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Extension of Torsors and Curved Maurer Cartan Equation

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Ka Laam Chan

Dissertation Committee:
Vladimir Baranovsky, Chair
Zhiqin Lu
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2019

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ABSTRACT OF THE DISSERTATION

Extension of Torsors and Curved Maurer Cartan Equation

By

Ka Laam Chan

Doctor of Philosophy in Mathematics

University of California, Irvine, 2019

Vladimir Baranovsky, Chair

In this thesis we will study the extension problem of (nilpotent) G-torsors.

In chapter 1, we will review the Maurer Cartan equation of a DGLA and go through some examples of Maurer Cartan equations in deformation of different algebraic structures.

In chapter 2, we will define L_∞ algebras, which are generalizations of DGLAs. We will also state the homotopy transfer of structure theorem and the formal Kuranishi theorem, which are used throughout the whole thesis.

In chapter 3, we will briefly go through Getzler's result on the unique horn filling of Deligne-Getzler ∞-groupoids, which gives us a generalization (on L_∞ algebras) of the Baker-Campbell-Hausdorff formula for a Lie algebra.

In chapter 4, we will state Hinich's result on descent of Deligne groupoid and go through Fiorenza, Manetti, and Martinengo's result on the special case when we have a semicosimplicial Lie algebra, i.e. the solutions to the group valued cocycle condition are exactly the Maurer Cartan solutions on the L_∞ total complex and equivalence of cocycles are exactly equivalence of Maurer Cartan solutions. We will then show an example of this result on deformation of (nilpotent) G-torsors.

In chapter 5, we will apply Fiorenza, Manetti, and Martinengo's result on the extension problem of (nilpotent) G-torsors and show that solutions to the curved cocycle condition that gives the G-torsors extensions are exactly the curved Maurer Cartan solutions of a curved L_∞ algebra and equivalence of extensions are exactly equivalence of curved Maurer Cartan solutions.

Chapter 1

Maurer Cartan Equation and Deformation

A differential graded Lie algebra (DGLA) is a graded vector space together with a Lie algebra and chain complex structure that are compatible. It is well known that many formal deformation problems in characteristic 0 can be described by the Maurer Cartan elements of some differential graded Lie algebra. In this chapter, we will review the Maurer Cartan equation of a differential graded Lie algebra and show some examples of Maurer Cartan equation in deformations of basic algebraic structures.

1.1 Differential Graded Lie Algebras and Maurer Cartan Equations

Definition 1.1. *A* ***Lie algebra*** *is a vector space* $\mathfrak{g}$ *over some characteristic* 0 *field* $\mathbb{K}$ *together with a binary operation* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *called the* ***Lie bracket*** *that satisfies:*

- *Anticommutativity,*

$$[x, y] = -[y, x]$$

for all $x, y,$ in $\mathfrak{g}$.

- *Bilinearity,*

$$[ax + by, z] = a[x, z] + b[y, z], \qquad [z, ax + by] = a[z, x] + b[z, y]$$

for all scalars a, b in $\mathbb{K}$ and all elements x, y, z in $\mathfrak{g}$.

- *The Jacobi identity,*

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

for all x, y, z in $\mathfrak{g}$.

Definition 1.2. *A Lie algebra $\mathfrak{g}$ is* ***nilpotent*** *if $\mathfrak{g}_n = 0$ for $n >> 0$, where $\mathfrak{g}_n$ is defined recursively by $\mathfrak{g}_1 = \mathfrak{g}$, $\mathfrak{g}_{n+1} = [\mathfrak{g}, \mathfrak{g}_n]$ (i.e. $\mathfrak{g}$ is nilpotent if its lower central series terminates).*

Lie algebras are closely related to Lie groups, which are smooth manifolds with a group structure. When $\mathfrak{g}$ is nilpotent (and connected) the exponential map $e : \mathfrak{g} \to G$ from the Lie algebra $\mathfrak{g}$ to the Lie group G and the logarithmic map $\log : G \to \mathfrak{g}$ from the Lie group G to the Lie algebra $\mathfrak{g}$ are mutually inverse bijections, thus we have a one-to-one correspondence between the Lie algebra $\mathfrak{g}$ and the Lie group G. In the case where $\mathfrak{g}$ is a finite dimension Lie algebra, then the image of $\mathfrak{g}$ in $\mathfrak{gl}_n(k)$ will be a subset of the upper triangular matrices with 0's on the diagonal. In this form, e and log are given by the standard matrix exponential and logarithmic maps and they clearly converge on $\mathfrak{g}$.

We can now use these maps to introduce a group structure on $\mathfrak{g}$.

Definition 1.3. *If $\mathfrak{g}$ is a nilpotent Lie algebra, then the **Baker-Campbell-Hausdorff product** is the solution to*

$$*(a,b) = \log(e^a e^b)$$

which give us a product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. This product induces a group structure on $\mathfrak{g}$.

In the case where $\mathfrak{g}$ is a matrix Lie algebra, we can easily work out for first three terms of the Baker-Campbell-Hausdorff formula by taking the power series expansion of log and e:

$$\begin{aligned}
\log(e^a e^b) =& \log(1 + a + b + \frac{a^2}{2} + ab + \frac{b^2}{2} + \frac{a^3}{6} + \frac{a^2b}{2} + \frac{ab^2}{2} + \frac{b^3}{6} + \dots) \\
=& (a + b + \frac{a^2}{2} + ab + \frac{b^2}{2} + \frac{a^3}{6} + \frac{a^2b}{2} + \frac{ab^2}{2} + \frac{b^3}{6} + \dots) \\
& - \frac{1}{2}(a^2 + ab + ba + b^2 + a^3 + \frac{3a^2b}{2} + \frac{3ab^2}{2} + \frac{ba^2}{2} + \frac{b^2a}{2} + aba + \\
& bab + b^3 + \dots) + \frac{1}{3}(a^3 + a^2b + aba + ab^2 + ab^2 + bab + b^2a + b^3 \\
& + \dots) + \dots \\
=& a + b + \frac{ab}{2} - \frac{ba}{2} + \frac{a^2b}{12} + \frac{ab^2}{12} + \frac{ba^2}{12} + \frac{b^2a}{12} - \frac{2aba}{12} - \frac{2bab}{12} + \dots \\
=& a + b + \frac{1}{2}[a,b] + \frac{1}{12}([a,[a,b]] + [b,[b,a]]) + \dots
\end{aligned}$$

A general Baker-Campbell-Hausdorff formula is given by [10]:

$$a * b = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{p_1+q_1>0 \\ p_n+q_n>0}} \frac{(\sum_{i=1}^{n}(p_i+q_i))^{-1}}{p_1!q_1!\dots p_n!q_n!} \operatorname{ad}(a)^{p_1} \operatorname{ad}(b)^{q_1} \dots \operatorname{ad}(a)^{p_n} \operatorname{ad}(b)^{q_n-1} b$$

where $\operatorname{ad}(a)(b) = [a,b]$. Note that this formula is non-unique due to the Jacobi identity.

Now we are ready to define differential graded Lie algebras.

Definition 1.4. *A* ***graded Lie algebra*** *is a Lie algebra* $\mathfrak{g}$ *with a grading on the vector space*

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}^i$$

such that the Lie bracket respects this grading

$$[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j}.$$

Definition 1.5. *A* ***differential graded Lie algebra (DGLA)*** *is a graded Lie algebra* $L = \bigoplus_{n \in \mathbb{Z}} L^n$ *together with a degree 1 linear map* $d : L \to L$ *such that:*

- $d[a, b] = [da, b] + (-1)^{|a|}[a, db]$
- $d \circ d = 0$

Definition 1.6. *The* ***Maurer Cartan equation*** *of a DGLA L is*

$$da + \frac{1}{2}[a, a] = 0, \qquad a \in L^1.$$

The set of solutions $\mathrm{MC}(L) \subset L^1$ *of the Maurer Cartan equation are called the* ***Maurer Cartan set*** *of the DGLA L and the elements in* $\mathrm{MC}(L)$ *are called* ***Maurer Cartan elements****.*

The exponential of the adjoint action on L', where $(L')^i = L^i$ for every $i \neq 1, (L')^1 = L^1 \oplus \mathbb{K}d$, d considered as a formal symbol of degree 1, induces the **gauge action** of L^0 over the set of solution of the Maurer Cartan equation:

$$\begin{aligned} a \cdot (w) = \phi^{-1}(e^{\mathrm{ad}(a)}\phi(w)) &= \sum_{n \geq 0} \frac{1}{n!} \mathrm{ad}(a)^n(w) - \sum_{n \geq 1} \frac{1}{n!} \mathrm{ad}(a)^{n-1}(da) \\ &= w + \sum_{n \geq 0} \frac{\mathrm{ad}(a)^n}{(n+1)!}([a, w] - da) \end{aligned}$$

Here $\phi : L^1 \to (L')^1$, $\phi(u) = u + d$, is the affine embedding and the group structure of L^0 is given by the Baker-Campbell-Hausdorff formula. See [10] for the proof that this is in fact an action.

Definition 1.7. *Let $\phi, \psi \in L^1$ be solutions to the Maurer Cartan equation of L. ϕ is* ***gauge equivalent*** *to ψ iff $\phi = a \cdot (\psi)$ for some $a \in L^0$. We can define the* ***equivalence classes of Maurer Cartan solutions*** *to be the Maurer Cartan set mod the gauge equivalence.*

1.2 Examples of Maurer Cartan Equations in Deformation Theory

1.2.1 Deformation of Associative Algebra

Suppose we have an associative algebra B with multiplication μ. Consider the deformation of the algebra by a local Artinian ring A, i.e. $B \otimes_{\mathbb{K}} A$, $A = \mathbb{K} \oplus m$, m the maximal ideal of A, such that reducing $B \otimes_{\mathbb{K}} A$ mod m, we should get the original structure on $B \otimes A/m = B$. Such deformation is governed by the DGLA $\mathcal{L} = L \otimes m$, where $L = \bigoplus_{k \in \mathbb{Z}} L^k[-k]$, $L^k = \mathrm{Hom}(B^{\otimes k+1}, B)$ the kth Hochschild complex for $k \geq -1$, $L^{\leq -1} = 0$. The differential of L is given by

$$\begin{aligned}(df)(a_0 \otimes \cdots \otimes a_{k+1}) =& a_0 \cdot f(a_1 \otimes \cdots \otimes a_{k+1}) \\ & - \sum_{i=0}^{k} (-1)^i f(a_0 \otimes \cdots \otimes (a_i \cdot a_{i+1}) \otimes \cdots \otimes a_{k+1}) \\ & + (-1)^k f(a_0 \otimes \cdots \otimes a_k) \cdot a_{k+1}, \qquad f \in L^k\end{aligned}$$

and the Lie bracket is given by the Gerstenhaber bracket, which is

$$[f_1, f_2] = f_1 \circ f_2 - (-1)^{k_1 k_2} f_2 \circ f_1, \qquad f_i \in L^{k_i}$$

where the (non-associative) product $\circ$ is defined as

$$(f_1 \circ f_2)(a_0 \otimes \cdots \otimes a_{k_1+k_2}) = \sum_{i=0}^{k_1} (-1)^{ik_2} f_1(a_0 \otimes \cdots \otimes a_{i-1} \otimes f_2(a_i \otimes \cdots \otimes a_{i+k_2}) \otimes a_{i+k_2+1} \otimes \cdots \otimes a_{k_1+k_2})$$

In this example, the deformations of μ will correspond to the set of Maurer Cartan solutions while equivalence of deformations will correspond to equivalence of Maurer Cartan solutions.

Consider $\mu_\epsilon = \mu + \beta$, $\beta \in L^1 \otimes m$, the multiplication in $B \otimes A$. Assume that μ_ϵ is associative, then we have:

$$[\mu_\epsilon, \mu_\epsilon](f, g, h) = 2((\mu_\epsilon(\mu_\epsilon(f, g), h) - (\mu_\epsilon(f, \mu_\epsilon(g, h))) = 0$$

where $f, g, h \in B$ and the bracket is the Gerstenhaber bracket inherited from the Hochschild complex and extended by k-linearity. If we expand the bracket, we get

$$[\mu_\epsilon, \mu_\epsilon] = [\mu + \beta, \mu + \beta] = [\mu, \mu] + 2[\mu, \beta] + [\beta, \beta].$$

But $[\mu, \mu] = 0$ as μ is our original multiplication (which is associative) and $[\mu, \beta]$ is precisely $d\beta$. So we get back (after dividing by 2) the Maurer Cartan equation on $\mathcal{L}$

$$d\beta + \frac{1}{2}[\beta, \beta] = 0$$

and we conclude that μ_ϵ is associative if and only if β satisfies the Maurer Cartan equation on $\mathcal{L}$.

We will now work out the case where $A = \mathbb{K}[\epsilon]/\epsilon^4$, i.e. the fourth power of epsilon vanishes. For $f, g \in B$, we have

$$\mu_\epsilon(f,g) = fg + \beta_1(f,g)\epsilon + \beta_2(f,g)\epsilon^2 + \beta_3(f,g)\epsilon^3.$$

Now apply the Maurer Cartan equation to $f, g, h \in B$, we have:

$$\begin{aligned}
(d\beta + \frac{1}{2}[\beta,\beta])(f,g,h) =& (d\beta + \beta\circ\beta)(f,g,h) \\
=& d\beta(f,g,h) + \beta\circ\beta(f,g,h) \\
=& d\beta(f,g,h) + \beta(\beta(f,g),h) - \beta(f,\beta(g,h)) \\
=& f\beta(g,h) - \beta(fg,h) + \beta(f,gh) - \beta(f,g)h \\
& + \beta(\beta(f,g),h) - \beta(f,\beta(g,h)) \\
=& f\beta_1(g,h)\epsilon + f\beta_2(g,h)\epsilon^2 + f\beta_3(g,h)\epsilon^3 \\
& - \beta_1(fg,h)\epsilon - \beta_2(fg,h)\epsilon^2 - \beta_3(fg,h)\epsilon^3 \\
& + \beta_1(f,gh)\epsilon + \beta_2(f,gh)\epsilon^2 + \beta_3(f,gh)\epsilon^3 \\
& - \beta_1(f,g)h\epsilon - \beta_2(f,g)h\epsilon^2 - \beta_3(f,g)h\epsilon^3 \\
& + \beta_1(\beta_1(f,g),h)\epsilon^2 + \beta_2(\beta_1(f,g),h)\epsilon^3 + \beta_1(\beta_2(f,g),h)\epsilon^3 \\
& - \beta_1(f,\beta_1(g,h))\epsilon^2 - \beta_2(f,\beta_1(g,h))\epsilon^3 - \beta_1(f,\beta_2(g,h))\epsilon^3
\end{aligned}$$

Group the terms according to power of ϵ and express them in terms of d and $[-,-]$, we get:

$$\begin{aligned}
& d\beta_1 = 0 \\
& d\beta_2 + \frac{1}{2}[\beta_1,\beta_1] = 0 \\
& d\beta_3 + [\beta_1,\beta_2] = 0
\end{aligned}$$

Note that these are precisely the equations for the deformation.

In the above computation, we are doing our computation in $L \otimes m$. However, there are more than one way for which $\mathcal{L}$ is isomorphic to $L \otimes m$. Consider $\gamma : B \otimes A \to B \otimes A$ an automorphism of $B \otimes A$ which is identity mod ϵ.

$$\gamma(x) = x + \gamma_1(x)\epsilon + \gamma_2(x)\epsilon^2 + \gamma_3(x)\epsilon^3 + \dots.$$

We can define a new multiplication μ'_ϵ in $B \otimes A$ by

$$\mu'_\epsilon(f, g) = \gamma^{-1}(\mu_\epsilon(\gamma(f), \gamma(g))).$$

The above definition can be view as an action of $\mathrm{Aut}(B \otimes A)$ on $L \otimes m$ sending β to $\gamma \cdot \beta$. Note that we can express the components of γ^{-1} in terms of the components of γ. In the case where $\epsilon^4 = 0$, we get:

$$\begin{aligned}
x =& \gamma^{-1}(\gamma(x)) \\
=& \gamma^{-1}(x + \gamma_1(x)\epsilon + \gamma_2(x)\epsilon^2 + \gamma_3(x)\epsilon^3) \\
=& \gamma^{-1}(x) + \gamma^{-1}(\gamma_1(x))\epsilon + \gamma^{-1}(\gamma_2(x))\epsilon^2 + \gamma^{-1}(\gamma_3(x))\epsilon^3 \\
=& x + \gamma_1^{-1}(x)\epsilon + \gamma_2^{-1}(x)\epsilon^2 + \gamma_3^{-1}(x)\epsilon^3 \\
& + (\gamma_1(x) + \gamma_1^{-1}(\gamma_1(x))\epsilon + \gamma_2^{-1}(\gamma_1(x))\epsilon^2)\epsilon \\
& + (\gamma_2(x) + \gamma_1^{-1}(\gamma_2(x))\epsilon)\epsilon^2 + \gamma_3(x)\epsilon^3 \\
=& x + (\gamma_1^{-1}(x) + \gamma_1(x))\epsilon + (\gamma_2^{-1}(x) + \gamma_1^{-1}(\gamma_1(x)) + \gamma_2(x))\epsilon^2 \\
& + (\gamma_3^{-1}(x) + \gamma_2^{-1}(\gamma_1(x)) + \gamma_1^{-1}(\gamma_2(x)) + \gamma_3(x))\epsilon^3
\end{aligned}$$

So we get the relations:

$$
\begin{aligned}
\epsilon : 0 =& \gamma_1^{-1}(x) + \gamma_1(x) \\
\epsilon^2 : 0 =& \gamma_2^{-1}(x) + \gamma_1^{-1}(\gamma_1(x)) + \gamma_2(x) \\
\epsilon^3 : 0 =& \gamma_3^{-1}(x) + \gamma_2^{-1}(\gamma_1(x)) + \gamma_1^{-1}(\gamma_2(x)) + \gamma_3(x)
\end{aligned}
$$

Solving for γ^{-1}'s in terms of γ's, we get:

$$
\begin{aligned}
&\gamma_1^{-1}(x) = -\gamma_1(x) \\
&\gamma_2^{-1}(x) = -\gamma_2(x) + \gamma_1 \circ \gamma_1(x) \\
&\gamma_3^{-1}(x) = -\gamma_3(x) + \gamma_2 \circ \gamma_1(x) + \gamma_1 \circ \gamma_2(x) - \gamma_1 \circ \gamma_1 \circ \gamma_1(x)
\end{aligned}
$$

We will now compute $\mu'_\epsilon(f,g) = \gamma^{-1}(\mu_\epsilon(\gamma(f), \gamma(g)))$:

$$
\begin{aligned}
\mu_\epsilon(\gamma(f), \gamma(g)) =& \mu_\epsilon(f + \gamma_1(f)\epsilon + \gamma_2(f)\epsilon^2 + \gamma_3(f)\epsilon^3, g + \gamma_1(g)\epsilon + \gamma_2(g)\epsilon^2 + \gamma_3(g)\epsilon^3) \\
=& \mu_\epsilon(f,g) + \mu_\epsilon(f, \gamma_1(g))\epsilon + \mu_\epsilon(f, \gamma_2(g))\epsilon^2 + \mu_\epsilon(f, \gamma_3(g))\epsilon^3 \\
& + \mu_\epsilon(\gamma_1(f), g)\epsilon + \mu_\epsilon(\gamma_1(f), \gamma_1(g))\epsilon^2 + \mu_\epsilon(\gamma_1(f), \gamma_2(g))\epsilon^3 \\
& + \mu_\epsilon(\gamma_2(f), g)\epsilon^2 + \mu_\epsilon(\gamma_2(f), \gamma_1(g))\epsilon^3 + \mu_\epsilon(\gamma_3(f), g)\epsilon^3 \\
=& fg + (\beta_1(f,g) + f\gamma_1(g) + \gamma_1(f)g)\epsilon + \\
& + (\beta_2(f,g) + \beta_1(f, \gamma_1(g)) + f\gamma_2(g) + \beta_1(\gamma_1(f), g) \\
& + \gamma_1(f)\gamma_1(g) + \gamma_2(f)g)\epsilon^2 \\
& + (\beta_3(f,g) + \beta_2(f, \gamma_1(g)) + \beta_1(f, \gamma_2(g)) + f\gamma_3(g) + \beta_2(\gamma_1(f), g) \\
& + \beta_1(\gamma_1(f), \gamma_1(g)) + \gamma_1(f)\gamma_2(g) + \beta_1(\gamma_2(f), g) + \gamma_2(f)\gamma_1(g) + \\
& \gamma_3(f)g)\epsilon^3
\end{aligned}
$$

Apply γ_{-1} to get:

$$
\begin{aligned}
\gamma^{-1}(\mu_\epsilon(\gamma(f),\gamma(g))) =& fg + (\gamma_1^{-1}(fg) + \beta_1(f,g) + f\gamma_1(g) + \gamma_1(f)g)\epsilon \\
&+ (\gamma_2^{-1}(fg) + \gamma_1^{-1}(\beta_1(f,g)) + \gamma_1^{-1}(f\gamma_1(g)) \\
&+ \gamma_1^{-1}(\gamma_1(f)g) + \beta_2(f,g) + \beta_1(f,\gamma_1(g)) + f\gamma_2(g) \\
&+ \beta_1(\gamma_1(f),g) + \gamma_1(f)\gamma_1(g) + \gamma_2(f)g)\epsilon^2 \\
&+ (\gamma_3^{-1}(fg) + \gamma_2^{-1}(\beta_1(f,g)) + \gamma_2^{-1}(f\gamma_1(g)) + \gamma_2^{-1}(\gamma_1(f)g) \\
&+ \gamma_1^{-1}(\beta_2(f,g)) + \gamma_1^{-1}(\beta_1(f,\gamma_1(g)) + \gamma_1^{-1}(f\gamma_2(g)) \\
&+ \gamma_1^{-1}(\beta_1(\gamma_1(f),g)) + \gamma_1^{-1}(\gamma_1(f)\gamma_1(g)) + \gamma_1^{-1}(\gamma_2(f)g) \\
&+ \beta_3(f,g) + \beta_2(f,\gamma_1(g)) + \beta_1(f,\gamma_2(g)) + f\gamma_3(g) + \beta_2(\gamma_1(f),g) \\
&+ \beta_1(\gamma_1(f),\gamma_1(g)) + \gamma_1(f)\gamma_2(g) + \beta_1(\gamma_2(f),g) + \gamma_2(f)\gamma_1(g) \\
&+ \gamma_3(f)g)\epsilon^3
\end{aligned}
$$

Since $\mu'_\epsilon(f,g) = fg + \beta'_1(f,g)\epsilon + \beta'_2(f,g)\epsilon^2 + \beta'_3(f,g)\epsilon^3$, by matching the coefficients of ϵ with the above, we get:

$$
\begin{aligned}
\beta'_1(f,g) =& -\gamma_1(fg) + \beta_1(f,g) + f\gamma_1(g) + \gamma_1(f)g \\
\beta'_2(f,g) =& -\gamma_2(fg) + \gamma_1 \circ \gamma_1(fg) - \gamma_1(\beta_1(f,g)) - \gamma_1(f\gamma_1(g)) - \gamma_1(\gamma_1(f)g) \\
&+ \beta_2(f,g) + \beta_1(f,\gamma_1(g)) + f\gamma_2(g) + \beta_1(\gamma_1(f),g) + \gamma_1(f)\gamma_1(g) + \gamma_2(f)g \\
\beta'_3(f,g) =& -\gamma_3(fg) + \gamma_2 \circ \gamma_1(fg) + \gamma_1 \circ \gamma_2(fg) - \gamma_1 \circ \gamma_1 \circ \gamma_1(fg) - \gamma_2(\beta_1(f,g)) \\
&+ \gamma_1 \circ \gamma_1(\beta_1(f,g)) - \gamma_2(f\gamma_1(g)) + \gamma_1 \circ \gamma_1(f\gamma_1(g)) - \gamma_2(\gamma_1(f)g) \\
&+ \gamma_1 \circ \gamma_1(\gamma_1(f)g) - \gamma_1(\beta_2(f,g)) - \gamma_1(\beta_1(f,\gamma_1(g))) - \gamma_1(f\gamma_2(g)) \\
&- \gamma_1(\beta_1(\gamma_1(f),g)) - \gamma_1(\gamma_1(f)\gamma_1(g)) - \gamma_1(\gamma_2(f)g) + \beta_3(f,g) + \beta_2(f,\gamma_1(g)) \\
&+ \beta_1(f,\gamma_2(g)) + f\gamma_3(g) + \beta_2(\gamma_1(f),g) + \beta_1(\gamma_1(f),\gamma_1(g)) + \gamma_1(f)\gamma_2(g) \\
&+ \beta_1(\gamma_2(f),g) + \gamma_2(f)\gamma_1(g) + \gamma_3(f)g
\end{aligned}
$$

This gives us the formula (for $\epsilon^4 = 0$) of $\gamma \cdot \beta$ and is precisely the gauge action of γ on β. Rewrite this in terms of brackets we will get

$$
\begin{aligned}
\beta_1' &= \beta_1 - d\gamma_1 \\
\beta_2' &= \beta_2 - d\gamma_2 + [\gamma_1, \beta_1] - \frac{1}{2}[\gamma_1, d\gamma_1] \\
\beta_3' &= \beta_3 - d\gamma_3 + [\gamma_1, \beta_2] + [\gamma_2, \beta_1] + \frac{1}{2}[\gamma_1, [\gamma_1, \beta_1]] - \frac{1}{2}[\gamma_1, d\gamma_2] - \frac{1}{2}[\gamma_2, d\gamma_1] - \frac{1}{6}[\gamma_1, [\gamma_1, d\gamma_1]]
\end{aligned}
$$

Thus by moding away equivalent μ_ϵ's, we will get the set of associative structures on the deformation.

