times}} \to \mathbb{R}.$$

Much like k-forms, a tensor can also be seen as associating, to each point $p \in M$, a multilinear map $T(p) : \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ times}} \times \underbrace{(T_pM)^* \times \cdots \times (T_pM)^*}_{l \text{ times}} \to \mathbb{R}$, and the association is defined to be smooth in similar terms, which means every k-form is an alternating $\binom{0}{k}$-tensor, and every vector field if a $\binom{1}{0}$-tensor.

If one considers the volume form dV under the light of the definition above,

A *Riemannian metric* on a smooth manifold M is a smooth 2-tensor g on M, satisfying

(i) *g is symmetric*: $g(X,Y) = g(Y,X), \forall$ smooth vector fields X, Y on M,

(ii) *g is positive definite*: $g(X,X) > 0$ if $X \neq 0$.

A metric g determines an inner product on each tangent space T_pM which is typically written as

$$\langle X,Y\rangle_p = \langle X(p),Y(p)\rangle \coloneqq g(X,Y)(p).$$

A Riemannian manifold is a pair (M,g), where M is a smooth manifold and g is a Riemannian metric on M. In a coordinate system $(x_1,\cdots,x_n)$, we can define

$$g_{ij} = \langle \partial_i, \partial_j\rangle \coloneqq \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right\rangle.$$

Given a function $f \in C^\infty(M)$, we can define its *differential at point* p as the 1-form determined on a coordinate frame $\{\partial_1,\cdots,\partial_n\}$ associated to a chart $\varphi : U \subset M \to \mathbb{R}^n$ around a point $p \in U$ by

$$df_p(\partial_i) = \frac{\partial}{\partial x^i}(f\circ\varphi^{-1})(\varphi(p)).$$

Using the differential above, we can apply the Riesz representation theorem to assert the existence of a unique vector field $\mathrm{grad}(f) \in \mathfrak{X}(M)$ satisfying

$$df_p(X) = \langle \mathrm{grad}(f), X\rangle\,,\ \forall X \in \mathfrak{X}(M), p \in M.$$

We can define the *exterior differential* operator $d : \Lambda^k M \to \Lambda^{k+1}M$ on a k-form $\omega = a_{i_1\cdots i_k} dx^{i_1}\wedge\cdots\wedge dx^{i_k}$ as

$$d\omega = da_{i_1\cdots i_k}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

Reexamining the volume form dV using the interior multiplication and the exterior differential, we can deduce that $d(i_X dV)$ is n-form, and therefore a multiple of dV, so we can define the *divergence of the vector field* X as the function satisfying

$$d(i_X dV) = \mathrm{div}(X)dV.$$

Finally we can define the most important operator to be used in this document. Given $f \in C^{\infty}(M)$, *the Laplacian of* f is the C^{∞} function

$$\Delta f = \text{div}(\text{grad}(f)).$$

0.2 Sobolev Spaces and the Sobolev Inequality

Definition 0.1. *We say that a function f on M is measurable if, for all compact sets K and $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subseteq K$ such that $\text{Vol}(K - K_\varepsilon) < \varepsilon$ and such that the restriction $f|_{K_\varepsilon}$ is continuous on K_ε.*

With this definition, we can recall the definition of the L^p spaces on M. The L^p norm of a function f on M is defined to be

$$||f||_{L^p(M)} := \left(\int |f|^p dV \right)^{\frac{1}{p}},$$

where dV is the volume form, when this quantity is well defined. For simplicity, we may write $||f||_p = ||f||_{L^p(M)}$ when there is no risk of confusion. For $p = \infty$, then define

$$||f||_{L^\infty(M)} := \inf\{\varepsilon > 0 : \text{Vol}(\{x : |f(x)| > \varepsilon\}) = 0\},$$

that is, $||f||_\infty$ is the smallest of the essential bounds of $|f|$. Similar to the case $p < \infty$, we may write $||f||_\infty = ||f||_{L^\infty(M)}$.

Recall that $C_0^\infty(M) = \{f \in C^\infty(M) : f \text{ has compact support }\}$. We can then define the space $L^p(M)$ to be the completion of $C_0^\infty(M)$ with respect to the L^P norm, i.e.,

$$L^p(M) := \overline{\{f \in C_0^\infty(M) : ||f||_p < \infty\}},$$

where the closure above is taken using equivalence classes of sequences much like when defining the set of all real numbers $\mathbb{R}$ through Dedekind cuts. For $L^\infty(M)$,

$$L^\infty(M) \coloneqq \{f : M \to \mathbb{R} : ||f||_\infty < \infty\}.$$

Also, for simplicity, define

$$\mathcal{F}_k^p \coloneqq \{\varphi \in C^\infty(M) : |\nabla^l \varphi| \in L^p(M), \forall\ 0 \le l \le k\}.$$

Finally, for any set S from the above, $S_0 \coloneqq \{f \in S : f \text{ has compact support }\}$. We are now ready to define the Sobolev spaces $H_k^p(M)$.

Definition 0.2 (Sobolev spaces)**.** *The Sobolev space $H_k^p(M)$ is the completion of $\mathcal{F}_k^p$ with respect to the norm*

$$||\varphi||_{H_k^p(M)} \coloneqq \sum_{l=0}^{k} ||\nabla^l \varphi||_p.$$

Also, define

$$H_k^{0,p}(M) = \{f \in H_k^p(M) : f \textit{ has compact support }\}.$$

As we will be interested in studying the spectrum of the Laplace-Beltrami operator acting on functions in M, it is important to carefully define its domain, which will depend on the geometry of the manifold, see [37] A more detailed discussion on the domain of a densely-defined operator can be found in chapter 3, where we deal with the Hodge Laplacian.

Consider the Laplace-Beltrami operator Δ acting on $C_0^\infty(M)$,

$$\Delta f \coloneqq \operatorname{div}(\nabla f) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x_j} f \right).$$

We are interested in solving eigenvalue problems as follows. When $\partial M =$ and M is compact, Δ is a self-adjoint elliptic operator on $H_i^2(M)$, and the spectral theory of self-adjoint operators guarantees the existence of *eigenvalues* $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_i \to \infty$ and of the corresponding *eigenfunctions* satisfying

$$\Delta\varphi_i = -\lambda_i\varphi_i,$$

where $\varphi_i \in C^\infty(M) \cap H_i^2(M)$ can be chosen so that $\{\varphi_i\}$ forms an orthonormal basis for $H_1^2(M)$.

When $\partial M \neq$, we must specify some boundary conditions so that Δ is self-adjoint. Usually, there are two types of boundary conditions.

When we say that *Dirichlet boundary condition* is posed, we mean that we consider $\mathfrak{Dom}(\Delta) = H_{0,1}^2(M)$. The eigenvalues are

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots,$$

and the corresponding eigenfunctions satisfy

$$\begin{cases} \Delta\varphi_i = -\lambda\varphi_i, \\ \varphi_i \in C^\infty(M), \, \varphi|_{\partial M} = 0, \end{cases}$$

and form an orthonormal basis for $H_{0,1}^2(M)$. This is the condition we will be using in most of this document.

When *Neumann boundary condition* is posed, $\mathfrak{Dom}(\Delta) = H_1^2(M)$ and its eigenvalues and eigenfunctions are

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots,$$

and $\{\varphi_i\}$ forms an orthonormal basis of $H_1^2(M)$, satisfying

$$\begin{cases} \Delta\varphi_i = -\lambda\varphi_i, \\ \varphi_i \in C^\infty(M),\ \frac{\partial\varphi_i}{\partial\nu}\Big|_{\partial M} = 0, \end{cases}$$

where ν is the outer normal vector along ∂M.

Since these boundary conditions will affect our computations, let us make the following definition.

Definition 0.3. *Let M be a Riemannian manifold. Define $H(M)$ as follows.*

- *If $\partial M = \emptyset$, define $H(M) := \left\{f \in H_1^2(M) : \int_M f = 0\right\}$.*
- *If $\partial M \neq \emptyset$ and Dirichlet boundary condition is posed, define $H(M) := H_{0,1}^2(M)$*
- *If $\partial M \neq \emptyset$ and Neumann boundary condition is posed, define*

$$H(M) := \left\{f \in H_1^2(M) : \int_M f = 0\right\}$$

.

Then, Δ is a self-adjoint operator on $H(M)$ and, by the Min-Max principle, we can find an orthonormal basis $\{\varphi_i\}$ for $H(M)$ with $\Delta\varphi_i = \lambda\varphi_i$, $\varphi_i \in C^\infty(M) \cap H(M)$ such that

$$\lambda_1 = \inf\left\{\frac{\int |\nabla f|^2}{\int f^2} : f \in H(M)\right\},$$

$$\lambda_i = \inf\left\{\frac{\int |\nabla f|^2}{\int f^2} : f \in H(M), \int f\varphi_j = 0, j = 1, \cdots, i-1\right\}.$$

In particular, the above property states that λ_1 is the largest constant $C \in \mathbb{R}$ for which

$$\int |\nabla f|^2 \geq C \int f^2, \forall f \in H(M).$$

The inequality above is called a *Poincaré inequality*. Another type of inequality we will make use of is the *Sobolev inequality* below.

If M^n is a compact Riemannian manifold with boundary, $n > 2$, then there exists a constant C such that

$$C\left(\int_M f^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \int_M |\nabla f|^2, \forall f \in H(M).$$

The constant C is called a *Sobolev constant*. To make its dependence on the metric clear, we sometimes write $C = C(M, g)$. If M is noncompact, however, the Sobolev inequality may not hold, but it can be shown that its validity is equivalent to that of the Isoperimetric inequality, see [37] for a proof. Similarly, one can show that the existence of such a constant C is equivalent to certain estimates on the heat kernel on the manifold, see [35].

Finally, since we will deform the metric in our manifolds in chapter 2, let us now briefly explore the effects of rescaling the metric the Sobolev inequality. Let M^n be a compact manifold.

Theorem 0.1 (Sobolev constant under scaling of the metric [10])**.** *Let (M^n, g) be a closed Riemannian manifold. Then the L^2 Sobolev Constant has the property that for any $\varepsilon > 0$,*

$$C_s(M, \varepsilon^2 g) = C_s(M, g).$$

Proof. Let $\tilde{g} = \varepsilon^2 g$ and $\tilde{\varphi} = \varepsilon^{-\frac{n}{2}}\varphi$. Then $d\mu_{\tilde{g}} = \varepsilon^n d\mu_g$, $\tilde{\varphi}^2 d\mu_{\tilde{g}} = \varphi^2 d\mu_g$, and

$$\begin{aligned}
\int_M \left|\tilde{\nabla}\tilde{\varphi}\right|_{\tilde{g}}^2 d\mu_{\tilde{g}} &= \varepsilon^{-2}\int_M |\nabla\varphi|_g^2\, d\mu_g \\
&\geq \varepsilon^{-2} C_s(M, g)\left(\int_M \varphi^{\frac{2n}{n-2}} d\mu_g\right)^{\frac{n-2}{n}} - \varepsilon^{-2}\operatorname{Vol}(g)^{-\frac{2}{n}}\int_M \varphi^2 d\mu_g \\
&= C_s(M, g)\left(\int_M \tilde{\varphi}^{\frac{2n}{n-2}} d\mu_{\tilde{g}}\right)^{\frac{n-2}{n}} - \operatorname{Vol}(\tilde{g})^{-\frac{2}{n}}\int_M \tilde{\varphi}^2 d\mu_{\tilde{g}}.
\end{aligned}$$

From this we can easily deduce that

$$C_s(M, \varepsilon^2 g) = C_s(M, g),$$

i.e., the L^2 Sobolev inequality is invariant under scaling the metric. □

0.3 Gromov-Hausdorff Convergence

In Chapter 2 we make use of the idea of "convergence" for Riemannian manifolds. The goal there is to clearly define what it means for a sequence of family of manifolds and more general spaces to converge to another manifold or space, and to draw conclusions about the spectrum of the sequence in terms of the limit space and vice-versa. We define the basics of Gromov-Hausdorff convergence. The main source for this section is [31], Chapter 10.

In 1914, F. Hausdorff first published his book titled "Basic Set Theory", the first know comprehensive introduction to set theory. The book also contains chapters on measure theory and topology, which were then still considered parts of set theory. In this book, he introduces a way to measure how far apart two subsets of a metric space are from each other, although a very close relative appeared in the doctoral thesis of M. Fréchet in 1906.

Definition 0.4 (Hausdorff Distance)**.** *Let (X, d) be a metric space and $U, V \subset X$. Then we define*

$$d'(U, V) = \inf\{d(a, b) : u \in U, v \in V\},$$

$$B(U, \varepsilon) = \{x \in X : d'(\{x\}, U) \leq \varepsilon\},$$

$$d_H(U, V) = \inf\{\varepsilon \in \mathbb{R} : U \subset B(V, \varepsilon), V \subset B(U, \varepsilon)\}.$$

$d_H(U, V)$ is called the Hausdorff distance between the subset U and V.

One can see that $d(U,V)$ is small if some points of the sets are close, while $d_H(U,V)$ is small if every point of A is close to a point in B and vice versa. The Hausdorff distance defines a metric on the closed subsets of X. This collection of all closed subsets of X is compact under this metric when X is compact.

It is however impossible to compare two different metric spaces using the Hausdorff distance unless they are subsets of a third, larger metric space. To remedy that, another tool was introduced by D. Edwards [15] in 1975 and later rediscovered and generalized by M. Gromov in 1981 [22], [21]. Gromov's construction is explained in the following paragraphs.

If X and Y are metric spaces, a metric on the disjoint union $X \sqcup Y$ is said to be *admissible* if it extends the given metrics on X and Y.

Definition 0.5 (Gromov-Hausdorff Distance)**.** *The Gromov-Hausdorff distance between compact metric spaces X and Y is defined as*

$$d_{GH}(X,Y) = \inf\{d_H(X,Y) : \ d \text{ is an admissible metric on } X \sqcup Y\}.$$

Intuitively, we try to put a metric on $X \sqcup Y$ such that Y and Y are as close as possible in the Hausdorff distance, while respecting the constraint that it extends the given metrics on X and Y.

The most important consequence of the definition of the Gromov-Hausdorff distance is that it defines a an equivalence relation between compact metric spaces. That is, if $d_{GH}(X,Y) = 0$ then X and Y are isometric. See [31]. It clear, therefore, that one should not attempt to compare a compact metric space to a noncompact one.

When comparing noncompact metric spaces, we define the *pointed Gromov-Hausdorff distance* so that we can speak about convergence in compact subsets of our space. We will write (X,x,d) for a *pointed metric space*, i.e., (X,d) is a metric space and $x \in X$. We may also omit the metric d when there is no risk of confusion.

Definition 0.6 (Pointed Gromov-Hausdorff Distance). *Let (X, x) and (Y, y) be pointed metric spaces. Define the pointed Gromov-Hausdorff distance between X and Y as*

$$d_{GH}((X,x),(Y,y)) \coloneqq \inf\{d_H(X,Y) + d(x,y) : \; d \text{ is an admissible metric on } X \sqcup Y\}.$$

We say that a sequence of noncompact metric spaces (X_i, x_i, d_i) converges to a noncompact metric space (X, x, d) in the pointed Gromov-Hausdorff topology if, for all $R > 0$, the closed metric balls satisfy

$$d_{GH}\left((\overline{B}(x_i,R),x_i,d_i),(\overline{B}(x,R),x,d)\right) \longrightarrow 0$$

It still holds that $d_{GH}((X,x),(Y,y)) = 0$ implies that the spaces are isometric. Many times, however, it is inconvenient to deal with balls centered at a point of the space, given the odd shape they may have in comparison to the ones in the limiting space (see figure 2.2). To make these cases more tractable, we will "embed" spaces into one another by using approximations. We are closely following the ideas in [23].

