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Uniform Stability of Switched Linear Systems: Extensions of LaSalle's Invariance Principle

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***Abstract*— This paper addresses the uniform stability of switched linear systems, where uniformity refers to the convergence rate of the multiple solutions that one obtains as the switching signal ranges over a given set. We provide a collection of results that can be viewed as extensions of LaSalle's Invariance Principle to certain classes of switched linear systems. Using these results one can deduce asymptotic stability using multiple Lyapunov functions whose Lie derivatives are only negative semi-definite. Depending on the regularity assumptions placed on the switching signals, one may be able to conclude just asymptotic stability or (uniform) exponential stability. We show by counter-example that the results obtained are tight.**

***Index Terms*— Switched systems, hybrid systems, LaSalle's Invariance Principle, Stability.**

I. INTRODUCTION

Switched systems are typically represented by equations of the form

$$\dot{x} = f_\sigma(x), \qquad x \in \mathbb{R}^n, \tag{1}$$

where $\sigma : [0, \infty) \to \mathcal{P}$ denotes a piecewise constant signal that effectively "switches" the right-hand-side of the differential equation by selecting different vector fields from a parameterized family $\{f_p : p \in \mathcal{P}\}$. The time instants at which σ is discontinuous are called *switching times*. The key distinction between the switched system (1) and the time-varying system

$$\dot{x} = g(x, t), \qquad x \in \mathbb{R}^n,\ t \geq 0, \tag{2}$$

with g defined by $g(x,t) := f_{\sigma(t)}(x)$, $\forall x \in \mathbb{R}^n,\ t \geq 0$, is that one typically associates a family of admissible switching signals $\mathcal{S}$ to (1) and studies the properties of the solutions to (1) as σ ranges over $\mathcal{S}$. Clearly, for a single switching signal σ, (1) and (2) represent exactly the same object.

The set of solutions to an (unswitched) system like (2) is parameterized solely by a set of initial conditions. However, the set of solutions to a switched linear system like (1) is parameterized both by a set of initial conditions and by an admissible set of switching signals $\mathcal{S}$ on which σ is assumed to lie. This poses important questions with respect to the uniformity of properties such as stability, convergence, etc., as σ ranges over $\mathcal{S}$. This paper addresses the uniform asymptotic stability of switched systems, where uniformity refers to the multiple solutions that one obtains as the switching signal ranges over a given set. We consider two notions of asymptotic stability for switched system. In the weaker one, no uniformity in the rate of convergence is required, whereas in the stronger one we do require it.

We take here a fairly broad definition of what is meant by a class of admissible switching signals. In particular, we consider families of switching signals that may be trajectory dependent. As a straightforward extension of previous results, we show that when the class of switching signals is trajectory independent, uniform asymptotic stability of linear switched systems actually implies exponential stability. However, this is not true in general, which underscores the fact that the class of linear switched systems is much richer than the class of linear systems (time-varying or not).

The main contribution of this paper is a collection of results inspired by LaSalle's Invariance Principle, which can be used to determine if a switched linear systems is asymptotically stable. The results cover (i) different structural assumptions placed on the systems being switched as well as (ii) distinct regularity assumptions placed on the class of switching signals considered. Depending on the structural assumptions, one may be able to conclude asymptotic stability or simply convergence to an invariant set. Different assumptions on the set of switching signals may or may not lead to uniformity.

LaSalle's Invariance Principle [1] addresses the asymptotic stability of a system described by a differential equation of the form

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \tag{3}$$

with f locally Lipschitz. We always consider systems for which the origin is an equilibrium point (i.e., $f(0) = 0$) and with some abuse say that a system is stable, meaning that the origin is a stable equilibrium point of the system. LaSalle's Invariance Principle states that when there exists a continuously differentiable, positive definite, and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ for which

$$L_f V(z) \leq 0, \qquad \forall z \in \mathbb{R}^n, \tag{4}$$

where $L_f V$ denotes the *Lie derivative* of V along the vector field f, then every solution x to (3) converges to the largest invariant set $\mathcal{M}$ contained in $\{z \in \mathbb{R}^n : L_f V(z) = 0\}$. When the set $\mathcal{M}$ only contains the origin, we conclude that (3) is globally asymptotically stable. For a linear system

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n, \tag{5}$$

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with a quadratic positive definite Lyapunov function $V(x) := x'Px$, $x \in \mathbb{R}^n$, the condition (4) is equivalent to requiring the matrix

$$Q := A'P + PA$$

to be negative semi-definite. When this condition is satisfied, we conclude from LaSalle's Invariance Principle that every solution to (5) converges to the largest invariant set $\mathcal{U}$ in the kernel of Q. From linear geometric theory (cf. [2]), it is well known that $\mathcal{U}$ is the largest A-invariant subspace[1] in the kernel of Q, which is precisely the unobservable subspace of the pair (Q, A). Note that when Q is negative semi-definite, we can write Q as $-C'C$, with $C \in \mathbb{R}^{m\times n}$ full rank, and conclude that the solution to (5) converges to the unobservable subspace of the pair (C, A). In case (Q, A) or (C, A) is observable then $\mathcal{U}$ is the singleton $\{0\}$ and (5) is globally asymptotically stable.

Some of the most useful tools used to prove stability of switched systems employ multiple Lyapunov functions [3], [4] and do not require the explicit computation of solutions to the switched system. The following result is of this type: Suppose that there exists a family $\{V_p : p \in \mathcal{P}\}$ of continuously differentiable, radially unbounded, positive definite functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that

$$L_{f_p} V_p(z) \leq W(z), \qquad \forall p \in \mathcal{P},\ z \in \mathbb{R}^n, \tag{6}$$

for some negative definite function $W : \mathbb{R}^n \to \mathbb{R}$ and

$$V_{p_2}\big(x(t)\big) \leq V_{p_1}\big(x(t)\big), \tag{7}$$

at every "switching time" t at which σ switches from p_1 to p_2. Then (1) is globally asymptotically stable. Note that, when σ is generated by a hybrid system, it is often possible to verify that (7) holds without actually computing the solution x to the switched system. This is because switching typically results from discrete transitions that are triggered by algebraic conditions on x.

As an extension of LaSalle's Invariance Principle, we will show that, for certain classes of switched linear systems, the function W in (6) need only be negative semi-definite to conclude asymptotic stability of the switched system (1). However, our results utilize different techniques than the ones used by LaSalle [1]. By exploring the switching structure of (1), we are able to conclude asymptotic stability from purely algebraic (observability) conditions and avoid the type of integral conditions found, e.g., in [5]–[7], [8, Chapter 4], which involve the solution to the differential equation. We also do not require checking for invariance over compact sets of functions as in [9]. Related to this research are also the results in [10], where the authors present an invariance principle for discrete-time systems that can be used to design switching controllers. In discrete-time, the closed-loop switched system can be viewed as a time-invariant nonlinear system so it is possible to use an argument similar to the one found in [1] to prove asymptotic stability. The paper [11] provides an invariance principle for deterministic time-invariant hybrid systems that is also relevant because often switched systems arise from abstractions of hybrid systems (cf. Section II). However, the results in [11] require checking set-invariance for a hybrid system, which is in general difficult. The use of switched systems as an abstraction to hybrid systems is, in fact, an attempt to obviate this.

In the context of our LaSalle-like Theorems, we study the impact of regularity assumptions on the switching signals on the type of asymptotic stability that is obtained (cf. Sections III and IV). In essence, to obtain uniform exponential stability of a switched linear system, we need all admissible switching signals to have infinitely many disjoint intervals of length no smaller than some scalar $\tau_D > 0$ and these intervals must be separated by no more than some scalar $T < \infty$. However, τ_D and T can be arbitrarily small and large, respectively. This requirement is tight in the sense that without it we can find counter-examples for which uniformity and even asymptotic stability are lost. These results also set us apart from the work mentioned above on LaSalle-like Theorems.

The remaining of this paper is organized as follows. In Section II we provide the basic mathematical framework under which we study switched systems and propose definitions for stability that capture the uniformity properties mentioned above. We also review existing results in light of this framework and highlight the importance of uniformity in the analysis of switched and hybrid systems. Section III contains the main result. Namely that, when a certain observability condition holds, we can conclude asymptotic stability of a switched linear system, even when W in (6) is only negative semi-definite. In Section IV, we relax the observability condition. We show in Section IV-A that when extra structure is available, observability can be relaxed to detectability. In Section IV-B, we show that when no extra structure is available we can only conclude that x converges to a certain invariant set—similarly to what happens in LaSalle's Invariance Principle. Section V contains concluding remarks and directions for future research. A subset of the results in this paper were presented at the 40th Conf. on Decision and Contr., Orlando, Florida [12].

Notation: A function $V : \mathbb{R}^n \to \mathbb{R}$ is called *positive definite* when $V(z) \geq 0$, $\forall z \in \mathbb{R}^n$ with equality just for $z = 0$ and V is called *radially unbounded* when $V(z)$ is always unbounded as $z \to \infty$. We say that a function $\alpha : [0,\infty) \to [0,\infty)$ is of *class* $\mathcal{K}$, and write $\alpha \in \mathcal{K}$, when α is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, then we say it is of *class* $\mathcal{K}_\infty$ and write $\alpha \in \mathcal{K}_\infty$. We say that a function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is of *class* $\mathcal{KL}$, and write $\beta \in \mathcal{KL}$ when $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s, t)$ decreases to 0 as $t \to \infty$ for each fixed $s \geq 0$.

II. Switched Systems

A *switched system* is defined by a parameterized family $\{f_p : p \in \mathcal{P}\}$ of locally Lipschitz vector fields from $\mathbb{R}^n$ to itself, together with a set $\mathcal{S}$ of piecewise constant *switching signals* from $[0,\infty)$ to $\mathcal{P}$. By a *piecewise constant* signal, we mean a signal that exhibits a finite number of discontinuities in any finite time interval and that is constant between consecutive discontinuities. By convention, we take piecewise

[1] A subspace $\mathcal{S}$ of $\mathbb{R}^n$ is called *A-invariant* when $A\mathcal{S} \subset \mathcal{S}$.

constant signals σ to be continuous from above, i.e., $\forall t \geq 0$ the limit from above of $\sigma(s)$ as $s \downarrow t$ is equal to $\sigma(t)$. The corresponding switched system is then represented by

$$\dot{x} = f_\sigma(x), \qquad \sigma \in \mathcal{S},\ t \geq 0. \tag{8}$$

When all the vector fields $\{f_p : p \in \mathcal{P}\}$ are linear, we say that (8) is a *switched linear system*. By a *solution to the switched system* (8), we mean a pair (x, σ) for which $\sigma \in \mathcal{S}$ and $x : [0, \infty) \to \mathbb{R}^n$ is a piecewise differentiable solution to the time-varying ordinary differential equation

$$\dot{x} = f_{\sigma(t)}(x), \qquad t \geq 0. \tag{9}$$

By a *piecewise differentiable* signal, we mean a signal whose derivative exhibits a finite number of discontinuities in any finite time interval.