1.2.2 Deformation of Lie Algebra

Now suppose we have a Lie algebra $\mathfrak{g}$ with Lie bracket l. Again consider the deformation of the algebra by a local Artinian ring A as before. Similar to the associative case, the deformation will be governed by the DGLA $\mathcal{L} = L \otimes m$, where $L^k = \mathrm{Hom}(\wedge^{k+1}\mathfrak{g}, \mathfrak{g})$ is the kth Chevalley-Eilenberg complex. The differential of L is given by

$$
\begin{aligned}
(df)(a_0 \wedge \cdots \wedge a_{k+1}) = &\sum_p (-1)^p [a_p, f(a_0 \wedge \cdots \wedge \hat{a}_p \wedge \cdots \wedge a_{k+1})] \\
&+ \sum_{0 \le p < q \le k+1} (-1)^{p+q} f([a_p, a_q] \wedge \ldots \hat{a}_p \wedge \cdots \wedge \hat{a}_q \wedge \cdots \wedge a_{k+1}) \qquad f \in L^k
\end{aligned}
$$

and the Lie bracket is given by the Richardson-Nijenhuis bracket, which is

$$
[f_1, f_2] = i_{f_1} f_2 - (-1)^{k_1 k_2} i_{f_2} f_1, \qquad f_i \in L^{k_i}
$$

where i_{f_1} is defined as

$$
i_{f_1} f_2(a_0 \wedge \cdots \wedge a_{p+q}) = \sum_{\sigma \in \mathrm{Sh}_{q+1,p}} \mathrm{sgn}(\sigma) f_1(f_2(a_{\sigma(0)} \wedge \cdots \wedge a_{\sigma(q)}) \wedge a_{\sigma(q+1)} \wedge \cdots \wedge a_{\sigma(p+q)})
$$

where is sum is over all the $(q+1,p)$-shuffles, i.e. permutation σ of $\{0,\dots,p+q\}$ such that $\sigma(0) < \cdots < \sigma(q)$ and $\sigma(q+1) < \cdots < \sigma(p+q)$.

Consider $l_\epsilon = l + \beta$, $\beta \in L^1 \otimes m$, the Lie bracket in $\mathfrak{g} \otimes A$, l_ϵ satisfies the Jacobi identity if and only if

$$[l_\epsilon, l_\epsilon](a,b,c) = 2(l_\epsilon(l_\epsilon(a,b),c) + l_\epsilon(l_\epsilon(b,c),a) - l_\epsilon(l_\epsilon(a,c),b)) = 0$$

where $a,b,c \in \mathfrak{g}$ and the bracket is the Richardson-Nijenhuis bracket inherited from the Chevalley-Eilenberg complex and extended by k-linearity. Note that l_ϵ is skewsymmetric as $l_\epsilon \in \mathrm{Hom}(\wedge^2\mathfrak{g},\mathfrak{g}) \otimes m$. Now if we expand the bracket like the associative case, we get

$$[l_\epsilon, l_\epsilon] = [l+\beta, l+\beta] = [l,l] + 2[l,\beta] + [\beta,\beta]$$

Since l satisfies the Jacobi identity, $[l,l] = 0$. On the other hand $[l,\beta]$ gives us exactly the differential, $d\beta$ of β. So we get back (after dividing by 2) the Maurer Cartan equation on $\mathcal{L}$ and thus we conclude that l_ϵ satisfies the Jacobi identity if and only if β satisfies the Maurer Cartan equation on $\mathcal{L}$.

We will now work out the case where $A = \mathbb{K}[\epsilon]/\epsilon^4$. For $a, b \in \mathfrak{g}$, we have

$$l_\epsilon(a,b) = l(a,b) + \beta_1(a,b)\epsilon + \beta_2(a,b)\epsilon^2 + \beta_3(a,b)\epsilon^3$$

Now apply the Maurer Cartan equation to $a, b, c \in \mathfrak{g}$, we have:

$$
\begin{aligned}
(d\beta + \frac{1}{2}[\beta, \beta])(a, b, c) =& (d\beta + \beta \bullet \beta)(a, b, c) \\
=& d\beta(a, b, c) + \beta \bullet \beta(a, b, c) \\
=& d\beta(a, b, c) + \beta(\beta(a, b), c) + \beta(\beta(b, c), a) + \beta(\beta(a, c), b) \\
=& l(a, \beta(b, c)) - l(b, \beta(a, c)) + l(c, \beta(a, b)) - \beta(l(a, b), c) \\
& - \beta(l(b, c), a) + \beta(l(a, c), b) + \beta(\beta(a, b), c) + \beta(\beta(b, c), a) \\
& + \beta(\beta(a, c), b) \\
=& l(a, \beta_1(b, c))\epsilon + l(a, \beta_2(b, c))\epsilon^2 + l(a, \beta_3(b, c))\epsilon^3 \\
& - l(b, \beta_1(a, c)) - l(b, \beta_2(a, c))\epsilon^2 - l(b, \beta_3(a, c))\epsilon^3 \\
& + l(c, \beta_1(a, b))\epsilon + l(c, \beta_2(a, b))\epsilon^2 + l(c, \beta_3(a, b))\epsilon^3 \\
& - \beta_1(l(a, b), c)\epsilon - \beta_2(l(a, b), c)\epsilon^2 - \beta_3(l(a, b), c)\epsilon^3 \\
& - \beta_1(l(b, c), a)\epsilon - \beta_2(l(b, c), a)\epsilon^2 - \beta_3(l(b, c), a)\epsilon^3 \\
& + \beta_1(l(a, c), b)\epsilon + \beta_2(l(a, c), b)\epsilon^2 + \beta_3(l(a, c), b)\epsilon^3 \\
& + \beta_1(\beta_1(a, b), c)\epsilon^2 + \beta_1(\beta_2(a, b), c)\epsilon^3 + \beta_2(\beta_1(a, b), c)\epsilon^3 \\
& + \beta_1(\beta_1(b, c), a)\epsilon^2 + \beta_1(\beta_2(b, c), a)\epsilon^3 + \beta_2(\beta_1(b, c), a)\epsilon^3 \\
& + \beta_1(\beta_1(a, c), b)\epsilon^2 + \beta_1(\beta_2(a, c), b)\epsilon^3 + \beta_2(\beta_1(a, c), b)\epsilon^3
\end{aligned}
$$

Group the terms according to power of ϵ, we get:

$$\begin{aligned}
\epsilon : 0 =& l(a,\beta_1(b,c)) - l(b,\beta_1(a,c)) + l(c,\beta_1(a,b)) \\
&- \beta_1(l(a,b),c) - \beta_1(l(b,c),a) + \beta_1(l(a,c),b) \\
\epsilon^2 : 0 =& l(a,\beta_2(b,c)) - l(b,\beta_2(a,c)) + l(c,\beta_2(a,b)) \\
&- \beta_2(l(a,b),c) - \beta_2(l(b,c),a) + \beta_2(l(a,c),b) \\
&+ \beta_1(\beta_1(a,b),c) + \beta_1(\beta_1(b,c),a) + \beta_1(\beta_1(a,c),b) \\
\epsilon^3 : 0 =& l(a,\beta_3(b,c)) - l(b,\beta_3(a,c)) + l(c,\beta_3(a,b)) \\
&- \beta_3(l(a,b),c) - \beta_3(l(b,c),a) + \beta_3(l(a,c),b) \\
&+ \beta_1(\beta_2(a,b),c) + \beta_1(\beta_2(b,c),a) + \beta_1(\beta_2(a,c),b) \\
&+ \beta_2(\beta_1(a,b),c) + \beta_2(\beta_1(b,c),a) + \beta_2(\beta_1(a,c),b)
\end{aligned}$$

Rewrite these in terms of the differential and bracket, we get

$$\begin{aligned}
&d\beta_1 = 0 \\
&d\beta_2 + \frac{1}{2}[\beta_1,\beta_1] = 0 \\
&d\beta_3 + [\beta_1,\beta_2] = 0
\end{aligned}$$

The equivalence solutions are computed exactly as in the associative case with μ replaced by l and γ an automorphism from $\mathfrak{g}\otimes A \to \mathfrak{g}\otimes A$. An easy check shows that we again have

$$\begin{aligned}
&\beta_1' = \beta_1 - d\gamma_1 \\
&\beta_2' = \beta_2 - d\gamma_2 + [\gamma_1,\beta_1] - \frac{1}{2}[\gamma_1,d\gamma_1] \\
&\beta_3' = \beta_3 - d\gamma_3 + [\gamma_1,\beta_2] + [\gamma_2,\beta_1] + \frac{1}{2}[\gamma_1,[\gamma_1,\beta_1]] - \frac{1}{2}[\gamma_1,d\gamma_2] - \frac{1}{2}[\gamma_2,d\gamma_1] - \frac{1}{6}[\gamma_1,[\gamma_1,d\gamma_1]]
\end{aligned}$$

1.2.3 Deformation of Modules over Algebras

Suppose we have a module M over an associative algebra B with B action $\cdot : B \otimes M \to M$. Again consider the deformation of the module by a local Artinian ring A as before. The deformation is governed by the DGLA $\mathcal{L} = L \otimes m$, where $L^k = \mathrm{Hom}(B^{\otimes k} \otimes M, M)$ is the kth Hochschild complex. The differential of L is given by

$$\begin{aligned}(df)(r_1 \otimes \cdots \otimes r_{k+1} \otimes m) =& r_1 \cdot f(r_2 \otimes \cdots \otimes r_{k+1} \otimes m)\\ &+ \sum_{i=1}^{k}(-1)^i f(r_1 \otimes \cdots \otimes (r_i \cdot r_{i+1}) \otimes \cdots \otimes r_{k+1} \otimes m)\\ &+ (-1)^{k+1} f(r_1 \otimes \cdots \otimes r_{k+1} \cdot m), \qquad f \in L^k\end{aligned}$$

and the Lie bracket is given by Gerstenhaber bracket, which is

$$[f_1, f_2] = f_1 \circ f_2 - (-1)^{k_1 k_2} f_2 \circ f_1, \qquad f_i \in L^{k_i}$$

where the (non-associative) product $\circ$ is defined as

$$(f_1 \circ f_2)(r_1 \otimes \cdots \otimes r_{k_1+k_2} \otimes m) = (-1)^{k_1 k_2 + 1} f_1(r_1 \otimes \cdots \otimes r_{k_1} \otimes f_2(r_{k_1+1} \otimes \cdots \otimes r_{k_1+k_2} \otimes m))$$

Consider $\cdot_\epsilon = \cdot + \beta$, $\beta \in L_1 \otimes m$, a new B action on $M \otimes A$. We will now examine the condition $r_1 \cdot_\epsilon (r_2 \cdot_\epsilon m) - (r_1 r_2) \cdot_\epsilon m = 0$.

$$\begin{aligned}r_1 \cdot_\epsilon (r_2 \cdot_\epsilon m) - (r_1 r_2) \cdot_\epsilon m =& r_1 \cdot_\epsilon (r_2 m + \beta(r_2, m)) - (r_1 r_2) m \cdot - \beta(r_1 r_2, m)\\ =& r_1 r_2 m + r_1 \beta(r_2, m) + \beta(r_1, r_2 m) + \beta(r_1, \beta(r_2, m))\\ &- (r_1 r_2) m \cdot - \beta(r_1 r_2, m)\\ =& r_1 \beta(r_2, m) + \beta(r_1, r_2 m) + \beta(r_1, \beta(r_2, m)) - \beta(r_1 r_2, m)\\ =& d\beta + \frac{1}{2}[\beta, \beta]\end{aligned}$$

Thus we get $r_1 \cdot_\epsilon (r_2 \cdot_\epsilon m) - (r_1 r_2) \cdot_\epsilon m = 0$ if and only if β satisfies the Maurer Cartan equation on $\mathcal{L}$.

We will now work out the case where $A = \mathbb{K}[\epsilon]/\epsilon^4$.

$$
\begin{aligned}
r_1 \cdot_\epsilon (r_2 \cdot_\epsilon m) - (r_1 r_2) \cdot_\epsilon m =& r_1 \cdot_\epsilon (r_2 m + \beta_1(r_2, m)\epsilon + \beta_2(r_2, m)\epsilon^2 + \beta_3(r_2, m)\epsilon^3) \\
&- (r_1 r_2) \cdot_\epsilon m \\
=& r_1 r_2 m + \beta_1(r_1, r_2 m)\epsilon + \beta_2(r_1, r_2 m)\epsilon^2 + \beta_3(r_1, r_2 m)\epsilon^3 \\
&+ r_1 \beta_1(r_2, m)\epsilon + \beta_1(r_1, \beta_1(r_2, m))\epsilon^2 + \beta_2(r_1, \beta_1(r_2, m))\epsilon^3 \\
&+ r_1 \beta_2(r_2, m)\epsilon^2 + \beta_1(r_1, \beta_2(r_2, m))\epsilon^3 + r_1 \beta_3(r_2, m)\epsilon^3 \\
&- r_1 r_2 m - \beta_1(r_1 r_2, m)\epsilon - \beta_2(r_1 r_2, m)\epsilon^2 - \beta_3(r_1 r_2, m)\epsilon^3
\end{aligned}
$$

If we simplify the terms and match the coefficients of ϵ, we get the following sets of equations:

$$
\begin{aligned}
\epsilon :& \beta_1(r_1, r_2 m) + r_1 \beta_1(r_2, m) - \beta_1(r_1 r_2, m) = 0 \\
\epsilon^2 :& \beta_2(r_1, r_2 m) + \beta_1(r_1, \beta_1(r_2, m)) - \beta_2(r_1 r_2, m) = 0 \\
\epsilon^3 :& \beta_3(r_1, r_2 m) + \beta_2(r_1, \beta_1(r_2, m)) + \beta_1(r_1, \beta_2(r_2, m)) + r_1 \beta_3(r_2, m) - \beta_3(r_1 r_2, m) = 0
\end{aligned}
$$

Rewrite these in terms of the differential and bracket, we get

$$
\begin{aligned}
& d\beta_1 = 0 \\
& d\beta_2 + \frac{1}{2}[\beta_1, \beta_1] = 0 \\
& d\beta_3 + [\beta_1, \beta_2] = 0
\end{aligned}
$$

Now suppose we have an automorphism γ of $M \otimes A$. We can define $\cdot'_\epsilon$ on $M \otimes A$ by slightly altering the way we define μ'_ϵ

$$
r \cdot'_\epsilon m = \gamma^{-1}(r \cdot_\epsilon \gamma(m))
$$

Continue to do the computation similar to the associative algebra case and we will again get

$$
\begin{aligned}
\beta_1' &= \beta_1 - d\gamma_1 \\
\beta_2' &= \beta_2 - d\gamma_2 + [\gamma_1, \beta_1] - \frac{1}{2}[\gamma_1, d\gamma_1] \\
\beta_3' &= \beta_3 - d\gamma_3 + [\gamma_1, \beta_2] + [\gamma_2, \beta_1] + \frac{1}{2}[\gamma_1, [\gamma_1, \beta_1]] - \frac{1}{2}[\gamma_1, d\gamma_2] - \frac{1}{2}[\gamma_2, d\gamma_1] - \frac{1}{6}[\gamma_1, [\gamma_1, d\gamma_1]]
\end{aligned}
$$

Chapter 2

L_∞ Algebras and Homotopy Transfer of Structure Theorem

In chapter 1, we have examined how the Maurer Cartan solutions of a DGLA gives the deformations of a few simple algebraic objects. However, in certain situations, DGLAs are not sufficient or not the most suitable object to describe the deformations. In this chapter, we will define L_∞ algebras which generalize DGLAs and show how we can obtain an L_∞ structure on a differential graded vector space through the homotopy transfer of structure theorem. We will also state the formal Kuranishi theorem which tells us how the Maurer Cartan elements behave under homotopy transfer.

2.1 L_∞ Algebras and L_∞ morphisms

There are a few ways to define L_∞ structures on a graded vector space V. One way is to define them using higher Lie brackets that satisfy the general Jacobi identities. Another way is to define L_∞ structures as codifferentials on the symmetric coalgebra $\overline{S(V[1])}$. Both of these definitions are equivalent [1], but in this thesis we will use the coalgebra definition as it will make the proof of homotopy transfer of structure theorem more intuitive.

Definition 2.1. *A (counital coassociative) coalgebra over a field $\mathbb{K}$ (or a $\mathbb{K}$-coalgebra) is a vector space C over $\mathbb{K}$ together with $\mathbb{K}$-linear maps $\Delta : C \to C \otimes C$ and $\epsilon : C \to \mathbb{K}$ such that*

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$$

and

$$(\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id} = (\epsilon \otimes \mathrm{id}) \circ \Delta$$

Notice that coalgebras are dual to algebras in the categorical sense. The map Δ is called the comultiplication of C and the map ϵ is called the counit of C. The first equality is the dual of associativity of algebraic multiplication, which is called the coassociativity of the comultiplication. The second equality is the dual of existence of multiplicative identity.

Definition 2.2. *Given a (counital coassociative) $\mathbb{K}$-coalgebra (C, Δ, ϵ), a* ***coderivation*** *is a $\mathbb{K}$-module map $d : C \to C$ satisfying the co-Leibniz rule*

$$\Delta \circ d = (d \otimes \mathrm{id} + \mathrm{id} \otimes d) \circ \Delta : C \to C \otimes C$$

A ***codifferential*** *is a coderivation that satisfies $d \circ d = 0$.*

The main example of coalgebra we will consider in this thesis will be the reduced symmetric coalgebra.

Definition 2.3. *Let V, W be a graded vector spaces over $\mathbb{K}$. The* ***twisting map*** *$T : V \otimes W \to W \otimes V$ is defined by*

$$T(v \otimes w) = (-1)^{|v||w|} w \otimes v$$

for every pair of homogeneous elements $v \in V$ and $w \in W$.

Definition 2.4. *Let V be a graded vector space over $\mathbb{K}$. The **tensor algebra** generated by V is*

$$T(V) = \bigoplus_{n \geq 0} \bigotimes^n V$$

*endowed with an associated product $(v_1 \otimes \cdots \otimes v_p)(v_{p+1} \otimes \cdots \otimes v_n) = (v_1 \otimes \cdots \otimes v_n)$. Let $I \subset T(V)$ be the homogeneous ideal generated by the elements $x \otimes y - T(x \otimes y)$, $x, y \in V$. The **symmetric algebra** generated by V is the quotient*

$$S(V) = T(V)/I = \bigoplus_{n \geq 0} \bigodot^n V, \qquad \bigodot^n V = \bigotimes^n V/(\bigotimes^n V \bigcap I)$$

The product in $S(V)$ is denoted by $\odot$. If $\pi : T(V) \to S(V)$ is the projection to the quotient, $\pi(v_1 \otimes \cdots \otimes v_n) = v_1 \odot \cdots \odot v_n \, \forall v_1, \ldots v_n \in V$. On the other hand, we have an injection $\mathbf{i}$ that maps $v_1 \odot \cdots \odot v_n$ to the symmetric tensors.

Definition 2.5. *The **reduced tensor coalgebra** is defined as*

$$\overline{T(V)} = \bigoplus_{n > 0} \bigotimes^n V$$

and is an ideal generated by V in $T(V)$. The coassociative coproduct is defined as $\Delta : \overline{T(V)} \to \overline{T(V)} \otimes \overline{T(V)}$,

$$\Delta = \sum_{n=1}^{\infty} \sum_{a=1}^{n-1} \Delta_{a,n-a}, \qquad \Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{r=1}^{n-1} (v_1 \otimes \cdots \otimes v_r) \otimes (v_{r+1} \otimes \cdots \otimes v_n)$$

*The **reduced symmetric coalgebra** is defined as*

$$\overline{S(V)} = \bigoplus_{n > 0} \bigodot^n V$$

and the coassociative coproduct is given by Δ on the symmetric tensors.

Definition 2.6. *Let V be a graded vector space; a codifferential Q of degree 1 on the (reduced) symmetric coalgebra $\overline{S(V[1])}$ is called an $\mathbf{L}_\infty$* ***structure*** *on V. The graded vector space V together with an L_∞ structure Q on V is called an $\mathbf{L}_\infty$* ***algebra****.*

Note that the codifferential Q is determined by $Q^1 : \overline{S(V[1])} \to V$, see Corollary VIII.34 in [10]. If we break the map apart, we will get maps $q_1 = Q^1_1 : V \to V$ (which squares to zero), $q_2 = Q^1_2 : V \odot V \to V$, $q_3 = Q^1_3 : V \odot V \odot V \to V$, etc. In particular, we have

$$Q(v_1 \odot \cdots \odot v_n) = \sum_{i=1}^{n-1} \sum_{\sigma \in \mathrm{Sh}(i,n-i)} \mathrm{sgn}(\sigma) q_i(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)}) \odot v_{\sigma(i+1)} \odot \cdots \odot v_{\sigma(n)}.$$

The map q_1 corresponds to the differential on $V[1]$; the map q_2 corresponds to the bracket on $V[1]$; the maps q_n, $n \geq 3$, corresponds to the higher brackets on $V[1]$. The fact that $Q^2 = 0$ implies that the q_i's have to satisfy a series of equations, which are called the general Jacobi identities.

Definition 2.7. *Let $Q = (q_1, q_2, \dots)$ be an L_∞ structure on V, then the complex $(V[1], q_1)$ is called the* ***tangent complex*** *of (V, Q).*

Definition 2.8. *An $\mathbf{L}_\infty$* ***morphism*** *(weak morphism) $F : (V, Q) \to (W, R)$ between L_∞ algebra is a morphism of dg-coalgebra $F : \overline{S(V[1])} \to \overline{S(W[1])}$ which is given by a family of degree zero maps $f_i = F^1_i : V[1]^{\odot i} \to W[1]$, $i \geq 1$, such that $F : \overline{S(V[1])} \to \overline{S(W[1])}$ commutes with Q and R.*

$$\begin{aligned} &F(v_1 \odot \cdots \odot v_n) \\ =&\sum_{k=1}^{n} \frac{1}{k!} \sum_{i_1+\cdots+i_k=n} \sum_{\sigma \in \mathrm{Sh}(i_1,\dots,i_k)} \mathrm{sgn}(\sigma) f_{i_1}(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i_1)}) \odot \cdots \odot f_{i_k}(v_{\sigma(n-i_k+1)} \odot \cdots \odot v_{\sigma(n)}). \end{aligned}$$

An L_∞ morphism F is called a ***strict morphism*** *if F is defined by $f_1 : V \to W$, i.e. $F = \overline{S(f_1)}$, $f_i = 0$ for $i \geq 2$. Note that since F commutes with Q and R, f_i's have to satisfy* *a series of compatibility equations involving the differentials and (higher) brackets on V and W.*

Definition 2.9. *A morphism $F = (f_1, f_2, \dots) : (V, Q) \to (W, R)$ of L_∞ algebras is a **weak equivalence ($\mathbf{L}_\infty$ quasi-isomorphism)** if the chain map $f_1 : (V[1], q_1) \to (W[1], r_1)$ is a quasi-isomorphism between the tangent complexes.*

Note that the coalgebra definition and the higher bracket definition of L_∞ algebras are related by the décalage isomorphism $d\acute{e}c : \bigodot^n(V[1]) \cong (\bigwedge^n V)[n]$. In particular, a DGLA $(L, d, [-,-])$ can be regarded as an L_∞ algebra with the coderivative $q_1(l) = -d(l)$, $q_2(l_1, l_2) = (-1)^{|l_1|}[l_1, l_2]$, $q_i = 0$ for $i \geq 3$, and $f : L \to M$ is a strict L_∞ morphism iff it is a morphism of DGLA.

Before moving on, we should stated that from now on we shall work in a complete setting. Completeness is not needed for the homotopy transfer of structure theorem, but it is essential in the proof of the formal Kuranishi theorem. Since the only homotopy transfer of structure we will need is between the polynomial differential forms and the cochain complexes with coefficients in complete L_∞ algebras (note the polynomial differential forms in general are not complete, but we can replace them with its complete version, see [1]), working in only a complete setting is not a problem for the purpose of this thesis.

Definition 2.10. *A **complete graded space** is a graded space V equipped with a descending filtration $F^\bullet V$,*

$$V = F^1 V \supset \cdots \supset F^p V \supset \dots$$

*such that V is complete in the induced topology, i.e. the natural $V \to \varprojlim V/F^\bullet V$ is an isomorphism of graded spaces. Given complete graded spaces $(W, F^\bullet W)$ and $(V, F^\bullet V)$, a **continuous** map of graded spaces is a map $f : W \to V$ such that $f(F^p W) \subset F^p V$ for all $p \geq 1$.*

Definition 2.11. *A **complete dg space** $(V, F^\bullet V, d)$ is a complete graded space $(V, F^\bullet V)$ together with a continuous differential d.*

Definition 2.12. *A **complete** $\mathbf{L}_\infty$ **algebra** is a complete graded space $(V, F^\bullet V)$ together with an L_∞ structure Q on V such that q_i's are continuous for the induced topology, i.e. $q_i(F^{p_1}V[1] \odot \cdots \odot F^{p_i}V[1]) \subset F^{p_1+\cdots+p_i}V[1]$, for all $i, p_1, \ldots, p_i \geq 1$.*

Definition 2.13. *A **continuous** $\mathbf{L}_\infty$ **morphism** $F : (W, F^\bullet W, R) \to (V, F^\bullet V, Q)$ between complete L_∞ algebras $(W, F^\bullet W, R)$ and $(V, F^\bullet V, Q)$ is an L_∞ morphism $(W, R) \to (V, Q)$ such that f_i's are continuous, i.e. $f_i(F^{p_1}W[1] \odot \cdots \odot F^{p_i}W[1]) \subset F^{p_1+\cdots+p_i}V[1]$, for all $i, p_1, \ldots, p_i \geq 1$.*

Remark. *From now on in this thesis, we will assume completeness useless otherwise specified.*

Now we can define the Maurer Cartan set of a complete L_∞ algebra in a similar fashion to the Maurer Cartan set of a DGLA.

