Title
CERENKOV RADIATION

Author
Tanaka, Katsumi.

Publication Date
1951-05-07

UNIVERSITY OF CALIFORNIA — BERKELEY

RADIATION LABORATORY

DISCLAIMER

UCRL-1286
Unclassified-Physics
Distribution

UNIVERSITY OF CALIFORNIA

CERENKOV RADIATION

Katsumi Tanaka

May 7, 1951

Berkeley, California

CERENKOV RADIATION

Katsumi Tanaka

Physics Department, University of California
Berkeley, California

May 7, 1951

Introduction

The radiation of light by an electron uniformly moving with the velocity of light in a optically isotropic body was observed by Cerenkov[1] and the theory was worked out by Frank and Tamm[2]. Later Ginsburg[3] investigated this problem for uniaxial crystals when the electron moves along and perpendicular to the optic axis.

The purpose of this paper is to investigate the case when the electron or proton moves in an arbitrary direction with respect to the optic axis. We will use the Hamilton method in the electrodynamics of anisotropic media developed by Ginsburg[4]. It will be shown that the present theory reduces to the special cases when the electron travels along and perpendicular to the optic axis.

Propagation Vector and Electric Field Vector.

Introducing the previous results by Ginsburg[4]

$$\sum_j \varepsilon_j (K_\lambda)_j \, (a_{\lambda i})_j = 0 \qquad i = 1, 2. \quad j = 1,2,3. \tag{1}$$

$$\sum_j \varepsilon_j^2 (a_{\lambda i})_j^2 = 1 \tag{2}$$

$$\sum_j \varepsilon_j (a_{\lambda_1})_j (a_{\lambda_2})_j = 0 \tag{3}$$

$$\sum_j \varepsilon_j^2 (a_{\lambda_1})_j (a_{\lambda_2})_j = 0 \tag{4}$$

$$(b_{\lambda_i})_j = \mathcal{E}_j (a_{\lambda_i})_j \qquad (5)$$

The i indices indicate the direction of polarization and the j indices refer to the direction of the system of principal axes of the polarization ellipsoid. $\mathcal{E}_j$ are the principal dielectric constants. a_{λ_i} are properly normalized vectors of electric field strength of plane waves which may propagate in the crystal. b_{λ_i} are related to (a_{λ_i}) by (5).

(1) follows from div $B = 0$ of Maxwell equations.

(2) is a normalization condition

(3) means that not arbitrary directions of vectors a_{λ_1} a_{λ_2} are chosen but definite ones relating to K_λ .

(4) follows from the fact we chose b_{λ_1}, b_{λ_2} which are orthogonal

It was also shown that

$v_o \cos \Theta_{oi} = \dfrac{c}{n_{\lambda i}}$ where Θ_{oi} is the angle between K_λ and V_o . (6)

The energy of radiated waves having polarization of kind i and within an angle range ϕ, $\phi + d\phi$

$$H_i^{str}(\phi) = \frac{e^2 t}{2\pi c^2 v_o} \int_{\nu} (\vec{V}_o \vec{a}_i (\Theta_o \phi, \nu))^2 n_i^2 (\Theta_o, \phi, \nu) \nu d\nu \qquad (7)$$

Radiation in Uniaxial Crystal.

With reference to Fig. 1,

$$\cos \theta_o = \ell\ell' + mm' + nn'$$

$$\cos \theta_o = \sin\gamma \sin\alpha \cos(\eta - \beta) + \cos\gamma \cos\alpha \qquad (8)$$