Definition 0.7 (Gromov-Hausdorff Approximations). *Let (X, d_X) and (Y, d_Y) be metric spaces and $\varepsilon > 0$. A pair of (not necessarily continuous) maps $f : X \to Y$ and $g : Y \to X$ is called an ε-Gromov-Hausdorff approximation or ε-approximations if for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$,*

$$|d_X(x_1,x_2) - d_Y(f(x_1),f(x_2))| < \varepsilon, \qquad d_X(g\circ f(x),x) < \varepsilon,$$
$$|d_Y(y_1,y_2) - d_X(g(y_1),g(y_2))| < \varepsilon, \qquad d_Y(f\circ g(y),y) < \varepsilon.$$

The set of all such pairs is denoted by $\mathrm{Appr}_\varepsilon(X,Y)$. In the pointed case, one restricts to pointed maps.

$$\mathrm{Appr}_\varepsilon((X,p),(Y,q)) \coloneqq \{(f,g) \in \mathrm{Appr}_\varepsilon(X,Y) : f(p) = q, g(q) = f(p)\}.$$

Such approximations sometimes are referred to as maps with *distortion* less than ε. The greatest use of this tool comes in the form of the next proposition.

Proposition 0.1 (Properties of Gromov-Hausdorff Approximations [23]). *Let (X, d_X) and (Y, d_Y) be compact metric spaces with base points $p \in X$ and $q \in Y$, and $\varepsilon > 0$. Then the following properties hold.*

(a) If $d_{GH}(X, Y) < \varepsilon$, then $\operatorname{Appr}_{2\varepsilon}(X, Y) \neq \emptyset$.

(b) If $\operatorname{Appr}_{\varepsilon}(X, Y) \neq \emptyset$, then $d_{GH}(X, Y) \leq 2\varepsilon$.

(c) If $d_{GH}((X, p), (Y, q)) < \varepsilon$, then $\operatorname{Appr}_{2\varepsilon}((X, p), (Y, q)) \neq \emptyset$.

(d) If $\operatorname{Appr}_{\varepsilon}((X, p), (Y, q)) \neq \emptyset$, then $d_{GH}((X, p), (Y, q)) \leq 2\varepsilon$.

Proof. We will only prove the last two items, as the first two can be proved similarly.

(c) Assume $d_{GH}((X, p), (Y, q)) < \varepsilon$. Let $0 < \delta < \varepsilon - d_{GH}((X, p), (Y, q))$ and choose an admissible metric d^{ε} on $X \sqcup Y$ such that

$$d_H^{\varepsilon}((X, p), (Y, q)) < d_{GH}((X, p), (Y, q)) + \delta < \varepsilon.$$

Then $d^{\varepsilon}(p, q), d_H^{\varepsilon}(X, Y) < \varepsilon$ and, from definition 0.4 , for each $x \in X$ we can choose $y_x \in Y$ such that $d^{\varepsilon}(x, y_x) < \varepsilon$. Similarly, for each $y \in Y$ there is $x_y \in X$ such that $d^{\varepsilon}(y, x_y) < \varepsilon$. Then, define $f : X \to Y$ as

$$f(x) = \begin{cases} q & \text{if } x = p, \\ x_y & \text{otherwise.} \end{cases}$$

Similarly, $g : Y \to X$ is defined by

$$g(y) = \begin{cases} p & \text{if } y = q, \\ y_x & \text{otherwise.} \end{cases}$$

Clearly $d^\varepsilon(f(x), x) < \varepsilon$ for all $x \in X$, and for all $x_1, x_2 \in X$ we have

$$|d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2))| \leq d^\varepsilon(f(x_1), x_1) + d^\varepsilon(f(x_2), x_2) < 2\varepsilon,$$

by triangle inequality. Similarly, $|d_X(g(y_1), g(y_2)) - d_Y(y_1, y_2))| < 2\varepsilon, \forall y_1, y_2 \in Y$. For the other inequalities, note that

$$\begin{aligned} d_X(g \circ f(x), x) &= d^\varepsilon(g \circ f(x), x) \\ &\leq d^\varepsilon(g(f(x)), f(x)) + d^\varepsilon(f(x), x) \\ &< 2\varepsilon, \end{aligned}$$

as well as $d_Y(f \circ g(y), y) < 2\varepsilon$ for all $y \in Y$, thus implying that $(f, g) \in \mathrm{Appr}_{2\varepsilon}((X, p), (Y, q)) \neq \emptyset$ which proves (c).

(d) Conversely, assume $(f, g) \in \mathrm{Appr}_\varepsilon((X, p), (Y, q)) \neq \emptyset$. We will construct an admissible metric d^ε on $X \sqcup Y$ such that $d^\varepsilon_H(X, Y) + d^\varepsilon(p, q) < 2\varepsilon$.

The definition of admissible metric requires that d^ε extends d_X and d_Y, as well as that d^ε is symmetric, and therefore it suffices to define $d^\varepsilon(x, y)$ for $x \in X$ and $y \in Y$ as to not violate the properties of a metric on the set $X \sqcup Y$. The identity of indiscernibles $d^\varepsilon(u, v) = 0 \iff u = v$ is not violated since d^ε extends the metrics on the sets X and Y and symmetry is by definition.

Define $d^\varepsilon : (X \sqcup Y) \times (X \sqcup Y) \to \mathbb{R}$ as

$$d^\varepsilon(x, y) \coloneqq \frac{\varepsilon}{2} + \inf\{d_X(x, x') + d_Y(f(x'), y) : x' \in X\}$$

for $x \in X, y \in Y$. From the discussion in the previous paragraph it remains to show that d^ε satisfies the triangle inequality. For $x_1, x^2 \in X$ and $y \in Y$,

$$\begin{aligned}
&d^\varepsilon(x_1, x_2) + d^\varepsilon(x_2, y)\\
&= d_X(x_1, x_2) + \frac{\varepsilon}{2} + \inf\{d_X(x_2, x') + d_Y(f(x'), y) : x' \in X\}\\
&= \frac{\varepsilon}{2} + \inf\{d_X(x_1, x_2) + d_X(x_2, x') + d_Y(f(x'), y) : x' \in X\}\\
&\geq \frac{\varepsilon}{2} + \inf\{d_X(x_1, x') + d_Y(f(x'), y) : x' \in X\}\\
&= d^\varepsilon(x_1, y)
\end{aligned}$$

as well as

$$\begin{aligned}
&d^\varepsilon(x_1, y) + d^\varepsilon(y, x_2)\\
&= \varepsilon + \inf\{d_X(x_1, x') + d_Y(f(x'), y) + d_X(x_2, x'') + d_Y(f(x''), y) : x', x'' \in X\}\\
&\geq \varepsilon + \inf\{d_X(x_1, x') + d_Y(f(x'), f(x'')) + d_X(x_2, x'') : x', x'' \in X\}\\
&\geq \varepsilon + \inf\{d_X(x_1, x') + d_X(x', x'') - \varepsilon + d_X(x_2, x'') : x', x'' \in X\}\\
&\geq \inf\{d_X(x_1, x_2) : x', x'' \in X\}\\
&= d^\varepsilon(x_1, x_2).
\end{aligned}$$

Note that we have used that $d_X(x', x'') - \varepsilon < d_Y(f(x'), f(x'')) < d_X(x', x'') + \varepsilon$ from definition 0.7 and the triangle inequality for d_X. The remaining configurations for the triangle inequality for d^ε, i.e., $d^\varepsilon(x, y_1) + d^\varepsilon(y_1, y_2) \geq d^\varepsilon(x, y_2)$ and $d^\varepsilon(y_1, x) + d(x, y_2) \geq d^\varepsilon(y_1, y_2)$

$\forall x \in X, y_1, y_2 \in Y$ can be verified in a similar manner, and d^ε is well defined.

Using this metric it is not hard to see that $d^\varepsilon(p,q) = \frac{\varepsilon}{2}$, since $q = f(p)$. Moreover

$$d^\varepsilon(x, f(x)) = \frac{\varepsilon}{2} + \inf\{d_X(x,x') + d_Y(f(x', f(x))) : x' \in X\} = \frac{\varepsilon}{2}$$

by taking $x' = x$. Applying the above to $y \in Y$ and using definition 0.7 once more gives

$$d^\varepsilon(y, g(y)) \leq d^\varepsilon(y, f \circ g(y)) + d^\varepsilon(f \circ g(y), g(y)) < \frac{3\varepsilon}{2}.$$

Therefore $X \subset B(f(X), \frac{3\varepsilon}{2}) \subset B(Y, \frac{3\varepsilon}{2})$ and $Y \subset B(X, \frac{3\varepsilon}{2})$ with respect to the metric d^ε, which implies $d^\varepsilon_H(X,Y) \leq \frac{3\varepsilon}{2}$ and that

$$d_{GH}((X,p),(Y,q)) \leq d^\varepsilon_H(X,Y) + d^\varepsilon(p,q) = 2\varepsilon.$$

□

To conclude, in chapter 2 and after, we will use Gromov-Hausdorff approximations to prove convergence on compact balls centered at a point in the space. See proposition 2.1.

0.4 The Schauder Interior Estimates

The main result of this section is well-known in our context (see for example [20], Chapter 6). The goal is to have interior estimates on eigenfunctions of the Laplacian. To keep our language general, throughout this section denote $Lu = f$ the equation

$$Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u = f(x), \;\; a^{ij} = a^{ji},$$

where the coefficients and f are defined in an open set $\Omega \subset \mathbb{R}^n$ and, unless otherwise stated, the operator L is *strictly elliptic*; that is

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2,\ \forall x \in \Omega,\ \xi \in \mathbb{R}^n,$$

for some positive constant λ.

We say that a real or complex-valued function function f on a bounded set $D \subset \mathbb{R}^n$ satisfies the *Hölder condition* at $x_0 \in D$, or that it is *Hölder continuous* with exponent α at x_0, or that it is *α-Hölder continuous at x_0* if there are nonnegative real constants $C, 0 < \alpha < 1$ such that

$$|f(x) - f(x_0)| \leq C||x - x_0||^\alpha,\ \forall x \in D.$$

When $\alpha = 1$ the function is said to be *Lipschitz Continuous.* In other words, f is Hölder continuous at x_0 if the quantity

$$[f]_{\alpha,x_0} := \sup_D \frac{|f(x) - f(x_0)|}{||x - x_0||^\alpha}$$

is finite.

This notion can be extended to a set D (not necessarily bounded) as a whole without referring to a fixed point. We say that f is *uniformly Hölder continuous* with exponent α in a set D if the quantity

$$[f]_{\alpha,D} := \sup_{x,y \in D,\ x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is finite.

If f is uniformly Hölder continuous with exponent α on compact subsets of D, then we say that f is *locally Hölder continuous* with exponent α in D. These coincide when D is compact. Moreover, local Hölder continuity is a stronger property than pointwise Hölder continuity in compact subsets. A locally Hölder continuous function will be pointwise Hölder continuous in D if it is also bounded in D.

To simplify out notation in what follows, given a *multi-index*, i.e., an m-tuple of non-negative integers $\beta = (\beta_1, ..., \beta_m)$, the norm of β is defined as $|\beta| \coloneqq \beta_1 + ... + \beta_m$.

If $\beta = (\beta_1, ..., \beta_m)$ is a multi-index, then

$$D^\beta f = \frac{\partial^m f}{\partial^{\beta_1} x_1 \cdots \partial^{\beta_m} x_m}.$$

The *Hölder space* $C^{k,\alpha}(\Omega)$, where $k \geq 0$ is a positive integer, consists of those functions on Ω having continuous derivatives up to order k and such that the k^{th} partial derivatives are Hölder continuous with exponent α, where $0 < \alpha < 1$.

Definition 0.8 (Hölder spaces). *Given $\Omega \subset \mathbb{R}^n$ an open subset of the euclidean space, $0 < \alpha < 1$ and $k \geq 0$ a positive integer, the Hölder space $C^{k,\alpha}(\Omega)$ is defined as*

$$C^{k,\alpha}(\Omega) \coloneqq \left\{ f \in C^k(\Omega) : D^\beta f \text{ is locally } \alpha\text{-Hölder continuous on } \Omega, \ 0 \leq |\beta| \leq k \right\},$$

and the space

$$C^{k,\alpha}(\overline{\Omega}) \coloneqq \left\{ f \in C^k(\overline{\Omega}) : [D^\beta f]_{\alpha,\Omega} < \infty, \ 0 \leq |\beta| \leq k \right\}$$

If we write $C^{k,0}(\Omega) = C^k(\Omega)$ and $C^{k,0}(\overline{\Omega}) = C^k(\overline{\Omega})$, we can include $\alpha = 0$ in our definition. When $k = 0$, we may also omit writing it in the notation, for simplicity. Likewise for when $\alpha = 0$. We also write $C_0^{k,\alpha}(\Omega)$ for the space of functions in $C^{k,\alpha}(\Omega)$ with compact support in

Ω. Let us then define some useful seminorms.

$$\begin{aligned} [u]_{k,0,\Omega} &= |D^k u|_{0,\Omega} = \sup_{\beta=|k|} \sup_{\Omega} |D^\beta u|, \\ [u]_{k,\alpha,\Omega} &= [D^k u]_{\alpha,\Omega} = \sup_{\beta=|k|} [D^\beta u]_{\alpha,\Omega}, \end{aligned} \tag{0.1}$$

for $k = 0, 1, 2, ...$, and their related norms

$$\begin{aligned} ||u||_{C^k(\overline{\Omega})} &= |u|_{k,\Omega} = |u|_{k,0,\Omega} = \sum_{j=0}^{k} [u]_{j,0,\omega} = \sum_{j=0}^{k} |D^j u|_{0,\Omega}, \\ ||u||_{C^{k,\alpha}(\overline{\Omega})} &= |u|_{k,\Omega} + [u]_{k,\alpha,\Omega} = |u|_{k,\Omega} + [D^k u]_{\alpha,\Omega}, \end{aligned} \tag{0.2}$$

on the spaces $C^k\left(\overline{\Omega}\right)$ and $C^{k,\alpha}\left(\overline{\Omega}\right)$, respectively.

There are also some other norms which will be useful in the proof of the Schauder estimates. For $x, y \in \Omega$, let $d_x = \text{dist}(x, \partial\Omega)$ and $d_{x,y} = \min\{d_x, d_y\}$. Then define

$$\begin{aligned} [u]^*_{k,0,\Omega} = [u]^*_{k,\Omega} &= \sup_{x\in\Omega, |\beta|=k} d_x^k |D^\beta u(x)|, \;\; k = 1, 2, ...; \\ |u|^*_{k,\Omega} = |u|^*_{k,0,\Omega} &= \sum_{j=0}^{k} [u]^*_{j,\Omega}; \\ [u]^*_{k,\alpha,\Omega} &= \sup_{x,y\in\Omega, \;\; |\beta|=k} d_{x,y}^{k+\alpha} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}, \;\; 0 < \alpha \leq 1; \\ |u|^*_{k,\alpha,\Omega} &= |u|^*_{k,\Omega} + [u]^*_{k,\alpha,\Omega}. \end{aligned} \tag{0.3}$$

Note that we have $[u]^*_{0,\Omega} = |u|^*_{0,\Omega} = |u|_{0,\Omega}$.

To obtain estimates for the interior norm $|u|^*_{2,\alpha,\Omega}$ of a solution to $Lu = f$ in Ω it suffices to bound $|u|_{0,\Omega}$ and the seminorm $[u]^*_{2,\alpha,\Omega}$, by using the following *interpolation inequalities.*

Lemma 0.1 (Interpolation inequalities [20])**.** *Let $u \in C^{2,\alpha}(\Omega)$, where Ω is an open subset of*

$\mathbb{R}^n$. *Then for any* $\varepsilon > 0$ *there is a constant* $C = C(\varepsilon)$ *such that*

$$[u]^*_{j,\beta,\Omega} \le C|u|_{0,\Omega} + \varepsilon[u]^*_{2,\alpha,\Omega},$$

$$|u|^*_{j,\beta,\Omega} \le C|u|_{0,\Omega} + \varepsilon[u]^*_{2,\alpha,\Omega},$$

$$j = 0, 1, 2;\ 0 \le \alpha, \beta \le 1,\ j + \beta < 2 + \alpha.$$

A proof can be found in [20], Appendix 1 to chapter 6, Lemma 6.32.