It is often convenient to restrict the set of admissible switching signals to be trajectory dependent. This can be achieved by defining $\mathcal{S}$ to be a relation between the set of piecewise differentiable signals x taking values in $\mathbb{R}^n$ and the set of piecewise constant signals σ taking values in $\mathcal{P}$. Thus the elements of $\mathcal{S}$ are actually admissible pairs (x, σ). With some abuse of notation and to avoid introducing additional symbols, we still use (8) to denote a switched system and call $\mathcal{S}$ the *set of switching signals*, with the understanding that a *solution to* (8) is a pair $(x, \sigma) \in \mathcal{S}$ for which (9) holds. This formalism still captures the case in which the set of admissible switching signals is not trajectory dependent, because we can choose $\mathcal{S}$ to have the property that if (x, σ) belongs to the set then so does $(\bar{x}, \sigma)$ for any other piecewise differentiable $\bar{x}$. When this happens, we say that we have *trajectory-independent switching* and write $\sigma \in \mathcal{S}$ to mean that $(x, \sigma) \in \mathcal{S}$ for every x. We review next a few sets of switching signals that will be used in the paper. We also contrast switched systems with time-varying and hybrid systems.

a) Sets of Switching Signals: All sets of switching signals considered here are subsets of the set $\mathcal{S}_{\text{non-chatt}}$ of pairs (x, σ) for which x and σ are piecewise differentiable and piecewise constant, respectively. It turns out that this set often does not exhibit sufficient regularity for our purposes so we need to consider "better-behaved" subsets of $\mathcal{S}_{\text{non-chatt}}$. These include: the set $\mathcal{S}_{\text{dwell}}[\tau_D]$, $\tau_D > 0$ for which any consecutive discontinuities of σ are separated by at least a "dwell-time" τ_D; the set $\mathcal{S}_{\text{average}}[\tau_D, N_0]$, $\tau_D, N_0 > 0$ for which the number of discontinuities of σ in any the open interval is bounded above by the length of the interval normalized by an "average dwell-time" τ_D plus a "chatter bound" N_0; the set $\mathcal{S}_{\text{p-dwell}}[\tau_D, T]$, $\tau_D > 0$, $T \in [0, \infty]$ for which there is an infinite number of disjoint intervals of length no smaller than a "persistent dwell-time" τ_D on which σ is constant, and consecutive intervals with this property are separated by no more than a "period of persistence" T. More precise definitions of these sets can be found in the Appendix. It is straightforward to check (cf. Appendix) that

$$\begin{aligned}\mathcal{S}_{\text{dwell}}[\tau_D] = \mathcal{S}_{\text{average}}[\tau_D, 1] = \mathcal{S}_{\text{p-dwell}}[\tau_D, 0]\\ \subset \mathcal{S}_{\text{average}}[\tau_D, N_0] \subset \mathcal{S}_{\text{p-dwell}}[\delta\tau_D, T],\end{aligned} \tag{10}$$

$\forall \tau_D > 0, N_0 \geq 1, \delta \in (0,1),\ T := \frac{N_0 - \delta}{1 - \delta}\delta\tau_D$. The sets described next are limiting cases of the previous ones. Although they lack "uniformity," they still exhibit sufficient regularity for our purposes. These sets include, the set $\mathcal{S}_{\text{finite}}$ where each σ is restricted to have a finite number of discontinuities; the set $\mathcal{S}_{\text{dwell}}$, where each σ is restricted to have a dwell-time bounded away from zero but this bound is not uniform over all switching signals; the set $\mathcal{S}_{\text{average}}$, where each σ is restricted to have an average dwell-time bounded away from zero and finite chatter bound but these bounds are not uniform over all switching signals; the set $\mathcal{S}_{\text{p-dwell}}$, where each σ is restricted to have a positive persistent dwell-time and finite period of persistence but these are not uniform over all switching signals; the set $\mathcal{S}_{\text{weak-dwell}}$, for which each σ is restricted to have a persistent dwell-time bounded away from zero but can have infinite period of persistence. Because of (10), one concludes that

$$\mathcal{S}_{\text{finite}} \subset \mathcal{S}_{\text{dwell}} \subset \mathcal{S}_{\text{average}} \subset \mathcal{S}_{\text{p-dwell}} \subset \mathcal{S}_{\text{weak-dwell}} \subset \mathcal{S}_{\text{non-chatt}},$$

where all the inclusions are strict. All the sets defined so far correspond to trajectory-independent switching but the following one does not: Given a covering $\chi := \{\chi_p : p \in \mathcal{P}\}$ of $\mathbb{R}^n$, we denote by $\mathcal{S}_{\text{cover}}[\chi]$ the set of pairs $(x, \sigma) \in \mathcal{S}_{\text{non-chatt}}$ for which

$$x(t) \in \chi_{\sigma(t)}, \qquad \forall t \geq 0. \tag{11}$$

Not much can be said, in general, whether or not $\mathcal{S}_{\text{cover}}[\chi]$ is contained in any of the previous sets. However, it is sometimes possible to prove containment without computing the solution to the switched system by using, e.g., Lipschitz continuity of the vector fields $\{f_p : p \in \mathcal{P}\}$ and/or invariance of the χ_p.

b) Switched Systems vs. Time-varying Systems: A question that typically arises in the context of switched systems is: *What is the difference between a switched system such as* (8) *and a time-varying system such as* (9)*?* Hopefully, the definition above made this clear: the time-varying system (9) admits a family of solutions that can be parameterized solely by the the initial condition $x(0)$, whereas the switched system (8) admits a family of solutions that is parameterized both by the initial condition $x(0)$ and the switching signal σ. This distinction is crucial when one studies the uniformity of properties (such as stability, convergence, etc.) over the whole family of solutions to the system. This is further explored in Section II-A.

c) Switched vs. Hybrid Systems: Switched systems typically arise in the context of hybrid systems, i.e., systems that combine continuous dynamics (typically modeled by differential of difference equations) and event-driven logic (typically modeled by finite or infinite-state automaton) [13]. A simple hybrid system can be represented as follows:

$$\dot{x} = f_q(x), \qquad\qquad q = \phi_{q^-}(x), \tag{12}$$

where $x \in \mathbb{R}^n$ is called the *continuous state*, $q \in \mathcal{P}$ the *discrete state*, the vector fields $\{f_p : p \in \mathcal{P}\}$ are as above, and the $\phi_p : \mathbb{R}^n \to \mathcal{P}$, $p \in \mathcal{P}$ are called the *discrete transition functions*. A solution to (12) is any pair $(x, q) \in \mathcal{S}_{\text{non-chatt}}$ such that x is a solution to the differential equation

$$\dot{x} = f_{q(t)}(x), \qquad t \geq 0,$$

and

$$q(t) = \phi_{q^-(t)}\big(x(t)\big), \qquad \forall t \geq 0,$$

where for each $t > 0$, $q^-(t)$ denotes the limit from below of $q(\tau)$ as $\tau \uparrow t$. Much more general models for hybrid systems exist (cf., e.g., [11], [13]–[15]) but this simple one is sufficient for our purposes.

In general, determining properties of (12) directly is difficult so a common technique used to analyze these systems is to embed them into a switched system that may have more solutions but is simpler to analyze. This is closely related to the concept of abstraction in [16]. The simplest switched system that abstracts (12) is defined by (8) with $\mathcal{S}$ equal to the set $\mathcal{S}_{\text{tight}}$ of pairs $(x, q) \in \mathcal{S}_{\text{non-chatt}}$ such that

$$q(t) = \phi_{q^-(t)}\big(x(t)\big), \qquad \forall t \geq 0.$$

Clearly, not much is gained from this particular abstraction because the set of solutions to the switched and the hybrid systems are exactly the same. More interesting abstractions arise when the hybrid system is of the form

$$\dot{x} = f_q(x), \qquad \dot{z} = g_q(x,z), \qquad q = \phi_{q^-}(x,z), \tag{13}$$

with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $q \in \mathcal{P}$ and one is able to find a set of switching signals $\mathcal{S}_{\text{loose}}$ such that all solutions to (13) are also solutions to (8) with $\mathcal{S} := \mathcal{S}_{\text{loose}}$. Typically, z would be a component of the state for which one does not seek to investigate convergence. This type of system arises, e.g., in supervisory control where one chooses $\mathcal{S}_{\text{loose}}$ to be $\mathcal{S}_{\text{dwell}}[\tau_D]$ for some $\tau_D > 0$ [17]–[19]; or $\mathcal{S}_{\text{average}}[\tau_D, N_0]$ for some $\tau_D, N_0 > 0$ [20], [21]; or even $\mathcal{S}_{\text{finite}}$ [22], [23]. The fact that any solution to the original hybrid system must necessarily belong to these sets of switching signals needs to be proved separately. However, this is often enforced by construction [17]–[19] or can be proved using relatively simple arguments [20]–[23]. This type of approach was also pursued in [24] to stabilize Linear Parameter Varying (LPV) systems, where the authors enforce by design that the discontinuities of σ are separated by a minimum time that guarantees stability of the switched system. Several other examples can be found in the literature.

d) Switched vs. Discontinuous Systems: Switched systems also provide a framework to study the properties of discontinuous systems of the type

$$\dot{x} = \begin{cases} f_{p_1}(x) & x \in \chi_{p_1} \\ f_{p_2}(x) & x \in \chi_{p_2} \\ \vdots \end{cases} \tag{14}$$

where $\chi := \{\chi_p : p \in \mathcal{P}\}$ is a disjoint covering of $\mathbb{R}^n$ (cf., e.g., [25]). This system can be viewed as the switched system (8) if one defines $\mathcal{S}$ to be the set $\mathcal{S}_{\text{cover}}[\chi]$ defined in (11). This illustrates that switched systems do not always have solutions (even for Lipschitz continuous vector fields f_p) because it is simple to produce systems like (14) that do not have any solution (at least in the sense of Carathéodory). One should therefore be careful when proving properties of *all* solutions to (8) as the statements can be vacuously true.

A. Stability

We say that the switched system (8) is *stable* if there exists a function α of class $\mathcal{K}$ such that

$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \qquad \forall t \geq t_0 \geq 0, \tag{15}$$

along every solution to (8). When (8) is stable and $x(t)$ converges to zero as $t \to +\infty$ we say that (8) is *asymptotically stable*. When this convergence is uniform over all switching signals, i.e., when there exists a function β of class $\mathcal{KL}$ such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t), \qquad \forall t \geq t_0 \geq 0, \tag{16}$$

along every solution to (8), we say that (8) is *uniformly asymptotically stable*. If β can be chosen of the form $\beta(s,t) = ce^{-\lambda t}s$, $\forall t, s \geq 0$ for given constants $c, \lambda > 0$ we say that (8) is *(uniformly) exponentially stable*. Although equations (15) and (16) appear similar to the ones in the corresponding definitions for non-switched systems, one must keep in mind that there is an universal quantification with respect to all solutions to the switched system and therefore the functions α and β must not depend on the switching signal.

