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Several Duality Theorems for Interlocking Ridge and Channel Networks

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Many fluviially eroded terrains display an intimate interaction of their patterns of channel and ridge lines. Under the assumption that there is exactly one outer ridge link positioned between any two neighboring outer channel links, the paper analyzes such interlocking networks by investigating their dual graphs. Several formal relationships are established which translate into a set of both existing and new theorems describing the close interdependency of channel and ridge parameters: Interlocking networks are of equal magnitude, the link numbers of the ridge and channel paths enclosing channel and ridge subnetworks have the same mean value, the link number of the boundary path enclosing a subnetwork is equal to the number of boundary paths passing through the root of the subnetwork, there exists a one-to-one correspondence between the inner nodes of the two networks as well as between their paths, and every such pair of corresponding ridge and channel paths delineates an area that is closed with regard to slope runoff.

INTRODUCTION

One of the most common and conspicuous landform patterns is the landscape of alternating ridges and valleys created by fluvial erosion. Its surface is formed by a continuous band of slopes which intersect at their upper end to form ridges and at their lower end to form channels. The three-dimensional distribution of both channel and ridge lines is completely controlled by the combined effects of channel and slope erosion.

In areas of roughly uniform environmental control the slope distribution along lines of steepest descent tends to be relatively simple and consistent. Consequently, the spatial distribution of ridges and their junctions display a strikingly close relationship to the distribution of channels and their bifurcations. While this phenomenon is neither unfamiliar nor unexpected, its description is largely limited to the cartographic display of particular terrains. That is, except for the few research efforts referred to below, intertwined patterns of channels and ridges are not described by any formal principles governing their distribution but rather through the presentation of the raw data. It is the purpose of this paper to unravel some of the systematic interdependencies that seem to exist between such interlocking ridge and channel patterns.

Well before, but especially since, *Horton's* [1945] famous paper on erosional stream development, natural drainage networks have been subjected to geometric and topologic analysis. These analyses [*Cayley*, 1859; *Horton*, 1932; *Strahler*, 1952; *Shreve*, 1966; *Wartzt*, 1968, 1975; *Schumm*, 1977] treated various spatial and connectivity properties like drainage density or network order as they relate to each other and to erosional and depositional processes which in turn depend on climatic regime and local geology. *Shreve's* [1966] definition of topological randomness in natural channel networks, in particular, gave rise to a large body of research that investigated network structure by itself as well as its dependency on external parameters [*Abrahams*, 1984]. Comparable studies of ridge patterns are remarkably few in number and the considerable conceptual problems associ-

ated with the formal description and explanation of ridge lines have barely been mastered. It is therefore not surprising that the peculiar interrelations governing the geometric layout and the internal connectivity of interlocking ridge and channel networks are, at this point, more a matter of generalized observations and impressions than of theoretical prediction and empirical verification. Probably the first researchers who recognized and described natural channel and ridge lines as interrelated formal structures were *Wartzt* [1966, 1968, 1975] and *Wartzt and Woldenberg* [1967]; however, the *Wartzt/Woldenberg* approach leads to both theoretical and practical difficulties and has not seen much application in the literature [*Pfaltz*, 1976; *Mark*, 1979]. *Mark* [1979, 1982] studied patterns of peaks interlinked by ridge lines and isolated specific statistical tendencies in the sets of ridge trees he had obtained from topographic maps.

The first quantitative study investigating the interdependency between selected parameters of channel (sub)networks and adjacent ridge lines that lead to specific empirical findings was *Abraham's* [1980] work on the relation between divide angles and the length distribution of channel links. *Werner's* [1982, 1986, 1988] approach to the channel/ridge dependency problem differs in that it addresses topological relations only; however, it does so for entire ridge and channel networks rather than their local building blocks. Applying the principal ideas of network delineation by *Krumbein and Shreve* [1970], *Werner* devised an operational procedure that permits, in a fairly unequivocal fashion, the identification of ridges on topographic maps. Tying the definition of each outer ridge link to the elevated terrain "between" neighboring outer channel links rendered interlocking ridge and channel networks as dual phenomena; in turn, the dual relationship permitted the deduction of several duality theorems establishing precise relations between topological ridge and channel network features.