We also want to be able to state the estimates in a sharp form, so the last norms in the spaces $C^k(\Omega), C^{k,\alpha}(\Omega)$. For $\sigma \in \mathbb{R}$ and k a nonnegative integer, we define

$$\begin{aligned}
[f]^{(\sigma)}_{k,0,\Omega} &= [f]^{(0)}_{k,\Omega} = \sup_{x\in\Omega,\ |\beta|=k} d_x^{k+\sigma}|D^\beta f(x)|;\\
[f]^{(\sigma)}_{k,\alpha,\Omega} &= \sup_{x,y\in\Omega,\ |\beta|=k} d_{x,y}^{k+\alpha+\sigma}\frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha},\ 0 < \alpha \le 1;\\
|f|^{(\sigma)}_{k,\Omega} &= \sum_{j=0}^{k}[f]^{(\sigma)}_{j,\Omega};\\
|f|^{(\sigma)}_{k,\alpha,\Omega} &= |f|^{(\sigma)}_{k,\Omega} + [f]^{(\sigma)}_{k,\alpha,\Omega}.
\end{aligned} \tag{0.4}$$

When $\sigma = 0$ these are the same norms as defined in equations 0.3 so that $[\,\cdot\,]^{(0)} = [\,\cdot\,]^*$ and $|\cdot| = |\cdot|^*$. Through a tedious computation one can also verify that

$$|fg|^{(\sigma+\tau)}_{0,\alpha,\Omega} \le |f|^{(\sigma)}_{0,\alpha,\Omega}|g|^{(\tau)}_{0,\alpha,\Omega} \text{ for } f, g \in C^{k,\alpha}(\Omega), \sigma + \tau \ge 0. \tag{0.5}$$

Finally we can state the promised estimate.

Proposition 0.2 (Schauder Interior Estimates [20]). *Let* Ω *be an open subset of* $\mathbb{R}^n$, *and let*

$u \in C^{2,\alpha}(\Omega)$ be a bounded solution in Ω of the equation

$$Lu = a^{ij}D_{ij}u + b^iD_iu + cu = f,$$

where $f \in C^\alpha(\Omega)$ and there are positive constants λ, Λ such that the coefficients satisfy

$$a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2,\ \forall x \in \Omega, \xi \in \mathbb{R}^n$$

and

$$|a^{ij}|^{(0)}_{0,\alpha,\Omega}, |b^i|^{(1)}_{0,\alpha,\Omega}, |c|^{(2)}_{0,\alpha,\Omega} \leq \Lambda,$$

then

$$|u|^*_{2,\alpha,\Omega} \leq C\left(|u|_{0,\Omega} + |f|^{(2)}_{0,\alpha,\Omega}\right) \tag{0.6}$$

where $C = C(n, \alpha, \lambda, \Lambda)$.

Proof. From lemma 0.1, we only need to bound $[u]^*_{2,\alpha,\Omega}$.Furthermore, assuming the estimate for compact subsets of Ω, let $\{\Omega_i\}$ be a sequence of open subset of Ω such that $\Omega_i \subset \Omega_{i+1} \subset \Omega$ and $\cup_i\Omega_i = \Omega$. Then $[u]^*_{2,\alpha,\Omega_i} < \infty,\ \forall i$ and, given any two points $x, y \in \Omega$, we can take i_0 such that $x, y \in \Omega_i, \forall i \geq i_0$, and for any second derivative D^2u of u we have

$$\begin{aligned}(d^{(i)}_{x,y})^{2+\alpha}\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} &\leq [u]^*_{2,\alpha,\Omega_i} \\ &= C\left(|u|_{0,\Omega_i} + |f|^{(2)}_{0,\alpha,\Omega_i}\right) \\ &\leq C\left(|u|_{0,\Omega} + |f|^{(2)}_{0,\alpha,\Omega}\right)\end{aligned}$$

where $d^{(i)}_{x,y} = \min\{\text{dist}(x, \partial\Omega_i), \text{dist}(\text{y}, \partial\Omega_\text{i})\}$. As $i \to \infty$, we get

$$[u]^*_{2,\alpha,\Omega} = \sup_{x,y\in\Omega,\ |\beta|=2} d^{2+\alpha}_{x,y}\frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha} \leq C\left(|u|_{0,\Omega} + |f|^{(2)}_{0,\alpha,\Omega}\right),\ \forall x, y \in \Omega.$$

Therefore, it is enough to prove the estimate for compact subsets of Ω. The above reasoning also shows that $[u]^*_{2,\alpha,\Omega} < \infty$, regardless of other estimates.

Let x_0, y_0 be any two distinct points in Ω.

Suppose $d_{x_0} = d_{x_0,y_0} = \min\{d_{x_0}, d_{y_0}\}$. Let $\mu \leq \frac{1}{2}$ be a positive constant to be specified later, and set $d = \mu d_{x_0}$, $B = B_d(x_0)$. Then rewrite $Lu = f$ as

$$a^{ij}(x_0)D_{ij}u = (a^{ij}(x_0) - a^{ij}(x))D_{ij}u - b^i D_i u - cu + f =: F(x).$$

This allows us to consider this as an equation with constant coefficients $a^{ij}(x)$ in B, which has a more established theory. We will then use Lemma 6.1 (a) from [20], which asserts that if $y_0 \in B_{d/2}(x_0)$, then for any second derivative D^2u we have

$$\left(\frac{d}{2}\right)^{2+\alpha} \frac{|D^2u(x_0) - D^2u(y_0)|}{|x_0 - y_0|^\alpha} \leq C\left(|u|_{0,B} + |F|^{(2)}_{0,\alpha,B}\right),$$

and then

$$d_{x_0}^{2+\alpha} \frac{|D^2u(x_0) - D^2u(y_0)|}{|x_0 - y_0|^\alpha} \leq \frac{C}{\mu^{2+\alpha}}\left(|u|_{0,B} + |F|^{(2)}_{0,\alpha,B}\right).$$

If $|x_0 - y_0| \geq d/2$ then we have

$$\begin{aligned} d_{x_0}^{2+\alpha} \frac{|D^2u(x_0) - D^2u(y_0)|}{|x_0 - y_0|^\alpha} &\leq \left(\frac{d_{x_0}}{|x_0 - y_0|}\right)^\alpha \left[d_{x_0}^2|D^2u(x_0)| + d_{y_0}^2|D^2u(y_0)|\right] \\ &\leq \left(\frac{2}{\mu}\right)^\alpha \left[d_{x_0}^2|D^2u(x_0)| + d_{y_0}^2|D^2u(y_0)|\right] \\ &\leq \frac{4}{\mu^\alpha}[u]^*_{2,\Omega}. \end{aligned}$$

Combining these two we get

$$d_{x_0}^{2+\alpha} \frac{|D^2u(x_0) - D^2u(y_0)|}{|x_0 - y_0|^\alpha} \leq \frac{C}{\mu^{2+\alpha}}\left(|u|_{0,\Omega} + |F|^{(2)}_{0,\alpha,B}\right) + \frac{4}{\mu^\alpha}[u]^*_{2,\Omega}. \tag{0.7}$$

Let us save this last inequality for later use and focus on estimating $|F|^{(2)}_{0,\alpha,B}$ in terms of $|u|_{0,\Omega}$ and $[u]^*_{2,\alpha,\Omega}$. First note that

$$|F|^{(2)}_{0,\alpha,B} \leq \sum_{i,j} | \left(a^{ij}(x_0) - a^{ij}(x)\right) D_{ij}u|^{(2)}_{0,\alpha,B} + \sum_{i} |b^i D_i u|^{(2)}_{0,\alpha,B}| + |cu|^{(2)}_{0,\alpha,B} + |f|^{(2)}_{0,\alpha,B} \quad (0.8)$$

and that, given $g \in C^\alpha(\Omega)$,

$$\begin{aligned} |g|^{(2)}_{0,\alpha,B} &\leq d^2|g|_{0,B} + d^{2+\alpha}[g]_{\alpha,B} \\ &\leq \frac{\mu^2}{(1-\mu)^2}[g]^{(2)}_{0,\Omega} + \frac{\mu^{2+\alpha}}{(1-\mu)^{2+\alpha}}[g]^{(2)}_{0,\alpha,\Omega} \\ &\leq 4\mu^2[g]^{(2)}_{0,\Omega} + 8\mu^{2+\alpha}[g]^{(2)}_{a,\alpha,\Omega} \\ &\leq 8\mu^2|g|^{(2)}_{0,\alpha,\Omega}, \end{aligned} \quad (0.9)$$

where we have used that $d_x = \text{dist}(x, \partial\Omega) > (1-\mu)d_{x_0} \geq \frac{1}{2}d_{x_0},\ \forall x \in B$.

Now for the first term in 0.8, omit the indices for each pair i, j and use 0.5 and 0.9 to obtain

$$\begin{aligned} |(a(x_0) - a(x))D^2u|^{(2)}_{0,\alpha,\Omega} &\leq |a(x_0) - a(x)|^{(0)}_{0,\alpha,B}|D^2u|^{(2)}_{0,\alpha,B} \\ &\leq |a(x_0) - a(x)|^{(0)}_{0,\alpha,B}\left(4\mu^2[u]^*_{2,\Omega} + 8\mu^{2+\alpha}[u]^*_{2,\alpha,\Omega}\right), \end{aligned}$$

and since

$$\begin{aligned} |a(x_0) - a(x)|^{(0)}_{0,\alpha,B} &\leq \sup_{x\in B}|a(x_0) - a(x)| + d^\alpha[a]_{\alpha,B} \\ &\leq 2d^\alpha[a]_{\alpha,B} \\ &\leq 2^{1+\alpha}\mu^\alpha[a]^*_{0,\alpha,\Omega} \\ &\leq 4\Lambda\mu^\alpha, \end{aligned}$$

we can estimate the first term in 0.8 by

$$\sum_{i,j} |(a^{ij}(x_0) - a^{ij}(x))D_{ij}u|^{(2)}_{0,\alpha,B} \leq 32n^2\Lambda\mu^{2+\alpha}\left([u]^*_{2,\Omega} + \mu^\alpha[u]^*_{2,\alpha,\Omega}\right)$$

$$\leq 32n^2\Lambda\mu^{2+\alpha}\left(C(\mu)|u|_{0,\Omega} + 2\mu^\alpha[u]^*_{2,\alpha,\Omega}\right),$$

where we have used $\varepsilon = \mu^\alpha$ in the first inequality of Lemma 0.1.

Now let us focus on the second term of 0.8. Omit the indices i, use inequality 0.9, lemma 0.1 with $\varepsilon = \mu^2$ and the a priori estimates on u to get

$$\begin{aligned} |bDu|^{(2)}_{0,\alpha,B} &\leq 8\mu^2|bDu|^{(2)}_{0,\alpha,\Omega} \\ &\leq 8\mu^2|b|^{(1)}_{0,\alpha,\Omega}|Du|^{(1)}_{0,\alpha,\Omega} \\ &\leq 8\mu^2\Lambda|u|^*_{1,\alpha,\Omega} \\ &\leq 8\mu^2\Lambda\left(C(\mu)|u|_{0,\Omega} + \mu^{2\alpha}[u]^*_{2,\alpha,\Omega}\right). \end{aligned}$$

For the next term of inequality 0.8 use inequalities 0.9 and 0.5, the a priori estimates on c and lemma 0.1 to get

$$|cu|^{(2)}_{0,\alpha,B} \leq 8\mu^2|c|^{(2)}_{0,\alpha,\Omega} \leq 8\ \Lambda\mu^2\left(C(\mu)|u|_{0,\Omega} + \mu^{2\alpha}[u]^*_{2,\alpha,\Omega}\right).$$

For the last term of inequality 0.8, from the bound in μ we obviously have

$$|f|^{(2)}_{0,\alpha,B} \leq 8\mu^2|f|^{(2)}_{0,\alpha,\Omega}.$$

We can now finally put all those estimates together in inequality 0.8 to get

$$|F|^{(2)}_{0,\alpha,B} \leq C\mu^{2+2\alpha}[u]^*_{2,\alpha,\Omega} + C(\mu)\left(|u|_{0,\Omega} + |f|^{(2)}_{0,\alpha,\Omega}.\right)$$

We now need to translate the estimate above into the originally promised Schauder estimate on a compact subset of Ω.

Going back to inequality 0.7 obtained previously, the above estimate yields

$$d_{x_0,y_0}^{2+\alpha}\frac{|D^2u(x_0)-D^2u(y_0)|}{|x_0-y_0|^\alpha}\leq C\mu^\alpha[u]^*_{2,\alpha,\Omega}+C(\mu)\left(|u|_{0,\Omega}+|f|^{(2)}_{0,\alpha,\Omega}\right).$$

Since the right-hand side of this inequality is independent of x_0, y_0, we can take the supremum over all pairs $x_0, y_0 \in \Omega$ satisfying $d_{x_0} = d_{y_0} = d_{x_0,y_0}$ to obtain

$$[u]^*_{2,\alpha,\Omega}\leq C\mu^\alpha[u]^*_{2,\alpha,\Omega}+C(\mu)\left(|u|_{0,\Omega}+|f|^{(2)}_{0,\alpha,\Omega}\right).$$

Finally, we can choose $\mu = \mu_0$ so that $C\mu_0^\alpha \leq \frac{1}{2}$ to arrive at the desired estimate

$$[u]^*_{2,\alpha,\Omega}\leq C\left(|u|_{0,\Omega}+|f|^{(2)}_{0,\alpha,\Omega}\right)$$

□

0.5 Moser Iteration Argument

This section's goal is to illustrate the use of Moser's iteration technique to obtain bounds on the L^∞ norm of eigenfunctions of the Laplacian. Throughout this section, $f \in C^\infty(M)$ is an eigenfunction of Δ on a compact manifold M, i.e.,

$$\Delta f+\lambda f=0,$$

for some nonnegative number λ. If $\partial M \neq \emptyset$, we assume the *Dirichlet condition*

$$f|_{\partial M} = 0.$$

Recall that *Hölder's inequality* states that

$$|(f,g)| \leq ||f||_{L^p(M)}||g||_{L^q(M)}.$$

for integer exponents $1 \leq p, q \leq \infty$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Using this inequality, taking $1 \leq r, s, t \leq \infty$ and choosing θ such that

$$\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}$$

we have

$$\begin{aligned} ||f||_{L^r(M)} &= \left(\int_M |f^r|\right)^{\frac{1}{r}} \\ &= \left(\int_M |f^{r\theta} f^{r(1-\theta)}|\right)^{\frac{1}{r}} \\ &\leq \left(||f^{r\theta}||_{L^{\frac{s}{r\theta}}(M)} ||f^{r(1-\theta)}||_{L^{\frac{t}{r(1-\theta)}}(M)}\right)^{\frac{1}{r}} \\ &= \left(\left(\int_M f^s\right)^{\frac{r\theta}{s}} \left(\int_M f^t\right)^{\frac{r(1-\theta)}{t}}\right)^{\frac{1}{r}} \\ &= ||f||^{\theta}_{L^s(M)} ||f||^{1-\theta}_{L^t(M)}. \end{aligned}$$

We also assume that $f \in H^1_0(M)$,i.e., that it is the limit of smooth compactly supported functions in M in the Sobolev norm

$$||f||_{H_0^1(M)} = \sqrt{\int_M |f|^2 + \int_M |\nabla f|^2},$$

and that the most basic Sobolev inequality

$$\left(\int |f|^{2q_n}\right)^{\frac{1}{q_n}} \leq C_n^2 \int |\nabla f|^2;\ q_n = \frac{n}{n-2}, \forall f \in H^1(M)$$

holds on M^n.