$$\mathcal{E}_x = \mathcal{E}_y = \mathcal{E}_\alpha = \mathcal{E}_o$$

$$\mathcal{E}_z = \mathcal{E}_e$$

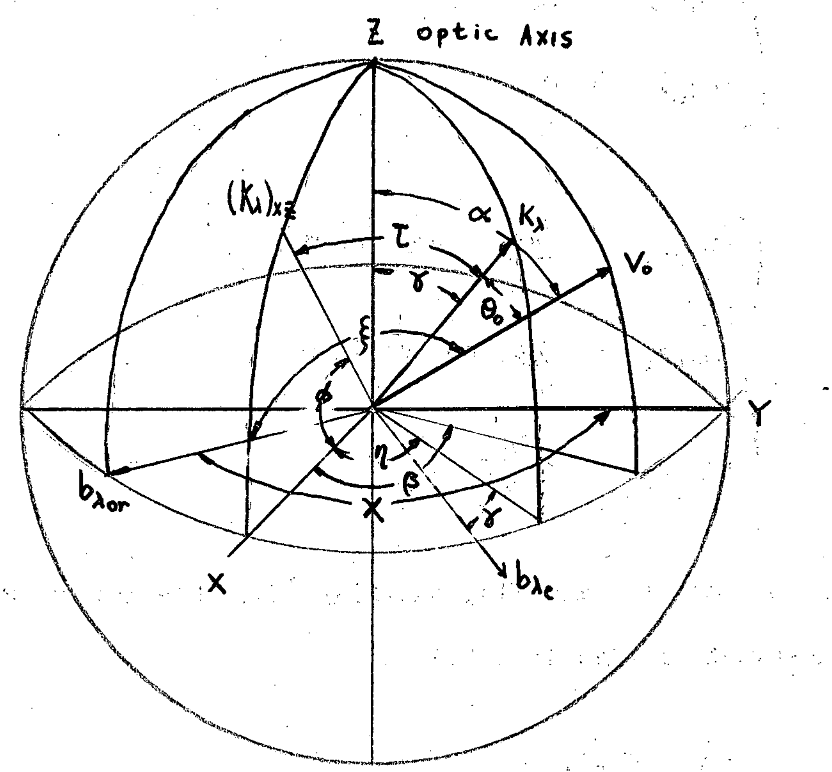

Fig 1

$(K_\lambda)_{xz}$ is the projection of K_λ on the xz plane.

$$(K_\lambda)_x = K_\lambda \sin\gamma \cos\eta = K_\lambda \cos\tau \cos\phi \tag{9}$$

$$(K_\lambda)_y = K_\lambda \sin\gamma \sin\eta = K_\lambda \sin\tau \tag{10}$$

$$(K_\lambda)_z = K_\lambda \cos\gamma = K_\lambda \cos\tau \sin\phi \tag{11}$$

z axis, K_λ, b_e are in the same plane because extraordinary waves are polarized in the principal plane. From (1) and (4), it follows that $b_{\lambda o}$ is in the xy plane.

Hence using (5) we have from the geometry

$$\begin{cases} (a_{\lambda_o})_x = a_{\lambda_o} \sin\chi \\ (a_{\lambda_o})_y = a_{\lambda_o} \cos\chi \\ (a_{\lambda_o})_z = 0 \end{cases} \tag{12}$$

$$\begin{cases} (a_{\lambda_e})_x = \frac{b_{\lambda e}}{o} \cos\gamma \cos\eta \\ (a_{\lambda_e})_y = \frac{b_{\lambda e}}{\varepsilon_o} \cos\gamma \sin\eta \\ (a_{\lambda e})_z = -\frac{b_{\lambda e}}{\varepsilon_e} \sin\gamma \end{cases} \tag{13}$$

It is obvious that (12) and (13) satisfay (3). Now let us consider the O ray. Substituting (12) into (2)

$$a^2_{\lambda or} = \frac{1}{\varepsilon_{or}} \tag{14}$$

From (6)

$$\cos\theta_{o\,or} = \cos\theta_{or} = \frac{c}{v_o\, n_{\lambda o}} = \frac{1}{\beta_o\, n_{\lambda o}} = \frac{1}{\beta_o\sqrt{\varepsilon_o}} \tag{15}$$

In order to find (7), we now shall calculate $(\vec{v}_o \cdot \vec{a}_{\lambda i})^2$

$$(\vec{v}_o \cdot \vec{a}_{\lambda o})^2 = v_o^2\, a_{\lambda o}^2 \cos^2\zeta = \frac{v_o^2}{\varepsilon_o}\cos^2\zeta \quad \text{using (14).}$$

From (8), by moving K_λ to $b_{\lambda o}$ which is in the same direction as $a_{\lambda o}$ by (5),

$$\cos\zeta = \sin\left(\frac{\pi}{2}\right)\sin\alpha\cos\left(x+\beta-\frac{\pi}{2}\right) + \cos\frac{\pi}{2}\cos\alpha\,,$$

$$\cos\zeta = \sin\alpha\sin(\chi+\beta) \qquad \text{as } \eta' = \chi - \frac{\pi}{2} \tag{16}$$

Subsituting the 2nd form of (9), (10) and (12) into (1),

$$\cos\gamma\cos\phi\sin\chi + \sin\gamma\cos\chi = 0$$

or

$$\cos^2\chi = \frac{\cos^2\gamma\cos^2\phi}{\sin^2\gamma + \cos^2\gamma\cos^2\phi} = \frac{\cos^2\gamma\cos^2\phi}{1-\cos^2\gamma\sin^2\phi} \tag{17}$$

Again with the aid of (8)

$$\cos\gamma = \sin\gamma\cos\eta\cos\phi + \cos\gamma\sin\phi \tag{18}$$

In order to write cos χ and hence cos ξ as a function of α β which determine the position vector of the velocity and ϕ and numerical constants β_o ε_{or}, we will eliminate the other variables using (9), (10), (11), (18) as well as (8).

