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NC Ball Maps and Changes of Variables

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ABSTRACT OF THE DISSERTATION

NC Ball Maps and Changes of Variables

by

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Professor J. William Helton, Chair

In this dissertation, we analyze problems set in a dimension-free, noncommutative setting. To be more specific, we use a class of functions defined by power series in noncommutative variables and evaluate these functions on sets of matrices of all sizes – hence the dimension-free term. These types of functions have recently been used in the study of dimension-free linear system engineering problems [HP07], [CHSY03].

Here we analyze a class of functions called NC analytic with the intention of understanding changes of variables in dimension-free classes of problems in matrix variables. To this end, we force geometric constraints on our analytic functions and ask how this affects the algebraic structure of the series defining them. In particular, we present a characterization of maps that send dimension-free matrix balls to dimension-free matrix balls and carry the boundary to the boundary. We study this problem in various cases restricting the variables (and matrices) to have additional symmetric structure and in cases where the variables (and matrices) have no such restrictions. These characterizations are then used to study the more general question of understanding when a dimension-free set is bianalytic to a dimension-free ball.

In addition to our study of NC analytic functions, we present a result on a representation of noncommutative rational expressions. Recently there have been studies linking convexity of noncommutative rational functions to linear matrix

inequalities, LMIs [HMV06]. The algorithm presented in the final chapter of the thesis presents a necessary step to automatically converting inequalities involving convex rational expressions into LMIs.

1 Introduction

Much of this thesis is dedicated to classifying maps defined by noncommutative variables which behave well on sets of matrices of all sizes. The hope of these classifications is to help us understand changes of variables in this noncommutative setting. This understanding may aid in the study of "dimension-free" systems engineering problems. Our attempt is to define and classify "analytic" noncommutative maps which respect some geometric constraints which hold on sets of matrices of all dimensions. In particular, we have studied classes of analytic maps that carry a dimension-free ball to a dimension-free ball with the boundary mapping to the boundary. We then use this classification to try to answer some basic questions regarding when a dimension-free set is bianalytic to a convex dimension-free set.

1.1 Ball Maps and Changes of Variables

One of the major developments in linear systems engineering in the 1990's was to utilize matrix inequalities in problem solving. Techniques for turning complicated matrix inequalities into computationally nicer matrix inequalities were required and have been studied [SIG98], [Par00]. This has sparked interest in understanding polynomial and rational functions in noncommutative variables and their use in matrix inequalities. More recently it has been shown, for instance, that many convex matrix inequalities in the dimension-free setting built from noncommutative rational functions are equivalent to linear matrix inequalities [HMV06]. This result links the computationally nice linear matrix inequality to the more complicated rational matrix inequalities. We think that the study of changes of variables from dimension-free sets to dimension-free convex sets may be useful in this area. See [HP07] or [CHSY03] for a more detailed description of dimension-free problems and their relation to linear systems and control theory.

Our study of analytic maps that carry a dimension-free ball to a dimension-free ball with the boundary mapping to the boundary is related to a classical question from complex analysis. D'Angelo and others have attempted to classify many of the proper holomorphic rational maps from $\mathbb{C}^N$ to $\mathbb{C}^M$ carrying the ball to the ball and mapping the sphere to the sphere [D'A92]. In the case where $N = M = 1$ these maps are all linear fractional transformations. When the dimension of $M > 2N-2$, there are more interesting examples of proper rational holomorphic maps from $\mathbb{C}^N$ to $\mathbb{C}^M$. It turns out that the geometric conditions in the dimension-free setting force a much more rigid structure and the resulting NC analytic maps are simpler [HKMS09].

As a first example, suppose that $(x_1, \ldots, x_G, x_1^T, \ldots, x_G^T)$ is a list of noncommuting variables. By noncommuting we mean that $x_i x_j \neq x_j x_i$. In this case, the variables $(x_1, \ldots, x_G, x_1^T, \ldots, x_G^T)$ are nonsymmetric since $x_j \neq x_j^T$. The case where some variables are symmetric will also be handled later. Suppose that w is in the free group generated by $2G$ generators $\{e_1, \ldots, e_{2G}\}$. We define

$$x^{e_j} := \begin{cases} x_j & \text{if } j \leq G, \\ x_{j-G}^T & \text{if } j > G. \end{cases}$$

and extend the definition so that if $w = e_{j_1} \cdots e_{j_K}$ then

$$x^w = x^{e_{j_1}} \cdots x^{e_{j_K}}.$$

By taking real linear combinations of terms x^w we thus create polynomials (and series) in the NC variables $(x_1, \ldots, x_G, x_1^T, \ldots, x_G^T)$. We will be substituting matrices of all compatible sizes for the variables to create polynomial functions. If a polynomial $p(x)$ is symmetric, we define its positivity domain $\mathcal{D}_p$ as the set of all matrix tuples X such that $p(X) \succ 0$.

We will say that a function f is NC analytic in the variables $(x_1, \ldots, x_G)$ if

$$f(x) := \sum_{w \in \mathcal{W}} a_w x^w$$

with $a_w \in \mathbb{R}^K$ and the sum absolutely convergent on matrix balls (we will use the operator norm) of a fixed radius for all sizes of matrices. The set $\mathcal{W}$ consists of all words in the free group generated by $\{e_1, \ldots, e_G\}$ and thus all terms x^w are free of any transposed letters x_j^T. We say that f is a unit NC ball map if it is NC analytic and carries the unit matrix ball to the unit matrix ball with boundary to boundary in each dimension.

Now we give the flavor of some of our results.

Result 1.1.1. *The map*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_G \end{pmatrix} \overset{f}{\mapsto} \begin{pmatrix} f_1(x) \\ \vdots \\ f_K(x) \end{pmatrix}$$

is a unit NC ball map with $f(0) = 0$ if and only if $f(x) = Ax$ for some $A \in \mathbb{R}^{K \times G}$ such that $A^T A = I$.

This Corollary will be stated formally and proved later in Section 3.7. The result says that a large class of the unit NC ball maps are simply linear isometries. We extend these results to the more general case that allows for some symmetric variables as well. The results in the mixed variable case are similar however the "isometries" that define the ball maps are a bit different than the case above.

These classifications of NC ball maps are used to study changes of variables in the dimension-free setting. The changes of variables happen on sets defined in terms of NC polynomials. In particular, we examine a procedure, the SoS Construction , for finding an NC bianalytic map between positivity domains of a nice class of NC polynomials and a canonical convex polynomial. The proof of the result hinges on the classification of unit NC ball maps stated previously. We will state the version of the result that deals with functions depending on nonsymmetric variables only. Later we will discuss the case when some symmetric variables are allowed as well.

We say that a polynomial in noncommuting variables is hereditary if all transposed letters appear to the left of all non-transposed letters in each term. As an example, the polynomial

$$3x_1^T x_2^T x_2 x_1 + x_1^T x_1$$

is hereditary. We say that a polynomial $p(x)$ is matrix positive if $p(X)$ is a positive definite matrix for each matrix tuple $X \in (\mathbb{R}^{N\times N})^G$ and for all dimensions N. There is a result that says all matrix positive polynomials in noncommuting variables can be written as a sum of squares of noncommuting polynomials [Hel02]. Moreover, if $p(x)$ is a matrix positive hereditary polynomial, then there exist analytic polynomials $r_1(x), \ldots, r_K(x)$ for some K so that

$$p(x) = \sum_{j=1}^{K} r_j(x)^T r_j(x).$$

The minimum number of terms, K, required in the decomposition is easily obtainable since the Gram Representation for hereditary polynomials is unique [Hel02] and K is the rank of the Gram matrix. The process of taking a positive hereditary polynomial $p(x)$ as an input and producing a minimal sized vector of NC analytic polynomials

$$r(x) = \begin{pmatrix} r_1(x) \\ \vdots \\ r_K(x) \end{pmatrix}$$

will be called the SoS Construction and is possible for all positive hereditary polynomials. See Theorem 4.0.5 for a formal statement and proof of how the SoS Construction will give a map from a positivity domain of an NC polynomial to a ball.

Result 1.1.2. *Consider the polynomial $p(x)$ in the NC variables*

$$(x_1, \ldots, x_G)$$

where $p(x)$ is an NC hereditary, matrix positive polynomial such that $p(0) = 0$ and suppose $a > 0$. Define $q(x) := a^2 - p(x)$ and

$$d(x) := a^2 - \sum_{j=1}^{G} x_j^T x_j.$$

Then the SoS Construction applied to $p(x)$ will construct an NC bianalytic map

$$r(x) = \begin{pmatrix} r_1(x) \\ \vdots \\ v_G(x) \end{pmatrix}$$

between $\mathcal{D}_q$ and $\mathcal{D}_d$ mapping 0 to 0 or one does not exist.

1.2 Symmetric and Mixed Variables and Changes of Variables

This thesis is a study of a class of functions that are inspired by work on dimensionless problems in the theory of linear systems and control. When we consider problems in control and systems theory we see that some of these noncommuting variables that we are trying to model and understand have additional structure. What we often see is a function depending on some symmetric and some nonsymmetric variables. We have considered problems similar to those described in the previous section but posed in the case of some variables being symmetric and some nonsymmetric – this is what we shall refer to as the mixed variable case. After a bit of study, we notice that analytic functions of nonsymmetric variables have more rigid structure than those of just symmetric variables. An analogy that usually works is to say that NC analytic functions of nonsymmetric variables are to complex analytic maps as NC analytic functions of symmetric variables are to real analytic maps.

To state a simple result on changes of variables, we will first define a certain "linear fractional" map in the NC variables $(u, v) = (u_1, \ldots, u_K, v_1, \ldots, v_K)$. Define

$$f(u, v) := v - (I - vv^T)^{1/2} u (1 - v^T u)^{-1} (1 - v^T v)^{1/2}. \tag{1.1}$$

This map is discussed in detail in Section 3.3. When fixing N and a $V \in (\mathbb{R}^{N\times N})^K$, the map

$$u \mapsto f(u, V)$$

maps the unit matrix ball to itself, maps the boundary to the boundary, and sends V to 0.

To proceed, we describe the generalization of the isometries used in Result 1.1.1. We have called this class of isometric maps, used to classify ball maps and changes of variables in the mixed NC variable case, NC full isometries. These NC full isometries are quite similar to the complete isometries of C^*-algebras [Pau02]. Suppose that

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \\ y_1 \\ \vdots \\ y_{G_2} \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} h_1(x,y) \\ \vdots \\ h_K(x,y) \end{pmatrix}$$

is an NC analytic map in $(x_1, \ldots, x_{G_1}, y_1, \ldots, y_{G_2})$ with $y_j = y_j^T$ and $h(x,y)$ is linear in x. Suppose that $h(X,Y)$ converges for all $\|Y\| < \beta$ and all X which are real-valued matrix tuples of compatible dimension. Suppose that $N \in \mathbb{N}$ and $Y \in (S\mathbb{R}^{N\times N})^{G_2}$ is such that $h(X,Y)$ converges for all $X \in (\mathbb{R}^{N\times N})^{G_1}$. For $n \in \mathbb{N}$ define the map

$$h_n(\cdot, Y) : \mathbb{R}^{nNG_1 \times nN} \to \mathbb{R}^{nNK \times nN} \tag{1.2}$$

so that if

$$\overline{X} = \begin{pmatrix} X(1,1) & \ldots & X(1,n) \\ \vdots & \ddots & \vdots \\ X(n,1) & \ldots & X(n,n) \end{pmatrix} \tag{1.3}$$

where $X(i,j) \in (\mathbb{R}^{N\times N})^{G_1}$ for all $1 \le i,j \le n$, then

$$h_n(\overline{X}, Y) := \begin{pmatrix} h(X(1,1),Y) & \ldots & h(X(1,n),Y) \\ \vdots & \ddots & \vdots \\ h(X(n,1),Y) & \ldots & h(X(n,n),Y) \end{pmatrix}. \tag{1.4}$$

If for all $N \in \mathbb{N}$ and for all $Y \in (S\mathbb{R}^{N\times N})^{G_2}$ with $\|Y\| < \beta$

$$\|h_n(\overline{X}, Y)\| = \|\overline{X}\| \tag{1.5}$$

for all $\overline{X} \in \mathbb{R}^{nNG_1 \times nN}$ we say that $h(x,y)$ is an NC n-full isometry. If $h(x,y)$ is an NC n-full isometry for all $n \in \mathbb{N}$ then $h(x,y)$ is an NC full isometry. Properties and characterizations of NC full isometries are investigated in Section 3.4. The following result is properly stated and proven in Section 4.4 as Proposition 4.4.1.

Result 1.2.1. *Consider a polynomial $p(x,y)$ in the NC mixed variables*

$$(x_1,\dots,x_{G_1},y_1,\dots,y_{G_2}).$$

Suppose that $p(x,y)$ is a hereditary, matrix positive polynomial such that

$$p(x,y) = \sum_{j=1}^{K} r_j^T(x,y) r_j(x,y) \tag{1.6}$$

for some K and some NC analytic polynomials

$$r_1(x,y),\dots,r_K(x,y).$$

Suppose $p(0,0)=0$. Suppose that $a,b>0$ are real numbers and

$$R : \mathcal{D}_{a^2-p(x,y)} \to M\mathcal{B}_b^{G_1,G_2} \tag{1.7}$$

is an NC bianalytic map. Then the map

$$(x,y) \mapsto f(\frac{1}{a}(W(bx(1-b^2y^Ty)^{1/2},by),\frac{1}{a}W(0,by))$$

is an NC full isometry, where f is the linear fractional map defined in Equation (1.1) and

$$W(x,y) := (r\circ R^{-1})(x,y). \tag{1.8}$$

With further assumptions on the bianalytic map in the previous result, we obtain a finer stucture. The result below is stated in detail in Theorem 4.4.6.

Result 1.2.2. *Suppose that*

$$r(x,y) = \begin{pmatrix} r_1(x,y) \\ \vdots \\ r_K(x,y) \end{pmatrix}$$

is a vector of analytic polynomials and that

$$p(x,y) := r^T(x,y)r(x,y).$$

Suppose that R is an NC bianalytic map defined on $(\mathbb{R}^{N\times N})^{G_1}\times(S\mathbb{R}^{N\times N})^{G_2}$ for all N such that R induces an NC bianalytic correspondence between $\mathcal{D}_{\alpha-p}$ and $M\mathcal{B}_\alpha(\mathcal{H})$ for all positive matrices α. Define

$$W := r\circ R^{-1} \quad \textit{and} \quad g(y) := W(0,y).$$

Under a few more technical assumptions we obtain the following results: Then

$$h(z,u) := W(z,u) - g(u)$$

is an NC full isometry in z. Also we have

$$r(x,y) = \widetilde{\Omega}(y)R(x,y)$$

where for $Y \in (S\mathbb{R}^{N\times N})^{G_2}$, the operator $\widetilde{\Omega}(Y)$ is an isometry on

$$\mathbb{R}^{NG_1} \oplus Range(R_{sym}(Y))$$

1.3 A Realization for Noncommmutative Rational Expressions

As mentioned earlier, there has been a connection made between many linear control theory problems and matrix inequalities. Much work was done in the 1990s to convert particular matrix inequalities into the computationally nicer linear matrix inequalities (LMIs). More recently in [HMV06], there was a result connecting LMIs to matrix inequalities from noncommutative convex rational expressions in symmetric variables. This more recent result gives hope that there may be some general algorithm to automatically convert noncommutative convex rational inequalities into LMIs. One of the results in this paper represents a first step in an algorithm to automatically convert noncommutative convex rational expressions in symmetric variables into LMIs.

Again, let $x = \{x_1, \ldots, x_G\}$ denote noncommuting variables. Let $\mathcal{N}_*(x)$ denote the free $\mathbb{R}$-algebra on the $2G$ generators $\{x, x^T\} = \{x_1, \ldots, x_G, x_1^T, \ldots, x_G^T\}$, i.e. the noncommutative polynomials on those $2G$ generators. The algebra has a natural involution determined by $x_j \mapsto x_j^T$, $x_j^T \mapsto x_j$, and

$$(x_{j_1} \cdots x_{j_n})^T = x_{j_n}^T \cdots x_{j_1}^T.$$

Let $p(x,x^T)^{-1}$ denote the inverse of $p(x,x^T)$ satisfying $p(x,x^T)^{-1}p(x,x^T) = 1 = p(x,x^T)p(x,x^T)^{-1}$. Let the NC rational expressions of $\{x, x^T\}$ with real coefficients, denoted by $\mathcal{R}_*(x)$, be the closure of the NC polynomials, $\mathcal{N}_*(x)$, under finite numbers of inversions, products, transposes, and sums. For a $d \times d$ matrix $L(x, x^T)$ call L a NC linear pencil if the entries of L are polynomials in $\mathcal{N}_*(x)$ of degree one or less. See Theorem 5.3.1 for a full statement and proof of the result below.

Result 1.3.1. *Suppose that $r(x, x^T)$ is an NC rational expression. Then there exists an NC linear pencil $L(x, x^T)$ such that $r(x, x^T)$ is the Schur complement of $L(x, x^T)$.*

The fact that all symmetric NC rational functions can be realized as the Schur complement of a symmetric NC linear pencil is not a new result. In [BR88], there is a proof of an equivalent realization using formal power series to represent the rational expressions. We will give a more constructive proof that all symmetric NC rational functions are Schur complements of NC linear pencils. From this easy proof an algorithm for constructing the pencil in question readily follows. The resulting algorithm has been implemented under the NC Algebra package for Mathematica.

2 Definitions and Basics

In this chapter we state definitions for many of the main objects in the thesis. In particular we introduce the class of NC analytic functions and define notation used for domains of NC analytic functions. In addition we prove several basic lemmas that are useful for understanding NC analytic functions.

Suppose that $(x_1, \ldots, x_G)$ is a list of noncommuting variables or **NC variables**. By noncommuting we mean that $x_i x_j \neq x_j x_i$. In this thesis we will consider classes of maps defined in terms of these noncommuting variables. These classes of maps will be defined on domains consisting of sets of matrix tuples. With this in mind, we define an involution on the variables consistent with the conjugate transpose involution defined on matrices with complex entries. The involution will be denoted by x_i^T since we will generally (but not always) be evaluating maps on tuples of matrices with real entries. At times we will consider **NC symmetric variables**, that is, variables y_j such that $y_j = y_j^T$. As one may suspect, maps defined in terms of symmetric variables will have their domains restricted to tuples of symmetric matrices. By **NC mixed variables** we mean a list of variables

$$(x_1, \ldots, x_{G_1}, y_1, \ldots, y_{G_2})$$

such that $(x_1, \ldots, x_G)$ are NC variables and $(y_1, \ldots, y_{G_2})$ are NC symmetric variables. As a rule in this thesis, the variables G, G_1, and G_2 will be reserved for the number of unknowns. The variables G and G_1 will represent the number of NC variables and G_2 will represent the number of symmetric NC variables.

2.1 Definitions and Notation for Balls and Multi-discs

We will be evaluating maps defined in terms of NC (symmetric, mixed) variables on sets of tuples of (symmetric, mixed) square matrices of all sizes. So when we use the term "ball" we mean the union over all size of balls of tuples of (symmetric, mixed) square matrices. The notation that we use to refer to these dimension-free matrix balls follows. We use the notation $\mathbb{R}^{N\times N}$ to refer to N by N real matrices and $S\mathbb{R}^{N\times N}$ to refer to symmetric N by N real matrices. Since the sets that we will be evaluating our maps on are disjoint unions of sets with different topologies, we will use the topology already associated with each set. In particular, if

$$(X_1,\ldots,X_{G_1},Y_1,\ldots,Y_{G_2}) \in (\mathbb{R}^{N\times N})^{G_1} \times (S\mathbb{R}^{N\times N})^{G_2}$$

the topology on $(\mathbb{R}^{N\times N})^{G_1} \times (S\mathbb{R}^{N\times N})^{G_2}$ is determined by the operator norm of

$$\begin{pmatrix} X_1 \\ \vdots \\ X_{G_1} \\ Y_1 \\ \vdots \\ Y_{G_2} \end{pmatrix}$$

viewed as a map from $\mathbb{R}^N$ to $\mathbb{R}^{N(G_1+G_2)}$. When we refer to the boundary or interior of a dimension-free set, we are referring to the disjoint union of boundaries or interiors taken over all sizes of square matrices.

The following is a list of some of the common sets that we will be using as domains for functions of NC variables. We will always consider balls centered at the 0 tuple.

- We will often restrict functions to be defined on matrices of a fixed size. The notation for a ball of G_1-tuples of $N \times N$ real matrices with total norm less than a is

$$\mathcal{B}_a^{G_1}(N) := \{X \in (\mathbb{R}^{N\times N})^{G_1} : \|X\| < a\}$$

- Our notation for a dimension-free matrix ball of G_1-tuples of real matrices with total norm less than a is

$$\mathcal{B}_a^{G_1} := \bigcup_{N=1}^{\infty} \mathcal{B}_a^{G_1}(N)$$

- For functions defined in terms of symmetric variables, we will need domains of symmetric matrix tuples. The notation for a ball of G_2-tuples of symmetric $N \times N$ real matrices with norm less than a is

$$S\mathcal{B}_a^{G_2}(N) := \{Y \in (S\mathbb{R}^{N\times N})^{G_2} : \|Y\| < a\}$$

- Our notation for a dimension-free matrix ball of G_2-tuples of real symmetric matrices with total norm less than a is

$$S\mathcal{B}_a^{G_2} := \bigcup_{N=1}^{\infty} S\mathcal{B}_a^{G_2}(N)$$

- The notation for G_1-tuples of $N \times N$ complex matrices with total norm less than a is

$$C\mathcal{B}_a^{G_1}(N) := \{X \in (\mathbb{C}^{N\times N})^{G_1} : \|X\| < a\}$$

- And similar to the earlier cases, our notation for G_1-tuples of complex matrices with total norm less than a is

$$C\mathcal{B}_a^{G_1} := \bigcup_{N=1}^{\infty} C\mathcal{B}_a^{G_1}(N)$$

- At times we will consider functions of some symmetric NC variables and some free NC variables. In this case, we will need to consider "mixed balls." Our notation for a mixed ball with G_1-tuples of $N \times N$ matrices and G_2-tuples of symmetric $N \times N$ matrices with total norm less than a is

$$M\mathcal{B}_a^{G_1,G_2}(N) := \{(X,Y) \in (\mathbb{R}^{N\times N})^{G_1} \times (S\mathbb{R}^{N\times N})^{G_2} : \|(X,Y)\| < a\}$$

- Our notation for a dimension-free mixed ball with G_1-tuples of free matrices and G_2-tuples of symmetric matrices with total norm less than a is

$$M\mathcal{B}_a^{G_1,G_2} := \bigcup_{N=1}^{\infty} M\mathcal{B}_a^{G_1,G_2}(N)$$

- When we use functions of some symmetric and some free NC variables we will at times consider domains that are similar to bi-discs. For us, a bi-disc will refer to the product of one dimension-free ball of G_1-tuples of free matrices with a dimension-free ball of G_2-tuples of symmetric matrices. We use the $\boxtimes$ symbol to mean the union over all dimensions of products taken in each dimension. In the case that the G_1-tuples of free matrices are real matrices with total norm less than 1 and the G_2-tuples of real symmetric matrices have total norm less that β, we use the notation

$$\mathcal{B}_1^{G_1} \boxtimes S\mathcal{B}_\beta^{G_2} := \bigcup_{N=1}^{\infty} \mathcal{B}_1^{G_1}(N) \times S\mathcal{B}_\beta^{G_2}(N)$$

  and we refer to this as the $(1, \beta)$ matrix bi-disc.

- In the case that the G_1-tuples of free matrices are complex matrices with total norm less than 1 and the G_2-tuples of real symmetric matrices have total norm less that β, we use the notation

$$C\mathcal{B}_1^{G_1} \boxtimes S\mathcal{B}_\beta^{G_2} := \bigcup_{N=1}^{\infty} C\mathcal{B}_1^{G_1}(N) \times S\mathcal{B}_\beta^{G_2}(N)$$

  and call this the $(1, \beta)$ complex matrix bi-disc.

- We will use a matrix multi-disk occasionally when studying functions of G NC variables. Our notation for the matrix multi-disc of G-tuples of $N \times N$ real matrices each component of norm less that ϵ is

$$\mathbb{D}_\epsilon^G(N) := \{X \in (\mathbb{R}^{N \times N})^G : \|X_j\| < \epsilon\}$$

- Our notation for the dimension-free matrix multi-disc of G-tuples of matrices where each component has norm less than ϵ is

$$\mathbb{D}_\epsilon^G := \bigcup_{N=1}^{\infty} \mathbb{D}_\epsilon^G(N)$$

2.2 Polynomials, Analytic Maps, and Their Domains

Suppose that $(x_1, \ldots, x_G)$ are NC variables. Suppose that w is in the free semigroup generated by $2G$ generators $\{e_1, \ldots, e_{2G}\}$. We say that

$$w = e_{j_1} \cdots e_{j_K}$$

is a **word** of length K. We denote the length of a word by $|w|$. We define

$$x^{e_j} := \begin{cases} x_j & \text{if } j \leq G, \\ x_{j-G}^T & \text{if } j > G. \end{cases}$$

Thus if $w = e_{j_1} \cdots e_{j_K}$ then

$$x^w = x^{e_{j_1}} \cdots x^{e_{j_K}}$$

Also we define $x^{\emptyset} = 1$. By taking real linear combinations of terms x^w we thus create polynomials in the NC variables $(x_1, \ldots, x_G, x_1^T, \ldots, x_G^T)$. We will refer to a polynomial in the NC variables $(x_1, \ldots, x_G, x_1^T, \ldots, x_G^T)$ with coefficients in $\mathbb{R}$ as an NC polynomial.

To define the involution in terms of these words, we first define an involution on the generators of the free group. To this end we define

$$e_j^T := \begin{cases} e_{j+G} & \text{if } j \leq G, \\ e_{j-G} & \text{if } j > G. \end{cases} \tag{2.1}$$

Thus if

$$x^w := x^{e_{j_1} \cdots e_{j_K}},$$

then

$$(x^w)^T := x^{e_{j_K}^T \cdots e_{j_1}^T}.$$

Now we extend the transpose linearly. The transpose involution is consistent with the transpose in matrix algebra. As an example, consider the polynomial

$$p(x) = 2x_1 x_2^T + x_2^T x_2^T x_3.$$

The polynomial $p^T(x) = p(x)^T$ is given by

$$p^T(x) = p(x)^T = 2x_2 x_1^T + x_3^T x_2^2.$$

If $p(x)$ is an NC polynomial whose words are made up only from the NC variables $(x_1, \ldots, x_G)$ (that is, there are no transposed variables in any terms) then we say the NC polynomial is **analytic**. If we are dealing with polynomials in NC symmetric variables only, all NC polynomials in symmetric variables would be analytic. If $p(x, y)$ is a polynomial in the NC mixed variables $(x_1, \ldots, x_{G_1}, y_1, \ldots, y_{G_2})$ (notice that no term has transposed factors x_j^T,) then $p(x, y)$ is also considered analytic.

If $p(x)$ is an NC polynomial whose terms are all of the form

$$(x^{w_1})^T x^{w_2}$$

for analytic x^{w_1} and x^{w_2}, then we say that $p(x)$ is **hereditary**. Another way to characterize hereditary NC polynomials is to say that any transposed letters that appear in a term must appear to the left of all non-transposed letters. For example, the polynomial

$$p(x) = x_1^T x_2^T x_1 - 3x_2^T x_1 x_2$$

is a hereditary NC polynomial. If $p(x, y)$ is a polynomial in the NC mixed variables

$$(x_1, \ldots, x_{G_1}, x_1^T, \ldots, x_{G_1}^T, y_1, \ldots, y_{G_2})$$

whose terms are of the form $((x, y)^{w_1})^T (x, y)^{w_2}$ for analytic $(x, y)^{w_1}$ and $(x, y)^{w_2}$, then $p(x, y)$ is also called hereditary.

There has been interest in the area of NC polynomials and real semialgebraic geometry of late. In particular, much research has been devoted to functions defined by NC polynomials on dimensionless sets such as those previously discussed [HMP05]. For a symmetric polynomial in the NC variables $(x_1, \ldots, x_G, x_1^T, \ldots, x_G^T)$, $p(x)$, we define the **positivity domain** of p, denoted $\mathcal{D}_p$, to be

$$\mathcal{D}_p := \bigcup_{N \geq 1} \{X \in (\mathbb{R}^{N \times N})^G : p(X) \succ 0\}.$$

Similarly, if $p(x,y)$ is a polynomial in the NC mixed variables

$$(x_1,\dots,x_{G_1},x_1^T,\dots,x_{G_1}^T,y_1,\dots,y_{G_2}),$$

then the positivity domain of p is defined to be

$$\mathcal{D}_p := \bigcup_{N\geq 1}\{(X,Y)\in(\mathbb{R}^{N\times N})^{G_1}\times(S\mathbb{R}^{N\times N})^{G_2} : p(X,Y)\succ 0\}.$$

For the rest of this paper we will use the convention that $p(X)\succ 0$ means $p(X)$ is a positive definite matrix and $p(X)\succeq 0$ means that $p(X)$ is a positive semi-definite matrix. We will use the term "binding" to describe a matrix inequality that is specifically non-strict. That is, an inequality $A\succeq 0$ is **binding** if there exists a vector v such that $Av=0$.