Taking $k_n = 1 + \frac{2}{n}$, with $\frac{1}{k_n} = \frac{\theta_n}{q_n} + (1 - \theta_n)$, where $\theta_n = \frac{n}{2+n}$, the second form of Hölders inequality above, with $r = 2k_n$, $s = 2q_n$ and $t = 2$ gives

$$\begin{aligned}
||f||^{2k_n}_{L^{2k_n}(M)} &= \int_M |f|^{2k_n} \\
&= ||f||^{2k_n}_{L^{2k_n}(M)} \\
&\leq ||f||^{2k_n\theta_n}_{L^{q_n}(M)} ||f||^{2k_n(1-\theta_n)}_{L^2(M)} \\
&= \left(\int_M |f|^{2q_n}\right)^{\frac{k_n\theta_n}{q_n}} \left(\int_M |f|^2\right)^{k_n(1-\theta_n)} \\
&\leq \left(\int_M |f|^{2q_n}\right)^{\frac{1}{q_n}} \left(\int_M |f|^2\right)^{\frac{2}{n}},
\end{aligned}$$

which together with the Sobolev inequality gives

$$\int_M |f|^{2(1+\frac{2}{n})} \leq C_n^2 \left(\int_M |\nabla f|^2\right) \left(\int |f|^2\right)^{\frac{2}{n}},\ \forall f \in H_0^1(M). \tag{0.10}$$

This type of inequality was addressed by V, Maz'ya in [30], chapter 2.

So far this discussion holds for all functions $f \in H_0^1(M)$. Now, using the fact that f is an eigenfunction, and that either $\partial M = \emptyset$ or f satisfies the Dirichlet condition on the boundary, taking $1 \leq p < \infty$, integrating by parts gives

$$\begin{aligned}
\lambda \int_M |f|^{2p} &= \lambda \int_M f \left(|f|^{2p-1} \frac{f}{|f|} \right) \\
&= - \int_M \Delta f \left(|f|^{2p-1} \frac{f}{|f|} \right) \\
&= \int_M \nabla f \cdot \nabla \left(|f|^{2p-1} \frac{f}{|f|} \right) \\
&= (2p-1) \int_M |f|^{2p-2} \frac{f^2}{|f|^2} |\nabla f|^2 \\
&= (2p-1) \int_M |f^{p-1} \nabla f|^2 \\
&= \frac{2p-1}{p^2} \int_M |\nabla |f|^p|^2
\end{aligned} \tag{0.11}$$

Using inequality 0.10 for the function $|f|^p$ and equation 0.11 for f, we get

$$\begin{aligned}
\int_M |f|^{2p(1+\frac{2}{n})} &\leq C_n^2 \left(\int_M |\nabla |f|^p|^2 \right) \left(\int |f|^{2p} \right)^{\frac{2}{n}} \\
&= \frac{p^2}{2p-1} C_n^2 \lambda \left(\int_M |f|^{2p} \right) \left(\int |f|^{2p} \right)^{\frac{2}{n}} \\
&\leq C_n^2 p \lambda \left(\int_M |f|^{2p} \right)^{1+\frac{2}{n}},
\end{aligned} \tag{0.12}$$

which can be rewritten as

$$||f||_{L^{2p(1+\frac{2}{n})}(M)} \leq (C_n^2 p \lambda)^{\frac{1}{2p(1+\frac{2}{n})}} ||f||_{L^{2p}(M)}.$$

Finally, the essence of the Moser iteration argument is to iterate the above inequality by applying it to $p = p_i = (1 + \frac{2}{n})^i$, to obtain

$$||f||^2_{L^{2p_i}(M)} \leq \left(1 + \frac{2}{n}\right)^{\sum_{j=1}^{i} \frac{j-1}{p_j}} (3C_n^2 \lambda)^{\sum_{j=1}^{i} \frac{1}{p_j}} ||f||^2_{L^2(M)}, \tag{0.13}$$

which we now prove by induction on i.

Clearly, for $i = 1$, inequality 0.13 reads

$$\left(\int_M |f|^{2(1+\frac{2}{n})}\right)^{\frac{1}{1+\frac{2}{n}}} \leq (3C_n^2\lambda)^{\frac{1}{1+\frac{2}{n}}} \int_M |f|^2,$$

which follows from inequality 0.12 with $p = 1$.

Assuming that inequality 0.13 holds for $i = i_0$, let us prove it for $i = i_0 + 1$. Starting from the left-hand side and using inequality 0.12 for suitable $p = p_i$ gives

$$\begin{aligned}
||f||^2_{L^{2p_{i_0+1}}(M)} &= ||f||^2_{L^{2p_{i_0}(1+\frac{2}{n})}(M)} \\
&\leq \left(C_n^2 p_{i_0}\lambda\right)^{\frac{1}{p_{i_0}(1+\frac{2}{n})}} ||f||^2_{L^{p_{i_0}}(M)} \\
&\leq \left(C_n^2 p_{i_0}\lambda\right)^{\frac{1}{p_{i_0}(1+\frac{2}{n})}} \left(1+\frac{2}{n}\right)^{\sum_{j=1}^{i_0}\frac{j-1}{p_j}} (3C_n^2\lambda)^{\sum_{j=1}^{i_0}\frac{1}{p_j}} ||f||^2_{L^2(M)} \\
&= \frac{\left(C_n^2 p_{i_0}\lambda\right)^{\frac{1}{p_{i_0}(1+\frac{2}{n})}}}{\left(1+\frac{2}{n}\right)^{\frac{i_0}{p_{i_0}(1+\frac{2}{n})}} (3C_n^2\lambda)^{\frac{1}{p_{i_0}(1+\frac{2}{n})}}} \left(1+\frac{2}{n}\right)^{\sum_{j=1}^{i_0+1}\frac{j-1}{p_j}} (3C_n^2\lambda)^{\sum_{j=1}^{i_0+1}\frac{1}{p_j}} ||f||^2_{L^2(M)} \\
&\leq \left(\frac{1}{3}\right)^{\frac{1}{(1+\frac{2}{n})^{i_0+1}}} \left(1+\frac{2}{n}\right)^{\sum_{j=1}^{i_0+1}\frac{j-1}{p_j}} (3C_n^2\lambda)^{\sum_{j=1}^{i_0+1}\frac{1}{p_j}} ||f||^2_{L^2(M)} \\
&\leq \left(1+\frac{2}{n}\right)^{\sum_{j=1}^{i_0+1}\frac{j-1}{p_j}} (3C_n^2\lambda)^{\sum_{j=1}^{i_0+1}\frac{1}{p_j}} ||f||^2_{L^2(M)},
\end{aligned}$$

which proves inequality 0.13.

We finish up the argument by noting that, as $i \to \infty$,

$$\sum_{j=1}^{i}\frac{1}{p_j} = \sum_{j=1}^{i}\frac{1}{\left(1+\frac{2}{n}\right)^j} \to \frac{n}{2}$$

and that $\sum_{j=1}^{i}\frac{j}{p_j} = \sum_{j=1}^{i}\frac{j}{(1+\frac{2}{n})^j}$ converges to a number depending only on n. Moreover, the limit of the norms $||f||_{p_j}$ can be computed as follows.

Fix $||f||_\infty > \varepsilon > 0$ and define $S_\varepsilon = \{x : |f(x)| \geq ||f||_\infty - \varepsilon\}$. Then

$$||f||_{p_j} \geq \left(\int_{S_\varepsilon} (||f||_\infty - \varepsilon)^{p_j} dV \right)^{\frac{1}{p_j}} = (||f||_\infty - \varepsilon) \operatorname{Vol}(S_\varepsilon)^{\frac{1}{p_j}},$$

which implies $\liminf_{p \to \infty} ||f||_{p_j} \geq ||f||_\infty$. Conversely, as $|f(x) \leq ||f||_\infty$ for almost every x, we have that

$$||f||_{p_{j+1}} \leq \left(\int_M |f(x)|^{p_{j+1} - p_j} |f(x)|^{p_j} dV \right)^{\frac{1}{p_{j+1}}} \leq ||f||_\infty^{\frac{p_{j+1} - p_j}{p_{j+1}}} ||f||_{p_j}^{\frac{p_j}{p_{j+1}}},$$

which in turn implies $\limsup_{j \to \infty} ||f||_{p_j} \leq ||f||_\infty$, and therefore

$$\lim_{j \to \infty} ||f||_{p_j} = ||f||_\infty,$$

and we have obtained an estimate on $||f||_\infty$, in the form of

$$||f||_\infty \leq \lambda C ||f||_{L^2(M)},$$

where C depends only on the geometry of M.

0.6 Elliptic Regularity

Let L be an elliptic operator of order $2k$

$$Lu = \sum_{|\alpha| \leq 2k} a_\alpha(x) \partial^\alpha u,$$

on a domain Ω, where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index with $\partial^\alpha u = \partial^{\alpha_1} \cdots \partial^{\alpha_n}$. Recall that L is said to be elliptic when

$$\sum_{|\alpha|=2k} a_\alpha(x)\partial^\alpha \xi^\alpha \neq 0,\ \forall \xi = (\xi_1, \cdots, \xi_n) \neq 0$$

Assume all coefficients a_α of L have continuous derivatives up to order $2k$. Given a function f, the *Dirichlet problem* for L consists of finding a function u such that

$$\begin{cases} Lu = f \\ u|_{\partial\Omega} = 0 \end{cases}.$$

Using Garding's inequality and the Lax-Milgram lemma, one can guarantee the existence of weak solutions $u \in H^k$ to this problem, i.e., we can find functions u with $||u||_{L^k(\Omega)} \leq \infty$, $u|_{\partial\Omega} = 0$ and

$$\sum_{|\alpha|\leq 2k} a_\alpha(x)\partial^\alpha u = f,$$

where each partial derivative is taken in the sense of distributions. Unfortunately, when we use this definition, the expression Lu may not make sense in the usual sense, since u may not have well-defined derivatives of every order.

To solve this issue, we need to use the Elliptic Regularity Theorem. We will not prove the theorem in this section, but a very detailed exposition can be found in [2].

The *Elliptic Regularity Theorem* guarantees that, if f is L^2-integrable, then u will have L^2-integrable weak derivative of up to order $2k$, for any $k \geq 0$. In particular, $f \in C^\infty(\Omega)$ imples that $u \in C^\infty(\Omega)$.

0.7 Tensors and Differential Forms

Recall that, if M is a smooth manifold, then ω is a *differential form of degree* k if it is a smooth section of the k^{th} exterior power of the cotangent bundle of M. In other words, $\omega(p) \in \Lambda^k T_p^* M$ is an alternating multilinear map

$$\omega(p) : T_pM \times \cdots \times T_pM \to \mathbb{R}$$

satisfying

$$\omega(p)(v_1, \cdots, v_k) = \operatorname{sign}(\sigma)\omega(p)(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

for any permutarion $\sigma : \{1, \cdots, k\} \to \{1, \cdots, k\}$ of the indices, and such that the coefficients

$$a_{i_1,\cdots,i_k}(p) = \omega(p)(e_{i_1}, \cdots, e_{i_k})$$

are smooth functions of p on M, for any local frame $\{e_1, \cdots, e_n\}$ and indices $i_1, \cdots, i_k \in \{1, \cdots, n\}$.

If ω is a k-form, then it can be written in local coordinates as

$$\omega = a_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} =: a_I dx_I$$

for unique smooth coefficients a_I, where $\{dx_1, \cdots, dx_n\}$ is the dual frame to $\left\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right\}$ and $I = (i_1, \cdots, i_n)$ is a multi-index.

The *exterior differential* of a k-form ω is defined as the $k+1$-form

$$dw := \sum_{j=0}^{n} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I.$$

The usual local coordinate frame

$$\left\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right\}$$

sometimes leads to tedious and/or ineffective computations and it it more advantageous to use a local orthonormal frame $\{e_1, \cdots, e_n\}$, which implies the existence of a dual frame of 1-forms $\{\omega_1, \cdots, \omega_n\} \subset \Lambda^1 M$ such that $\omega_i(e_j) = \delta_{ij}$ is the Kronecker delta. If M is a compact orientable manifold then $\omega_1 \wedge \cdots \wedge \omega_n$ is the volume form.

If M is an orientable manifold and ∇ is the Levi-Civita connection, we extend ∇ to all tensor fields as follows.

Definition 0.9. *Let X, Y be vector fields on M. Then*

1. *If $f \in C^\infty(M)$, then $\nabla_X f = Xf$.*
2. *$\nabla_X Y$ is the usual Levi-Civita connection.*
3. *If ω is a 1-form, define the 1-form $\nabla\omega$ as*

$$(\nabla_X \omega)Y := X(\omega(Y)) - \omega(\nabla_X Y).$$

4. *In general, let $\{e_1, \cdots, e_n\}$ be a local frame of M and $\{\omega_1, \cdots, \omega_n\}$. Given*

$$T = a_{i_1, \cdots, i_r, j_1, \cdots, j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \omega_{j_1} \otimes \cdots \otimes \omega_{j_s},$$

define

$$\begin{aligned}\nabla_X T =& X(a_{i_1,\cdots,i_r,j_1,\cdots,j_s})e_{i_1}\otimes\cdots\otimes e_{i_r}\otimes\omega_{j_1}\otimes\cdots\otimes\omega_{j_s}\\ &+\sum_{k=1}^{r} a_{i_1,\cdots,i_r,j_1,\cdots,j_s}e_{i_1}\otimes\cdots\otimes\nabla_X e_{i_k}\otimes\cdots\otimes e_{i_r}\otimes\omega_{j_1}\otimes\cdots\otimes\omega_{j_s}\\ &+\sum_{l=1}^{s} a_{i_1,\cdots,i_r,j_1,\cdots,j_s}e_{i_1}\otimes\cdots\otimes e_{i_r}\otimes\omega_{j_1}\otimes\cdots\otimes\nabla_X\omega_{j_l}\otimes\cdots\otimes\omega_{j_s}\end{aligned}$$

It is convenient to note how the connection acts on the frame by defining the forms ω_{ij} through

$$\nabla_X e_j = \omega_{ij}(X)e_i$$

end the curvature tensor as

$$R_{ijkl} = \left\langle \nabla_{e_k}\nabla_{e_l}e_j - \nabla_{e_l}\nabla_{e_k}e_j - \nabla_{[e_k,e_l]}e_j, e_i\right\rangle$$

where $[X,Y] = XY - YX$ is the *Lie bracket*. The following are useful formulas in this setting.

Lemma 0.2 (Cartan's formulas)**.** *Using the definitions above, the following hold:*

(i) $d\omega_j + w_i \wedge \omega_{ij} = 0$*,*

(ii) $d\omega_{ij} + \omega_{is}\wedge\omega_{sj} = \frac{1}{2}R_{ijkl}\omega_k\omega_l$*.*

After a straightforward computation, this implies that given a tensor $\eta = a_{i_1\cdots i_k}\omega_{i_1}\otimes\cdots\otimes\omega_{i_k}$, we have

$$\nabla_{e_l}\eta = \left(e_l(a_{i_1\cdots i_k}) - \sum_{s=1}^{k} a_{a_1\cdots r\cdots i_k}\omega_{ri_s}(e_l)\right)\omega_{i_1}\otimes\cdots\otimes\omega_{i_k},$$

where r appears replacing the s^{th} index i_s. For simplicity we also define the following.

Definition 0.10. *We define*

$$a_{i_1\cdots i_k,l} := e_l(a_{i_1\cdots i_k}) - \sum_{s=1}^{k} a_{a_1\cdots r\cdots i_k}\omega_{ri_s}(e_l)$$

and call it the covariant derivative of the coefficients $a_{i_1\cdots i_k}$.

As a consequence, given $\omega = a_{i_1,\cdots i_k}\omega_{i_1}\wedge\cdots\wedge\omega_{i_k}$, we can write

$$d\omega = (-1)^k a_{i_1\cdots i_k,l}\omega_{i_1}\wedge\cdots\wedge\omega_{i_k}\wedge\omega_l$$

after a tedious computation.

It is clear that the exterior differential d is a map $d:\Lambda^k M\to\Lambda^{k+1}M$. Its adjoint, the *codifferential operator* $\delta:\Lambda^{k+1}M\to\Lambda^k M$ is the operator satisfying

$$(d\omega,\eta) = (\omega,\delta\eta),$$

$\forall\omega\in\Lambda^k M, \eta\in\Lambda^{k+1}M$.