The current paper continues this work in two directions: It develops a new methodology of mathematical investigation, and on this basis, it proceeds to prove both new theorems (theorems 1–3) and theorems reported in earlier papers (theorems 4–5). While applicable definitions and concepts will be briefly reiterated below, the reader is referred to *Werner* [1988] for a more detailed review and explication of the subject matter.

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Fig. 1. (a) Section of a pair of interlocking networks of unspecified magnitude. In line with the definition of such networks, their outer links are strictly alternating. (b) Interlocking networks of finite magnitude. Most natural networks selected for research purposes are subnetworks of much larger networks in which they are embedded and are, in this sense, incomplete. (c) A pair of complete and finite interlocking networks covering a sphere.

A BRIEF SKETCH OF ASSUMPTIONS, METHODS, AND LIMITATIONS

Since a major part of the paper consists of somewhat tedious graph theoretical concepts and arguments, it might be helpful to temporarily relax the standards of exactitude and completeness and, instead, to try to communicate the main premises and methods of reasoning through a more transparent language supported by illustrations.

As in any modeling research approach, we first simplify and idealize the object of investigation and, in doing so, introduce appropriate operational definitions, most of which come from the existing literature on channel networks and the theory of graphs.

Definition of Interlocking Networks

Like channel networks, ridge networks are graph theoretically coded as trivalent planar labeled trees. A channel network and the network of ridges delineating its basins and subbasins are said to be interlocking if there is one and only one outer ridge link positioned between any two neighboring outer channel links (Figures 1a and 1b). As such, interlocking networks are idealized representations of natural networks that provide an essentially accurate rendition of the main ridges and channels but only a rough approximation at the microlevel of individual outer links.

Duality Relations

The interlocking and alternating positioning of their outer links creates a reversible interdependence relation between interlocking networks, causing any statement about one in reference to the other to apply equally in the reverse direction. This peculiar duality relation between interlocking networks is one of the main concepts of the paper and will be referred to as type II duality to distinguish it from the familiar (and entirely different) graph theoretical concept of duality denoted as type I. To emphasize its generality we will frequently label two interlocking networks with X and Y without specifying which one is the ridge and which one is the channel network.

Methodology

To investigate the interdependence of interlocking networks we construct and examine their (type I) dual graphs (Figures 2 and 3). All proofs and results are first formulated in the language of these dual structures, then the conclusions are translated back into the context of the primal graphs, that is, the interlocking networks, producing several theorems describing the interrelatedness of interlocking networks.

Domain of Applicability

Since, in the case of natural networks, every ridge is positioned between channels and vice versa, the interlocking networks shown in Figure 1b are necessarily incomplete. Applying the theorems in a morphologically "meaningful" way therefore requires an additional assumption that insures their completeness. This can be accomplished in two ways: either we assume that the networks are of infinite magnitude and cover the entire plane (Figure 1a) or that they are finite and cover the surface of a finite sphere (Figure 1c). While the second assumption is of mathematical interest only, the first comes fairly close to the topographic reality because most natural channel and ridge networks that are the subject of empirical research are subnetworks of much larger networks whose magnitudes, for all practical purposes, approach infinity. Thus, keeping in mind the definition of interlocking networks, the theorems apply to all natural networks and network paths as long as they are recognized as being embedded in interlocking ridge and channel networks of infinite magnitude. For large but finite networks the theorems apply only in approximation; they apply less as magnitude decreases.

GRAPH THEORETICAL DESCRIPTION OF INTERLOCKING NETWORKS

We will start the investigation by simplifying both channel and ridge patterns to the point where they can be subjected to graph theoretical analysis. Researchers investigating the connectivity structure of channel networks found it convenient to represent them as trivalent planar rooted trees [Shreve, 1966; Smart, 1972] because this assumption greatly facilitates mathematical analysis and is also quite compatible with the cartographic representation of most natural channel networks.

With regard to the patterns of ridges we will consider all and only those ridge lines that delineate the basins and subbasins of a given channel network (Figures 1a and 1b).