The idea of NC analytic maps extend the ideas of maps defined by NC analytic polynomials. We consider series in the NC variables $(x_1,\dots,x_G)$, that is sums of the form

$$\sum_{|w|\geq 0} a_w x^w$$

where $a_w\in\mathbb{R}^K$ for some K. Alternatively, we could write the series as

$$\sum_{|w|\geq 0} a_w\otimes x^w$$

to emphasize that

$$a_w x^w := \begin{pmatrix} a_{w,1}x^w \\ \vdots \\ a_{w,K}x^w \end{pmatrix}.$$

Notice that there are no transposed variables in these series. We define an **NC analytic function**, f, in the NC variables $(x_1,\dots,x_G)$ to be a function

$$f:\mathcal{B}_\epsilon^G\to\bigcup_{N\geq 1}(\mathbb{R}^{N\times N})^K$$

for some $\epsilon>0$ and $K\in\mathbb{N}$ where

$$f(x)=\sum_{|w|\geq 0} a_w x^w$$

and this series converges absolutely on $\mathcal{B}_\epsilon^G$. We can also consider series in NC symmetric and mixed variables and define a similar notion of analyticity. For the NC symmetric variable case, we consider sums of the form

$$\sum_{|w|\geq 0} a_w y^w$$

where $(y_1, \ldots, y_{G_2})$ are NC symmetric variables and $a_w \in \mathbb{R}^K$ for some K. If such a series converges absolutely on $S\mathcal{B}_\epsilon^{G_2}$ for some $\epsilon > 0$, then the function

$$f : S\mathcal{B}_\epsilon^{G_2} \to \bigcup_{N\geq 1} (\mathbb{R}^{N\times N})^K$$

by

$$f(y) = \sum_{|w|\geq 0} a_w y^w$$

is called an NC analytic function as well. In the mixed variable case, we consider sums of the form

$$\sum_{|w|\geq 0} a_w (x, y)^w$$

with the same convergence criteria for mixed balls as above. The important feature of analytic functions in NC mixed variables is that the terms in the series expansion have no transposed x_j^T factors. An NC analytic function,f, from G variables to G variables is **NC bianalytic** if it has an inverse function that is NC analytic. This idea applies in an obvious way also to the cases of NC symmetric and mixed variables as well. It turns out that properties of NC analytic functions in symmetric variables are similar to properties of real analytic functions; whereas properties of NC analytic functions are much nicer and follow along the lines of holomorphic functions of complex variables.

2.3 Some Basic Facts

The following results state that if an NC analytic function is identically zero on some dimensionless open set, then its series expansion must be identically zero. This result holds regardless of the case of symmetric, mixed, or unrestricted NC variables.

Lemma 2.3.1. *Suppose that*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_G \end{pmatrix} \stackrel{p}{\mapsto} \begin{pmatrix} p_1(x) \\ \vdots \\ p_K(x) \end{pmatrix}$$

where p is NC analytic on $S\mathcal{B}_\delta^{G_2}$. Suppose that $J \subseteq S\mathcal{B}_\delta^{G_2}$ is a dimension-free open set in the sense that for all N the set

$$J \cap (S\mathbb{R}^{N\times N})^G$$

is open in $(S\mathbb{R}^{N\times N})^G$. If $p(X) \equiv 0$ on J, then $p(x) = 0$. Instead, we may define the dimension-free open set J to be an open set of tuples of self-adjoint operators on a Hilbert space and obtain the same result that $p(x) = 0$.

Proof. Suppose that $X \in J \cap (S\mathbb{R}^{N\times N})^G$. Consider the function

$$P(t) := p(tX) = \sum_{k=0}^{\infty} h_k(X)t^k$$

where h_k are homogeneous polynomials of degree k who sum to p. Since $J \cap (S\mathbb{R}^{N\times N})^G$ is open, there is a neighborhood of 1, $\mathcal{W}_1 \subseteq \mathbb{R}$, such that $P(t) = p(tX) = 0$ whenever $t \in \mathcal{W}_1$. Thus the coefficients of the series expansion for P are equal to 0; that is,

$$h_k(X) = 0$$

for all k. Since $h_k(X) = 0$ for all $X \in J$ and all k, we conclude $h_k(x) = 0$ for all k. So

$$p(x) = \sum_{k=0}^{\infty} h_k(x) = 0.$$

Notice that if we instead fix X to be a tuple of operators on a Hilbert space, the argument is still valid. ●●

Corollary 2.3.2. *Suppose that*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \end{pmatrix} \stackrel{p}{\mapsto} \begin{pmatrix} p_1(x) \\ \vdots \\ p_K(x) \end{pmatrix}$$

where p is NC analytic on $\mathcal{B}_\delta^{G_1}$*. Suppose that* $J \subseteq \mathcal{B}_\delta^{G_1}$ *is a dimensionless open set in the sense that for all* N *the set*

$$J \cap (\mathbb{R}^{N\times N})^G$$

is open in $(\mathbb{R}^{N\times N})^G$*. If* $p(X) \equiv 0$ *on* J*, then* $p(x) = 0$*.*

Corollary 2.3.3. *Suppose that*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \\ y_1 \\ \vdots \\ y_{G_2} \end{pmatrix} \stackrel{p}{\mapsto} \begin{pmatrix} p_1(x) \\ \vdots \\ p_K(x) \end{pmatrix}$$

where p is NC analytic on $M\mathcal{B}_\delta^{G_1,G_2}$*. Suppose that* $J \subseteq M\mathcal{B}_\delta^{G_1,G_2}$ *is a dimensionless open set in the sense that for all* N *the set*

$$J \cap (\mathbb{R}^{N\times N})^{G_1} \times (S\mathbb{R}^{N\times N})^{G_2}$$

is open in $(\mathbb{R}^{N\times N})^{G_1} \times (S\mathbb{R}^{N\times N})^{G_2}$*. If* $p(X) \equiv 0$ *on* J*, then* $p(x) = 0$*.*

The next result says that NC analytic maps are continuous when restricted to matrices of a fixed size.

Lemma 2.3.4. *Suppose that*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \end{pmatrix} \stackrel{p}{\mapsto} \begin{pmatrix} p_1(x) \\ \vdots \\ p_K(x) \end{pmatrix}$$

where p is NC analytic on $\mathcal{B}_\delta^{G_1}$*. Then p restricted to* $\mathcal{B}_\delta^{G_1}(N)$ *is continuous for all* N*.*

Proof. Clearly, if p were a polynomial function the result would hold. Write p as a series of homogeneous polynomials

$$p(x) = \sum_{n=1}^{\infty} h_n(x).$$

Let $\epsilon > 0$. Fix $\|X^0\| = \delta$ and consider any sequence $\{X^j\} \subseteq \overline{\mathcal{B}^{G_1}_{\delta'}(N)} \subseteq \mathcal{B}^{G_1}_{\delta}(N)$ such that $X^j \to X^0$. Since the series defining p converges uniformly on $\overline{\mathcal{B}^{G_1}_{\delta'}(N)}$, there is some M such that

$$\| \sum_{n \geq M} h_n(X)\| < \frac{\epsilon}{3}$$

for all $X \in \overline{\mathcal{B}^{G_1}_{\delta'}(N)}$. Since $\sum_{n=0}^{M-1} h_n(x)$ is continuous, there is some M' such that

$$\| \sum_{n=0}^{M-1} h_n(X^0) - \sum_{n=0}^{M-1} h_n(X^j)\| < \frac{\epsilon}{3}$$

when $j > M'$. Thus

$$\|p(X^0) - p(X^j)\| \leq \| \sum_{n=0}^{M-1} h_n(X^0) - \sum_{n=0}^{M-1} h_n(X^j)\| + \| \sum_{n \geq M} h_n(X^0)\| + \| \sum_{n \geq M} h_n(X^j)\| < \epsilon$$

when $j > M'$. ●●

The study of directional derivatives of functions between Banach spaces is classical and many preliminaries can be found in [HP74]. This can be useful when dealing with NC analytic functions as well – see the proof of Lemma 4.4.4. To define the directional derivative for an NC analytic function, suppose that $(x_1, \ldots, x_G)$ are NC variables and

$$f(x) = \sum_w f_w x^w$$

with $f_w \in \mathbb{R}^K$ and $f(X)$ converging absolutely when $X \in \mathcal{B}^G_\epsilon$. Then the **directional derivative** of f at $X \in (\mathbb{R}^{N \times N})^G$ in direction $H \in (\mathbb{R}^{N \times N})^G$ is defined to be

$$\partial_H f(X) := \frac{d}{dt}\Big|_{t=0} f(X + tH)$$

when the limit exists.

Proposition 2.3.5. *If f is NC analytic then there is a dimensionless neighborhood of 0, U, such that $\partial_H f(X)$ exists for all $H \in (\mathbb{R}^{N \times N})^G$, whenever $X \in U(N)$.*

Proof. For a fixed $X \in \mathcal{B}^G_\epsilon$ and any corresponding direction H define

$$F(t) := f(X + tH).$$

Notice that there is an interval, J, containing 0 such that F converges absolutely and uniformly on J. Thus $F'(0)$ exists and is equal to $\partial_H f(X)$. ●●

Lemma 2.3.6. *Suppose that*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \end{pmatrix} \overset{f}{\mapsto} \begin{pmatrix} f_1(x) \\ \vdots \\ f_K(x) \end{pmatrix}$$

where f is NC analytic and

$$f(x) = \sum_w f_w x^w = C + Ax + \sum_{n=2}^{\infty} p_n(x)$$

where $C \in \mathbb{R}^K, A \in \mathbb{R}^{K\times G}$ and p_n are K-vectors of homogeneous polynomials of degree n. Then

1. $\partial_h f(0) = Ah.$

2. *For any fixed N, the function $f \in C^1((\mathbb{R}^{N\times N})^{G_1}, (\mathbb{R}^{N\times N})^K)$*

Proof. Notice that for any $X \in \mathcal{B}_\epsilon^{G_1}(N)$, any direction $H \in (\mathbb{R}^{N\times N})^{G_1}$ and any vector v we have

$$f(X + tH)v = \sum_{j=0}^{\infty} p^j(X + tH)v = \sum_{j=0}^{\infty} t^j s^j(X, H)v$$

for some absolutely convergent series $s^j(X, H)$ and for all $t \in J$ where J is some interval containing 0. We notice that

$$t \mapsto \sum_{j=0}^{\infty} t^j s^j(X, H)v$$

is C^∞ and that $\partial_H f(X) = s^1(X, Y)$. Notice that $s^1(0, H) = (A \times I_N)H = \partial_H f(0)$. Notice also that the map

$$S : (\mathbb{R}^{N\times N})^{G_1} \mapsto L((\mathbb{R}^{N\times N})^{G_1}, (\mathbb{R}^{N\times N})^K)$$

by

$$\partial_H f(X) = S(X)H$$

is continuous by Lemma 2.3.4. ●●

2.4 Compositions of NC Analytic Functions

One difficulty we have encountered while studying NC analytic functions involves the basic idea of compositions of two NC analytic functions. In particular, we would like to say that "the composition of two NC analytic functions is NC analytic." However, this has been a difficult claim to verify.[1]

It is worthy to note that the NC analytic functions on a fixed dimension-free domain form an algebra. It helps prove a useful corollary about a ceratin type of composition of NC analytic functions.

Lemma 2.4.1. *Suppose that*

$$F : \mathcal{D} \to \mathcal{W}_F \quad \textit{and} \quad H : \mathcal{D} \to \mathcal{W}_H$$

are NC analytic where $\mathcal{D}$ is a dimension-free set containing 0. Then

1. *the function $F + H$ is NC analytic on $\mathcal{D}$*
2. *and the function FH is NC analytic on $\mathcal{D}$.*

Proof. The proofs of these two facts are quite easy. Suppose that

$$F(x) = \sum_w F_w x^w \quad \text{and} \quad G(x) = \sum_w G_w x^w.$$

Notice that for a fixed $X \in \mathcal{D}$ we have

$$\sum_w \|F_w X^w + G_w X^w\| \leq \sum_w \|F_w X^w\| + \sum_w \|G_w X^w\| < \infty.$$

For the second fact, we will follow an elementary argument for a similar fact about series of numbers [Rud76]. Pick an order for the words w (which order does not matter since the sums converge absolutely) and define

$$A := \sum_{n \geq 0} a_n := \sum_w F_w X^w \text{ and } B := \sum_{n \geq 0} b_n := \sum_w G_w X^w.$$

[1] A recent result of V. Vinikov tells us that compositions of compatible NC analytic functions converge on some dimension-free neighborhood of the origin. However, the series given by the composition $G \circ F$ is not guaranteed to converge on the entire domain of F by Vinikov's result.

Now define for each $k \geq 0$,

$$c_k = \sum_{n=0}^{k} a_n b_{k-n}.$$

Clearly,

$$\|c_k\| \leq \sum_{n=0}^{k} \|a_n\| \|b_{k-n}\|$$

and therefore

$$\sum_{k \geq 0} \|c_k\|$$

converges since the series

$$\sum_{n \geq 0} a_n \text{ and } \sum_{n \geq 0} b_n$$

converge absolutely [Rud76]. To see that

$$\sum_{n \geq 0} c_n = AB$$

define

$$A_n := \sum_{k=0}^{n} a_k, \; B_n := \sum_{k=0}^{n} b_k, \; C_n := \sum_{k=0}^{n} c_k, \; \beta_n = B_n - B.$$

Then

$$\begin{aligned} C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \ldots + (a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \ldots + a_n B_0 \\ &= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \ldots + a_n (B + \beta_0) \\ &= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \ldots + a_n \beta_0 \end{aligned}$$

Define

$$\gamma_n := a_0 \beta_n + a_1 \beta_{n-1} + \ldots + a_n \beta_0.$$

It suffices to show that

$$\lim_{n \to \infty} \gamma_n = 0.$$

Let $\epsilon > 0$ and define

$$\alpha = \sum_{n \geq 0} \|a_n\|.$$

We know that $\beta_n \to 0$ so choose N such that $\|\beta_n\| < \epsilon$ for $n \geq N$. Now notice

$$\begin{aligned}\|\gamma_n\| &\leq \|a_n\beta_0 + \ldots + a_{n-N}\beta_N\| + \|a_{n-N-1}\beta_{N+1} + \ldots + a_0\beta_n\| \\ &\leq \|a_n\beta_0 + \ldots + a_{n-N}\beta_N\| + \epsilon\alpha.\end{aligned}$$

Keep N fixed and let $n \to \infty$ to see

$$\limsup_{n\to\infty} \|\gamma_n\| \leq \epsilon\alpha.$$

Thus $\gamma_n \to 0$ and

$$C_n \to AB.$$

••

Here is a useful corollary. This says that composing an NC analytic function with an NC analytic polynomial results in an NC analytic map.

Corollary 2.4.2. *Suppose that*

$$F : \mathcal{D}_F \to \mathcal{W}_F \quad \textit{and} \quad H : \mathcal{W}_F \subseteq \mathcal{D}_H \to \mathcal{W}_H$$

where F and H are NC analytic, $0 \in \mathcal{D}_F$, and $H(x)$ is a polynomial. Then the composition $H \circ F$ has a series expansion that converges absolutely on $\mathcal{D}_F$ and is thus NC analytic.

Proof. Notice that the composition of a polynomial with an NC analytic function results in some finite combination of sums and products of NC analytic functions, which is again NC analytic by Lemma 2.4.1 ••

In lieu of a general result on compositions, we will make some assumptions to guarantee that our NC analytic functions compose to produce an NC analytic function. For exact details of the assumptions, referred to as **analytically composing** functions, see Section 3.5.

3 Classification of NC Ball Maps and NC Mixed Ball Maps

One pursuit in classical several complex variables has involved the issue of when a domain in $\mathbb{C}^n$ is bianalytic to a ball in $\mathbb{C}^n$. Results here are fragmentary for the general case. A more successful pursuit has been in the classification of analytic maps of the ball in $\mathbb{C}^n$ into the ball in $\mathbb{C}^m$ which map the boundary into the boundary [D'A92]. In this chapter we pursue some similar classification questions. In particular, we give classifications of NC ball maps and NC mixed ball maps.

An NC ball map is an NC analytic map defined in terms of non-symmetric variables that maps a matrix ball to a matrix ball and carries the boundary to the boundary for all sizes of square matrix tuples. A NC mixed ball map is an NC analytic map defined in terms of some symmetric and some non-symmetric variables that maps a mixed matrix ball to a mixed matrix ball and carries the boundary to the boundary for all sizes of square matrix tuples.

The classification shows that the structure of NC ball maps are quite rigid in that they can be classified by a fairly small class of isometries. To this end, in Section 3.4, we introduce the notion of and characterize NC full isometries – these are similar to the complete isometries of C^*-algebras [Pau02]. In Section 3.6 we introduce the class of maps defined as NC bi-disc maps and show that a subclass of these maps are equivalent to NC full isometries. In the end, we obtain our classification of NC ball maps (and NC mixed ball maps) by using a change of variables to create an NC bi-disc map that is also an NC full isometry.

Parts of this chapter are included in [HKMS09] which deals with the classification of NC ball maps which are matrix valued functions of non-symmetric variables.

3.1 Definitions, Examples and a Simply Stated NC Ball Map Result

In order to define NC mixed ball maps specifically, suppose that

$$(x_1, \ldots, x_{G_1}, y_1, \ldots, y_{G_2})$$

are NC mixed variables with $y_j = y_j^T$ for all $1 \leq j \leq G_2$. Suppose that the map

$$(x_1, \ldots, x_{G_1}, y_1, \ldots, y_{G_2}) \overset{p}{\mapsto} (r_1, \ldots, r_{K_1}, q_1, \ldots, q_{K_2})$$

is NC analytic with $q_j = q_j^T$ for all $1 \leq j \leq K_2$. We will call such maps p **NC mixed ball maps from the β-ball to the α-ball** if the following properties hold:

1. For all N, if $X \in (\mathbb{R}^{N \times N})^{G_1}$ and $Y \in (S\mathbb{R}^{N \times N})^{G_2}$ with $\|(X, Y)\| < \beta$, then $p(X, Y)$ converges absolutely and $\|p(X, Y)\| < \alpha$.

2. For all N, if $X \in (\mathbb{R}^{N \times N})^{G_1}$ and $Y \in (S\mathbb{R}^{N \times N})^{G_2}$ with $\|(X, Y)\| = \beta$ and $\|Y\| < \beta$, then $\lim_{\delta \uparrow 1} p(\delta X, Y)$ exists and $\|\lim_{\delta \uparrow 1} p(\delta X, Y)\| = \alpha$.

Suppose that the map

$$(x_1, \ldots, x_G) \overset{p}{\mapsto} (p_1, \ldots, p_K) \tag{3.1}$$

is NC analytic. We will call such a map p an **NC ball map from the β-ball to the α-ball** if the following properties hold:

1. For all N, if $X \in (\mathbb{R}^{N \times N})^G$ with $\|X\| < \beta$, then $p(X)$ converges absolutely and $\|p(X)\| < \alpha$.

2. For all N, if $X \in (\mathbb{R}^{N \times N})^G$ with $\|X\| = \beta$, then $\lim_{\delta \uparrow 1} p(\delta X)$ exists and $\|\lim_{\delta \uparrow 1} p(\delta X)\| = \alpha$.

In the cases that $\alpha = \beta = 1$, we will refer to unit NC mixed ball maps and unit NC ball maps. In this chapter we will classify NC mixed ball maps. We will perform this classification of mixed ball maps via a classification of something called bi-disc maps found in Section 3.6.

We will also classify NC ball maps. A unit NC ball map, p, such that $p(0) = 0$ is a linear isometry as formulated in the following Corollary.

Corollary 3.1.1. *The map*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_G \end{pmatrix} \overset{p}{\mapsto} \begin{pmatrix} p_1(x) \\ \vdots \\ p_K(x) \end{pmatrix}$$

is a unit NC ball map with $p(0) = 0$ *if and only if* $p(x) = Ax$ *for some* $A \in \mathbb{R}^{K \times G}$ *such that* $A^T A = I$.

This Corollary will be proved later in Section 3.7.

When the map, p, does depend on some symmetric variables y_j the maps can be more complicated. An example is the following NC analytic map

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{p}{\mapsto} \begin{pmatrix} \sin(y)x \\ \cos(y)x \\ y \end{pmatrix}$$

where $y = y^T$ and $\sin(y)$ and $\cos(y)$ are defined by their Taylor series expansions about 0. Notice that

$$\|p(X, Y)\| = \|(X, Y)\|$$

for all matrix pairs $(X, Y) \in \mathbb{R}^{N \times N} \times S\mathbb{R}^{N \times N}$ and all N.

3.2 Equivalence of the Bi-disc and the Mixed Matrix Ball

We define the (α, β) **matrix bi-disc**, $\mathcal{B}_\alpha^{G_1} \boxtimes S\mathcal{B}_\beta^{G_2}$, in the following way

$$\mathcal{B}_\alpha^{G_1} \boxtimes S\mathcal{B}_\beta^{G_2} := \bigcup_{N=1}^{\infty} \mathcal{B}_\alpha^{G_1}(N) \times S\mathcal{B}_\beta^{G_2}(N).$$

That is, the matrix bi-disc is the union over all dimensions of the product of a non-symmetric matrix ball and a symmetric matrix ball in each dimension. Similarly we define the (α, β) **complex matrix bi-disc**, $C\mathcal{B}_\alpha^{G_1} \boxtimes S\mathcal{B}_\beta^{G_2}$, in the following way

$$C\mathcal{B}_\alpha^{G_1} \boxtimes S\mathcal{B}_\beta^{G_2} := \bigcup_{N=1}^{\infty} C\mathcal{B}_\alpha^{G_1}(N) \times S\mathcal{B}_\beta^{G_2}(N).$$

In the following lemma we will need to include some of the "boundary" of the bi-disc. Namely, let

$$\overline{\mathcal{B}_\alpha^{G_1}} \boxtimes S\mathcal{B}_\beta^{G_2} := \bigcup_{N=1}^{\infty} \overline{\mathcal{B}_\alpha^{G_1}(N)} \times S\mathcal{B}_\beta^{G_2}(N).$$

So we have taken the union over all dimensions of the product of a non-symmetric, closed matrix ball and a symmetric ball in each dimension.

The following lemma gives an example of an NC bi-analytic map between a matrix bi-disc and a matrix ball carrying a piece of the boundary of the bi-disc to a piece of the boundary of the ball. This change of variables is key to the classification of NC mixed ball maps via the classification of bi-disc maps.

For a positive definite matrix A, we refer to $A^{1/2}$ as the unique positive definite matrix such that $A^{1/2}A^{1/2} = A$.

Lemma 3.2.1. *Let $N \in \mathbb{N}$ and $\beta > 0$. Define the set*

$$\Gamma_\beta^N := \{(X, Y) \in (\mathbb{R}^{N\times N})^{G_1} \times (S\mathbb{R}^{N\times N})^{G_2} : \|(X, Y)\| \leq \beta \text{ and } \|Y\| < \beta\}. \quad (3.2)$$

Define the map

$$\phi_{N,\beta} : \overline{\mathcal{B}_1^{G_1}(N)} \times S\mathcal{B}_\beta^{G_2}(N) \to \Gamma_\beta^N \quad (3.3)$$

by

$$\phi_{N,\beta}(X, Y) = (X(\beta^2 I - Y^T Y)^{1/2}, Y). \quad (3.4)$$

1. *If $(X, Y) \in \partial\mathcal{B}_1^{G_1}(N) \times S\mathcal{B}_\beta^{G_2}(N)$, then*

$$\phi_{N,\beta}(X, Y) \in \Gamma_N \text{ and } \|\phi_{N,\beta}(X, Y)\| = \beta.$$

2. *If $(X, Y) \in \mathcal{B}_1^{G_1}(N) \times S\mathcal{B}_\beta^{G_2}(N)$, then*

$$\phi_{N,\beta}(X, Y) \in \Gamma_N \text{ and } \|\phi_{N,\beta}(X, Y)\| < \beta.$$

3. *The map* $\phi_{N,\beta}$ *has a power series expansion about* $(0,0)$ *converging absolutely on* $\overline{\mathcal{B}_1^{G_1}(N)} \times S\mathcal{B}_\beta^{G_2}(N)$ *and this series expansion is the same for all* N. *I.e. the map* $\phi_{N,\beta}$ *is the restriction of some NC analytic map* ϕ_β *that converges absolutely on* $\overline{\mathcal{B}_1^{G_1}} \boxtimes S\mathcal{B}_\beta^{G_2}$

4. *The map* ϕ_β *is invertible and the inverse is NC analytic.*

Proof. 1. Suppose that $(X,Y) \in \partial\mathcal{B}_1^{G_1}(N) \times S\mathcal{B}_\beta^{G_2}(N)$. Then

$$I - X^T X \succeq 0 \tag{3.5}$$

with binding. Now since

$$\beta^2 I - Y^T Y \succ 0 \tag{3.6}$$

then

$$I \succeq X^T X \tag{3.7}$$

with binding is equivalent to

$$\beta^2 I - Y^T Y \succeq (\beta^2 I - Y^T Y)^{1/2} X^T X (\beta^2 I - Y^T Y)^{1/2} \tag{3.8}$$

with binding. This is equivalent to

$$\beta^2 I \succeq (\beta^2 I - Y^T Y)^{1/2} X^T X (\beta^2 I - Y^T Y)^{1/2} + Y^T Y$$

with binding. Thus

$$\|(X(\beta^2 I - Y^T Y)^{1/2}, Y)\| = \beta \text{ and } \phi_{N,\beta}(X,Y) \in \Gamma_\beta^N.$$

2. Suppose that $(X,Y) \in \mathcal{B}_{1,G_1}^N \times S\mathcal{B}_{\beta,G_2}^N$. Then

$$I - X^T X \succ 0. \tag{3.9}$$

Now since

$$\beta^2 I - Y^T Y \succ 0 \tag{3.10}$$

then

$$I \succ X^T X \tag{3.11}$$

is equivalent to

$$\beta^2 I - Y^T Y \succ (\beta^2 I - Y^T Y)^{1/2} X^T X (\beta^2 I - Y^T Y)^{1/2}. \tag{3.12}$$

Thus $\|(X(I - Y^T Y)^{1/2}, Y)\| < \beta$ and $\phi_{N,\beta}(X,Y) \in \Gamma_\beta^N$.

3. To see that $\phi_{N,\beta}$ has a convergent power series expansion on Γ_β^N we need only show that $G(Y) = \sqrt{\beta^2 I - Y^T Y}$ has a convergent power series on $S\mathcal{B}_\beta^{G_2}$.

   Suppose $g : (-\beta^2, \beta^2) \subseteq \mathbb{R} \to (0, \beta\sqrt{2}) \subseteq \mathbb{R}$ and $g(t) = \sqrt{\beta^2 - t}$. Recall the Taylor series expansion of g about 0 is

$$g(t) = \beta - \sum_{k=1}^{\infty} \frac{(2(k-1))!}{2^{2k-1}(k-1)!k!} \beta^{-(2k-1)} t^k. \tag{3.13}$$

   The radius of convergence for this series is β^2. Now if $Y \in S\mathcal{B}_\beta^{G_2}$, then the series $g(Y^T Y)$ converges absolutely.

   To see that $g(Y^T Y) \succ 0$, first let U be a real orthogonal matrix such that

$$U^T D U = Y^T Y \text{ with } D = diag(\lambda_1, \ldots, \lambda_N). \tag{3.14}$$

   Then

$$g(Y^T Y) = U^T g(D) U \tag{3.15}$$

   with $g(D) = diag(g(\lambda_1), \ldots, g(\lambda_N))$. Since $g(\lambda_j) > 0$ for each j,

$$g(Y^T Y) \succ 0.$$

   Notice that

$$g(Y^T Y)^2 = U^T g(D)^2 U = U^T(\beta^2 I - D)U = \beta^2 I - Y^T Y.$$

   Thus the series expansion for $G(Y)$ is given by a rearrangement of

$$G(Y) = \beta I - \sum_{k=1}^{\infty} \frac{(2(k-1))!}{2^{2k-1}(k-1)!k!} \beta^{-(2k-1)} (Y_1^2 + \ldots + Y_{G_2}^2)^k. \tag{3.16}$$

   Noting that this sum is free of the matrix size N gives the desired result.

4. Notice that

$$\phi_\beta^{-1}(X, Y) = (X(\beta^2 I - Y^T Y)^{-1/2}, Y).$$

   Consider the power series for

$$g : (-\beta, \beta) \to (\frac{1}{2\beta}, \infty) \tag{3.17}$$

where

$$g(t) = \frac{1}{\beta - t} \tag{3.18}$$

expanded about 0. This series has radius of convergence β. Now we have for $Y \in S\mathcal{B}_{\beta}^{G_2}(N)$ that

$$\beta^2 I \succ Y^T Y. \tag{3.19}$$

So

$$\beta I \succeq (\beta^2 I - Y^T Y)^{1/2} \succ 0. \tag{3.20}$$

Thus $\|\beta I - (\beta^2 I - Y^T Y)^{1/2}\| < \beta$. Thus

$$(\beta^2 I - Y^T Y)^{-1/2} = (\beta I - (\beta I - (\beta^2 I - Y^T Y)^{1/2}))^{-1} \tag{3.21}$$

and the series $g(\beta I - (\beta^2 I - Y^T Y)^{1/2}))$ converges absolutely.

To see the series converges to $(\beta^2 I - Y^T Y)^{-1/2}$ first choose an orthogonal U such that

$$U^T D U = (\beta^2 I - Y^T Y)^{1/2}$$

and where D is a diagonal matrix of the eigenvalues of $(\beta^2 I - Y^T Y)^{1/2}$. Then

$$\begin{aligned} g(\beta I - (\beta^2 I - Y^T Y)^{1/2})) &= g(U^T(\beta I - D)U) \\ &= U^T g(\beta I - D)U = U^T(\beta I - (\beta I - D))^{-1}U \\ &= U^T D^{-1} U = (\beta^2 I - Y^T Y)^{-1/2}. \end{aligned} \tag{3.22}$$

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3.3 Linear Fractional Transformation of a Ball

Let $K \in \mathbb{N}$. Suppose that $u = (u_1, \dots, u_K)$ and $v = (v_1, \dots, v_K)$ are K-vectors of NC variables. Define now the function

$$f(u, v) = v - (I - vv^T)^{1/2} u (1 - v^T u)^{-1} (1 - v^T v)^{1/2}. \tag{3.23}$$

Notice that if $K = 1$, $v = v^T$, and u and v commute, then

$$f(u, v) = \frac{v - u}{1 - uv}. \tag{3.24}$$

Now fix $v \in \mathbb{C}$ where $|v| < 1$ and consider the map

$$\mathbb{C} \ni u \mapsto f(u, v) \in \mathbb{C}. \tag{3.25}$$

The map in Equation (3.25) is a linear fractional map that maps the unit disc to the unit disc, maps the unit circle to the unit circle, and maps v to 0.