From (9) and (11)

$$\cos \gamma = \frac{\sin \phi \sin \gamma \cos \eta}{\cos \phi}$$

Substituting in (18)

$$\cos \Upsilon = \sin \gamma \cos \eta \left(\frac{\cos^2 \phi + \sin^2 \phi}{\cos \phi} \right) = \frac{\sin \gamma \cos \eta}{\cos \phi} \tag{19}$$

From (19) and (10),

$$\sin^2 \gamma \sin^2 \eta + \frac{(\sin \gamma \cos \eta)^2}{\cos^2 \phi} = 1 \tag{20}$$

Now we have from (8), (9), (10)

$$\begin{aligned} \cos \theta_o &= \sin \alpha \cos \beta \sin \gamma \cos \eta + \sin \alpha \sin \beta \sin \gamma \sin \eta \\ &\quad + \tan \phi \cos \alpha \sin \gamma \cos \eta . \\ &= (\sin \alpha \cos \beta + \tan \phi \cos \alpha) \sin \gamma \cos \eta \\ &\quad + \sin \alpha \sin \beta \sin \gamma \sin \eta \end{aligned}$$

$$\cos \theta_o = A x + B y \tag{21}$$

where $A = \sin \alpha \cos \beta + \tan \phi \cos \alpha$

$B = \sin \alpha \sin \beta$

$x = \sin \gamma \cos \eta$

$y = \sin \gamma \sin \eta$

Substituting x y in (19), (20)

$$\cos \Upsilon = \frac{x}{\cos \phi} \tag{19'}$$

$$\frac{x^2}{\cos \phi^2} + y^2 = 1 \tag{20'}$$

From (19)' and (17)

$$\cos^2 \chi = \frac{\chi^2}{1 - \chi^2 \tan^2\phi} \tag{17'}$$

Solving (21) and (20)'

$$\chi = \frac{A \cos^2\phi \cos \theta_o \pm \sqrt{A^2 \cos\theta_o \cos^4\phi + \cos^2\phi(B^2 - \cos^2\theta_o)(B^2 + A^2\cos^2\phi)}}{B^2 + A^2 \cos^2 \phi}$$

$$= \frac{A \cos^2\phi \cos \theta_o \pm B \cos \phi \sqrt{B^2 + A^2 \cos^2 \phi - \cos^2 \theta_o}}{B^2 + A^2 \cos^2\phi} \tag{22}$$

Substituting for A, $\cos \theta_o$, B, we finally have

$$\chi = \frac{(\sin\alpha \cos\beta \cos \phi + \cos\alpha \sin \phi)^2 \dfrac{\cos \phi}{\beta_0 \sqrt{\mathcal{E}_0}}}{(\sin\alpha \sin\beta)^2 + (\sin\alpha \cos\beta \cos \phi + \cos\alpha \sin \phi)^2}$$

$$\pm \frac{\sin\alpha \sin \beta \cos\phi \sqrt{\sin^2\alpha (\cos^2\phi - \cos^2\beta \sin^2\phi) + \sin^2\phi + \sin2\alpha\cos\beta\sin\phi\cos\phi - \dfrac{1}{\beta_o^2 \mathcal{E}_o}}}{(\sin\alpha \sin\beta)^2 + (\sin\alpha \cos \beta \cos \phi + \cos\alpha \sin \phi)^2} \tag{22'}$$

From equation (16)

$$\cos^2 \zeta = \sin^2\alpha \, \sin^2(\chi + \beta) = \sin^2\alpha \, \cos^2\chi \, \sin^2\beta (\tan\chi \cot\beta + 1)^2$$

$$\cos^2 \zeta = \sin^2\alpha \, \cos^2\chi \, \sin^2\beta \left(\frac{\sqrt{1 - \cos^2\chi} \cot\beta}{\cos\chi} + 1 \right)^2 \qquad (16')$$

where $\cos^2 \chi = \dfrac{X^2}{1 - X^2 \tan^2 \phi}$

and X is given by (22').

Substituting (14), (16') into (7) and we have $H_o^{str}(\phi)$. Now let us consider the two particular cases

(a) Electron moves along the optic axis

$\alpha = \beta = 0$ (Fig. 1)

$\cos^2 \zeta = 0$

$\therefore (\vec{v}_o \cdot \vec{a}_{\lambda or}) = 0$

$\therefore H_o^{str}(\phi) = 0$ (3

(b) Electron moves perpendicular to the optic axis.

$\alpha = \beta = \frac{\pi}{2}$ (Fig. 1) $A = 0$, $B = 1$

$$X = \pm\sqrt{1 - \frac{1}{\beta_o^2 \varepsilon_o}} \cdot \cos\phi$$

$$\cos^2 \chi = \frac{\cos^2\phi \left(1 - \frac{1}{\beta_o^2 \varepsilon_o}\right)}{1 - \left(1 - \frac{1}{\beta_o^2 \varepsilon_o}\right)\sin^2\phi} = \frac{\cos^2\phi \left(1 - \frac{1}{\beta_o^2 \varepsilon_o}\right)}{\cos^2\phi + \frac{1}{\beta_o^2 \varepsilon_o} \sin^2\phi}$$

(23)

$$\cos^2 \xi = \sin^2\left(\chi + \frac{\pi}{2}\right) = \cos^2\left(\frac{\pi}{2} - \chi - \frac{\pi}{2}\right) = \cos^2\chi$$

Using (14), (23)

$$(\vec{v}_o \cdot \vec{a}_{\lambda o})^2 = v_o^2 \frac{\cos^2\phi \left(1 - \frac{c^2}{v_o^2 \mathcal{E}_o}\right)}{\mathcal{E}_o\left(\cos^2\phi + \frac{c^2}{v_o^2 \mathcal{E}_o} \sin^2\phi\right)} \qquad (24)$$

Substituting (15) and (24) into (7)

$$H_o^{str}(\phi) = \frac{e^2 v_o t}{2\pi c^2} \int \nu \frac{\cos^2\phi \left(1 - \frac{c^2}{v_o^2 \mathcal{E}_o}\right)}{\cos^2\phi + \frac{c^2}{v_o^2 \mathcal{E}_o} \sin^2\phi} \, d\nu \quad (3 \qquad (7')$$

We shall now consider the e ray. (See Fig. 2).