Their identification on a topographic map follows, in principle, the procedure developed by *Krumbein and Shreve* [1970] for channels and adapted for ridge lines by *Werner* [1988]. In graph theoretical terms, these ridge lines also form a tree, and at least for the time being, we will assume that this ridge tree is also trivalent; this can be achieved by decomposing each node that appears to be of degree $x > 3$ into an interlinked set of $x - 2$ nodes of degree 3. The justification for this assumption lies ultimately in the impossibility of its empirical verification: There is simply no way to decide, with absolute certainty and mathematical exactitude, the valency of many network nodes in the field. Thus we adopt an assumption that is mathematically convenient and simultaneously compatible with an imprecisely perceived reality. If a ridge network is a subnetwork of a larger network of ridges, its root is, of course, the connecting link (the root of the larger network will be defined in a subsequent section).

We will also stipulate that the drainage basin boundaries of any two neighboring outer channel links share exactly one outer ridge link in common (Figure 1). Thus there exists "between" any two neighboring outer channel links exactly one outer ridge link. Unlike the previous specifications, this assumption is sometimes difficult to reconcile with the delineation of ridge lines on the basis of contour cusps. For the following analysis and its results we therefore have to keep in mind that the ridge network as defined here is often no more than an approximation at the local level, for the relatively small areas located "between" neighboring outer channel links. At the "macrolevel," however, the ridge and channel networks, whether observed in the field or delineated by the contour crenulation method of *Krumbein and Shreve* [1970] and *Werner* [1988], fit the axiomatic assumptions spelled out above quite well. For the balance of this paper we will call two networks interlocking if and only if they are both trivalent planar rooted trees whose outer links occupy alternating positions, as described above.

Following the established terminologies in geology and graph theory, we call the nodes of channel and ridge networks outer or inner nodes depending on whether they are of degree 1 or 3, we call links with at least one outer node as end point outer links and links with inner nodes as end points inner links, and we distinguish between outer, intermediate, and inner network paths depending on whether both, one, or none of the end links of a path are outer network links. The magnitude of a network is defined as its number of outer links minus the outlet link or root (the root of a ridge network will be defined in a subsequent section).

Each network link separates the network into two components; the component which does not contain the network root will be called the subnetwork defined by the link in question. The magnitude of a network link is defined as the magnitude of its associated subnetwork. For the particular case of the outlet link we define the entire network as the associated network. Hence the magnitude of a network's outlet link is equal to the magnitude of the network.

Let C be a channel network with $n + 1$ outer links, one of which serves as the network outlet. Since C is a rooted planar tree, the outer links form an ordered sequence and can be labeled c_i , where $i = 0, 1, \dots, n$ with c_0 being the outlet link. We denote the sequence $\{c_i | i = 0, 1, \dots, n\}$ by $S(C)$. Associated with each outer link c_i is exactly one outer node p_i ; we will use the same symbol for the sequence of

outer nodes of C . We also introduce the convention that for any two elements X, Y in any sequence S the symbolic relation $X < Y$ means that X occupies a position before the position of Y if we read the elements of S from left to right.

Let R be the interlocking ridge network of C as defined earlier. Note that the definition of outer ridge links as strictly alternating with outer channel links insures that the two networks are of equal magnitude n ; furthermore, the definition imposes the sequential ordering of the outer links of the channel network C onto the set of outer links of the ridge network R . That permits us to identify a particular outer ridge link, the one positioned between the neighboring outer channel links c_0 and c_1 , as the first outer ridge link which we will label r_0 . As a matter of convention, r_0 is designated as the root of R (Figure 1b). Generally, we will designate the outer ridge link located between the outer channel links c_i and c_{i+1} as r_i and designate its outer node as q_i . We use $S(R)$ as the symbol for the sequence $\{r_i | i = 0, \dots, n\}$ of outer links (or outer nodes q_i) of R and denote the double sequence of alternating outer ridge and channel links (nodes) of C and R by $S(C, R)$. The order relation $X < Y$ for elements of the sequence $S(C)$, as established above, also applies to the elements of the sequences $S(R)$ and $S(C, R)$.