There is a gap between asymptotic stability of switched systems and uniform asymptotic stability in the sense that one can find switched systems that are asymptotically stable but not uniformly so. Moreover, this gap exists both for state-dependent and state-independent switching. This will be explicitly shown later in Example 2, once we have the tools needed to prove stability. For now, we present two results (Lemmas 1 and 2) that underscore the importance of uniformity.

Lemma 1: For linear switched systems with trajectory-independent switching, uniform asymptotic stability is equivalent to exponential stability. □

The proof of this Lemma (given in the Appendix) follows closely the proof of the well-known fact that for time-varying (nonswitched) linear systems, uniform asymptotic stability is equivalent to exponential stability [8, Chapter 3]. This result was proved in [27] for switched linear systems over the class $\mathcal{S}_{\text{non-chatt}}$ of all piecewise constant switching signals and extended in [28] to switched homogeneous (not necessarily linear) systems over the same class of switching signals. Lemma 1 provides a straightforward extension of these results to other classes of switching signals. However, it is important to notice that in general it cannot be extended to state-dependent switching (cf. Example 1 below). This attests to the fact that the class of state-dependent switched linear systems is significantly richer than the class of linear system (time-varying or not).

Example 1: Consider the following switched system

$$\dot{x} = -\sigma x, \qquad \sigma \in \mathcal{S},$$

where $\mathcal{S}$ contains all pairs $(x, \sigma) \in \mathcal{S}_{\text{non-chatt}}$, with x taking values in $\mathbb{R}$ and σ in $\mathcal{P} := [0, \infty)$, such that

$$\sigma(t) = \begin{cases} 0 & x(t) = 0 \\ 2^n & |x(t)| \in [2^{-n-1}, 2^{-n}),\ n \in \mathbb{Z} \end{cases} \tag{17}$$

where $\mathbb{Z}$ denotes the set of (positive and negative) integers. This set of switching signals $\mathcal{S}$ is actually the set $\mathcal{S}_{\text{cover}}[\chi]$ that we encountered before, for the covering $\chi := \{\chi_p : p \in \mathcal{P}\}$ of $\mathbb{R}$ defined by

$$\chi_p := \begin{cases} \{0\} & p = 0 \\ [2^{-n-1}, 2^{-n}) & p = 2^n, n \in \mathbb{Z} \\ \emptyset & \text{otherwise} \end{cases}$$

From (17), we can see that $-\frac{1}{2}x|x| \le \dot{x} \le -x|x|$, and therefore

$$\frac{x(t_0)}{1 + (t - t_0)|x(t_0)|} \le x(t) \le \frac{2x(t_0)}{2 + (t - t_0)|x(t_0)|},$$

$\forall t \ge t_0,\ \sigma \in \mathcal{S}$, which shows that the convergence of x to zero is uniform but not exponential. It would be straightforward to modify this example so that σ would take values in a compact set because the problems arise as σ takes values in a small neighborhood of the origin and not as σ takes large values. □

So far we considered autonomous switched systems. Consider now the following switched system with inputs:

$$\dot{x} = f_\sigma(x, u), \qquad \sigma \in \mathcal{S},\ t \ge 0, \tag{18}$$

where $\{f_p : p \in \mathcal{P}\}$ denotes a family of locally Lipschitz vector fields from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$ and the set $\mathcal{S}$ of switching signals consists of triples (x, u, σ). A solution to (18) is now a triple $(x, u, \sigma) \in \mathcal{S}$ that satisfies the ordinary differential equation

$$\dot{x} = f_{\sigma(t)}(x, u), \qquad t \ge 0.$$

We say that (18) has *$\mathcal{L}_2$-induced norm* $\mathfrak{g}$ if there exists a constant $\mathfrak{g}_0$ such that

$$\Big(\int_0^t \|x(\tau)\|^2\Big)^{\frac{1}{2}} \le \mathfrak{g}\Big(\int_0^t \|u(\tau)\|^2\Big)^{\frac{1}{2}} + \mathfrak{g}_0\|x(0)\|, \tag{19}$$

$\forall t \ge 0$, along every solution to (18). When (19) is replaced by

$$\|x(t)\| \le \mathfrak{g} \sup_{[0,t)} \|u(\tau)\| + \mathfrak{g}_0\|x(0)\|, \qquad \forall t \ge 0,$$

we say that (18) has *$\mathcal{L}_\infty$-induced norm* $\mathfrak{g}$, and if (19) is replaced by

$$\|x(t)\| \le \mathfrak{g}\Big(\int_0^t \|u(\tau)\|^2\Big)^{\frac{1}{2}} + \mathfrak{g}_0\|x(0)\|, \qquad \forall t \ge 0,$$

we say that (18) has *$\mathcal{L}_2$-to-$\mathcal{L}_\infty$-induced norm* $\mathfrak{g}$. The following lemma also depends crucially on trajectory-independent switching and uniformity. Proving it is straightforward once exponential stability has been established (cf. [29] for details).

Lemma 2: Suppose that the linear switched system (18) has linear maps f_p uniformly bounded over $\mathcal{P}$. For trajectory-independent switching, if (18) is uniformly asymptotically stable, then it has finite $\mathcal{L}_2$, $\mathcal{L}_\infty$, and $\mathcal{L}_2$-to-$\mathcal{L}_\infty$ induced norms. □

The results in this section are the basis of essentially every argument that we are aware of to prove robust stability of switched and hybrid systems. Indeed, they provide the main motivation to study the uniformity of convergence to the origin for stable linear switched systems. Examples of robust stability arguments that use these results can be found, e.g., in the proofs of robust stability and performance for supervisory control schemes [20], [21], [30].

B. Lyapunov Stability Theorems

Several Lyapunov-like theorems that can be found in the literature allow one to establish the stability of a switched system such as (8), without explicitly solving the ordinary differential equations (9). The result presented below is based on the idea of multiple Lyapunov functions in [3], [4]. It is less general than the ones in [31], [32] but it has the advantage that, in the spirit of Lyapunov's direct method, the stability test can be performed without solving (9) because it relies solely on the Lie derivative of the multiple Lyapunov functions and not on the evolution of x between switching times. The price paid is of course a more conservative result than those, e.g., in [31], [32].

Theorem 3: Suppose that there exists a family $\{V_p : p \in \mathcal{P}\}$ of continuously differentiable, radially unbounded, positive definite functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that, $\forall (x, \sigma) \in \mathcal{S}$,

$$V_{\sigma(t)}\big(x(t)\big) \le V_{\sigma(t^-)}\big(x(t)\big), \qquad \forall t \ge 0, \tag{20}$$

and, $\forall z \in \mathbb{R}^n,\ p \in \mathcal{P}$,

$$\alpha_1(\|z\|) \le V_p(z) \le \alpha_2(\|z\|), \quad L_{f_p}V_p(z) \le W_p(z), \tag{21}$$

for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and negative semi-definite functions $W_p : \mathbb{R}^n \to \mathbb{R}$, $p \in \mathcal{P}$. Then (8) is stable. Moreover, if there exists an $\alpha_3 \in \mathcal{K}$ such that

$$W_p(z) \le -\alpha_3(\|z\|), \qquad \forall z \in \mathbb{R}^n,\ p \in \mathcal{P}, \tag{22}$$

then (8) is uniformly asymptotically stable. □

Note that (20) is only non-trivially satisfied at points of discontinuity of σ and is trivially true when all the V_p are equal (common Lyapunov function). The condition (21) could be relaxed by restricting the quantification on z and p to pairs $(z, p) \in \mathbb{R}^n \times \mathcal{P}$ such that there exists some $(x, \sigma) \in \mathcal{S}$ such that $x(t) = z$ and $\sigma(t) = p$ for some $t \ge 0$. This is particularly useful for sets of switching signals such as $\mathcal{S}_{\text{cover}}[\chi]$, where the quantification would be over the set of pairs (z, p) such that $z \in \chi_p$.

Although we could not quite find Theorem 3 in the literature (taking into account the "uniformity" built into our definition of stability) it is straightforward to adapt to our formulation the stability proofs, e.g., in [3], [4], [31], [32] so this is not really a new result. The uniformity needed comes from the assumption that the functions α_1, α_2, and α_3 in (21) and (22) do not depend on p.

For switched linear systems such as

$$\dot{x} = A_\sigma x, \qquad \sigma \in \mathcal{S},\ t \ge 0, \tag{23}$$

and quadratic Lyapunov functions, we obtain the following corollary of Theorem 3:

Corollary 1: Suppose that there exists a compact family $\{P_p \in \mathbb{R}^{n\times n} : p \in \mathcal{P}\}$ of symmetric positive definite matrices such that, for every $(x, \sigma) \in \mathcal{S}$,

$$x(t)' P_{\sigma(t)} x(t) \le x(t)' P_{\sigma(t^-)} x(t), \qquad \forall t \ge 0, \tag{24}$$

and

$$A_p' P_p + P_p A_p \le -C_p' C_p \quad (\le 0), \qquad \forall p \in \mathcal{P}, \tag{25}$$

for appropriately defined matrices $C_p \in \mathbb{R}^{m\times n}$. Then (23) is stable. Moreover, if

$$C_p' C_p \ge \rho I > 0, \qquad \forall p \in \mathcal{P}, \tag{26}$$

for some $\rho > 0$, then (23) is uniformly asymptotically stable. □

In the following sections we discard (26) and investigate which convergence properties still hold for the switched system.