Lemma 0.3 (The codifferential)**.** *The codifferential of a k-form* $\omega = a_{i_1\cdots i_k}\omega_{i_1}\wedge\cdots\wedge\omega_{i_k}$ *can be computed as*

$$\delta\omega = (-1)^k k a_{i_1\cdots i_k,i_k}\omega_{i_1}\wedge\cdots\wedge\omega_{i_{k-1}}$$

Proof. Let $\eta = b_{i_1\cdots i_{k-1}}\omega_{i_1}\wedge\cdots\wedge\omega_{i_{k-1}}$ be a $(k-1)$-form. Note that the inner products can be written as

$$\big((-1)^k k a_{i_1\cdots i_k,i_k}\omega_{i_1}\wedge\cdots\wedge\omega_{i_{k-1}},\eta\big) = (-1)^k k!\int_M a_{i_1\cdots i_k,i_k}b_{i_1\cdots i_{k-1}}$$

and

$$(\omega,d\eta) = (-1)^{k-1}k!\int_M a_{i_1\cdots i_k}b_{i_1\cdots i_{k-1},i_k}.$$

Subtracting the two yields

$$\begin{aligned}\big((-1)^k k a_{i_1\cdots i_k,i_k}\omega_{i_1}\wedge\cdots\wedge\omega_{i_{k-1}},\eta\big)-(\omega,d\eta)=&(-1)^k k!\int_M a_{i_1\cdots i_k,i_k}b_{i_1\cdots i_{k-1}}+a_{i_1\cdots i_k}b_{i_1\cdots i_{k-1},i_k}\ dv\\ &=\int_M\beta\\ &=\int_M d\alpha\\ &=0\end{aligned}$$

by Stokes theorem. Above we have defined the differential form

$$\beta=\big(a_{i_1\cdots i_k,i_k}b_{i_1\cdots i_{k-1}}+a_{i_1\cdots i_k}b_{i_1\cdots i_{k-1},i_k}\big)\ \omega_1\wedge\cdots\wedge\omega_n,$$

recalling that $\omega_1\wedge\cdots\wedge\omega_n=dv$ is he volume form, and

$$\alpha=\sum_{l=1}^{n}(-1)^{l-1}a_{i_1\cdots i_{k-1}l}b_{i_1\cdots i_{k-1},i_k}\omega_1\wedge\cdots\wedge\omega_{l-1}\wedge\omega_{l+1}\wedge\cdots\wedge\omega_n.$$

□

0.8 The Lie Derivative

When doing differential geometry on a Riemannian manifold, the Levi Civita connection is the usual alternative to the derivative in $\mathbb{R}^n$. In many cases, however, we would like to extend the idea of directional derivative of other geometric objects,e.g. vector fields. For that, we use the *Lie derivative.*

First, recall that given a smooth vector field X on a Riemannian manifold M, there is a unique *maximal flow* $\theta: M\times I\to M$, $I\subset\mathbb{R}$ satisfying

(i) $\theta_0(p) = p, \ \forall p \in M$,

(ii) $\theta_t \circ \theta_s(p) = \theta_{t+s}(p), \ \forall p \in M \ \forall t, s, s+t \in I$,

for which X is its *infinitesimal generator*, in the sense that

$$X(p) = \left.\frac{d}{dt}\right|_{t=0} \theta_t(p).$$

The usual definition for the directional derivative of a vector field Y with respect to X in $\mathbb{R}^n$ takes advantage of the fact that $T_p\mathbb{R}^n = \mathbb{R}^n, \ \forall x \in \mathbb{R}^n$, which allows you to simply add any two tangent vectors, regardless of the base point, as

$$D_X Y(x) = \lim_{t \to 0} \frac{Y(\theta_t(p)) - Y(p)}{t},$$

where θ is a maximal flow generated by X. Unfortunately, however, when dealing with this concept in an arbitrary Riemannian manifold, the numerator of the above fraction does not make sense as tangent vectors at different base points cannot be added. To remedy that, we use the derivative of the function θ_{-t} at $\theta_t(p)$, which is a map

$$d(\theta_{-t})_{\theta_t(p)} : T_{\theta_t(p)}M \to T_pM.$$

We can then define the Lie derivative.

Definition 0.11. *Let X, Y be smooth vector fields on a Riemannian manifold M and θ a maximal flow generated by X. The Lie derivative of Y with respect to X at point p is the smooth vector field defined as*

$$\begin{aligned}\mathcal{L}_X Y(p) &:= \left.\frac{d}{dt}\right|_{t=0} d((\theta_{-t})_{\theta_t(p)})\,(Y(\theta_t(p))) \\ &= \lim_{t \to 0} \frac{d(\theta_{-t})_{\theta_t(p)}\,(Y(\theta_t(p))) - Y(p)}{t}.\end{aligned}$$

By using the *Lie bracket* $[X, Y] \coloneqq XY - YX$, it can be proven that, if X, Y are smooth vector fields then the Lie derivative satisfies the following properties.

Lemma 0.4 ([24]). *The Lie derivative of vector fields satisfies the following:*

(i) $\mathcal{L}_X Y = [X, Y]$,

(ii) $\mathcal{L}_X Y = -\mathcal{L}_Y X$,

(iii) $\mathcal{L}_X [Y, Z] = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X, Z]$,

(iv) $\mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y Z - \mathcal{L}_Y \mathcal{L}_X Z$,

(v) If $f \in C^\infty(M)$, *then* $\mathcal{L}_X(fY) = (Xf)Y + f\mathcal{L}_X Y$,

(vi) If $F : M \to N$ *is a diffeomorphism, then* $F_*(\mathcal{L}_X Z) = \mathcal{L}_{F_* X} F_* Z$.

A proof can be found in [24], but shouldn't be difficult to produce. It is also interesting to extend this idea to tensor fields of arbitrary rank. Let $A \in T^k M$ be a tensor field of rank k. Then a flow θ generated by vector field X on M can be used to pull back the tensor A. Given $v_1, \cdots, v_k \in T_p M$, then $d(\theta_t)_p v_1, \cdots, d(\theta_t)_p v_k \in T_{\theta_t(p)} M$ and we can construct the tensor $d(\theta_t)^*_x A$ through the formula

$$d(\theta_t)^*_p A \ (p)(v_1, \cdots, v_k) = A(\theta_t(p))(d(\theta_t)_p v_1, \cdots, d(\theta_t)_p v_k).$$

For simplicity we will also write $\theta^*_t A = d(\theta_t)^*_p A$. With that in mind and in the same spirit as the previous definition, we can define $\mathcal{L}_X A$.

Definition 0.12. *Let* $A \in T^k M$ *be a smooth tensor field of rank* k, X *be smooth vector fields on a Riemannian manifold* M *and* θ *a maximal flow generated by* X. *The Lie derivative of*

A with respect to X at point p is the smooth tensor defined as

$$\begin{aligned}\mathcal{L}_X A(p) &:= \frac{d}{dt}\bigg|_{t=0} \theta_t^* A \\ &= \lim_{t\to 0} \frac{\theta_t^* A(p) - A(p)}{t}.\end{aligned}$$

And similar properties to the lemma above can be deduced.

Lemma 0.5 ([24])**.** *The Lie derivative of tensor fields satisfies the following:*

(i) $\mathcal{L}_X f = Xf$*,*

(ii) $\mathcal{L}_X(fA) = (\mathcal{L}_X F)A + f\mathcal{L}_X A$*,*

(iii) $\mathcal{L}_X(A \otimes B) = (\mathcal{L}_X A) \otimes B + A \otimes \mathcal{L}_X B$*,*

(iv) If $X_1, \cdots, X_k$ *are smooth vector fields and* A *is a smooth* k*-tensor field, then*

$$\mathcal{L}_X(A(X_1, \cdots, X_k)) = (\mathcal{L}_X A)(X_1, \cdots, X_k) + A(\mathcal{L}_X X_1, \cdots, X_k) + \cdots + A(X_1, \cdots, \mathcal{L}_X X_k).$$

Finally, we can use the above to compute some important properties of $\mathcal{L}$ when acting on differential forms. Recall that a differential form $\omega \in \Lambda^k M$ is an alternating tensor in the sense that

$$\omega(p)(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = (\operatorname{sgn}\sigma)\ \omega(p)(v_1, \cdots, v_k)$$

for any $p \in M$, $v_1, \cdots, v_k \in T_pM$ and any permutation $\sigma \in S_k$ of the indices $1, \cdots, k$.

Since $\Lambda^k M \subset T^k M$, then $\mathcal{L}_X \omega$ is already defined for a differential form ω, and we have the equivalent of the Leibniz's product rule:

$$\mathcal{L}_X(\omega \wedge \eta) = (\mathcal{L}_X \omega) \wedge \eta + \omega \wedge (\mathcal{L}_X \eta).$$

On a smooth manifold, we can define canonical 1-forms in a neighborhood by using a coordinate chart $\varphi : U \subset M \to \mathbb{R}^n$. The local frame $\partial_1, \cdots, \partial_n$ is defined by

$$\partial_i : C^\infty(U) \to \mathbb{C}^\infty(U)$$
$$f \mapsto \partial_i f$$

where $\partial_i f(p) = \frac{\partial(\varphi \circ f)}{\partial x_i}(p)$. Its dual frame $dx^1, \cdots, dx^n$ forms a basis to $\Lambda^1 M$ and any k-form $\omega \in \Lambda^k M$ can be written as

$$\omega = a_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} = a_I dx^I.$$

The exterior differential of ω is defined as

$$d\omega := \partial_l a_{i_1 \cdots i_k} dx^l \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \partial_l a_I dx^l \wedge dx^I.$$

Another important operation for differential forms is the *interior multiplication*. Given a vector field X, then define the map $i_X : \Lambda^k M \to \Lambda^{k-1} M$ as

$$i_X \omega(v_1, \cdots, v_k) = \omega(x)(X(x), v_1, \cdots, v_k),$$

for all $x \in M$, $v_1, \cdots, v_k \in T_x M$. By convention, we interpret $i_X \omega = 0$ when $\omega \in \Lambda^0 M = \mathbb{R}$. Another common notation is

$$i_X \omega = X \lrcorner\, \omega,$$

which often reads "X into ω ". Note that $i_X \circ i_X = 0$ and if $\omega \in \Lambda^k M, \eta \in \Lambda^l M$, then

$$i_X(\omega \wedge \eta) = (i_X \omega) \wedge \eta + (-1)^k \omega \wedge (i_X \eta).$$

As per the relation between interior multiplication and the Lie derivative, we have *Cartan's magic formula:*

$$\mathcal{L}_V\omega = V \lrcorner\, (d\omega) + d(V \lrcorner\, \omega).$$

Another important property is the relation between d and $\mathcal{L}$.

Proposition 0.3 (The Lie derivative Commutes with d)**.** *If V is a smooth vector field and ω is a smooth differential form, then*

$$\mathcal{L}_V(d\omega) = d(\mathcal{L}_V\omega).$$

Proof. By Cartan's magic formula and the fact that $d \circ d = 0$ we have

$$\mathcal{L}_V d\omega = V \lrcorner\, d(d\omega) + d(V \lrcorner\, d\omega) = d(V \lrcorner\, \omega);$$

$$d\mathcal{L}_V\omega = d(V \lrcorner\, d\omega) + d(dV \lrcorner\, \omega) = d(V \lrcorner\, d\omega).$$

□

The last important property in this section, which is used when dealing with the Hodge Laplacian in chapter 3, gives a useful formula for the Lie derivative of a differential form with respect to a vector field.

If $dv \in \Lambda^n(M)$ is the volume form on a smooth manifold M and X is a smooth vector field on M then, recalling the definition of the divergence of a vector field from the beggining of the chapter, we have

$$\mathcal{L}_X dv = (\mathrm{div}(X))dv.$$

Chapter 1

Introduction

Let M be a complete noncompact Riemannian manifold of dimension n and denote by Δ the Laplacian acting on $C_0^\infty(M)$,

$$\Delta f := \operatorname{div}(\nabla f) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x_j} f \right).$$

It is well known that the self-adjoint extension of Δ on $L^2(M)$ exists and is a unique non-positive densely-defined linear operator.

The spectrum of $-\Delta$, written as $\sigma(-\Delta)$, consists of all points $\lambda \in \mathbb{C}$ for which $\Delta + \lambda I$ fails to be invertible,

$$\sigma(-\Delta) := \{\lambda \in \mathbb{C} : \dim \ker(\Delta + \lambda I) \neq 0\}.$$

A nonzero function $f \in \mathfrak{Dom}(-\Delta)$ for which $(\Delta + \lambda I) f = 0$ is called an *eigenfunction* of $-\Delta$ with *eigenvalue* λ.

Since $-\Delta$ is nonegative, its L^2-spectrum is contained in $[0, \infty)$. The essential spectrum of $-\Delta$, consists of the cluster points in the spectrum and of isolated eigenvalues of infinite multiplicity, i.e., complex numbers λ such that $(\Delta + \lambda I)f_j = 0$ for infinitely many, linearly independent $f_j \in \mathfrak{Dom}(-\Delta)$,

$$\sigma_{ess}(-\Delta) \coloneqq \{\lambda \in \mathbb{C} : \lambda = \lim \lambda_j, \lambda_j \in \sigma(-\Delta)\} \cup \{\lambda \in \sigma(-\Delta) : \dim \ker(\Delta + \lambda I) = \infty\}.$$

The pure point spectrum of $-\Delta$ is defined as $\sigma_{\mathrm{pp}}(\Delta) \coloneqq \sigma(-\Delta) \backslash \sigma_{\mathrm{ess}}(\Delta)$.

When M is compact, one can find an orthonormal basis of eigenfunctions $\{f_i\} \in \mathfrak{Dom}(-\Delta) \cap C^\infty(M)$ with $(\Delta + \lambda_i I)f_i = 0$ such that

$$\lambda_1 = \inf \left\{ \frac{\int_M |\nabla f|^2}{\int_M f^2} \mid f \in \mathrm{Dom}(-\Delta) \right\},$$

$$\lambda_i = \inf \left\{ \frac{\int_M |\nabla f|^2}{\int_M f^2} \mid f \in \mathrm{Dom}(\Delta), \ \int f \cdot f_j = 0, \ j = 1, ..., i-1 \right\}.$$

Unlike the compact case, noncompact manifolds may have nonempty essential spectrum.

Further, on a fixed manifold, if two metrics differ only on a compact set, their essential spectrum σ_{ess} is the same.

1.1 Classical Weyl Criterion

Now we turn to the spectrum of noncompact, complete Riemannian manifolds. H. Donnelly studied the essential spectrum using the Weyl criterion. In 1981, he proved certain initial results for ,noncompact manifolds, by using the following:

Theorem 1.1 (Classical Weyl Criterion [13])**.** *Let $\delta > 0$. If there exists an infinite dimensional subspace G in the domain of Δ such that*

$$||\Delta u + \lambda u||_{L^2} \leq \delta ||u||_{L^2} \tag{1.1}$$

for all $u \in G$, then

$$\sigma_{ess}(-\Delta) \cap (\lambda - \delta, \lambda + \delta) \neq \emptyset.$$

The functions u are referred to as the *approximate eigenfunctions* corresponding to the eigenvalue λ. The above criterion is simple to apply and has directed the study of the essential spectrum of the Laplacian for the last three decades. A related result of the above is as follows: let u be a nonzero smooth function with compact support. If (1.1) is satisfied, then

$$\sigma(-\Delta) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset.$$

Example 1.1. *In $\mathbb{R}^n$, $\Delta \approx \frac{\partial^2}{\partial r^2}$, where r is the distance function from a fixed point (say, the origin). Then $u = e^{i\lambda r}\varphi(r)$ are approximate eigenfunctions for $\lambda \geq 0$, where φ is an appropriate cutoff function. We can then use the Weyl criterion to conclude that $\sigma_{ess}(-\Delta) = [0, \infty)$.*

Example 1.2. *An important example is the n-sphere S^n with its standard round metric. To find eigenvalues, we embed S^n inside $\mathbb{R}^{n+1} - \{0\}$ in the usual way, consider a positive homogeneous function $f \in C^\infty(\mathbb{R}^{n+1} - \{0\})$ of degree s, and then take the restriction to the sphere of the Laplacian Δ on $\mathbb{R}^{n+1} - \{0\}$ applied to the function $|x| - sf$. The result is that if f is harmonic relative to the Laplacian on $\mathbb{R}^{n+1} - 0$, then the restriction to S^n of $\Delta(|x| - sf)$ is a scalar multiple of the restriction of f to S^n, with the scalar being $s(s + n - 2)$, i.e., $\sigma(-\Delta) = \{s(s + n - 2); s \in \mathbb{N}\}$.*

Example 1.3. *In the hyperbolic space $\mathbb{H}^{n+1}$, $\sigma(-\Delta) = \left[\frac{n^2}{4}, \infty\right)$. See [12]*

A question that follows naturally is the following: For which manifolds, under which geometric conditions, is the spectrum maximal, i.e., $\sigma(-\Delta) = [0, \infty)$?