The geometric interpretation of the map in NC variables in Equation (3.23) is similar. Suppose we fix $N \in \mathbb{N}$ and $V \in (\mathbb{R}^{N\times N})^K$ with $\|V\| < 1$ and consider the map

$$u \mapsto f(u, V). \tag{3.26}$$

The first part of Lemma 3.3.1 tells us that the map defined in Equation(3.26) maps the unit ball of K-tuples of $N \times N$ matrices to the unit ball of K-tuples of $N \times N$ matrices carrying the boundary to the boundary. The third part of Lemma 3.3.1 tells us that $f(V, V) = 0$; that is, the map given in Equation (3.26) takes V to 0.

Lemma 3.3.1. *Suppose that $N \in \mathbb{N}$ and $V \in (\mathbb{R}^{N\times N})^K$ with $\|V\| < 1$.*

1. *Then $U \mapsto f(U, V)$ maps the unit ball in $(\mathbb{R}^{N\times N})^K$ to the unit ball in $(\mathbb{R}^{N\times N})^K$ with boundary to boundary.*

2. *If $U \in (\mathbb{R}^{N\times N})^K$ with $\|U\| \leq 1$, then $f(f(U, V), V) = U$.*

3. *$f(V, V) = 0$*

Proof. The proof is motivated by linear system theory but an understanding of system theory is not needed to read the proof.

1. Suppose that

$$i = \begin{pmatrix} i_1 \\ \vdots \\ i_{K+1} \end{pmatrix} \quad \text{and} \quad o = \begin{pmatrix} o_1 \\ \vdots \\ o_{K+1} \end{pmatrix} \tag{3.27}$$

   where $o_j, i_j \in \mathbb{R}^N$ for all j. Suppose that

   (a) $Mi = o$

(b) $-Uo_1 = \begin{pmatrix} i_2 \\ \vdots \\ i_{K+1} \end{pmatrix}$

where

$$M := \begin{pmatrix} (I - V^T V)^{1/2} & -V^T \\ V & (I - VV^T)^{1/2} \end{pmatrix} \tag{3.28}$$

and $U \in (\mathbb{R}^{N \times N})^K$ with $\|U\| \leq 1$. Notice that by the definition of M we have $M^T M = I$. So

$$\|i_1\|^2 + \cdots + \|i_{K+1}\|^2 = \|o_1\|^2 + \cdots + \|o_{K+1}\|^2 \tag{3.29}$$

for any input i. Also we may solve, since $I - V^T U$ is invertible, for $(o_2, \ldots, o_{K+1})$ in terms of i_1 to obtain

$$\begin{pmatrix} o_2 \\ \vdots \\ o_{K+1} \end{pmatrix} = f(U, V) i_1. \tag{3.30}$$

Case 1. *Suppose* $\|U\| < 1$. Then

$$\|o_1\|^2 > \|i_2\|^2 + \cdots + \|i_{K+1}\|^2$$

and thus from Equation (3.29) we have

$$\|o_2\|^2 + \cdots + \|o_{K+1}\|^2 < \|i_1\|^2.$$

Thus $\|f(U, V)\| < 1$.

Case 2. *Suppose* $\|U\| = 1$. Now since

$$\|o_1\|^2 \geq \|i_2\|^2 + \cdots + \|i_{K+1}\|^2$$

we have also that

$$\|o_2\|^2 + \cdots + \|o_{K+1}\|^2 \leq \|i_1\|^2.$$

So $\|f(U, V)\| \leq 1$. Note that there exists some nonzero y such that $\|Uy\| = \|y\|$. By Equation (3.29) we have that

$$\|i_1\|^2 = \|o_2\|^2 + \cdots + \|o_{K+1}\|^2$$

whenever $\|Uo_1\|^2 = \|o_1\|^2$. Let

$$i = \begin{pmatrix} (I - V^TV)^{-1/2}(I - V^TU)y \\ -Uy \end{pmatrix}. \tag{3.31}$$

Then

$$Mi = \begin{pmatrix} y \\ V(I - V^TV)^{-1/2}(I - V^TU)y - (I - VV^T)^{1/2}Uy \end{pmatrix}. \tag{3.32}$$

So $o_1 = y$ and $\|Uo_1\| = \|o_1\|$ since y was chosen such that $\|y\| = \|Uy\|$. Thus

$$\|i_2\|^2 + \cdots + \|i_{K+1}\|^2 = \|o_1\|^2 \neq 0$$

and $\|f(U, V)\| = 1$.

2. Define

$$F := f(U, V) = V - (I - VV^T)^{1/2}U(I - V^TU)^{-1}(I - V^TV)^{1/2}$$

First notice that

$$\begin{aligned} I - V^TF &= I - V^TV + V^T(I - VV^T)^{1/2}U(I - V^TU)^{-1}(I - V^TV)^{1/2} \\ &= (I - V^TV) + (I - V^TV)^{1/2}V^TU(I - V^TU)^{-1}(I - V^TV)^{1/2} \\ &= (I - V^TV)^{1/2}(I - V^TU)(I - V^TU)^{-1}(I - V^TV)^{1/2} + \ldots \\ &\ldots (I - V^TV)^{1/2}V^TU(I - V^TU)^{-1}(I - V^TV)^{1/2} \\ &= (I - V^TV)^{1/2}(I - V^TU)^{-1}(I - V^TV)^{1/2} \end{aligned}$$

So

$$(I - V^TF)^{-1} = (1 - V^TV)^{-1/2}(I - V^TU)(I - V^TV)^{-1/2}.$$

We use this and elementary calculations to obtain

$$\begin{aligned} f(F, V) &= V - (I - VV^T)^{1/2}F(I - V^TF)^{-1}(I - V^TV)^{1/2} \\ &= V - (I - VV^T)^{1/2}F(I - V^TV)^{-1/2}(I - V^TU) \\ &= V - (I - VV^T)^{1/2}V(I - V^TV)^{-1/2}(I - V^TU) + (I - VV^T)U \\ &= V - V(I - V^TU) + U - VV^TU = U \end{aligned}$$

3. Notice that

$$\begin{aligned} f(V,V) &= V - (I - VV^T)^{1/2}V(I - V^TV)^{-1}(I - V^TV)^{1/2} \\ &= V - (I - VV^T)^{1/2}V(I - V^TV)^{-1/2} \\ &= V - V(I - V^TV)^{1/2}(I - V^TV)^{-1/2} = 0. \end{aligned}$$

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3.4 NC Full Isometries

Our classification of NC ball maps will be done in terms of NC full isometries. An NC full isometry is an NC analytic function in some NC variables $x = (x_1, \dots, x_{G_1})$ and some NC symmetric variables $y = (y_1, \dots, y_{G_2})$. An NC full isometry is like a complete isometry of C^*-algebras in that, for each fixed Y, we define a sequence of functions such that

- each map is linear in x
- each map has an isometric mapping property
- the dimension of the domain (for x) increases without bound as we move forward in the sequence.

We present a few classifications of NC full isometries. The first classification involves the form of an NC full isometry when a fixed value of Y is chosen. The second classification works for all values of y and says more about the structure of the series defining the NC full isometry. Both depend on previous classifications of complete isometries of C^*-algebras due to their similarities [Arv76], [Pau02].

3.4.1 NC Full Isometry Definition and Preliminaries

Suppose that

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \\ y_1 \\ \vdots \\ y_{G_2} \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} h_1(x,y) \\ \vdots \\ h_K(x,y) \end{pmatrix}$$

is an NC analytic map in $(x_1, \ldots, x_{G_1}, y_1, \ldots, y_{G_2})$ with $y_j = y_j^T$ and $h(x,y)$ is linear in x. Suppose that $h(X,Y)$ converges for all $Y \in S\mathcal{B}_\beta^{G_2}$ and all X which are real-valued matrix tuples of compatible dimension. Suppose that $N \in \mathbb{N}$ and $Y \in (S\mathbb{R}^{N\times N})^{G_2}$ is such that $h(X,Y)$ converges for all $X \in (\mathbb{R}^{N\times N})^{G_1}$. For $n \in \mathbb{N}$ define the map

$$h_n(\cdot, Y) : \mathbb{R}^{nNG_1\times nN} \to \mathbb{R}^{nNK\times nN} \tag{3.33}$$

so that if

$$\overline{X} = \begin{pmatrix} X(1,1) & \ldots & X(1,n) \\ \vdots & \ddots & \vdots \\ X(n,1) & \ldots & X(n,n) \end{pmatrix} \tag{3.34}$$

where $X(i,j) \in (\mathbb{R}^{N\times N})^{G_1}$ for all $1 \leq i,j \leq n$, then

$$h_n(\overline{X}, Y) := \begin{pmatrix} h(X(1,1),Y) & \ldots & h(X(1,n),Y) \\ \vdots & \ddots & \vdots \\ h(X(n,1),Y) & \ldots & h(X(n,n),Y) \end{pmatrix}. \tag{3.35}$$

If for all $N \in \mathbb{N}$ and for all $Y \in S\mathcal{B}_\beta^{G_2} \cap (S\mathbb{R}^{N\times N})^{G_2}$

$$\|h_n(\overline{X}, Y)\| = \|\overline{X}\| \tag{3.36}$$

for all $\overline{X} \in \mathbb{R}^{nNG_1\times nN}$ we say that $h(x,y)$ is an **NC n-full isometry**. If $h(x,y)$ is an NC n-full isometry for all $n \in \mathbb{N}$ then $h(x,y)$ is an **NC full isometry**.

We can classify NC full isometries without any dependence on symmetric y very easily. Notice that, with no dependence on symmetric y, an NC full isometry is at least a linear isometry from $\mathbb{R}^{G_1}$ to $\mathbb{R}^K$. Due to the series expansion, NC full isometries with no dependence on symmetric y are simply represented by linear isometries mapping $\mathbb{R}^{G_1} \to \mathbb{R}^K$. Thus the classification of NC ball maps (no dependence on symmetric y) is simpler than the classification of NC mixed ball maps since it is accomplished using linear isometries of $\mathbb{R}^{G_1}$ to $\mathbb{R}^K$ rather than NC full isometries.

Notice the similarity between NC full isometries and complete isometries of C^*-algebras [Pau02]. Suppose A and B are C^*-algebras and $\phi : A \to B$ is a map. We can define a new map $\phi_n : A^{n\times n} \to B^{n\times n}$ by

$$\phi_n((a_{i,j})) = (\phi(a_{i,j}))$$

where $A^{n\times n}$ and $B^{n\times n}$ are $n\times n$ matrices with components from A or B respectively. A map ϕ is a **complete isometry** if $\|\phi_n\| = 1$ for all n. If $\sup_n \|\phi_n\|$ is finite, we say that ϕ is **completely bounded**. Also, there is a norm on the completely bounded maps between two C^*-algebras defined by $\|\phi\|_{cb} = \sup_n \|\phi_n\|$.

We will use these NC full isometries in our classification of NC bi-disc maps. In particular, all normalized NC bi-disc maps are NC full isometries as we shall see later in Theorem 3.6.1. As a concrete example of an NC full isometry consider

$$\begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} (\sin y)x_1 \\ (\cos y)x_1 \\ x_2 \end{pmatrix} \tag{3.37}$$

where $\sin y$ and $\cos y$ are defined by their series expansions about 0 and where $y^T = y$.

Here are two results that we need from the literature. We would like for these results to be stated for real C^*-algebras but they unfortunately have not been stated this way. However each result does carry over to an analogous result for real C^*-algebras. For examples of treatments of real C^*-algebras see [Goo82], [HMP06]. The first is a result from Paulsen's book (Theorem 8.4) [Pau02].

Theorem 3.4.1. *Let A be a C^*-algebra with unit, and let $\phi : A \to B(\mathcal{H})$ be a completely bounded map. Then there exists a Hilbert space $\mathcal{K}$, a $*$-homomorphism $\pi : A \to B(\mathcal{K})$, and bounded operators $V_i : \mathcal{H} \to \mathcal{K}$, $i = 1, 2$, with $\|\phi\|_{cb} = \|V_1\|\|V_2\|$*

such that

$$\phi(a) = V_1^* \pi(a) V_2$$

for all $a \in A$. *Moreover if* $\|\phi\|_{cb} = 1$, *then the* V_i*'s can be taken to be isometries.*

We also need a result classifying $*$-homomorphisms of a particular type from Arveson's book [Arv76].

Theorem 3.4.2. *Every representation of* $\mathcal{C}(\mathcal{H})$, *the compact operators on a Hilbert space* $\mathcal{H}$, *is equivalent to a multiple of the identity representation.*

Here we include a Lemma that shows that we can use real or complex numbers when defining an NC full isometry and each definition results in the same class of functions. Suppose that $h(x, y)$ is defined by a convergent series expansion as earlier. Suppose that $N \in \mathbb{N}$ and $Y \in (S\mathbb{R}^{N\times N})^{G_2}$ is such that $h(X, Y)$ converges for all $X \in (\mathbb{C}^{N\times N})^{G_1}$. For $n \in \mathbb{N}$ define the map

$$h_n(\cdot, Y) : \mathbb{C}^{nNG_1 \times nN} \to \mathbb{C}^{nNK\times nN} \tag{3.38}$$

so that if

$$\overline{X} = \begin{pmatrix} X(1,1) & \dots & X(1,n) \\ \vdots & \ddots & \vdots \\ X(n,1) & \dots & X(n,n) \end{pmatrix} \tag{3.39}$$

where $X(i, j) \in (\mathbb{C}^{N\times N})^{G_1}$ for all $1 \le i, j \le n$, then

$$h_n(\overline{X}, Y) := \begin{pmatrix} h(X(1,1),Y) & \dots & h(X(1,n),Y) \\ \vdots & \ddots & \vdots \\ h(X(n,1),Y) & \dots & h(X(n,n),Y) \end{pmatrix}. \tag{3.40}$$

If for all $N \in \mathbb{N}$ and for all $Y \in S\mathcal{B}_\beta^{G_2} \cap (S\mathbb{R}^{N\times N})^{G_2}$

$$\|h_n(\overline{X}, Y)\| = \|\overline{X}\| \tag{3.41}$$

for all $\overline{X} \in \mathbb{C}^{nNG_1\times nN}$ we say that $h(x, y)$ is an **NC** $\mathbb{C}, n$**-full isometry**. If $h(x, y)$ is an NC n-full isometry for all $n \in \mathbb{N}$ then $h(x, y)$ is an **NC** $\mathbb{C}$**-full isometry**. As a consequence of the next lemma, we will not need to distinguish between NC full isometries and NC $\mathbb{C}$-full isometries.

Lemma 3.4.3. *NC full isometries are NC $\mathbb{C}$-full isometries and vice versa.*

Proof. Clearly if h is an NC $\mathbb{C}$-full isometry, then h is an NC full isometry. Now suppose, in order to reach a contradiction, that h is not an NC $\mathbb{C}$-full isometry and h is an NC full isometry. That is, suppose there exists an n such that h is not an NC $\mathbb{C}, n$-full isometry. Recall that for any N there is an isomorphism

$$\pi : \mathbb{C}^{N\times N} \to \left\{ \left(\begin{pmatrix} a_{k,j} & b_{k,j} \\ -b_{k,j} & a_{k,j} \end{pmatrix} \right)_{k=1,j=1}^{N,N} : a_{k,j}, b_{k,j} \in \mathbb{R} \right\} \subset \mathbb{R}^{2N\times 2N}.$$

Suppose there exists a matrix $Z \in \mathbb{C}^{NG_1 n\times Nn}$ and $Y \in S\mathcal{B}_\beta^{G_2}$ such that

$$\|Z\| \neq \|h_n(Z, Y)\|.$$

But we quickly reach a contradiction when we note

$$\|Z\| = \left\| \begin{pmatrix} \pi(Z_1) \\ \vdots \\ \pi(Z_{G_1}) \end{pmatrix} \right\| = \left\| h_n \left(\begin{pmatrix} \pi(Z_1) \\ \vdots \\ \pi(Z_{G_1}) \end{pmatrix}, \begin{pmatrix} \pi(Y_1) \\ \vdots \\ \pi(Y_{G_2}) \end{pmatrix} \right) \right\| = \|h_n(Z, Y).\|$$

●●

We can now present a preliminary classification result of NC full isometries that follows from the above classification theorems of complete isometries. This is a first step towards our main classification of NC full isometries, see Theorem 3.4.9.

Lemma 3.4.4. *Suppose that*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \\ y_1 \\ \vdots \\ y_{G_2} \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} h_1(x, y) \\ \vdots \\ h_K(x, y) \end{pmatrix} \tag{3.42}$$

is an NC full isometry for $Y \in S\mathcal{B}_\delta^{G_2}$. Then for each $Y \in S\mathcal{B}_\delta^{G_2}(N)$ there exist

1. $M_1, \ldots, M_{G_1} \in \mathbb{N}$ *such that* $M_i \geq K$

2. *matrices* $U_{i,left} \in \mathbb{R}^{M_iN \times NK}$ *and* $U_{i,right} \in \mathbb{R}^{M_iN \times N}$ *such that* $U_{i,left}^T U_{i,left} = I$ *and* $U_{i,right}^T U_{i,right} = I$ *for each* $i = 1, \ldots, G_1$

such that

$$h(X, Y) = \sum_{i=1}^{G_1} U_{i,left}^T (I_{M_i} \otimes X_i) U_{i,right} \tag{3.43}$$

for all $X \in (\mathbb{R}^{N \times N})^{G_1}$.

Proof. Fix $Y \in S\mathcal{B}_\delta^{G_2}(N)$. Define for $i = 1, \ldots, G_1$ the linear map

$$\mathcal{J}_{i,Y}(X_i) := h\left(\begin{pmatrix} 0 \\ \vdots \\ X_i \\ \vdots \\ 0 \end{pmatrix}, Y\right). \tag{3.44}$$

Notice that we have

$$h(X, Y) = \sum_{i=1}^{G_1} \mathcal{J}_{i,Y}(X_i) \tag{3.45}$$

Define

$$\mathcal{P}_{i,Y}(X_i) := \begin{pmatrix} \mathcal{J}_{i,Y}(X_i) & 0 & \cdots & 0 \end{pmatrix}. \tag{3.46}$$

We see that each $\mathcal{P}_{i,Y} : \mathbb{R}^{N \times N} \to \mathbb{R}^{NK \times NK}$ is a complete isometry of real C^*-algebras and thus $\|\mathcal{P}_{i,Y}\|_{cb} = 1$. Notice also

$$\sum_{i=1}^{G_1} \mathcal{P}_{i,Y}(X_i) = \begin{pmatrix} h(X, Y) & 0 & \cdots & 0 \end{pmatrix}. \tag{3.47}$$

Now we use Theorem 3.4.1 on each $\mathcal{P}_{i,Y}(X_i)$; i.e. there exist $M_1, \ldots, M_{G_1} \geq K$, *-homomorphisms $\pi_1, \ldots, \pi_{G_1}$ such that

$$\pi_i : \mathbb{R}^{N \times N} \to \mathbb{R}^{M_iN \times M_iN}$$

and isometries $V_{i,1}, V_{i,2} \in \mathbb{R}^{M_iN \times KN}$ for $i = 1, \ldots, G_1$ such that

$$\sum_{i=1}^{G_1} \mathcal{P}_{i,Y}(X_i) = \sum_{i=1}^{G_1} V_{i,1}^T \pi_i(X_i) V_{i,2}. \tag{3.48}$$

By a theorem of Arveson, stated as Theorem 3.4.2, for each $i = 1, \dots, G_1$, there exists an isometry $W_i \in \mathbb{R}^{M_i N \times M_i N}$ such that

$$\pi_i(X_i) = W_i^T (I_{M_i} \otimes X_i) W_i. \tag{3.49}$$

Define

$$U_{i,j} := W_i V_{i,j} \in \mathbb{R}^{M_i N \times KN} \tag{3.50}$$

and notice that $U_{i,j}^T U_{i,j} = I_{KN}$. Now from Equation (3.48) we have

$$\begin{pmatrix} h(X,Y) & 0 & \dots & 0 \end{pmatrix} = \sum_{i=1}^{G_1} \mathcal{P}_{i,Y}(X_i) = \sum_{i=1}^{G_1} U_{i,1}^T (I_{M_i} \otimes X_i) U_{i,2}. \tag{3.51}$$

●●

Notice that not all maps of the form given by Equation 3.78 are NC full isometries. As an example consider the map

$$\begin{pmatrix} X \\ Y \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} = 0 \tag{3.52}$$

where $Y = Y^T$.

The next Lemma states that NC full isometries map isometries to isometries.

Lemma 3.4.5. *Suppose that $h(x, y)$ is an NC full isometry on $S\mathcal{B}_\beta^{G_2}$. If $X^*X = I$, then*

$$h(X,Y)^* h(X,Y) = I.$$

Proof. Suppose that $Y \in S\mathcal{B}_\beta^{G_2}(N)$and that $X \in (\mathbb{C}^{N \times N})^{G_1}$ is such that $X^*X = I$. Define $\{v_j\}_{j=1}^N \subseteq \mathbb{C}^{NG_1}$ to be the set of vectors satisfying

$$X = \begin{pmatrix} v_1 & \cdots & v_N \end{pmatrix}. \tag{3.53}$$

Notice that $\{v_j\}_{j=1}^N$ is an orthonormal set. Now define matrices $M_1, \dots, M_N \in (\mathbb{C}^{N \times N})^K$ such that

$$M_1 := h(\begin{pmatrix} v_1 & 0 & \cdots & 0 \end{pmatrix}, Y), M_2 := h(\begin{pmatrix} 0 & v_2 & 0 & \cdots & 0 \end{pmatrix}, Y), etc. \tag{3.54}$$

Notice that

$$h(X,Y) = \sum_{j=1}^{N} M_j \tag{3.55}$$

since h is linear in X and that $\|M_j\| = 1$.

For $N \geq 2$, define $\mathbb{D}^{N-1} \subseteq \mathbb{C}^{N-1}$ by

$$\mathbb{D}^{N-1} := \{(z_1, \ldots, z_{N-1}) : |z_j| \leq 1, j = 1, \ldots, N-1\}. \tag{3.56}$$

Define the map

$$F : \mathbb{D}^{N-1} \to (\mathbb{C}^{N\times N})^K \tag{3.57}$$

by

$$F(z) := \sum_{j=1}^{N-1} z_j M_j + M_N = h(\begin{pmatrix} z_1 v_1 & \ldots & z_{N-1} v_{N-1} & v_N \end{pmatrix}, Y). \tag{3.58}$$

Notice that for $z \in \mathbb{D}^{N-1}$

$$\| \begin{pmatrix} z_1 v_1 & \ldots & z_{N-1} v_{N-1} & v_N \end{pmatrix} \| = 1 \tag{3.59}$$

and so $\|F(z)\| = 1$ for all $z \in \mathbb{D}^{N-1}$.

Since $\|M_N\| = 1$ we may choose $u_N \in \mathbb{C}^N$ such that

$$\|u_N\| = 1 \quad \text{and} \quad \|M_N u_N\| = \|F(0)\| = \|M_N\| = 1.$$

Now note that

$$g(z) := F(z) u_N \tag{3.60}$$

obtains a maximum at 0 in the interior of $\mathbb{D}^{N-1}$ and is thus constant on $\mathbb{D}^{N-1}$ [Sha92]. In particular note that $M_j u_N = 0$ for $j \neq N$.

Similarly define unit vectors $u_1, \ldots, u_{N-1}$ such that $M_j u_k = 0$ for $j \neq k$ and such that $\|M_j u_j\| = 1$. We now show that $\{u_j\}_{j=1}^N$ is an orthogonal set. To see that $u_j^* u_k = 0$ for $j \neq k$ write $u_j = w_1 + w_2$ for $w_1 \in \text{nullspace}(M_j)$ and $w_2 \in \text{nullspace}(M_j)^\perp$. Then

$$\|u_j\|^2 = 1 = \|w_1\|^2 + \|w_2\|^2$$

and, in particular, $\|w_2\| \leq 1$. Also we have

$$\|M_j u_j\| = 1 = \|M_j w_2\| \leq \|M_j\| \cdot \|w_2\| = \|w_2\|. \tag{3.61}$$

Thus $\|w_2\| = 1$ and so $\|w_1\| = 0$. Thus $u_j = w_2 \in \text{nullspace}(M_j)^{\perp}$. Since for $k \neq j$, $M_j u_k = 0$ we have that $u_j \perp u_k$. Thus the matrix U defined by

$$U := \begin{pmatrix} u_1 & \dots & u_N \end{pmatrix} \tag{3.62}$$

is unitary.

Now notice that

$$h(X,Y)U = \begin{pmatrix} M_1 u_1 & \dots & M_N u_N \end{pmatrix}. \tag{3.63}$$

Suppose $j \neq k$. Then

$$h((0,\dots,u_j,\dots,u_k,\dots,0),Y) = M_j + M_k. \tag{3.64}$$

So since $\|(0,\dots,u_j,\dots,u_k,\dots,0)\| = 1$ then $\|M_j + M_k\| = 1$. Thus, to see that $M_j u_j \perp M_k u_k$, notice that

$$1 = \|(M_j + M_k)U\| = \| \begin{pmatrix} M_j u_j & M_k u_k \end{pmatrix} \| \tag{3.65}$$

since

$$M_j u_l = 0 = M_k u_l$$

when $l \neq j$ and $l \neq k$. Now

$$\begin{pmatrix} M_j u_j & M_k u_k \end{pmatrix}^* \begin{pmatrix} M_j u_j & M_k u_k \end{pmatrix} = \begin{pmatrix} 1 & (M_j u_j)^* M_k u_k \\ (M_k u_k)^* M_j u_j & 1 \end{pmatrix}. \tag{3.66}$$

Thus

$$\text{Eigs} \begin{pmatrix} 1 & (M_j u_j)^* M_k u_k \\ (M_k u_k)^* M_j u_j & 1 \end{pmatrix} = \{1 \pm |(M_j u_j)^* M_k u_K|^2\} \tag{3.67}$$

and

$$1 = \|(M_j + M_k)U\| = \sqrt{1 + |(M_j u_j)^* M_k u_k|^2}. \tag{3.68}$$

Thus $M_j u_j \perp M_k u_k$ for $j \neq k$ and so

$$U^* h(X,Y)^* h(X,Y)U = \begin{pmatrix} M_1 u_1 & \dots & M_N u_N \end{pmatrix}^* \begin{pmatrix} M_1 u_1 & \dots & M_N u_N \end{pmatrix} = I. \tag{3.69}$$

To complete the proof we note that

$$(h(X,Y))^* h(X,Y) = U(U^*(h(X,Y))^* h(X,Y)U)U^* = UU^* = I. \tag{3.70}$$

••

The following lemma will aid in proving the uniqueness of our later classifications.

Lemma 3.4.6. *Suppose nonzero matrices* $A, U \in \mathbb{R}^{NK \times NG_1}$ *and* $B, V \in \mathbb{R}^{N \times N}$ *satisfy*

$$AXB = UXV \tag{3.71}$$

for all $X \in (\mathbb{R}^{N \times N})^{G_1}$*. Then*

$$A = tU \quad \text{and} \quad B = \frac{V}{t}$$

for some scalar t*.*

Proof. Consider vectors

$$v_1 \in \mathbb{R}^{NK}, \quad v_2 \in \mathbb{R}^{NG_1}, \quad v_3, v_4 \in \mathbb{R}^N.$$

Take $X = v_2 v_3^T$. Then

$$v_1^T A v_2 v_3^T B v_4 = v_1^T U v_2 v_3^T V v_4.$$

So

$$\frac{v_1^T A v_2}{v_1^T U v_2} = \frac{v_3^T V v_4}{v_3^T B v_4} =: t.$$

●●

3.4.2 NC Full Isometry Classification

The following lemma gives our main NC full isometry classification for the one x variable case.

Lemma 3.4.7. *Suppose that*

$$\begin{pmatrix} x \\ y_1 \\ \vdots \\ y_{G_2} \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} h_1(x,y) \\ \vdots \\ h_K(x,y) \end{pmatrix}$$

Fix $Y \in S\mathcal{B}_\delta^{G_2}(N)$*. Then*

$$h(X,Y) = U_l^T X U_r$$

where U_r *an isometry on* $\mathbb{R}^N$ *and where* U_l^T *is an isometry from* $\mathbb{R}^N$ *to* $\mathbb{R}^{KN}$ *for all* $X \in (\mathbb{R}^{N\times N})^{G_1}$*. This representation is unique up to multiplication by some scalar,* α*. That is, if*

$$h(X,Y) = U_{l,1}^T X U_r^1 = U_{l,2}^T X U_r^2,$$

then

$$U_{l,1}^T = \alpha U_{l,2}^T \quad \textit{and} \quad \alpha U_r^1 = U_r^2.$$

Proof. Fix $Y \in S\mathcal{B}_\delta^{G_2}(N)$. Then by Lemma 3.4.4

$$h(X,Y) = \tilde{U}_{left}^T (I_M \otimes X) \tilde{U}_{right} \tag{3.72}$$

where $\tilde{U}_{left}$ in $\mathbb{R}^{MN\times KN}$, $\tilde{U}_{right} \in \mathbb{R}^{MN\times N}$ and

$$\tilde{U}_{left}^T \tilde{U}_{left} = I_{KN} \text{ and } \tilde{U}_{right}^T \tilde{U}_{right} = I_N.$$

To this point we only used that h is a "full contraction." Now we use the 1-full isometry property.