Definition 2.14. *Given a complete L_∞ algebra $(V, F^\bullet V, Q)$, its **curvature** is the map of sets $\mathcal{R}_V : V^1 \to V^2$ given by*

$$\mathcal{R}_V(x) = \sum_{i \geq 1} \frac{1}{i!} q_i(\overbrace{x \odot \cdots \odot x}^{i}) \qquad \forall x \in V^1$$

Note that the infinite sum above converges because $(V, F^\bullet V, Q)$ is complete.

Definition 2.15. *The **Maurer Cartan set** of a complete L_∞ algebra $(V, F^\bullet V, Q)$ is the set*

$$\mathrm{MC}(V) := \{x \in V^1 \ s.t. \ \mathcal{R}_V(x) = 0\}.$$

Given a continuous L_∞ morphism $F : (W, F^\bullet W, R) \to (V, F^\bullet V, Q)$ of complete L_∞ algebras, the associated morphism of Maurer Cartan sets is given by $\mathrm{MC}(F) := F_*|_{\mathrm{MC}(W)} : \mathrm{MC}(W) \to \mathrm{MC}(V)$ where

$$F_*(x) = \sum_{i \geq 1} \frac{1}{i!} f_i(\overbrace{x \odot \cdots \odot x}^{i}) \qquad \forall x \in W^1.$$

An easy computation [2] shows that $\mathrm{MC}(F)$ does in fact map Maurer Cartan set to Maurer Cartan set.

Notice that gauge equivalence is not defined on Maurer Cartan solutions of an L_∞ algebra L as L^0 is not a Lie algebra in general. Instead, we will use the following equivalence for Maurer Cartan solutions of L_∞ algebras:

Definition 2.16. *Two Maurer Cartan solutions $a, a' \in \mathrm{MC}(V)$ are **(homotopy) equivalent** if there exist $z \in \mathrm{MC}(V \otimes \mathbb{K}[s, ds])$ such that*

$$z|_{s=0} = a, \qquad\qquad z|_{s=1} = a'$$

where the evaluation map is given by $\mathrm{Eval}_{s=s_0} : V \otimes \mathbb{K}[s, ds] \to V$ is given by

$$\mathrm{Eval}_{s=s_0}(x(s) + y(s)ds) = x(s_0)$$

It may not seem obvious at first, but in the case where V is a DGLA this definition is exactly the same as the gauge equivalence.

Theorem 2.17 ([9])**.** *For a DGLA L, gauge equivalence on $\mathrm{MC}(L)$ is exactly the same as homotopy equivalence on $\mathrm{MC}(L)$.*

The advantage of using homotopy equivalence instead of gauge equivalence is that we can define this on L_∞ algebras (or even curved L_∞ algebras) and thus provides a more general version of equivalence on Maurer Cartan solutions.

2.2 Homotopy Transfer and Formal Kuranishi Theorem

In this section we will review the homotopy transfer of structure theorem and observe how the Maurer Cartan set behave under homotopy transfer.

Definition 2.18. *A **complete contraction***

$$W \underset{g}{\overset{f}{\rightleftarrows}} V \circlearrowleft K$$

is a complete dg space $(V, F^\bullet V, d_V)$ and a dg space (W, d_W), together with dg morphisms $f : (W, d_W) \to (V, d_V)$, $g : (V, d_V) \to (W, d_W)$ and a contracting (degree minus one) homotopy $K : V \to V$, such that

- *g is a left inverse to f, that is, $gf = \mathrm{id}_W$*
- *K is a homotopy between fg and id_V, that is, $Kd_V + d_V K = fg - \mathrm{id}_V$*
- *K satisfies the side conditions $Kf = K^2 = gK = 0$*
- *K and fg are continuous with respect to the filtration $F^\bullet V$ on V.*

In this case W is equipped with the induced filtration $F^p W = f^{-1}(F^p V)$. The last condition ensures that (W, d_W) is complete, and f, g are continuous morphisms.

We can now state the homotopy transfer of structure theorem, which we cite directly from [1]. Using this theorem, we can easily obtain an L_∞ structure on a differential graded space.

Theorem 2.19. *Given a complete contraction* $W[1] \underset{g_1}{\overset{f_1}{\rightleftarrows}} V[1] \circlearrowleft K$ *and a complete* L_∞ *algebra structure* Q *on* $(V, F^\bullet V)$ *with linear part* $q_1 = d_{V[1]}$*, there is an induced complete* L_∞ *algebra structure* R *on* $(W, F^\bullet W)$ *with linear part* $r_1 = d_{W[1]}$*, together with continuous* L_∞ *morphisms* $F : (W, R) \to (V, Q)$*,* $G : (V, Q) \to (W, R)$ *with linear parts* f_1*,* g_1 *respectively. Denoting by* F_i^k *the composition* $W[1]^{\odot i} \hookrightarrow \overline{S(W[1])} \xrightarrow{F} \overline{S(V[1])} \twoheadrightarrow V[1]^{\odot k}$*,* F *and* R *are determined recursively by*

$$f_i = \sum_{k=2}^{i} K q_k F_i^k \qquad \text{for } i \geq 2,$$

$$r_i = \sum_{k=2}^{i} g_1 q_k F_i^k \qquad \text{for } i \geq 2.$$

We denote by $K_i^\Sigma : V[1]^{\odot i} \to V[1]^{\odot i}$ *the degree minus one map defined by*

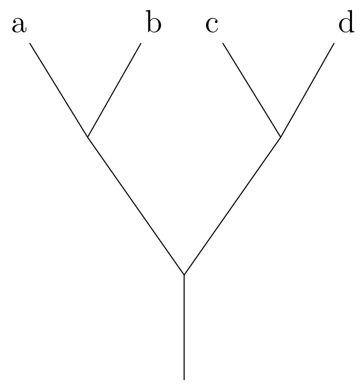
$$K_i^\Sigma(v_1 \odot \cdots \odot v_i) = \frac{1}{i!} \sum_{\sigma \in S_i} \sum_{j=1}^{i} \pm f_1 g_1(v_{\sigma(1)}) \odot \cdots \odot f_1 g_1(v_{\sigma(j-1)}) \odot K(v_{\sigma(j)}) \odot v_{\sigma(j+1)} \odot \cdots \odot v_{\sigma(i)},$$

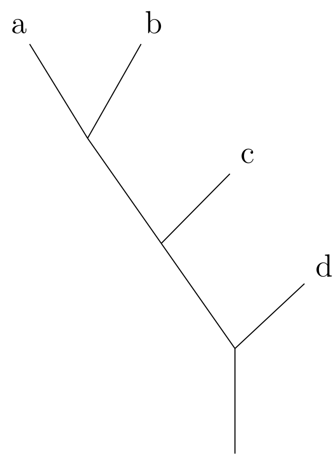
where $\pm$ *is the appropriate Koszul sign (taking into account that* $|K| = -1$*). Denoting by* Q_i^k *the composition* $V[1]^{\odot i} \hookrightarrow \overline{S(V[1])} \xrightarrow{Q} \overline{S(V[1])} \twoheadrightarrow V[1]^{\odot k}$*, the* L_∞ *morphism* G *is determined recursively by*

$$g_i = \sum_{k=1}^{i-1} g_k Q_i^k K_i^\Sigma \qquad \text{for } i \geq 2.$$

Essentially, the brackets of W is given by a combinatorical formula involving summation over rooted trees. In the case where the V is a complete DGLA, i.e. $q_i = 0$ for $i \geq 3$, we start by lifting the elements of W into V through f_1, take the brackets in V and connect the branches using the homotopy K, and send the resulting element back to W through g_1.

For example, the 4-bracket in W, $[a, b, c, d]$, is given by the following combinatoric root trees (up to permutation on a,b,c,d)





The nodes on the trees represents bracketing in V, the edges connecting the leaves represents the map $f_1 : W \to V$, the edge connecting the root represents the map $g_1 : V \to W$, and the edges connecting between nodes represents the homotopy map $K : V \to V$.

The following theorem is called formal Kuranishi theorem by Bandiera. It tells us what happens to the Maurer Cartan set after a homotopy transfer and gives us a bijection between a subset of our original Maurer Cartan set and the new Maurer Cartan set after the transfer, which is essential to the proof of the main theorem of this thesis. The proof of this theorem is essentially due to Getzler [7], and Bandiera stated the theorem explicitly in its current form in his paper [1].

Theorem 2.20. *Under the hypothesis of the homotopy transfer of structure theorem, the correspondence*

$$\rho : \mathrm{MC}(V) \to \mathrm{MC}(W) \times K(V^1) : x \to (\mathrm{MC}(G)(x), K(x))$$

is bijective. The inverse ρ^{-1} admits the following recursive construction: given $y \in \mathrm{MC}(W)$ and $K(v) \in K(V^1)$, we define a succession of elements $x_n \in V^1$, $n \geq 0$, by $x_0 = 0$ and

$$x_{n+1} = f_1(y) - q_1 K(v) + \sum_{i \geq 2} \frac{1}{i!} (K q_i - f_1 g_i) \left(x_n^{\odot i} \right).$$

This succession converges (with respect to the complete topology induced by the filtration on V) to a well defined $x \in V^1$, and we have $\rho^{-1}(y, K(v)) = x$. Finally, $\rho^{-1}(-, 0) = \mathrm{MC}(F) : \mathrm{MC}(W) \to \mathrm{MC}(V)$ is a bijective correspondence between the sets $\mathrm{MC}(W)$ and $\mathrm{Ker}\ K \bigcap \mathrm{MC}(V)$, whose inverse is the restriction of g_1.

Chapter 3

Deligne-Getzler ∞-Groupoid and Baker-Campbell-Hausdorf Product

Suppose we have a nilpotent Lie algebra $\mathfrak{g}$, we have a bijection between $\mathfrak{g}$ and the corresponding Lie group G and thus a Baker-Campbell-Hausdorff product for $\mathfrak{g}$. Our natural question is then to ask what is the generalization of the Lie group G and the Baker-Campbell-Hausdorf product when we have a nilpotent L_∞ algebra L. For a nilpotent DGLA L, L^0 is a nilpotent Lie algebra and thus we can obtain a Baker-Campbell-Hausdorff product for L^0 from e^{L^0}. However, in the case where L is an L_∞ algebra, L^0 is not a Lie algebra as the bracket does not satisfy the Jacobi identity. We need an object that generalizes the Lie group G for an L_∞ algebra L while getting back our usual Baker-Campbell-Hausdorff product when L is just a Lie algebra or DGLA. It turns out that the natural object to consider will be Kan complexes.

In his paper [7], Getzler shows us how to integrate a nilpotent DGLA (L_∞ algebras) to an ∞-groupoid (Kan complex), which generalizes the way a nilpotent Lie algebra integrates to its exponential group. General Baker-Campbell-Hausdorff product can then be seen as the horn filling of Kan complex. A first model for such ∞-groupoid, $\mathrm{MC}_\infty(L) := \mathrm{MC}(\Omega^*(\Delta_\bullet; L))$, was introduced by Sullivan and studied in depth by Hinich [8]. The problem with $\mathrm{MC}_\infty(L)$ is that $\mathrm{MC}_\infty(L)$ is way larger than what we needed (and it is not a ∞-groupoid in a strict sense [2] but it is irrelevant for the purpose of this thesis). In the case where $\mathfrak{g}$ is a nilpotent Lie algebra, the nerve $\mathcal{N}(e^{\mathfrak{g}})$ is only a deformation retraction of $\mathrm{MC}_\infty(\mathfrak{g})$. Getzler introduced a smaller model $\gamma_\bullet$ that is homotopy equivalent to $\mathrm{MC}_\infty(L)$ as a Kan complex to solve this problem [7]. Bandiera rewrites $\gamma_\bullet$ as $\mathrm{Del}_\infty(L) := \mathrm{MC}(C^*(\Delta_\bullet; L))$ using the formal Kuranishi theorem [1], which is the notation we are going to use in this thesis.

3.1 ∞ Structure on Cochain Complexes

3.1.1 Groupoids and ∞-groupoids

We start the section by defining simplicial sets and Kan complexes. Kan complexes provide an important combinatoric tool to study homotopy theory. Unlike the simplicial sets, Kan complexes provides an extension condition that is analogous to the extension property in topology and will be the object we use to generalize Lie groups and thus Baker-Campbell-Hausdorff product for a DGLA (L_∞ algebra).

Let Δ be the simplex category which has ordinals $[n] = (0 < 1 < \cdots < n)$ as objects and has non decreasing maps as morphisms. It is generated by the face maps $d_k : [n-1] \to [n]$, $0 \le k \le n$, which are injective maps

$$d_k(i) = \begin{cases} i & i < k \\ i+1 & i \ge k \end{cases}$$

and the degeneracy maps $s_k : [n] \to [n-1]$, $0 \leq k \leq n-1$, which are surjective maps

$$s_k(i) = \begin{cases} i & i \leq k \\ i-1 & i > k \end{cases}$$

The face and degeneracy maps satisfy the following simplicial identities:

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i, & i < j \\
d_i s_j &= s_{j-1} d_i, & i < j \\
d_i s_j &= id, & i = j \text{ or } i = j+1 \\
d_i s_j &= s_j d_{i-1}, & i > j+1 \\
s_i s_j &= s_{j+1} s_i, & i \leq j
\end{aligned}$$

Definition 3.1. *A simplicial set X is a contravariant functor from Δ to the category of sets. This gives us a sequence of sets $X_n = X([n])$ indexed by the natural numbers $n \in \{0, 1, 2, \dots\}$, and the maps*

$$\begin{aligned}
\delta_k &= X(d_k) : X_n \to X_{n-1}, & 0 \leq k \leq n \\
\sigma_k &= X(s_k) : X_{n-1} \to X_n, & 0 \leq k \leq n
\end{aligned}$$

satisfying the simplicial identities.

Given a category C, we denote the opposite category C^{op}. Given a small category S, we denote by C^S the category of functors $S \to C$. Using this notation, the category of simplicial sets can be denoted by $\text{SSet} = \text{Set}^{\Delta^{\text{op}}}$. We can define simplicial objects in other categories by taking the contravariant functor from Δ to that particular category and denote them using similar notations.

Now before we can define the Kan complex, we need to define the horn.

Definition 3.2. *Let* $\Delta_n = \Delta(-, [n]) \in \mathrm{SSet}$ *be the standard simplicial* n*-simplex in* SSet*. For* $0 \leq i \leq n$*, let* $\Lambda_n^i \subset \Delta_n$ *be simplicial set defined as the union of the faces* $d_k[\Delta_{n-1}] \subset \Delta_n$*,* $k \neq i$*. An* ***n-horn*** *in* X *is a simplicial map from* Λ_n^i *to* X

Definition 3.3. *The simplicial object* X *satisfies the* ***Kan condition*** *if any morphism of n-horn can be extended to a simplicial morphism* $\Delta_n \to X$*. Such* X *is called a* ***Kan complex****.*

Now let's define groupoids.

Definition 3.4. *A* ***groupoid*** *is a small category (i.e. the collection of objects and morphisms are actual sets) in which every morphism is an isomorphism. More precisely, a groupoid* G *is:*

• *A set* G_0 *of objects;*

• *For each pair of objects* x *and* y *in* G_0*, there exists a (possibility empty) set* $G(x, y)$ *of morphisms from* x *to* y*. An element* $f \in G(x, y)$ *is denoted* $f : x \to y$*;*

• *For every object* x*, we have an identity element* id_x *of* $G(x, x)$*;*

• *For every triple objects* x*,*y*, and* z*, we have the composition function*

$$\mathrm{comp}_{x,y,z} : G(x, y) \times G(y, z) \to G(x, z) : (g, f) \mapsto gf;$$

• *For every pair of object* x *and* y*, we have an inverse function*

$$\mathrm{inv} : G(x, y) \to G(y, x) : f \mapsto f^{-1};$$

satisfying , for any $f : x \to y$*,* $g : y \to z$*, and* $h : z \to w$*:*

•$f\,\mathrm{id_x} = f$ *and* $\mathrm{id}_y\, f = f$*;*

•$(hg)f = h(gf)$*;*

$\bullet ff^{-1} = \mathrm{id}_y$ *and* $f^{-1}f = \mathrm{id}_{\mathrm{x}}$.

Using this definition, a ***group*** *is a groupoid with a single object.*

For the purpose of this thesis, the most important example of groupoid is the Deligne groupoid of a DGLA L.

Definition 3.5. *For any DGLA L, the Deligne groupoid of L is the groupoid whose objects* $\mathrm{Del}(L)_0$ *are given by the Maurer Cartan elements of L,* $\mathrm{MC}(L)$*, and morphisms between Maurer Cartan elements x and x' by*

$$\mathrm{Del}(L)_1 = \mathrm{Hom}_{\mathrm{Del}(L)}(x, x') = \{a \in L^0 | a \cdot x = x'\}$$

where $\cdot$ is the gauge action of L^0 on $\mathrm{MC}(L)$.

It easy to check that Deligne groupoids are groupoids; in fact, they are the action groupoid (groupoids whose the objects are G-set and morphisms given by G actions on the G-set) of L^0 on $\mathrm{MC}(L)$. Deligne groupoids play an important role in the theory of descent of Deligne groupoids, which relates global deformation problems to the Maurer Cartan solutions of a Deligne groupoid of a DGLA. We will study descent of Deligne groupoids in more detail in the next chapter.

We will now introduce the nerve functor, which associate each groupoid (group) a corresponding simplicial set.

Definition 3.6. *Given a groupoid G, the* ***nerve*** $\mathcal{N}(G)$ *of G is a simplicial set whose 0-simplices are objects of G, 1-simplices morphisms of G, and n-simplices n-tuples of composable morphisms of G, i.e.*

$$x_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} x_n$$

where x_i is an object in G and the $f_i : x_{i-1} \to x_i$ is a morphism from x_{i-1} to x_i. The face maps

$$d_i : \mathcal{N}(G)_k \to \mathcal{N}(G)_{k-1}$$

are given by composition of morphisms at the i-th object, i.e. d_i sends

$$x_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i-1}} x_{i-1} \xrightarrow{f_i} x_i \xrightarrow{f_{i+1}} x_{i+1} \xrightarrow{f_{i+2}} \dots \xrightarrow{f_n} x_n$$

to

$$x_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i-1}} x_{i-1} \xrightarrow{f_{i+1}f_i} x_{i+1} \xrightarrow{f_{i+2}} \dots \xrightarrow{f_n} x_n$$

The degeneracy maps

$$s_i : \mathcal{N}(G)_k \to \mathcal{N}(G)_{k+1}$$

are given by inserting identity morphism at the object x_i, i.e. s_i sends

$$x_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i-1}} x_{i-1} \xrightarrow{f_i} x_i \xrightarrow{f_{i+1}} x_{i+1} \xrightarrow{f_{i+2}} \dots \xrightarrow{f_n} x_n$$

to

$$x_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i-1}} x_{i-1} \xrightarrow{f_i} x_i \xrightarrow{\mathrm{id}} x_i \xrightarrow{f_{i+1}} x_{i+1} \xrightarrow{f_{i+2}} \dots \xrightarrow{f_n} x_n$$

Proposition 3.7. *Given a groupoid G, the nerve $\mathcal{N}(G)$ is a Kan complex.*

See [2] for proof. The above proposition shows us that Kan complexes give us a good generalization for groupoids (groups) and thus models the ∞-groupoids. In this thesis, we will use the terms Kan complexes and ∞-groupoids interchangeably.

Definition 3.8. *Two parallel 1-simplices f and g of a Kan complex X are* ***homotopic*** *if and only if there exist a 2-simplex in X of either of the following form*

$$\begin{tikzcd} & X_1 \arrow[dr, equal] & \\ X_0 \arrow[ur, "f"] \arrow[rr, "g"'] & & X_1 \end{tikzcd} \qquad \begin{tikzcd} & X_0 \arrow[dr, "f"] & \\ X_0 \arrow[ur, equal] \arrow[rr, "g"'] & & X_1 \end{tikzcd}$$

This defines an equivalence relation on the 1-simplices of X [12].

The left adjoint to the nerve functor, $\mathcal{N} : \mathrm{Grpd} \to \mathrm{Kan}$, which takes Kan complexes back to groupoids is called the fundamental groupoid functor.

Definition 3.9. *Given a Kan complex X, the* ***fundamental groupoid****, $\pi_{\leq 1}X$, is the groupoid with the following properties:*

- *the set of objects are 0-simplices in X*
- *the morphisms are homotopy classes of 1-simplices in X*
- *the identity morphism of $x \in X_0$ is represented by the degenerate 1-simplex $s_0(x)$*
- *a composition relation $h = g \circ f$ in $\pi_{\leq 1}X$ if and only if for any choices of 1-simplices representing these morphisms, there exist a 2-simplex in X with boundary*

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & x_2 \end{tikzcd}$$

The fundamental groupoid of a Kan complex mimics the fundamental groupoid of a topological space. The following proposition tells us that the $\pi_{\leq 1}$ functor preserves the homotopy relation.

Proposition 3.10. *If X and Y are homotopy equivalent Kan complexes, then $\pi_{\leq 1}X$ and $\pi_{\leq 1}Y$ are equivalent as groupoids.*

See [12] for proof.

In the case where the 0-simplices in our Kan complexes X and Y are the same, we have an isomorphism of groupoids, i.e. the objects and the morphisms of the groupoids are exactly the same.

Proposition 3.11. *If X and Y are homotopy equivalent Kan complexes and that $\pi_{\leq 1}X$ and $\pi_{\leq 1}Y$ have the same set of objects, then $\pi_{\leq 1}X$ and $\pi_{\leq 1}Y$ are isomorphic groupoids.*

Proof. Equivalent groupoids with the same set of objects are isomorphic. □

Remark. *The definition for the fundamental groupoid of a Kan complex is a special case of the fundamental category of a simplicial set. In fact, the fundamental category of a simplicial set X is a groupoid if and only if X is a Kan complex [12].*

3.1.2 Deligne-Getzler ∞-groupoids

We are now ready to present Getzler's result [7]. We will first introduce two important complexes and from them construct the Deligne-Getzler ∞-groupoid that gives us the general Baker-Campbell-Hausdorff product of a DGLA (L_∞ algebra).

Definition 3.12. *For every $n \geq 0$, the* ***differential graded commutative algebra of polynomial differential forms on the standard*** $\mathbf{n}$***-simplex*** $\mathbf{\Delta_n}$ *is:*

$$\Omega_n^* = \frac{\mathbb{K}[t_0, \ldots, t_n, dt_0, \ldots, dt_n]}{(\sum t_i - 1, \sum dt_i)}.$$

where the differential is induced by the usual differential for differential forms that sends $t_i \to dt_i$. Notice that $\Omega_\bullet^*$ has a natural structure of simplicial dg commutative algebra. The face map is given by

$$\partial_i : \Omega_n \to \Omega_{n-1}$$
$$\omega(t_0, \ldots, t_i, \ldots, t_n, dt_0, \ldots, dt_i, \ldots, dt_n) \mapsto \omega(t_0, \ldots, 0, \ldots, t_{n-1}, dt_0, \ldots, 0, \ldots, dt_{n-1})$$

and the degeneracy map is given by

$$s_i : \Omega_n \to \Omega_{n+1}$$
$$\omega(t_0, \ldots, t_i, \ldots, t_n, dt_0, \ldots, dt_i, \ldots, dt_n) \mapsto$$
$$\omega(t_0, \ldots, t_i + t_{i+1}, \ldots, t_{n+1}, dt_0, \ldots, dt_i + dt_{i+1}, \ldots, dt_{n+1})$$

Given a simplicial set X, the ***space of polynomial*** $\mathbf{k}$***-forms*** *on X is $\Omega^k(X) := \mathrm{SSet}(X, \Omega^k)$, i.e. the simplicial set morphisms from X to Ω^k, and $\Omega^*(X) := \oplus_{k\geq 0}\Omega^k(X)$. In particular, when X is $\Delta_\bullet$, we have $\Omega^*(\Delta_\bullet) = \Omega^*$.*

Definition 3.13. *The* ***complex of non-degenerate simplicial*** $\mathbb{K}$***-cochains on*** $\mathbf{X}$ *is $C^*(X) := C^*(X; \mathbb{K}) = \oplus_{k\geq 0} C^k(X)$ where $C^k(X)$ is the space of $\mathbb{K}$-valued k-cochains $\alpha : X_k \to \mathbb{K} : \sigma \mapsto \alpha_\sigma$ on X vanishing on degenerate simplices. The differential is given by*

$$d\alpha(\sigma) = (d\alpha)_\sigma = \sum_{i=0}^{k+1} (-1)^i \alpha_{\partial_i \sigma}$$

where $\partial_i : X_{k+1} \to X_k, i = 0, \ldots, k+1$, are the face maps of X.

Definition 3.14. *Given a simplicial set X and a dg space L, we defined $\Omega^*(X; L) = \Omega^*(X) \otimes L$* ***the complex of polynomial differential forms on X with coefficients in L****, and $C^*(X; L) = C^*(X) \otimes L$* ***the complex of non-degenerate simplicial cochains on X with coefficients in L****.*

Note that $\Omega^*(X; L)$ inherits any algebraic structure L has as $\Omega^*(X)$ is commutative. The complex $\Omega(X; L)$ does not inherit a complete structure in general, but we can replace $\Omega(X; L)$ with its completion $\widehat{\Omega}^*(X; L) := \varprojlim \Omega^*(X; L/F^pL)$ which will also have the same algebraic structure as L.

On the other hand, given a complete dg space L, the complex $C^*(X; L)$ is complete with respect to the filtration $F^pC^*(X; L) = C^*(X; F^pL)$. However, in general $C^*(X; L)$ does not inherit the algebraic structure of L. Fortunately, we do have a standard contraction from $\Omega^*(X; L)$ to $C^*(X; L)$ (and thus from $\hat{\Omega}^*(X; L)$ to $C^*(X; L)$) and we can use the homotopy transfer of structure theorem to induce an algebraic structure on $C^*(X; L)$ using the structure on $\hat{\Omega}^*(X; L)$.