Within the channel network C each link c of magnitude $m(c)$ defines a subnetwork C_c of magnitude $m(c) < n$ (unless $c = c_0$ in which case $m(c) = n$ and $C_c = C$). Excluding the outlet link, the outer links of C_c form a subsequence of consecutive links within the sequence $S(C)$ of all outer links of C . Let c_i and c_j denote the first and last outer link of C_c within $S(C)$ so that $m(c) = |j - i + 1|$. By construction there exist exactly two outer ridge links r_{i-1} and r_j adjacent to the sequence of outer links of C_c within the double sequence $S(C, R)$. Since (1) the subnetwork C_c and its outlet channel link c uniquely define each other, (2) the same is true for the subnetwork C_c and the two ridge links r_{i-1}, r_j enclosing the sequence of outer channel links of C_c within $S(C, R)$, and (3) r_{i-1} and r_j are the end links of exactly one ridge path (r_{i-1}, r_j) in R because R is a tree, it follows that the channel link c and the ridge path (r_{i-1}, r_j) are one-to-one related. Hence there exists a one-to-one relationship between the $2n - 1$ links of C and a corresponding subset of $2n - 1$ ridge paths in R . We will denote as b_c the ridge path (r_{i-1}, r_j) associated with the channel link c . To restate the chain of one-to-one mappings in symbols,

$$c \Leftrightarrow C_c \Leftrightarrow \{c_i, c_{i+1}, \dots, c_j\} \Leftrightarrow \{r_{i-1}, r_j\} = b_c$$

While in graph theoretical terms the one-to-one mapping between the elements of the set of all links of C and a corresponding subset from the set of all paths in R is simply a formal consequence of the definition of interlocking networks, it has important substantive meaning in physical terms. Since any outer channel path originating in an outer link c_k of the subnetwork C_c and terminating outside of C_c must pass through the link c , and since, by construction, $r_{i-1} < c_i \leq c_k \leq c_j < r_j$, it follows that b_c , the ridge path connecting the outer ridge links r_{i-1} and r_j , is the boundary of the area drained by the channel subnetwork C_c . Hence the subset $\{b_c\}$ of ridge paths identified above constitutes the set of boundaries of the basin and subbasins drained by the network C and its subnetworks C_c . We therefore call b_c the boundary path associated with the link c .

To summarize, in interlocking networks C, R there exists for each channel link c a ridge path b_c ; this path encloses the

subnetwork C_c defined by c in the particular sense that its two outer links directly precede and follow the first and last outer link of C_c within the double sequence $S(C, R)$ of the outer links of C and R . In substantive terms the path b_c forms the boundary of the basin drained by the subnetwork C_c defined by the link c .

SOME SIMPLE (TYPE II) DUALITY RELATIONS BETWEEN INTERLOCKING NETWORKS

At this point it is essential to note that the particular properties of interlocking networks utilized in the arguments developed above are the same for each network, namely, both networks are trivalent planar rooted trees, and their respective relationships to each other follow the same definition which stipulates that the outer links of one form an alternating sequence with the outer links of the other. Thus the entire argument presented in the previous section remains valid if we exchange the concepts of ridge and channel and their derivatives. This, of course, means that the concepts of ridge and channel are dual concepts. It should be clear, however, that the duality applies only to the specific network properties identified above and to concepts derived from them; thus the internal connectivity of the two networks is not part of this duality relation.

Since a well established definition of dual graphs exists that is quite different from the one discussed here, we will refer to the duality between interlocking networks as type II duality and call the common notion of dual graphs type I. Both duality types are of prime importance for this paper, type I constituting the main tool for proving duality relations of type II.

As a consequence of the type II duality of the concepts of ridge and channel in interlocking networks C, R , we can duplicate the forgoing argument linking channel links to ridge boundary paths after exchanging the two concepts. Thus there exists exactly one channel path in C for every ridge link r in R . This path, denoted by b_r , forms the boundary of the area that contains all of and only the ridges that make up the ridge subnetwork R_r defined by the ridge link r . We will call this area the drainage area of the ridge subnetwork R_r . In strictly formal terms this area and its ridge network correspond to the familiar concepts of a drainage basin and the channel network by which it is drained. To acknowledge their equivalence we will drop the term basin altogether and simply refer to them as channel drainage area and ridge drainage area, respectively.

The intimate correspondence of the two types of drainage areas becomes readily apparent, in both their forms and functions, in the following examples.