Define $\mathcal{R}_{right} := range\ \tilde{U}_{right}$ and $\mathcal{R}_{left} := range\ \tilde{U}_{left}$. Given $t = (t_1, \cdots, t_M)$ and $t_j \in \mathbb{R}$ the subspace $diag_M(t, \mathbb{R}^N)$ is defined by

$$diag_M(t, \mathbb{R}^N) := \{ \begin{pmatrix} t_1\mu \\ t_2\mu \\ \vdots \\ t_M\mu \end{pmatrix} : \mu \in \mathbb{R}^N \}.$$

Lemma 3.4.8. *Suppose that* $\mathcal{R}_{right}$ *and* $\tilde{U}_{right}$ *are as described above in the proof of Lemma 3.4.7.*

1. $\mathcal{R}_{right}$ *has the form* $\mathcal{R}_{right} = diag_M(t, \mathbb{R}^N)$ *for some* $t = (t_1, \ldots, t_M) \in \mathbb{R}^M$, $t_j \in \mathbb{R}$*, with* t *normalized to make* $\sum_j^M |t_j|^2 = 1$*.*

2. $\tilde{U}_{right}$ has the form

$$\tilde{U}_{right} = \begin{pmatrix} t_1 U_r \\ t_2 U_r \\ \vdots \\ t_M U_r \end{pmatrix} \tag{3.73}$$

with U_r an isometry in $\mathbb{R}^{N\times N}$ and each $t_j \in \mathbb{R}$, with $\sum_j |t_j|^2 = 1$.

Proof. Pick $\nu \in \mathcal{R}_{right}$ with $\|\nu\| = 1$ and decompose it compatibly with $I_M \otimes X$, namely, $\nu = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_M \end{pmatrix}$. Let ρ be the rank one partial isometry which takes a given (arbitrary) vector η in $\mathbb{R}^N$ to a given (arbitrary) vector, μ, of the same length. That is, $\|\rho\eta\| = \|\eta\|$ and $\rho\eta = \mu$. Observe that $\|(I_M \otimes \rho)\nu\|$ is less than $\|\nu\|$ unless $\nu = e(t,\eta)$ for some η in $\mathbb{R}^N$, with e defined by $e(t,\eta) = \begin{pmatrix} t_1\eta \\ \vdots \\ t_M\eta \end{pmatrix}$ and $t = (t_1, \ldots, t_M)$ where t_j are some scalars such that

$$t_1^2 + \ldots + t_M^2 = 1.$$

This follows because $\|\rho\nu_j\| < \|\nu_j\|$ unless ν_j is collinear with η. This implies that if we maximize $\|U_{left}^T(I_M \otimes \rho)\nu\|$ over norm 1 vectors ν in $\mathcal{R}_{right}$ we get $\|\tilde{U}_{left}^T(I_M \otimes \rho)\tilde{U}_{right}\| < 1$ unless $e(t,\eta) \in \mathcal{R}_{right}$. Since $h(\rho, Y)$ is a partial isometry,

$$1 = \|h(\rho, Y)\| = \|\tilde{U}_{left}^T(I_M \otimes \rho)\tilde{U}_{right}\|,$$

we have proved: *if $\eta \in \mathbb{R}^N$, then there is vector $t(\eta) \neq 0$ in $\mathbb{R}^M$ for which* $e(t(\eta), \eta) \in \mathcal{R}_{right}$. That is, $\mathcal{R}_{right}$ contains all scalar multiples of vectors of the form

$$e(t(\eta), \eta) = \begin{pmatrix} t_1(\eta)\ \eta \\ \vdots \\ t_M(\eta)\ \eta \end{pmatrix}, \tag{3.74}$$

with some nonzero $t_j(\eta)$, for each η.

Define $\mathcal{R}^j$ to be the range of the component $[\tilde{U}_{right}]_j$ of $\tilde{U}_{right}$. The sentence above implies that if η is in $\mathbb{R}^N$, then it is in $\mathcal{R}^j$ for some j; thus $\mathbb{R}^N$ is the union of the subspaces $\mathcal{R}^j,\ j = 1, \cdots, M$. This can only happen if one of the spaces is itself all of $\mathbb{R}^N$; wlog we take $\mathcal{R}^1 = \mathbb{R}^N$. Thus $[\tilde{U}_{right}]_1$ is invertible, denote its inverse by B.

Given $\eta \in \mathbb{R}^N$ there exists $w \in \mathbb{R}^N$ such that $\eta = [\tilde{U}_{right}]_1 w$ and

$$\begin{pmatrix} t_1(\eta)\ B\eta \\ t_2(\eta)\ B\eta \\ \vdots \\ t_M(\eta)\ B\eta \end{pmatrix} = (I_M \otimes B)\ e(t(\eta), \eta) = \begin{pmatrix} w \\ B[\tilde{U}_{right}]_2 w \\ \vdots \\ B[\tilde{U}_{right}]_M w \end{pmatrix}. \tag{3.75}$$

Here $t_1(\eta)$ is not zero for any $\eta \neq 0$, since $\mathcal{R}^1 = \mathbb{R}^N$. From which we get for each j

$$t_j(\eta)\ B\eta = t_1(\eta)\ B[\tilde{U}_{right}]_j\ B\eta$$

for all η. Equivalently,

$$\frac{t_j(\eta)}{t_1(\eta)}\ \eta = \ [\tilde{U}_{right}]_j\ B\eta.$$

Conspicuously, every vector is an eigenvector of the matrix $[\tilde{U}_{right}]_j\ B$. Thus $[\tilde{U}_{right}]_j\ B = t_j I_N$, so $[\tilde{U}_{right}]_j = t_j B^{-1} = t_j [\tilde{U}_{right}]_1$, where we emphasize that t is independent of η. Rescaling the t and B simultaneously gives us the desired representation (3.73). That U_r is unitary comes from U_{right} being unitary. ●●

To finish the proof of Lemma 3.4.7, notice

$$\begin{aligned} h(X,Y) &= \tilde{U}_{left}^T (I_M \otimes X) \begin{pmatrix} t_1 U_r \\ \vdots \\ t_M U_r \end{pmatrix} \\ &= \tilde{U}_{left}^T \begin{pmatrix} t_1 X U_r \\ \vdots \\ t_M X U_r \end{pmatrix} = \tilde{U}_{left}^T \begin{pmatrix} t_1 I_N \\ \vdots \\ t_M I_N \end{pmatrix} X U_r. \end{aligned} \tag{3.76}$$

So defining

$$U_l^T := \tilde{U}_{left}^T \begin{pmatrix} t_1 I_N \\ \vdots \\ t_M I_N \end{pmatrix}$$

gives the desired result.

The uniqueness of the representation follows directly from Lemma 3.4.6. ●●

The following theorem generalizes the previous lemma to the case of several x-variables and is our main classification result of this section.

Theorem 3.4.9. *Suppose that*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \\ y_1 \\ \vdots \\ y_{G_2} \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} h_1(x,y) \\ \vdots \\ h_K(x,y) \end{pmatrix} \tag{3.77}$$

is an NC full isometry for $Y \in S\mathcal{B}_\delta^{G_2}$. *Then for each* $Y \in S\mathcal{B}_\delta^{G_2}(N)$ *there exist matrices* $U_{i,left} \in \mathbb{R}^{N \times NK}$ *and* $U_r \in \mathbb{R}^{N \times N}$ *with the properties for each* $i = 1, \ldots, G_1$*:*

1. $U_{i,left}^T$ *is an isometry*
2. $U_{i,left}\, U_{j,left}^T = 0$ *if* $i \neq j$
3. $U_r\, U_r^T = U_r^T\, U_r = I_N$

producing the representation

$$h(X,Y) = \sum_{i=1}^{G_1} U_{i,left}^T X_i U_r = U_{left}^T X U_r \tag{3.78}$$

for all $X \in (\mathbb{R}^{N \times N})^{G_1}$ *and where*

$$U_{left}^T = [U_{1,left}^T \ \ldots \ U_{G_1,left}^T]. \tag{3.79}$$

This representation is unique in the following sense. If

$$h(X,Y) = U_{left,1}^T X U_r^1 = U_{left,2}^T X U_r^2, \tag{3.80}$$

then there exists a scalar, α*, such that*

$$U_{left,1}^T = \alpha U_{left,2}^T \quad \text{and} \quad \alpha U_r^1 = U_r^2.$$

Conversely, every such h *is a full isometry.*

Proof. First we prove the converse. Consider

$$h(X,Y)^T h(X,Y) = \sum_{i=1}^{G_1} (X_i U_r)^T U_{i,left} U_{i,left}^T X_i U_r$$

since $U_{j,left} U_{i,left}^T = 0$ when $i \neq j$. Since $U_{i,left}^T$ is an isometry on $\mathbb{R}^{NM_i}$ is

$$h(X,Y)^T h(X,Y) = U_r^T \left[\sum_{i=1}^{G_1} X_i^T X_i \right] U_r \tag{3.81}$$

From this we see that the inequality $\sum_{i=1}^{G_1} X_i^T X_i \leq I_N$ (and its binding case) implies the same for $h(X,Y)^T h(X,Y)$. Thus

$$\|h(X,Y)\| = \|X\|.$$

This proves the converse direction.

To prove the forward direction, use Lemma 3.4.7 to obtain for h the representation for a fixed $Y \in S\mathcal{B}_\delta^{G_2}(N)$ for $N \geq 2$

$$h(X,Y) = \sum_{i=1}^{G_1} U_{i,left}^T X_i U_r^i \tag{3.82}$$

with $U_{i,left}^T$ and U_r^i both isometries. (Note that the $N = 1$ case is trivial.)

Set $X_i = s_i U_r^{iT}$ for each i, where s_i is a complex scalar. Then for any w in $\mathbb{R}^N$.

$$h(X,Y)w = \sum_{i=1}^{G_1} s_i U_{i,left}^T w \tag{3.83}$$

Thus if $\sum_{i=1}^{G_1} |s_i|^2 = 1$, we have

$$\sum_{i=1}^{G_1} X_i^* X_i = \sum_{i=1}^{G_1} |s_i|^2 I_N = I_N.$$

So by Lemma 3.4.5

$$\|w\|^2 = w^T h(X,Y)^* h(X,Y) w = \sum_{i,j=1}^{G_1} \overline{s_i} s_j \ w^T U_{i,left} U_{j,left}^T w.$$

Which, using that $U_{i,left}U_{i,left}^T = I_N$, says

$$\|w\|^2 = \sum_{i=1}^{G_1} |s_i|^2 \|w\|^2 + \sum_{i,j=1 \ and \ i \neq j}^{G_1} \overline{s_i} s_j \ w^T U_{i,left} U_{j,left}^T w.$$

So we have

$$\sum_{i,j=1 \ and \ i \neq j}^{G_1} \overline{s_i} s_j \ w^T U_{i,left} U_{j,left}^T w = 0.$$

Since this holds for all normalized s, we have proved $w^T U_{i,left} U_{j,left}^T w = 0$ for $i \neq j$, that is, $U_{i,left} U_{j,left}^T = 0$.

Next we analyze U_r^i. Calculate

$$h(X,Y)^T h(X,Y) = \sum_{i=1}^{G_1} (U_r^i)^T X_i^T X_i U_r^i \tag{3.84}$$

Suppose that $X_1 = P_1$ and $X_2 = P_2$ are projections such that $X_1 + X_2 = I$. Let $X_j = 0$ for all other j. Define

$$Q_1 = (U_r^1)^T P_1 U_r^1 \quad \text{and} \quad Q_2 = (U_r^2)^T P_2 U_r^2.$$

Then by Lemma 3.4.5

$$h(X,Y)^T h(X,Y) = Q_1 + Q_2 = I.$$

By definition, Q_1 and Q_2 are projections. Thus

$$Q_1 Q_2 = Q_1(I - Q_1) = 0. \tag{3.85}$$

Now the range of Q_1 is orthogonal to the range of Q_2 iff the range of P_1 is orthogonal to the range of $U_r^1 (U_r^2)^T P_2 U_r^2 (U_r^1)^T$. Thus

$$I - P_1 = P_2 = U_r^1 (U_r^2)^T P_2 U_r^2 (U_r^1)^T.$$

That is, for all projections P

$$P = U_r^1 (U_r^2)^T P U_r^2 (U_r^1)^T.$$

Suppose that A is a symmetric matrix. Then using the spectral decomposition of A we have that

$$A = U_r^1 (U_r^2)^T A U_r^2 (U_r^1)^T.$$

From this, we have that the isometric matrix $U_r^2(U_r^1)^T$ is a scalar multiple of the identity. Thus $U_r^1 = \alpha_2 U_r^2$ for some α_2 such that $|\alpha_2| = 1$.

Notice that the uniqueness here follows directly from Lemma 3.4.6.

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3.4.3 Algebraic NC Full Isometries

In the previous section we classified maps $h(\cdot, Y)$ on matrices for any substituted value Y. Now we go from matrix structure to algebraic structure. The following proposition gives a sufficient condition for an NC full isometry.

Proposition 3.4.10. *Suppose that*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \\ y_1 \\ \vdots \\ y_{G_2} \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} h_1(x,y) \\ \vdots \\ h_K(x,y) \end{pmatrix} \tag{3.86}$$

and that

$$h(x,y) = U_{left}^T(y) x U_{right}(y) \tag{3.87}$$

where $U_{left}^T(y)$ is a $K \times G_1$ matrix of series convergent on $S\mathcal{B}_\delta^{G_2}$ such that

$$U_{left}(Y) U_{left}^T(Y) = I$$

for all $Y \in S\mathcal{B}_\delta^{G_2}(N)$ and $U_{right}(y)$ is a series convergent on $S\mathcal{B}_\delta^{G_2}$ such that

$$U_{right}^T(Y) U_{right}(Y) = I$$

for all $Y \in S\mathcal{B}_\delta^{G_2}(N)$. Then $h(x,y)$ is an NC full isometry.

Proof. It suffices to show that $h(x,y)$ is an NC n-full isometry for each n. Suppose $Y \in S\mathcal{B}_\delta^{G_2}(N)$. Suppose

$$\overline{X} = \begin{pmatrix} X_{1,1} & \dots & X_{1,n} \\ \vdots & \ddots & \vdots \\ X_{n,1} & \dots & X_{n,n} \end{pmatrix} \tag{3.88}$$

where each $X_{i,j} \in (\mathbb{R}^{N\times N})^{G_1}$. Then

$$\begin{aligned}
h_n(\overline{X}, Y) &= \begin{pmatrix} h(X_{1,1}, Y) & \dots & h(X_{1,n}, Y) \\ \vdots & \ddots & \vdots \\ h(X_{n,1}, Y) & \dots & h(X_{n,n}, Y) \end{pmatrix} \\
&= \begin{pmatrix} U_{left}^T(Y) X_{1,1} U_{right}(Y) & \dots & U_{left}^T(Y) X_{1,n} U_{right}(Y) \\ \vdots & \ddots & \vdots \\ U_{left}^T(Y) X_{n,1} U_{right}(Y) & \dots & U_{left}^T(Y) X_{n,n} U_{right}(Y) \end{pmatrix} \\
&= (I_n \otimes U_{left}^T(Y)) \overline{X} (I_n \otimes U_{right}(Y)).
\end{aligned}$$

Thus

$$h_n^T(\overline{X}, Y) h_n(\overline{X}, Y) = (I_n \otimes U_{right}(Y))^T \overline{X}^T \overline{X} (I_n \otimes U_{right}(Y)).$$

Since

$$(I_n \otimes U_{right}(Y))^T (I_n \otimes U_{right}(Y)) = I$$

and

$$(I_n \otimes U_{right}(Y)) \in \mathbb{R}^{nN\times nN}$$

we have that

$$\|h_n(\overline{X}, Y)\| = \|\overline{X}\|.$$

••

We would like to say that the hypotheses of the previous result are necessary and sufficient for NC full isometries. However, we unfortunately have to add a few assumptions to prove the next result. It is still possible that the hypotheses of the previous result are necessary and sufficient for NC full isometries.

Let us revisit a concrete example of an NC full isometry and show that it can be written in the form of Equation 3.78 and in the form of Equation 3.87.. Consider

$$\begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} (\sin y) x_1 \\ (\cos y) x_1 \\ x_2 \end{pmatrix} \tag{3.89}$$

where $\sin y$ and $\cos y$ are defined by their series expansions about 0 and where $y^T = y$. Suppose that $Y \in S\mathbb{R}^{N\times N}$ and $X \in (\mathbb{R}^{N\times N})^2$. Notice that

$$
\begin{aligned}
h(X,Y) &= \begin{pmatrix} \sin Y & -\cos Y & 0 \\ \cos Y & \sin Y & 0 \\ 0 & 0 & I \end{pmatrix} (I_3 \otimes X_1) \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{pmatrix} (I_3 \otimes X_2) \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \sin Y & -\cos Y & 0 \\ \cos Y & \sin Y & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_1 & 0 \\ 0 & 0 & X_1 \end{pmatrix} \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{pmatrix} \begin{pmatrix} X_2 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_2 \end{pmatrix} \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \sin Y & 0 \\ \cos Y & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.
\end{aligned}
$$

Notice that this map $h(x,y)$ has been written in the form of Equation 3.78 and in the form of Equation 3.87.

Lemma 3.4.11. *Suppose that*

$$h(x,y) = \sum_{w \in \mathcal{W}_y} \left(a_{i,j}^w(y)\right)_{i,j=1}^{K,G_1} x y^w \tag{3.90}$$

is an NC full isometry. Here $\mathcal{W}_y$ is the set of all words in y and each $a_{i,j}^w(y)$ is a convergent series on $S\mathcal{B}_\beta^{G_2}$. Then

$$H(x,y) := A_\phi(y) := \left(a_{i,j}^\phi(y)\right)_{i,j=1}^{K,G_1} x$$

is an NC full isometry – here ϕ denotes the empty word.

Proof. Suppose

$$(X,Y) \in (\mathbb{R}^{N\times N})^{G_1} \times S\mathcal{B}_\beta^{G_2}(N)$$

and define

$$\widetilde{X}_j := \begin{pmatrix} 0 & 0 \\ X_j & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{Y}_k := \begin{pmatrix} 0 & 0 \\ 0 & Y_k \end{pmatrix}$$

for all $1 \leq j \leq G_1$ and $1 \leq k \leq G_2$. Notice that $\widetilde{X}_j \widetilde{Y}_k = 0$ for all j, k. So

$$h\left(\widetilde{X}, \widetilde{Y}\right) = \begin{pmatrix} 0 & 0 \\ H(X,Y) & 0 \end{pmatrix}.$$

Now notice

$$\|X\| = \left\|\widetilde{X}\right\|$$

by construction,

$$\left\|\widetilde{X}\right\| = \left\|h\left(\widetilde{X}, \widetilde{Y}\right)\right\|$$

since $h(x,y)$ is an NC full isometry in x, and

$$\left\|h(\widetilde{X}, \widetilde{Y})\right\| = \|H(X,Y)\|.$$

Thus $\|H(X,Y)\| = \|X\|$ and $H(x,y)$ is an NC 1-full isometry in x.

Let $n \in \mathbb{N}$. Suppose $Y \in S\mathcal{B}^{G_2}(N)$. Suppose

$$\overline{X} = \begin{pmatrix} X(1,1) & \dots & X(1,n) \\ \vdots & \ddots & \vdots \\ X(n,1) & \dots & X(n,n) \end{pmatrix} \in \mathbb{R}^{nNG_1 \times nN}.$$

Define

$$\widetilde{\overline{X}} := \left(\left(\widetilde{X}_k(i,j)\right)_{k=1}^{G_1}\right)_{i,j=1}^{n,n} := \left(\left(\begin{pmatrix} 0 & 0 \\ X_k(i,j) & 0 \end{pmatrix}\right)_{k=1}^{G_1}\right)_{i,j=1}^{n,n}$$

and

$$\widetilde{Y}_j := \begin{pmatrix} 0 & 0 \\ 0 & Y_j \end{pmatrix}.$$

Then

$$h_n\left(\widetilde{\overline{X}}, \widetilde{Y}\right) = \left(h\left(\widetilde{X}(i,j), \widetilde{Y}\right)\right)_{i,j=1}^{n,n} = \left(\begin{pmatrix} 0 & 0 \\ H(X(i,j),Y) & 0 \end{pmatrix}\right)_{i,j=1}^{n,n}.$$

Thus

$$\left\|h_n\left(\widetilde{\overline{X}}, \widetilde{Y}\right)\right\| = \left\|H_n\left(\overline{X}, Y\right)\right\|.$$

Noting that

$$\left\|h_n\left(\widetilde{\overline{X}}, \widetilde{Y}\right)\right\| = \left\|\widetilde{\overline{X}}\right\| = \left\|\overline{X}\right\|$$

since $h_n(x,y)$ is an NC n-full isometry in x, we have that

$$\left\| H_n\left(\overline{X}, Y\right)\right\| = \left\|\overline{X}\right\|.$$

Therefore $H(x,y)$ is an NC full isometry in x.

Alternatively, we could prove the lemma by simply showing that $A_\phi^T(Y)A_\phi(Y) = I$ for each $Y \in S\mathcal{B}_\delta^{G_2}$ and therefore $H(x,y) = A_\phi(y)x$ is an NC full isometry in x. ●●

To prove a more general result about the series structure of NC full isometries, we will need a few definitions. Recall we consider $h(x,y)$ which are power series in x, y, and are full isometries, thus linear in x. We can express h as

$$h(x,y) = \sum_{i,j}^{J} a_j^i(y) x_i b_j^i(y) \tag{3.91}$$

where $J = \infty$. In the case where $J < \infty$ and the functions $a_j^i(y)$ and $b_j^i(y)$ are given by NC rational functions, we say that h has an **NC mixed finite rational expansion.**

Let $\mathcal{H}$ be an infinite dimensional Hilbert space. We will use $\mathcal{L}(\mathcal{H})$ to denote bounded linear operators on $\mathcal{H}$ and $S\mathcal{L}(\mathcal{H})$ to denote self-adjoint bounded linear operators. Define $M\mathcal{L}(\mathcal{H})_\delta^{G_1,G_2}$ to be the set of all tuples (X,Y) such that $X_i \in \mathcal{L}(\mathcal{H})$ for $1 \leq i \leq G_1$, $Y_j \in S\mathcal{L}(\mathcal{H})$ for $1 \leq j \leq G_2$ and

$$\|(X,Y)\| < \delta$$

where (X,Y) is considered an operator from $\mathcal{H}$ to $\bigoplus_{j=1}^{G_1+G_2} \mathcal{H}$ and the norm is the corresponding operator norm.

Now we will define a condition similar to the geometric mapping condition on NC full isometries. Suppose that for G_2-tuples of symmetric operators,Y, on $\mathcal{H}$ with norm less than β, $h(X,Y)$ converges absolutely, is linear in x, and that

$$\|h(X,Y)\| = \|X\|$$

for all G_1-tuples of bounded operators X. Then we will say h is a **mixed $\mathcal{H}$-space isometry**.

It may be the case that assuming that h is an NC full isometry is equivalent to h being a mixed $\mathcal{H}$-space isometry. A similar condition has been shown for the noncommutative Schur-Agler class [AKV06] and may be true for this class of functions.

We will present a few lemmas before the main result of the section.

Lemma 3.4.12. *Suppose that*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \\ y_1 \\ \vdots \\ y_{G_2} \end{pmatrix} \mapsto g(x,y) := \sum_w g_w(x,y)^w \tag{3.92}$$

for some $g_w \in \mathbb{R}^K$ *and that* $g(X,Y)$ *converges absolutely for* $(X,Y) \in M\mathcal{L}(\mathcal{H})_\delta^{G_1,G_2}$. *Suppose that* $\{(X^n,Y^n)\}_n$ *is a sequence in* $M\mathcal{L}(\mathcal{H})_\delta^{G_1,G_2}$ *converging in the strong operator topology to* $(X^0,Y^0) \in M\mathcal{L}(\mathcal{H})_\delta^{G_1,G_2}$. *Then*

$$g(X^n,Y^n) \to g(X^0,Y^0)$$

in the strong operator topology. Also, if $\{(X^n,Y^n)\}_n$ *is a sequence in* $M\mathcal{L}(\mathcal{H})_\delta^{G_1,G_2}$ *converging in the operator topology to* $(X^0,Y^0) \in M\mathcal{L}(\mathcal{H})_\delta^{G_1,G_2}$, *then*

$$g(X^n,Y^n) \to g(X^0,Y^0)$$

in the operator topology.

Proof. Since finite sums and products of strongly convergent sequences converge in the strong operator topology, the result holds for polynomials. Let $\epsilon > 0$. Now suppose that $v \in \mathcal{H}$. Then

$$\|\sum_w (g_w(X^n,Y^n)^w - g_w(X^0,Y^0)^w)v\| \leq \sum_w \|(g_w(X^n,Y^n)^w - g_w(X^0,Y^0)^w)v\|. \tag{3.93}$$

There exists M such that

$$\sum_{|w|>M} \|(g_w(X^n,Y^n)^w - g_w(X^0,Y^0)^w)v\| < \frac{\epsilon}{2}.$$

Let C_M be equal to the number of words less than or equal to length M. By SOT continuity of polynomials, there is an N such that for all w

$$\|(g_w(X^n, Y^n)^w - g_w(X^0, Y^0)^w)v\| < \frac{\epsilon}{2C_M}$$

for all $n > N$. Thus for $n > N$ we have

$$\|\sum_w (g_w(X^n, Y^n)^w - g_w(X^0, Y^0)^w)v\| < \epsilon.$$

The proof of the result in the operator topology case uses the same estimates as above. ●●

To prove the main result of this section, we need a result about series which commute with all other series. Namely, they are trivial.

Lemma 3.4.13. *Suppose that*

$$g(y) := \sum_{w \in \mathcal{W}_y} g_w y^w \tag{3.94}$$

where $y = (y_1, \ldots y_{G_2})$, $G_2 > 1$, *and* $y_j = y_j^T$. *Suppose that for* $Y_j \in S\mathcal{L}(\mathcal{H})$ *with* $\|Y\| < \alpha$ *the series* $g(Y)$ *converges absolutely. If there exists some* Y^0 *and* $\beta > 0$ *such that* $Y_j^0 \in S\mathcal{L}(\mathcal{H})$, $\|Y^0\| < \alpha$, *and*

$$Y_j g(Y) = g(Y) Y_j$$

for all j *and* $Y \in (S\mathcal{L}(\mathcal{H}))^{G_2}$ *such that* $\|Y - Y^0\| < \beta$ *and* $\|Y\| < \alpha$, *then* g *is constant.*

Proof. Suppose that v is a nonempty word such that $g_v \neq 0$. W.l.o.g. suppose that v ends in 2. So $v = (v', 2)$ for some word v'. Since

$$y_1 g(y) - g(y) y_1 = 0$$

on some dimension-free, open set, then by Lemma 2.3.1

$$\sum_w a_w y^w = y_1 g(y) - g(y) y_1 = 0$$

and $a_w = 0$ for all w. But notice that says

$$a_{(1,v)} = a_{(1,v',2)} = g_v = 0$$

and we reach a contradiction. ●●

Now follows the main result of this section on the algebraic form of NC full isometries. Notice that the extra hypotheses that h is a mixed $\mathcal{H}$-space isometry and that it has an NC mixed finite rational expansion may be weakened.

Theorem 3.4.14. *Suppose*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \\ y_1 \\ \vdots \\ y_{G_2} \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} h_1(x,y) \\ \vdots \\ h_K(x,y) \end{pmatrix}$$

is a mixed $\mathcal{H}$-space isometry for $Y \in (S\mathcal{L}(\mathcal{H}))^{G_2}$ with $\|Y\| < \beta$. Suppose that $h(x,y)$ has an NC mixed finite rational expansion. Then

$$h(x,y) = a(y)xb(y)$$

where $b(y)b(y)^* = I = b(y)^*b(y)$ and $a(y)a(y)^* = I$. Here $b(y)$ is a series in y while $a(y)$ is a $K \times G_1$ matrix of series in y.

Proof. We will prove the result in two cases based on the number of variables: either $G_1 = 1$ or $G_1 > 1$.

Case 1. Suppose $x = x_1$. In our notation h is

$$h(x,y) = \sum_j^J a_j(y)xb_j(y) \tag{3.95}$$

Suppose J is the smallest possible number of terms over all representations of the form (3.95) for h and the functions making up $a_j(y)$ and $b_j(y)$ are rational.

Lemma 3.4.15. *Let Y be a G_2-tuple of symmetric operators on $\mathcal{H}$. Then*

$$h(X,Y) = U_l(Y)XU_r(Y)$$

for some isometric operator on $\mathcal{H}$, $U_r(Y)$, and some isometry $U_l(Y)$ mapping $\mathcal{H}$ to $\mathcal{H}^K$.

Proof. Suppose that $Y^{(n)}$ are symmetric finite rank operators converging to Y in the SOT. Then by Lemma 3.4.12, $h(X, Y^{(n)})$ converge to $h(X, Y)$ in the SOT for each X. By Theorem 3.4.9, there exist $U_l(Y^{(n)})$ and $U_r(Y^{(n)})$ such that $h(X, Y^{(n)}) = U_l(Y^{(n)})XU_r(Y^{(n)})$. Let $u \in \mathcal{H}$ such that $\|u\| = 1$.