Theorem 3.15 (Getzler, [7])**.** *There is a standard contraction from $\hat{\Omega}^*(X; L)$ to $C^*(X; L)$ given by integrating forms over simplices in one direction, inclusion of Whitney's elementary forms in the other direction, and Dupont homotopy as the contracting homotopy.*

In particular when L is a complete DGLA (L_∞ algebra), so is $\widehat{\Omega}^*(X; L)$ by extension of scalars. There is an induced complete L_∞ algebra structure on $C^*(X; L)$ via homotopy transfer along Dupont's contraction.

We can now define the Deligne-Getzler ∞-groupoid of a complete DGLA (L_∞ algebra) L. Denote

$$\Delta_\bullet : \Delta_0 \rightrightarrows \Delta_1 \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \Delta_2 \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} \cdots$$

the standard cosimplicial simplex in SSet^{Δ}, i.e. the set of covariant functors from Δ to SSet.

Definition 3.16. *Given a complete DGLA (L_∞ algebra) L, the **Deligne-Getzler ∞-groupoid** of L is the simplicial set $\mathrm{Del}_\infty(L)_n := \mathrm{MC}(C^*(\Delta_n; L))$ of Maurer-Cartan cochains on $\Delta_\bullet$ with coefficients in L. In other words, the functor $\mathrm{Del}_\infty(-) : \widehat{\mathbf{L}_\infty} \to \mathbf{SSet}$ is the composition*

$$\mathrm{Del}_\infty(-) : \widehat{\mathbf{L}_\infty} \xrightarrow{C^*(\Delta_\bullet;-)} \widehat{\mathbf{L}_\infty}^{\Delta^{op}} \xrightarrow{\mathrm{MC}(-)} \mathbf{SSet}.$$

In the paper [7], Getzler shows that $\mathrm{Del}_\infty(L)$ is in fact a Kan complex and is a suitable model as our ∞-groupoid. (To be more precise, Getzler shows that an isomorphic simplicial set $\gamma_\bullet(L)$ is a Kan complex and Bandiera rewrite it as $\mathrm{Del}_\infty(L)$. The two are isomorphic through formal Kuranishi theorem.). In fact, if L is a non negatively graded nilpotent DGLA (L_∞ algebras), we have

Proposition 3.17 ([7]). *Let L be a non negatively graded nilpotent DGLA (L_∞ algebras), then $\mathrm{Del}_\infty(L)$ is isomorphic to the nerve $\mathcal{N}(\mathrm{Del}^{\mathrm{op}}(L))$.*

Getzler prove the above proposition by showing that when L is a non negatively graded nilpotent DGLA (L_∞ algebras), then $\mathrm{Del}_\infty(L)$ is a T-complex of rank 2 and thus the nerve of a groupoid. The proof is out of the scope of this thesis, see [7] for details.

Notice that we have only defined Del for a DGLA in Definition 3.5. For a non negatively graded L_∞ algebra, we can extend the definition of Del using the following proposition:

Proposition 3.18 ([2]). *Let L be a non negatively graded L_∞ algebra, then $\mathrm{Del}_\infty(L)$ is the nerve of a groupoid G.*

We will then define $\mathrm{Del}(L) = G^{\mathrm{op}}$. This definition coincides with the definition of Del for DGLAs (see Theorem 5.2.36 in [2]).

For our purpose, the main advantage in working with cochains rather than with forms, i.e. $\gamma_\bullet$, is that Del_∞ gives us an easy and clear presentation of the Baker-Campbell-Hausdorff product.

3.2 Baker-Campbell-Hausdorff Product and Horn Filling

In this section, we will show how $\mathrm{Del}_\infty(L)$ gives us the general Baker-Campbell-Hausdorff product on L by horn filling. We will start by looking at a complete contraction given by Bandiera in [1].

Let L be a complete L_∞ algebra. For $i = 0, \ldots, n$, we define a homotopy $h^i : C^*(\Delta_n; L) \to C^{*-1}(\Delta_n; L)$ by

$$h^i(\alpha)_{i_0 \cdots i_k} = \begin{cases} 0 & \text{if } i \in \{i_0, \cdots, i_k\} \\ (-1)^j \alpha_{i_0 \cdots i_{j-1} i i_j \cdots i_k} & \text{if } 0 \leq i_0 < \cdots < i_{j-1} < i < i_j < \cdots < i_k \leq n \end{cases}$$

where we denote by $\beta_{i_0 \cdots i_k} \in L^{i-k}$, $0 \leq i_0 < \cdots < i_k \leq n$, the evaluation of a cochain $\beta \in C^i(\Delta_n; L)$ on the k-simplex of Δ_n spanned by the vertices $i_0, \ldots, i_k$. We denote by $e_i : \Delta_0 \to \Delta_n$ the inclusion of the i-th vertex of Δ_n and by $\pi : \Delta_n \to \Delta_0$ the final morphism. The above operator h^i give us a homotopy on the complete contraction

$$\begin{tikzcd} L = C^*(\Delta_0; L) \arrow[r, shift left, "\pi^*"] & C^*(\Delta_n; L) \arrow[l, shift left, "e_i^*"] \arrow[loop right, "h^i"] \end{tikzcd}$$

If $\partial_i : \Delta_{n-1} \to \Delta_n$ is the inclusion of the i-th face of the simplex Δ_n, then ∂_i^* sends $h^i(C^1(\Delta_n; L))$ isomorphically to $C^0(\Delta_{n-1}; L)$.

Apply the formal Kuranishi theorem by choosing $W = L$, $V = C^*(\Delta_n; L)$, and choosing $K = h^i$. We will obtain the following proposition [1]:

Proposition 3.19 ([2]). *For all $i = 0, \dots, n$, the correspondence*

$$\rho^i : \mathrm{MC}(C^*(\Delta_n; L)) = \mathrm{Del}_\infty(L)_n \to \mathrm{MC}(L) \times h^i(C^1(\Delta_n; L)) : \alpha \to (e_i^*(\alpha), h^i(\alpha))$$

is bijective.

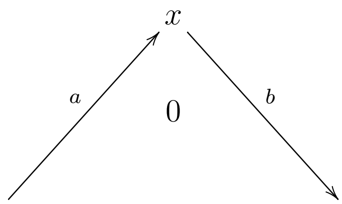
This proposition basically tells us that if we fix a Maurer Cartan element in L and a cochain in $h^i(C^1(\Delta_n; L))$, we can recover the unique cochain in $\mathrm{Del}_\infty(L)_n$ by using the recursive formula provided in the formal Kuranishi theorem. We will now use this proposition to show that $\mathrm{Del}_\infty(L)$ gives us the general Baker-Campbell-Hausdorff product on L.

The first thing we want to note is that given a Maurer Cartan solution $x \in \mathrm{MC}(L)$ and $a \in L^0$, we can create a 1-horn in $\mathrm{Del}_\infty(L)_1 = \mathrm{MC}(C(\Delta_1; L))$ where the left vertex is x and the edge is a. Since $\mathrm{Del}_\infty(L)$ is a Kan complex, there is a unique filling x' to this horn. The resulting 1-simplex is then the (opposite) morphism between the Maurer Cartan solutions x and x', i.e.

$$x \xrightarrow{\quad a \quad} x'$$

is the morphism from x' to x. When L is a DGLA, this is exactly the gauge action of a on x [2] and thus this horn filling generalizes the gauge action of L^0 on $\mathrm{MC}(L)$ for L_∞ algebras.

Now let us consider a Maurer Cartan solution $x \in \mathrm{MC}(L)$ (it does not matter which x we pick) and $a, b \in L^0$. We want to get a (generalized) the Baker-Campbell-Hausdorff product of a and b (i.e. composition of morphisms) through the horn filling in $\mathrm{Del}_\infty(L)_2$. Before we do that, let us label the lower left vertex of the 2-simplex [0], the top vertex [1], and the lower right vertex [2]. Consider the following 2-horn: Put x on the [1] vertex, a on the [01] edge, b on the [12] edge, and 0 on [012]



Now consider the cochain α' in $h^1(C^1(\Delta_2; L))$ (it is easy to check that α is in fact in $h^1(C^1(\Delta_2; L)))$ where α is given by

$$0 = \alpha'([1]) = \alpha'([01]) = \alpha'([12]) = \alpha'([012]),$$

i.e., all simplices in Δ_2 containing vertex [1] (the vertex where we put x in the above diagram) has value zero, and

$$\alpha'([0]) = a, \qquad \alpha'([2]) = b, \qquad \alpha'([02]) = 0,$$

i.e. all simplices that does not contain the vertex [1] get their value from the simplices with [1] inserted into the indexes in the horn we created above (for example, $\alpha'([0])$ will get the value from [01] in our horn).

We can then apply the recursive formula from the formal Kuranishi theorem and get an unique cochain $\alpha \in \mathrm{MC}(C^*(\Delta_2; L)) = \mathrm{Del}_\infty(L)_2$. The (general) Baker-Campbell-Hausdorff product $\rho_2^x(-)$ between the morphism a and b is then defined by evaluating α on the face

$\partial_1\Delta_2$ opposite to the vertex [1].

$$
\begin{tikzcd}
& x \arrow[dr, "b"] \arrow[d, phantom, "0"] & \\
{} \arrow[ur, "a"] & {} & {}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
& x \arrow[dr, "b"] \arrow[d, phantom, "0"] & \\
\rho_1^x(a) = x' \arrow[ur, "a"] \arrow[rr, "\rho_2^x(a,b,0)"'] & {} & \rho_1^x(b) = x''
\end{tikzcd}
$$

When L is a complete DGLA, we can recover the Gauge action and the Baker-Campbell-Hausdorff product on L. In particular, $x' = \rho_1^x(a) = a \cdot x$, $x'' = \rho_1^x(b) = -b \cdot x$, and $\rho_2^x(a, b, 0) = a * b$ [2].

Notice that we can get a general Baker-Campbell-Hausdorff product between $a, b \in L^0$ and $\eta \in L_{-1}$ by simply replacing 0 with η for the 2-simplex [012] in our horn. Repeating the same procedure as above and we will get

$$
\begin{tikzcd}
& x \arrow[dr, "b"] \arrow[d, phantom, "\eta"] & \\
{} \arrow[ur, "a"] & {} & {}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
& x \arrow[dr, "b"] \arrow[d, phantom, "\eta"] & \\
\rho_1^x(a) = x' \arrow[ur, "a"] \arrow[rr, "\rho_2^x(a,b,\eta)"'] & {} & \rho_1^x(b) = x''
\end{tikzcd}
$$

where $\rho_2^x(a, b, \eta)$ is the general Baker-Campbell-Hausdorff product of a, b, and η.

In general when we have $n \geq 0$ and $0 \leq i \leq n$, we can create a horn by assign the Maurer Cartan element where the composition of morphisms take place to the ith vertex $[i]$ of Δ_n and elements in L^{1-k} that are involved in the product to the k-simplices of Δ_n containing $[i]$. Recover a cochain α' in $h^i(C^1(\Delta_n; L))$ from this horn using the same method as above and apply the recursive formula from the formal Kuranishi theorem, and we should get the higher general Baker-Campbell-Hausdorff product.

Remark. *For the purpose of this thesis, we will only need to worry about the Baker-Campbell-Hausdorff product between elements in* L^0 *as our* L_∞ *algebra will be concentrated in non negative degrees.*

3.3 Del, Del_∞ and MC_∞ for Non-negatively Graded L_∞ Algebras

Recall that for a DGLA L, two Maurer Cartan solutions of L, $x, x' \in \mathrm{MC}(L)$, are equivalent if and only if there exist $a \in L^0$ such that $a \cdot x = x'$. However, in the case where L is an L_∞ algebra, gauge equivalence does not make sense as L^0 is not a Lie algebra in general. Instead we define that for an L_∞ algebra L, two Maurer Cartan solutions of L, $x, x' \in \mathrm{MC}(L)$, are equivalent if and only if there exist $z \in \mathrm{MC}(L \otimes \mathbb{K}[s, ds])$ such that $z|_{s=0} = x$ and $z|_{s=1} = x'$. There are several ways (for example [7]) to show that this (homotopic) equivalence relation generalizes the gauge equivalence and are exactly the same in the case where L is a DGLA. For the purpose of this thesis though, it is enough for us to just investigate the relations between Del, Del_∞ and MC_∞ for non-negatively graded L_∞ algebras (which will be used throughout this thesis) and get the above result for a non negatively graded L as a corollary.

We will start out by defining MC_∞.

Definition 3.20. *Given an* L_∞ *algebra* L, $\mathrm{MC}_\infty(L)$ *is the simplicial set* $\mathrm{MC}(L \otimes \Omega_\bullet)$ *with induced face and degeneracy maps from* $\Omega_\bullet$.

It is well known that

Proposition 3.21 ([8])**.** $\mathrm{MC}_\infty(L)$ *is a Kan complex.*

Notice that in our definition of homotopy equivalence between Maurer Cartan solutions $x, x' \in \mathrm{MC}(L)$, we have $z \in \mathrm{MC}(L \otimes \mathbb{K}[s, ds])$ such that $z|_{s=0} = x$ and $z|_{s=1} = x'$. In other words, two Maurer Cartan solutions $x, x' \in \mathrm{MC}(L)$ are homotopy equivalent if and only if there exist a 1-simplex $z \in \mathrm{MC}(L \otimes \Omega_1)$ such that $d_0(z) = x$ and $d_1(z) = x'$.

The fundamental groupoid of $\mathrm{MC}_\infty(L)$ has the Maurer Cartan solutions of L as objects and homotopy classes of 1-simplices, i.e. $z \in \mathrm{MC}(L \otimes \Omega_1)$ as morphisms. Getzler shows us in [7] that $\mathrm{MC}_\infty(L)$ is homotopy equivalent to $\mathrm{Del}_\infty(L)$, so after applying Proposition 3.11, we will get

Lemma 3.22. *$\pi_{\leq 1} \mathrm{MC}_\infty(L)$ is isomorphic to $\pi_{\leq 1} \mathrm{Del}_\infty(L)$ as groupoids.*

Proposition 3.17 tells us that for non negatively graded DGLA (L_∞ algebras) $\pi_{\leq 1} \mathrm{Del}_\infty(L) = \pi_{\leq 1}\mathcal{N}(\mathrm{Del}^{\mathrm{op}}(L)) = \mathrm{Del}^{\mathrm{op}}(L)$, so we have

Proposition 3.23 ([2])**.** *Given a non negatively graded DGLA (L_∞ algebra) L, $\pi_{\leq 1} \mathrm{MC}_\infty(L)$ is isomorphic to $\mathrm{Del}^{\mathrm{op}}(L)$ as groupoids.*

The homotopy class of $z \in \mathrm{MC}(L \otimes \mathbb{K}[s, ds])$ that gives us an equivalence between two Maurer Cartan solutions $x, x' \in \mathrm{MC}(L)$ is identified with a (opposite) morphism between x and x' in $\mathrm{Del}(L)$ and thus a 1-simplex of $\mathrm{Del}_\infty(L)$ as $\mathrm{Del}_\infty(L) = \mathcal{N}(\mathrm{Del}^{\mathrm{op}}(L))$. From last section, we know that for a non negatively graded DGLA L, the (opposite) gauge action of L^0 on $\mathrm{MC}(L)$ is precisely the unique horn filling on 1-simplices of $\mathrm{Del}_\infty(L)$. It follows naturally that the homotopy equivalence is the same as gauge equivalence on a non negatively graded DGLA, i.e. for $x, x' \in \mathrm{MC}(L)$, x is gauge equivalent to x' if and only if x is homotopy equivalent to x'. Furthermore, we have established that

Corollary 3.24. *Given a non negatively graded DGLA (L_∞ algebra) L, equivalence classes of $z \in \mathrm{MC}(L \otimes \mathbb{K}[s, ds])$ giving us the homotopy equivalence on $\mathrm{MC}(L)$, i.e. morhpisms in $\pi_{\leq 1} \mathrm{MC}_\infty(L)$, is in bijection with 1-simplices of $\mathrm{Del}_\infty(L)$.*

Proof. $\pi_{\leq 1} \mathrm{MC}_\infty(L)$ is isomorphic to $\mathrm{Del}^{\mathrm{op}}(L)$ and morphisms in $\mathrm{Del}^{\mathrm{op}}(L)$ are given by 1-simplices of $\mathrm{Del}_\infty(L)$ as $\mathrm{Del}_\infty(L) = \mathcal{N}(\mathrm{Del}^{\mathrm{op}}(L))$. □

Chapter 4

Descent of Deligne groupoids

Suppose we have a nilpotent Lie algebra $\mathfrak{g}$ such that $e^{\mathfrak{g}} = G$, a unipotent Lie group, and $P \to X$ a G-bundle over X. In the paper [8], Hinich introduced the theorem on descent of Deligne groupoids which allows us to use combinatoric tools to solve the formal deformation problem on P. The idea is to construct a Thom-Whitney complex out of the Čech (semi)cosimplicial DGLA and the equivalence classes of formal deformations of P are then given by the Deligne groupoid of the Thom-Whitney complex.

In the case where we have a Čech semicosimplicial Lie algebra, i.e. concentrated in degree 0, Fiorenza, Manetti, and Martinengo use homotopy transfer of structure to obtain an L_∞ structure on the Čech complex from the Thom-Whitney complex. They show in [3] that the solutions to the deformation equation (i.e. cocycle condition on transition functions) are exactly the Maurer Cartan solutions of the L_∞ Čech complex and the equivalences of deformations are exactly the the equivalence of Maurer Cartan solutions.

While Hinich's theorem only works when we have a sheaf $\mathcal{L}$ of non-negatively graded DGLA over X, Bandiera extended the theorem by replacing the Deligne groupoid with the Deligne-Getlzer ∞-groupoid, which works for negatively graded DGLAs [1]. Since we are only going to deal with semicosimplicial Lie algebras, Deligne groupoid is sufficient for us and there is no need to use the Deligne-Getzler ∞-groupoid. But as in the chapters before, we are going to use Bandiera's reformulation of Fiorenza, Manetti, and Martinengo's result from [3] using Deligne-Getzler ∞-groupoids as it is more intuitive for what we are doing.

4.1 Semicosimplicial DGLA and $\check{C}$ech Complex

We will define the semicosimplicial DGLAs in this section. In particular, we are interested in the case where the DGLAs are concentrated in degree 0, i.e. Lie algebras, and introduce the $\check{C}$ech semicosimplicial Lie algebra of a sheaf of Lie algebras over a space X.

Definition 4.1. *$L_\bullet$ is a semicosimplicial differential graded Lie algebra if $L_\bullet$ is a covariant functor $\underrightarrow{\Delta} \to$ DGLAs from the category $\underrightarrow{\Delta}$, whose objects are finite ordinals and whose morphisms are order-preserving injective maps, to the category of DGLAs. In other words,*

$$L_\bullet : L_0 \rightrightarrows L_1 \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; L_2 \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; \cdots$$

is a diagram where each L_i is a DGLA and each ∂_k, the coface morphism whose image misses k, is a morphism of DGLA, i.e. for each $i \geq 0$ there are $i+1$ morphisms of DGLAs

$$\partial_{k,i} : L_{i-1} \to L_i, \qquad k = 0, \dots, i,$$

such that $\partial_{k+1,i+1}\partial_{l,i} = \partial_{l,i+1}\partial_{k,i}$ for any $k \geq l$.

We now want to introduce the $\check{C}$ech cochains. Let X be a topological space and let $\mathcal{F}$ be a presheaf of abelian groups on X. Let $\mathcal{U}$ be an open cover of X.

Definition 4.2. *A* **q-*simplex*** *σ* ***of*** *$\mathcal{U}$ is an ordered collection of $q+1$ sets chosen from $\mathcal{U}$ such that the intersect of these sets is nonempty, i.e. $\sigma = (U_i)_{i\in\{0,\dots,q\}}$. The intersection of* *these sets is called the* ***support*** *of σ and is denoted $|\sigma|$. The* ***j-th partial boundary*** *of σ is the $(q-1)$-simplex obtained by removing the j-th set from σ, i.e.:*

$$\partial_j \sigma := (U_i)_{i \in \{0,\dots,q\} \setminus \{j\}}$$

The ***boundary*** *of σ is then defined to be the alternating sum of the partial boundaries viewed as an element of the free abelian group spanned by the simplices of $\mathcal{U}$.*

$$\partial \sigma := \sum_{j=0}^{q} (-1)^{j+1} \partial_j \sigma$$

Definition 4.3. *A* ***q-cochain*** *of $\mathcal{U}$ with coefficients in a presheaf $\mathcal{F}$ over X is a map which associates with each q-simplex σ an element $\mathcal{F}(|\sigma|)$. We denote the set of all q-cochains of $\mathcal{U}$ with coefficients in $\mathcal{F}$ by $\prod_{i_0 < \dots < i_q} \mathcal{F}(U_{i_0 \dots i_q})$ and the abelian group of all cochains $\mathcal{F}(\mathcal{U})$.*

For the purpose of this thesis, we are interested in the case where $\mathcal{F}$ is a sheaf of Lie algebras over X and $\mathcal{U}$ an affine open cover of X. We can now define our Čech semicosimplicial Lie algebra.

Definition 4.4. *Let X be a topological space, $\mathfrak{g}$ be a sheaf of Lie algebras on X, and $\mathcal{U}$ be an affine open cover of X. The* ***Čech semicosimplicial Lie algebra associated with the sheaf $\mathfrak{g}$ and affine open cover $\mathcal{U}$ of X*** *is the semicosimplicial Lie algebra*

$$\mathfrak{g}(\mathcal{U})_\bullet : \mathfrak{g}(\mathcal{U})_0 \rightrightarrows \mathfrak{g}(\mathcal{U})_1 \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathfrak{g}(\mathcal{U})_2 \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

where $\mathfrak{g}(\mathcal{U})_q := \prod_{i_0 < \dots < i_q} \mathfrak{g}(U_{i_0 \dots i_q})$ and the arrows given by the pullback of partial boundaries. These arrows naturally gives a semicosimplical structure given how we defined them above.

Example 4.5. *When X is a smooth algebraic variety over an algebraically closed field $\mathbb{K}$ of characteristics 0, $\mathcal{U} = \{U_i\}$ an affine open covering of X, and $\mathcal{T}_X$ the tangent sheaf, then $\mathcal{T}_X(|\sigma|)$ is a section of the tangent bundle over the $(q+1)$-tuple intersection $|\sigma|$. Note that*

$\mathcal{T}_X(\mathcal{U})_q = \prod_{i_0<\cdots<i_q} \mathcal{T}_X(U_{i_0\ldots i_q})$*, the set of all* q*-cochains, is a Lie algebra. We can thus define the* $\check{C}$*ech semicosimplicial Lie algebra* $\mathcal{T}_X(\mathcal{U})_\bullet$ *associated with the tangent sheaf and an affine open cover* $\mathcal{U}$ *of* X *as [3]:*

$$\mathcal{T}_X(\mathcal{U})_\bullet : \prod_i \mathcal{T}_X(U_i) \rightrightarrows \prod_{i<j} \mathcal{T}_X(U_{ij}) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \prod_{i<j<k} \mathcal{T}_X(U_{ijk}) \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} \cdots$$

We can also turn the abelian group of all cochains $\mathfrak{g}(\mathcal{U})$ into a cochain complex, i.e. the $\check{C}$ech complex, by introducing a grading $\mathfrak{g}(\mathcal{U}) = \bigoplus_q \mathfrak{g}(\mathcal{U})_q[-q]$ and a coboundary operator:

Definition 4.6. *The* ***coboundary operator*** $\delta_q : \mathfrak{g}(\mathcal{U})_q \to \mathfrak{g}(\mathcal{U})_{q+1}$ *is given by*

$$(\delta_q f)(\sigma) := \sum_{j=0}^{q+1} (-1)^j \operatorname{res}_{|\sigma|}^{|\partial_j \sigma|} f(\partial_j \sigma),$$

where $\operatorname{res}_{|\sigma|}^{|\partial_j\sigma|}$ *is the restriction morphism from the q-tuple intersection to* $(q+1)$*-tuple intersection. Note that* $\delta_{q+1} \circ \delta_q = 0$*, so* δ *is in fact a differential for the complex. The abelian group* $\mathfrak{g}(\mathcal{U})$ *together with the grading* $\mathfrak{g}(\mathcal{U}) = \bigoplus_q \mathfrak{g}(\mathcal{U})_q[-q]$ *and the coboundary operator* δ *is called the* **Čech complex associated with the sheaf** $\mathfrak{g}$ ***and affine open cover*** $\mathcal{U}$ ***of X***

Definition 4.7. *A* q*-cochain* $f \in \mathfrak{g}(\mathcal{U})_q$ *is called a* **q-*cocycle*** *if it is in the kernel of* δ_q*. In particular, a* 1*-cocycle* $\alpha \in \mathfrak{g}(\mathcal{U})_1$ *satisfies, for every non-empty* $U = A \cap B \cap C$ *with* $A, B, C \in \mathcal{U}$

$$\alpha(B \cap C)|_U - \alpha(A \cap C)|_U + \alpha(A \cap B)|_U = 0.$$

4.2 Totalization and Homotopy Limit

There are a few homotopy equivalent ways of defining the homotopy limit of a semicosimplical DGLA. One way is through the Thom-Whitney-Sullivan construction [3]. The resulting object will be a DGLA, which means the Maurer Cartan equation will be simpler, but at the cost that the complex is significantly larger and contains less algebraic information. Bandiera introduced another version of the homotopy limit based on the cochain complexes, which obtains an L_∞ structure through homotopy transfer. This model is smaller and we can directly retrieve the Baker-Campbell-Hausdorff formula from the chain complex as discussed from chapter 3, but we will have to deal with L_∞ Maurer Cartan equations. In the case where we have a $\check{C}$ech semicosimplicial Lie algebra, Bandiera's construction is isomorphic to the $\check{C}$ech complex.