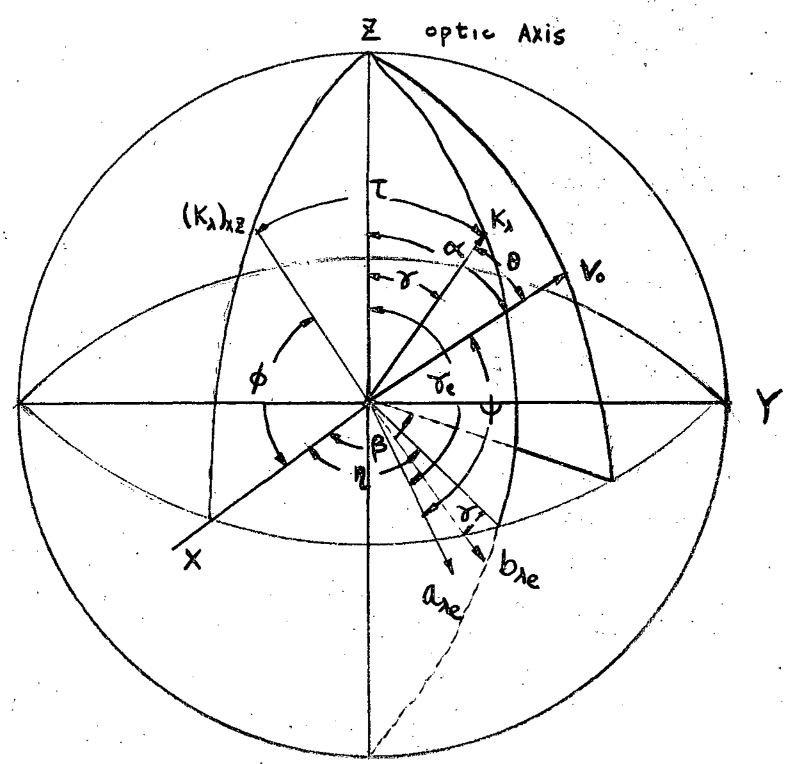

Fig 2

We will first find cosine of the angle θ_{oe} between v_o and a normal to the radiated waves.

Refractive index of extraordinary waves may be expressed as

$$\frac{1}{\eta^2_{\lambda e}(\tau, \phi)} = \frac{\cos^2\tau \cos^2\phi + \sin^2\tau}{\varepsilon_e} + \frac{\cos^2\tau \sin^2\phi}{\varepsilon_o} \qquad (25)$$

Substituting (19')

$$\frac{1}{\eta^2_{\lambda e}} = \frac{1 - \chi^2 \tan^2\phi}{\varepsilon_e} + \frac{\chi^2 \tan^2\phi}{\varepsilon_o} \qquad (25')$$

where $\chi = \sin\gamma \cos\eta$

From (6)

$$\frac{1}{\eta^2_{\lambda e}} = \beta_o^2 \cos\theta_{oe} \qquad (6')$$

Combining (25'), (6'), and (22)

$$\chi^2 = \frac{\beta_o^2 \cos^2\theta_{oe} - \frac{1}{\varepsilon_e}}{\tan^2\phi \left(\frac{1}{\varepsilon_o} - \frac{1}{\varepsilon_e}\right)}$$

$$= \left\{ \frac{A^2\cos^4\phi\,\cos^2\theta_{oe} \pm 2AB\cos^3\phi\,\cos\theta_{oe}\sqrt{B^2 + A^2\cos^2\phi - \cos^2\theta_{oe}}}{(B^2 + A^2\cos^2\phi)^2} \right.$$

$$\left. + \frac{B^2\cos^2\phi(B^2 + A^2\cos^2\phi - \cos^2\theta_o)}{(B^2 + A^2\cos^2\phi)^2} \right\}$$

$$\underbrace{\left(\frac{A^2\cos^4\phi - B^2\cos^2\phi}{(B^2 + A^2\cos^2\phi)^2} - \frac{\beta_o^2}{\tan^2\phi\left(\frac{1}{\varepsilon_o} - \frac{1}{\varepsilon_e}\right)} \right)}_{L}\cos^2\theta_{oe}$$

$$+ \underbrace{\left(\frac{B^2\cos^2\phi}{B^2 + A^2\cos^2\phi} + \frac{\frac{1}{\varepsilon_e}}{\tan^2\phi\left(\frac{1}{\varepsilon_o} - \frac{1}{\varepsilon_e}\right)} \right)}_{M}$$