Remark 1: In hybrid control systems, σ is often generated by a supervisory logic that guarantees, by construction, that (20) or (24) hold. Typically, these equations only needs to hold over subsets of the state space that are defined by simple algebraic conditions (often linear or affine subspaces of $\mathbb{R}^n$, cf. [33] and references therein). □

III. Main Result

Consider the switched linear system

$$\dot{x} = A_\sigma x, \qquad \sigma \in \mathcal{S},\ t \ge 0, \tag{27}$$

and suppose that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric positive definite $n \times n$ matrices for which (24) and (25) hold for every $(x, \sigma) \in \mathcal{S}$. Defining

$$v(t) := x'(t) P_{\sigma(t)} x(t), \qquad \forall t \ge 0,$$

we conclude from (25) that between switching times we have

$$\dot{v}(t) \le -\|y(t)\|^2 \le 0, \qquad \forall t \ge 0, \tag{28}$$

where $y(t) := C_{\sigma(t)} x(t)$, $t \ge 0$. Moreover, at switching times $v(t)$ may be discontinuous but it is non-increasing because of (24). This means that v is uniformly bounded by $v(0)$ and therefore, for every $t \ge 0$,

$$\|x(t)\| \le \mu \|x(0)\|, \qquad \mu := \frac{\max_{p\in\mathcal{P}} \sigma_{\max}[P_p]}{\min_{p\in\mathcal{P}} \sigma_{\min}[P_p]}, \tag{29}$$

where $\sigma_{\max}[P_p]$ and $\sigma_{\min}[P_p]$ denote the largest and smallest singular values of P_p, respectively. Stability of (27) follows directly from (29). From (28) we can also conclude that $y(t)$ is an $\mathcal{L}_2$ signal, i.e., $\int_0^\infty \|y(t)\|^2 dt < \infty$. Indeed, because of (28)

$$v(t) \le v(\tau) + \int_\tau^t \dot{v}(s) ds \le v(\tau) - \int_\tau^t \|y(s)\|^2 ds,$$

$\forall t \ge \tau \ge 0$ and therefore

$$\int_\tau^t \|y(s)\|^2 ds \le v(\tau) - v(t) \le v(\tau), \qquad \forall t \ge \tau \ge 0, \tag{30}$$

which shows that $y \in \mathcal{L}_2$. Although $y \in \mathcal{L}_2$ and its derivative is bounded wherever it exists, we cannot use Barbalat's Lemma [34], [35] to conclude that $y(t) \to 0$ as $t \to \infty$ because y is not continuous. However, this lemma can easily be extended to discontinuous signals, provided that the interval between consecutive discontinuities is uniformly bounded below by a positive constant. This happens, e.g, when $\sigma \in \mathcal{S}_{\text{dwell}}$. It turns out that even if we only have $\sigma \in \mathcal{S}_{\text{weak-dwell}}$ ($\supset \mathcal{S}_{\text{dwell}}$) but in addition, every pair (C_p, A_p), $p \in \mathcal{P}$ is observable, we can actually conclude that the whole state $x(t)$ converges to zero as $t \to \infty$ and not just $y(t) \to 0$. This is stated in the following theorem:

Theorem 4: Suppose that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric positive definite matrices such that, for every $(x, \sigma) \in \mathcal{S}$,

$$x(t)' P_{\sigma(t)} x(t) \le x(t)' P_{\sigma(t^-)} x(t), \qquad \forall t \ge 0, \tag{31}$$

and

$$A_p' P_p + P_p A_p \le -C_p' C_p \quad (\le 0), \qquad \forall p \in \mathcal{P}, \tag{32}$$

for an appropriately defined set of matrices $\{C_p : p \in \mathcal{P}\}$ for which $\{(C_p, A_p) : p \in \mathcal{P}\}$ is compact. Then (27) is stable. Moreover, when every pair (C_p, A_p), $p \in \mathcal{P}$ is observable,

(i) if $\mathcal{S} \subset \mathcal{S}_{\text{weak-dwell}}$ then (27) is asymptotically stable,

(ii) if $\mathcal{S} \subset \mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ for some $\tau_D > 0$, $T < \infty$ then (27) is (uniformly) exponentially stable. □

Here and in all the results that follow, the condition (32) can be relaxed to

$$z'(A_p' P_p + P_p A_p + C_p' C_p) z \le 0,$$

for all pairs $(z, p) \in \mathbb{R}^n \times \mathcal{P}$ for which there exists some $(x, \sigma) \in \mathcal{S}$ such that $x(t) = z$ and $\sigma(t) = p$ for some $t \ge 0$.

For the non-uniform stability result (i) and when $\mathcal{P}$ is finite, the condition (31) can be relaxed to simply demanding that for every consecutive intervals $[\tau_1, t_1)$ and $[\tau_2, t_2)$ on which σ takes the same value $p \in \mathcal{P}$, we have

$$x(\tau_2)' P_p x(\tau_2) \le x(t_1)' P_p x(t_1)$$

[36]. However, this condition does not seem to be sufficient for uniform stability so we do not pursue it here.

The observability of the pairs (C_p, A_p) automatically guarantees the existence of positive definite matrices P_p that satisfy (32) (even with equality). The challenge in applying this theorem—as with essentially any theorem based on multiple Lyapunov functions (e.g., Theorem 3 or other versions of it in the literature)—is to find matrices P_p that also satisfy (31). This is often done numerically by finding solutions to systems of Matrix Linear Inequalities (LMIs) (cf., e.g., [25]) or constructively by selecting appropriate "switching surfaces" (cf., e.g., [3]). The reader is referred to [33] for a more detailed discussion on this issue.

Before proving Theorem 4 we would like to point-out that some form of regularity in the switching signals is needed to conclude asymptotic stability. In fact, the requirement that $\mathcal{S} \subset \mathcal{S}_{\text{weak-dwell}}$ (or $\mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ for the uniform case) is not an artifact of the proof and without it we could construct counter-examples to the result above. This is shown through the following example.

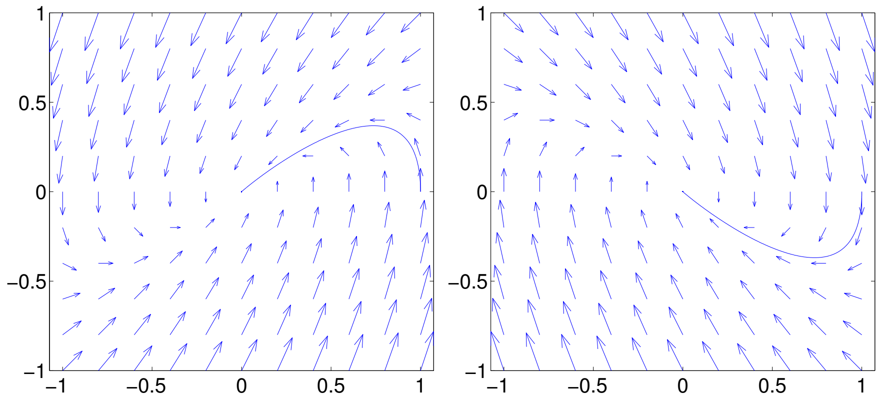

Fig. 1. Vector fields and typical trajectory for the matrices A_1 (left) and A_2 (right) defined in (33).

Example 2: Consider the switched system with $\mathcal{P} := \{1, 2\}$ and

$$A_1 := \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}, \qquad A_2 := \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \tag{33}$$

whose vector fields are drawn in Figure 1. For these matrices we have

$$A_p' P_p + A_p P_p = -C_p' C_p, \qquad \forall p \in \mathcal{P}, \tag{34}$$

with $P_1 = P_2 = I$ and $C_1 = C_2 = [0 \quad 2]$. Equation (32) holds because of (34) and, since all the P_p are equal (common Lyapunov function) the inequality (31) is also trivially satisfied. Moreover, both pairs (C_p, A_p), $p \in \mathcal{P}$ are observable. It turns out that it is possible to construct a piecewise constant switching signal σ for which x does not converge to zero. Of course this σ does not satisfy the regularity imposed by $\mathcal{S} \subset \mathcal{S}_{\text{weak-dwell}}$. To construct such a switching signal suppose that at time t_k the state $x(t_k)$ is over the horizontal axis (i.e., $x_2(t_k) = 0$) by setting first $\sigma = 1$ and then $\sigma = 2$ it is possible to compute another time $t_{k+1} > t_k$ such that $x(t_{k+1})$ is again over the horizontal axis. It is straightforward to check that this will occur if we set $\sigma = 1$ for some time δ_k followed by $\sigma = 2$ for some time $\delta_k/(1 - 2\delta_k)$, and therefore

$$t_{k+1} = t_k + \delta_k + \frac{\delta_k}{1 - 2\delta_k} = t_k + \frac{2\delta_k(1 - \delta_k)}{1 - 2\delta_k}. \tag{35}$$

Moreover, solving the differential equations, we conclude that

$$x_1(t_{k+1}) = \frac{e^{-2\delta_k(1-\delta_k)/(1-2\delta_k)}}{1 - 2\delta_k} x_1(t_k). \tag{36}$$

Although $e^{-2\delta_k(1-\delta_k)/(1-2\delta_k)}/(1-2\delta_k) < 1$ for every $\delta_k > 0$, it is possible to chose the δ_k so that $t_k \to \infty$ (which guarantees that σ is piecewise constant because it only has a finite number of discontinuities in finite time) and yet $x_1(t_k)$ does not converge to zero. Indeed, iterating (35) and (36) from 0 to k and taking logarithms of the latter, we conclude that

$$t_k = \sum_{i=0}^{k-1} \frac{2\delta_i(1 - \delta_i)}{1 - 2\delta_i}, \tag{37}$$

$$\log x_1(t_k) = \log x_1(t_0) - \sum_{i=0}^{k-1} \Big(\frac{2\delta_i(1 - \delta_i)}{1 - 2\delta_i} - \log(1 - 2\delta_i) \Big). \tag{38}$$

Since

$$\frac{2\delta_k(1 - \delta_k)}{1 - 2\delta_k} = o(\delta_k), \ \frac{2\delta_k(1 - \delta_k)}{1 - 2\delta_k} - \log(1 - 2\delta_k) = o(\delta_k^3),$$

it is possible to select the δ_k (e.g., $\delta_k = \frac{1}{k}$) such that the series in (37) diverges (and therefore $t_k \to \infty$) and yet the series in (38) converges (and therefore $x_1(t_k) \not\to 0$). Clearly, the δ_k must converge to zero, which means that the corresponding switching signal is not in $\mathcal{S}_{\text{weak-dwell}}$. Using the same ideas one could also construct a switching signal σ in $\mathcal{S}_{\text{weak-dwell}}$ but not in any $\mathcal{S}_{\text{p-dwell}}[\tau_D, T]$, $\tau_D > 0$, $T < \infty$ for which $x \to 0$ but not exponentially fast. In fact, one could make the convergence arbitrarily slow. This could be achieved by making $\delta_k = \frac{1}{k}$ for most values of k, interlaced by $\delta_{\bar{k}} = \tau_D > 0$, for values of $\bar{k}$ increasingly spread apart, so that there is no finite period of persistency. This would be an example of a state-independent switched system that is asymptotically state but not uniformly. This example illustrates how the requirements on $\mathcal{S}$ in the statements (i) and (ii) are tight. □

We start by proving (ii) and leave the proof of (i) for Section IV-B. To prove (ii) we need the following result (proved in the Appendix), which is a consequence of the Squashing Lemma in [37]:

Lemma 5: Assume given positive finite constants τ_D, T, λ and a compact set of matrix pairs $\{(A_p, C_p) : p \in \mathcal{P}\}$ such that every pair (C_p, A_p), $p \in \mathcal{P}$ is observable. Then, there exist constants $c, k > 0$ such that for every $\sigma \in \mathcal{S}_{\text{p-dwell}}[\tau_D, T]$,

$$\|\Phi_\sigma(t, \tau)\| \le c e^{-\lambda(t-\tau)}, \qquad \forall t \ge \tau \ge 0, \tag{39}$$

where $\Phi_\sigma(t, \tau)$ denotes the state transition matrix of the time-varying system

$$\dot{z} = \big(A_{\sigma(t)} + K(t) C_{\sigma(t)}\big) z, \qquad t \ge 0,$$

for some appropriately chosen time-varying output-injection matrix K whose norm is uniformly bounded by the constant k. □

Proof: [Theorem 4 (ii)] We have already shown that (27) is stable. We show next that when every pair (C_p, A_p), $p \in \mathcal{P}$ is observable and $\mathcal{S} \subset \mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ for some $\tau_D > 0, T < \infty$, then (27) is exponentially stable. To this effect, let λ be an arbitrary positive constant and $(x, \sigma) \in \mathcal{S}$ a solution to (27).