1.2 Examples and History

With additional assumptions on the curvature and geometry of the manifold we can locate the essential spectrum (see for example [41],[25],[9],[14], [16], [17]) by comparing the manifold to the n-dimensional Euclidean space.

The works of J. F. Escobar [16] in 1986 and J. F. Escobar and A. Freire [17] in 1992, together with the Soul theorem prove that $\sigma(M) = [0, +\infty)$, provided that the manifold has nonnegative sectional curvature and satisfies some additional conditions. Later, in 1994, D. Zhou [41] proved that those "additional conditions" are superfluous.

In a similar spirit to the Weyl criterion, one can define the L^p essential spectrum as follows.

Definition 1.1. *We say $\lambda \in \sigma_p(-\Delta)$ if, $\forall \delta > 0$, there exists an infinite dimensional subspace G in the domain of Δ such that, $\forall u \in G$,*

$$||\Delta u + \lambda u||_{L^p} \leq \delta ||u||_{L^p}. \tag{1.2}$$

On a very influential paper from 1993, K. T. Sturm [38] proved the following.

Theorem 1.2 ([38]). *If the volume of (M, g) grows uniformly subexponentially, then $\sigma_p(M)$ is independent of $p \in [1, +\infty]$. In particular, it is a subset of the real line. Every isolated eigenvalue in $\sigma_p(M)$ with finite algebraic multiplicity for some $p \in [1, +\infty]$ is also an isolated eigenvalue in $\sigma_q(M)$ with the same algebraic multiplicity for all $q \in [1, +\infty]$.*

Using Sturm's result, Wang [39] proved in 1997 that if $\mathrm{Ric}(M) \geq -\frac{\delta}{r^2}$, then $\sigma_p(M) = [0, +\infty)$

for any $p \in [1, +\infty]$.

Later in 2010 this was generalized by Z. Lu and D. Zhou [28] as follows:

Theorem 1.3 ([28]). *Let M be a complete non-compact Riemannian manifold. Assume that*

$$\liminf_{x\to\infty} Ric_M(x) \geq 0.$$

Then the L^p essential spectrum of M is $[0, +\infty)$ for any $p \in [1, +\infty]$.

It has been less clear how the spectrum behaves for manifolds without asymptotically non-negative Ricci curvature. Z. Lu and D. Zhou conjectured that $\sigma_p(M) = [0, +\infty)$ for any $p \in [1, +\infty]$ if M is a complete non-compact Riemannian manifold with Ricci curvature bounded below and uniformly sub-exponential volume growth.

In 2015, R. Schoen and H. Tran disprove this conjecture by finding a certain class of manifolds that admit a large finite number of gaps in their essential spectra. An example of manifolds in this class is a Delaunay surface with small neck size, so that each component of N is an annulus of small area. Recall that the Delaunay surfaces are the surfaces of revolution of constant mean curvature in $\mathbb{R}^3$ (see [4]). They look roughly like a singly periodic tower of spheres connected by necks (see figure 1.1).

Theorem 1.4. *([36]) Let (M, g_0) be a complete noncompact Riemannian manifold of bounded curvature and positive injectivity radius. Given any positive integer G there is a metric g on M such that (M, g) has bounded curvature and positive injectivity radius, the eigenvalues of g with respect to g_0 are bounded above and below by positive constants, and the L^2 essential spectrum of g has at least G gaps.*

The main difficulty in applying Donnelly's Weyl criterion stems from the fact that it requires canonical smooth functions on a manifold (specifically functions for which the left-hand side

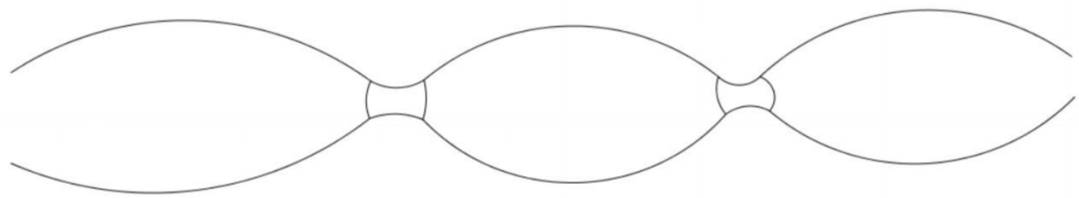

Figure 1.1: Surface with small necks

of (1.1) is well-defined). There are many canonical smooth functions on a manifold, including the heat kernel and the Green's function. However, on a general manifold we do not have an explicit expression for these functions. It is possible to give upper and/or lower bounds for these functions, but those require another canonical function, namely the distance function on the manifold. Due to the presence of cut-loci, the distance function is in general not smooth.

Example 1.4. *Take, for instance, $M = S^1 \times (-\infty, \infty)$, letting (θ, x) be the coordinates. Then the radial function r which gives the distance of (θ, x) to $(0, 0)$ is*

$$r(\theta, x) = \sqrt{x^2 + (\min(\theta, 2\pi - \theta))^2}$$

A straightforward computation gives

$$\Delta r = -\frac{2\pi}{\sqrt{x^2 + \pi^2}} \delta_{\{\theta = \pi\}} + \textit{a smooth function},$$

Therefore Δr is not locally L^2.

However, r is Lipschitz and locally L^1 (cf. Cheeger [7, Chapter 4], also see [39], [29]). Thus in order to use the Weyl criterion, we must be in a setting where the distance function is smooth, or the manifold has a pole.

1.3 Generalized Weyl Criterion

In 2014 N. Charalambous and Z. Lu [4] introduce a new method to compute the spectrum of a self-adjoint operator on a Hilbert space, which has the following application in the case of the Laplacian:

Theorem 1.5 (The New Weyl Criterion [4]). *If, for $\lambda \in \mathbb{R}^+$, there exists a nonzero function u in the domain of Δ such that*

$$||u||_{L^\infty} \cdot ||\Delta u + \lambda u||_{L^1} \leq \delta ||u||_{L^2}^2 \tag{1.3}$$

for some $\delta > 0$, then

$$\sigma(-\Delta) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset,$$

where $\varepsilon = \min\{1, (\lambda + 2)\delta^{\frac{1}{3}}\}$. Moreover

$$\sigma_{ess}(-\Delta) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset,$$

if, for any compact subset K of of the manifold there exists a nonzero function u in the domain of Δ satisfying (1.3) *whose support is outside K.*

Using this theorem they were able to compute the spectrum of a Ricci nonnegative manifold directly, using functions constructed from the distance function. The above theorem also allowed them to further generalize the result of Z. Lu and D. Zhou and find the most general conditions possible so that the spectrum of the Laplacian on functions is maximal, in other words it is $[0, \infty)$.

Theorem 1.6 ([4]). *Let X be a complete noncompact Riemannian manifold. Take a fixed point x_o, and let $r(x) = d(x, x_o)$ be the radial distance to x_o. Assume that the radial Ricci curvature away from x_o is asymptotically nonnegative, in other words, there exists a contin-*

uous positive function $\delta(r)$ *on* $\mathbb{R}^+$ *such that*

(i). $\lim_{r\to\infty} \delta(r) = 0$ *and*

(ii). $\mathrm{Ric}_X(\partial r, \partial r) \geq -(n-1)\delta(r)$ *away from the cut-locus of* x_o*.*

If the volume of the manifold is finite we additionally assume that its volume does not decay exponentially at x_o*. Then the* L^2 *spectrum of the Laplacian is* $[0, \infty)$*.*

Chapter 2

Spectral Continuity and a gap theorem

The text of this chapter relies in the paper submitted and accepted for publication in "Geometry of Submanifolds", the Contemporary Mathematics volume, [3].

In the previous sections we have cited results that allow us to find large sets of noncompact manifolds whose essential spectrum is a connected subset of the real line. There are, however, many known cases where the essential spectrum has an arbitrary number of gaps, see [1],[26],[27],[32],[36].

In this chapter we are interested in further exploring this set of manifolds. We will first turn our attention to spectral continuity and then study the evolution of the spectrum of a manifold under Gromov-Hausdorff convergence. We will then use these ideas to prove the existence of gaps in the essential spectrum of a periodic manifold, which is close in spirit to a recent result by R. Schoen and H. Tran [36].

The first natural case to consider is the evolution of eigenvalues under the continuous deformation of a manifold or its Riemannian metric. J. Dodziuk proved the following result in [11].

Theorem 2.1 ([11]). *Let X be a compact manifold and let g_t be a family of Riemannian metrics on X. Assume that*

$$g_t \to g$$

in the $\mathcal{C}^0$ topology. Then the spectrum (eigenvalues) of g_t converges to the spectrum of g.

In the recent paper [5] (also see [33] for related results), N. Charalambous and Z. Lu generalized spectral continuity to the case when the quadratic forms of two self-adjoint operators are ε-close. In the following paragraphs we describe their results.

Let $\mathcal{H}$ be a Hilbert space with two inner products $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_1$. Consider two densely defined nonnegative operators H_0 and H_1 on $\mathcal{H}$ that are self-adjoint with respect to the inner products $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_1$ respectively. Let Q_0, Q_1 be their respective quadratic forms and denote the two norms on $\mathcal{H}$ by $\|\cdot\|_0$ and $\|\cdot\|_1$. Note that both Q_0 and Q_1 are nonnegative.

Denote the domain of the Friedrichs extension of H_0 and H_1 by $\mathfrak{Dom}(H_0)$ and $\mathfrak{Dom}(H_1)$ respectively. We assume that there exists a dense subspace $\mathcal{C} \subset \mathcal{H}$ such that $\mathcal{C}$ is contained in $\mathfrak{Dom}(H_0) \cap \mathfrak{Dom}(H_1)$ (in the case of the Laplacian, $\mathcal{C}$ will be the space of smooth functions/forms with compact support).

Definition 2.1. *We say that the operators H_0, H_1 are ε-close, if there exists a positive constant $0 < \varepsilon < 1$ such that for all $u \in \mathcal{C}$ the following two inequalities hold*

$$(1-\varepsilon)\,\|u\|_0^2 \le \|u\|_1^2 \le (1+\varepsilon)\,\|u\|_0^2; \tag{2.1}$$

$$(1-\varepsilon)\,Q_0(u,u) \le Q_1(u,u) \le (1+\varepsilon)\,Q_0(u,u). \tag{2.2}$$

We note that if H_0, H_1 are ε-close, then for any $u, v \in \mathcal{C}$

$$|(u,v)_1 - (u,v)_0| \leq \varepsilon(\|u\|_0 \, \|v\|_0); \tag{2.3}$$

$$|Q_1(u,v) - Q_0(u,v)| \leq \varepsilon \, [Q_0(u,u) \, Q_0(v,v)]^{1/2}. \tag{2.4}$$

Moreover, it can be shown that the resolvents of the two operators are also ε close (see [5] for the details).

It has been shown in [5] that two ε-close operators have nearby spectra. This result has an important application in the context of the Hodge Laplacian on k-forms over a Riemannian manifold with two ε-close metrics over it. In particular, it allows for the proof of the following theorem which holds even in the noncompact case, thus generalizing the result of Dodziuk.

Theorem 2.2 ([5]). *Let X^n be an orientable manifold, and let g_0, g_1 be two smooth complete Riemannian metrics on X that are ε-close for some $0 < \varepsilon < 1/2$.*

Fix $A > 0$. Then for any $\lambda \in \sigma(k, \Delta_1) \cap [0, A]$,

$$\mathrm{dist}(\lambda, \sigma(k, \Delta_0)) < c(A, n) \, \varepsilon^{\frac{1}{3}}$$

for some constant $c(A)$ depending only on A. A similar result holds for the essential spectra of the operators. In particular,

$$d_{\mathfrak{h}}(\sigma(k, \Delta_1), \sigma(k, \Delta_0)) = o(1),$$

where $o(1) \to 0$, as $\varepsilon \to 0$.

To clarify the notation in the above theorem, $d_{\mathfrak{h}}$ denotes the pointed Gromov-Hausdorff distance between the spectra as subsets of the real line with a common fixed point -1. $\sigma(k, \Delta_i)$ denotes the spectrum of nonnegative definite Hodge Laplacian Δ_i actong on k-

forms which corresponds to the metric g_i for $i = 0, 1$.

In contrast, in the setting of a family of compact Riemannian manifolds which is convergent in the Gromov-Hausdorff sense, we have the following important results. For definitions and notation regarding the Gromov-Hausdorff distance, check section 0.3. The first result is due to K. Fukaya [19].

Theorem 2.3 ([19]). *Let X_t be a family of compact Riemannian manifolds which is Gromov-Hausdorff convergent to a compact metric space X. We assume that X is not a point. Assume that the curvatures of the manifolds X_t are uniformly bounded. Then the eigenvalues of X_t converge to those of X.*

The above result was later generalized by J. Cheeger and T. H. Colding [8].

Theorem 2.4 ([8]). *Let X_t be a family of compact Riemannian manifolds which is Gromov-Hausdorff convergent to a compact metric space X. We assume that X is not a point. Assume that the Ricci curvatures of the manifolds X_t are uniformly bounded below. Then the eigenvalues of X_t converge to those of X.*

There is no known common generalization of theorems 2.1, 2.2. 2.3 and 2.4. For the remaining of this chapter we will study a special case, which will allow us to find manifolds with gaps in their L^2 essential spectrum.

2.1 Main construction

Our goal is to construct a manifold similar to that of Figure 1.1. Let (X_1, g_1), (X_2, g_2) be complete Riemannian manifolds, $x_i \in X_i$ and $\varepsilon > 0$.

$$N := \left(S^{n-1} \times (-2, 2), \varepsilon^2 g_0\right)$$

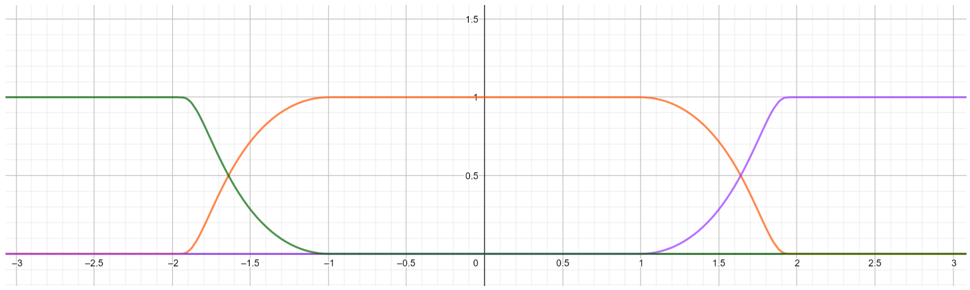

Figure 2.1: Partition of Unity

where g_0 is the standard product metric on $S^{n-1} \times (-2, 2)$. Define

$$f_1 \colon S^{n-1} \times (-2, -1) \to X_1$$
$$(\theta, t) \mapsto f_1(\theta, t) = \exp_{x_1}(-t\varepsilon\theta)$$

and

$$f_2 \colon S^{n-1} \times (1, 2) \to X_2$$
$$(\theta, t) \mapsto f_2(\theta, t) = \exp_{x_2}(t\varepsilon\theta).$$

Define the manifold X_ε as the quotient

$$X_\varepsilon \coloneqq \frac{(X_1 \setminus B_{x_1}(\varepsilon)) \cup (X_2 \setminus B_{x_2}(\varepsilon)) \cup N}{\sim}$$

where $(\theta, t) \sim f_i(\theta, t)$, $i = 1, 2$ in their respective domains.

Outside the neck region N the metric is not changed, i.e., $g_\varepsilon = g_i$ on $X_\varepsilon \setminus N$.