Definition 4.8. *Given a semicosimplcial complete DGLA $L_\bullet \in \widehat{\mathbf{DGLA}}^{\underrightarrow{\Delta}}$, its **Thom-Whitney complex** is the complete DGLA*

$$\mathrm{Tot}_{\mathrm{TW}}(L_\bullet) = \left\{ (\alpha_0, \ldots, \alpha_n, \ldots) \in \prod_{n\geq 0} \widehat{\Omega}^*(\Delta_n; L_n) \text{ s. t. } \partial_*^j(\alpha_{n-1}) = \delta_j^*(\alpha_n) \right\}$$

where the morphism $\partial_*^j : \widehat{\Omega}^*(\Delta_{n-1}; L_{n-1}) \to \widehat{\Omega}^*(\Delta_{n-1}; L_n)$ is the push-forward by the j-th cofaces of $L_\bullet$ and $\delta_j^* : \widehat{\Omega}^*(\Delta_n; L_n) \to \widehat{\Omega}^*(\Delta_{n-1}; L_n)$ is the pull back by the j-th coface of $\underrightarrow{\Delta}$.

Definition 4.9. *Given a semicosimplcial complete L_∞ algebra $L_\bullet \in \widehat{\mathbf{L}_\infty}^{\underrightarrow{\Delta}}$, its **totalization** $\mathrm{Tot}(L_\bullet)$ is the complete L_∞ algebra*

$$\mathrm{Tot}(L_\bullet) = \left\{ (\alpha_0, \ldots, \alpha_n, \ldots) \in \prod_{n\geq 0} C^*(\Delta_n; L_n) \text{ s. t. } \partial_*^j(\alpha_{n-1}) = \delta_j^*(\alpha_n) \right\}$$

where the morphism $\partial_*^j : C^*(\Delta_{n-1}; L_{n-1}) \to C^*(\Delta_{n-1}; L_n)$ is the push-forward by the j-th cofaces of $L_\bullet$ and $\delta_j^* : C^*(\Delta_n; L_n) \to C^*(\Delta_{n-1}; L_n)$ is the pull back by the j-th coface of $\underrightarrow{\Delta}$.

Suppose $L_\bullet$ is a semicosimplcial complete DGLA, then $\mathrm{Tot}_{\mathrm{TW}}(L_\bullet)$ and $\mathrm{Tot}(L_\bullet)$ are homotopy equivalent as differential complexes and both are models for homotopy limit of $L_\bullet$ in the category of complexes. $\mathrm{Tot}(L_\bullet)$ is smaller than $\mathrm{Tot}_{\mathrm{TW}}(L_\bullet)$ and in the case where $L_\bullet$ is a Čech semicosimplicial (complete) Lie algebra, $\mathrm{Tot}(L_\bullet)$ is precisely the Čech complex. The problem with $\mathrm{Tot}(L_\bullet)$ is that in general $\mathrm{Tot}(L_\bullet)$ is not a DGLA. To solve this, we can consider $\mathrm{Tot}(L_\bullet)$ in the category of L_∞ algebra, i.e. viewing the DGLA L_i's in $L_\bullet$ as L_∞ algebra and take the limit in the category of L_∞ algebra. Notice that we can obtain an L_∞ structure on $\mathrm{Tot}(L_\bullet)$ using the homotopy transfer of structure from $\mathrm{Tot}_{TW}(L_\bullet)$ to $\mathrm{Tot}(L_\bullet)$ and the L_∞ structure on $\mathrm{Tot}(L_\bullet)$ obtained this way is the same as the L_∞ structure defined in [3].

Remark. *Consider the case where $L_\bullet$ is a semicosimplicial (complete) Lie algebra. Suppose we have a degree n element $\alpha = (\alpha_0, \dots, \alpha_n, \dots) \in \mathrm{Tot}(L_\bullet)$. Because of the way $\mathrm{Tot}(L_\bullet)$ is defined, for $m \geq n$, the degree n element $\alpha_m \in C^n(\Delta_m, L_m)$ are uniquely determined by $\alpha_n \in C^n(\Delta_n; L_n)$. The evaluation of α_m, a degree n cochain, on a k-simplex in Δ_m is an element in L_m^{n-k}. Since L_m is a Lie algebra, the evaluation of α_m on any k-simplex in Δ_m is 0 for $k \neq n$. Thus, α_m is determined by α_n. In particular, when $n = 1$, any degree one elements $\alpha_m \in C^1(\Delta_m, L_m)$ is determined after we fixed an element in $\alpha_1 \in C^1(\Delta_1; L_1)$. In fact, we can recover the total complex from the totalization.*

Proposition 4.10. *Given a semicosimplicial Lie algebra $L_\bullet$, the total complex $\bigoplus_n L_n[-n]$ of $L_\bullet$ is isomorphic to $\mathrm{Tot}(L_\bullet)$ as a dg space, and thus obtains the same L_∞ structure from homotopy transfer. When we have a Čech semicosimplicial Lie algebra $\mathfrak{g}(\mathcal{U})_\bullet$ of a sheaf of Lie algebras, $\mathrm{Tot}(\mathfrak{g}(\mathcal{U})_\bullet)$ is precisely the Čech complex $\mathfrak{g}(\mathcal{U})$.*

Proof. Suppose we have a degree 1 element α in $\mathrm{Tot}(L_\bullet)$, and we denote L_i the Lie algebras in the semicosimplical Lie algebra $L_\bullet$, then α is of the form $\alpha = (\alpha_0, \dots, \alpha_n, \dots)$, where α_i is a degree 1 cochain in $C^1(\Delta_i; L_i)$.

Notice that $\alpha_0 = 0$ as $C^1(\Delta_0, L_0) = 0$, and all the vertices of α_i are images of combinations of coface maps of α_0 by the construction of Tot. As with the vertices, the edges in $\alpha_i \in C^1(\Delta_i; L_i)$ are are images of combinations of coface maps of α_1, which are given by elements in L_1. Since the evaluation of α_i on a j-simplex of Δ_i is an element in L_i^{1-j} and all our L_i's are concentrated in degree zero, the evaluation of α_i on all the j-simplices are 0 for $j \geq 2$, i.e. α_i is determined by its values on the edges. So we get that α_1 uniquely determines α, i.e. the degree 1 elements of $\mathrm{Tot}(L_\bullet)$ are in bijection with L_1.

In general, for a degree k element α in $\mathrm{Tot}(L_\bullet)$, $\alpha_i = 0$ for $i < k$. For $i \geq k$, the evaluation of α_i on k-simplices of Δ_i is given by the pullback of combinations of coface maps of α_k, and the evaluation of α_i on all the j-simplices of Δ_i will be 0 for $j > k$. Thus α_k uniquely determines a degree k α. We can thus identify $\mathrm{Tot}(L_\bullet)$ as the total complex, where the underlying space is $\bigoplus\limits_n L_n[-n]$.

Now let's check that our bijection is in fact an isomorphism of dg space. It is obvious that our bijection preserves the graded vector space structure, so the only thing to check is the differential. Suppose we have an element l of degree k in $\bigoplus\limits_n L_n[-n]$, i.e. l is an element in L_k, then $d(l) = \sum_{j=0}^{k+1}(-1)^j \partial_j(l) \in L_{k+1}$. Now consider α an element of degree k in $\mathrm{Tot}(L_\bullet)$ whose evaluation of the k- simplices in Δ_k is l. α is of the form $\alpha = (0, \ldots, 0, \alpha_k, \alpha_{k+1}, \ldots)$ and $d(\alpha) = (\delta^*(0), \ldots, \delta^*(0), \delta^*(\alpha_k), \delta^*(\alpha_{k+1}), \ldots)$.

Note that $\delta^*(\alpha_i) = \partial_*(\alpha_{i-1})$ by the construction of Tot and α_{k-1} is the 0 cochain, so $\delta^*(\alpha_k) = 0$. By the same reasoning we have $\delta^*(\alpha_{k+1}) = \partial_*(\alpha_k)$, whose evaluation at the $k+1$ simplex in Δ_{k+1} is $\sum_{j=0}^{k+1}(-1)^j \partial_j(l) = d(l)$, so we have $d(\alpha) = (0, \ldots, 0, 0, \delta^*(\alpha_{k+1}) = d(l), \ldots)$. Thus by our previous discussion, $d(\alpha)$ must be a degree $k+1$ element in $\mathrm{Tot}(L_\bullet)$ which under our bijection will precisely be $d(l)$ in $\bigoplus\limits_n L_n[-n]$. $\square$

4.3 Hinich's Theorem on Descent of Deligne Groupoids

Now we are in position to state Hinich's theorem on descent of Deligne groupoid. We will also show Fiorenza, Manetti, and Martinengo's result that in the case where the semicosimplical DGLA is concentrated in degree 0, i.e. a semicosimplical Lie algebra, instead of an equivalence, we will get an isomorphism of groupoids.

Theorem 4.11 (Hinich, [8]). *For semicosimplicial DGLAs $L_\bullet$ concentrated in non negative degrees, the Deligne functor commutes with homotopy limits, i.e., there is a natural equivalence of groupoids*

$$\mathrm{Del}(\mathrm{Tot}(L_\bullet)) \simeq \mathrm{Tot}(\mathrm{Del}(L_\bullet)).$$

$\mathrm{Tot}(\mathrm{Del}(L_\bullet))$ (the left hand side) is called the groupoid of descent data on $L_\bullet$. In the case where $L_\bullet$ is a secosimplicial Lie algebra, its objects are the nonabelian 1-cocycles

$$Z^1(\exp(L_1)) = \{m \in L_1 | e^{\partial_0(m)} e^{-\partial_1(m)} e^{\partial_2(m)} = 1\}$$

and its morphisms between two cocycles m_0 and m_1 are

$$\{a \in L_0 | e^{-\partial_1(a)} e^{m_1} e^{\partial_0(a)} = e^{m_0}\}$$

Remark. *Note that since the L_i's in $L_\bullet$ are all concentrated in non negative degrees, we have an equivalence between* $\mathrm{Del}(L_i)$ *and* $\mathrm{Del}_\infty(L_i)$ *through the nerve functor, $\mathcal{N}$, and fundamental groupoid functor, $\pi_{\leq 1}$. We will give a definition of* Tot *in the simplicial set sense below. For now, think of* $\mathrm{Tot}(\mathrm{Del}(L_\bullet))$ *as the homotopy limit of the Deligne groupoids* $\mathrm{Del}(L_i)$*'s.*

$\mathrm{Del}(\mathrm{Tot}(L_\bullet))$ (the right hand side) is the groupoid where the objects are Maurer Cartan solutions of the L_∞ algebra $\mathrm{Tot}(L_\bullet)$ and the morphisms are the morphisms between Maurer

Cartan solutions. Notice that $\mathrm{Del}(\mathrm{Tot}(L_\bullet))$ make sense because $L_\bullet$ is non negatively graded and thus $\mathrm{Tot}(L_\bullet)$ must also be non negatively graded.

Hinich's theorem basically tells us that the groupoid of descent data, $\mathrm{Tot}(\mathrm{Del}(L_\bullet))$, is homotopy equivalent to the Deligne groupoid of the totalization of our semicosimplicial DGLA, $\mathrm{Del}(\mathrm{Tot}(L_\bullet))$, when the DGLAs are concentrated in non negative degrees. In the case where we have a semicosimplicial Lie algebra, Fiorenza, Manetti, and Martinengo proved in [3] that instead of just equivalence, we are getting an isomorphism of groupoids, that is, the nonabelian 1-cocyles, as a subset of $\mathrm{Tot}(L_\bullet)$, is the same as the solution of the L_∞ Maurer Cartan equation on $\mathrm{Tot}(L_\bullet)$, and that two nonabelian cocyles are equivalent if and only if they are equivalent Maurer Cartan elements.

While we will take a more direct approach by directly comparing the simplices in $\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet))$ and $\mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet))$ in the proof of the following theorem in comparison to [3], the main idea is essentially the same.

Theorem 4.12 (Fiorenza, Manetti, and Martinengo, [3]). *For semicosimplicial Lie algebra $L_\bullet$, there is an isomorphism of ∞-groupoids*

$$\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet)) \cong \mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet)).$$

and thus an isomorphism of groupoids

$$\mathrm{Del}(\mathrm{Tot}(L_\bullet)) \cong \mathrm{Tot}(\mathrm{Del}(L_\bullet)).$$

Before we start the proof, we should define the totalization of the semicosimplical simplicial set $\mathrm{Del}_\infty(L_\bullet)$ (the right hand side of our isomorphism). The totalization of semicosimplical simplicial sets is defined the same way as the totalization of semicosimplicial complete L_∞ algebras by simply replacing $C^*(\Delta_i; L_i)$ with $\underline{\mathrm{SSet}}(\Delta_i, L_i)$. This totalization holds similar universal property in analogues to the totalization of semicosimplicial DGLA. Using this definition, we have

$$\mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet)) = \left\{ (\alpha_0, \dots, \alpha_n, \dots) \in \prod_{n\geq 0} \underline{\mathrm{SSet}}(\Delta_n, \mathrm{Del}_\infty(L_n)) \ \text{s.t.}\ \partial_*^j(\alpha_{n-1}) = \delta_j^*(\alpha_n) \right\}.$$

Notice that $\underline{\mathrm{SSet}}(X,Y)_n = \mathrm{SSet}(\Delta_n \times X, Y)$, so $\mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet))_i = \{(\alpha_0, \dots, \alpha_n, \dots) \in \prod_{n\geq 0} \mathrm{SSet}(\Delta_i \times \Delta_n, \mathrm{Del}_\infty(L_n)) \ \text{s.t.}\ \partial_*^j(\alpha_{n-1}) = \delta_j^*(\alpha_n)\}$.

Proof. Due to the fact that $\mathrm{Tot}(L_\bullet)$ is concentrated in non negative degrees, by Proposition 3.17 $\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet)) = \mathcal{N}(\mathrm{Del}^{\mathrm{op}}(\mathrm{Tot}(L_\bullet)))$. This means that $\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet))$ is uniquely determined by $\mathrm{Del}(\mathrm{Tot}(L_\bullet))$, i.e. the objects (0-simplices in $\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet))$), the morphisms (1-simplices in $\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet))$), and the composition of morphisms (2-simplices in $\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet))$). On the other hand, since L_i's are all concentrated in degree 0, $\mathrm{Del}_\infty(L_i) = \mathcal{N}(\mathrm{Del}^{\mathrm{op}}(L_i))$. Using the same reasoning as above, $\mathrm{Del}_\infty(L_\bullet)$ is uniquely determined by the 0,1,2-simplices and hence also its homotopy limit $\mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet))$.

Thus we will be comparing the 0,1,2-simplices for $\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet))$ and $\mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet))$. This will give us a comparison of Maurer-Cartan elements, morphisms between Maurer-Cartan elements, and composition of morphisms between Maurer-Cartan elements.

0-simplices

0-simplices of $\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet))$:

The 0-simplices of $\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet))$ are the Maurer-Cartan elements on $\mathrm{Tot}(L_\bullet)$. Suppose $\alpha = (\alpha_0, \alpha_1, \dots)$ satisfies the Maurer-Cartan equation on $\mathrm{Tot}(L_\bullet)$. Since the L_∞ structure on $\mathrm{Tot}(L_\bullet)$ is given by the individual L_∞ structure on $C^*(\Delta_i; L_i)$, each α_i must satisfy the Maurer-Cartan equation on $C^*(\Delta_i; L_i)$. α_1 trivially satisfies the Maurer-Cartan equation on $C^*(\Delta_1; L_1)$ as it is the top degree element. For α_2 that satisfies the Maurer-Cartan equation in $C^*(\Delta_2; L_2)$, by definition it is an element of $\mathrm{Del}_\infty(L_2)_2 = \mathrm{MC}(C^*(\Delta_2; L_2))$, where its vertices are images of 0 under the coface morphisms and the 2-simplex is 0 as L_2 is concentrated in degree 0. α_2 gives us a Baker-Campbell-Hausdorff product on the three edges of α_2, and by the construction of Tot, each edge of α_2 is the image of the coface morphisms ∂_*^j of the 1-simplex α_1, i.e.

$$\partial_*^0\alpha_1 * -\partial_*^1\alpha_1 * \partial_*^2\alpha_1 = 0,$$

which is precisely the nonabelian 1-cocycle in $\mathrm{Tot}(L_\bullet)$.

0-simplices of $\mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet))$:

Suppose $\alpha = (\alpha_0, \alpha_1, \dots)$ is a 0-simplex for $\mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet))$, then $\alpha_0 \in \mathrm{SSet}(\Delta_0 \times \Delta_0, \mathrm{Del}_\infty(L_0))$, i.e. the set of simplicial morphisms between $\Delta_0 \times \Delta_0$ and $\mathrm{Del}_\infty(L_0)$. α_0 must be 0 because the simplicial morphisms between $\Delta_0 \times \Delta_0$ and $\mathrm{Del}_\infty(L_0)$ are identified by the 0-simplices of $\mathrm{Del}_\infty(L_0)$, which in turn must be 0 because the only Maurer-Cartan elements of L_0 is 0. $\alpha_1 \in \mathrm{SSet}(\Delta_0 \times \Delta_1, \mathrm{Del}_\infty(L_1))$, which can be identified with the 1-simplices of $\mathrm{Del}_\infty(L_1)$ and thus identified with L_1. $\alpha_2 \in \mathrm{SSet}(\Delta_0 \times \Delta_2, \mathrm{Del}_\infty(L_2))$, which can be identified as the 2-simplices in $\mathrm{Del}_\infty(L_2)$. Note that since $\partial_*^j(\alpha_{n-1}) = \delta_j^*(\alpha_n)$, the edges of α_2 are precisely the images of ∂_*^j of α_1, and since α_2 is uniquely determined by the edges, we conclude that α_1 must be a nonabelian 1-cocycle in $\mathrm{Tot}(L_1)$. In general, cochains in $\mathrm{Del}_\infty(L_i)$ are determined by their edges as L_i is concentrated in degree 0, and the edges are uniquely determined by the 1-cochains in $\mathrm{Del}_\infty(L_1)$. Thus, the 0-simplex of $\mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet))$ are in bijection with the nonabelian 1-cocycles in $\mathrm{Tot}(L_\bullet)$.

1-simplices

1-simplices of $\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet))$:

1-simplex of $\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet))$ is a 1-cochain $\alpha = (\alpha_0, \alpha_1, \dots)$ of the following form:

$$a \xrightarrow{\quad l \quad} -l \cdot a$$

where a is a nonabelian 1-cocycle of $\mathrm{Tot}(L_\bullet)$, $-l \cdot a$ the resulting 1-cocycle from $-l$ acting on a, and l a morphism from $-l \cdot a$ to a. Since $a = (a_0, a_1, \dots)$ and $l = (l_0, l_1, \dots)$, this induces a morphism between $(-l \cdot a)_i$ and a_i, i.e. the component terms of $-l \cdot a$ and a, in $C^1(\Delta_i; L_i)$. (For simplicity on notations, we will abuse the notation and denote the evaluation of the 1-cochain β on the edge of Δ_1 β.) For $i = 0$, we have

$$0 \xrightarrow{l_0} 0$$

where l_0 is an element in L_0. For $i = 1$, we have a morphism between the two 1-cochains in $C^1(\Delta_1; L_1)$,

$$0 \xrightarrow{a_1} 0$$

$$0 \xrightarrow{(-l \cdot a)_1} 0$$

The condition $\partial_*^j(a_0) = \delta_j^*(a_1)$ and $\partial_*^j((-l * a)_0) = \delta_j^*((-l * a)_1)$, $j = 0, 1$ ensures that the edge connecting the j-th vertex of each cochain must be $\partial_*^j(l_0)$, i.e. the diagram

$$\begin{tikzcd}
\partial_*^0(0) \arrow[r, "a_1"] \arrow[d, "\partial_*^0(l_0)"'] & \partial_*^1(0) \arrow[d, "\partial_*^1(l_0)"] \\
\partial_*^0(0) \arrow[r, "(-l \cdot a)_1"] & \partial_*^1(0)
\end{tikzcd}$$

must commutate. Note that a_i as a cochain is uniquely determined by its edges and these edges are determined by a_1, so the 1-simplices are in bijection with the set of $a_1, (-l \cdot a)_1 \in L_1$ and $l_0 \in L_0$ such that the above diagram commutes.

1-simplices of $\mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet))$:

Suppose $\alpha = (\alpha_0, \alpha_1, \dots)$ is a 1-simplex in $\mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet))$, then $\alpha_0 \in \mathrm{SSet}(\Delta_1 \times \Delta_0, \mathrm{Del}_\infty(L_0))$ can be represented by a 1-cochain in $\mathrm{Del}_\infty(L_0)$.

$$0 \xrightarrow{\quad l \quad} 0$$

This 1-cochain has 0 on both of its vertices and an element $l \in L_0$ on the edge. $\alpha_1 \in \mathrm{SSet}(\Delta_1 \times \Delta_1, \mathrm{Del}_\infty(L_1))$ can be view as two 1-cochains in $\mathrm{Del}_\infty(L_1)$ where all their vertices are 0 and each of their edges is an element in L_1.

$$0 \xrightarrow{\quad m_0 \quad} 0$$

$$0 \xrightarrow{\quad m_1 \quad} 0$$

The condition that $\partial_*^j(\alpha_0) = \delta_j^*(\alpha_1)$ where $j = 0, 1$ implies that the edge connecting the j-th vertex of each cochain must be $\partial_*^j(\alpha_0)$, i.e. the diagram

$$\begin{tikzcd}
\partial_*^0(0) \arrow[r, "m_0"] \arrow[d, "\partial_*^0(l)"'] & \partial_*^1(0) \arrow[d, "\partial_*^1(l)"] \\
\partial_*^0(0) \arrow[r, "m_1"] & \partial_*^1(0)
\end{tikzcd}$$

must be commutative. Since the i-cochains, $i \geq 2$, in $\mathrm{Del}_\infty(L_i)$ are uniquely determined by its edges and elements in $\mathrm{SSet}(\Delta_1 \times \Delta_i, \mathrm{Del}_\infty(L_i))$ are identified by pairs of i-cochains in $\mathrm{Del}_\infty(L_i)$, the condition $\partial_*^j(\alpha_{i-1}) = \delta_j^*(\alpha_i)$ implies that α_i's are uniquely determined by the images of the two 1-cochains in $\mathrm{Del}_\infty(L_1)$ and all the connecting edges between the two i-cochains of α_i are images of α_0. Thus the 1-simplices are in bijection with the set of $m_0, m_1 \in L_1$ and $l \in L_0$ such that the above diagram commutes.

2-simplices

2-simplices of $\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet))$:

The 2-simplices of $\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet))$ are 2-cochains of the following form:

$$\begin{tikzcd} & a \arrow[dr, "l_1"] & \\ l_0 \cdot a \arrow[ur, "l_0"] \arrow[rr, "l_2 = l_0 * l_1"'] & & -l_1 \cdot a \end{tikzcd}$$

where a is a 1-cocycle of the $\mathrm{Tot}((L_\bullet)$, $l_0 \cdot a$ and $-l_1 \cdot a$ are the resulting 1-cocycles when l_0 and $-l_1$ act on a, $l_0, l_1 \in L_0$ and their composition is given by the Baker-Campbell-Hausdorff formula in L_0.

2-simplices of $\mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet))$:

Suppose $\alpha = (\alpha_0, \alpha_1, \dots)$ is a 2-simplex in $\mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet))$, then $\alpha_0 \in \mathrm{SSet}(\Delta_2 \times \Delta_0; \mathrm{Del}_\infty(L_0))$ is represented by 2-cochains in $\mathrm{Del}_\infty(L_0)$, i.e.

$$\begin{tikzcd} & 0 \arrow[dr, "l_1"] & \\ 0 \arrow[ur, "l_0"] \arrow[rr, "l_2"'] & & 0 \end{tikzcd}$$

where $l_0 * l_1 = l_2$ through the Baker-Campbell-Hausdorff formula in L_0. $\alpha_1 \in \mathrm{SSet}(\Delta_2 \times \Delta_1; \mathrm{Del}_\infty(L_1))$ is a prism of the following form:

$$\begin{tikzcd}
& 0 \arrow[dr, "\partial^0_*(l_1)"] \arrow[dd, "a"] & \\
0 \arrow[ur, "\partial^0_*(l_0)"] \arrow[rr, "\partial^0_*(l_2)"] \arrow[dd, "l_0 \cdot a"'] & & 0 \arrow[dd, "-l_1 \cdot a"'] \\
& 0 \arrow[dr, "\partial^1_*(l_1)"] & \\
0 \arrow[ur, "\partial^1_*(l_0)"] \arrow[rr, "\partial^1_*(l_2)"'] & & 0
\end{tikzcd}$$

where a is a 1-cochain in $\mathrm{Del}_\infty(L_1)$

$$0 \xrightarrow{a} 0$$

and $l_0 \cdot a, -l_1 \cdot a$ are the 1-cochains in $\mathrm{Del}_\infty(L_1)$ that satisfies the commutative diagram with a similar to the one discussed in the 1-simplex case. The top and bottom triangles are images of α_0 through ∂^j_* and $a \in \mathrm{Del}_\infty(L_1)$. Similar to the 1-simplex case, α_0 and α_1 determines all of the α_i for $i \geq 2$, because α_i is uniquely determined by its edges and the choice of α_0 and a in α_1 uniquely determines all the edges in α_i. Hence the prism determines $\alpha \in \mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet))$.

Notice that the rectangular faces of the prism is exactly the commutative diagram discussed in 1-simplices of $\mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet))$ case. Thus, through the bijection we have established in the 0 and 1-simplices cases, each face can be identified as a 1-simplex in $\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet))$ and the prism can be identified as

$$\begin{tikzcd}
 & a \arrow[dr, "l_1"] & \\
-l_0 \cdot a \arrow[ur, "l_0"] \arrow[rr, "l_2=l_0*l_1"'] & & l_1 \cdot a
\end{tikzcd} .$$

in $\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet))$.