1. While a channel drainage area (i.e., a drainage basin) can be visualized as an overall concave form subdivided by convex forms with ridge lines at their top, a ridge drainage area can be visualized as an overall convex form subdivided by concave forms (valleys) with channel lines at their bottom.

2. Just as surface runoff does not cross the ridge path forming the boundary of a channel drainage area (i.e., a drainage basin), it does not cross the channel path forming the boundary of a ridge drainage area, that is, the area enclosing a ridge network.

3. While in a channel drainage area slope runoff moves from the ridge boundary to the channel network, in a ridge

Fig. 2. A network N (dashed curve) and its type I dual graph N^* (solid curve). The vertices X^*, Y^* and the connecting edge e^* in N^* are the type I dual elements of the regions X, Y and their common link e in N .

drainage area it moves from the ridge network to the channel boundary.

4. While a channel drainage area contains a sequence of ridge subnetworks that merge with the area boundary and subdivide the area into a set of ridge drainage areas, a ridge drainage area contains a sequence of channel subnetworks that merge with the area boundary and subdivide the area into a set of channel drainage areas.

The dual relations between interlocking ridge and channel networks sketched out previously will be expanded in a later section of this paper. At this point it should be reiterated that they represent a type of duality (type II) that is, in principle, different from the familiar graph theoretical type I duality [e.g., Price, 1971] that will be the subject matter of the following sections.

DUAL GRAPHS OF INTERLOCKING RIDGE AND CHANNEL NETWORKS

To further investigate the relationships that govern interlocking ridge and channel networks we will continue the analysis by studying their type I dual graphs. This strategy will render as trivial certain ridge-channel relationships which otherwise require rather lengthy and complicated proofs when derived in the context of the original interlocking ridge and channel networks [Werner 1982, 1986]. More importantly, it also permits comparatively easy recognition and proof of new and unexpected interdependencies.

It might be useful to briefly restate the principles which relate type I primal and dual graphs to each other. Inasmuch as the concept of type I dual graphs will be applied to both channel and ridge networks, we use "neutral" symbols rather than the particular ridge/channel symbols introduced earlier. Assume for the moment that the outer links of a network N are connected in a single node.

1. To each region X of N assign a vertex X^* of a new graph N^* (keeping in mind that the area outside the graph is itself a region, see Figure 2).

2. If X, Y are two regions of N sharing a link e in common, connect the vertices X^*, Y^* by an edge e^* .

The resulting set of vertices X^* and edges e^* is the type I dual graph N^* of the network N . Note that, in line with the principle of type I duality, the same construction procedure when applied to N^* will render the original network N .

As a consequence of joining the outer network links of N in a new node, both the type I primal and the dual graph consist of "closed" regions only, that is, the regions are completely delineated by cycles of the respective graph, with each region in one being the image of a node (vertex) in the other. In particular, the new node has as its image the region consisting of the complement to the finite regions of the type I dual graph, that is, "the rest of the universe."

Our treatment of the "open" regions between neighboring outer network links as if they were regions enclosed by cycles is simply a matter of mathematical convenience, as it permits the direct application of the concept of type I dual graphs and has no adverse consequences for our line of reasoning later on. Henceforth we will eschew this constructional detail and simply keep in mind the difference between the open regions of N located between neighboring outer links of N and the closed regions of N^* as we refer to the respective regions of N and N^* .

Let C^* and R^* denote the type I dual graphs of the interlocking ridge and channel networks C and R , respectively. Each of these graphs can be represented without loss of (topological) generality as a regular triangulated $(n + 1)$ -gon consisting of vertices and edges, as shown in Figure 2, where n is again the network magnitude. For any given network (whether C or R) and its type I dual image the dual elements are obvious: outer edges are the images of outer links, inner edges correspond to inner links, and polygonal regions (i.e., triangles) are the images of inner nodes, just as the vertices of the polygon are the images of the (open) regions between neighboring outer links. Likewise, the image of a path is a sequence of edges in which consecutive edges belong to the same triangle and no more than two edges belong to the same triangle; furthermore, subnetworks correspond to subpolygons whose vertices form a subsequence of consecutive vertices within the sequence of the polygon's vertices. Since all of these relations are mappings between type I dual elements, they are, of course, reversible: the image of the image of any object is that object itself.