From Lemma 5, we known that there exists a time-varying output-injection matrix K such that the state transition matrix Φ_σ of the time-varying system

$$\dot{z} = \big(A_{\sigma(t)} + K(t)C_{\sigma(t)}\big)z, \qquad t \geq 0,$$

satisfies (39) and $\|K(t)\| \leq k$, $\forall t \geq 0$. Suppose now that we re-write (27) as

$$\dot{x} = (A_\sigma + KC_\sigma)x - Ky. \tag{40}$$

By the variation of constants formula we then obtain

$$x(t) = \Phi_\sigma(t,\tau)x(\tau) - \int_\tau^t \Phi_\sigma(t,s)K(s)y(s)ds,$$

$\forall t \geq \tau \geq 0$. Taking norms, and using the Cauchy-Schwartz inequality together with (39), we conclude that

$$\begin{aligned}\|x(t)\| &\leq \|\Phi_\sigma(t,\tau)\|\,\|x(\tau)\| \\ &\quad + \Big(\int_\tau^t \|\Phi_\sigma(t,s)K(s)\|^2 ds\Big)^{\frac{1}{2}} \Big(\int_\tau^t \|y(s)\|^2 ds\Big)^{\frac{1}{2}} \\ &\leq ce^{-\lambda(t-\tau)}\|x(\tau)\| \\ &\quad + ck\Big(\int_\tau^t e^{-2\lambda(t-s)}ds\Big)^{\frac{1}{2}} \Big(\int_\tau^t \|y(s)\|^2 ds\Big)^{\frac{1}{2}}.\end{aligned}$$

Moreover, since

$$\int_\tau^t e^{-2\lambda(t-s)}ds = \int_0^{t-\tau} e^{-2\lambda s}ds \leq \int_0^\infty e^{-2\lambda s}ds = \frac{1}{2\lambda},$$

$\forall t \geq \tau \geq 0$, we conclude that

$$\|x(t)\| \leq ce^{-\lambda(t-\tau)}\|x(\tau)\| + \frac{ck}{\sqrt{2\lambda}}\Big(\int_\tau^t \|y(s)\|^2 ds\Big)^{\frac{1}{2}}. \tag{41}$$

Since the set $\{P_p : p \in \mathcal{P}\}$ of positive definite matrices is compact, there exist constants $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 I \leq P_p \leq \alpha_2 I, \qquad \forall p \in \mathcal{P}. \tag{42}$$

Therefore, for every $t \geq \tau \geq 0$,

$$\begin{aligned}v(t) &:= x'(t)P_{\sigma(t)}x(t) \leq \alpha_2\|x(t)\|^2 \\ &\leq 2\alpha_2 c^2 e^{-2\lambda(t-\tau)}\|x(\tau)\|^2 + \bar{k}\int_\tau^t \|y(s)\|^2 ds \\ &\leq \frac{2\alpha_2 c^2}{\alpha_1}e^{-2\lambda(t-\tau)}v(\tau) + \bar{k}\int_\tau^t \|y(s)\|^2 ds,\end{aligned} \tag{43}$$

where $\bar{k} := \alpha_2 c^2 k^2/\lambda$. Here, we used (41) and the fact that for any positive scalars a, b, we have $(a+b)^2 \leq 2a^2 + 2b^2$. Combining (43) with (30) we obtain

$$v(t) \leq \frac{2\alpha_2 c^2}{\alpha_1}e^{-2\lambda(t-\tau)}v(\tau) + \bar{k}\big(v(\tau) - v(t)\big),$$

$\forall t \geq \tau \geq 0$, or equivalently

$$v(t) \leq \frac{\frac{2\alpha_2 c^2}{\alpha_1}e^{-2\lambda(t-\tau)} + \bar{k}}{1 + \bar{k}}v(\tau), \qquad \forall t \geq \tau \geq 0.$$

Suppose now that we pick a constant L such that

$$\rho := \frac{\frac{2\alpha_2 c^2}{\alpha_1}e^{-2\lambda L} + \bar{k}}{1 + \bar{k}} < 1,$$

and therefore v contracts by at least ρ in any interval of length L. From this, it is straightforward to conclude that

$$v(t) \leq \bar{c}\rho^{(t-\tau)/L}\, v(\tau), \qquad \forall t \geq \tau \geq 0,$$

where $\bar{c} := (\bar{k} + 2\alpha_2 c^2/\alpha_1)/\rho/(1+\bar{k})$. Together with (42), this leads to

$$\|x(t)\|^2 \leq \frac{v(t)}{\alpha_1} \leq \frac{\bar{c}\rho^{(t-\tau)/L}}{\alpha_1}v(\tau) \leq \frac{\bar{c}\alpha_2\rho^{(t-\tau)/L}}{\alpha_1}\|x(\tau)\|^2,$$

$\forall t \geq \tau \geq 0$, and therefore x converges to zero exponentially fast. Since the bound constructed above is independent of $(x,\sigma) \in \mathcal{S} \subset \mathcal{S}_{\text{p-dwell}}[\tau_D, T]$, we conclude that (27) is (uniformly) exponentially stable, which proves (ii). ■

IV. Relaxing the Observability Assumption

In this section we present two alternative methods to relax the observability assumption in Theorem 4. The first explores the case where the matrices A_p, $p \in \mathcal{P}$ have additional structure. The second does not require extra structure but we are no longer able to conclude that the state x of the switched system converges to the origin. Instead, we conclude that x converges to a specific "invariant" set.

A. Switched State Feedback

We consider here the case where all the matrices $A_p, p \in \mathcal{P}$ only differ by a state feedback matrix gain, i.e., all the A_p, $p \in \mathcal{P}$ are of the form

$$A_p = A + BF_p, \qquad \forall p \in \mathcal{P}, \tag{44}$$

where A and B are given matrices and $\{F_p : p \in \mathcal{P}\}$ is a compact set of state feedback matrix gains. This type of structure arises, e.g., when a fixed time-invariant process is controlled using a switched state feedback gain.

The following result explores the added structure provided by (44) to relax the observability assumption and simply demands detectability. However, now the right-hand-side of (32) is also required to be constant. This result is inspired by the Switching Theorem in [17].

Theorem 6: Suppose that (44) holds and that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric positive definite $n \times n$ matrices such that, for every $(x,\sigma) \in \mathcal{S}$,

$$x(t)'P_{\sigma(t)}x(t) \leq x(t)'P_{\sigma(t^-)}x(t), \qquad \forall t \geq 0,$$

and

$$A_p' P_p + P_p A_p \leq -C'C \quad (\leq 0), \qquad \forall p \in \mathcal{P},$$

for an appropriately defined matrix C. Then (27) is stable. Moreover, when (A,B,C) is a left-invertible system and every pair (C, A_p), $p \in \mathcal{P}$ is detectable,

(i) if $\mathcal{S} \subset \mathcal{S}_{\text{weak-dwell}}$ then (27) is asymptotically stable,

(ii) if $\mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ for some $\tau_D > 0$, $T < \infty$ then (27) is (uniformly) exponentially stable. □

Also here we start by proving (ii) and leave the proof of (i) for Section IV-B. To prove (ii) we need the following decomposition from [17, Lemma 5]:

Lemma 7: Let (A, B, C) be a left-invertible system. Then, there exist matrices A_-, $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{K}$, $\bar{F}$, Q, with Q nonsingular such that

$$QAQ^{-1} = \begin{bmatrix} \bar{A} & \bar{B}\bar{F} \\ \bar{K}\bar{C} & A_- \end{bmatrix}, \tag{45}$$

$$QB = \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix}, \qquad CQ^{-1} = \begin{bmatrix} \bar{C}' \\ 0 \end{bmatrix}, \tag{46}$$

where A_-'s spectrum is the set of transmission zeros of (A, B, C) with negative real part, and $(\bar{A}, \bar{B}, \bar{C})$ does not have any transmission zero with negative real part. □

Proof: [Theorem 6 (ii)] We have already shown for the general case that (27) is stable. We show next that under the assumptions of this theorem and $\mathcal{S} \subset \mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ for some $\tau_D > 0, T < \infty$, we have exponential stability for (27).

Let $A_-, \bar{A}, \bar{B}, \bar{C}, \bar{K}, \bar{F}, Q$ be the matrices whose existence is guaranteed by Lemma 7. Since detectability is invariant under state-coordinate transformations and $(C, A + BF_p)$, $p \in \mathcal{P}$ is detectable then so is the pair

$$\left(CQ^{-1}, Q(A+BF_p)Q^{-1}\right) = \Bigg(\begin{bmatrix} \bar{C} & 0 \end{bmatrix}, \begin{bmatrix} \bar{A} + \bar{B}\bar{F}_p & \bar{B}(\bar{F} + \tilde{F}_p) \\ 0 & A_- \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{K} \end{bmatrix} \begin{bmatrix} \bar{C} & 0 \end{bmatrix} \Bigg),$$

where $[\bar{F}_p \quad \tilde{F}_p] := F_p Q^{-1}$, $p \in \mathcal{P}$. Moreover, since detectability is also invariant under output-injection transformations and A_- is a stability matrix, we conclude that each pair $(\bar{C}, \bar{A} + \bar{B}\bar{F}_p)$, $p \in \mathcal{P}$ must be detectable. But $(\bar{A}, \bar{B}, \bar{C})$ does not have any transmission zero with negative real part, therefore $(\bar{C}, \bar{A} + \bar{B}\bar{F}_p)$ must actually be observable. On the other hand, defining $[z_1' \quad z_2']' := Qx$, from (27), (44), and (45)–(46) we conclude that

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \bar{A} + \bar{B}\bar{F}_\sigma & \bar{B}(\bar{F} + \tilde{F}_\sigma) \\ \bar{K}\bar{C} & A_- \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \tag{47}$$

$$y = \begin{bmatrix} \bar{C} & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \tag{48}$$