We have shown that the 0,1,2-simplices for $\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet))$ and $\mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet))$ match and the bijection obviously respects the face and degeneracy maps. Using Getzler's result in [7] (see Remark below), we conclude that there is an isomorphism of ∞-groupoids

$$\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet)) \cong \mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet))$$

Now since $\mathrm{Tot}(L_\bullet)$ and the L_i's are all non negatively graded, by Proposition 3.17, we have

$$\mathrm{Del}_\infty(\mathrm{Tot}(L_\bullet)) \cong \mathcal{N}(\mathrm{Del}^{\mathrm{op}}(\mathrm{Tot}(L_\bullet)))$$

and

$$\mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet)) \cong \mathrm{Tot}(\mathcal{N}(\mathrm{Del}^{\mathrm{op}}(\mathrm{Tot}(L_i))))$$

But since Tot and $\mathcal{N}$ commutes (easy check using the definition), we have

$$\mathrm{Tot}(\mathrm{Del}_\infty(L_\bullet)) \cong \mathcal{N}(\mathrm{Tot}(\mathrm{Del}^{\mathrm{op}}(\mathrm{Tot}(L_\bullet))))$$

Apply $\pi_{\leq 1}$ and op, we get an isomorphism of groupoids

$$\mathrm{Del}(\mathrm{Tot}(L_\bullet)) \cong \mathrm{Tot}(\mathrm{Del}(L_\bullet)),$$

Remark. *Getzler shows in [7] that if L is an L_∞ algebra concentrated in non negative degrees,* $\mathrm{Del}_\infty(L)$ *is a T-complex of rank 2. For our purposes, this means that* $\mathrm{Del}_\infty(L)$ *is determined by the 0,1, and 2 simplices and thus we get the statement of Proposition 3.17, i.e.* $\mathrm{Del}_\infty(L)$ *is the nerve of a groupoid.*

$\square$

4.4 Deformation of Principal G-bundles

The theory on descent of Deligne groupoids was introduced to solve deformation problems on principal G-bundles. In this section, we will use Theorem 4.12 to show that deformations of a principal G-bundle P are given by Maurer Cartan solutions of its Čech L_∞ algebra and equivalence of deformations is precisely the equivalence of Maurer Cartan solutions.

Let us start by defining principal G-bundles (G-torsors).

Remark. *In some literature there is a slight difference between the definition of principal G-bundles and G-torsors. However, for the purpose of our thesis, we will define them as the same object and use the terms principal G-bundle and G-torsor interchangeably.*

Definition 4.13. *Let X be a scheme, G algebraic group over $\mathbb{K}$, a **principal G-bundle** (**G-torsor**) is a scheme P with a free regular action of G and a morphism to X such that $X \cong P/G$. More precisely, we require the action map $\alpha : G \times P \to P : (g,p) \mapsto g(p)$ such* *that*

$$\begin{aligned} G \times P &\to P \times_X P \\ (g,p) &\mapsto (\alpha(g,p),p) \end{aligned}$$

is an isomorphism. Associativity of the action α is expressed by the requirement that the diagram

$$\begin{array}{ccc} G \times G \times P & \xrightarrow{\mathrm{id}\times\alpha} & G \times P \\ \Big\downarrow{\scriptstyle m\times\mathrm{id}} & & \Big\downarrow{\scriptstyle \alpha} \\ G \times P & \xrightarrow{\quad\alpha\quad} & P \end{array}$$

commutes.

In the case where G is unipotent, it is known that any such torsor is trivial over any affine $U \subset X$. In particular, it is locally trivial on X: for any affine covering $X = \bigcup U_i$, we can choose an isomorphism $P_{U_i} = \pi^{-1}(U_i) \cong G \times U_i$. Thus P can then be described by transition functions $\Phi_{ij} : U_i \cap U_j \to G$ which can viewed as exponent of $\varphi_{ij} : U_i \cap U_j \to \mathfrak{g}$.

Now suppose we have a principal G-bundle P over the base space X, $\pi : P \to X$, and $G = \exp(\mathfrak{g})$, where G is unipotent and $\mathfrak{g}$ is nilpotent. The structural group, i.e. the group of automorphism on the fibers, of P is precisely G. The transition function on the overlapping charts U_i and U_j is thus a section $\Phi_{ij} : U_{ij} \to G$, i.e. $\Phi_{ij} \in \Gamma(U_{ij}, G) = \exp(\Gamma(U_{ij}, \mathfrak{g}))$, $U_{ij} = U_i \cap U_j$. The deformations of a principal G-bundles P over a local Artinian algebra, (A, m_A), with residue field $\mathbb{K}$ are then determined by the sets of transition functions $\{\tilde{\Phi}_{ij}\}$ satisfying the cocycle condition

$$\tilde{\Phi}_{ij}\tilde{\Phi}_{jk} = \tilde{\Phi}_{ik}, \qquad \tilde{\Phi}_{ab} \in \exp(\Gamma(U_{ab}, \mathfrak{g} \otimes m_A)).$$

Or written in terms of Baker-Campbell-Hausdorff product on $\mathfrak{g} \otimes m_A$, we have

$$\tilde{\varphi}_{ij} * \tilde{\varphi}_{jk} = \tilde{\varphi}_{ik}, \qquad \tilde{\varphi}_{ab} \in \Gamma(U_{ab}, \mathfrak{g} \otimes m_A)$$

where $\exp(\tilde{\varphi}_{ab}) = \tilde{\Phi}_{ab}$.

Change of trivialization for P is given by

$$\tilde{\Phi}_{ij} \mapsto \tilde{\Sigma}_i^{-1} \tilde{\Phi}_{ij} \tilde{\Sigma}_j$$

where $\tilde{\Sigma}_a \in \exp(\Gamma(U_a, \mathfrak{g} \otimes m_A))$. We say that two cocycles $\{\tilde{\Phi}_{ij}\}$ and $\{\tilde{\Phi}'_{ij}\}$ are equivalent if there is a change of trivialization for P such that

$$\tilde{\Phi}'_{ij} = \tilde{\Sigma}_i^{-1} \tilde{\Phi}_{ij} \tilde{\Sigma}_j$$

Denote the $(\mathfrak{g} \otimes m_A)(\mathcal{U})_\bullet$ the Čech semicosimplicial Lie algebra associated with the sheaf of section in $\mathfrak{g} \otimes m_A$ and affine open cover $\mathcal{U}$ of the base space X. We have

$$(\mathfrak{g} \otimes m_A)(\mathcal{U})_\bullet : (\mathfrak{g} \otimes m_A)(\mathcal{U})_0 \rightrightarrows (\mathfrak{g} \otimes m_A)(\mathcal{U})_1 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} (\mathfrak{g} \otimes m_A)(\mathcal{U})_2 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots$$

The set of objects in the groupoid of descent data on $(\mathfrak{g} \otimes m_A)(\mathcal{U})_\bullet$, $\mathrm{Tot}(\mathrm{Del}((\mathfrak{g} \otimes m_A)(\mathcal{U})_\bullet))$, gives us precisely the set of transition functions $\{\tilde{\varphi}_{ab}\}$ that satisfies the cocycle condition

$$\tilde{\Phi}_{ij} \tilde{\Phi}_{jk} = \tilde{\Phi}_{ik}, \qquad \tilde{\Phi}_{ab} \in \exp(\Gamma(U_{ab}, \mathfrak{g} \otimes m_A))$$

as

$$Z^1(\exp((\mathfrak{g} \otimes m_A)(\mathcal{U})_1)) = \{\tilde{\varphi} \in (\mathfrak{g} \otimes m_A)(\mathcal{U})_1 | e^{\partial_0(\tilde{\varphi})} e^{-\partial_1(\tilde{\varphi})} e^{\partial_2(\tilde{\varphi})} = 1\}.$$

Two cocycles $\{\tilde{\Phi}_{ij}\}$ (identify as $\tilde{\varphi} \in (\mathfrak{g} \otimes m_A)(\mathcal{U})_1$) and $\{\tilde{\Phi}'_{ij}\}$ (identify as $\tilde{\varphi}' \in (\mathfrak{g} \otimes m_A)(\mathcal{U})_1$) are equivalent if there is a morphism between them in $\mathrm{Tot}(\mathrm{Del}((\mathfrak{g} \otimes m_A)(\mathcal{U})_\bullet))$, i.e. there exist $\sigma \in (\mathfrak{g} \otimes m_A)(\mathcal{U})_0$ such that

$$e^{\tilde{\varphi}'} = e^{-\partial_1(\sigma)} e^{\tilde{\varphi}} e^{\partial_0(\sigma)}$$

Thus the groupoid $\mathrm{Tot}(\mathrm{Del}((\mathfrak{g} \otimes m_A)(\mathcal{U})_\bullet))$ gives us all the nonabelian cocycles and their equivalence. Apply Theorem 4.12 to get an isomorphism of groupoids between $\mathrm{Tot}(\mathrm{Del}((\mathfrak{g} \otimes m_A)(\mathcal{U})_\bullet))$ and $\mathrm{Del}(\mathrm{Tot}((\mathfrak{g} \otimes m_A)(\mathcal{U})_\bullet))$, we get

Corollary 4.14. *Suppose we have a principal G-bundle P over the base space X, $\pi : P \to X$, and $G = \exp(\mathfrak{g})$, where G is unipotent and $\mathfrak{g}$ is nilpotent. The deformations of the principal G-bundle P are given by the Maurer Cartan solutions $\mathrm{MC}(\mathrm{Tot}((\mathfrak{g} \otimes m_A)(\mathcal{U})_\bullet))$ and the equivalence of deformations is precisely the equivalence of Maurer Cartan solutions, i.e. the solution for the cocycle condition*

$$\tilde{\Phi}_{ij}\tilde{\Phi}_{jk} = \tilde{\Phi}_{ik}$$

is in bijection with the Maurer Cartan solutions for $\mathrm{Tot}((\mathfrak{g} \otimes m_A)(\mathcal{U})_\bullet)$ and the equivalence (change of trivialization)

$$\tilde{\Phi}_{ij} \mapsto \tilde{\Sigma}_i^{-1} \tilde{\Phi}_{ij} \tilde{\Sigma}_j$$

is in bijection with the morphisms between $\mathrm{MC}(\mathrm{Tot}((\mathfrak{g} \otimes m_A)(\mathcal{U})_\bullet)))$ and this bijection respects composition of morphisms (change of trivialization).

Chapter 5

Extension of Principal G-bundles

Suppose we have a principal G-bundle P and $G = \exp(\mathfrak{g})$, where G is unipotent and $\mathfrak{g}$ is nilpotent, and suppose we have a Lie algebra extension $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ by another nilpotent Lie algebra $\mathfrak{h}$

$$0 \to \mathfrak{h} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

where $c : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{h}$ and $b : \mathfrak{g} \wedge \mathfrak{h} \to \mathfrak{h}$ determines the extension (and thus also the Lie group extension). Suppose $\tilde{G} = \exp(\tilde{\mathfrak{g}})$, our question is to ask what are the principal $\tilde{G}$-bundles $\tilde{P}$ that extend P.

From Manetti and Bandiera's work, Corollary 4.14 in this thesis, we know that the transition functions for the principal $\tilde{G}$-bundle satisfy the nonabelian cocycle condition

$$\tilde{\Phi}_{ij}\tilde{\Phi}_{jk}\tilde{\Phi}_{ki} = 1, \qquad \tilde{\Phi}_{ab} \in \Gamma(U_{ab}, \tilde{G})$$

if and only if the corresponding degree 1 element a in the L_∞ $\check{C}$ech complex, $\mathrm{Tot}(\tilde{\mathfrak{g}}(\mathcal{U})_\bullet)$,

$$\tilde{\mathfrak{g}}(\mathcal{U})_\bullet : \prod_i \Gamma(U_i, \tilde{\mathfrak{g}}) \rightrightarrows \prod_{i<j} \Gamma(U_{ij}, \tilde{\mathfrak{g}}) \Rrightarrow \prod_{i<j<k} \Gamma(U_{ijk}, \tilde{\mathfrak{g}}) \Rrightarrow \cdots$$

satisfies the L_∞ Maurer Cartan equation and that the equivalence of cocycles condition is the same as equivalence of Maurer Cartan solutions.

In this chapter, we will apply the above result together with results from [6] to show that the cocycle condition on the transition functions of the principal $\tilde{G}$-bundle $\tilde{P}$ lifting P (that are compatible with our extension) is the same as the curved Maurer Cartan equation on $\mathrm{Tot}(\mathfrak{h}(\mathcal{U})_\bullet)$ and the equivalence of cocycles is the same as the equivalence of curved Maurer Cartan solutions.

Remark. *To simplify notations, given a semicosimplicial Lie algebra* $\mathfrak{f}(\mathcal{U})_\bullet$ *we will use* $\mathcal{L}(\mathfrak{f})$ *to denote* $\mathrm{Tot}(\mathfrak{f}(\mathcal{U})_\bullet)$*,* $\hat{\mathcal{L}}(\mathfrak{f})$ *to denote* $\mathrm{Tot}_{\mathrm{TW}}(\mathfrak{f}(\mathcal{U})_\bullet)$*. We will also use* $[-,-]_{\mathfrak{f}}$ *to denote* $[-,-]_{\mathrm{Tot}_{\mathrm{TW}}(\mathfrak{f}(\mathcal{U})_\bullet)}$*, and* $[-,-]_{\hat{\mathfrak{f}}}$ *to denote* $[-,-]_{\mathrm{Tot}_{\mathrm{TW}}(\mathfrak{f}(\mathcal{U})_\bullet)\otimes\mathbb{K}[s,ds]}$*. In the case where it is clear from the context what bracket we are using, we will omit the subscripts all together.*

5.1 Lie Algebra Extensions

Before we can talk about extension of torsors, we need to definite a Lie algebra extension.

Definition 5.1. *Let* $\mathfrak{g}$ *and* $\mathfrak{h}$ *be two Lie algebras. An* ***extension*** $\tilde{\mathfrak{g}}$ *of* $\mathfrak{g}$ *by* $\mathfrak{h}$ *is a short exact sequence of the form*

$$0 \to \mathfrak{h} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0.$$

Definition 5.2. *Let* $\tilde{\mathfrak{g}}$ *and* $\tilde{\mathfrak{g}}'$ *be two extensions of* $\mathfrak{g}$ *by* $\mathfrak{h}$*.* $\tilde{\mathfrak{g}}$ *and* $\tilde{\mathfrak{g}}'$ *are said to be equivalent if there exits a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h} \arrow[r] \arrow[d, equal] & \tilde{\mathfrak{g}} \arrow[r] \arrow[d, "\varphi"] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathfrak{h} \arrow[r] & \tilde{\mathfrak{g}}' \arrow[r] & \mathfrak{g} \arrow[r] & 0 \end{tikzcd}$$

Definition 5.3. *A* ***non-abelian 2-cocycle*** *on* $\mathfrak{g}$ *with values in* $\mathfrak{h}$ *is a couple* (c, b) *of linear maps* $c : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{h}$ *and* $b : \mathfrak{g} \to \mathrm{Der}(h)$ *satisfying*

$$[b(x), b(y)] - b([x, y]) = \mathrm{ad}(c(x, y))$$

and

$$\sum b(x, c(y, z)) - c(b(x, y), z) = 0$$

where the sum is taken over cyclic permutations of x*,* y*, and* z*. We denote the set of non-abelian 2-cocycles* $\mathrm{Z}^{\mathbf{2}}_{\mathrm{nab}}(\mathfrak{g}, \mathfrak{h})$*. Two non-abelian 2-cocyles are equivalent,* $(c, b) \sim (c', b')$ *if there exist* $\beta : \mathfrak{g} \to \mathfrak{h}$ *satisfying*

$$b'_x = b_x + \mathrm{ad}_{\beta(x)}$$

and

$$c'(x, y) = c(x, y) + b_x(\beta(y)) - b_y(\beta(x)) - \beta([x, y]) + [\beta(x), \beta(y)]$$

Non-abelian cohomology $\mathrm{H}^{\mathbf{2}}_{\mathrm{nab}}(\mathfrak{g}, \mathfrak{h})$ *will be the quotient of* $\mathrm{Z}^2_{\mathrm{nab}}(\mathfrak{g}, \mathfrak{h})$ *by the equivalence relation.*

Notice that by choosing a vector space splitting $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{h}$ compatible with the embedding of $\mathfrak{h}$ and projection to $\mathfrak{g}$, the bracket of $\tilde{\mathfrak{g}}$ gives us a cocycle. For $x \in \mathfrak{g}$ and $y \in \mathfrak{h}$, $b : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ is given by $b(x)(y) = [x, y]_{\tilde{\mathfrak{g}}}$, and for $x, x' \in \mathfrak{g}$, $c : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{h}$ is given by the $\mathfrak{h}$ component of $[x, x']_{\tilde{\mathfrak{g}}}$.

A straight forward computation shows that

Theorem 5.4. *Extensions of $\mathfrak{g}$ by $\mathfrak{h}$ are classified by $\mathrm{H}^2_{\mathrm{nab}}(\mathfrak{g}, \mathfrak{h})$.*

In the paper [4], it is shown that $\mathrm{H}^2_{\mathrm{nab}}(\mathfrak{g}, \mathfrak{h})$ can be identified as equivalence classes of Maurer Cartan solutions for some DGLA L. For our purpose, it is enough for us to know that the maps b and c fully describe an extension.

5.2 Twisted Cocycle Condition and Twisted Cocycle Equivalence

We can now state the main problem of this thesis more precisely. Suppose we have a principal G-bundle P over the base space X, $\pi : P \to X$, and $G = \exp(\mathfrak{g})$, where G is unipotent and $\mathfrak{g}$ is nilpotent. Suppose we have a Lie algebra extension $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ by another nilpotent Lie algebra $\mathfrak{h}$

$$0 \to \mathfrak{h} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

where $c : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{h}$ and $b : \mathfrak{g} \wedge \mathfrak{h} \to \mathfrak{h}$ determines the extension. Since $\mathfrak{g}$ and $\mathfrak{h}$ are both nilpotent, we obtain a Lie group extension $\tilde{G}$ of G by H, $G = \exp(\mathfrak{g})$ and $H = \exp(\mathfrak{h})$,

$$1 \to H \to \tilde{G} \to G \to 1.$$

We can view $\tilde{G} = G \times H$ as a product of varieties by choosing a splitting of $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{h}$ (vector space splitting) and thus get an embedding $G \hookrightarrow \tilde{G}$ (embedding of variety) by embedding $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ and using the exponential map. The multiplication of $\tilde{G}$ is not the regular multiplication of $G \times H$ and is determined by c and b. Our question is then what are the principal $\tilde{G}$-bundles $\tilde{P}$ that extend P.

The structural group, i.e. the group of automorphism on the fibers, of $\tilde{P}$ is precisely $\tilde{G}$. The transition function on the overlapping charts U_i and U_j is thus a section $\tilde{\Phi}_{ij} : U_{ij} \to \tilde{G}$, i.e. $\tilde{\Phi}_{ij} \in \Gamma(U_{ij}, \tilde{G})$. The extensions of principal bundles $\tilde{P}$ over P are then determined by the sets of transition functions $\{\tilde{\Phi}_{ij}\}$ which extensions the transition functions $\{\Phi_{ij}\}$ of P while satisfying the cocycle condition

$$\tilde{\Phi}_{ij}\tilde{\Phi}_{jk} = \tilde{\Phi}_{ik}, \qquad \tilde{\Phi}_{ab} \in \Gamma(U_{ab}, \tilde{G})$$

Or written in terms of Baker-Campbell-Hausdorff product on $\tilde{\mathfrak{g}}$, we have

$$\tilde{\varphi}_{ij} * \tilde{\varphi}_{jk} = \tilde{\varphi}_{ik}, \qquad \tilde{\varphi}_{ab} \in \Gamma(U_{ab}, \tilde{\mathfrak{g}})$$

where pointwise $*$ is given by the multiplication of $\tilde{G}$ and $\exp(\tilde{\varphi}_{ab}) = \tilde{\Phi}_{ab}$.

By construction, $\tilde{\Phi}_{ij} = \Phi_{ij}\Psi_{ij} = \exp(\varphi_{ij})\exp(\psi_{ij})$ where $\Phi_{ij} \in \Gamma(U_{ij}, G)$, $\Psi_{ij} \in \Gamma(U_{ij}, H)$, $\varphi_{ij} \in \Gamma(U_{ij}, \mathfrak{g})$, and $\psi_{ij} \in \Gamma(U_{ij}, \mathfrak{h})$. The group value cocycle condition can then be rewritten as (product is taken in $\tilde{G}$)

$$\begin{aligned}
\exp(\varphi_{ij})\exp(\psi_{ij})\exp(\varphi_{jk})\exp(\psi_{jk}) &= \exp(\varphi_{ik})\exp(\psi_{ik}) \\
(\exp(\varphi_{ij})\exp(\varphi_{jk}))(\exp(-\varphi_{jk})\exp(\psi_{ij})\exp(\varphi_{jk}))\exp(\psi_{jk}) &= \exp(\varphi_{ik})\exp(\psi_{ik})
\end{aligned}$$

When $\{\varphi_{ab}\}$ are transition functions of P, then we have

$$
\begin{aligned}
&\exp(\varphi_{ij}) \cdot_{\tilde{G}} \exp(\varphi_{jk}) \\
=&(\exp(\varphi_{ij}) \cdot_G \exp(\varphi_{jk}))\mathcal{C}(\varphi_{ij}, \varphi_{jk}) \\
=&\exp(\varphi_{ik})\mathcal{C}(\varphi_{ij}, \varphi_{jk})
\end{aligned}
$$

where $\mathcal{C}(\varphi_{ij}, \varphi_{jk})$ is the H component of $\exp(\varphi_{ij})\cdot_{\tilde{G}}\exp(\varphi_{jk})$. If we rewrite $\exp(\varphi_{ij})\cdot_{\tilde{G}}\exp(\varphi_{jk})$ in Lie algebra terms using the Baker-Campbell-Hausdorff formula on $\tilde{\mathfrak{g}}$, i.e. $\varphi_{ij} *_{\tilde{\mathfrak{g}}} \varphi_{jk}$, then $\mathcal{C}(\varphi_{ij}, \varphi_{jk})$ is precisely the exponent of the $\mathfrak{h}$ component of $\varphi_{ij} *_{\tilde{\mathfrak{g}}} \varphi_{jk}$, which has junior terms

$$
\frac{1}{2}c(\varphi_{ij}, \varphi_{jk})+\frac{1}{12}(c(\varphi_{ij}, [\varphi_{ij}, \varphi_{jk}])+b(\varphi_{ij}, c(\varphi_{ij}, \varphi_{jk}))+c(\varphi_{jk}, [\varphi_{jk}, \varphi_{ij}])+b(\varphi_{jk}, c(\varphi_{jk}, \varphi_{ij})))+\ldots
$$

Combining this with the fact that

$$
\begin{aligned}
&\exp(-\varphi_{jk}) \exp(\psi_{ij}) \exp(\varphi_{jk}) \\
&= \mathrm{Ad}_{\exp(-\varphi_{jk})} \exp(\psi_{ij}) \\
&= \exp(\mathrm{ad}_{-\varphi_{jk}}) \exp(\psi_{ij}) \\
&= \exp((-1)^s \sum_{s=0}^{\infty} \frac{(\mathrm{ad}_{\varphi_{jk}})^s}{s!}(\psi_{ij})),
\end{aligned}
$$

the twisted group valued cocycles that give us the extensions of P can then we rewritten as

$$
\mathcal{C}(\varphi_{ij}, \varphi_{jk}) \exp((-1)^s \sum_{s=0}^{\infty} \frac{(\mathrm{ad}_{\varphi_{jk}})^s}{s!}(\psi_{ij})) \exp(\psi_{jk}) = \exp(\psi_{ik}).
$$

Change of trivialization for $\tilde{P}$ (without changing the trivialization for P) is given by

$$
\tilde{\Phi}_{ij} \mapsto \Sigma_i^{-1}\tilde{\Phi}_{ij}\Sigma_j
$$

where $\Sigma_a \in \Gamma(U_a, H)$. We can rewrite this change of cocycle as

$$\begin{aligned}
&\Sigma_i^{-1}\tilde{\Phi}_{ij}\Sigma_j \\
=&\Sigma_i^{-1}\Phi_{ij}\Psi_{ij}\Sigma_j \\
=&\Phi_{ij}(\Phi_{ij}^{-1}\Sigma_i^{-1}\Phi_{ij})\Psi_{ij}\Sigma_j
\end{aligned}$$

Let $\Sigma_a = \exp(\sigma_a)$ and rewrite the H component of the above in Lie algebra terms, we get

$$\exp((-1)^{s+1}\sum_{s=0}^{\infty}\frac{(\mathrm{ad}_{\varphi_{ij}})^s}{s!}(\sigma_i))\exp(\psi_{ij})\exp(\sigma_j)$$

Thus two twisted cocyles $\{\exp(\psi_{ij})\}$ and $\{\exp(\psi'_{ij})\}$ are equivalent iff there exist $\{\sigma_a \in \Gamma(U_a, \mathfrak{h})\}$ such that

$$\exp(\psi'_{ij}) = \exp((-1)^{s+1}\sum_{s=0}^{\infty}\frac{(\mathrm{ad}_{\varphi_{ij}})^s}{s!}(\sigma_i))\exp(\psi_{ij})\exp(\sigma_j)$$

for all overlapping charts U_{ij}.

The main result of [3], Corollary 4.14 in this thesis, is that the cocycle condition on an affine open cover $\mathcal{U}$ is essentially the same as the Maurer Cartan equation on the L_∞ Čech complex. It gives us a way to turn a non-abelian problem on the Lie group level to a problem on the Lie algebra (L_∞) level. Applying this theorem to our problem, we get that the transition functions for the principal $\tilde{G}$-bundle satisfies the nonabelian cocycle condition on $\mathcal{U}$ if and only if the corresponding degree 1 element $\tilde{a}$ in the Čech complex, $\mathcal{L}(\tilde{\mathfrak{g}})$,

$$\tilde{\mathfrak{g}}(\mathcal{U})_\bullet : \prod_i \Gamma(U_i, \tilde{\mathfrak{g}}) \rightrightarrows \prod_{i<j} \Gamma(U_{ij}, \tilde{\mathfrak{g}}) \rightrightarrows \prod_{i<j<k} \Gamma(U_{ijk}, \tilde{\mathfrak{g}}) \rightrightarrows \cdots$$

satisfies the L_∞ Maurer Cartan equation.