$$= \pm \underbrace{\frac{2AB\cos^3\phi}{(B^2 + A^2\cos^2\phi)^2}}_{N}\cos\theta_{oe}\sqrt{B^2 + A^2\cos^2\phi - \cos^2\theta_{oe}} \qquad (26)$$

$$L z^2 + M = N z\sqrt{B^2 + A^2\cos^2\phi - z^2} \qquad \text{where} \quad z = \cos\theta_{oe}$$

squaring

$$\underbrace{(L^2 + N^2)}_{P} z^4 + \left\{ \underbrace{2LM - N^2(B^2 + A^2\cos^2\phi)}_{Q} \right\} z^2 + M^2 = 0 \qquad (27)$$

$$P z^4 + Q z^2 + M^2 = 0$$

$$\therefore \cos^2 \theta_{oe} = \frac{-Q \pm \sqrt{Q^2 - 4PM^2}}{2P} \tag{28}$$

where QP are given by (27); MNL by (26); and AB by (21).

Now let us check the two particular cases.

(a) Electron moves along optic axis.

$$\alpha = \beta = 0 \qquad \therefore A = \tan \phi \qquad B = 0$$

$$L = \frac{\cos^2\phi}{\sin^2\phi} - \frac{\beta_o^2}{\tan^2\phi \left(\frac{1}{\varepsilon_o} - \frac{1}{\varepsilon_e}\right)}$$

$$M = \frac{\frac{1}{\varepsilon_e}}{\tan^2\phi \left(\frac{1}{\varepsilon_o} - \frac{1}{\varepsilon_e}\right)}$$

$$N = 0$$

$$P = L^2$$

$$Q = 2LM$$

$$\therefore \cos^2\theta_{oe} = -\frac{M}{L} = \frac{c^2}{\varepsilon_e v_o^2} \sqrt{\frac{1}{1 + \frac{c^2}{\varepsilon_e v_o^2}\left(1 - \frac{\varepsilon_e}{\varepsilon_o}\right)}} \quad (3 \tag{28'}$$

(b) Electron moves perpendicular to the optic axis

$$\alpha = \beta = \frac{\pi}{2} \qquad A = 0 \qquad B = 1$$

$$L = -\cos^2\phi - \frac{\beta_o^2}{\tan^2\phi \left(\frac{1}{\varepsilon_o} - \frac{1}{\varepsilon_e}\right)}$$

$$M = \cos^2\phi + \frac{1}{\mathcal{E}_e \tan^2\phi\left(\frac{1}{\mathcal{E}_o} - \frac{1}{\mathcal{E}_e}\right)}$$

$$P = L^2$$

$$Q = 2LM$$

$$\therefore \cos^2\theta_{oe} = -\frac{M}{L} = \frac{c^2\left(\cos^2\phi + \frac{\mathcal{E}_e}{\mathcal{E}_o}\sin^2\phi\right)}{v_o^2\mathcal{E}_e\left\{1 - \frac{c^2}{v_o^2\mathcal{E}_e}\left(1 - \frac{\mathcal{E}_e}{\mathcal{E}_o}\right)\sin^2\phi\right\}} \quad (3 \qquad (28'')$$

These results follow from the fact that the equation giving the index of refraction of the e ray reduces to the correct forms respectively for the two particular cases.

From the geometry and the fact $b_{\lambda e}$ lies in plane z, k_λ and perpendicular to k_λ

$$\left.\begin{aligned}
(a_{\lambda e})_x &= \frac{b_{\lambda e}}{\mathcal{E}_o}\cos\gamma\cos\eta = a_{\lambda_e}\sin\gamma_e\cos\eta \\
(a_{\lambda e})_y &= \frac{b_{\lambda e}}{\mathcal{E}_o}\cos\gamma\sin\eta = a_{\lambda e}\sin\gamma_e\sin\eta \\
(a_{\lambda_e})_z &= -\frac{b_{\lambda e}}{\mathcal{E}_e}\sin\gamma = a_{\lambda_e}\cos\gamma_e
\end{aligned}\right\} \quad (29)$$

Using (8)

$$\cos\psi = \sin\gamma_e\sin\alpha\cos(\eta - \beta) + \cos\gamma_e\cos\alpha \qquad (30)$$

Now, from (29) 2nd form and (2)

$$a_{\lambda_e}^2 = \frac{1}{\mathcal{E}_o \sin^2 \gamma_e + \mathcal{E}_e \cos^2 \gamma_e} \qquad (31)$$

also

$$\tan \gamma = -\frac{\mathcal{E}_e}{\mathcal{E}_o} \operatorname{ctg} \gamma_e \qquad (32)$$

Substituting (29), (12) into (3) and (4)

$$\tan x = -\tan \eta$$

since the quantity appears in the squared form in the final answers, we will let

$$\left.\begin{aligned} \sin x &= \sin \eta \\ \cos x &= \cos \eta \end{aligned}\right\} \qquad (33)$$

We will express $\cos \psi$ in terms of α , β $\cos \theta_{oe}$ by reducing the unknowns to the previous case.