Since each pair $(\bar{C}, \bar{A} + \bar{B}\bar{F}_p)$, $p \in \mathcal{P}$ is observable, from Lemma 5 we known that there exists a time-varying output-injection matrix $\tilde{K}$ such that the state transition matrix $\bar{\Phi}_\sigma$ of the time-varying system

$$\dot{z} = \left(\bar{A} + \bar{B}\bar{F}_{\sigma(t)} + \tilde{K}(t)\bar{C}\right)z, \qquad t \geq 0,$$

satisfies

$$\|\bar{\Phi}_\sigma(t, \tau)\| \leq \bar{c}e^{-\bar{\lambda}(t-\tau)}, \qquad \forall t \geq \tau \geq 0,$$

with $\|\tilde{K}(t)\| \leq \bar{k}$, $\forall t \geq 0$ for $\bar{\lambda}, k > 0$ independent of σ. Suppose now that we re-write (47)–(48) as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \bar{A} + \bar{B}\bar{F}_\sigma + \tilde{K}\bar{C} & \bar{B}(\bar{F} + \tilde{F}_\sigma) \\ 0 & A_- \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} - \begin{bmatrix} \tilde{K} \\ -\bar{K} \end{bmatrix} y, \tag{49}$$

$$y = \begin{bmatrix} \bar{C} & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \tag{50}$$

Because of the diagonal structure of

$$\begin{bmatrix} \bar{A} + \bar{B}\bar{F}_\sigma + \tilde{K}\bar{C} & \bar{B}(\bar{F} + \tilde{F}_\sigma) \\ 0 & A_- \end{bmatrix} \tag{51}$$

and the fact that both matrices in the diagonal are exponentially stable, we conclude that the state transition matrix Φ_σ associated with (51) satisfies (39) for appropriately defined c and λ. From this point on we can replicate the proof of Theorem 4, by using (49) in place of (40). ■

B. Convergence to an Invariant Set

We now generalize Theorem 4 to the case when the observability assumption fails and we do not have additional structure. In this case, we are not able to conclude that the state x of the switched system converges to zero. However, we show that it converges to a particular "invariant" set:

Theorem 8: Suppose that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric positive definite matrices such that, for every $(x, \sigma) \in \mathcal{S}$,

$$x(t)'P_{\sigma(t)}x(t) \leq x(t)'P_{\sigma(t^-)}x(t), \qquad \forall t \geq 0,$$

and

$$A_p'P_p + P_pA_p \leq -C_p'C_p \quad (\leq 0), \qquad \forall p \in \mathcal{P},$$

for an appropriately defined compact set of matrices $\{C_p : p \in \mathcal{P}\}$ for which $\{(C_p, A_p) : p \in \mathcal{P}\}$ is compact. Then (27) is stable. Moreover, if $\mathcal{S} \subset \mathcal{S}_{\text{p-dwell}}$ then, along solutions to (27), x converges to the smallest subspace $\mathcal{M}$ that is A_p-invariant for all $p \in \mathcal{P}$ and contains the unobservable subspaces of all pairs (C_p, A_p), $p \in \mathcal{P}$. □

When all pairs (C_p, A_p), $p \in \mathcal{P}$ are observable, the set $\mathcal{M}$ simply contains the origin and we obtain a result similar to the statement (i) in Theorem 4, except that here we need $\mathcal{S} \subset \mathcal{S}_{\text{p-dwell}}$, which is a stronger requirement than $\mathcal{S} \subset \mathcal{S}_{\text{weak-dwell}}$. Also, Theorem 8 makes no uniformity claims regarding the convergence to $\mathcal{M}$. In view of the statement (ii) in Theorem 4, one could expect uniformity when $\mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ for some $\tau_D > 0$, $T < \infty$. We conjecture that even in this case, the convergence to $\mathcal{M}$ will not be uniform, however so far we were unable to find a counter-example.

Proof: [Theorem 8] Since we have already shown that (27) is stable, we only need to show that x converges to $\mathcal{M}$. To this effect, let $(x, \sigma) \in \mathcal{S}$ be a solution to (27). Since $\mathcal{S} \subset \mathcal{S}_{\text{p-dwell}}$ we known that $(x, \sigma) \in \mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ for some $\tau_D > 0$, $T < \infty$. Pick then a time interval $[\tau_k, t_k)$ of length no smaller than τ_D on which $\sigma = p$. From the Kalman's Decomposition Theorem [38], we known that there exists a (p-dependent) coordinate transformation $[x_u' \quad x_o']' := Q_p x$, with Q_p nonsingular for which the system (27) can be represented as

$$\begin{bmatrix} \dot{x}_u \\ \dot{x}_o \end{bmatrix} = \begin{bmatrix} A^u & X \\ 0 & A^o \end{bmatrix} \begin{bmatrix} x_u \\ x_o \end{bmatrix}, \qquad y = \begin{bmatrix} 0 & C^o \end{bmatrix} \begin{bmatrix} x_u \\ x_o \end{bmatrix}, \tag{52}$$

$\forall t \in [\tau_k, t_k)$, with the pair (C^o, A^o) observable. Moreover, since $Q_p^{-1}[x_u' \quad 0]'$ belongs to the unobservable subspace of the pair (C_p, A_p) and therefore to $\mathcal{M}$, the distance from x to $\mathcal{M}$ is determined by x_o. Picking some $\lambda > 0$, from the

Squashing Lemma 9 in the Appendix, with $\tau_0 := \tau_D$ and $\lambda_0 := \lambda$, we conclude that there exists an output-injection matrix K_δ for which

$$\|e^{(A^o+K_\delta C^o)t}\| \leq \delta e^{-\lambda(t-\tau_D)}, \quad \forall t \geq \tau_D,\ p \in \mathcal{P}. \tag{53}$$

For the time being we do not specify a particular value for δ but we recall that K_δ in (53) will depend on the particular δ to be selected later. Since we can also write (52) as

$$\begin{bmatrix}\dot{x}_u\\ \dot{x}_o\end{bmatrix} = \begin{bmatrix}A^u & X\\ 0 & A^o+K_\delta C^o\end{bmatrix}\begin{bmatrix}x_u\\ x_o\end{bmatrix} - \begin{bmatrix}0\\ K_\delta\end{bmatrix} y,$$

we conclude that

$$x_o(t) = e^{(A^o+K_\delta C^o)(t-\tau_k)}x_o(\tau_k) - \int_{\tau_k}^{t} e^{(A^o+K_\delta C^o)(t-\tau)}K_\delta y(\tau)d\tau, \qquad \forall t \in [\tau_k, t_k].$$

Taking norms and using the Cauchy-Schwartz inequality together with (53), we conclude that, for every $t \in [\tau_k+\tau_D, t_k]$,

$$\begin{aligned}\|x_o(t)\| &\leq \|e^{(A^o+K_\delta C^o)(t-\tau_k)}\|\,\|x_o(\tau_k)\| \\ &\quad + \Big(\int_{\tau_k}^{t}\|e^{(A^o+K_\delta C^o)(t-\tau)}K_\delta\|^2 d\tau\Big)^{\frac{1}{2}} \\ &\quad \Big(\int_{\tau_k}^{t}\|y(\tau)\|^2 d\tau\Big)^{\frac{1}{2}} \\ &\leq \delta\|x_o(\tau_k)\| + k_\delta\Big(\int_{\tau_k}^{t_k}\|y(\tau)\|^2 d\tau\Big)^{\frac{1}{2}},\end{aligned}$$

where

$$k_\delta := \Big(\int_0^\infty \|e^{(A^o+K_\delta C^o)s}K_\delta\|^2 ds\Big)^{\frac{1}{2}}$$

is finite because $A^o + K_\delta C^o$ is exponentially stable [cf. (53)]. Since we already established that the overall state of the system is bounded, $\|x_o(\tau_k)\|$ is bounded and therefore we can make the term $\delta\|x_o(\tau_k)\|$ arbitrarily small by choosing δ sufficiently small. Since it has also been established that $y \in \mathcal{L}_2$, once δ is chosen (and therefore k_δ takes a specific finite value), the term $\int_{\tau_k}^{t_k}\|y(\tau)\|^2 d\tau$ can be made arbitrarily small by considering an interval that starts at a time τ_k sufficiently large. We therefore conclude that, given any $\epsilon > 0$, there is a time T_ϵ sufficiently large so that for any $\tau_k \geq T_\epsilon$, the distance from $x(\tau_k+\tau_D)$ to the unobservable subspace of the pair (C_p, A_p) is smaller than ϵ. Because of the compactness of $\{A_p : p \in \mathcal{P}\}$ and $\{C_p : p \in \mathcal{P}\}$, the time T_ϵ can be selected independently of $p := \sigma(t)$, $t \in [\tau_k, t_k)$.

Because $\mathcal{M}$ contains all the unobservable subspaces of the pairs (C_p, A_p), $p \in \mathcal{P}$ the distance from x to $\mathcal{M}$ at the times $\{\tau_k + \tau_D\}$ is smaller than ϵ. Moreover, since the separation between the $\tau_k + \tau_D$ is bounded by $T + \tau_D$ and between them x evolves according to flows for which $\mathcal{M}$ is invariant, the distance from x to $\mathcal{M}$ is bounded by $e^{a(T+\tau_D)}\epsilon$, $a := \max_{p\in\mathcal{P}}\|A_p\|$ between the $\tau_k + \tau_D$ (cf. Lemma 10 in the Appendix). Finally, as we can make ϵ arbitrarily small, we conclude that the distance from x to $\mathcal{M}$ actually converges to zero. ∎

We are now ready to complete the proofs of Theorems 4 and 6, by adapting the proof of Theorem 8.

Proof: [Theorem 4 (i)] Consider the proof of Theorem 8 with the following two modifications:

1) all the pairs (C_p, A_p), $p \in \mathcal{P}$ are observable and therefore $\mathcal{M}$ only contains the origin; and
2) the solution (x, σ) to (27) is in $\mathcal{S}_{\text{p-dwell}}[\tau_D, +\infty]$ instead of $\mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ for some $\tau_D > 0$, $T < \infty$.