Notice that the transition functions of our principal G-bundle P give us a Maurer Cartan solution a in $\mathcal{L}(\mathfrak{g})$ and we have a surjection $\mathcal{L}(\tilde{\mathfrak{g}}) \to \mathcal{L}(\mathfrak{g})$. The extension problem of P is thus the same as the problem of lifting a to $\mathcal{L}(\tilde{\mathfrak{g}})$. Lifting a Maurer Cartan solution over a surjection is difficult in general but we have a vector space splitting $\mathcal{L}(\tilde{\mathfrak{g}}) = \mathcal{L}(\mathfrak{h}) \oplus \mathcal{L}(\mathfrak{g})$ and on $\mathcal{L}(\mathfrak{h})$ we can define a curved L_∞ algebra structure.

The **goal of this thesis** is to show that the twisted cocycle condition that gives us the extensions of the principal G-bundle P

$$\mathcal{C}(\varphi_{ij}, \varphi_{jk}) \exp((-1)^s \sum_{s=0}^{\infty} \frac{(\mathrm{ad}_{\varphi_{jk}})^s}{s!}(\psi_{ij})) \exp(\psi_{jk}) = \exp(\psi_{ik})$$

is the same as the curved Maurer Cartan equation on $\mathcal{L}(\mathfrak{h})$ and the equivalence of twisted cocycles given by

$$\exp(\psi'_{ij}) = \exp((-1)^{s+1} \sum_{s=0}^{\infty} \frac{(\mathrm{ad}_{\varphi_{ij}})^s}{s!}(\sigma_i)) \exp(\psi_{ij}) \exp(\sigma_j)$$

is the same as the equivalence of curved Maurer Cartan solutions on $\mathcal{L}(\mathfrak{h})$.

We will break down the proof of this result into two parts. We will start out by showing that there is a bijection between Maurer Cartan solutions of the Thom Whitney Complex of $\tilde{\mathfrak{g}}(\mathcal{U})_\bullet$, $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$, after fixing an element $a \in \mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{g}))$ and the curved Maurer Cartan solutions of the curved Thom Whitney Complex of $\mathfrak{h}(\mathcal{U})_\bullet$, $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$, and and also an agreement between equivalence of the respective solutions.

5.3 Curved Maurer Cartan Solutions for Thom Whitney Complex

5.3.1 Bijection between Maurer Cartan Solutions

Consider the Thom-Whitney complex of $\tilde{\mathfrak{g}}(\mathcal{U})_\bullet$, $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$. By construction, $\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) = \hat{\mathcal{L}}(\mathfrak{g}) \oplus \hat{\mathcal{L}}(\mathfrak{h})$ as a complex (induced by the vector space splitting $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{h}$). The Maurer Cartan equation for $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$ is given by

$$d(a+\alpha) + \frac{1}{2}[a+\alpha, a+\alpha]_{\tilde{\mathfrak{g}}} = 0$$

where $a \in \hat{\mathcal{L}}(\mathfrak{g})^1$, $\alpha \in \hat{\mathcal{L}}(\mathfrak{h})^1$. When we expand the left hand side, we get

$$\begin{aligned} & da + d\alpha + \frac{1}{2}([a,a]_{\tilde{\mathfrak{g}}} + [a,\alpha]_{\tilde{\mathfrak{g}}} + [\alpha,a]_{\tilde{\mathfrak{g}}} + [\alpha,\alpha]_{\tilde{\mathfrak{g}}}) \\ = \; & da + \frac{1}{2}[a,a]_{\tilde{\mathfrak{g}}} + d\alpha + \frac{1}{2}(2[a,\alpha]_{\tilde{\mathfrak{g}}} + [\alpha,\alpha]_{\tilde{\mathfrak{g}}}) \\ = \; & da + \frac{1}{2}[a,a]_{\mathfrak{g}} + \frac{1}{2}c(a,a) + (d + \mathrm{ad}_a)(\alpha) + \frac{1}{2}[\alpha,\alpha]_{\mathfrak{h}} \end{aligned}$$

Assume that a satisfies the Maurer Cartan equation for $\hat{\mathcal{L}}(\mathfrak{g})$, i.e., $da + \frac{1}{2}[a,a]_{\mathfrak{g}} = 0$ and define $C = \frac{1}{2}c(a,a)$, then the the Maurer Cartan equation for $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$ reduces to

$$C + (d + \mathrm{ad}_a)(\alpha) + \frac{1}{2}[\alpha,\alpha]_{\mathfrak{h}} = 0.$$

This equation gives us the Maurer Cartan solutions of $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$ of the form $a + \alpha$ where $a \in \mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{g}))$ and $\alpha \in \hat{\mathcal{L}}(\mathfrak{h})$, both $\hat{\mathcal{L}}(\mathfrak{g})$ and $\hat{\mathcal{L}}(\mathfrak{h})$ viewed as a subcomplex using the decomposition $\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) = \hat{\mathcal{L}}(\mathfrak{g}) \oplus \hat{\mathcal{L}}(\mathfrak{h})$.

We will now define a new (curved) differential $d_{\mathfrak{h}}$ on $\hat{\mathcal{L}}(\mathfrak{h})$ where $d_{\mathfrak{h}}$ is the restriction of $d + \mathrm{ad}_a$ (defined in $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$) on $\hat{\mathcal{L}}(\mathfrak{h}) \subset \hat{\mathcal{L}}(\tilde{\mathfrak{g}}) = \hat{\mathcal{L}}(\mathfrak{g}) \oplus \hat{\mathcal{L}}(\mathfrak{h})$. We claim that $(\hat{\mathcal{L}}(\mathfrak{h}), d_{\mathfrak{h}}, [-,-]_{\mathfrak{h}})$ is a curved

DGLA (see Appendix). Note that this new differential respects the brackets (i.e. it obey the product rule) by the nature of ad_a, so all we have to check is $d_{\mathfrak{h}}^2 = [C, -]$.

Consider $d_{\mathfrak{h}}^2$ on $u \in \hat{\mathcal{L}}(\mathfrak{h}) \subset \hat{\mathcal{L}}(\tilde{\mathfrak{g}}) = \hat{\mathcal{L}}(\mathfrak{g}) \oplus \hat{\mathcal{L}}(\mathfrak{h})$.

$$\begin{aligned}
& d_{\mathfrak{h}}(d_{\mathfrak{h}}(u)) \\
= \quad & d_{\mathfrak{h}}(du + [a, u]_{\tilde{\mathfrak{g}}}) \\
= \quad & d^2 u + [a, du]_{\tilde{\mathfrak{g}}} + d[a, u]_{\tilde{\mathfrak{g}}} + [a, [a, u]_{\tilde{\mathfrak{g}}}]_{\tilde{\mathfrak{g}}}
\end{aligned}$$

Note that $d^2 = 0$ and $d[a, u]_{\tilde{\mathfrak{g}}} = [da, u]_{\tilde{\mathfrak{g}}} + (-1)^{|a|}[a, du]_{\tilde{\mathfrak{g}}} = [da, u]_{\tilde{\mathfrak{g}}} - [a, du]_{\tilde{\mathfrak{g}}}$ as d is a differential in $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$. Since we assume that a satisfies the Maurer Cartan equation in $\hat{\mathcal{L}}(\mathfrak{g})$, we also have $da = -\frac{1}{2}[a, a]_{\mathfrak{g}}$. Combining these with the Jacobi identity $[a, [a, u]_{\tilde{\mathfrak{g}}}]_{\tilde{\mathfrak{g}}} = \frac{1}{2}[[a, a]_{\tilde{\mathfrak{g}}}, u]_{\tilde{\mathfrak{g}}}$ we get

$$\begin{aligned}
& d_{\mathfrak{h}}^2(u) \\
= \quad & [da, u]_{\tilde{\mathfrak{g}}} + [a, [a, u]_{\tilde{\mathfrak{g}}}]_{\tilde{\mathfrak{g}}} \\
= \quad & -\frac{1}{2}[[a, a]_{\mathfrak{g}}, u]_{\tilde{\mathfrak{g}}} + \frac{1}{2}[[a, a]_{\tilde{\mathfrak{g}}}, u]_{\tilde{\mathfrak{g}}} \\
= \quad & \frac{1}{2}[c(a, a), u]_{\tilde{\mathfrak{g}}} \\
= \quad & [C, u]_{\mathfrak{h}}
\end{aligned}$$

Thus we conclude that $(\hat{\mathcal{L}}(\mathfrak{h}), d_{\mathfrak{h}}, [-, -]_{\mathfrak{h}})$ is a curved DGLA. Since the curved Maurer Cartan equation for $\hat{\mathcal{L}}(\mathfrak{h})$ is precisely

$$C + (d + \mathrm{ad}_a)(\alpha) + \frac{1}{2}[\alpha, \alpha]_{\mathfrak{h}} = 0,$$

the set of curved Maurer Cartan solutions for $\hat{\mathcal{L}}(\mathfrak{h})$ is in bijection with the set of Maurer Cartan solutions for $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$ after fixing a Maurer Cartan solution a for $\hat{\mathcal{L}}(\mathfrak{g})$. Denote $\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}))$

the set of Maurer Cartan solutions $a+\alpha$ for $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$ after fixing $a \in \mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{g}))$, we thus get

$$\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}})) \cong \mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{h}))$$

5.3.2 Bijection between Equivalence of Maurer Cartan Solutions

Recall that two Maurer Cartan solutions $z, z' \in \mathrm{MC}(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}))$ are equivalent if and only if there exits $\tilde{z} \in \mathrm{MC}(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \mathbb{K}[s, ds])$ such that

$$\tilde{z}|_{s=0} = z, \qquad \tilde{z}|_{s=1} = z'$$

Elements of $\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \mathbb{K}[s, ds]$ are sum of simple tensors of $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$ and $\mathbb{K}[s, ds]$, i.e. they are polynomials of s and ds with coefficients over $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$.

The differential $\tilde{d}$ of $\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \mathbb{K}[s, ds]$ is given by

$$\begin{aligned}
& \tilde{d}(x(s) + y(s)ds) \\
= \;& d(x(s)) - d(y(s))ds - \frac{dx}{ds}(s)ds \\
= \;& d(x(s)) - (d(y(s)) + \frac{dx}{ds}(s))ds
\end{aligned}$$

$\tilde{d}^2$ is then given by

$$\begin{aligned}
& \tilde{d}^2(x(s) + y(s)ds) \\
= \;& d^2(x(s)) - \frac{d}{ds}(d(x(s))) + d(d(y(s)) + \frac{dx}{ds}(s))ds \\
= \;& d^2(x(s)) - \frac{d}{ds}(d(x(s)) + d^2(y(s))ds + d(\frac{dx}{ds}(s))ds
\end{aligned}$$

Since d and $\frac{d}{ds}$ commutes by the way we construct $\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \mathbb{K}[s, ds]$, we get

$$\tilde{d}^2(x(s) + y(s)ds) = d^2(x(s)) + d^2(y(s))ds = 0$$

The bracket of $\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \mathbb{K}[s, ds]$ is given by

$$\begin{aligned} & [x_1(s) + y_1(s)ds, x_2(s) + y_2(s)ds] \\ = \quad & [x_1(s), x_2(s)] + (-1)^{|x_1(s)|}[x_1(s), y_2(s)]ds - [y_1(s), x_2(s)]ds \end{aligned}$$

The evaluation map $\mathrm{Eval}_{s=s_0} : \hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \mathbb{K}[s, ds] \to \hat{\mathcal{L}}(\tilde{\mathfrak{g}})$ is given by

$$\mathrm{Eval}_{s=s_0}(x(s) + y(s)ds) = x(s_0)$$

It can be easily checked that this is a DGLA and that the evaluation map is a DGLA morphism [11].

Now let's check that if $a+\alpha$ and $a+\alpha'$ are equivalent Maurer Cartan solution for $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$, then α and α' are equivalent curved Maurer Cartan solution for $\hat{\mathcal{L}}(\mathfrak{h})$.

Notice that $\tilde{z} \in \hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \mathbb{K}[s, ds]$ can be written as $\tilde{a} + \tilde{\alpha}$ where $\tilde{a} \in \hat{\mathcal{L}}(\mathfrak{g}) \otimes \mathbb{K}[s, ds]$ and $\tilde{\alpha} \in \hat{\mathcal{L}}(\mathfrak{h}) \otimes \mathbb{K}[s, ds]$ as we have a splitting

$$\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \mathbb{K}[s, ds] = \hat{\mathcal{L}}(\mathfrak{g}) \otimes \mathbb{K}[s, ds] \oplus \hat{\mathcal{L}}(\mathfrak{h}) \otimes \mathbb{K}[s, ds].$$

But since we are not changing the trivialization of P, we are only interested in the case $\tilde{a} = a \in \hat{\mathcal{L}}(\mathfrak{g}) \otimes \mathbb{K}[s, ds]$. If $\tilde{z}|_{s=0} = a + \alpha$ and $\tilde{z}|_{s=1} = a + \alpha'$, it is clear that

$$a|_{s=0} = a, \qquad a|_{s=1} = a$$

$$\tilde{\alpha}|_{s=0} = \alpha, \qquad \tilde{\alpha}|_{s=1} = \alpha'$$

and that a (as a constant polynomial) satisfies the Maurer Cartan equation for $\hat{\mathcal{L}}(\mathfrak{g}) \otimes \mathbb{K}[s, ds]$.

Now in order to show that α and α' are equivalent curved Maurer Cartan solutions, we need to show that $\tilde{\alpha}$ is a curved Maurer Cartan solution for $\hat{\mathcal{L}}(\mathfrak{h}) \otimes \mathbb{K}[s, ds]$.

The first thing to check is that $(\hat{\mathcal{L}}(\mathfrak{h}) \otimes \mathbb{K}[s, ds], \tilde{d}_{\mathfrak{h}}, [-,-]_{\hat{\mathfrak{h}}})$ does in fact have a curved DGLA structure. Note that the differential, $\tilde{d}_{\mathfrak{h}}$, for $\hat{\mathcal{L}}(\mathfrak{h}) \otimes \mathbb{K}[s, ds]$ is defined as

$$\begin{aligned} & \tilde{d}_{\mathfrak{h}}(x(s) + y(s)ds) \\ = \; & d_{\mathfrak{h}}(x(s)) - (d_{\mathfrak{h}}(y(s)) + \frac{dx}{ds}(s))ds \end{aligned}$$

and $\tilde{d}_{\mathfrak{h}}{}^2$ is then given by

$$\begin{aligned} & \tilde{d}_{\mathfrak{h}}{}^2(x(s) + y(s)ds) \\ = \; & d_{\mathfrak{h}}^2(x(s)) + d_{\mathfrak{h}}^2(y(s))ds \end{aligned}$$

Define $\tilde{C} = C + 0\, ds = C$, then

$$\begin{aligned} & [\tilde{C}, x(s) + y(s)ds] \\ = \; & [C, x(s)] + [C, y(s)]ds \\ = \; & d_{\mathfrak{h}}^2(x(s)) + d_{\mathfrak{h}}^2(y(s))ds. \end{aligned}$$

Thus $\tilde{d}_{\mathfrak{h}}^2 = [\tilde{C}, -]_{\hat{\mathfrak{h}}}$ and $\hat{\mathcal{L}}(\mathfrak{h}) \otimes \mathbb{K}[s, ds]$ is a curved DGLA.

Consider the Maurer Cartan equation for $\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \mathbb{K}[s, ds]$ and $a + \tilde{\alpha} \in \mathrm{MC}(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \mathbb{K}[s, ds])$. We have

$$
\begin{aligned}
& \tilde{d}(a+\tilde{\alpha}) + \frac{1}{2}[a+\tilde{\alpha}, a+\tilde{\alpha}]_{\hat{\tilde{g}}} \\
= \quad & \tilde{d}(a) + \frac{1}{2}[a,a]_{\hat{\tilde{g}}} + (\tilde{d} + \mathrm{ad}_a)(\tilde{\alpha}) + \frac{1}{2}[\tilde{\alpha}, \tilde{\alpha}]_{\hat{\tilde{g}}} \\
= \quad & \tilde{d}(a) + \frac{1}{2}[a,a]_{\hat{\mathfrak{g}}} + \frac{1}{2}c(a,a) + (\tilde{d} + \mathrm{ad}_a)(\tilde{\alpha}) + \frac{1}{2}[\tilde{\alpha}, \tilde{\alpha}]_{\hat{\mathfrak{h}}} \\
= \quad & \tilde{C} + (\tilde{d} + \mathrm{ad}_a)(\tilde{\alpha}) + \frac{1}{2}[\tilde{\alpha}, \tilde{\alpha}]_{\hat{\mathfrak{h}}} \\
= \quad & 0
\end{aligned}
$$

But $\tilde{d}_{\mathfrak{h}} = \tilde{d} + \mathrm{ad}_a$ as

$$
\begin{aligned}
& \tilde{d}_{\mathfrak{h}}(x(s) + y(s)ds) \\
= \quad & (d + \mathrm{ad}_a)(x(s)) - ((d + \mathrm{ad}_a)(y(s)) + \frac{dx}{ds}(s))ds \\
= \quad & d(x(s)) - (d(y(s)) + \frac{dx}{ds}(s))ds + (\mathrm{ad}_a(x(s)) - \mathrm{ad}_a(y(s)))ds \\
= \quad & \tilde{d}(x(s) + y(s)ds) + \mathrm{ad}_a(x(s) + y(s)ds) \\
= \quad & (\tilde{d} + \mathrm{ad}_a)(x(s) + y(s)ds),
\end{aligned}
$$

so the curved Maurer Cartan equation for $\hat{\mathcal{L}}(\mathfrak{h}) \otimes \mathbb{K}[s, ds]$ is exactly the same as the Maurer Cartan equation for $\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \mathbb{K}[s, ds]$ and we conclude that $\tilde{\alpha} \in \mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{h}) \otimes \mathbb{K}[s, ds])$.

Thus, if $a + \alpha$ and $a + \alpha'$ are equivalent Maurer Cartan solutions in $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$ given by $a + \tilde{\alpha}$

$$
a|_{s=0} = a, \qquad a|_{s=1} = a
$$

$$
\tilde{\alpha}|_{s=0} = \alpha, \qquad \tilde{\alpha}|_{s=1} = \alpha',
$$

then α and α' are equivalent curved Maurer Cartan solutions in $\hat{\mathcal{L}}(\mathfrak{h})$ given by $\tilde{\alpha}$

$$\tilde{\alpha}|_{s=0} = \alpha, \qquad\qquad \tilde{\alpha}|_{s=1} = \alpha'$$

Define $\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1)$ as the subset of $\mathrm{MC}(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1)$ where elements are of the form $a + \tilde{\alpha}$. We then obtain a bijection between $\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1)$ and $\mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{h}) \otimes \Omega_1)$ that respects the evaluation maps.

The (opposite) morphisms between Maurer Cartan solutions $a + \alpha$ and $a + \alpha'$ in $\mathrm{Del}(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}))$ given by elements of the form $a + \tilde{\alpha} \in \mathrm{MC}(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1)$ are precisely the homotopy classes of $a + \tilde{\alpha}$ by Proposition 3.23.

In fact, as we have discussed in Corollary 3.24, the (opposite) morphisms between Maurer Cartan solutions for an L_∞ algebra L can be identified as the 1-simplices of $\mathrm{Del}_\infty(L)$. We can then apply formal Kuranishi on the contraction

$$\begin{tikzcd} C^*(\Delta_1; L) \arrow[r, shift left] & L \otimes \Omega_1 \arrow[l, shift left] \arrow[loop right, "K"] \end{tikzcd}$$

where K is the Dupont homotopy, to identify the 1-simplices of $\mathrm{Del}_\infty(L)$ with the set

$$\mathrm{MC}(L \otimes \Omega_1, K) = \{x \in \mathrm{MC}(L \otimes \Omega_1) | K(x) = 0\}$$

Define $\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K)$ as the subset of $\mathrm{MC}(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K)$ where elements are of the form $a + \tilde{\alpha}$. Note $\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K)$ gives us precisely the (opposite) morphisms (that does not change the trivalization of P) between elements of the form $a + \alpha$.

Since $K(a) = 0$, $K(a + \tilde{\alpha}) = 0$ if and only if $K(\tilde{\alpha}) = 0$. We have a bijection

$$\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K) \cong \mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{h}) \otimes \Omega_1, K).$$

Now represent (opposite) morphisms between curved Maurer Cartan solutions with $\mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{h})\otimes \Omega_1, K)$. Using this identification, it is clear that the bijection between the (opposite) morphisms respects composition of morphisms, i.e. if $a+\tilde{\alpha}\circ a+\tilde{\alpha}' = a+\tilde{\alpha}''$, then $\tilde{\alpha}\circ\tilde{\alpha}' = \tilde{\alpha}''$.

Denote $\mathrm{Del}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}))$ the groupoid with objects $\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}))$ and morphisms $\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}})\otimes\Omega_1, K)$, $\mathrm{Del}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}))$ is then a subgroupoid of $\mathrm{Del}(\hat{\mathcal{L}}(\tilde{\mathfrak{g}})))$. The identification of $a+\alpha$ with α clearly gives us a morphism of groupoids, so we have an isomorphism of groupoids

$$\mathrm{Del}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}})) \cong \mathrm{Del}(\hat{\mathcal{L}}(\mathfrak{h})).$$

Remark. *In general if L is a curved DGLA (L_∞ algebra),* $\mathrm{Del}(L)$ *might not be well defined and* $\mathrm{Del}_\infty(L)$ *might not be a Kan complex. (It is not in the literature that they are well defined.) However, in our situation, if we define* $\mathrm{Del}(\hat{\mathcal{L}}(\mathfrak{h}))$*, by mimicking* Del *in the non curved case, as the groupoid whose objects are* $\mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{h}))$ *and whose (opposite) morhpisms* $\mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{h})\otimes\Omega_1, K)$*, we will get a proper groupoid as this groupoid will be isomorphic to the subgroupoid of* $\mathrm{Del}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}))$*. Furthermore, we will define* $\mathrm{Del}_\infty(\hat{\mathcal{L}}(\mathfrak{h}))$ *as* $\mathcal{N}(\mathrm{Del}^{\mathrm{op}}(\hat{\mathcal{L}}(\mathfrak{h}))$*. This is also well defined as* $\mathrm{Del}(\hat{\mathcal{L}}(\mathfrak{h}))$ *is a groupoid.*

5.4 Cocycle Condition for Extensions of Principal G-bundles and Curved L_∞ Maurer Cartan Solutions

In the previous section, we have established that the Maurer Cartan solutions for $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$ (after fixing their image $a\in\mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{g}))$ under the projection $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})\to\hat{\mathcal{L}}(\mathfrak{g})$) are in bijection with the set of curved Maurer Cartan solutions for $\hat{\mathcal{L}}(\mathfrak{h})$ and that the equivalences (morphisms) of Maurer Cartan solutions in $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$ (that preserve the image $a\in\mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{g}))$) are exactly the equivalence (morphisms) of curved Maurer Cartan solutions in $\hat{\mathcal{L}}(\mathfrak{h})$. We can now put every piece together by using formal Kuranishi to relate $\mathrm{MC}(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}))$ to the cocycles for extensions of principal G-bundles P and relate $\mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{h}))$ to the Maurer Cartan set of the curved L_∞ algebra $\mathcal{L}(\mathfrak{h})$.

Given an affine cover $\mathcal{U}$, the transition functions of P give us a Maurer Cartan element $a \in \mathrm{MC}(\mathcal{L}(\mathfrak{g}))$. Notice that by applying the formal Kuranishi theorem on the Dupont contraction, we will have a unique lift of a in $\mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{g}))$. To simplify the notation, we will denote both elements a. Denote $\mathrm{MC}_a(\mathcal{L}(\tilde{\mathfrak{g}}))$ the set of Maurer Cartan solutions $a+\alpha$ for $\mathcal{L}(\tilde{\mathfrak{g}})$ after fixing $a \in \mathrm{MC}(\mathcal{L}(\mathfrak{g}))$, $\mathrm{Del}_a(\mathcal{L}(\tilde{\mathfrak{g}}))$ the groupoid with objects $\mathrm{MC}_a(\mathcal{L}(\tilde{\mathfrak{g}}))$ and morphisms the morphisms between the Maurer Cartan elements in the Deligne groupoid $\mathrm{Del}(\mathcal{L}(\tilde{\mathfrak{g}}))$ that is identity on a.

Theorem 5.5. *There is an isomorphism of groupoids*

$$\mathrm{Del}_a(\mathcal{L}(\tilde{\mathfrak{g}})) \cong \mathrm{Del}(\mathcal{L}(\mathfrak{h}))$$

where $\mathcal{L}(\mathfrak{h})$ has a curved L_∞ structure induced from the curved DGLA $\hat{\mathcal{L}}(\mathfrak{h})$ whose differential is given by $d + \mathrm{ad}_a$ and curvature $\frac{1}{2}c(a,a)$.

Before we start the proof, recall that we can obtain an L_∞ structure on a Čech complex of $\mathfrak{f}(\mathcal{U})_\bullet$ (which can be identified as $\mathcal{L}(\mathfrak{f})$) from $\hat{\mathcal{L}}(\mathfrak{f})$ with homotopy $\tilde{K}$ using the homotopical transfer of structure theorem where $\tilde{K}$ is termwise the Dupont homotopy [7].

$$\begin{tikzcd} \mathcal{L}(\mathfrak{f}) \arrow[r, shift left] & \hat{\mathcal{L}}(\mathfrak{f}) \arrow[l, shift left] \arrow[loop right, "\tilde{K}"] \end{tikzcd}$$

The formal Kuranishi theorem gives us [1]

$$\mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{f}), \tilde{K}) \cong \mathrm{MC}(\mathcal{L}(\mathfrak{f}))$$

where $\mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{f}), \tilde{K}) = \{z \in \mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{f})) | \tilde{K}(z) = 0\}$.