Equations (9), (11), and (33) give

$$\tan \gamma = \operatorname{ctg} \phi \frac{1}{\cos x} \qquad (34)$$

From (32) and (34)

$$\cos \gamma_e = \sqrt{\frac{\left(\frac{\operatorname{ctg} \phi}{\cos \chi}\right)^2}{\left(\frac{\operatorname{ctg} \phi}{\cos \chi}\right)^2 + \left(\frac{\mathcal{E}_e}{\mathcal{E}_o}\right)^2}} \qquad (35)$$

Combining (30), (32), (33), (34), and (35)

$$\cos \psi = \cos \gamma_e (\sin\alpha \cos\beta \cos \chi \tan \gamma_e + \sin\alpha \sin\beta \sin \chi \tan \gamma_e + \cos\alpha)$$

$$\cos\psi = \sqrt{\frac{\left(\frac{\text{ctg}\,\phi}{\cos\chi}\right)^2}{\left(\frac{\text{ctg}\,\phi}{\cos\chi}\right)^2+\left(\frac{\mathcal{E}_e}{\mathcal{E}_o}\right)^2}}\left(-\sin\alpha\cos\beta\tan\phi\frac{\mathcal{E}_e}{\mathcal{E}_o}\cos^2\chi - \sin\alpha\sin\beta\tan\phi\frac{\mathcal{E}_e}{\mathcal{E}_o}\cos\chi\sqrt{1-\cos^2\chi}+\cos\alpha\right) \tag{36}$$

where

$$\cos^2\chi = \frac{X^2}{1 - X^2\tan^2\phi} \tag{17'}$$

and X is given by (22). Cos θ_{oe} in X is from (28).

Substituting (35) into (31)

$$a^2_{\lambda_e} = \frac{1}{\mathcal{E}_o + (\mathcal{E}_e - \mathcal{E}_o)\cos^2\gamma_e} = \frac{\left(\frac{\text{ctg}\,\phi}{\cos\chi}\right)^2+\left(\frac{\mathcal{E}_e}{\mathcal{E}_o}\right)^2}{\mathcal{E}_e\left\{\frac{\mathcal{E}_e}{\mathcal{E}_o}+\left(\frac{\text{ctg}\,\phi}{\cos\chi}\right)^2\right\}} \tag{31'}$$

Hence from (31') and (36)

$$(\vec{v}_o \cdot \vec{a}_{\lambda e})^2 = v_o^2\, a^2_{\lambda e}\cos^2\psi$$

$$= \frac{v_o^2\left(\frac{\text{ctg}\,\phi}{\cos\chi}\right)^2}{\mathcal{E}_e\left\{\frac{\mathcal{E}_e}{\mathcal{E}_o}+\left(\frac{\text{ctg}\,\phi}{\cos\chi}\right)^2\right\}}\left(-\sin\alpha\cos\beta\tan\phi\frac{\mathcal{E}_e}{\mathcal{E}_o}\cos^2\chi - \sin\alpha\sin\beta\tan\phi\frac{\mathcal{E}_e}{\mathcal{E}_o}\cos\chi\sqrt{1-\cos^2\chi}+\cos\alpha\right) \tag{37}$$

Substituting (37) and (28) into (7) and we have $H_e^{str}(\phi)$.

Let us consider particular cases.

(a) Electron moves along optic axis.

$$\alpha = \beta = 0 \qquad A = \tan\phi,\ B = 0,\ \chi = \mathrm{ctg}\phi\cos\theta_{oe}$$

$$\cos^2 \chi = \frac{\mathcal{E}_o}{\mathcal{E}_e \tan^2\phi(\mathcal{E}_o\beta_o^2 - 1)}$$

$$(\vec{v}_o \cdot \vec{a}_{\lambda e})^2 = v_o^2 \frac{\cos^2\gamma_e}{\mathcal{E}_o + (\mathcal{E}_e - \mathcal{E}_o)\cos^2\gamma_e}$$

$\cos^2\gamma_e$ is given by (35)

$$\therefore (\vec{v}_o \cdot \vec{a}_{\lambda e})^2 = v_o^2 \frac{1 - \frac{c^2}{v_o^2 \mathcal{E}_{or}}}{\mathcal{E}_e} \qquad (3$$

(b) Electron moves perpendicular to optic axis.