For the remainder of this paper the term correspondence will be used only in the restricted sense of a one-to-one mapping between the elements of sets of equal cardinality. We will abbreviate any one-to-one correspondence between objects x, y by $y = x^*$ or $x = y^*$; such correspondences include, in particular, the type I duality relations between networks and the triangulated polygons that are their type I dual images. The main object of this paper, the type II duality features of interlocking ridge and channel networks, will be explicitly described as such.

To represent the graphs C^* and R^* simultaneously and in such a way that the interlocking quality of their type I duals C and R is also represented, we superimpose the two such that their vertices, and therefore their outer edges, form an alternating sequence $S(C^*, R^*)$ in exact correspondence to the sequence $S(C, R)$ of outer links and nodes of the interlocking networks C and R . Starting with a pair of interlocking networks, Figure 3 illustrates the construction and superposition of the corresponding type I dual graphs.

Fig. 3. Construction of the type I dual graph (B) of two interlocking networks (A). Each of the networks is first replaced by its corresponding type I dual graph, as shown in Figure 2. The superposition of these graphs forms a double triangulation that is the type I dual graph of the original pair of interlocking networks. The type I duality of A and B guarantees that any statement that applies to B can be translated into an equally valid statement about A. Since it is much easier to prove certain statements about B rather than A, this type I duality relation provides the main methodology for this paper.

Inasmuch as the orientation of the labeling of the outer links of the ridge network introduced earlier is in the opposite direction of the labeling orientation of the outer channel links, we will actually work with the mirror image of the triangulated polygon that is the type I dual of R ; note that in that transformation, information is neither lost nor altered provided, of course, we keep in mind that from now on the symbol R^* refers to the mirror image of the type I dual of the ridge network R .

As a matter of convenient illustration, the vertices of C^* and R^* in the graph B of Figure 3 are positioned on a circle. We refer to the superposition of the two triangulated polygons C^* and R^* as a double triangulation and denote it by the symbol (C^*, R^*) .

Inasmuch as each open region between neighboring outer links of one network contains, by construction, exactly one outer link and node of the other network, it is evident that the polygonal vertices of C^* and R^* are not only the dual images of these regions but that they also correspond to the outer links and nodes contained in these regions. Specifically, if r_i is an outer ridge link and q_i is the associated outer ridge node located between the outer channel links c_i and c_{i+1} of C , then the vertex of C^* , which is the type I dual image of the region located between c_i and c_{i+1} , corresponds to the outer ridge node q_i of the ridge network R . A

parallel statement applies to the outer nodes p_j of C and the vertices of R^* . The particular design of the double triangulation (C^*, R^*) developed above guarantees that the sequence of alternating outer nodes of the two networks is mapped onto the sequence of alternating vertices of the polygons such that sequential order is preserved.

To capture the correspondences that map the outer node (link) x of a network X onto the open region $y(x)$ of the interlocking network Y , which in turn is mapped by type I duality onto the vertex $z(y(x))$ of the polygon Y^* , we take advantage of the transitivity of the correspondence relations y, z and label the vertex $z(y(x))$ by x^* . We recall the construction principle for type I dual graphs, which specifies that the image of a network link is the edge connecting the vertices which are the images of the two network regions sharing that link in common. Thus if the outer channel link c_k is shared by the two regions containing the outer ridge links r_{k-1}, r_k with outer ridge nodes q_{k-1}, q_k , then the end points of the type I dual image of c_k , that is, the edge c_k^* , are the vertices q_{k-1}^* and q_k^* . Similarly, if the inner channel link c is shared by the two regions containing the outer ridge links r_w, r_z with outer ridge nodes q_w, q_z , then $c^* = (q_w^*, q_z^*)$.

We now combine this result with the earlier finding that there exists for each network link a path in the interlocking network that forms the boundary of the drainage area associated with the subnetwork defined by that link. Specifically, if c is a channel link and c_{w+1} through c_z are the outer links (but excluding the outlet link c) of the subnetwork C_c defined by c , then the ridge path defined by the outer ridge links r_w, r_z with outer nodes q_w, q_z is the boundary path associated with c .