Using exactly the same argument, we conclude that for any given $\epsilon > 0$ there exists a time $\tau_k + \tau_D$ sufficiently large so that the distance from $x(\tau_k + \tau_D)$ to the origin is smaller than ϵ. This means that there is a sequence of times along which x converges to zero. Since stability of (27) has already been established, we conclude that x must actually converge to zero as $t \to \infty$, which proves (i) in Theorem 4. ∎

Proof: [Theorem 6 (i)] In the proof of (ii) in Theorem 6, we saw that there exists a coordinate transformation that allows us to write (27) as

$$\dot{z}_1 = (\bar{A} + \bar{B}\bar{F}_\sigma)z_1 + \bar{y}, \qquad y = \bar{C}z_1 \tag{54}$$

$$\dot{z}_2 = A_- z_2 + \bar{K}y, \qquad \bar{y} = \bar{B}(\bar{F} + \tilde{F}_\sigma)z_2 \tag{55}$$

where A_- is a stability matrix and each pair $(\bar{C}, \bar{A} + \bar{B}\bar{F}_p)$, $p \in \mathcal{P}$ is observable [cf. equation (47)–(48)]. Since A_- is a stability matrix and $y \in \mathcal{L}_2$, we conclude that $z_2 \to 0$ as $t \to \infty$. Moreover, z_2 and $\bar{y}$ are also in $\mathcal{L}_2$. It then remains to show that $z_1 \to 0$. To do this, note that the differential equation (54) driving z_1 is similar to the differential equation (52) driving x^o in the proof of Theorem 8. The only difference being the exogenous input $\bar{y}$. However, since it has already been established that this input is $\mathcal{L}_2$, we could replicate the proof of Theorem 8 and still conclude that there is a sequence of times along which z_2 converges to zero. Since z_1 converges to zero as $t \to \infty$ and the system is stable, we conclude that the whole state converges to zero as $t \to \infty$, which proves (i) in Theorem 6. ∎

V. Conclusion

We extended LaSalle's Invariance Principle to certain classes of switched linear systems. In particular, we illustrated how to prove asymptotic stability using multiple Lyapunov functions whose Lie derivatives are only negative semi-definite and investigated under which conditions the convergence is uniform and exponential. We showed that uniformity of convergence depends critically on the class of switching signals considered. In particular, on the existence of a "persistent" dwell-time. We are currently extending these results to nonlinear switched systems. Preliminary results can be found in [36].

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Appendix

a) Sets of Switching Signals: All sets of switching signals considered in this paper are subsets of the set $\mathcal{S}_{\text{non-chatt}}$ of pairs (x, σ) for which x and σ are piecewise differentiable and piecewise constant, respectively. Other sets considered include:

- The set $\mathcal{S}_{\text{dwell}}[\tau_D]$, $\tau_D > 0$ of pairs $(x, \sigma) \in \mathcal{S}_{\text{non-chatt}}$ for which any consecutive discontinuities of σ are separated by no less than τ_D. The constant τ_D is called the *dwell-time*.
- The set $\mathcal{S}_{\text{average}}[\tau_D, N_0]$, $\tau_D, N_0 > 0$ of pairs $(x, \sigma) \in \mathcal{S}_{\text{non-chatt}}$ for which
$$N_\sigma(t, \tau) \le N_0 + \frac{t - \tau}{\tau_D}, \qquad \forall t \ge \tau \ge 0,$$
where $N_\sigma(t, \tau)$ denotes the number of discontinuities of σ in the open interval (τ, t). The constant τ_D is called the *average dwell-time* and N_0 the *chatter bound*.
- The set $\mathcal{S}_{\text{p-dwell}}[\tau_D, T]$, $\tau_D > 0$, $T \in [0, \infty]$ of pairs $(x, \sigma) \in \mathcal{S}_{\text{non-chatt}}$ for which there is an infinite number of disjoint intervals of length no smaller than τ_D on which σ is constant, and consecutive intervals with this property are separated by no more than T. The constant τ_D is called the *persistent dwell-time* and T the *period of persistence*.

The following sets are limiting cases of the above.

- The set $\mathcal{S}_{\text{finite}} := \bigcup_{N_0>0} \mathcal{S}_{\text{average}}[\infty, N_0]$, where each σ is restricted to have a finite number of discontinuities.
- The set $\mathcal{S}_{\text{dwell}} := \bigcup_{\tau_D>0} \mathcal{S}_{\text{dwell}}[\tau_D]$, where each σ is restricted to have a dwell-time bounded away from zero but this bound is not uniform over all switching signals.
- The set $\mathcal{S}_{\text{average}} := \bigcup_{\tau_D>0, N_0>0} \mathcal{S}_{\text{average}}[\tau_D, N_0]$, where each σ is restricted to have an average dwell-time bounded away from zero and finite chatter bound but these bounds are not uniform over all switching signals.
- The set $\mathcal{S}_{\text{p-dwell}} := \bigcup_{\tau_D>0, T<\infty} \mathcal{S}_{\text{p-dwell}}[\tau_D, T]$, where each σ is restricted to have a positive persistent dwell-time and finite period of persistence but these are not uniform over all switching signals.
- The set $\mathcal{S}_{\text{weak-dwell}} := \bigcup_{\tau_D>0} \mathcal{S}_{\text{p-dwell}}[\tau_D, +\infty]$, for which each σ is restricted to have a persistent dwell-time bounded away from zero but can have infinite period of persistence.

Proof: [Equation (10)] For any switching signal with interval between consecutive discontinuities no smaller then τ_D, there can be at most $1 + L/\tau_D$ discontinuities on an interval of length L, therefore $\mathcal{S}_{\text{dwell}}[\tau_D] \subset \mathcal{S}_{\text{average}}[\tau_D, 1]$. The converse inclusion is a consequence of the fact that if $\sigma \in \mathcal{S}_{\text{average}}[\tau_D, 1]$, there can be at most one discontinuity of σ on any interval of length smaller than τ_D therefore the interval between consecutive discontinuities of σ is larger or equal to τ_D. This proves that $\mathcal{S}_{\text{dwell}}[\tau_D] = \mathcal{S}_{\text{average}}[\tau_D, 1]$.

The fact that $\mathcal{S}_{\text{dwell}}[\tau_D] = \mathcal{S}_{\text{p-dwell}}[\tau_D, 0]$ is a trivial consequence of the definition of these sets. The fact that $\mathcal{S}_{\text{dwell}}[\tau_D] \subset \mathcal{S}_{\text{average}}[\tau_D, N_0]$, $\forall N_0 \ge 1$ is a consequence of $\mathcal{S}_{\text{dwell}}[\tau_D] = \mathcal{S}_{\text{average}}[\tau_D, 1]$ together with $\mathcal{S}_{\text{average}}[\tau_D, 1] \subset \mathcal{S}_{\text{average}}[\tau_D, N_0]$, $\forall N_0 \ge 1$.

Suppose now that $\sigma \in \mathcal{S}_{\text{average}}[\tau_D, N_0]$ and suppose that there exist n consecutive discontinuities of σ separated by less than $\delta\tau_D$, for some $\delta \in (0, 1)$. This means that there must exist an interval of length smaller than $\delta\tau_D(n-1)$ on which there are n discontinuities of σ. But since $\sigma \in \mathcal{S}_{\text{average}}[\tau_D, N_0]$, we conclude that
$$n \le N_0 + \frac{\delta\tau_D(n-1)}{\tau_D} = N_0 + \delta(n-1)$$
and therefore $n \le (N_0 - \delta)/(1 - \delta)$. This means that there can be at most $(N_0 - \delta)/(1 - \delta)$ consecutive discontinuities of σ separated by less than $\delta\tau_D$ and therefore two intervals on which σ remains constants for at least $\delta\tau_D$ cannot be separated by more than $\delta\tau_D(N_0 - \delta)/(1 - \delta)$, i.e., $\sigma \in \mathcal{S}_{\text{p-dwell}}[\delta\tau_D, \delta\tau_D(N_0 - \delta)/(1 - \delta)]$. ∎

Proof: [Lemma 1] The fact that exponential stability implies uniform asymptotic stability is trivial so we only need to prove the converse. To this effect, assume that (8) is uniformly asymptotically stable and therefore that (16) holds for some $\beta \in \mathcal{KL}$. Moreover, let T be a constant sufficiently large so that $\beta(1, T) \le e^{-\lambda}$, for some $\lambda > 0$. Suppose that we pick an arbitrary solution $(x, \sigma) \in \mathcal{S}$ to (8) and denote by $\Phi_\sigma(t, \tau)$ the state transition matrix of the time-varying linear system (9) (recall that the maps f_p are linear for switched linear systems). Because of the semi-group property of the state transition matrix we can write
$$\Phi_\sigma(t, t_0) = \Phi_\sigma(t, t_k)\Phi_\sigma(t_k, t_{k-1}) \cdots \Phi_\sigma(t_1, t_0),$$
$\forall t \ge t_0 \ge 0$, where $\{t_1, t_2, \dots t_k\}$ is an ascending sequence of times in (t_0, t) such that $t_{k+1} \ge t_k + T$, $k \ge 0$. The integer k can be chosen smaller than $(t - t_0)/T$. We then conclude that
$$\begin{aligned}\|\Phi_\sigma(t, t_0)\| \le \|\Phi_\sigma(t, t_k)\| \, \|\Phi_\sigma(t_k, t_{k-1})\| \cdots \\ \cdots \|\Phi_\sigma(t_1, t_0)\|. \quad (56)\end{aligned}$$
We show next that
$$\|\Phi_\sigma(\tau_2, \tau_1)\| \le \beta(1, \tau_2 - \tau_1), \qquad \forall \tau_2 \ge \tau_1 \ge 0. \quad (57)$$
Take an arbitrary vector $z \in \mathbb{R}^n$ with $\|z\| = 1$ and let $\bar{x} : [0, \infty) \to \mathbb{R}^n$ be the solution to (9) with initial condition $\bar{x}(\tau_1) = z$. Because we have trajectory-independent switching, $(\bar{x}, \sigma)$ is also a solution to the switched system and therefore
$$\begin{aligned}\|\bar{x}(\tau_2)\| &= \|\Phi_\sigma(\tau_2, \tau_1) z\| \\ &\le \beta(\|z\|, \tau_2 - \tau_1) = \beta(1, \tau_2 - \tau_1). \quad (58)\end{aligned}$$
Since z was an arbitrary unit-norm vector, we conclude that (57) holds. From this and (56) we obtain
$$\begin{aligned}\|\Phi_\sigma(t, t_0)\| &\le \beta(1, t - t_k)\beta(1, t_k - t_{k-1}) \cdots \beta(1, t_1 - t_0) \\ &\le \beta(1, 0)\beta(1, T)^k \le \beta(1, 0)e^{-\lambda k} \le \beta(1, 0)e^{\frac{\lambda(t - t_0)}{T}}.\end{aligned}$$
This means that
$$\begin{aligned}\|x(t)\| &= \|\Phi_\sigma(t, t_0)x(t_0)\| \le \|\Phi_\sigma(t, t_0)\| \, \|x(t_0)\| \\ &\le \beta(1, 0)e^{\frac{\lambda(t - t_0)}{T}}\|x(t_0)\|, \qquad \forall t \ge t_0 \ge 0,\end{aligned}$$
which provides the desired exponential bound. ∎

To prove Lemma 5 we need the following result, which is essentially the Squashing Lemma in [37].