$$\alpha = \beta = \frac{\pi}{2} \qquad A = 0 \qquad B = 1$$

$$x = \cos\phi\sqrt{1 - \cos^2\theta_{oe}}$$

$$\cos^2 x = \frac{1 - \cos^2\theta_{oe}}{1 + \tan^2\phi\cos^2\theta_{oe}}$$

$$(\vec{v}_o \cdot \vec{a}_{\lambda e})^2 = \frac{v_o^2 (1 - \cos^2 \chi)}{\mathcal{E}_o \left(1 + \frac{\mathcal{E}_o}{\mathcal{E}_e} \frac{\cos^2 \phi}{\cos^2 x}\right)}$$

$$= \frac{v_o^2 (1 - \cos^2\theta_{oe}) \cos^2\theta_{oe}}{\mathcal{E}_o \cos^2\phi (1 - \cos^2\theta_{oe} + \frac{\mathcal{E}_o}{\mathcal{E}_e} \operatorname{ctg}^2\phi + \frac{\mathcal{E}_o}{\mathcal{E}_e} \cos^2\theta_{oe})(1 + \tan^2\phi \cos^2\theta_{oe})}$$

$$(\vec{v}_o \cdot \vec{a}_{\lambda e})^2 = \frac{v_o^2 \left(1 - \frac{c^2}{\mathcal{E}_e v_o^2}\right) \frac{c^2}{v_o^2 \mathcal{E}_o} \sin^2\phi}{\mathcal{E}_o (\cos^2\phi + \frac{c^2}{\mathcal{E}_o v_o^2} \sin^2\phi)} \qquad (3$$

Substituting $(\vec{v}_o \cdot \vec{a}_{\lambda e})^2$ and $\eta_{\lambda e}^2$ from (6') and (28) in (7), we have $H_i^{str}(\phi)$ for the general case.

Acknowledgment.

I am very grateful to Professor Robert Serber who suggested and supervised this research. I also wish to thank Mr. Robert Mather for his remarks on the experimental aspects of the problem.

References

1. P. A. Cerenkov, Phys. Rev. 52, 378 (1937).

2. Frank and Tamm, Journal of Physics, U.S.S.R. 1 - 2, 1939-40.

3. V. L. Ginsburg, Journal of Physics, U.S.S.R. 3, 1940, p. 101.

4. V. L. Ginsburg, Journal of Physics, U.S.S.R. 3, 1940, p. 95.

Appendix

The general result by Ginsburg

$$H_i^{str} = \frac{c^2 t}{2\pi c^2 v_o} \int_{\gamma} \int_{\phi}^{2\pi} (\vec{v}_o \, \vec{a}_i(\theta, \phi))^2 \, \eta_i^2(\theta_o, \phi) \gamma d\gamma d\phi \qquad \text{uniaxial crystal}$$

For isotropic body, assuming vector $\vec{a}_1$ to be complanate to the vectors $\vec{v}_o$ and $\vec{k}_\lambda$ we get,

$$H_1^{str} = \frac{e^2 v_o t}{2\pi c^2} \int_{\gamma} \gamma \left(1 - \frac{c^2}{v_o^2 \eta_1^2(\gamma)}\right) d\gamma$$

which is Tamms formula radiation only if $1 - \frac{c^2}{v_o^2 n_1^2(\gamma)} \geq 0$

With an angle range ϕ, $\phi + d\phi$ the energy radiated

$$H_i^{str} = \frac{e^2 t}{2\pi c^2 v_o} \int_{\gamma} (\vec{v}_o \, \vec{a}_i (\theta, \phi, \gamma))^2 \, n_i^2 (\theta, \phi) \gamma) \gamma d\gamma$$

Ratio of H_i^{str} by Ginsburg and Tamm

Special cases:

(a) Electron moves along optic axis.

$$H_{or}^{str} = 0$$

$$H_e^{str} = \frac{e^2 v_o t}{2\pi c^2} \int_{\gamma} \left(1 - \frac{c^2}{v_o^2 \varepsilon_{or}(\gamma)}\right) \left\{ 1 + \frac{c^2}{v_o^2 \varepsilon_e(\gamma)} \left(1 - \frac{\varepsilon_e(\gamma)}{\varepsilon_{or}(\gamma)}\right)\right\} d\gamma$$

We now want to integrate this expressing $\mathcal{E}_e$ and $\mathcal{E}_{or}$ as a function of ν. However, we may neglect dispersion and let $\mathcal{E}_e$ and $\mathcal{E}_{or}$ be constants. Then the percentage difference of Ginsburg and Tamm results will be

$$\gamma_{(\nu)} = \frac{\left| \frac{dH^{str}_{Gin}}{d\nu} - \frac{dH^{str}_{Tamm}}{d\nu} \right|}{\frac{dH^{str}_{Tamm}}{d\nu}} = \frac{\left| \left(1 - \frac{c^2}{v_o^2 \mathcal{E}_{or}}\right) \left[\left\{ 1 + \frac{c^2}{v_o^2 \mathcal{E}_e} \left(1 - \frac{\mathcal{E}_e}{\mathcal{E}_{or}}\right) \right\} - 1 \right] \right|}{\left(1 - \frac{c^2}{v_o \mathcal{E}_{or}}\right)} \times 100$$

which can easily be seen.

$$\gamma = 100 \cdot \frac{c^2}{v_o^2 \mathcal{E}_e} \left| 1 - \frac{\mathcal{E}_e}{\mathcal{E}_{or}} \right|$$

for 340 Mev $\beta = \frac{v_o}{c} = 0.68.$

Rutile TiO_2: positive crystal

$n_e = 2.9029$ $\qquad$ $\mathcal{E}_e = 8.41$

$n_o = 2.6158$ $\qquad$ $\mathcal{E}_{or} = 6.81$

$$= 100 \frac{\left| 1 - \frac{8.41}{6.81} \right|}{(0.68)^2 \; 8.41} = 6\%.$$

(b) Electron perpendicular to optic axis.