Notice it is not possible that

$$\lim_{n\to\infty} < u, U_r(Y^{(n)})v >= 0$$

for all v. Suppose that for all v we have

$$\lim_{n\to\infty} < u, U_r(Y^{(n)})v >= 0.$$

Then for $w \in \mathcal{H}$ such that $\|wu^*\| = 1$,

$$h(wu^*, Y)v = \lim_{n\to\infty} U_l(Y^{(n)})wu^*U_r(Y^{(n)})v = 0$$

for all v and thus $\|h(wu^T, Y)\| = 0$, a contradiction.

Choose $v \in \mathcal{H}$ such that

$$\lim_{n\to\infty} < u, U_r(Y^{(n)})v >\neq 0.$$

We will now restrict our attention to a subsequence for which $< u, U_r(Y^{(n)})v >$ converges to a number other than 0. That is,

$$\lim_{k\to\infty} < u, U_r(Y^{(n_k)})v >= c.$$

Suppose that for $w \in \mathcal{H}$ we define

$$L(w) := \frac{1}{c}h(wu^T, Y)v = \frac{1}{c}\lim_{k\to\infty} U_l(Y^{(n_k)})wu^TU_r(Y^{(n_k)})v.$$

Then

$$L(w) = U_l(Y)w$$

since

$$\lim_{k\to\infty} U_l(Y^{(n_k)})w\frac{u^TU_r(Y^{(n_k)})v}{c} = \lim_{k\to\infty} U_l(Y^{(n_k)})w = U_l(Y)w.$$

Notice that since each $U_l(Y^{(n_k)})$ is unitary, the limit, $U_l(Y)$, is also unitary.

By taking a subsequence of this subsequence and reworking the above argument for h^T, we see that there is a unitary $U_r(Y)$ such that

$$\lim_{j\to\infty} U_r(Y^{(n_{k_j})}) = U_r(Y).$$

In particular, we see that

$$\lim_{j\to\infty} h(X, Y^{(n_{k_j})}) = U_l(Y)XU_r(Y) = h(X,Y).$$

●●

Fix $Y \in (S\mathcal{L}(\mathcal{H}))^{G_2}$. Start by using Lemma 3.4.15 to obtain

$$\begin{aligned}&[\sum_i a_i(Y)Xb_i(Y)][\sum_j b_j(Y)^*X^*a_j(Y)^*] \\ &= h(X,Y)h(X,Y)^* = U_l(Y)XX^*U_l(Y)^*\end{aligned} \tag{3.96}$$

where $U_l(Y)$ is an isometry (whose dependence on Y may not be algebraic). We now require a few more lemmas.

Lemma 3.4.16. *For all vectors $w \in \mathcal{H}^K$*

$$U_l(Y)^*w \in span\{a_j(Y)^*w\}_{j=1}^J.$$

Proof. Suppose not for some $w \in \mathcal{H}^K$. Notice that $U_l(Y)^*w \neq 0$. Suppose that

$$\text{span}\{a_j(Y)^*w\}_{j=1}^J \neq \{0\}.$$

Then for any vectors $\mu_1, \mu_2 \in \mathcal{H}$, there exists an operator X^* such that

$$\mu_1 = X^*U_l(Y)^*w \quad \text{and} \quad \mu_2 = X^*a_j(Y)^*w$$

for each j where $a_j(Y)^*w \neq 0$. Now by Equation (3.96) we see that

$$< \sum_{j=1}^J b_j(Y)^*\mu_2, \sum_{j=1}^J b_j(Y)^*\mu_2 >= \|\mu_1\|^2.$$

We have reached a contradiction since this equation must hold for all vectors $\mu_1, \mu_2 \in \mathcal{H}$. Suppose that

$$\text{span}\{a_j(Y)w\}_{j=1}^J = \{0\}.$$

Then for any vector $\mu \in \mathcal{H}$, there exists an operator X such that

$$\mu = X^* U_l(Y)^* w.$$

Now by Equation (3.96) we see that

$$0 = \|\mu\|^2.$$

This is a contradiction. ●●

So for each $w \in \mathcal{H}^K$ there exist scalars $\lambda_j(Y, w)$ making

$$U_l(Y)^* w = \sum_{j=1}^{J} \lambda_j(Y, w) a_j(Y)^* w. \tag{3.97}$$

Next define the set

$$\mathfrak{N} := \{Y \in (S\mathcal{L}(\mathcal{H}))^{G_2} : \exists w \in \mathcal{H}^K \text{ s.t. } \{a_j(Y)^* w\}_{j=1}^{J} \text{ are linearly independent}\}. \tag{3.98}$$

Case a. Suppose $\mathfrak{N}$ is not empty and that the vectors $a_1(Y)^* w, \ldots, a_J(Y)^* w$ are linearly independent. Then for any set of vectors $\mu_j,\ j = 1, \cdots, J$ in $\mathcal{H}$ we can find an operator X^* which makes

$$\mu_1 = X^* a_1(Y)^* w, \cdots, \mu_J = X^* a_J(Y)^* w.$$

We now use Equation (3.96) to say

$$0 = w^* U_l(Y) X X^* U_l(Y)^* w - \sum_{i,j=1}^{J} w^* a_i(Y) X b_i(Y) b_j(Y)^* X * a_j(Y)^*$$

Now using Equation (3.97), we have

$$0 = \sum_{i,j=1}^{J} w^* a_i(Y) X \left[\overline{\lambda_i(Y, w)} \lambda_j(Y, w) I - b_i(Y) b_j(Y)^* \right] X^* a_j(Y)^* w.$$

So

$$\sum_{i,j=1}^{J} \mu_i^* [\overline{\lambda_i(Y, w)} \lambda_j(Y, w) I - b_i(Y) b_j(Y)^*] \mu_j = 0.$$

This implies for all $1 \leq i, j \leq J$

$$\overline{\lambda_i(Y, w)} \lambda_j(Y, w) I - b_i(Y) b_j(Y)^* = 0. \tag{3.99}$$

Lemma 3.4.17. *The set $\mathfrak{N}$ is open.*

Proof. Suppose that $Y \in \mathfrak{N}$. W.L.O.G. take $|w| = 1$. Suppose that $a > 0$ is such that

$$\min_{|\lambda|=1} \|\sum_{j=1}^{J} \lambda_j a_j(Y)^* w\| = a.$$

Since the a_j^* are continuous there exists $\alpha > 0$ such that

$$\|a_j(Y)^* - a_j(\tilde{Y})^*\| < \frac{a}{2} \text{ whenever } \|Y - \tilde{Y}\| < \alpha$$

for all $j = 1, \ldots, J$. Now let $\tilde{Y} \in (S\mathcal{L}(\mathcal{H}))^{G_2}$ such that $\|Y - \tilde{Y}\| < \alpha$ and $\|Y\| < \delta$. Let $\lambda \in \mathbb{R}^J$ such that $|\lambda| = 1$. Then

$$\|\sum_{j=1}^{J} \lambda_j a_j(\tilde{Y})^* w\| \geq \|\sum_{j=1}^{J} \lambda_j a_j(Y)^* w\| \geq a \tag{3.100}$$

or

$$\|\sum_{j=1}^{J} \lambda_j a_j(\tilde{Y})^* w\| < \|\sum_{j=1}^{J} \lambda_j a_j(Y)^* w\|. \tag{3.101}$$

Suppose that

$$\|\sum_{j=1}^{J} \lambda_j a_j(\tilde{Y})^* w\| < \|\sum_{j=1}^{J} \lambda_j a_j(Y)^* w\|. \tag{3.102}$$

Then

$$\|\sum_{j=1}^{J} \lambda_j a_j(Y)^* w\| - \|\sum_{j=1}^{J} \lambda_j a_j(\tilde{Y})^* w\| \leq \|\sum_{j=1}^{J} \lambda_j [a_j(Y)^* - a_j(\tilde{Y})^*] w\| \tag{3.103}$$

and

$$\|\sum_{j=1}^{J} \lambda_j a_j(Y)^* w\| - \sum_{j=1}^{J} |\lambda_j| \, \|[a_j(Y)^* - a_j(\tilde{Y})^*]\| \, |w| \leq \|\sum_{j=1}^{J} \lambda_j a_j(\tilde{Y})^* w\|. \tag{3.104}$$

Notice

$$0 < a - \frac{a}{2} \leq \|\sum_{j=1}^{J} \lambda_j a_j(Y)^* w\| - \sum_{j=1}^{J} |\lambda_j| \, \|[a_j(Y)^* - a_j(\tilde{Y})^*]\| \, |w|. \tag{3.105}$$

Thus for all $\lambda \in \mathbb{R}^J$ with $|\lambda| = 1$ we have that

$$0 < \|\sum_{j=1}^{J} \lambda_j a_j(\tilde{Y})^* w\|. \tag{3.106}$$

Thus $\mathfrak{N}_N$ is open. ●●

Thus the equation

$$\overline{\lambda_i(Y,w)}\lambda_j(Y,w)I = b_i(Y)b_j(Y)^*$$

implies the function given by $b_i(y)b_j(y)^*$ commutes with all series $f(Y)$ for all G_2-tuples of symmetric operators with norm less than β. Thus the function given by $b_i(y)b_j(y)^*$ is constant by Lemma 3.4.13. We now have

$$cb_j(y) = b_i(y)$$

for some constant c and rescaling gives the desired result.

Case b. Suppose that $\mathfrak{N}_N$ is empty. Then for $Y \in S\mathcal{B}_\beta^{G_2}(N)$ every w makes $a_1(Y)^*w, \cdots, a_J(Y)^*w$ linearly dependent. Then from [HMV06], we see that $a_1(y), \ldots, a_J(y)$ are linearly dependent. This contradicts the minimality of J.

This finishes the proof of Theorem 3.4.14 for the one variable case.

Case 2. Suppose that $x = (x_1, \ldots, x_{G_1})$ for $G_1 > 1$. Start by applying the one variable case of this result to each $h(x_j, y)$ to obtain

$$h(x,y) = \sum_i^J a^i(y)x_i b^i(y) \tag{3.107}$$

with $a^i(Y)a^i(Y)^* = I$ and $b^i(Y)^*b^i(Y) = I$ for each tuple $Y \in (S\mathcal{L}(\mathcal{H}))^{G_2}$. We need to show that

$$a^i(Y)a^j(Y)^* = 0 \quad \text{for } i \neq j \text{ and } \quad b^i(Y) = t_i b^1(Y)$$

where t_i are scalars.

Fix $Y \in (S\mathcal{L}(\mathcal{H}))^{G_2}$. Set $X_i = s_i b^i(Y)^*$ for each i, where s_i is a complex scalar. Then for any w in $\mathcal{H}$

$$h(X,Y)w = \sum_{i=1}^{G_1} s_i a^i(Y)w \tag{3.108}$$

Thus if $\sum_{i=1}^{G_1} |s_i|^2 = 1$, we have

$$\sum_{i=1}^{G_1} X_i^* X_i = \sum_{i=1}^{G_1} |s_i|^2 I_N = I_N.$$

Using Lemma 3.4.5, we can see that $h(X,Y)$ is an isometry. Thus

$$\|w\|^2 = < h(X,Y)w, h(X,Y)w > = \sum_{i,j=1}^{G_1} < s_i a^i(Y)w, s_j a^j(Y)w > .$$

Which, using that $a^i(Y)a^i(Y)^* = I$, says

$$\|w\|^2 = \sum_{i=1}^{G_1} s_i^2 \|w\|^2 + \sum_{i,j=1 \ and \ i\neq j}^{G_1} < s_i a^i(Y)w, s_j a^j(Y)w > .$$

We now have

$$\sum_{i,j=1}^{G_1} < s_i a^i(Y)w, s_j a^j(Y)w >= 0.$$

Since this holds for all normalized s, we have proved $< a^i(Y)w, a^j(Y)w >= 0$ for $i \neq j$, that is, $a^i(Y)a^j(Y)^* = 0$.

Next we analyze $b^i(Y)$. Calculate

$$h(X,Y)^* h(X,Y) = \sum_{i=1}^{G_1} b^i(Y)^* X_i^* X_i b^i(Y) \tag{3.109}$$

Suppose that $X_1 = P_1$ and $X_2 = P_2$ are projections such that $P_1 + P_2 = I$. Let $X_j = 0$ for all other j. Define

$$Q_1 = b^1(Y)^* P_1 b^1(Y) \quad \text{and} \quad Q_2 = b^2(Y)^* P_2 b^2(Y).$$

Then by Lemma 3.4.5

$$h(X,Y)^* h(X,Y) = Q_1 + Q_2 = I.$$

By definition, Q_1 and Q_2 are projections. Thus

$$Q_1 Q_2 = Q_1(I - Q_1) = 0. \tag{3.110}$$

Now the range of Q_1 is orthogonal to the range of Q_2 iff the range of P_1 is orthogonal to the range of $b^1(Y)b^2(Y)^* P_2 b^2(Y) b^1(Y)^*$. Thus

$$I - P_1 = P_2 = b^1(Y)b^2(Y)^* P_2 b^2(Y) b^1(Y)^*.$$

That is, for all projections P

$$P = b^1(Y)b^2(Y)^*Pb^2(Y)b^1(Y)^*.$$

Suppose that A is a symmetric operator on $\mathcal{H}$. Then using the spectral decomposition of A we have that

$$A = b^1(Y)b^2(Y)^*A\, b^2(Y)b^1(Y)^*.$$

From this, we have that the isometric operator $b^2(Y)b^1(Y)^*$ is a scalar multiple of the identity. Thus $\alpha_2 b^2(Y) = b^1(Y)$ for some α_2 such that $|\alpha_2| = 1$.

••

3.5 Analytically Composing Functions

At this point we will need to make a few assumptions which guarantee that certain compositions of NC analytic functions are again NC analytic. We will call this condition **analytically composing** and will refer to the analytically composing condition by number. Please refer to the list below for details on the assumptions. Here the map f refers to the linear fractional map defined in Equation (3.23.)

1. An NC analytic function $h(x, y)$ is **analytically composing - 1** if the map

$$(x, y) \mapsto f(h(x, y), h(0, y))$$

is NC analytic whenever h is an NC mixed analytic map from the (1,1) matrix bi-disc to the unit ball $\mathcal{B}_1^K$.

2. An NC analytic function $h(x, y)$ is **analytically composing - 2** if the map

$$(x, y) \mapsto f(h(x(1 - y^T y)^{-1/2}, y), g(y))$$

is NC analytic for every NC analytic g mapping $S\mathcal{B}_1^{G_2}$ to $\mathcal{B}_1^K$ and whenever h maps the $(1, 1)$ matrix bi-disc to the unit ball $\mathcal{B}_1^K$.

3. An NC analytic function $p(x, y)$ is **analytically composing - 3** if the map

$$(x, y) \mapsto f(p(x(1 - y^T y)^{1/2}, y), p(0, y))$$

is NC analytic when p is a unit mixed NC ball map.

4. An NC analytic function $p(x)$ is **analytically composing - 4** if the map

$$x \mapsto p\left(af\left(\frac{x}{a}, c\right)\right)$$

is NC analytic when p is an NC analytic map defined on $\mathcal{B}_a^{G_1}$ for some $a > 0$ and any constant c.

It is likely that as the subject progresses the assumptions will be removed or bypassed because of more sophisticated definitions.

3.6 NC Bi-disc Map Classification

In order to classify NC mixed ball maps we need to first understand the following class of maps called NC bi-disc maps. These maps take the matrix bi-disc to the ball and have a condition on mapping part of the boundary of the matrix bi-disc to the boundary of the ball. Noting this mapping condition, it may be more appropriate to name this class of maps "NC bi-disc to ball maps" but we will use the shorter name "NC bi-disc maps."

It will be shown later that we may view NC mixed ball maps as NC bi-disc maps due to a change of variables. We classify the NC bi-disc maps in terms of NC full isometries.

Suppose that

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \\ y_1 \\ \vdots \\ y_{G_2} \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} h_1(x, y) \\ \vdots \\ h_K(x, y) \end{pmatrix} \tag{3.111}$$

is an NC analytic map on $\mathcal{B}_1^{G_1} \boxtimes S\mathcal{B}_\beta^{G_2}$. We say that $h(x, y)$ is an **NC bi-disc map** when $h(x, y)$ has the following properties:

1. For all N and $Y \in S\mathcal{B}_\beta^{G_2}(N)$ the map

$$x \mapsto h(x, Y)$$

   maps $\mathcal{B}_1^{G_1}(N)$ into $\mathcal{B}_1^K(N)$.

2. For all N and $Y \in S\mathcal{B}_\beta^{G_2}(N)$ the map

$$x \mapsto h(x, Y)$$

   has the property that if $X \in \partial\mathcal{B}_1^{G_1}(N)$, then

$$\lim_{\delta \uparrow 1} h(\delta X, Y) \text{ exists and is in } \partial\mathcal{B}_1^K(N).$$

If $h(x, y)$ is an NC bi-disc map and $h(0, y) = 0$, then we say that $h(x, y)$ is a **normalized NC bi-disc map**.

As an example of such a normalized NC bi-disc map, consider

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{p}{\mapsto} \begin{pmatrix} \sin(y)x \\ \cos(y)x \end{pmatrix}$$

where $y = y^T$. Notice that

$$\|p(X, Y)\| = \|X\|$$

for all pairs (X, Y) in the $(1, 1)$ matrix bi-disc. Another example is the class of NC ball maps sending 0 to 0.

For an NC bi-disc map

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \\ y_1 \\ \vdots \\ y_{G_2} \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} h_1(x, y) \\ \vdots \\ h_K(x, y) \end{pmatrix} \tag{3.112}$$

and each $n \geq 0$, define K-vectors whose entries are series homogeneous of degree n in x, $h^n(x, y)$ such that

$$h(x, y) =: \sum_{n=0}^{\infty} h^n(x, y). \tag{3.113}$$

Theorem 3.6.1. *1. Suppose that*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \\ y_1 \\ \vdots \\ y_{G_2} \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} h_1(x, y) \\ \vdots \\ h_K(x, y) \end{pmatrix} \tag{3.114}$$

is an NC bi-disc map defined on the $(1, \beta)$ *matrix bi-disc. Suppose that* $h(x, y)$ *is analytically composing - 1. Define the map*

$$H(x, y) = f(h(x, y), h(0, y)) \tag{3.115}$$

where f *is defined in Equation (3.23). Then* $H(x, y)$ *is a normalized NC bi-disc map.*

2. Suppose $h(x, y)$ *is a normalized NC bi-disc map defined on the* $(1, \beta)$ *matrix bi-disc. Then*

(a) $h(x, y)$ *is homogeneous degree 1 in* x*, that is* $h(x, y) = h^1(x, y)$

(b) $h(x, y)$ *is an NC full isometry.*

In particular, normalized NC bi-disc maps are NC full isometries.

We will prove this theorem in Section 3.6.3

3.6.1 Definition of NC Complex Bi-disc Maps

We will now give an equivalent definition of NC bi-disc maps. Suppose that the map

$$(x_1, \ldots, x_{G_1}, y_1, \ldots, y_{G_2}) \overset{h}{\mapsto} (h_1, \ldots, h_K)$$

is NC analytic. We will call such a map h an NC $\mathbb{C}$-**bi-disc map** if the following properties hold for some $\beta > 0$:

1. For all N and $Y \in S\mathcal{B}_\beta^{G_2}(N)$ the map
$$x \mapsto h(x, Y)$$
maps $C\mathcal{B}_1^{G_1}(N)$ into $C\mathcal{B}_1^K(N)$.

2. For all N and $Y \in S\mathcal{B}_\beta^{G_2}(N)$ the map
$$x \mapsto h(x, Y)$$
has the property that if $X \in \partial C\mathcal{B}_1^{G_1}(N)$, then
$$\lim_{\delta \uparrow 1} h(\delta X, Y) \text{ exists and is in } \partial C\mathcal{B}_1^K.$$

Lemma 3.6.2. *NC bi-disc maps are NC* $\mathbb{C}$*-bi-disc maps and vice versa.*

Proof. Clearly if h is an NC $\mathbb{C}$-bi-disc map, then h is an NC bi-disc map. Now suppose, in order to reach a contradiction, that h is an NC bi-disc map and not an NC $\mathbb{C}$-bi-disc map. Define K-vectors of homogeneous degree n polynomials $h^n(x, y)$ so that
$$h(x, y) = \sum_{n=0}^{\infty} h^n(x, y).$$
Recall that for any N there is an isomorphism
$$\pi : \mathbb{C}^{N\times N} \to \left\{ \left(\begin{pmatrix} a_{k,j} & b_{k,j} \\ -b_{k,j} & a_{k,j} \end{pmatrix} \right)_{k=1,j=1}^{N,N} : a_{k,j}, b_{k,j} \in \mathbb{R} \right\} \subset \mathbb{R}^{2N\times 2N}.$$
So if $H \in \mathbb{C}^{N\times N}$ is Hermitian, then $H = U\Lambda U^*$ where U is unitary and $\Lambda =$diag$\{\lambda_1, \ldots, \lambda_N\}$. Thus $\pi(H) = \pi(U)\pi(\Lambda)\pi^T(U)$ and eigs$(H) =$eigs$(\pi(H))$.

Case 1. *Suppose that there exists some* $Z \in (\mathbb{C}^{N\times N})^{G_1}$ *and* $W \in (S\mathbb{R}^{N\times N})^{G_2}$ *such that* $\|Z\| < 1, \|W\| < 1$ *and* $\|h(Z, W)\| \geq 1$. Define
$$X := (\pi(Z_1), \ldots, \pi(Z_{G_1})) \text{ and } Y := (\pi(W_1), \ldots, \pi(W_{G_2}))$$

Then notice

$$X^T X = \pi(Z^* Z) \text{ and } Y^T Y = \pi(W^* W).$$

Thus

$$\|X\| = \|Z\| \text{ and } \|Y\| = \|W\|.$$

Now we would like to show that $\|h(X,Y)\| = \|h(Z,W)\|$. Let $\epsilon > 0$. Choose M so that if we define

$$g(x,y) := \begin{pmatrix} g_1(x,y) \\ \vdots \\ g_K(x,y) \end{pmatrix} := \sum_{j=0}^{M} h^j(x,y),$$

then $|\ \|h(X,Y)\| - \|g(X,Y)\|\ | < \epsilon/2$ and $|\ \|h(Z,W)\| - \|g(Z,W)\|\ | < \epsilon/2$. Notice that $g_i(X,Y) = \pi(g_i(Z,W))$ for all i. So

$$\sum_{i=1}^{K} g_i^T(X,Y) g_i(X,Y) = \pi\left(\sum_{i=1}^{K} g_i^*(Z,W) g_i(Z,W)\right).$$

Thus we have that

$$\|g(X,Y)\| = \|g(Z,W)\| \text{ and } |\ \|h(X,Y)\| - \|h(Z,W)\|\ | < \epsilon.$$

Thus $\|h(X,Y)\| = \|h(Z,W)\| \geq 1$. We now have a contradiction with the assumption that h is an NC bi-disc map.

Case 2. *Suppose that there exists some $Z \in (\mathbb{C}^{N\times N})^{G_1}$ and $W \in (S\mathbb{R}^{N\times N})^{G_2}$ such that $\|W\| < 1, \|Z\| = 1$ and $\lim_{\delta\uparrow 1} \|h(\delta Z, W)\| \neq 1$.* Suppose that $\delta < 1$ and define $X := (\pi(Z_1), \ldots, \pi(Z_{G_1}))$ and $Y = (\pi(W_1), \ldots, \pi(W_{G_2}))$. Then just as in Case 1 we have that $\|X\| = \|Z\| = 1, \|Y\| = \|W\|$ and $\|h(\delta X, Y)\| = \|h(\delta Z, W)\|$ for all $\delta < 1$. Thus $\lim_{\delta\uparrow 1} \|h(\delta X, Y)\| \neq 1$ and we reach a contradiction with the assumption that h is a bi-disc map.

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3.6.2 Lemmas for Proving Theorem 3.6.1

We will need a few lemmas to prove Theorem 3.6.1.

Lemma 3.6.3. *Suppose* $(X,Y) \in \mathcal{B}_1^{G_1} \boxtimes S\mathcal{B}_\beta^{G_2}$ *and* $X \neq 0$*. Define the pair* $(\overline{X},\overline{Y})$ *by*

$$\overline{X}_j := \begin{pmatrix} 0 & \frac{X_j}{\|X\|} \\ 0 & 0 \end{pmatrix} \tag{3.116}$$

and

$$\overline{Y}_j := \begin{pmatrix} Y_j & 0 \\ 0 & Y_j \end{pmatrix}. \tag{3.117}$$

Suppose that w *is a word in* (x,y) *with degree in* x *greater than or equal to 2. Then* $(\overline{X},\overline{Y})^w = 0$.

Proof. Suppose that w is a word in (x,y) with degree in x greater than or equal to 2. Write $w = w_1w_2$ where w_1 is degree 2 in x. Then

$$w_1 = \alpha_1 x_j \alpha_2 x_k \alpha_3$$

for some α_l words in y and some j,k. So then

$$(\overline{X},\overline{Y})^{w_1} = \overline{Y}^{\alpha_1}\overline{X}_j\overline{Y}^{\alpha_2}\overline{X}_k\overline{Y}^{\alpha_3} \tag{3.118}$$

$$= \begin{pmatrix} Y^{\alpha_1} & 0 \\ 0 & Y^{\alpha_1} \end{pmatrix}\begin{pmatrix} 0 & \frac{X_j}{\|X\|} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Y^{\alpha_2} & 0 \\ 0 & Y^{\alpha_2} \end{pmatrix}\begin{pmatrix} 0 & \frac{X_k}{\|X\|} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Y^{\alpha_3} & 0 \\ 0 & Y^{\alpha_3} \end{pmatrix} = 0. \tag{3.119}$$

Thus

$$(\overline{X},\overline{Y})^w = (\overline{X},\overline{Y})^{w_1}(\overline{X},\overline{Y})^{w_2} = 0. \tag{3.120}$$

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Lemma 3.6.4. *Suppose that* $h(x,y)$ *is a normalized NC bi-disc map on* $\mathcal{B}_1^{G_1} \boxtimes S\mathcal{B}_\beta^{G_2}$*.Then*

$$\|h^1(X,Y)\| = \|X\| \tag{3.121}$$

for all $(X,Y) \in \mathcal{B}_1^{G_1} \boxtimes S\mathcal{B}_\beta^{G_2}$*. In particular,* $h^1(x,y)$ *is a normalized NC bi-disc map on* $\mathcal{B}_1^{G_1} \boxtimes S\mathcal{B}_\beta^{G_2}$*.*

Proof. Suppose $(X,Y) \in \mathcal{B}_1^{G_1} \boxtimes S\mathcal{B}_\beta^{G_2}$ and $X \neq 0$. Now define the pair $(\overline{X}, \overline{Y})$ by

$$\overline{X}_j := \begin{pmatrix} 0 & \frac{X_j}{\|X\|} \\ 0 & 0 \end{pmatrix} \tag{3.122}$$

and

$$\overline{Y}_j := \begin{pmatrix} Y_j & 0 \\ 0 & Y_j \end{pmatrix}. \tag{3.123}$$

Notice that

$$\|\overline{X}\| = 1 \quad \text{and} \quad \|\overline{Y}\| = \|Y\|$$

and so $\|\lim_{\delta \uparrow 1} h(\delta\overline{X}, \overline{Y})\| = 1$. Also note that

$$h^n(\delta\overline{X}, \overline{Y}) = 0 \tag{3.124}$$

for $n \geq 2$ by Lemma 3.6.3. Thus

$$\lim_{\delta \uparrow 1} h(\delta\overline{X}, \overline{Y}) = \begin{pmatrix} \begin{pmatrix} 0 & \frac{h_1^1(X,Y)}{\|X\|} \\ 0 & 0 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} 0 & \frac{h_K^1(X,Y)}{\|X\|} \\ 0 & 0 \end{pmatrix} \end{pmatrix}. \tag{3.125}$$

Thus

$$\|h^1(X,Y)\| = \|X\|.$$

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Lemma 3.6.5. *Suppose that $h(x,y)$ is a normalized NC bi-disc map defined on $C\mathcal{B}_1^{G_1} \boxtimes S\mathcal{B}_\beta^{G_2}$, $N \in \mathbb{N}$, and that $Y \in S\mathcal{B}_\beta^{G_2}(N)$. If $X^*X = I$, then*

$$h^1(X,Y)^* h^1(X,Y) = I.$$

Proof. Suppose that $Y \in S\mathcal{B}_\beta^{G_2}(N)$and that $X \in (\mathbb{C}^{N\times N})^{G_1}$ is such that $X^*X = I$. Define $\{v_j\}_{j=1}^N \subseteq \mathbb{C}^{NG_1}$ to be the set of vectors satisfying

$$X = \begin{pmatrix} v_1 & \cdots & v_N \end{pmatrix}. \tag{3.126}$$

Notice that $\{v_j\}_{j=1}^N$ is an orthonormal set. Now define matrices $M_1, \ldots, M_N \in (\mathbb{C}^{N\times N})^K$ such that

$$M_1 := h^1(\begin{pmatrix} v_1 & 0 & \cdots & 0 \end{pmatrix}, Y), M_2 := h^1(\begin{pmatrix} 0 & v_2 & 0 & \cdots & 0 \end{pmatrix}, Y), etc. \tag{3.127}$$

Notice that

$$h^1(X,Y) = \sum_{j=1}^N M_j \tag{3.128}$$

since h^1 is linear in X and that $\|M_j\| = 1$ by Lemma 3.6.4.