On $N = S^{n-1} \times (-2, 2)$, Let ρ_0, ρ_1, ρ_2 be a smooth partition of unit of $\mathbb{R}$ such that $0 \leq \rho_i \leq 1$, $\rho_1(t) = 1$ for $t \leq -2$, $\rho_0(t) = 1$ for $-1 \leq t \leq 1$, $\rho_2(t) = 1$ for $2 \leq t$ and $\sum_{i=0}^{2} \rho_i = 1$. See figure 2.1.

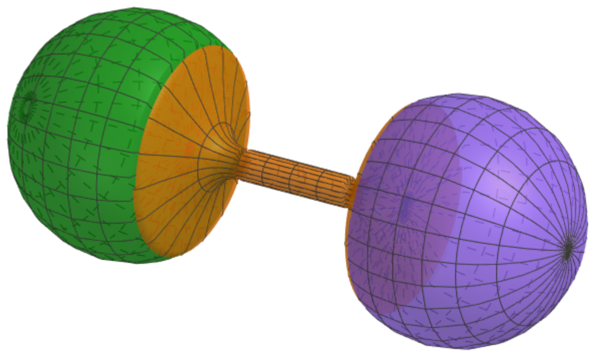

Figure 2.2: Dumbbell of Calabi

Then we define

$$g_\varepsilon = \rho_0 g_N + \rho_1 g_1 + \rho_2 g_2.$$

The manifold $(X_\varepsilon, g_\varepsilon)$ looks like the manifolds X_1 and X_2 connected by a thin neck from x_1 and x_2. Figure 2.2 is an example of this construction with $X_1 = X_2 = S^2$, immersed in R^3. Such surface is referred to as the "dumbbell", and was first considered by E. Calabi and studied by J. Cheeger in [6].

Let us fix notations for the following results.

Definition 2.2. *We use M to denote the manifold obtained by taking $X_1 = X_2 = \mathbb{R}^n$, $x_1 = x_2 = 0 \in \mathbb{R}^n$ and $\varepsilon = 1$ in the process above. Let $x_0 = (1, 0) \in N$ be a fixed middle point of M. In order words, M consists of two copies of $\mathbb{R}^n$ joint by a tube of radius 1 and length 2.*

We will prove that

Proposition 2.1. *Let X_1, X_2 be two compact Riemannian manifolds. Using the above notations, we have*

(i) Define the metric space $X_0 = X_1 \sqcup X_2$, where we identify x_1 and x_2. Then X_ε converges to X_0 in the pointed Gromov-Hausdorff topology.

(ii) $(X_\varepsilon, \varepsilon^{-2} g_\varepsilon, x_0) \longrightarrow (M, x_0)$ *in the pointed Gromov-Hausdorff topology.*

Proof. We use Gromov-Hausdorff approximations as per proposition 0.1. For (1), consider the approximations (φ, ψ), $\varphi : X_\varepsilon \to X_0$ and $\psi : X_0 \to X_\epsilon$ defined by

$$\varphi(x) = \begin{cases} x & \text{if } x \in X_1 \cup X_2 \\ x_1 = x_2 & \text{if } x \in N/(X_1 \cup X_2) \end{cases}$$

and

$$\psi(x) = \begin{cases} x & \text{if } x \neq x_1 \text{ and } x \neq x_2 \\ x_0 & \text{if } x = x_1 = x_2 \end{cases}$$

It's easy to see that

$$|d_{X_0}(\varphi(x), \varphi(y)) - d_{X_\varepsilon}(x, y)| \leq 4\varepsilon, |d_{X_\varepsilon}(\psi(a), \psi(b)) - d_{X_0}(a, b)| \leq 4\varepsilon$$

$\forall x, y \in X_\varepsilon, a, b, \in X_0$, where L_ε is the length of the neck N. As $\lim_{\varepsilon \to 0} L_\varepsilon = 0$, by restricting the approximations to compact balls, X_ε is Gromov-Hausdorff convergent to X_0. Note that in this setting the collar region (where the cylinder is glued to the manifold) shrinks to a point.

For (2), it is easy to see convergence on the neck $N \setminus (X_1 \cup X_2)$, so let's consider the congergence $(X_i, \varepsilon^{-2} g_i) \to \mathbb{R}^n$. Let $\delta > 0$ be less than the injectivity radius of (X_i, g_i) at x_i. We know that

$$d(\exp_{x_i})_x = I + o(\delta^2),$$

where $\exp_{x_i}$ is the exponential map with respect to the metric g_i, $d(x, x_i) < \delta$.

Given $R > 0$, choose $\varepsilon < \frac{\delta}{R}$ so that the ball $B_{x_i}(R) \subset (X_i, \varepsilon^{-2} g_i)$ is a subset of $B_{x_i}(\delta) \subset (X_i, g_i)$, and therefore

$$|d(exp_{x_i})^* g - I| = o(\delta^2)$$

in $B_{x_i}(R) \subset (X_i, \varepsilon^{-2} g_i)$, which proves the convergence. □

Lemma 2.1. *The Sobolev constants for both $(X_\varepsilon, g_\varepsilon)$ and $(X_\varepsilon, \varepsilon^{-2} g_\varepsilon)$ are uniformly bounded.*

Proof. The limit of $(X_\varepsilon, \varepsilon^{-2} g_\varepsilon, x_\varepsilon)$ is the space M, which is obtained by connecting two copies of $\mathbb{R}^n$ by a neck with fixed size. The Sobolev constant of X_ε doesn't not change with ε, see theorem 0.1. By continuity, in order to prove the uniform bound for the Sobolev space, it suffices to prove the Sobolev inequality on the limiting space.

Let f be a smooth function of compact support on the limiting space M. As before, we can write

$$f = f_1 + f_2 + f_0$$

where f_1, f_2 has their support within one copy of $\mathbb{R}^n$ and f_0 has its support within a fixed geodesic ball. Denote by $M_i \subset M$ the support of f_i for $i = 0 \dots 2$. Since on Euclidean space we have uniform Sobolev constants, we have

$$\left(\int_{M_i} |f_i|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} dv \leq C \int_{M_i} |\nabla f_i|^2 v \, dv$$

for $i = 1, 2$. On the other hand, we have the usual Sobolev inequality on a compact manifold, thus we have

$$\left(\int_{M_0} |f_0|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} v \, dv \leq C' \int_{M_0} |\nabla f_0|^2 \, dv$$

with a possibly different Sobolev constant C'. Combining the above we get

$$\left(\int_{M_i} |f_i|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq \max(C, C') \sum_{i=0}^{2} \int_{M_i} |\nabla f_i|^2 \leq C \left(\int_M (|\nabla f|^2 + |f|^2) \, dv \right).$$

Since the Sobolev constant is independent of scaling, we have also proved the existence of a uniform Sobolev constant for $(X_\varepsilon, g_\varepsilon)$. □

It is also not hard to see that

2.2 A Gap Theorem

In what follows we prove the main technical theorem of this chapter. We will make use of usual Schauder Interior estimates for eigenfunctions of the Laplacian.

Proposition 0.2 (Schauder Interior Estimate)**.** *Let X be a Riemannian manifold (not necessarily compact or complete). Suppose that f satisfies the equation*

$$\Delta f + \lambda f = 0$$

on X. Then for any $k \geq 0$ and any compact subset X' of X, there exists constant $C = C(k, X', \lambda)$ depending in the manifold X such that

$$\|f\|_{C^k(X')} \leq C\|f\|_{L^2(X)}.$$

For a proof in the Euclidean space, see section 0.4, which heavily relies on [20].

In the context of Gromov-Hausdorff convergence the limit space might not always be a smooth manifold, but it is a measured length space. Nevertheless, one may still define the spectrum of the generalized Laplace operator on it [19]. We denote the spectrum of the generalized Laplacian on metric space X by $\mathrm{Spec}(X)$.

Theorem 2.5. *Let X_1, X_2 be two compact Riemannian manifolds and take $\lambda \notin \mathrm{Spec}(X_1) \cup \mathrm{Spec}(X_2)$. Consider the manifold $(X_\varepsilon, g_\varepsilon)$ defined above. Set $2\delta = dist(\lambda, \mathrm{Spec}(X_1) \cup \mathrm{Spec}(X_2))$ and take $\lambda' \in (\lambda - \delta, \lambda + \delta)$. Then, for $\varepsilon > 0$ small enough, $\lambda' \notin \mathrm{Spec}(X_\varepsilon)$.*

Proof. We will prove the theorem by contradiction. Assume that for any $\varepsilon > 0$, there is a $\lambda_\varepsilon \in (\lambda - \delta, \lambda + \delta)$ such that λ_ε is an eigenvalue of X_ε. Let f_ε be the corresponding eigenfunction. Then

$$\Delta_\varepsilon f_\varepsilon + \lambda_\varepsilon f_\varepsilon = 0.$$

Using the above proposition, we get that the norms $\|f_\varepsilon\|_{\mathcal{C}^k}$ are uniformly bounded away from the singular point of $X_1 \cup X_2$. As a result, there is a subsequence f_{ε_i} such that $f_{\varepsilon_i} \to \xi$ in the $\mathcal{C}^\infty$ topology. We shall prove that $\xi \neq 0$.

By a standard Moser iteration argument using the uniform Sobolev inequality, see section 0.5 for more details, we have

$$\|f_\varepsilon\|^2_{L^\infty} \leq C \frac{1}{\mathrm{Vol}(X_\varepsilon)} \int_{X_\varepsilon} |f_\varepsilon|^2 \, dV.$$

As a result, if we normalize the L^2 norm of f_ε to be 1 and given that $\mathrm{vol}(X_\varepsilon)$ is uniformly bounded, we get that the f_ε are uniformly bounded in L^∞. As a result, ξ is bounded. Let $\xi_i = \xi|_{X_i \setminus \{x_i\}}$ for $i = 1, 2$. By the above argument, since ξ is bounded, it follows that each ξ_i extends to a smooth function on X_i. Moreover, we have

$$\Delta \xi_i + \lambda_0 \xi_i = 0$$

for $i = 1, 2$ with $\lambda_0 \in [\lambda - \delta, \lambda + \delta]$. Finally, let V be a small neighborhood of the singular point of X. Since f_{ε_i} is bounded, we have

$$\|f_{\varepsilon_i}\|_{L^2(X_{\varepsilon_i} \setminus V)} \geq 1 - C \, \mathrm{vol}(V) > 1/2.$$

Given that outside V the convergence is in the $\mathcal{C}^\infty$ topology, we conclude that

$$\|\xi\|_{L^2(X_{\varepsilon_i} \setminus V)} > 1/2.$$

In particular, at least one of ξ_1, ξ_2 are not zero. □

The above theorem can be interpreted as a spectral continuity result: let $\lambda_k(X_\varepsilon)$ be the k-th eigenvalue of X_ε. Then a subsequence $\lambda_k(X_{\varepsilon_i})$ is convergent to the corresponding eigenvalue of the limit space.

Since the Ricci curvature of X_ε has no lower bound, Theorem 2.5 is not a special case of the theorem of Cheeger-Colding. It is neither a special case of the theorem of Dodziuk because the limit space is singular.

Let X be a fixed compact manifold. Let $x_1, x_2 \in X$ two distinct points. Construct a metric space by first making $\mathbb{Z}$ copies of X, and labelling them X_j. Then glue the point x_2 of X_j onto the point x_1 of X_{j+1} for each $j \in \mathbb{Z}$. Denote by M the metric space obtained through this gluing process.

Definition 2.3. *A smoothing X_{eps} of M is a smooth manifold constructed as in (2.1) at each x_1, x_2. Under the Gromov-Hausdorff convergence, we have*

$$\lim_{\varepsilon \to 0} X_\varepsilon = M.$$

Similar to to Lemma 2.1, we have

Lemma 2.2. *The Sobolev constant for X_ε is independent to ε.*

Using Theorem 2.5 we can prove the following.

Theorem 2.6. *Let X_ε be a smoothing of M constructed in a similar process as in the beginning of the chapter. That is, X_ε is smooth and the Gromov-Hausdorff limit of X_ε is M. Then, for ε small enough, the essential spectrum of X_ε has gaps.*

Proof. Note that since X_ε is a periodic manifold, its spectrum must coincide with its essential spectrum.

Our proof is a generalization of the method used in the proof of Theorem 2.5. Let $\lambda_\varepsilon \in \mathrm{Spec}(X_\varepsilon, g_\varepsilon)$. We shall prove that a subsequence λ_{ε_i} should be convergent to $\lambda_0 \in \mathrm{Spec}(X)$. Fix $\delta > 0$. Let f_ε be the approximating eigenfunction by the Weyl criterion, Theorem 1.5, such that

$$\|\Delta f_\varepsilon + \lambda_\varepsilon f_\varepsilon\|_{L^2} \leq \delta \|f_\varepsilon\|_{L^2}.$$

It is not difficult to see that there exist λ'_ε such that $\lambda_\varepsilon - \lambda'_\varepsilon = o(1)$ and for which we can find a function f'_ε that is the Dirichlet eigenfunction corresponding to λ'_ε on the support of f_ε. That is,

$$\Delta f'_\varepsilon + \lambda'_\varepsilon f'_\varepsilon = 0.$$

We normalize f'_ε so that the L^2 norm of f'_ε on K is 1. By using the uniform Sobolev inequality, we can prove that the f'_ε are uniformly bounded.

Let y_ε be the maximum point of f_ε. By the periodic property of X_ε, by translating if necessary, we may assume that y_ε is within a fixed copy of X. Normalizing f'_ε so that the maximum of it is 1. Then a subsequence of f'_ε will be convergent to a *non-zero* function f_0 on the copy X which, by elliptic estimates, must be smooth on the smooth part of M. Since f is also bounded, f must be an eigenfunction of X. Therefore $\lambda_0 \in \mathrm{Spec}(X)$.

□

It should be noted that the convergence rate depends on λ. For larger λ the rate of convergence may in fact be slower. It would therefore be interesting to know whether there is an infinite number of gaps in the spectrum of X_ε for ε small enough.

Chapter 3

The spectrum of the Laplacian on forms

One would like to extend the results obtained in the previous discussions, especially those in section 2.2, to the setting of differential forms. For that we need to extend the definition of the Laplace operator, so that it can act on the space of differential forms. Check section 0.7 for notations and some useful computations.

3.1 The Hodge Laplacian

Given $\omega = a_{i_1 \cdots i_k} \omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$ and $\eta = b_{i_1 \cdots i_k} \omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$, the L^2 *inner product* in $\Lambda^k M$ is defined as

$$(\omega, \eta) = k! \int_M a_I b_I \; dV$$

where dV is the volume form, $\{e_1, \cdots, e_n\}$ is a global orthonormal frame and $\{\omega_1, \cdots, \omega_n\}$ its dual frame, and I and J are the corresponding multi-indices.

Recall that the exterior differential of $\omega = a$ is the $k+1$-form, whose expression is

$$d\omega = (-1)^k a_{i_1 \cdots i_k, l} \omega_{i_1} \wedge \cdots \wedge \omega_{i_k} \wedge \omega_l$$

. The *covariant derivative* $a_{i_1 \cdots i_k, l}$ can be found in Definition 0.10.

The adjoint δ of d, the *codifferential operator* $\delta : \Lambda^{k+1} M \to \Lambda^k M$, is the operator satisfying

$$(d\omega, \eta) = (\omega, \delta\eta),$$

$\forall \omega \in \Lambda^k M, \eta \in \Lambda^{k+1} M$. Lemma 0.3 calculates a formula for δ.

We are finally ready to extend the Laplace operator to differential forms. The *Hodge Laplacian*, also known as the *Laplace–de Rham* operator, is defined as

$$\Delta := d\delta + \delta d = (d+\delta)^2.$$

As defined above, Δ is a nonnegative definite operator on $L^2(\Lambda^k M)$. It is also not hard to see that it extends the Laplace-Beltrami operator up to a negative sign when acting on a scalar function, which is a 0-form by definition.