Lemma 9 (Squashing Lemma): Given any observable matrix pair (C, A) and positive constants $\tau_0, \delta, \lambda_0$, it is possible to find an output-injection matrix K for which

$$\|e^{(A+KC)t}\| \leq \delta e^{-\lambda_0(t-\tau_0)}, \qquad \forall t \geq \tau_0. \tag{59}$$

□

Proof: [Lemma 9] The statement of the Squashing Lemma in [37] asserts the existence of a λ and K for which (59) holds (for arbitrary τ_0 and δ). However, in [37] it is actually proved that for a sufficiently large λ, a gain K_λ can always be found so that (59) holds with $K = K_\lambda$. Denoting by λ^* the the smallest value of λ for which the desired output-injection matrix can be found, two cases can be considered: (1) if $\lambda_0 \geq \lambda^*$ then the output injection matrix K in (59) is simply chosen equal to K_{λ_0}, and (2) if $\lambda_0 < \lambda^*$ then we can choose $K = K_{\lambda^*}$ and (59) will hold with λ^* on the right-hand-side and consequently also with $\lambda_0 < \lambda^*$ on the right-hand-side. ■

Proof: [Lemma 5] Since the set $\{A_p : p \in \mathcal{P}\}$ is compact, there must exist a finite positive constant a such that $\|A_p\| \leq a$, $\forall p \in \mathcal{P}$. Therefore, given any piecewise constant switching signal σ, we have that

$$\|\Psi_\sigma(t, \tau)\| \leq e^{a(t-\tau)}, \qquad \forall t \geq \tau \geq 0, \tag{60}$$

where $\Psi_\sigma(t, \tau)$ denotes the state transition matrix of $\dot{z} = A_{\sigma(t)} z$. This can be verified, by defining $v(t) := \|x(t)\|^2$ and observing that $\dot{v} = x'(A_\sigma + A'_\sigma)x \leq 2av$. From the Bellman-Gronwall Lemma [39, p. 346] one concludes that $v(t) \leq e^{2a(t-\tau)} v(\tau)$, $t \geq \tau \geq 0$, from which (60) follows. We thus conclude that in an interval of length T the norm of the state of (27) can grow, at most by e^{aT}.

Because all the pairs (C_p, A_p), $p \in \mathcal{P}$ are observable, we conclude from Lemma 9 with $\delta := e^{-aT}$, $\tau_0 := \frac{\tau_D}{2}$, and $\lambda_0 := 2\lambda$ that there exist output-injection matrices $\{K_p : p \in \mathcal{P}\}$ for which

$$\|e^{(A_p+K_pC_p)t}\| \leq e^{-aT-2\lambda(t-\frac{\tau_D}{2})} = e^{-aT+\lambda\tau_D-2\lambda t}, \tag{61}$$

$\forall t \geq \frac{\tau_D}{2}$, $p \in \mathcal{P}$. Moreover, the compactness of $\{A_p : p \in \mathcal{P}\}$ and $\{C_p : p \in \mathcal{P}\}$ guarantee that we can choose $\{K_p : p \in \mathcal{P}\}$ also compact and therefore one can pick k such that $k \geq \|K_p\|$, $\forall p \in \mathcal{P}$.

We are now ready to define the time-varying output-injection matrix $K(t)$ and prove (39). To this effect pick some $\sigma \in \mathcal{S}_{\text{p-dwell}}[\tau_D, T]$. We will call *long*, those intervals of time with length no smaller than τ_D on which σ is constant and define $K(t) = K_p$ on long intervals on which $\sigma = p \in \mathcal{P}$ and $K(t) = 0$ at any other times. Given two time instants $t \geq \tau \geq 0$, let $\mathcal{T} := \{t_1, \tau_1, t_2, \tau_2, t_3, \dots, t_k, \tau_k\} \subset (\tau, t)$ denote an increasing sequence of times in the interval (τ, t) such that the intervals $[t_i, \tau_i)$ are long with $\sigma = p_i$ and the intervals between these have length no longer than T, i.e.,

$$\tau_i \geq t_i + \tau_D, \quad \forall i \in \{1, 2, \dots, k\}, \tag{62}$$

$$t_{i+1} \leq \tau_i + T, \quad \forall i \in \{1, 2, \dots, k-1\}, \tag{63}$$

$$t \leq \tau_k + T, \;\; t_1 \leq \tau + T. \tag{64}$$

The sequence $\mathcal{T}$ can actually be empty ($k = 0$) if there is no long interval in (τ, t). This is only possible if $t - \tau < T + \tau_D$. The above definition of $K(t)$ leads to

$$\Phi_\sigma(t, \tau) = \Psi_\sigma(t, \tau_k) e^{(A_{p_k}+K_{p_k}C_{p_k})(\tau_k-t_k)} \Psi_\sigma(t_k, \tau_{k-1}) \cdots \Psi_\sigma(t_2, \tau_1) e^{(A_{p_1}+K_{p_1}C_{p_1})(\tau_1-t_1)} \Psi_\sigma(t_1, \tau).$$

From this, (60), and (61) we conclude that

$$\begin{aligned} \|\Phi_\sigma(t, \tau)\| &\leq e^{a(t-\tau_k)} e^{-aT+\lambda\tau_D-2\lambda(\tau_k-t_k)} e^{a(t_k-\tau_{k-1})} \dots \\ &\quad \dots e^{a(t_2-\tau_1)} e^{-aT+\lambda\tau_D-2\lambda(\tau_1-t_1)} e^{a(t_1-\tau)} \\ &\leq e^{aT-\lambda\left((\tau_k-t_k)+\cdots+(\tau_1-t_1)\right)}. \end{aligned}$$

Here, we also used the facts that $t - \tau_k \leq T$, $t_1 - \tau \leq T$, $t_{i+1} - \tau_i \leq T$, $i \in \{1, 2, \dots, k-1\}$, and also $\tau_D \leq \tau_i - t_i$, $i \in \{1, 2, \dots, k\}$, which are a consequence of (62)–(64). Moreover, since

$$\begin{aligned} t - \tau &= (t - \tau_k) + (\tau_k - t_k) + (t_k - \tau_{k-1}) + \cdots \\ &\quad + (\tau_1 - t_1) + (t_1 - \tau) \leq (\tau_k - t_k) + \cdots + (\tau_1 - t_1), \end{aligned}$$

we conclude that (39) holds with $c = e^{aT}$. ■

In the following, given a vector $x \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times n}$, let us denote by $\|x\|_\mathcal{M}$ the *distance from x to $\mathcal{M}$*, i.e., $\|x\|_\mathcal{M} := \inf_{z\in\mathcal{M}} \|x - z\|$; and by $\|A\|_\mathcal{M}$ the *$\mathcal{M}$-distance induced gain of A*, i.e., $\|A\|_\mathcal{M} := \sup_{z\in\mathbb{R}^n\setminus\mathcal{M}} \frac{\|Az\|_\mathcal{M}}{\|z\|_\mathcal{M}}$. The $\mathcal{M}$-distance induced gain satisfies the *submultiplicative property*, i.e., given two matrices $A, B \in \mathbb{R}^{n\times n}$, $\|AB\|_\mathcal{M} \leq \|A\|_\mathcal{M} \|B\|_\mathcal{M}$.

Lemma 10: Given an $n \times n$ matrix A and a subspace $\mathcal{M}$ of $\mathbb{R}^n$ that is A-invariant,

$$\|e^{At}\|_\mathcal{M} \leq e^{at}, \qquad \forall t \geq 0, \tag{65}$$

with $a := \|A\|_\mathcal{M} < \|A\|$. □

It is well known that, if the state of $\dot{x} = Ax$, starts at time τ inside an A-invariant subspace $\mathcal{M}$ (and therefore $\|x(\tau)\|_\mathcal{M} = 0$), then x remains there for all $t \geq \tau$. From Lemma 10 we further conclude that if x, starts close to $\mathcal{M}$ at time τ then $x(t) = e^{A(t-\tau)} x(\tau)$, $t \geq \tau$ remains close to it for some time. In fact, according to (65), the distance $\|x(t)\|_\mathcal{M} \leq e^{at} \|x(\tau)\|_\mathcal{M}$ increases at most exponentially fast with time.

Proof: [Lemma 10] Let $M \in \mathbb{R}^{n\times m}$ and $M^\perp \in \mathbb{R}^{n\times m-n}$ be matrices whose columns form orthonormal bases of $\mathcal{M}$ and its orthogonal complement $\mathcal{M}^\perp$, respectively. This means that

$$\|z\|_\mathcal{M} = \|{M^\perp}' z\|, \qquad \forall z \in \mathbb{R}^n. \tag{66}$$

Since $\mathcal{M}$ is A-invariant we have

$$A \begin{bmatrix} M & M^\perp \end{bmatrix} = \begin{bmatrix} M & M^\perp \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix},$$

with $A_1 \in \mathbb{R}^{m\times m}$, $A_2 \in \mathbb{R}^{m\times(n-m)}$, $A_3 \in \mathbb{R}^{(n-m)\times(n-m)}$ and therefore

$${M^\perp}' A = {M^\perp}' \begin{bmatrix} M & M^\perp \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} M' \\ {M^\perp}' \end{bmatrix} = A_3 {M^\perp}'.$$

This means that

$$a := \|A\|_{\mathcal{M}} := \sup_{z \notin \mathcal{M}} \frac{\|Az\|_{\mathcal{M}}}{\|z\|_{\mathcal{M}}} = \sup_{z \notin \mathcal{M}} \frac{\|M^{\perp\prime} Az\|}{\|M^{\perp\prime} z\|}$$
$$= \sup_{z \notin \mathcal{M}} \frac{\|A_3 M^{\perp\prime} z\|}{\|M^{\perp\prime} z\|} = \sup_{z \in \mathbb{R}^{n-m}} \frac{\|A_3 z\|}{\|z\|} = \|A_3\|,$$

because $M^{\perp\prime}$ is full rank. Note also that

$$\|A\| = \left\| \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \right\| \geq \|A_3\| = \|A\|_{\mathcal{M}} =: a.$$

Suppose now that we define $V := \|M^{\perp\prime} x\|^2$. Along trajectories of $\dot{x} = Ax$, we have

$$\dot{V} = 2x' M^{\perp} M^{\perp\prime} Ax = 2x' M^{\perp} A_3 M^{\perp\prime} x \leq 2aV,$$

and therefore $V(t) \leq e^{2a(t-\tau)} V(\tau)$, $t \geq \tau \geq 0$. From this and (66) we conclude that

$$\|x(t)\|_{\mathcal{M}} = \|e^{A(t-\tau)} x(\tau)\|_{\mathcal{M}} \leq e^{a(t-\tau)} \|x(\tau)\|_{\mathcal{M}},$$

$\forall t \geq \tau \geq 0$, from which (65) follows. ■

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