For $N \geq 2$, define $\mathbb{D}^{N-1} \subseteq \mathbb{C}^{N-1}$ by

$$\mathbb{D}^{N-1} := \{(z_1, \ldots, z_{N-1}) : |z_j| \leq 1, j = 1, \ldots, N-1\}. \tag{3.129}$$

Define the map

$$F : \mathbb{D}^{N-1} \to (\mathbb{C}^{N\times N})^K \tag{3.130}$$

by

$$F(z) := \sum_{j=1}^{N-1} z_j M_j + M_N = h^1(\begin{pmatrix} z_1 v_1 & \ldots & z_{N-1}v_{N-1} & v_N \end{pmatrix}, Y). \tag{3.131}$$

Notice that for $z \in \mathbb{D}^{N-1}$

$$\| \begin{pmatrix} z_1 v_1 & \ldots & z_{N-1}v_{N-1} & v_N \end{pmatrix} \| = 1 \tag{3.132}$$

and so $\|F(z)\| = 1$ for all $z \in \mathbb{D}^{N-1}$ by Lemma 3.6.4. Since $\|M_N\| = 1$ we may choose $u_N \in \mathbb{C}^N$ such that

$$\|u_N\| = 1 \quad \text{and} \quad \|M_N u_N\| = \|F(0)\| = \|M_N\| = 1.$$

Now note that

$$g(z) := F(z)u_N \tag{3.133}$$

obtains a maximum at 0 in the interior of $\mathbb{D}^{N-1}$ and is thus constant on $\mathbb{D}^{N-1}$ [Sha92]. In particular note that $M_j u_N = 0$ for $j \neq N$.

Similarly define unit vectors $u_1, \ldots, u_{N-1}$ such that $M_j u_k = 0$ for $j \neq k$ and such that $\|M_j u_j\| = 1$. We now show that $\{u_j\}_{j=1}^N$ is an orthogonal set. To see that $u_j^* u_k = 0$ for $j \neq k$ write $u_j = w_1 + w_2$ for $w_1 \in \text{nullspace}(M_j)$ and $w_2 \in \text{nullspace}(M_j)^\perp$. Then

$$\|u_j\|^2 = 1 = \|w_1\|^2 + \|w_2\|^2$$

and, in particular, $\|w_2\| \leq 1$. Also we have

$$\|M_j u_j\| = 1 = \|M_j w_2\| \leq \|M_j\| \cdot \|w_2\| = \|w_2\|. \tag{3.134}$$

Thus $\|w_2\| = 1$ and so $\|w_1\| = 0$. Thus $u_j = w_2 \in \text{nullspace}(M_j)^\perp$. Since for $k \neq j$, $M_j u_k = 0$ we have that $u_j \perp u_k$. Thus the matrix U defined by

$$U := \begin{pmatrix} u_1 & \ldots & u_N \end{pmatrix} \tag{3.135}$$

is unitary.

Now notice that

$$h^1(X,Y)U = \begin{pmatrix} M_1 u_1 & \ldots & M_N u_N \end{pmatrix}. \tag{3.136}$$

Suppose $j \neq k$. Then

$$h^1((0,\ldots,u_j,\ldots,u_k,\ldots,0),Y) = M_j + M_k. \tag{3.137}$$

So since $\|(0,\ldots,u_j,\ldots,u_k,\ldots,0)\| = 1$ then $\|M_j + M_k\| = 1$. Thus, to see that $M_j u_j \perp M_k u_k$, notice that

$$1 = \|(M_j + M_k)U\| = \| \begin{pmatrix} M_j u_j & M_k u_k \end{pmatrix} \| \tag{3.138}$$

since

$$M_j u_l = 0 = M_k u_l$$

when $l \neq j$ and $l \neq k$. Now

$$\begin{pmatrix} M_j u_j & M_k u_k \end{pmatrix}^* \begin{pmatrix} M_j u_j & M_k u_k \end{pmatrix} = \begin{pmatrix} 1 & (M_j u_j)^* M_k u_k \\ (M_k u_k)^* M_j u_j & 1 \end{pmatrix}. \tag{3.139}$$

Thus

$$\text{Eigs} \begin{pmatrix} 1 & (M_j u_j)^* M_k u_k \\ (M_k u_k)^* M_j u_j & 1 \end{pmatrix} = \{1 \pm |(M_j u_j)^* M_k u_K|^2\} \tag{3.140}$$

and

$$1 = \|(M_j + M_k)U\| = \sqrt{1 + |(M_j u_j)^* M_k u_k|^2}. \tag{3.141}$$

Thus $M_j u_j \perp M_k u_k$ for $j \neq k$ and so

$$U^* h^1(X,Y)^* h^1(X,Y) U = \begin{pmatrix} M_1 u_1 & \dots & M_N u_N \end{pmatrix}^* \begin{pmatrix} M_1 u_1 & \dots & M_N u_N \end{pmatrix} = I. \tag{3.142}$$

To complete the proof we note that

$$h^1(X,Y)^* h^1(X,Y) = U(U^* h^1(X,Y)^* h^1(X,Y) U)U^* = UU^* = I. \tag{3.143}$$

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3.6.3 Proof of NC Bi-disc Map Classification: Theorem 3.6.1

The proof of this result uses a technique found in [Phi68].

The analyticity of H follows from the assumption that h is analytically composing - 1. From the third part of Lemma 3.3.1, we see that $H(0,y) = 0$.

Now we fix $N \in \mathbb{N}$ and $Y \in S\mathcal{B}_\beta^{G_2}(N)$ and consider the map

$$X \mapsto H(X,Y) \tag{3.144}$$

for $X \in \mathcal{B}_1^{G_1}(N)$. By Lemma 3.3.1, we see that if $\|X\| < 1$, then $\|H(X,Y)\| < 1$. Similarly by Lemma 3.3.1, if $\|X\| = 1$, then $\lim_{\delta \uparrow 1} \|H(\delta X, Y)\| = 1$. Thus H is a normalized NC bi-disc map.

Suppose that

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \\ y_1 \\ \vdots \\ y_{G_2} \end{pmatrix} \overset{h}{\mapsto} \begin{pmatrix} h_1(x,y) \\ \vdots \\ h_K(x,y) \end{pmatrix} \tag{3.145}$$

is a normalized NC bi-disc map.

1. *We will show that h is homogeneous degree one in x.* Fix $N \geq 1$. Let $X \in (\mathbb{R}^{N\times N})^{G_1}$ and $Y \in S\mathcal{B}_\beta^{G_2}(N)$ so that $\|X\| = 1$. Then for all $0 < \delta < 1$ and $0 \leq \theta \leq 2\pi$ we have

$$0 \leq I - h^*(\delta e^{i\theta}X, Y)h(\delta e^{i\theta}X, Y).$$

Here we are using Lemma 3.6.2; the equivalence of NC bi-disc maps and NC $\mathbb{C}$-bi-disc maps. So for each $0 < \delta < 1$,

$$\begin{aligned} 0 &\leq \int_0^{2\pi} (I - h^*(\delta e^{i\theta}X, Y)h(\delta e^{i\theta}X, Y))d\theta \\ &= I - \delta^2 h^1(X,Y)^T h^1(X,Y) - \sum_{n=2}^{\infty} h^n(\delta X, Y)^T h^n(\delta X, Y). \end{aligned}$$

Case 1. *Suppose that* $X^TX = I$. Then we have that

$$0 \leq \lim_{\delta\uparrow 1} (I - \delta^2 h^1(X,Y)^T h^1(X,Y) - \sum_{n=2}^{\infty} h^n(\delta X, Y)^T h^n(\delta X, Y)).$$

By Lemma 3.4.5

$$h^1(X,Y)^T h^1(X,Y) = I \tag{3.146}$$

and so

$$0 \leq -\lim_{\delta\uparrow 1} \sum_{n=2}^{\infty} h^n(\delta X, Y)^T h^n(\delta X, Y).$$

So $\lim_{\delta\uparrow 1} h^n(\delta X, Y) = 0$ for all $n > 1$ when X is an isometry.

Case 2. *Suppose that* $X^TX \neq I$. Write X in polar form so that $X = ZJ$ where $Z^*Z = I$ and $J \geq 0$. Then by the Spectral Theorem we may write for some M that

$$J = c_1E_1 + \ldots + c_ME_M$$

where E_i is a projection for all i, $\sum_{i=1}^{M} E_i = I$, the numbers c_i are the eigenvalues of J (and thus the c_i are the eigenvalues of X), and $E_lE_i = 0$ for all $i \neq l$. So defining $W_i := ZE_i$ for $1 \leq i \leq M$ we have $X = \sum_{i=1}^{M} c_iW_i$ where $\sum_i W_i^*W_i = I$. Define $\phi_n : \mathbb{C}^M \to \mathbb{C}^{NK\times N}$ by

$$\phi_n(z_1, \ldots, z_M) = \lim_{\delta\uparrow 1} h^n(\sum_{i=1}^{M} \delta z_i W_i, Y) \text{ for each } n \geq 2.$$

Notice that ϕ_n is analytic in the poly-disk,

$$D = \{z \in \mathbb{C}^M : |z_i| \leq 1, \forall i\}.$$

Suppose that $z \in \partial D$ and $|z_j| = 1$ for all j, then

$$(\sum_{i=1}^{M} z_i W_i)^* \sum_{i=1}^{M} z_i W_i = I.$$

As in Case 1, we have that

$$0 \leq -\lim_{\delta \uparrow 1} \sum_{n=2}^{\infty} h^n \left(\sum_{i=1}^{M} \delta z_i W_i, Y \right)^* h^n \left(\sum_{i=1}^{M} \delta z_i W_i, Y \right) = -\sum_{n=2}^{\infty} \phi_n^*(z)\phi_\alpha(z).$$

Thus $\phi_n(z) \equiv 0$ on the skeleton of D for all $n > 1$. So $\phi_n \equiv 0$ in D [Sha92]. Since $c = (c_1, \ldots, c_M) \in D$ we have that $0 = \phi_n(c) = h^n(X, Y)$ for all $n > 1$.

By the two cases above we have that $h^n(x, y) = 0$ for all $n > 1$.

2. *We will now show that h is an NC full isometry.* Suppose $n, N \in \mathbb{N}$ and $Y \in S\mathcal{B}_\beta^{G_2}(N)$. Then for all $X \in (\mathbb{C}^{N \times N})^{G_1}$ the series $h(X, Y)$ is absolutely convergent. Now write

$$h(X, Y) = \sum_{k=1}^{\infty} A_k(Y) X b_k(Y) \tag{3.147}$$

for some block $K \times G_1$ matrices $A_k(Y)$ whose entries are absolutely convergent series in Y and for some absolutely convergent series in Y, $b_k(Y)$. Now suppose that

$$\overline{X} = \begin{pmatrix} X(1,1) & \ldots & X(1,n) \\ \vdots & \ddots & \vdots \\ X(n,1) & \ldots & X(n,n) \end{pmatrix} \in \mathbb{C}^{nNG_1 \times nN}. \tag{3.148}$$

Define

$$\overline{Y} = \begin{pmatrix} I_n \otimes Y_1 \\ \vdots \\ I_n \otimes Y_{G_2} \end{pmatrix} \tag{3.149}$$

and notice that for all k

$$A_k(\overline{Y}) = I_n \otimes A_k(Y) \text{ and } b_k(\overline{Y}) = I_n \otimes b_k(Y) \tag{3.150}$$

Then

$$\begin{aligned} h_n(\overline{X}, Y) &= \begin{pmatrix} h(X(1,1),Y) & \dots & h(X(1,n),Y) \\ \vdots & \ddots & \vdots \\ h(X(n,1),Y) & \dots & h(X(n,n),Y) \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=1}^{\infty} A_k(Y)X(1,1)b_k(Y) & \dots & \sum_{k=1}^{\infty} A_k(Y)X(1,n)b_k(Y) \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{\infty} A_k(Y)X(n,1)b_k(Y) & \dots & \sum_{k=1}^{\infty} A_k(Y)X(n,n)b_k(Y) \end{pmatrix} \qquad (3.151) \\ &= \sum_{k=1}^{\infty} (I_n \otimes A_k(Y))\overline{X}(I_n \otimes b_k(Y)) \\ &= \sum_{k=1}^{\infty} A_k(\overline{Y})\overline{X}b_k(\overline{Y}) = h(\overline{X}, \overline{Y}). \end{aligned}$$

Since h is a normalized NC bi-disc map and $\|\overline{Y}\| < 1$, by Lemma 3.6.4 and using part (a) we have that

$$\|h_n(\overline{X}, Y)\| = \|h(\overline{X}, \overline{Y})\| = \|\overline{X}\|. \tag{3.152}$$

Thus h is an NC full isometry.

3.7 NC Ball Map Classification

In this section we will classify NC ball maps. The key to this classification is to view NC ball maps as NC bi-disc maps and to use the classification in Theorem 3.6.1.

Corollary 3.7.1. *The map*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_G \end{pmatrix} \overset{p}{\mapsto} \begin{pmatrix} p_1(x) \\ \vdots \\ p_K(x) \end{pmatrix}$$

is an NC ball map from the β-ball to the α-ball mapping 0 to 0 if and only if there exists some matrix $A \in \mathbb{R}^{K\times G}$ such that $A^T A = I$ and

$$p(x) = \frac{\alpha}{\beta}Ax.$$

Proof. Define the map

$$\begin{pmatrix} x_1 \\ \vdots \\ x_G \\ y \end{pmatrix} \overset{h}{\mapsto} \frac{p(\beta x)}{\alpha} \tag{3.153}$$

for $y = y^T$. Notice $h(x, y)$ is a normalized NC bi-disc map. Thus by Theorem 3.6.1, the map $h(x, y)$ is an NC full isometry. So $h(x, y) = Ax$ for some $A \in \mathbb{R}^{K\times G}$ with $A^T A = I$. So

$$\frac{p(\beta x)}{\alpha} = Ax$$

and thus

$$p(x) = \frac{\alpha}{\beta}Ax$$

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3.8 NC Mixed Ball Map Classification

In this section we classify NC mixed ball maps. The classification is in terms of NC bi-disc maps and uses the linear fractional map discussed in Section 3.3

Theorem 3.8.1. *Suppose the map*

$$(x_1, \ldots, x_{G_1}, y_1, \ldots, y_{G_2}) \overset{p}{\mapsto} (r_1, \ldots, r_K)$$

where $y_j^T = y_j$ for all j is a NC mixed ball map from the β ball to the α ball with the analytically composing - 3 property. Then

$$h(x, y) := f\left(\frac{1}{\alpha}p(\beta x(1 - y^T y)^{1/2}, \beta y), \frac{1}{\alpha}p(0, \beta y)\right) \tag{3.154}$$

is an NC full isometry for $Y \in S\mathcal{B}_1^{G_2}$ where

$$f(u, v) = v - (I - vv^T)^{1/2}u(1 - v^T u)^{-1}(1 - v^T v)^{1/2}. \tag{3.155}$$

Conversely, if $h(x,y)$ *is an NC full isometry with the analytically composing property - 2 and* $g(y)$ *is a symmetric unit ball map, then*

$$p(x,y) = f(h(x(1-y^Ty)^{-1/2}, y), g(y))$$

is an NC mixed unit ball map.

Proof. $(\Longrightarrow)$

The NC analyticity of h follows since p is analytically composing - 3. Suppose that $p(x,y)$ is an NC mixed ball map and $(X,Y) \in \overline{\mathcal{B}_1^{G_1}(N)} \times S\mathcal{B}_1^{G_2}(N)$ for some N.

Case 1. Suppose $\|X\| < 1$. Then by Lemma 3.2.1, $\|(\beta X\sqrt{I-Y^TY}, \beta Y)\| < \beta$. Thus $\left\|\frac{1}{\alpha}p(\beta X\sqrt{I-Y^TY}, \beta Y)\right\| < 1$ and by Lemma 3.3.1

$$\left\| f\left(\frac{1}{\alpha}p(\beta X\sqrt{I-Y^TY}, \beta Y), \frac{1}{\alpha}p(0,Y)\right)\right\| < 1.$$

Case 2. Suppose $\|X\| = 1$. Then by Lemma 3.2.1, $\|(\beta X\sqrt{I-Y^TY}, \beta Y)\| = \beta$. So

$$\lim_{\delta\uparrow 1} h(\delta X, Y) = f\left(\lim_{\delta\uparrow 1}\frac{1}{\alpha}p(\delta\beta X\sqrt{I-Y^TY}, \beta Y), \frac{1}{\alpha}p(0,\beta Y)\right). \quad (3.156)$$

Since p is an NC mixed ball map $\|\lim_{\delta\uparrow 1}\frac{1}{\alpha}p(\delta\beta X\sqrt{I-Y^TY}, \beta Y)\| = 1$. Thus by Lemma 3.3.1 ,

$$\lim_{\delta\uparrow 1} h(\delta X, Y)$$

exists and

$$\|\lim_{\delta\uparrow 1} h(\delta X, Y)\| = 1.$$

$(\Longleftarrow)$

Suppose that $h(x,y)$ is an NC full isometry. Then $h(x,y)$ is an NC bi-disc map and by Lemma 3.3.1. Define

$$p(x,y) = f(h(x(1-y^Ty)^{-1/2}, y), g(y)).$$

Analyticity of p follows from the analytically composing - 2 property of h. The claim is that p is an NC unit mixed ball map.

Case 1. Suppose that $\|(X,Y)\| < 1$. Then by Lemma 3.2.1

$$(X(I-Y^TY)^{-1/2},Y) \in \mathcal{B}_1^{G_1}(N) \times S\mathcal{B}_1^{G_2}(N). \quad (3.157)$$

Thus

$$\|h(X(I-Y^TY)^{-1/2},Y)\| < 1$$

since h is an NC bi-disc map and

$$\|p(X,Y)\| = \|f(h(X(I-Y^TY)^{-1/2},Y),g(Y))\| < 1 \quad (3.158)$$

by Lemma 3.3.1.

Case 2. Suppose that $\|(X,Y)\| = 1$ and $\|Y\| < 1$. Then

$$X^TX + Y^TY \preceq I \quad (3.159)$$

with binding. Thus

$$X^TX \preceq I - Y^TY \quad (3.160)$$

with binding. Since $Y^TY \prec I$,

$$(I-Y^TY)^{-1/2}X^TX(I-Y^TY)^{-1/2} \preceq I \quad (3.161)$$

with binding. Thus

$$(X(I-Y^TY)^{-1/2},Y) \in \partial\mathcal{B}_1^{G_1}(N) \times S\mathcal{B}_1^{G_2}(N). \quad (3.162)$$

So

$$\lim_{\delta\uparrow 1} p(\delta X,Y) = f(\lim_{\delta\uparrow 1} h(\delta X(I-Y^TY)^{-1/2},Y),g(Y)).$$

Since h is an NC bi-disc map

$$\|\lim_{\delta\uparrow 1} h(\delta X(I-Y^TY)^{-1/2},Y)\| = 1$$

and so

$$\|\lim_{\delta\uparrow 1} p(\delta X,Y)\| = \|f(\lim_{\delta\uparrow 1} h(\delta X(I-Y^TY)^{-1/2},Y),g(Y))\| = 1 \quad (3.163)$$

by Lemma 3.3.1.

●●

3.8.1 NC Mixed Ball Maps from the β-ball to the β-ball for all β

In this section we investigate the property of NC analytic maps that act as mixed ball maps on all sizes of balls.

Lemma 3.8.2. *Suppose that*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \\ y_1 \\ \vdots \\ y_{G_2} \end{pmatrix} \overset{W}{\mapsto} \begin{pmatrix} W_1(x,y) \\ \vdots \\ W_K(x,y) \end{pmatrix} \tag{3.164}$$

is an NC mixed ball map from $M\mathcal{B}_\beta^{G_1,G_2}$ to $\mathcal{B}_\beta^K$ for all $\beta > 0$. Define

$$g(y) := W(0,y). \tag{3.165}$$

Suppose that for each $\beta > 0$

$$(x,y) \mapsto \frac{1}{\beta} W(\beta x, \beta y)$$

is analytically composing - 3. For fixed $\beta > 0$, define

$$\begin{aligned} h[\beta](x,y) &:= f(\frac{1}{\beta} W(\beta x(1 - y^T y)^{1/2}, \beta y), \frac{1}{\beta} g(\beta y)) \\ &= f(\frac{1}{\beta} W(x(\beta^2 - \beta^2 y^T y)^{1/2}, \beta y), \frac{1}{\beta} g(\beta y)). \end{aligned} \tag{3.166}$$

Then

$$\lim_{\beta\to\infty} h[\beta](\beta X(\beta^2 - Y^T Y)^{-1/2}, \frac{Y}{\beta}) = g(Y) - W(X,Y).$$

Proof. Suppose that $\beta > \|Y\|$. Notice that $h[\beta](x,y)$ is an NC full isometry by Theorem 3.8.1. Then we have from Equation (3.166) that

$$\begin{aligned} &\beta h[\beta] \left(X(\beta^2 - Y^T Y)^{-1/2}, \frac{Y}{\beta} \right) = h[\beta] \left(\beta X(\beta^2 - Y^T Y)^{-1/2}, \frac{Y}{\beta} \right) \\ &= \beta f \left(\frac{W(X,Y)}{\beta}, \frac{g(Y)}{\beta} \right) \\ &= g(Y) - \left(I - \frac{g(Y) g(Y)^T}{\beta^2} \right)^{\frac{1}{2}} W(X,Y) \left(I - \frac{g(Y)^T W(X,Y)}{\beta^2} \right)^{-1} \left(I - \frac{g(Y)^T (Y)}{\beta^2} \right)^{\frac{1}{2}}. \end{aligned} \tag{3.167}$$

Now take the limit as $\beta \to \infty$ in Equation (3.167) above to arrive at

$$\lim_{\beta\to\infty} h[\beta]\left(\beta X(\beta^2 - Y^TY)^{-1/2}, \frac{Y}{\beta}\right) = g(Y) - W(X,Y).$$

●●

Theorem 3.8.3. *Suppose that*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{G_1} \\ y_1 \\ \vdots \\ y_{G_2} \end{pmatrix} \overset{W}{\mapsto} \begin{pmatrix} W_1(x,y) \\ \vdots \\ W_K(x,y) \end{pmatrix} \tag{3.168}$$

is an NC mixed ball map from $M\mathcal{B}_\beta^{G_1,G_2}$ *to* $\mathcal{B}_\beta^K$ *for all* $\beta > 0$*. Define*

$$g(y) := W(0,y).$$

Suppose that for each $\beta > 0$

$$(x,y) \mapsto \frac{1}{\beta}W(\beta x, \beta y)$$

is analytically composing - 3. Then

$$h(x,y) := g(y) - W(x,y)$$

is an NC full isometry. Moreover, if we assume that $g(y) - W(x,y)$ *has an NC mixed finite rational expansion and is a mixed* $\mathcal{H}$*-space isometry, then*

$$W(x,y) = g(y) - U(y)xV(y) \tag{3.169}$$

where $U(y)$ *is a* $K \times G_1$ *matrix whose entries are series in* y *with the property that for each* $Y \in S\mathcal{B}_\beta^{G_2}$

$$U^T(Y)U(Y) = I$$

and $V(y)$ *is analytic on* $S\mathcal{B}_\beta^{G_2}$ *and maps* G_2 *symmetric variables to* G_1 *variables such that for each* $Y \in S\mathcal{B}_\beta^{G_2}$,

$$V^T(Y)V(Y) = I.$$

Proof. For fixed $\beta > 0$, define

$$\begin{aligned} h[\beta](x,y) &:= f\left(\frac{1}{\beta}W(\beta x(1-y^Ty)^{1/2}, \beta y), \frac{1}{\beta}g(\beta y)\right) \\ &= f\left(\frac{1}{\beta}W(x(\beta^2-\beta^2y^Ty)^{1/2}, \beta y), \frac{1}{\beta}g(\beta y)\right). \end{aligned} \tag{3.170}$$

By Theorem 3.8.1 $h[\beta](x,y)$ is an NC full isometry in x for each $\beta > 0$.

Suppose that $N \in \mathbb{N}, Y \in (S\mathbb{R}^{N\times N})^{G_2}$, and $X \in (\mathbb{R}^{N\times N})^{G_1}$. By Lemma 3.8.2 we have

$$\lim_{\beta\to\infty} h[\beta]\left(\beta X(\beta^2 - Y^TY)^{-1/2}, \frac{Y}{\beta}\right) = g(Y) - W(X,Y) = h(X,Y).$$

Thus

$$\lim_{\beta\to\infty} \left\| h[\beta]\left(\beta X(\beta^2 - Y^TY)^{-1/2}, \frac{Y}{\beta}\right)\right\| = \|h(X,Y)\| \tag{3.171}$$

Since each $h[\beta](x,y)$ is an NC full isometry,

$$\lim_{\beta\to\infty} \left\| h[\beta]\left(\beta X(\beta^2 - Y^TY)^{-1/2}, \frac{Y}{\beta}\right)\right\| = \lim_{\beta\to\infty} \|\beta X(\beta^2 - Y^TY)^{-1/2}\| = \|X\|. \tag{3.172}$$

So from Equations (3.171) and (3.172) we see that $\|h(X,Y)\| = \|X\|$. Note that $h(0,y) = 0$ so that $h(x,y)$ is a normalized NC bi-disc map. Thus $h(x,y)$ is an NC full isometry by Theorem 3.6.1.

The remaining conclusion in the result is a direct corollary to Theorem 3.4.14.

••

3.8.2 Example: Computing h for a Simple NC Mixed Ball Map

Consider the NC mixed ball map given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{W}{\mapsto} \begin{pmatrix} \sin(y)x \\ \cos(y)x \\ y \end{pmatrix}$$

where $y = y^T$. Here we emphasize that we have one NC variable and one symmetric NC variable. Notice that W is an NC mixed ball map from the β-ball to the β-ball for all $\beta > 0$. Thus for each $\beta > 0$, there exists an NC full isometry $h[\beta](x, y)$ such that

$$\begin{aligned} h[\beta](x,y) &:= f(\frac{1}{\beta}W(\beta x(1-y^Ty)^{1/2}, \beta y), \frac{1}{\beta}g(\beta y)) \\ &= f(\frac{1}{\beta}W(x(\beta^2-\beta^2 y^Ty)^{1/2}, \beta y), \frac{1}{\beta}g(\beta y)) \end{aligned} \tag{3.173}$$

where we define

$$g(y) := W(0,y) = \begin{pmatrix} 0 \\ 0 \\ y \end{pmatrix}.$$

Now we compute $h[\beta]$ for a fixed $\beta > 0$. To begin we compute

$$\begin{aligned} &\frac{1}{\beta}W(\beta x(1-y^Ty)^{1/2}, \beta y)[1-\frac{1}{\beta^2}g(\beta y)^T W(\beta x(1-y^Ty)^{1/2}, \beta y)]^{-1} \\ &= \begin{pmatrix} \sin(\beta y)x(1-y^2)^{1/2} \\ \cos(\beta y)x(1-y^2)^{1/2} \\ y \end{pmatrix} \left[1 - \begin{pmatrix} 0 & 0 & y \end{pmatrix} \begin{pmatrix} \sin(\beta y)x(1-y^2)^{1/2} \\ \cos(\beta y)x(1-y^2)^{1/2} \\ y \end{pmatrix} \right]^{-1} \\ &= \begin{pmatrix} \sin(\beta y)x(1-y^2)^{1/2}(1-y^2)^{-1} \\ \cos(\beta y)x(1-y^2)^{1/2}(1-y^2)^{-1} \\ y(1-y^2)^{-1} \end{pmatrix}. \end{aligned} \tag{3.174}$$

Also notice that

$$\begin{aligned} (I - \frac{g(\beta y)}{\beta}\frac{g(\beta y)^T}{\beta})^{1/2} &= \left(I - \begin{pmatrix} 0 \\ 0 \\ y \end{pmatrix} \begin{pmatrix} 0 & 0 & y \end{pmatrix} \right)^{1/2} \\ = \left(I - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y^2 \end{pmatrix} \right)^{1/2} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1-y^2)^{1/2} \end{pmatrix} \end{aligned} \tag{3.175}$$

while

$$(1 - \frac{g(\beta y)^T}{\beta}\frac{g(\beta y)}{\beta})^{1/2} = (1-y^2)^{1/2}. \tag{3.176}$$

Therefore

$$h[\beta](x,y) = f(\frac{1}{\beta}W(\beta x(1-y^Ty)^{1/2}, \beta y), \frac{1}{\beta}g(\beta y))$$
$$= \begin{pmatrix} 0 \\ 0 \\ y \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1-y^2)^{1/2} \end{pmatrix} \begin{pmatrix} \sin(\beta y)x(1-y^2)^{1/2}(1-y^2)^{-1}(1-y^2)^{1/2} \\ \cos(\beta y)x(1-y^2)^{1/2}(1-y^2)^{-1}(1-y^2)^{1/2} \\ y(1-y^2)^{-1}(1-y^2)^{1/2} \end{pmatrix}. \tag{3.177}$$

So we conclude

$$h[\beta](x,y) = \begin{pmatrix} -\sin(\beta y)x \\ -\cos(\beta y)x \\ 0 \end{pmatrix}.$$

Now notice for any fixed (X, Y) that

$$\lim_{\beta\to\infty} h[\beta](\beta X(\beta^2 - Y^TY)^{-1/2}, \frac{Y}{\beta}) = \lim_{\beta\to\infty} \begin{pmatrix} -\sin(Y)X(I - \frac{Y^TY}{\beta^2})^{-1/2} \\ -\cos(Y)X(I - \frac{Y^TY}{\beta^2})^{-1/2} \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -\sin(Y)X \\ -\cos(Y)X \\ 0 \end{pmatrix} = g(Y) - W(X,Y) \tag{3.178}$$

as predicted in Theorem 3.8.3.

Chapter 3, in part, is a reprint of materials from "Noncommutative Ball Maps" in Journal of Functional Analysis, Volume 257, July 2009, Helton, J. William; Klep, Igor; McCollough, Scott; Slinglend, Nick [HKMS09]. Written permission for the use of this material from the co-authors has been received.

4 NC Bianalytic Maps

A major pursuit in the study of several complex variables has been to understand when a set is bianalytic to a ball. In this section we examine some noncommutative versions of this problem.