The differential *df* of a function f can be identified with a vector field by

$$df = \frac{\partial f}{\partial x_i} dx^i \longleftrightarrow g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}.$$

Under the above identification, the adjoint of d is the usual negative divergent operator, since, when f has compact support,

$$\begin{aligned}
(df, X) + (f, \operatorname{div} X) &= \int_M (df(X) + f \operatorname{div} X)\ dV \\
&= \int_M (X(f)dV + f\mathcal{L}_X dV) \\
&= \int_M \mathcal{L}_X(f\ dV) \\
&= \int d\ i_X(fdV) \\
&= 0
\end{aligned}$$

where we are using that the *Lie derivative* satisfies

$$\mathcal{L}_X(\omega) = d\ i_X(\omega) + i_X d\omega$$

as defined in section 0.8.

Since that $\delta f = 0,\ \forall f \in C^\infty(M)$, we have

$$\Delta f = d\delta f + \delta df = \delta df = \delta \partial_i f dx^i = -div(\nabla f) = -\Delta f.$$

Since there is a sign difference we will define the spectrum the Hodge-Laplacian on k-forms slightly differently from the on on functions. We leave this discussion for the following sections, but the idea is to use the sign convention that yields nonnegative eigenvalues.

When studying the Laplacian, we would like to clearly define its domain. For simplicity, assume M is a compact manifold in the following discussion.

Let $L^2(\Lambda^k(M))$ be the metric space completion of the space $\Lambda^k(M)$ of k-forms on M equipped with the inner product defined above. Since $L^2(\Lambda^k(M))$ is complete, one would like to define the Hodge-Laplacian Δ on $L^2(\Lambda(M)$, but unfortunately one cannot do so without losing some of the key properties of Δ, namely the fact that it is self-adjoint, and we explain the reasoning below.

First, note that Δ is a *closed-graph operator*, meaning that if $\eta_j \to \eta$ and $\Delta\eta_j \to \eta'$ in $L^2(\Lambda^k(M))$, we have

$$(\eta' - \Delta\eta, \omega) = \lim_{j\to\infty} (\Delta\eta_j, \omega) - \lim_{j\to\infty} (\eta_j, \Delta\omega) = 0, \ \forall \omega \in L^2(\Lambda^k(M)) \Rightarrow \Delta\eta = \eta'.$$

Now, assume by contradiction that $\mathfrak{Dom}(\Delta) = L^2(\Lambda^k(M))$. By the *Closed Graph Theorem*, Δ must be bounded. Considering $k = 0$ and $M = S^1 \approx [0,1] \subset \mathbb{R}$ we get that the Hodge Laplacian is $\Delta = -\frac{\partial^2}{\partial x^2}$ is bounded, but this is a contradiction since we can choose $f(x) = C\cos(2\pi x)$, $C > 0$, to see that

$$|\Delta f|^2 = (\Delta f, \Delta f) = 4\pi^2 C \int_0^1 f(t)^2 dt = 4\pi^2 C|f|^2.$$

From the discussion above, it becomes clear that we need to carefully define the domain of Δ. Our goal is to extend Δ to a *densely-defined* self adjoint operator, in the sense that $\overline{\mathfrak{Dom}(\Delta)} = L^2(\Lambda^k(M))$.

Definition 3.1. *Let H be a densely-defined linear operator on a Hilbert space $\mathcal{H}$. It's adjoint H^* is defined as follows.*

(i) The domain of H^* consists of vectors $x \in \mathcal{H}$ for which

$$f_x : y \mapsto \langle x, Hy \rangle$$

is a bounded linear functional, for any $y \in \mathfrak{Dom}(H)$*;*

(ii) By the Riesz Representation Theorem for a linear functional, if x *is in the domain of* H^**, there is a unique vector* $z \in \mathcal{H}$ *such that*

$$f_x(y) = \langle x, Hy \rangle = \langle z, y \rangle \, , \ \forall y \in \mathfrak{Dom}(H).$$

This vector z *is defined to be* H^*x*. It can be shoen that the dependence of* z *on* x *is linear.*

If $H^* = H$*, (which implies* $\mathfrak{Dom}(H^*) = \mathfrak{Dom}(H)$*), then* H *is said to be self-adjoint.*

Define the Sobolev space $H_0^1(k, M)$ as the completion of $\Lambda^k(M)$ under the norm

$$||\eta||_{H_0^1(k,M)} := \sqrt{\int_M |\eta|^2 \ dv + \int_M |\nabla \eta|^2 \ dv}$$

and the quadratic form Q on $H_0^1(k, M)$ by

$$Q(\omega, \eta) = \int_M \left(\langle d\omega, d\eta \rangle + \langle \delta\omega, \delta\eta \rangle \right) \ dv$$

for any $\omega\eta \in H_0^1(k, M)$. Then we define

$$\mathfrak{Dom}(\Delta) = \left\{ \varphi \in H_0^1(k, M) : \forall \psi \in \Lambda^k(M), \exists \xi \in L^2(\Lambda^k(M)) \text{ such that } Q(\varphi, \psi) = -(f, \psi) \right\}.$$

Lemma 3.1. *Using the above notation, we have*

$$\mathfrak{Dom}(\Delta) = \mathfrak{Dom}(\Delta^*).$$

Proof. First note that

$$Q(\varphi, \psi) = (d\varphi, d\psi) + (\delta\varphi, \delta\psi) = (\varphi, \delta d\psi) + (\varphi, d\delta\psi) = (\varphi, \Delta\psi).$$

Now, given $\varphi \in \mathfrak{Dom}(\Delta)$, consider the functional $f_\varphi : \psi \mapsto (\Delta\psi, \varphi)$. From the definition of $\mathfrak{Dom}(\Delta)$, given $\psi \in \Lambda^k(M)$, there is $\xi \in L^2(\Lambda^k(M))$ such that

$$|f_\varphi(\psi)| = |(\Delta\psi, \varphi)| = |(\xi, \psi)| \leq ||\xi|| \cdot ||\psi||.$$

The above shows that f_φ is a bounded linear functional, and therefore $\varphi \in \mathfrak{Dom}(\Delta)$.

Conversely, assume $\varphi \in \mathfrak{Dom}(\Delta^*)$. Then

$$h_\varphi : \psi \mapsto (\varphi, \Delta\psi)$$

is bounded, by definition 3.1. By the Riesz Representation Theorem, there is a unique $\xi \in L^2(\Lambda^k(M))$ such that $(\Delta\psi, \varphi) = (\xi, \psi)$, $\forall \psi \in L^2(\Lambda^k(M))$. Then we must have

$$Q(\psi, \psi) = (\Delta\varphi, \psi) = (\xi, \psi),$$

which implies $\varphi \in \mathfrak{Dom}(\Delta)$. □

The following intends to reinforce the importance of the above discussion. In it, we examine a situation where the domain of the Laplacian behaves in an unexpected manner.

Example 3.1. *Consider $M = [0, 1]$ and let $C^\infty(M)$ be the set of functions that are smooth on $(0, 1)$ and continuous on $[0, 1]$. Take $\Delta = -\frac{\partial^2}{\partial x^2}$. It can be shown that*

$$C^\infty(M) \not\subseteq \mathfrak{Dom}(\Delta^*).$$

3.2 The Spectrum of the Laplacian on forms

Now that we have precisely defined the Hodge Laplacian on k-forms, we are able to define its spectrum, in a fashion similar to the function case. Let $L^2(\Lambda^k M)$ denote the space of L^2 integrable k-forms as well as its Friedrichs extension on L^2. We denote the domain of the Laplacian on k-forms by $\mathfrak{Dom}(k, \Delta)$. The k-form spectrum of the Laplacian on M is the set of complex numbers λ for which the operator $\Delta - \lambda I$ fails to be invertible, where

$$\Delta = \Delta_k = d\delta + \delta d = (d + \delta)^2$$

is the Hodge Laplacian $\Delta : \Lambda^k M \to \Lambda^k M$. For technical reasons (see [5]), we abuse notation and use $\sigma(k, \Delta)$ to refer to the pointed metric space

$$(\sigma(k, \Delta) \cup \{-1\}, -1)\,.$$

Each $\lambda \in \sigma(k, \Delta) \setminus \{-1\}$ is called an eigenvalue of Δ. The essential spectrum of Δ is the set consisting of cluster points of $\sigma(k, \Delta)$ and eigenvalues of infinite multiplicity. Abusing notation, we write $\sigma_{ess}(k, \Delta)$ for the pointed metric space

$$(\sigma_{ess}(k, \Delta) \cup \{-1\}, -1)\,.$$

We will omit the degree k of the differential form when there is no risk of confusion. Note that these definitions imply

$$\sigma(-1, \Delta) = \sigma(n + 1, \Delta) = \sigma_{ess}(-1, \Delta) = \sigma_{ess}(n + 1, \Delta) = \emptyset,$$

meaning that the sets consists of a single point metric space $\{-1\}$.

Directly computing the essential spectrum of the Laplacian on forms, however, has been a more complicated task, even for the case of 1-forms, due to their stronger connection to the curvature of the manifold.

The generalized Weyl criterion, theorem 1.5, has an important application:

Theorem 3.1 (Charalambous-Lu [4]). *Let M be a complete Riemannian manifold. Suppose that $\delta > 0$ belongs to the essential spectrum of the Laplacian on k-forms, $\sigma_{ess}(k, \Delta)$. Then one of the following holds:*

(i) $\lambda \in \sigma_{ess}(k-1, \Delta)$ *or*

(ii) $\delta \in \sigma_{ess}(k+1, \Delta)$.

This theorem, in its turn, immediately gives an important corollary on the 1-form essential spectrum of the Laplacian.

Theorem 3.2 (Charalambous-Lu [4]). *Under the above conditions, we have*

$$\sigma_{ess}(1, \Delta) \supset \sigma_{ess}(0, \Delta) \setminus \{0\}.$$

In other words, the 1-form essential spectrum contains the function essential spectrum, and we can conclude that, in $\mathbb{R}^n$, $\sigma_{ess}(1, \Delta) = [0, \infty)$.

In the same article, Charalambous and Lu study the spectrum of the Hodge Laplacian under continuous deformations of metrics. We say that two metrics are ε-close if, for some $0 < \varepsilon < 1$,

$$(1-\varepsilon)g_0 < g_1 < (1+\varepsilon)g_0.$$

We can then write Δ_1 for the Hodge Laplacian with respect to the metric g_i. They prove the following.

Theorem 3.3 (Charalambous-Lu [5]). *Let M be a manifold, and let g_0, g_1 be two complete Riemannian metrics on M that are ε-close for some $0 < \varepsilon < 1/2$. Fix $A > 0$. Then, for any $\lambda \in \sigma(k, \Delta_1) \cap [0, A]$,*

$$\operatorname{dist}(\lambda, \sigma(k, \Delta_0)) < c(A)\varepsilon^{\frac{1}{3}}$$

for some constant $c(A)$ depending only on A. A similar result holds for the essential spectra of the operators. In particular,

$$d_{GH}(\sigma(k, \Delta_1), \sigma(k, \Delta_0)) = o(1),$$

where $o(1) \to 0$ as $\varepsilon \to 0$.

3.3 Removable Singularity of Eigenforms

We now briefly discuss the removable singularity theorem for eigenforms, which is a well known result.

Let $D = B(0,1) \setminus \{0\} \subset \mathbb{R}^n$. Consider a harmonic function $u : D \to \mathbb{R}$ with a possible singularity at 0, i.e.,

$$\begin{cases} \Delta u = 0 & \text{on } D, \\ \int_D u^2 \leq C. \end{cases}$$

Let us prove that 0 is a removable singularity. Fix $x_0 \in B(0, 1/2)$ and let $r = |x_0|$. Then $B(x_0, r/2) \subset D$. By a mean value property for subharmonic functions, see [34], we have

$$|u(x_0)| \leq \sqrt{\frac{C}{\operatorname{Vol} B(x_0, r/2)} \int_{B(x_0, r/2)} u^2(x)dx} \leq \frac{C}{r^{n/2}},$$

where $C = C(n)$. As x_0 is arbitrary, this implies that, for $n \geq 3$,

$$\lim_{x \to 0} |u(x)||x|^{n-2} = 0.$$

Now, let v be the unique solution to the Dirichlet problem

$$\begin{cases} \Delta v = 0 & \text{on } B(0,1), \\ v|_{\partial B(0,1)} = u \end{cases}$$

Note that v is bounded since, from the maximum principle, $|v| \leq \max\{u(x)|x \in \partial B(0,1)\}$. To show $u = v$, consider the function $w = u - v$. It is easy to see that $\Delta w = 0$ in D and that

$$\frac{w(x)}{|x|^{2-n}} = u(x)|x|^{n-2} - v(x)|x|^{n-2} \to 0$$

as $x \to 0$. Therefore we can extend w to a harmonic function on $B(0,1)$, and $u = v$.

For a k-form ω, since we are making this computation in $\mathbb{R}^n$, for the moment, note that if $\omega = a_I dx^I$ and $\Delta\omega = 0$, then we have

$$0 = \Delta\omega = \Delta(a^I dx^I) = (\Delta a_I)dx^I,$$

and therefore we can repeat the previous argument for the coefficients a_I to conclude that any isolated singularities of a harmonic form ω on $\mathbb{R}^n$ are removable.

Let us try and apply the same reasoning to eigenfunctions. Let $D = B(0,1) \setminus \{0\}$ and $f : D \to \mathbb{R}$ satisfying

$$\begin{cases} \Delta f = -\lambda f \text{ on } D \\ \int_{B(0,1)} f^2 \leq C \end{cases}$$

for some positive constant C.

To obtain a smooth solution h to the problem

$$\begin{cases} \Delta h = -\lambda f \text{ on } D \\ h|_{\partial B(0,1)} = f \\ h \in C^2(B(0,1)). \end{cases},$$

consider the functional

$$I(h) = \int \frac{1}{2}|\nabla h|^2 - \lambda f h$$

on $C^2(B(0,1))$. One can check that I satisfies both the convexity and the coerciveness conditions that guarantee the existence of a minimizer, see [18], sections 8.1, 8.2 and 8.4. The minimizer is the solution to the equation above.

Then, following the same reasoning as for harmonic k-forms, we can conclude that any isolated singularities of an L^2 bounded eigenform of the Hodge Laplacian on $B(0,1) \in \mathbb{R}^n$ is removable. The parallel we draw between eigenfunctions/eigenforms and harmonic functions/forms in a small neighborhood is not new, and a detailed exposition can be found in [40].

When our L^2 bounded eigenform has a possible isolated singularity at a point x in a Riemannian manifold X, as in theorem 2.5, we can use the exponential map at x_i, that satisfies

$$d(\exp_{x_i})_y = I + o(\delta^2),$$

for $y \in B(x, \delta)$, and rescale the metric in X as in proposition 2.1, to obtain the following result.

Proposition 3.1. *Let X be a compact Riemannian manifold and $x \in X$. If ω is a differential* *k-form on* $X \setminus \{x\}$ *satisfying*

$$\begin{cases} \Delta\omega = \lambda\omega \ \text{ on } X \setminus \{x\} \\ ||\omega||_{L^2(X)} \leq C \end{cases},$$

then x *is a removable singularity of* ω*, that is, there exists a differential* k*-form* $\tilde{\omega}$ *on* X*, with* $\omega = \tilde{\omega}$ *in* $X \setminus \{x\}$ *satisfying*

$$\Delta\tilde{\omega} = \lambda\tilde{\omega} \ \text{ on } X.$$

This discussion shows that theorem 2.5 is also valid for differential forms, and we can state it in its most general form. To simplify notation, write

$$\sigma(k, \Delta, M)$$

for the k-form spectrum of the Hodge Laplacian on a complete Riemannian manifold M. Then, the following holds.

Theorem 3.4. *Let* X_1, X_2 *be two compact Riemannian manifolds and take* $\lambda \notin \sigma(k, \Delta, X_1) \cup \sigma(k, \Delta, X_2)$*. Consider the manifold* $(X_\varepsilon, g_\varepsilon)$ *defined in chapter 2. Set* $2\delta = dist(\lambda, \sigma(k, \Delta, X_1) \cup \sigma(k, \Delta, X_2))$ *and take* $\lambda' \in (\lambda - \delta, \lambda + \delta)$*. Then, for* $\varepsilon > 0$ *small enough,* $\lambda' \notin \sigma(k, \Delta, X_\varepsilon)$*.*

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