$$H^{str}_{or}(\phi) = \frac{e^2 v_o t}{2\pi c^2} \int \nu \left(1 - \frac{c^2}{\mathcal{E}_{or} v_o^2}\right) \frac{\cos^2\phi}{\cos^2\phi + \frac{c^2}{v_o^2 \mathcal{E}_o} \sin\phi} \, d\nu$$

$$H_e^{str}(\phi) = \frac{e^2 v_o t}{2\pi c^2}\int \gamma\left(1 - \frac{c^2}{v_o^2 \mathcal{E}_e}\right) \frac{\mathcal{E}_e^2 \frac{c^2}{v_o^2 \mathcal{E}_e} \sin^2\phi \left\{1 - \frac{c^2}{v_o^2 \mathcal{E}_e}\left(1 - \frac{\mathcal{E}_e}{\mathcal{E}_{or}}\right)\sin^2\phi\right\}}{\mathcal{E}_{or}^2\left(\cos^2\phi + \frac{c^2}{v_o^2 \mathcal{E}_{or}}\sin^2\phi\right)\left(\cos^2\phi + \frac{\mathcal{E}_e}{\mathcal{E}_{or}}\sin^2\phi\right)}$$

(i) $\phi = 0$

$$H_{or}^{str}(0) = \frac{e^2 v_o t}{2\pi c^2}\int \gamma \left(1 - \frac{c^2}{\mathcal{E}_{or} v_o^2}\right) d\gamma$$

$$H_e^{str}(0) = 0 \; .$$

(ii) $\phi = \frac{\pi}{2}$

$$H_{or}^{str}\left(\frac{\pi}{2}\right) = 0$$

$$H_e^{str}\left(\frac{\pi}{2}\right) = \frac{e^2 v_o t}{2\pi c^2}\int \left(1 - \frac{c^2}{v_o^2 \mathcal{E}_e}\right)\left\{1 - \frac{c^2}{v_o^2 \mathcal{E}_e}\left(1 - \frac{\mathcal{E}_e}{\mathcal{E}_{or}}\right)\right\}$$

$$= \frac{e^2 v_o t}{2\pi c^2}\int \gamma\left(1 - \frac{c^2}{v_o^2 \mathcal{E}_e}\right)\left\{1 - \frac{c^2}{v_o^2 \mathcal{E}_e}\left(1 - \frac{\mathcal{E}_e}{\mathcal{E}_{or}}\right)\right\}$$

(i) $\phi = 0$

$\gamma = 0$

(ii) $\phi = \frac{\pi}{2}$ 340 Mev $\beta = 0.68$

$$\gamma = 100 \left| \frac{\left(1 - \frac{1}{\beta_o^2 \varepsilon_e}\right) \left\{\left(1 - \frac{1}{\beta_o^2 \varepsilon_e}\left(1 - \frac{\varepsilon_e}{\varepsilon_{or}}\right)\right\} - \left(1 - \frac{1}{\beta_o^2 \varepsilon_{or}}\right)}{\left(1 - \frac{1}{\beta_o^2 \varepsilon_{or}}\right)} \right|$$

$$= 22\%$$

<u>Angle</u>

I. Tamm

$$\cos \theta_o = \frac{1}{\beta_o n} = \frac{1}{0.68 \times 2.616} = 0.562 \qquad \theta_o = 55^\circ\, 45'$$

II. Ginsburg

(a) $\cos \theta_{oo} = 0.562 \qquad \theta_{oo} = 55^\circ\, 45'$

$$\cos \theta_{oe} = \frac{1}{\beta_o \sqrt{\varepsilon_e}} \sqrt{\frac{1}{1 + \frac{1}{\varepsilon_e \beta_o^2}\left(1 - \frac{\varepsilon_e}{\varepsilon_o}\right)}} = 0.612 \qquad \theta_{oe} = 52^\circ 16'$$

(b)(i) $\phi = 0$

$\cos \theta_{oo} = 0.562 \qquad \theta_{oo} = 55^\circ\, 45'$

$\cos \theta_{oe} = \frac{1}{\beta_o \sqrt{\varepsilon_e}} = 0.507 \qquad \theta_{oe} = 59^\circ\, 32'$

(ii) $\phi = \frac{\pi}{2}$

$\cos \theta_{oo} = 0.562 \qquad \theta_{oo} = 55^\circ\, 45'$

$$\cos\theta_{oe} = \frac{1}{\beta_o\sqrt{\mathcal{E}_e}}\sqrt{\frac{\frac{\mathcal{E}_e}{\mathcal{E}_o}}{1+\frac{1}{\beta_o^2\mathcal{E}_e}\left(\frac{\mathcal{E}_e}{\mathcal{E}_o}-1\right)}} = 0.5459$$

$$\theta_{oe} = 56^\circ\ 55' .$$