To properly state our versions of this problem, we will recall some definitions. We will say an NC analytic map, $p(x)$, is **NC bianalytic** if $p : \mathcal{C} \to \mathcal{D}$ has an NC analytic inverse and there exists $\epsilon, \delta > 0$ such that $\mathcal{B}^G_\epsilon \subseteq \mathcal{C}$ and $\mathcal{B}^G_\delta \subseteq \mathcal{D}$. Suppose that $p(x), q(x)$ are NC analytic polynomials. Then we say that $p(x)$ is **0-local bianalytic** to $q(x)$ if there exists some $\epsilon > 0$ and an NC bianalytic map $R : \mathcal{B}^G_\epsilon \to \mathcal{D}$ such that $R(0) = 0$ and $p(X) = (q \circ R)(X)$ for all $X \in \mathcal{B}^G_\epsilon$.

We are interested in knowing when the positivity set $\mathcal{D}_p$ of a given symmetric noncommutative polynomial p is bianalytic to a noncommutative ball $\mathcal{B}^G_\epsilon$. This is a special case of the important question of understanding bianalyticity to a positivity set $\mathcal{D}_q$ which is convex.

For an NC analytic function h in G variables $x = (x_1, \ldots, x_G)$ to be convex, we require that

$$th(X) + (1-t)h(W) \succeq h(tX + (1-t)W) \quad \text{for} \quad 0 \leq t \leq 1$$

on all tuples $X, W \in (\mathbb{R}^{n\times n})^G$ and all dimensions n. "Convexity" in this dimension-free setting is a very rigid constraint.

For example, it has been shown that convex polynomial functions must have degree two or less [HM04]. It has also been shown that dimension-free matrix inequalities defined by convex rational expressions in symmetric noncommuting variables are equivalent to linear matrix inequalities [HMV06]. Recently it was proved [HM09] that if q is a noncommutative symmetric polynomial, irreducible in a certain sense and if the positivity set $\mathcal{D}_q$ is convex, then the set is a "generalized ball", namely, it is a set of the form

$$\mathcal{D}_q = \underset{n}{\cup}\{X \in \mathbb{R}^{nG\times n} : 1 - X^* Q_{1,n} X - X^t Q_{2,n} (X^t)^* \succeq 0\}, \tag{4.1}$$

where $Q_{i,n} \in \mathbb{R}^{nG\times nG}, Q_{i,n} \succeq 0,\ X = \begin{pmatrix} X_1 \\ \vdots \\ X_G \end{pmatrix},\ X^* = (X_1^*, \dots, X_G^*)$ and $X^t = (X_1, \dots, X_G)$ is the unadjointed transpose. Note the case where $Q_{1,n} = I_{nG}$ and $Q_{2,n} = 0$ collapses to the ordinary unit dimension-free ball, $\mathcal{B}_1^G$. Thus determining which $\mathcal{D}_p$ are bianalytic to a mixed ball settles a major convex case.

In this chapter we present an answer to the following question.

Q_{map}: *When is the domain $\mathcal{D}_q$ NC bianalytic to a matrix ball, B, and what is the map between the domains?*

In solving the above problem we use some theory that is analogous to the aforementioned classification problem in several complex variables. The classical problem asks what is the structure of a map from the unit ball $\mathbb{C}^m$ to the unit ball in $\mathbb{C}^n$ that is holomorphic and maps 0 to 0 and the boundary sphere of one ball to the boundary sphere of the other ball [D'A92]. The analogue to this problem involves classifying NC ball maps. This classification of NC ball maps can be found in Section 3.7. The NC ball map result that we have found useful in answering the earlier questions follows.

Corollary 4.0.4. *An NC analytic map*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_G \end{pmatrix} \overset{p}{\mapsto} \begin{pmatrix} p_1(x) \\ \vdots \\ p_K(x) \end{pmatrix}$$

is a ball map from the β-ball to the α-ball such that $p(0) = 0$ iff $p(x) = \frac{\alpha}{\beta} Ax$ for some $A \in \mathbb{R}^{K\times G}$ such that $A^T A = I$.

Note that this is a restatement of Corollary 3.1.1.

Playing a key role in our solution to the above problem Q_{map} is a procedure, we call the SoS Construction . Consider the given polynomial $p(x)$ in the NC variables

$$(x_1, \dots, x_G)$$

where $p(x)$ is a matrix positive, hereditary polynomial and $p(0) = 0$. Note that there is no dependence on symmetric variables in this case. Since $p(x)$ is hereditary and matrix positive, there are algorithms to write $p(x)$ as the sum of squares of analytic polynomials

$$r(x) := \begin{pmatrix} r_1(x) \\ \vdots \\ r_K(x) \end{pmatrix}$$

for some minimal number of terms K [Hel02]. This process of writing $p(x)$ as a minimal sum of squares will be referred to by the term the SoS Construction

Our result to Q_{map} without symmetric variables is as follows

Theorem 4.0.5. *Consider the polynomial $p(x)$ in the NC variables*

$$(x_1, \ldots, x_G)$$

where $p(x)$ is an NC hereditary, matrix positive polynomial such that $p(0) = 0$ and suppose $a > 0$. Define $q(x) := a^2 - p(x)$ and $d(x) := a^2 - c_G(x)$ where

$$c_G(x) := \sum_{j=1}^{G} x_j^T x_j.$$

Then the SoS Construction will construct a polynomial NC bianalytic map, $R : \mathcal{D}_q \to \mathcal{D}_d$, mapping 0 to 0 or one does not exist. This polynomial map R is unique up to some unitary transformation.

The proof follows in Section 4.2.

One further consideration for the question Q_{map} involves the dependence upon symmetric variables $y_j = y_j^T$. We may ask: for a polynomial $q(x, y)$, when is $\mathcal{D}_q$ NC bianalytic to a mixed matrix ball?

In Section 4.4, we present some results similar to the above for the mixed variable case. In Proposition 4.4.1 we relate the NC bianalytic maps between positivity domains to NC full isometries. While the SoS Construction has not been generalized to the mixed variable case completely, Proposition 4.4.2 relates the question of bianalytic positivity domains to the SoS Construction . Further, if we add stronger hypotheses on the bianalytic maps between positivity domains in the mixed variable case, we obtain Proposition 4.4.6.

4.1 Preliminary Results

Here are a few Lemmas to help prove some of the other results in the chapter.

Referring to Theorem 4.0.5, the condition that the NC bianalytic map takes 0 to 0 may seem restrictive but the restriction can be overcome.

Lemma 4.1.1. *Consider the polynomial $p(x)$ in the NC variables*

$$(x_1, \ldots, x_G)$$

where $p(x)$ is an NC hereditary, matrix positive polynomial such that $p(0) = 0$ and suppose $a > 0$. Define $q(x) := a^2 - p(x)$ and $d(x) := a^2 - c_G(x)$ where

$$c_G(x) := \sum_{j=1}^{G} x_j^T x_j.$$

Suppose that

$$R : \mathcal{D}_q \to \mathcal{D}_d = \mathcal{B}_a^{G_1} \tag{4.2}$$

is an NC bianalytic map that is analytically composing -1. Suppose that R^{-1} is analytically composing-4. Then the map

$$\hat{R} : \mathcal{D}_q \to \mathcal{D}_d = \mathcal{B}_a^{G_1} \tag{4.3}$$

defined by

$$\hat{R}(x) = af\left(\frac{R(x)}{a}, \frac{R(0)}{a}\right) \tag{4.4}$$

is an NC bianalytic map such that $\hat{R}(0) = 0$, where f is defined in Equation (3.23).

Proof. Notice that

$$\hat{R}^{-1}(x) = R^{-1}\left(af\left(\frac{x}{a}, \frac{R(0)}{a}\right)\right). \tag{4.5}$$

To see that

$$(\hat{R} \circ \hat{R}^{-1})(x) = x \quad \text{and} \quad (\hat{R}^{-1} \circ \hat{R})(x) = x \tag{4.6}$$

use Lemma 3.3.1. The NC analyticity of $\hat{R}$ and $\hat{R}^{-1}$ follows from Lemma 3.2.1 and the analytically composing assumptions. ••

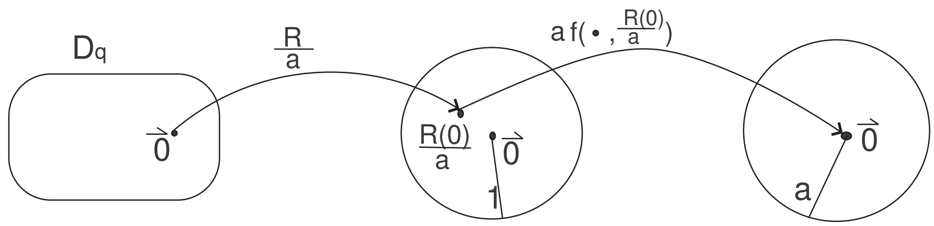

Figure 4.1: This figure shows the composition of two NC bianalytic maps to obtain $\hat{R}$.

In fact the NC bianalytic maps between $\mathcal{D}_q$ and $\mathcal{D}_d$ which map 0 to 0 are unique up to an isometric linear transformation.

Lemma 4.1.2. *Consider the polynomial $p(x)$ in the NC variables*

$$(x_1, \ldots, x_G)$$

where $p(x)$ is an NC hereditary, matrix positive polynomial such that $p(0) = 0$ and suppose $a > 0$. Define $q(x) := a^2 - p(x)$ and $d(x) := a^2 - c_G(x)$ where

$$c_G(x) := \sum_{j=1}^{G} x_j^T x_j.$$

Suppose that $R(x)$ and $\hat{R}(x)$ are two bianalytic maps between $\mathcal{D}_q$ and $\mathcal{D}_d$ such that $R(0) = 0$ and $\hat{R}(0) = 0$. Suppose that $R \circ \hat{R}^{-1}$ is NC analytic.

1. *Then there exists some matrix $A \in \mathbb{R}^{G \times G}$ such that $A^T A = I$ and $R(x) = A\hat{R}(x)$.*

2. *If $A \in \mathbb{R}^{G \times G}$ such that $A^T A = I$, then $AR(x)$ is a bianalytic map between $\mathcal{D}_q$ and $\mathcal{D}_d$ and $AR(0) = 0$.*

Proof. 1. Notice that the map

$$R \circ \hat{R}^{-1} : \mathcal{D}_d \to \mathcal{D}_d \tag{4.7}$$

is an NC ball map. Thus by Corollary 4.0.4 there exists a matrix $A \in \mathbb{R}^{G \times G}$ such that $A^T A = I$ and

$$(R \circ \hat{R}^{-1})(x) = Ax. \tag{4.8}$$

Thus

$$R(x) = A\hat{R}(x).$$

2. Notice that

$$R^T(x)R(x) = R^T(x)A^TAR(x)$$

and that

$$(AR)^{-1}(x) = R^{-1}(A^Tx).$$

••

Similarly we will need the following lemma relating a sum of squares decomposition to NC bianalytic maps. This is again a simple result of Corollary 4.0.4.

Lemma 4.1.3. *Consider the polynomial $p(x)$ in the NC variables*

$$(x_1, \ldots, x_G)$$

where $p(x)$ is an NC hereditary, matrix positive polynomial such that $p(0) = 0$ and suppose $a > 0$. Define $q(x) := a^2 - p(x)$. Suppose that

$$R : \mathcal{D}_q \to \mathcal{B}_a^{G_1}$$

gives a bianalytic map between the two domains. Suppose that there exist analytic polynomials

$$r_1(x), \ldots, r_G(x)$$

such that

$$p(x) = \sum_{j=1}^{G} r_j^T(x)r_j(x) = r^T(x)r(x).$$

Then there exists some matrix $A \in \mathbb{R}^{G\times G}$ such that $A^TA = I$ and $r(x) = AR(x)$. In particular, Lemma 4.1.2 says that $r(x)$ gives an NC bianalytic map between $\mathcal{D}_q$ and $\mathcal{B}_a^{G_1}$.

Proof. Notice that $r \circ R^{-1}$ is an NC ball map from $\mathcal{B}_a^{G_1}$ to $\mathcal{B}_a^{G_1}$ – the analyticity follows from Corollary 2.4.2. Thus by Corollary 4.0.4 there exists a matrix $A \in \mathbb{R}^{G\times G}$ such that $A^TA = I$ and

$$(r \circ R^{-1})(x) = Ax. \tag{4.9}$$

Thus

$$r(x) = AR(x).$$

••

4.2 Sets Bianalytic to a Ball: Proof of Theorem 4.0.5

Consider the polynomial $p(x)$ in the NC variables

$$(x_1, \ldots, x_G)$$

where $p(x)$ is a hereditary, matrix positive polynomial and $p(0) = 0$. Suppose that $a > 0$ and $\mathcal{D}_{a^2 - p(x)}$ is bianalytic to $\mathcal{B}_a^G = \mathcal{D}_{a - c_G(x)}$. Suppose that

$$R : \mathcal{D}_{a^2 - p(x)} \to \mathcal{B}_a^G \tag{4.10}$$

is an NC bianalytic map such that $R(0) = 0$. The assumption that $R(0) = 0$ is justified by Lemma 4.1.1. Since $p(x)$ is hereditary and matrix positive, there exist analytic polynomials $r_1(x), \ldots, r_K(x)$ such that

$$p(x) = \sum_{j=1}^{K} r_j^T(x) r_j(x) = r^T(x) r(x)$$

for some K [HMP05].

Then the map

$$r \circ R^{-1} : \mathcal{B}_a^G \to \mathcal{B}_a^K \tag{4.11}$$

is NC analytic by Corollary 2.4.2, is an NC ball map, and $(r \circ R^{-1})(0) = 0$. Thus

$$(r \circ R^{-1})(x) = Ax \tag{4.12}$$

for some $A \in \mathbb{R}^{K \times G}$ such that $A^T A = I$ by Corollary 4.0.4. Thus

$$r(x) = AR(x) \tag{4.13}$$

and

$$p(x) = R^T(x) R(x). \tag{4.14}$$

To see that $R(x)$ is a vector of polynomials, first, for each $j = 1, \ldots, G$ and each $n \in \mathbb{N}$, define homogeneous polynomials of degree n, $H_{j,n}(x)$, such that

$$R_j(x) =: \sum_{n=0}^{\infty} H_{j,n}(x). \tag{4.15}$$

For each n, let $H_n(x)$ denote the vector of polynomials

$$H_n(x) = \begin{pmatrix} H_{1,n}(x) \\ \vdots \\ H_{G,n}(x) \end{pmatrix}. \tag{4.16}$$

Suppose that

$$M := \max\{\text{degree}(r_j(x)) : j = 1, \ldots, K\}. \tag{4.17}$$

Then for $n > M$ we have by Equation (4.13) that

$$0 = AH_n(x). \tag{4.18}$$

Thus $H_n(x) \equiv 0$ since A is an isometry and $R(x)$ is a vector of polynomials. The uniqueness now follows from Lemma 4.1.3.

4.3 Invertible NC Analytic Functions

The output of the SoS Construction is an NC analytic function. Theorem 4.0.5 says that this output is a bianalytic map if one exists. One question that remains when looking at the output of the SoS Construction is whether or not the output is invertible "near 0." As a partial answer to this question we have the following result.

Proposition 4.3.1. *Suppose that $x = (x_1, \ldots, x_G)$ is a vector of NC variables and that*

$$p(x) = \begin{pmatrix} p_1(x) \\ \vdots \\ p_G(x) \end{pmatrix}$$

is an NC analytic map such that $p(0) = 0$. *Write*

$$p(x) = \sum_{j=1}^{\infty} H_j(x) = Ax + \sum_{j=2}^{\infty} H_j(x)$$

for some vectors of homogeneous degree j *polynomials* $H_j(x)$ *and some matrix* $A \in \mathbb{R}^{G \times G}$. *If* $p : \mathcal{B}_\epsilon^G \to \mathcal{D}$ *is invertible for some* $\epsilon > 0$, *then* A *is invertible.*

Proof: Suppose that $p : \mathcal{B}_\epsilon^G \to \mathcal{D}$ is invertible for some $\epsilon > 0$ and that A is not invertible. Let $v \in \mathbb{R}^G$ be a null vector for A such that $0 < \|v\| < \epsilon$. Define $X \in (\mathbb{R}^{2\times 2})^G$ such that

$$X_i := \begin{pmatrix} 0 & v_i \\ 0 & 0 \end{pmatrix}.$$

Then $0 < \|X\| < \epsilon$ and $p(X) = 0 = p(0)$. I.e. $p(x)$ is not one-to-one on $\mathcal{B}_\epsilon^G$ and we have reached a contradiction. ••

4.4 Maps Involving Symmetric Variables

In this section we will examine the Q_{map} question posed in this chapter's introduction concerning the effect of having some dependence on symmetric variables. In particular, consider a polynomial $p(x, y)$ in the NC mixed variables

$$(x_1, \ldots, x_{G_1}, y_1, \ldots, y_{G_2})$$

where $y_j = y_j^T$ for all j. Also suppose that $p(x, y)$ is a matrix-positive, hereditary polynomial and $p(0, 0) = 0$. We would like to determine when $\mathcal{D}_{a^2 - p(x,y)}$ is NC bianalytic to the mixed matrix ball $M\mathcal{B}_b^{G_1,G_2}$. The following results give an answer to this question.

Proposition 4.4.1. *Consider the polynomial* $p(x, y)$ *in the NC mixed variables*

$$(x_1, \ldots, x_{G_1}, y_1, \ldots, y_{G_2})$$

where $p(x, y)$ *is a hereditary, matrix positive polynomial and* $p(0, 0) = 0$. *Suppose that* $a, b > 0$ *and*

$$R : \mathcal{D}_{a^2 - p(x,y)} \to M\mathcal{B}_b^{G_1,G_2} \tag{4.19}$$

is an NC bianalytic map. Then there are NC analytic polynomials

$$r_1(x,y),\ldots,r_K(x,y)$$

such that

$$p(x,y) = \sum_{j=1}^{K} r_j^T(x,y) r_j(x,y) = r^T(x,y) r(x,y) \tag{4.20}$$

for some K and the map

$$(x,y) \mapsto f\left(\frac{1}{a}(W(bx(1-y^Ty)^{1/2}, by), \frac{1}{a}W(0,by)\right)$$

is an NC full isometry, where f is the linear fractional map defined in Equation (3.155) and

$$W(x,y) := (r \circ R^{-1})(x,y) \tag{4.21}$$

is analytically composing - 3.

Proof. Notice that

$$M\mathcal{B}_b^{G_1,G_2} = \mathcal{D}_{b^2 - c_{G_1+G_2}(x,y)}$$

where $c_{G_1+G_2}$ is defined by

$$c_{G_1+G_2}(x,y) := \sum_{j=1}^{G_1} x_j^T x_j + \sum_{j=1}^{G_2} y_j^2$$

Since $p(x,y)$ is hereditary and matrix positive, there exist analytic polynomials

$$r_1(x,y),\ldots,r_K(x,y)$$

such that

$$p(x,y) = \sum_{j=1}^{K} r_j^T(x,y) r_j(x,y) = r^T(x,y) r(x,y) \tag{4.22}$$

for some K [HMP05]. Define an NC analytic function

$$W(x,y) := (r \circ R^{-1})(x,y)$$

– note W is NC analytic by Corollary 2.4.2. To see that $W(x,y)$ is an NC mixed ball map, suppose that $(X,Y) \in \mathcal{B}_b^{G_1,G_2}$. Then

$$a^2 I - (p \circ R^{-1})(X,Y) \succ 0$$

and

$$\|(r \circ R^{-1})(X,Y)\| < a.$$

Now suppose that $(X,Y) \in \partial\mathcal{B}_b^{G_1,G_2}$ and $\|Y\| < 1$. Then

$$a^2 I - (p \circ R^{-1})(X,Y) \succeq 0$$

with binding and

$$\|\lim_{\delta \uparrow 1} (r \circ R^{-1})(\delta X, Y)\| = a.$$

So $W(x,y)$ is a NC mixed ball map from the b-ball to the a-ball. By Theorem 3.8.1, the map

$$(x,y) \mapsto f\left(\frac{1}{a}W(bx(1-y^T y)^{1/2}, by), \frac{1}{a}W(0,by)\right)$$

is a normalized bi-disc map – and hence an NC full isometry – for all vectors of NC analytic polynomials $r(x,y)$ satisfying Equation (4.20) , where f is the linear fractional map defined in Equation (3.155). ●●

Proposition 4.4.2. *Suppose that $p(x,y)$ in the NC mixed variables*

$$(x_1, \ldots, x_{G_1}, y_1, \ldots, y_{G_2})$$

is a matrix positive, hereditary polynomial. Suppose that $\mathcal{D}_{1-p(x,y)}$ is NC bianalytic to $M\mathcal{B}_1^{G_1,G_2}$. Define the matrix positive, hereditary polynomial $r(x) = p(x,0)$. Then $\mathcal{D}_{1-r(x)}$ is bianalytic to $\mathcal{B}_1^{G_1}$.

The proof follows in Section 4.4.2.

As a Corollary, we have that the SoS Construction can be applied to this mixed NC variable case. Recall that the SoS Construction will find an NC bianalytic map between $\mathcal{D}_{1-r(x)}$ and $\mathcal{B}_1^{G_1}$ or one does not exist. If no such map exists, then $\mathcal{D}_q$ is not NC bianalytic to $M\mathcal{B}_1^{G_1,G_2}$.

4.4.1 Lemmas for Proving Proposition 4.4.2

Lemma 4.4.3. *Suppose that $p(x,y)$ is NC analytic in the NC mixed variables*

$$(x_1, \ldots, x_{G_1}, y_1, \ldots, y_{G_2})$$

and that

$$p(X,Y) = (p(X,Y))^T$$

for all $(X,Y) \in \mathcal{B}_\epsilon^{G_1,G_2}$. *Then there exists some NC analytic* $g(y)$ *such that* $p(x,y) = g(y)$.

Proof. First write

$$p(x,y) = q(x) + g(y) + h(x,y)$$

where

$$h(x,y) = \sum_{w \in \mathcal{W}} a_w \{x,y\}^w$$

and $\mathcal{W}$ is the set of all words with at least one letter in x and at least one letter in y. Then there exists for each $n \geq 1$ a series $h_n(x,y)$ that is homogeneous degree n in x and so that

$$h(x,y) = \sum_{n=1}^{\infty} h_n(x,y).$$

We now prove strong restrictions on $q(x)$ and $h(x,y)$.

Claim 1. $q(x) = 0$

Suppose not. Then there is some word of minimal degree, w, so that $a_w \neq 0$. Let $d = |w|$. Thus we may choose $\overline{X} \in (\mathbb{R}^{N \times N})^{G_1}$ such that

$$\sum_{|w|=d} a_w \overline{X}^w \neq 0. \tag{4.23}$$

Now define the block $(d+1) \times (d+1)$ nilpotent matrix for each j by

$$X_j := \begin{pmatrix} 0 & \overline{X}_j & 0 & \dots & & 0 \\ 0 & 0 & \overline{X}_j & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & & \\ 0 & & \dots & & & \overline{X}_j \\ 0 & & \dots & & & 0 \end{pmatrix}. \tag{4.24}$$

Then, since $p(x,0)$ contains no transposes,

$$p(X,0) = \begin{pmatrix} g(0) & & \sum_{|w|=d} a_w \overline{X}^w \\ \vdots & \ddots & \\ 0 & \dots & g(0) \end{pmatrix}.$$

Since

$$\sum_{|w|=d} a_w \overline{X}^w \neq 0$$

we contradict that $p(x,y) = (p(x,y))^T$.

Claim 2. $h(x,y) = 0$

Suppose not. Then there is a minimal d so that $h_d(x,y) \neq 0$. Define the block $(d+1) \times (d+1)$ matrices

$$X_j = \begin{pmatrix} 0 & \overline{X}_j & 0 & \dots & & 0 \\ 0 & 0 & \overline{X}_j & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & & \\ 0 & & \dots & & & \overline{X}_j \\ 0 & & \dots & & & 0 \end{pmatrix}, \quad Y_i = \begin{pmatrix} \overline{Y}_i & & 0 \\ & \ddots & \\ 0 & & \overline{Y}_i \end{pmatrix}$$

where $\overline{Y}_i = \overline{Y}_i^T$ for each $1 \leq j \leq G_1$ and $1 \leq i \leq G_2$. Then

$$p(X,Y) = \begin{pmatrix} g(\overline{Y}) & & h_d(\overline{X},\overline{Y}) \\ & \ddots & \\ 0 & & g(\overline{Y}) \end{pmatrix}.$$

Since $h_d(\overline{X},\overline{Y}) \neq 0$ we contradict that $p(x,y) = (p(x,y))^T$.

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Lemma 4.4.4. *Suppose that $p(x,y)$ is a matrix positive hereditary polynomial. Define $q(x,y) = 1 - p(x,y)$. Suppose that $\mathcal{D}_q$ is NC bianalytic to $M\mathcal{B}_1^{G_1,G_2}$ and that*

$$H(x,y) = \begin{pmatrix} H_1(x,y) \\ \vdots \\ H_{G_1}(x,y) \\ H_{G_1+1}(y) \\ \vdots \\ H_{G_1+G_2}(y) \end{pmatrix} = \begin{pmatrix} \widetilde{H}(x,y) \\ \widehat{H}(y) \end{pmatrix}$$

where $H_j(y) = H_j(y)^T$ for $j > G_1$ is such an NC bianalytic map. If $\widehat{H}(Y) = 0$, then $Y = 0$.

Proof. First define

$$\mathcal{Z}_N := \{Y \in (S\mathbb{R}^{N\times N})^{G_2} : \widehat{H}(Y) = 0\}.$$

Claim 1. For each N, the set $\mathcal{Z}_N$ is connected.

Suppose not. Then there exist some open sets U_1 and U_2 and some N such that

$$U_1 \cap U_2 = \emptyset, \quad \mathcal{Z}_N \subseteq U_1 \cup U_2, \quad U_1 \cap \mathcal{Z}_N \neq \emptyset, \quad U_2 \cap \mathcal{Z}_N \neq \emptyset.$$

Now for $i = 1, 2$ define

$$V_i = \{(X,Y) : (X,Y) \in \mathcal{D}_q \text{ and } Y \in U_i\}.$$

Fix i. Let $(X,Y) \in V_i$. Then since $\mathcal{D}_q \cap (\mathbb{R}^{N\times N})^{G_1} \times (S\mathbb{R}^{N\times N})^{G_2}$ is open and U_i is open there is some $\delta > 0$ such that

$$\mathcal{B}_\delta(X,Y) \subseteq \mathcal{D}_q \cap (\mathbb{R}^{N\times N})^{G_1} \times (S\mathbb{R}^{N\times N})^{G_2} \text{ and } \mathcal{B}_\delta(Y) \subseteq U_i.$$

Thus $\mathcal{B}_\delta(X,Y) \subseteq V_i$ and each V_i is open.

Notice $V_1 \cap V_2 = \emptyset$. Since H^{-1} is continuous, H is an open map. Thus

$$H(V_1) \cap H(V_2) = \emptyset, \quad \mathcal{B}_1^{G_1}(N) \times \{0\} \subseteq H(V_1) \cup H(V_2),$$
$$H(V_1) \cap \mathcal{B}_1^{G_1}(N) \times \{0\} \neq \emptyset, \quad H(V_2) \cap \mathcal{B}_1^{G_1}(N) \times \{0\} \neq \emptyset.$$

Thus $\mathcal{B}_1^{G_1}(N) \times \{0\}$ is disconnected and we have a contradiction.

Now write

$$H(x,y) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \sum_{m=2}^{\infty} h_m(x,y)$$

for some $A \in \mathbb{R}^{G_1\times G_1}, B \in \mathbb{R}^{G_1\times G_2}, C \in \mathbb{R}^{G_2\times G_2}$, and vectors of polynomials,h_m, homogeneous degree m in (x,y). Suppose that for some N the set $\mathcal{Z}_N \neq \{0\}$.

Claim 2. The matrix A is invertible.

Suppose not. Then let $v \neq 0$ be a null vector for A. Define

$$X_i := \begin{pmatrix} 0 & tv_i \\ 0 & 0 \end{pmatrix}, \quad Y_j := 0$$

for $1 \le i \le G_1$ and $1 \le j \le G_2$. Then $H(X, Y) = 0$ for all t. Thus H is not one-to-one in any neighborhood of $(0, 0)$. This contradicts that H is invertible.