Now consider simplicial DGLA $\hat{\mathcal{L}}(\mathfrak{f}) \otimes \Omega_\bullet$. Since Ω_n is commutative, we have an induced contraction [2] given by

$$\begin{tikzcd} \mathcal{L}(\mathfrak{f}) \otimes \Omega_\bullet \arrow[r, shift left] & \hat{\mathcal{L}}(\mathfrak{f}) \otimes \Omega_\bullet \arrow[l, shift left] \arrow[loop right, "\tilde{K}\otimes \mathrm{id}"] \end{tikzcd}$$

Applying formal Kuranishi and we get an isomorphism of simplicial sets

$$\mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{f}) \otimes \Omega_\bullet, \tilde{K} \otimes \mathrm{id}) \cong \mathrm{MC}(\mathcal{L}(\mathfrak{f}) \otimes \Omega_\bullet)$$

In particular,

$$\mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{f}) \otimes \Omega_1, \tilde{K} \otimes \mathrm{id}) \cong \mathrm{MC}(\mathcal{L}(\mathfrak{f}) \otimes \Omega_1)$$

Note that curved homotopy transfer of structure theorem and thus curved formal Kuranishi theorem only apply when we have a correct filtration on our complexes. We will discuss this in more detail as a remark when we apply the curved homotopy transfer of structure theorem in the proof.

Now let us prove our theorem.

Proof. Given a principal G-bundle P and an affine cover $\mathcal{U}$, Corollary 4.14 told us that the set of transition functions gives us an element a in $\mathrm{MC}(\mathcal{L}(\mathfrak{g}))$ which can then be uniquely identified as an element in $\mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{g}))$ using formal Kuranishi. This a will remain fixed for the rest of the proof.

Define $\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}), \tilde{K})$ as the set of Maurer Cartan solutions $a + \alpha \in \mathrm{MC}(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}))$, where $a \in \mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{g}))$ is fixed and $\alpha \in \hat{\mathcal{L}}(\mathfrak{h})$, such that $\tilde{K}(a + \alpha) = 0$. Notice that the bijection respects the splitting of $\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) = \hat{\mathcal{L}}(\mathfrak{g}) \oplus \hat{\mathcal{L}}(\mathfrak{h})$, i.e. the image of $a \in \hat{\mathcal{L}}(\mathfrak{g})$ is in $\mathcal{L}(\mathfrak{g})$ and the image of $\alpha \in \hat{\mathcal{L}}(\mathfrak{h})$ is in $\mathcal{L}(\mathfrak{h})$ after we use the splitting of $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{h}$ to induce a splitting in both $\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) = \hat{\mathcal{L}}(\mathfrak{g}) \oplus \hat{\mathcal{L}}(\mathfrak{h})$ and $\mathcal{L}(\tilde{\mathfrak{g}}) = \mathcal{L}(\mathfrak{g}) \oplus \mathcal{L}(\mathfrak{h})$. This is because the Dupont contraction contracts between polynomial differential forms and cochains and has nothing to do with the coefficients, and thus we have

$$\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}), \tilde{K}) \cong \mathrm{MC}_a(\mathcal{L}(\tilde{\mathfrak{g}}))$$

But after fixing a, we also get that the Maurer Cartan solutions for $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$ is bjiective to the curved Maurer Cartan solution for $\hat{\mathcal{L}}(\mathfrak{h})$ (where $d_{\mathfrak{h}} = d + \mathrm{ad}_a$), i.e. $\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}})) \cong \mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{h}))$. We thus get

$$\mathrm{MC}_a(\mathcal{L}(\tilde{\mathfrak{g}})) \cong \mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{h}), \tilde{K})$$

Now apply the curved version of homotopical transfer of structure theorem [5] and curved version of formal Kuranishi theorem [6] to the contraction

$$\begin{tikzcd} \hat{\mathcal{L}}(\mathfrak{h}) \arrow[r, shift left] & \mathcal{L}(\mathfrak{h}) \arrow[l, shift left] \arrow[loop right, "\tilde{K}"] \end{tikzcd}$$

We get

$$\mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{h}), \tilde{K}) \cong \mathrm{MC}(\mathcal{L}(\mathfrak{h}))$$

Combine this with the above bijection, we get

$$\mathrm{MC}_a(\mathcal{L}(\tilde{\mathfrak{g}})) \cong \mathrm{MC}(\mathcal{L}(\mathfrak{h}))$$

which tells us that the Maurer Cartan solutions are in bijection.

For the arrows (equivalences) consider $\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K, \tilde{K} \otimes \mathrm{id})$, which is defined as the restriction of $\mathrm{MC}(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K, \tilde{K} \otimes \mathrm{id})$ to elements of the form $a + \tilde{\alpha}$. a is viewed as a constant polynomial in $\hat{\mathcal{L}}(\mathfrak{g}) \otimes \Omega_1$, $\tilde{\alpha} \in \hat{\mathcal{L}}(\mathfrak{h}) \otimes \Omega_1$, and

$$\mathrm{MC}(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K, \tilde{K} \otimes \mathrm{id}) = \{x \in \mathrm{MC}(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1) | K(x) = 0, \tilde{K} \otimes \mathrm{id}(x) = 0\}$$

where K is the contraction homotopy for

$$\begin{tikzcd} C^*(\Delta_1; \hat{\mathcal{L}}(\tilde{\mathfrak{g}})) \arrow[r, shift left] & \hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1 \arrow[l, shift left] \arrow[loop right, "K"] \end{tikzcd}$$

and $\tilde{K} \otimes \mathrm{id}$ is the contraction homotopy for

$$\begin{tikzcd} \mathcal{L}(\tilde{\mathfrak{g}}) \otimes \Omega_1 \arrow[r, shift left] & \hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1 \arrow[l, shift left] \arrow[loop right, "\tilde{K}\otimes\mathrm{id}"] \end{tikzcd}$$

Notice that K is a contraction that contracts the Ω_1 part without changing the coefficients and $\tilde{K} \otimes \mathrm{id}$ contracts the coefficients without changing the $s, ds \in \Omega_1 = \mathbb{K}[s, ds]$.

Recall that $\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K)$ give us the (opposite) morphisms between Maurer Cartan solutions of $\hat{\mathcal{L}}(\tilde{\mathfrak{g}})$ of the form $a + \alpha$ that is identity on a. Since $\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K, \tilde{K} \otimes \mathrm{id})$ can be viewed as a subset of $\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K)$, i.e.

$$\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K, \tilde{K} \otimes \mathrm{id}) = \mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K) \bigcap \mathrm{Ker}(\tilde{K} \otimes \mathrm{id}),$$

$\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K, \tilde{K} \otimes \mathrm{id})$ can thus be viewed as the set of morphisms that are killed by the contraction homotopy $\tilde{K} \otimes \mathrm{id}$.

But

$$\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, \tilde{K} \otimes \mathrm{id}) \cong \mathrm{MC}_a(\mathcal{L}(\tilde{\mathfrak{g}}) \otimes \Omega_1)\}$$

using formal Kuranishi on the contraction

$$\begin{tikzcd} \mathcal{L}(\tilde{\mathfrak{g}}) \otimes \Omega_1 \arrow[r, shift left] & \hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1 \arrow[l, shift left] \arrow[loop right, "\tilde{K}\otimes\mathrm{id}"] \end{tikzcd}$$

and restricting elements of the form $a + \tilde{\alpha} \in \mathcal{L}(\tilde{\mathfrak{g}})$ (we are abusing the notation here as a and $\tilde{\alpha}$ here should be the image of $a, \tilde{\alpha} \in \hat{\mathcal{L}}(\tilde{\mathfrak{g}})$). We get

$$\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K, \tilde{K} \otimes \mathrm{id}) \cong \mathrm{MC}_a(\mathcal{L}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K)$$

where the right hand side gives us the morphisms between Maurer Cartan solutions of the form $a + \alpha \in \mathcal{L}(\tilde{\mathfrak{g}}) = \mathcal{L}(\mathfrak{g}) \oplus \mathcal{L}(\mathfrak{h})$ that is identity on a. Notice although the K on both sides are technically different (i.e. the one on the right is the homotopy on $\mathcal{L}(\tilde{\mathfrak{g}}) \otimes \Omega_1$ and the one on the left is the homotopy on $\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1$), this is well defined as K only applies to Ω_1 and ignores the coefficients.

But we know from last section that

$$\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K, \tilde{K} \otimes \mathrm{id}) \cong \mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{h}) \otimes \Omega_1, K, \tilde{K} \otimes \mathrm{id}).$$

Apply curved homotopy transfer and curved formal Kuranishi on the right and restrict to the subset of $\mathrm{Ker}(K)$, we get

$$\mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{h}) \otimes \Omega_1, K, \tilde{K} \otimes \mathrm{id}) \cong \mathrm{MC}(\mathcal{L}(\mathfrak{h}) \otimes \Omega_1, K)$$

Remark. *Note that for a semicosimplicial Lie algebra $L_\bullet$, $\mathrm{Tot}_{\mathrm{TW}}(L_\bullet)$ and $\mathrm{Tot}(L_\bullet)$ are equipped with the filtrations*

$$F^{-1}\,\mathrm{Tot}_{\mathrm{TW}}(L_\bullet) \subset F^0\,\mathrm{Tot}_{\mathrm{TW}}(L_\bullet) \subset F^1\,\mathrm{Tot}_{\mathrm{TW}}(L_\bullet) \subset \dots$$

and

$$F^{-1}\,\mathrm{Tot}(L_\bullet) \subset F^0\,\mathrm{Tot}(L_\bullet) \subset F^1\,\mathrm{Tot}(L_\bullet) \subset \dots$$

where $F^i\,\mathrm{Tot}_{\mathrm{TW}}(L_\bullet)$ and $F^i\,\mathrm{Tot}(L_\bullet)$ are $\bigoplus_{k=i}^{\infty} \mathrm{Tot}_{\mathrm{TW}}(L_\bullet)[1]^k$ and $\bigoplus_{k=i}^{\infty} \mathrm{Tot}(L_\bullet)[1]^k$. Notice that the degree in $\mathrm{Tot}_{\mathrm{TW}}(L_\bullet)[1]$ and $\mathrm{Tot}(L_\bullet)[1]$ is precisely the number of overlapping open sets minus 2. These filtrations are complete as $\mathrm{Tot}_{\mathrm{TW}}(L_\bullet)[1] = \bigoplus_{k=-1}^{\infty} \mathrm{Tot}_{\mathrm{TW}}(L_\bullet)[1]^k$ and $\mathrm{Tot}(L_\bullet)[1] = \bigoplus_{k=-1}^{\infty} \mathrm{Tot}(L_\bullet)[1]^k$.

The curved L_∞ structure on $\hat{\mathcal{L}}(\mathfrak{h})$, in Getzler's sense, must then be in $F^1S^1(\hat{\mathcal{L}}(\mathfrak{h}), \hat{\mathcal{L}}(\mathfrak{h}))$ as C is an element in the triple intersection, d and $[-,-]$ are degree 1 maps in $\hat{\mathcal{L}}(\mathfrak{h})[1]$. This means that $\hat{\mathcal{L}}(\mathfrak{h})$ is pro-nilpotent and thus we can apply curved homotopy transfer of structure theorem and curved formal Kuranishi theorem. See Appendix for details on curved L_∞ algebras.

And thus we get a bijection between the two Maurer Cartan sets

$$\mathrm{MC}_a(\mathcal{L}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K) \cong \mathrm{MC}(\mathcal{L}(\mathfrak{h}) \otimes \Omega_1, K)$$

Now suppose we have two equivalent Maurer Cartan solutions $a+\alpha$, $a+\alpha' \in \mathrm{MC}_a(\mathcal{L}(\tilde{\mathfrak{g}}))$ and the equivalence is given by $a+\tilde{\alpha} \in \mathrm{MC}_a(\mathcal{L}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K)$ such that

$$\tilde{\alpha}|_{s=0} = \alpha, \qquad \tilde{\alpha}|_{s=1} = \alpha'$$

Because of the bijections $\mathrm{MC}_a(\mathcal{L}(\tilde{\mathfrak{g}})) \cong \mathrm{MC}(\mathcal{L}(\mathfrak{h}))$ and $\mathrm{MC}_a(\mathcal{L}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K) \cong \mathrm{MC}(\mathcal{L}(\mathfrak{h}) \otimes \Omega_1, K)$, we can uniquely lift $a+\alpha$, $a+\alpha'$, and $a+\tilde{\alpha}$ in $\mathrm{MC}(\mathcal{L}(\mathfrak{h}))$ and $\mathrm{MC}(\mathcal{L}(\mathfrak{h}) \otimes \Omega_1, K)$ respectively. Note that after the lift we still have $\tilde{\alpha}|_{s=0} = \alpha$ and $\tilde{\alpha}|_{s=1} = \alpha'$. The lifts agree because Getzler defines them as solutions to differential equations with initial conditions. Since the bijection between

$$\mathrm{MC}_a(\hat{\mathcal{L}}(\tilde{\mathfrak{g}}) \otimes \Omega_1, K) \cong \mathrm{MC}(\hat{\mathcal{L}}(\mathfrak{h}) \otimes \Omega_1, K)$$

respects composition as shown in the last section and the fact that homotopy transfer induces morphisms between Deligne groupoids, i.e. the composition of morphisms are respected, our bijection also respects composition of morphisms. (Again the Deligne Groupoid of a curved DGLA (L_∞ algebra) might not exist, but it does for our case.) Thus we have an isomorphism of groupoids

$$\mathrm{Del}_a(\mathcal{L}(\tilde{\mathfrak{g}})) \cong \mathrm{Del}(\mathcal{L}(\mathfrak{h}))$$

□

The left hand side of the isomorphism gives us the extensions of our principal G-bundle P to a principal $\tilde{G}$-bundle $\tilde{P}$, i.e. the different sets of transition functions on $\tilde{P}$ that satisfies the cocycle condition for $\tilde{P}$ while preserving the transition functions on P (over $\mathcal{U}$) and the equivalences of such extensions. The right hand side gives the set of curved Maurer

Cartan solutions for the curved L_∞ algebra $\mathcal{L}(\mathfrak{h})$ obtained from homotopy transfer and the equivalences of curved Maurer Cartan solutions.

Corollary 5.6. *Suppose we have a principal G-bundle P over the base space X, $\pi : P \to X$, and $G = \exp(\mathfrak{g})$, where G is unipotent and $\mathfrak{g}$ is nilpotent. Suppose we have a Lie algebra extension $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ by another nilpotent Lie algebra $\mathfrak{h}$*

$$0 \to \mathfrak{h} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0,$$

then extensions of the principal G-bundle P are given by the curved Maurer Cartan solutions $\mathrm{MC}(\mathcal{L}(\mathfrak{h}))$ and the equivalence of extension is precisely the equivalence of curved Maurer Cartan solutions, i.e. the solution for the twisted cocycle condition

$$\mathcal{C}(\varphi_{ij}, \varphi_{jk}) \exp((-1)^s \sum_{s=0}^{\infty} \frac{(\mathrm{ad}_{\varphi_{jk}})^s}{s!}(\psi_{ij})) \exp(\psi_{jk}) = \exp(\psi_{ik})$$

is in bijection with the curved Maurer Cartan solutions for $\mathcal{L}(\mathfrak{h})$ and the twisted equivalence (change of trivialization)

$$\exp(\psi'_{ij}) = \exp((-1)^{s+1} \sum_{s=0}^{\infty} \frac{(\mathrm{ad}_{\varphi_{ij}})^s}{s!}(\sigma_i)) \exp(\psi_{ij}) \exp(\sigma_j)$$

is in bijection with the morphisms between curved Maurer Cartan solution for $\mathcal{L}(\mathfrak{h})$ and this bijection respects composition of morphisms (change of trivialization).

Proof. Result follows directly from Theorem 5.5 and 4.12. □

Example 5.7. *In the case where the image of c is in the center of $\mathfrak{h}$, i.e. $c(x,y) \in \mathrm{Z}(\mathfrak{h})\, \forall x, y \in \mathfrak{g}$, we will have an honest action of $\mathfrak{g}$ (G) on $\mathfrak{h}$. The extensions of the bundle P are then given by the curved L_∞ Čech complex, $\mathcal{L}(\mathfrak{h})$, of the associated bundle $P_{\mathfrak{h}} = (P \times \mathfrak{h})/G$ (which a bundle of Lie algebras) where its curvature is given by c.*

Appendix A

Curved L_∞ Algebras and Curved Homotopy Transfer of Structure Theorem

We will introduced the curved DGLAs and curved L_∞ algebras and state the curved homotopy transfer of structure theorem and curved version of formal Kuranishi theorem in this appendix. Readers should go through chapter 1 and chapter 2 before reading this section.

A.1 Curved DGLAs and Curved L_∞ Algebras

We will start the section by defining curved DGLAs.

Definition A.1. *A* ***curved differential graded Lie algebra (curved DGLA)*** *is a graded Lie algebra* $L = \bigoplus_{n\in\mathbb{Z}} L^n$ *together with a degree 1 linear map* $d : L \to L$ *such that:*

- $d[a,b] = [da,b] + (-1)^{|a|}[a,db]$
- $d \circ d = [C,-]$ *where* C *is a degree 2 element in* L *called the* ***curvature element*** *of* L.

Since $d^2 \neq 0$, L does not have a differential complex structure. In particular, the cohomology groups of L are not well defined. Now in analogy with the non curved case, we will define the Maurer Cartan equation of a curved DGLA L.

Definition A.2. *The* ***Maurer Cartan equation*** *of a curved DGLA L with curvature element C is*

$$C + da + \frac{1}{2}[a,a] = 0, \qquad\qquad a \in L^1.$$

The set of solutions $\mathrm{MC}(L) \subset L^1$ *of the Maurer Cartan equation are called the* ***Maurer Cartan set*** *of the curved DGLA L and the elements in* $\mathrm{MC}(L)$ *are called* ***Maurer Cartan elements****.*

Notice that unlike the non curved case, $\mathrm{MC}(L)$ might be empty because 0 is no longer a Maurer Cartan element when $C \neq 0$.

Curved DGLAs can be generalized to curved L_∞ algebras just like DGLAs can be generalized to L_∞ algebras.

Definition A.3. *Let L be a complete graded vector space; a codifferential Q of degree 1 on the symmetric coalgebra* $S(L[1]) = \bigoplus_{n\geq 0} \bigodot^n L$ *is called a* ***curved*** $\mathbf{L}_\infty$ ***structure*** *on L. A curved L_∞ algebra is a complete graded space* $(L, F^\bullet L, d)$ *together with a curved L_∞ structure Q on L.*

Notice that like the non curved case, Q is determined by $Q^1 : S(L[1]) \to L$. The maps $q_i = Q^1_i : \bigodot^n L \to L$ give us the (higher) brackets on $L[1]$; $q_0 : \mathbb{K} \to L$ in particular gives us the **curvature** element of our curved L_∞ algebra. The series of equations (general Jacobi identities) given by $Q^2 = 0$ are different from the non curved case as we have to take into account the q_0 component of Q. In particular, we have

$$\begin{aligned} q_1^2(x) =& q_2(q_0, x) \\ q_1(q_0) =& 0 \qquad\qquad x \in L. \end{aligned}$$

This means that q_1 is no longer a differential for L and the cohomology for L is not defined. The general Jacobi identities for a curved L_∞ algebra L is given by

$$\sum_{i+j=n+1, i,j\in\mathbb{N}} \sum_{\sigma\in \mathrm{Sh}(i,j)} \mathrm{sgn}(\sigma)(-1)^{i(j-1)} q_j(q_i(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)}) \odot v_{\sigma(i+1)} \odot \cdots \odot v_{\sigma(n)}) = 0.$$

Definition A.4. *A (counital coassociative) coalgebra C is* ***coaugmented*** *if there exist a coalgebra morphism $\eta : \mathbb{K} \to C$. η is called an* ***coaugmentation*** *for C.*

$S(L[1])$ is equipped with a natural coaugmentation $\eta : \mathbb{K} \to S(L[1])$ induced from the embedding map $\eta : \mathbb{K} \to T(L[1])$. When Q agrees with the coaugmentation, i.e. $Q\eta = 0$, we have $q_0 = 0$ and we recover a (non curved) L_∞ algebra.

Definition A.5. *A curved L_∞ morphism (sometimes called a shmap) $F : (L, Q) \to (M, R)$ between curved L_∞ algebras is a morphism $F : S(L[1]) \to S(M[1])$ that commutes with the coproducts, counits, and codifferentials Q and R.*

Like the non curved case, F is determined by $F_i^1 = f_i : \bigodot^i L[1] \to M[1]$ and F can be computed in a fashion similar to the non curved case (the index start with 0 instead of 1).

As in the non curved case, curved DGLAs and curved L_∞ algebras are related through the décalage isomorphism. We have $q_0 = C$, $q_1(l) = -d(l)$, $q_2(l_1, l_2) = (-1)^{|l_1|}[l_1, l_2]$, $q_i = 0$ for $i \geq 3$.

Definition A.6. *Given a curved L_∞ algebra (L, Q), the Maurer Cartan equation on L is*

$$\sum_{n=0}^{\infty} \frac{1}{n!} q_n(x, \dots, x) = 0 \qquad x \in L^1.$$

*The set of solutions $\mathrm{MC}(L) \subset L^1$ of the Maurer Cartan equation are called the **Maurer Cartan set** of the curved L_∞ algebra L and the elements in $\mathrm{MC}(L)$ are called **Maurer Cartan elements**.*

The homotopy equivalence between Maurer Cartan elements of a curved L_∞ algebra L is defined exactly the same way as the non curved case.

Definition A.7. *Two Maurer Cartan solutions $a, a' \in \mathrm{MC}(L)$ are **(homotopy) equivalent** if there exist $z \in \mathrm{MC}(L \otimes \mathbb{K}[s, ds])$ such that*

$$z|_{s=0} = a, \qquad\qquad z|_{s=1} = a'$$

where the evaluation map is given by $\mathrm{Eval}_{s=s_0} : L \otimes \mathbb{K}[s, ds] \to L$ is given by

$$\mathrm{Eval}_{s=s_0}(x(s) + y(s)ds) = x(s_0)$$

Remark. *Before we move onto the next section, we should note that Getzler in [6] used a different definition of curved L_∞ algebra. Instead of using the coalgebra definition, Getzler define the curved L_∞ structure as an element in the filtered complex of degree 1 inhomogeneous multilinear maps*

$$S^1(L, L) = F^1 L \times \prod_{n=1}^{\infty} S^{n,1}(L, L),$$

where

$$S^{n,1}(L, L) = \{\textit{filtered graded symmetric } n\textit{-linear maps from } L \textit{ to } L \textit{ of degree 1 }\},$$

with filtration

$$F^kS^1(L,L) = \{(a_0, a_1, \dots) \in S^1(L,L) |\, a_n(F^{k_1}L, \dots, F^{k_n}L) \subset F^{k_1+\cdots+k_n+k}L\}$$

that satisfies conditions equivalent to the general Jacobi identities. Our coalgebra definition is equivalent to Getzler's definition by viewing $Q^1 = (q_0, q_1, \dots)$ as an element in $S^1(L,L)$.

Although we define the curved L_∞ algebra using the coalgebra definition in this section so that our definition of curved L_∞ algebra is consistent with the non curved case, we will use Getzler's definition in the next section as it gives us a clearer presentation of the curved homotopy transfer of structure theorem.

A.2 Curved Homotopy Transfer of Structure and Curved Formal Kuranishi Theorem

In this section, we will state the curved homotopy transfer of structure using Getzler's terminology, i.e. a curved L_∞ algebra is a complete graded vector space L together with a curved L_∞ structure $\lambda \in S^1(L,L)$ where λ satisfies conditions equivalent to the general Jacobi identity, and state the main result of Getzler's paper [6], the curved version of formal Kuranishi theorem. We will start with the following definitions.

Definition A.8. *A curved L_∞ algebra (L,λ) is **pro-nilpotent** if $\lambda \in F^1S^1(L,L)$.*

Definition A.9. *Given $a \in S^i(L, M)$ and $b \in S^0(K, L)$, define the composition $a \bullet b \in S^i(K, M)$ by*

$$(a \bullet b)_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \\ \sum_{k=0}^{n} \frac{1}{k!} \sum_{n_1+\cdots+n_k=n} \frac{1}{n_1! \ldots n_k!} a_k(b_{n_1}(x_{\sigma(1)}, \ldots), \ldots, b_{n_k}(\ldots, x_{\sigma(n)}))$$

Note that in order for $\bullet$ to be well defined, we need to restrict $b_0 \in F^1L$. We will now state the curved homotopy transfer of structure theorem, which is originally shown by Fukaya and stated in the current form by Getzler:

Theorem A.10 (Fukaya, [5][6])**.** *Given a complete contraction*

$$M \underset{g}{\overset{f}{\rightleftarrows}} L \circlearrowleft h$$

between a pair of complete filtered cochain complexes $(M, F^\bullet M, d)$ and $(L, F^\bullet L, \delta)$ with continuous morphism f and g. Suppose L is equipped with a pro-nilpotent curved L_∞ structure λ. Then there is a unique solution in $S^0(M, L)$ of the fixed-point equation

$$F = f - h\lambda \bullet F.$$

Furthermore, $\mu = g\lambda \bullet F \in S^1(M, M)$ is a curved L_∞ structure on M, and F is a curved L_∞ morphism from $(M, F^\bullet M, d, \mu)$ to $(L, F^\bullet L, \delta, \lambda)$.

Notice the pro-nilpotence is needed for the theorem to hold. We need $\lambda \in F^1S^1(L, L)$ to make the map $F \mapsto f - h\lambda \bullet F$ a contraction mapping under the metric

$$d_c(x, y) = \inf\{c^{-k} | \, x - y \in F^kL\}$$

where c may be any real number greater than 1. See [6] for details of the proof.

We can now state the curved version of formal Kuranishi theorem:

Theorem A.11 (Getzler, [6])**.** *Under the same setting as in Theorem A.10, the morphism g induces a bijection from* $\mathrm{MC}(L, h) \to \mathrm{MC}(M)$.

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