Since A is invertible and using Lemma 2.3.6, there exists some open set,U, such that $0 \in U \subseteq (\mathbb{R}^{N\times N})^{G_1}$ and $\widetilde{H}(U \times \{0\})$ is open by the inverse mapping theorem [Ma95]. Now assuming $\mathcal{Z}_N \neq \{0\}$ there exist $y_k \in \mathcal{Z}_N - \{0\}$ such that $y_k \to 0$. Notice that for all k we have $\widetilde{H}(0, y_k) \notin \widetilde{H}(U \times \{0\})$. But then since $\widetilde{H}$ is continuous $\widetilde{H}(0, y_k) \to (0, 0) \in \widetilde{H}(U \times \{0\})$, a contradiction. ••

4.4.2 Proof of Proposition 4.4.2

Since $\mathcal{D}_q$ is NC bianalytic to $M\mathcal{B}_1^{G_1,G_2}$, there exists an NC analytic map H defined on $\mathcal{D}_q$ with an NC analytic inverse and with the property that $H(0) = 0$. Suppose

$$H(x, y) = \begin{pmatrix} H_1(x, y) \\ \vdots \\ H_{G_1}(x, y) \\ H_{G_1+1}(y) \\ \vdots \\ H_{G_1+G_2}(y) \end{pmatrix} = \begin{pmatrix} \widetilde{H}(x, y) \\ \widehat{H}(y) \end{pmatrix}$$

where $H_j(y) = H_j(y)^T$ for $j > G_1$. Now define

$$h(x) := \begin{pmatrix} H_1(x, 0) \\ \vdots \\ H_{G_1}(x, 0) \end{pmatrix} = \widetilde{H}(x, 0).$$

By Lemma 4.4.4 we know that if $\widehat{H}(Y) = 0$ then $Y = 0$. Recall $r(x) := p(x, 0)$. So we have that h maps $\mathcal{D}_{1-r(x)}$ onto $\mathcal{B}_1^{G_1}$. Thus h is a bijection between $\mathcal{D}_{1-r(x)}$ and $\mathcal{B}_1^{G_1}$, $h(0) = 0$, and h has an NC analytic inverse. ••

4.4.3 Stronger Assumptions Lead to a More Rigid Structure

We will need to define a new type of dimension-free set to proceed with the proof of the following theorem. Suppose that $\alpha \in \mathbb{R}^{N\times N}$ is a positive definite matrix. Define the set $M\mathcal{B}_\alpha^{G_1,G_2}$ as follows. Let

$$M\mathcal{B}_\alpha^{G_1,G_2}(n) := \{(X,Y) \in (\mathbb{R}^{nN\times nN})^{G_1} \times S(\mathbb{R}^{nN\times nN})^{G_2} : X^TX + Y^TY \prec I_n \otimes \alpha\}$$

and define

$$M\mathcal{B}_\alpha^{G_1,G_2} := \bigcup_{n\geq 1} M\mathcal{B}_\alpha^{G_1,G_2}(n)$$

Similarly we can define a new type of positivity domain for an NC polynomial. Suppose that $p(x,y)$ is a symmetric NC polynomial. Let

$$\mathcal{D}_{\alpha-p(x,y)}(n) := \{(X,Y) \in (\mathbb{R}^{nN\times nN})^{G_1} \times S(\mathbb{R}^{nN\times nN})^{G_2} : p(X,Y) \prec I_n \otimes \alpha\}.$$

Now define

$$\mathcal{D}_{\alpha-p(x,y)} := \bigcup_{n\geq 1} \mathcal{D}_{\alpha-p(x,y)}(n).$$

Before stating and proving the last result of this chapter, we will need the following lemma.

Lemma 4.4.5. *Let B be a symmetric positive operator on real Hilbert space H. Fix $v \in H$ and define $\alpha_{t,M} := tvv^T + M(I - vv^T)$. Then there is a $t^* \in \mathbb{R}$ such that if $t > t^*$, there is M_t such that*

$$B \preceq \alpha_{t,M_t} \quad \textit{and}$$

$$\lim_{t\downarrow t^*} < \alpha_{t,M_t} w, w > - < Bw, w >= 0$$

iff $w \in span\{v\}$.

Proof. Partition B with respect to $span\{v\}$ and $span\{v\}^\perp$, with $B = \begin{pmatrix} a & \beta \\ \beta^T & \gamma \end{pmatrix}$ and note

$$\alpha_{t,M_t} - B = \begin{pmatrix} t-a & -\beta \\ -\beta^T & M_t I - \gamma \end{pmatrix}.$$

Take $t > t^* = a$ and $M_t > \gamma$, then $E_t \succeq 0$ iff the Schur complement inequality

$$t - a - \beta(M_t I - \gamma)^{-1}\beta^T \succeq 0$$

holds; which can always be achieved by selecting large enough M_t. Clearly, as $t \downarrow t^*$ we have $M_t \to \infty$. Partition w as

$$w = sv + \begin{pmatrix} 0 \\ w_2 \end{pmatrix} = \begin{pmatrix} s \\ w_2 \end{pmatrix}$$

for some $s \in \mathbb{R}$ and $\begin{pmatrix} 0 \\ w_2 \end{pmatrix} \in span\{v\}^\perp$, and observe

$$e_t(w) := \left\langle (\alpha_{t,M_t} - B)\begin{pmatrix} s \\ w_2 \end{pmatrix}, \begin{pmatrix} s \\ w_2 \end{pmatrix} \right\rangle = (t-\alpha)\|sv\|^2 - 2s < w_2, \beta^T > + < (M_t I - \gamma)w_2, w_2 >$$

Clearly $\lim_{t \downarrow t^* = \alpha} e_t(w) = \begin{cases} 0 & \text{if } w_2 = 0 \\ \infty & \text{if } w_2 \neq 0. \end{cases}$

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Theorem 4.4.6. *Suppose that*

$$r(x,y) = \begin{pmatrix} r_1(x,y) \\ \vdots \\ r_K(x,y) \end{pmatrix}$$

is a vector of analytic polynomials and that

$$p(x,y) := r^T(x,y)r(x,y).$$

Suppose that R is an NC bianalytic map defined on $(\mathbb{R}^{N\times N})^{G_1} \times (S\mathbb{R}^{N\times N})^{G_2}$ for all N such that R induces an NC bianalytic correspondence between $\mathcal{D}_{\alpha - p(x,y)}$ and $M\mathcal{B}_\alpha^{G_1,G_2}$ for all positive matrices α. Define

$$W := r \circ R^{-1} \quad \text{and} \quad g(y) := W(0,y).$$

Suppose that for each $\beta > 0$

$$(x,y) \mapsto \frac{1}{\beta} W(\beta x, \beta y)$$

is analytically composing - 3. Then

$$h(z,u) := W(z,u) - g(u)$$

is an NC full isometry in z. Moreover, suppose that $h(z,u) := W(z,u) - g(u)$ has finite mixed expansion (this is the same as assuming that R^{-1} has a finite mixed expansion) and is a mixed H-space isometry. Then

$$r(x,y) = \widetilde{\Omega}(y)R(x,y)$$

where for $Y \in (S\mathbb{R}^{N\times N})^{G_2}$, the operator $\widetilde{\Omega}(Y)$ is an isometry on

$$\mathbb{R}^{NG_1} \oplus Range(R_{sym}(Y))$$

Proof. The function W is NC analytic by Corollary 2.4.2. Notice that W is a mixed NC ball map from $M\mathcal{B}_\beta^{G_1,G_2}$ to $M\mathcal{B}_\beta^{G_1,G_2}$ for all $\beta > 0$. So by Theorem 3.8.3 we see that h is a full isometry in z.

Now we use the stronger ball map property based on symmetric α. Given unit vector v in $\mathbb{R}^N$; define $\alpha_{t,M} := tvv^T + M(I - vv^T)$. Select t^* such that for $t > t^*$ there exist $M_t \in \mathbb{R}$ so that

$$U^TU + Z^TZ \preceq \alpha_{t,M_t}$$

and

$$[U^TU + Z^TZ]w \cdot w = \lim_{t\downarrow t^*} \alpha_{t,M_t} w \cdot w$$

if $w \in span\{v\}$. This is possible due to Lemma 4.4.5. Thus for given v we have binding for α_{t,M_t} in the limit only at v. The ball map property implies if $(Z,U) \in M\mathcal{B}_{\alpha_{t,M_t}}^{G_1,G_2}(N)$ for all $t > t^*$, then $W^T(Z,U)W(Z,U) \preceq \alpha_{t,M_t}$ for all $t > t^*$ and

$$W^T(Z,U)W(Z,U) \preceq \lim_{t\downarrow t^*} \alpha_{t,M_t}$$

binds exactly at v. Thus

$$< [g(U)^Tg(U) + g(U)^Th(Z,U) + h(Z,U)^Tg(U) \\ + h(Z,U)^Th(Z,U)]v, v >=< [U^TU + Z^TZ]v, v >$$

Replace Z by $-Z$, which, since h is linear in z, gives

$$< [g(U)^T h(U,Z) + h(U,Z)^T g(U)]v, v >= 0.$$

This holds for any v, U, and Z. So $g^T h + h^T g = 0$.

$$< [g(U)^T g(U) + h(Z,U)^T h(Z,U)]v, v >=< [U^T U + Z^T Z]v, v > \tag{4.25}$$

The hypotheses we set on the full isometry h and Theorem 3.8.3 yield that h has the form

$$h(u,z) = S(u)zV(u) \tag{4.26}$$

where $S(U)^T S(U) = I$ and $V^T(U)V(U) = I$ for all $U \in (S\mathbb{R}^{N\times N})^{G_2}$. Now we set $z = 0$ and obtain from Equation (4.25)

$$< g(U)^T g(U)v, v >=< U^T Uv, v >,$$

that is, $\|g(U)v\| = \|Uv\|$ for all $U \in (S\mathbb{R}^{N\times N})^{G_2}, v \in \mathbb{R}^N$. Also we have from Equation (4.25)

$$\|g(U)v\|^2 + < V(U)^T Z^T Z V(U)v, v >= \|Uv\|^2 + < Z^T Zv, v > \tag{4.27}$$

So $V(U)^T Z^T Z V(U) = Z^T Z$ for all Z. Since $V(U)^T = V(U)^{-1}$ we have $V(U)$ is a scalar multiple of the identity.

Now we " process" $g(u)$. Write it as $g(u) = \begin{pmatrix} g_1(u) \\ \vdots \\ g_K(u) \end{pmatrix}$ where each $g_i(u)$ is a power series. Consider a $g_i(u)$ and write it as

$$g_i(u) = \sum_{j=1}^{G_2} g_{i,j}(u)u_j.$$

Notice that $g_i(0) = 0$ since $g(0) = 0$. Set

$$C(u) := (g_{i,j}(u))_{\substack{i=1,\ldots,K \\ j=1,\ldots,G_2}}.$$

Set

$$\Omega(u) := \begin{pmatrix} S(u) & C(u) \end{pmatrix}$$

where $S(u)$ is defined in Equation (4.26). Then $W(u,z) = \Omega(u)\begin{pmatrix} z \\ u \end{pmatrix}$. Also by Equation (4.27) we have

$$\left\| \Omega(U) \begin{pmatrix} Z \\ U \end{pmatrix} v \right\| = \left\| \begin{pmatrix} Z \\ U \end{pmatrix} v \right\|$$

for all v.

For any v we can take Z to make Zv equal to any vector. Thus $\Omega(U)$ is an isometry on $\mathbb{R}^{G_1 \cdot N} \oplus RangeU$.

Define

$$R_{sym}(x,y) := \begin{pmatrix} R_{G_1+1}(x,y) \\ \vdots \\ R_{G_1+G_2}(x,y) \end{pmatrix}$$

By Lemma 4.4.3 $R_{sym}(x,y) = R_{sym}(y)$ since each component of $R_{sym}(x,y)$ is symmetric-matrix-valued. Thus we know

$$r(x,y) = \Omega(R_{sym}(y))R(x,y)$$

and

$$\widetilde{\Omega} = \Omega \circ R_{sym}.$$

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5 A Realization for Noncommutative Rational Expressions

In control theory applications of optimization problems, one comes across constraints on noncommuting variables in the form of matrix inequalities involving rational functions of matrix variables. The inequality

$$PA + A^T P + (PB + C^T D)R^{-1}(PB + C^T D)^T + C^T C < 0$$

is an example of an inequality involving a rational function of several matrix variables. Since matrix multiplication does not commute, we must deal with these rational functions of matrix variables as rational functions in noncommuting indeterminants when manipulating these expressions algebraically. When these inequalities involving rational functions of matrix variables can be written in the form of linear matrix inequalities, there are reliable numerical algorithms for finding optimal solutions in a feasibility domain. To this end, it is in our interest to study the relationships between positivity sets of rational functions in noncommuting variables and LMIs.

Consider $x = (x_1, \ldots, x_G)$ a vector of NC variables. Define an **NC linear pencil** as a function, $L(x, x^T)$, that can be written as

$$L(x, x^T) = A_0 + \sum_{i=1}^{g} A_i x_i + F_i x_i^T$$

for some matrices $A_0, A_i, F_i \in \mathbb{R}^{m\times m}$ for some m. For example if $x = (x_1, x_2)$, then

$$L(x, x^T) := \begin{pmatrix} x_1 + x_1^T & x_2 \\ x_2^T & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_1 + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_1^T + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x_2^T + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is an NC linear pencil. Also notice that a Schur complement of $L(x, x^T)$ is

$$r(x, x^T) := x_1 + x_1^T - x_2 x_2^T.$$

Using some basic facts about Schur complements we can conclude that for any tuple of $m \times m$ matrices (X_1, X_2),

$$L(X, X^T) > 0 \text{ iff } r(X, X^T) > 0.$$

The types of realizations that we will be discussing involve writing NC rational expressions as Schur complements of NC linear pencils. See Theorem 5.3.1

Result 5.0.7. *Suppose that $r(x, x^T)$ is a symmetric NC rational expression. Then there exists a symmetric NC linear pencil $L(x, x^T)$ such that $r(x, x^T)$ is the Schur complement of $L(x, x^T)$. In the case that all of the x_i are symmetric, there exists a symmetric NC linear pencil $L(x)$ whose Schur complement is $r(x)$.*

The fact that all symmetric NC rational expressions can be realized as the Schur complement of a symmetric NC linear pencil is not a new result [BR88]. The question of how one can construct such a linear pencil is answered in this chapter. We will give a more constructive proof that all symmetric NC rational expressions are Schur complements of NC linear pencils. From this easy proof an algorithm for constructing the pencil in question readily follows. The resulting algorithm has been implemented under the NC Algebra package for Mathematica.

To prove that all symmetric NC rational functions are realizable as Schur complements of linear pencils, we will first need a different type of realization. We say that a rational function, $r(x, x^T)$, has a **CKB representation** if there exist some $d \in \mathbb{N}$, $B, C \in \mathbb{R}^d$ and a $d \times d$ NC linear pencil $K(x, x^T)$ so that

$$r(x, x^T) = C^T K(x, x^T)^{-1} B.$$

For example consider again

$$r(x, x^T) := x_1 + x_1^T - x_2 x_2^T.$$

Using Lemma 2.1 one can see that

$$r(x, x^T) = C^T K(x, x^T)^{-1} B$$

when

$$C = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad K(x, x^T) = \begin{pmatrix} x_1 + x_1^T & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & x_2^T & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Result 5.0.8. *All NC rational expressions have a CKB representation.*

See Lemma 5.2.2 for a proof of the above result. Given a symmetric rational function we use this CKB representation to find a symmetric NC linear pencil whose Schur complement is the given rational function.

Again, let $x = \{x_1, \ldots, x_G\}$ denote NC variables. Let $\mathcal{N}_*(x)$ denote the free $\mathbb{R}$-algebra on the $2G$ generators $\{x, x^T\} = \{x_1, \ldots, x_G, x_1^T, \ldots, x_G^T\}$, i.e. the non-commutative polynomials on those $2G$ generators. The algebra has a natural involution determined by $x_j \mapsto x_j^T$, $x_j^T \mapsto x_j$, and

$$(x_{j_1} \cdots x_{j_n})^T = x_{j_n}^T \cdots x_{j_1}^T.$$

Let $p(x, x^T)^{-1}$ denote the inverse of $p(x, x^T)$ satisfying $p(x, x^T)^{-1} p(x, x^T) = 1 = p(x, x^T) p(x, x^T)^{-1}$ and thus $(p(x, x^T)^{-1})^T = (p(x, x^T)^T)^{-1}$. We say that $p(x, x^T) \in \mathcal{N}_*(x, x^T)$ is **symmetric** if $p(x, x^T) = (p(x, x^T))^T$. Let the **NC rational expressions of** $\{x, x^T\}$ **with real coefficients**, [1] denoted by $\mathcal{R}_*(x)$, be the closure of $\mathcal{N}_*(x)$ under finite numbers of inversions, products, transposes, and sums. Define

[1] We do not address the complicated issue of when two rational expressions are the same. Fortunately it is not needed here since there may be many different realizations for a single rational function.

$r(x, x^T)$ to be **symmetric** if $(r(x, x^T))^T = r(x, x^T)$. When the x_i are assumed to be symmetric denote the polynomials as $\mathcal{N}(x)$ and the rational expressions as $\mathcal{R}(x)$.

This chapter describes elementary constructions of "system realizations" of multivariable noncommutative symmetric rational functions. The type of system realization that we produce is most easily defined as a Schur complement of a symmetric NC linear pencil where these new terms are defined as below.

We say that a $d \times d$ matrix $L(x)$ (or $L(x, x^T)))$ is a **NC linear pencil** if the entries of L are polynomials in $\mathcal{N}(x)$ (or $\mathcal{N}^*(x)$) of degree one or less.

Recall that if $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a block 2×2 matrix, then

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS_1^{-1}CA^{-1} & -A^{-1}BS_1^{-1} \\ -S_1^{-1}CA^{-1} & S_1^{-1} \end{pmatrix} \tag{5.1}$$

when A and $S_1 = D - CA^{-1}B$ are invertible. Or equivalently,

$$M^{-1} = \begin{pmatrix} S_2^{-1} & -S_2^{-1}BD^{-1} \\ -D^{-1}CS_2^{-1} & D^{-1} + D^{-1}CS_2^{-1}BD^{-1} \end{pmatrix} \tag{5.2}$$

when D and $S_2 = A - BD^{-1}C$ are invertible. The matrices S_1 and S_2 are called **Schur complements** of M.

5.1 Symmetric Rational Functions in Symmetric Variables and LMIs

In many practical optimization problems when one can write inequality constraints in the form of linear matrix inequalities, there are very reliable numerical algorithms to solve the optimization problem. One big problem of interest would be to write a set of inequality constraints given as positivity domains of convex rational functions in the form of an LMI.

Given a rational function $r(x)$ in symmetric NC variables $x = (x_1, \ldots, x_g)$ we can define the **positivity domain of** r to be

$$\mathcal{D}_r := \{X \in \mathbb{R}^{NG \times N} : N \geq 1, r(X) > 0 \text{ and } X_i^T = X_i\}.$$

The component of $\mathcal{D}_r$ containing 0 we will denote as $\mathcal{D}_r^0$. Given a symmetric NC linear pencil $L(x)$ define similarly

$$\mathcal{D}_L := \{X \in \mathbb{R}^{NG\times N} : N \geq 1, L(X) > 0 \text{ and } X_i^T = X_i\}.$$

Finding a symmetric NC linear pencil $L(x)$ so that $\mathcal{D}_L = \mathcal{D}_r^0$ is what is meant by writing the positivity set of a rational function as an LMI.

In [HMV06], the authors give a complete classification of matrix convex rational expressions (see their Theorem 3.3) by representing such r in terms of a symmetric linear pencil

$$\mathcal{L}_\gamma(x) := I_d - \sum_j \mathcal{A}_j x_j + \begin{pmatrix} 0_{d-1} & 0 \\ 0 & -1+\gamma- \ r(0) \end{pmatrix}$$

in the noncommuting variables x_j, where $\mathcal{A}_j$ are symmetric $D \times D$ matrices. Namely, for γ a real number, $\gamma - r$ is a Schur complement of the linear pencil $\mathcal{L}_\gamma$. Moreover, given a matrix convex r, the set consisting of G tuples X of $\mathbb{R}^{N\times N}$ symmetric matrices

$$\{X : r(X) - \gamma I \text{ is negative definite}\} \tag{5.3}$$

has component containing 0 which is the same as the "negativity set",

$$\{X : \mathcal{L}_\gamma(X) \text{ is negative definite}\} \tag{5.4}$$

for $\mathcal{L}_\gamma$.

Given a symmetric convex rational function $r(x)$ it is not difficult to show that the algorithm presented in this paper will produce a Schur complement realization for $r(x)$. If that realization were "minimal" and "unpinned," then as a consequence of the above result the sublevel sets for $r(x)$ will be equivalent to LMIs. So we faced two clear problems with this initial realization.

1. Our realization is not necessarily minimal. The issue of taking our realization and writing an equivalent minimal realization is not difficult however.

2. After we find a minimal realization it may well be a pinned realization. There is an unpinning algorithm.

Algorithms to minimize and unpin the realizations for symmetric convex rational expressions of symmetric variables have been implemented in NCAlgebra.

5.2 A Representation of Functions in $\mathcal{R}_*(x)$

In this section we will show that for every $r(x, x^T) \in \mathcal{R}_*(x)$ there exist vectors C and B in $\mathbb{R}^d$ and a $d \times d$ NC linear pencil $K(x, x^T)$ such that $r(x, x^T) = C^T K(x, x^T)^{-1} B$. We will call such a function $r(x, x^T)$ **CKB representable** and let $Z(x, x^T)$ denote the set of CKB representable functions in $\mathcal{R}_*(x)$.

5.2.1 Some Properties of $Z(x, x^T)$

Lemma 5.2.1. *1. Suppose that $r(x, x^T) \in \mathcal{N}_*(x)$ is a degree one or less polynomial.*
Then $r(x, x^T) \in Z(x, x^T)$.

2. Suppose that

$$r_1(x, x^T) = C_1^T K_1(x, x^T)^{-1} B_1 \in Z(x, x^T)$$

and

$$r_2(x, x^T) = C_2^T K_2(x, x^T)^{-1} B_2 \in Z(x, x^T)$$

for some NC linear pencils $K_1(x, x^T)$ and $K_2(x, x^T)$. Then

(a) $r_1(x, x^T) + r_2(x, x^T) \in Z(x, x^T)$,

(b) $r_1(x, x^T) r_2(x, x^T) \in Z(x, x^T)$, and

(c) $(r_1(x, x^T))^T \in Z(x, x^T)$.

3. Suppose that $r(x, x^T) = C^T K(x, x^T)^{-1} B$ for some NC linear pencil $K(x, x^T)$ and that $r(x, x^T) \neq 0$. Then $r(x, x^T)^{-1} \in Z(x, x^T)$.

Proof. The proof gives constructions for each of the above items.

1. Suppose that $r(x, x^T) \in \mathcal{N}_*(x)$ and r is degree one or less. Then

$$r(x, x^T) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -r(x, x^T) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5.5}$$

   So $r \in Z(x, x^T)$.

2. (a) Notice that $r_1(x, x^T) + r_2(x, x^T) =$

$$\begin{pmatrix} C_1^T & C_2^T \end{pmatrix} \begin{pmatrix} K_1(x, x^T) & 0 \\ 0 & K_2(x, x^T) \end{pmatrix}^{-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}. \tag{5.6}$$

So $r_1(x, x^T) + r_2(x, x^T) \in Z(x, x^T)$.

(b) Notice that

$$\begin{pmatrix} -K_1(x, x^T) & B_1 C_2^T \\ 0 & K_2(x, x^T) \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} -K_1(x, x^T)^{-1} & K_1(x, x^T)^{-1} B_1 C_2^T K_2(x, x^T)^{-1} \\ 0 & K_2(x, x^T)^{-1} \end{pmatrix}.$$

So then

$$\begin{pmatrix} C_1^T & 0 \end{pmatrix} \begin{pmatrix} -K_1(x, x^T) & B_1 C_2^T \\ 0 & K_2(x, x^T) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ B_2 \end{pmatrix} \tag{5.7}$$
$$= C_1^T (K_1(x, x^T)^{-1} B_1 C_2^T K_2(x, x^T)^{-1}) B_2 = r_1(x, x^T) r_2(x, x^T).$$

Thus $r_1(x, x^T) r_2(x, x^T) \in Z(x, x^T)$.

(c) Since

$$(r_1(x, x^T))^T = B_1^T K_1(x, x^T)^{-T} C_1, \tag{5.8}$$

we have that $(r_1(x, x^T))^T \in Z(x, x^T)$.

3. Notice that by equation 5.1,

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -K(x, x^T) & B \\ C^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (C^T K(x, x^T)^{-1} B)^{-1} = r(x, x^T)^{-1} \tag{5.9}$$

when $r(x, x^T) \neq 0$. Thus when $r(x, x^T) \neq 0$, we have $r(x, x^T)^{-1} \in Z(x, x^T)$.

••

5.2.2 The Existence Theorem

Theorem 5.2.2. *Every $r(x,x^T) \in \mathcal{R}_*(x)$ is CKB representable.*

Proof. By Lemma 2.1.1 we see that all NC polynomials of degree one or less are CKB representable. Thus since each $p(x) \in \mathcal{N}_*(x)$ can be written as a finite sum of finite products of degree one or less NC polynomials and by Lemma 2.1.2, we have that $\mathcal{N}_*(x) \subseteq Z(x,x^T)$. Therefore by the closure properties of Lemma 2.1.2 and Lemma 2.1.3 and the construction of $\mathcal{R}_*(x)$ we have that $\mathcal{R}_*(x) \subseteq Z(x,x^T)$. ●●

5.3 The Pencil Result

In this section we will prove that any symmetric $r(x,x^T) \in \mathcal{R}_*(x)$ can be written as a Schur complement of some symmetric NC linear pencil.

5.3.1 The Realization Theorem

Theorem 5.3.1. *Suppose that $r(x,x^T) \in \mathcal{R}_*(x)$ is symmetric. Then there exists a symmetric NC linear pencil $L(x,x^T)$ such that $r(x,x^T)$ is the Schur complement of $L(x,x^T)$. In the case that all of the x_i are symmetric, there exists a symmetric NC linear pencil $L(x)$ whose Schur complement is $r(x)$.*

Proof. From Proposition 5.2.2, there exist $C, B \in \mathbb{R}^d$ and a $d \times d$ NC linear pencil $G(x,x^T)$ such that $\frac{1}{2}r(x,x^T) = C^T G(x,x^T)^{-1}B$. Define now

$$L(x,x^T) = \begin{pmatrix} 0 & B^T & C^T \\ B & 0 & G(x,x^T) \\ C & (G(x,x^T))^T & 0 \end{pmatrix}. \tag{5.10}$$

Notice that

$$\begin{pmatrix} B^T & C^T \end{pmatrix} \begin{pmatrix} 0 & G(x,x^T) \\ (G(x,x^T))^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} B \\ C \end{pmatrix}$$
$$= C^T G(x,x^T)^{-1}B + B^T (G(x,x^T))^{-T} C = r(x,x^T)$$

and $L(x,x^T)$ is symmetric. ●●

5.4 Matrix Valued Rational Functions

In this section we will show that both Theorem 5.2.2 and Theorem 5.3.1 generalize to the case of matrix valued rational functions. By matrix valued rational functions we mean $m \times n$ matrices $W(x, x^T)$ with coefficents in $\mathcal{R}_*(x)$.

5.4.1 Generalizing Theorem 5.2.2

To begin, we must extend what is meant by CKB representable. If $W(x, x^T)$ is a $m \times n$ matrix valued rational function, then $W(x, x^T)$ is called CKB representable when there exist $d \in \mathbb{N}, \mathfrak{C}^T \in \mathbb{R}^{m\times d}, \mathfrak{B} \in \mathbb{R}^{d\times n}$, and a $d \times d$ NC linear pencil $\mathfrak{S}(x, x^T)$ such that $W(x, x^T) = \mathfrak{C}^T \mathfrak{S}(x, x^T)^{-1}\mathfrak{B}$.

Theorem 5.4.1. *Every matrix valued rational function is CKB representable.*

Proof. Suppose that $W(x, x^T) = (r_{i,j}(x, x^T))_{i,j=1}^{m,n}$ for some $r_{i,j}(x, x^T) \in \mathcal{R}_*(x)$. Then by Theorem 5.2.2 for each i, j there exist $C_{i,j}, B_{i,j} \in \mathbb{R}^{d_{i,j}}$ and a $d_{i,j} \times d_{i,j}$ NC linear pencil $G_{i,j}(x, x^T)$ such that $r_{i,j}(x, x^T) = C_{i,j}^T G_{i,j}(x, x^T)^{-1}B_{i,j}$. Now define

$$\mathfrak{C}^T := \begin{pmatrix} C_{1,1}^T & \cdots & C_{1,n}^T & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & C_{2,1}^T & \cdots & C_{2,n}^T & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & & \ddots & & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & C_{m,1}^T & \cdots & C_{m,n}^T \end{pmatrix},$$

$$\mathfrak{S}(x, x^T) := \begin{pmatrix} G_{1,1} & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & G_{1,n} & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & G_{2,1} & \cdots & 0 & & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & G_{2,n} & & 0 & \cdots & 0 \\ & \vdots & & & \vdots & & \ddots & & \vdots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & G_{m,1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & G_{m,n} \end{pmatrix},$$

$$
\mathfrak{B} := \begin{pmatrix} B_{1,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_{1,n} \\ B_{2,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_{2,n} \\ & \vdots & \\ B_{m,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_{m,n} \end{pmatrix}.
$$

Let $d = \sum_{i,j=1}^{m,n} d_{i,j}$. Notice that $\mathfrak{C}^T \in \mathbb{R}^{m\times d}, \mathfrak{B} \in \mathbb{R}^{d\times n}$, and $\mathfrak{S}(x, x^T)$ is a $d \times d$ NC linear pencil. In addition it is an easy computation to show that $W(x, x^T) = \mathfrak{C}^T \mathfrak{S}(x, x^T)^{-1}\mathfrak{B}$. Thus we have that every matrix valued rational function is CKB representable. ●●

5.4.2 Generalizing Theorem 3.1

Theorem 5.4.2. *Suppose that $W(x, x^T) \in M_m(\mathcal{R}_*(x))$ is symmetric. Then there exists a symmetric NC linear pencil $L(x, x^T)$ such that*

$$
W(x, x^T) = \textit{Schur Complement}(L(x, x^T)).
$$

In the case that all of the x_i are symmetric, there exists a symmetric NC linear pencil $L(x)$ whose Schur complement is $W(x)$.

Proof. By Theorem 5.4.1 there exist $d \in \mathbb{N}, C_1^T \in \mathbb{R}^{m\times d}, B_1 \in \mathbb{R}^{d\times m}$, and a $d \times d$ NC linear pencil $G_1(x, x^T)$ such that $\frac{1}{2}W(x, x^T) = C_1^T G_1(x, x^T)^{-1}B_1$. From this point the proof follows identically to that of Proposition 5.3.1. ●●

5.5 Symbolic Computation of Realizations

The theorems in this chapter convert readily to computer algorithms for finding realizations. In fact they were enhanced and implemented by John Shopple under the NC Algebra package for Mathematica.

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