Let c be a channel link, and let r_w, r_z and q_w, q_z be the outer ridge links and nodes located in the two (open) regions U, V that share the link c in common, so that in line with the foregoing discussion we have $c^* = (q_w^*, q_z^*)$. Then the paths delineating these two regions have as their end links the outer channel links c_w, c_{w+1} and c_z, c_{z+1} (Figure 4). Both of these paths include c as one of their links, and we can therefore assume without loss of generality that c_{w+1} and c_z are outer links of the subnetwork C_c defined by c while c_w and c_{z+1} are not. Moreover, c_{w+1} and c_z are the first and last outer links of C_c within the sequence $S(C)$ because their neighbors in $S(C)$, c_w and c_{z+1} , are not links of C_c . Hence the outer ridge links r_w, r_z directly precede and follow the sequence of outer channel links $c_{w+1}, c_{w+2}, \dots, c_z$ within the double sequence $S(C, R)$, which means that the ridge nodes q_w, q_z define the boundary path b_c associated with the link c . At the same time, however, these nodes correspond to the vertices q_w^*, q_z^* in (C^*, R^*) that are the end points of the edge c^* . Thus c^* is not only the dual image of the channel link c but also corresponds to the boundary path b_c associated with that link.

The preceding discussion could be considered as superfluous, as its result is a direct consequence of earlier findings: since c and its boundary path b_c are one-to-one related, and since the same is true for c and its dual image c^* , it follows that b_c and c^* are also one-to-one related. What matters here for use in later analysis is the graph theoretical manifestation of the two correspondences and their transitivity: the linkage of the ridge nodes q_w, q_z in R constitutes the boundary path of c , and the linkage of their images q_w^*, q_z^* in R^* is the image c^* of c . Or, in symbols,

Fig. 4. A subnetwork and its boundary path, both embedded in a pair of interlocking networks. The subnetwork is defined by its root link c , and the boundary path is defined by its two end links, r_w and r_z , which are positioned on either side of c .

$$\begin{array}{ccc} q_w, q_z \Leftrightarrow b_c \Leftrightarrow c & & \\ \Downarrow & & \Downarrow \\ q_w^*, q_z^* \Leftrightarrow & & c^* \end{array}$$

But, to avoid any confusion, it might be useful to clearly state that the dual image of the boundary path b_c is not the edge c^* but rather a sequence of edges in R^* that are the images of the links of b_c . These edges, incidentally, are all and only the edges of R^* intersecting the edge c^* , as we will prove later.

That the double triangulation (C^*, R^*) created through the superposition of the type I dual images of C and R is in fact the type I dual structure of the pair of interlocking networks C, R is evident from the fact that C and R can be reconstructed from (C^*, R^*) as interlocking networks through the same process by which the double triangulation was constructed from C and R . Reading Figure 3 from B to A provides an example of the transformation of a double triangulation back into a pair of interlocking networks.

SEVERAL DUALITY II RELATIONS BETWEEN INTERLOCKING NETWORKS

The discussion in this section will repeatedly take advantage of the fact that each edge in (C^*, R^*) is not only the dual image of a particular link in (C, R) but also corresponds to the boundary path associated with (the drainage area of the subnetwork defined by) that link.

The following lemma establishes an isomorphism between links in (C, R) and edges in (C^*, R^*) that provides the base for the proofs of several subsequent theorems.

Lemma. Let c and r be two links of the interlocking networks C and R , and let c^* and r^* refer to the edges of the triangulated polygons C^* and R^* that are the dual images of c and r . Then c is a link of the channel boundary path

associated with r , and r is a link of the ridge boundary path associated with c if and only if the edges c^* and r^* in the double triangulation (C^*, R^*) intersect.

For a proof see the appendix. The lemma has several noteworthy consequences which we will state as a set of theorems.

Theorem 1. In interlocking networks, channel and ridge boundary paths have the same mean number of links.

Multiple application of the lemma shows that every edge c^* in C^* is intersected by as many edges of R^* as the boundary path b_c associated with c has links. Specifically, the edges of R^* intersecting c^* are the duals of all and only the links of the boundary paths b_c . Since the number of intersections of edges from C^* by edges from R^* is, of course, equal to the number of intersections of edges from R^* by edges from C^* , the above theorem follows.

Theorem 1 is a nice example of a statement that in the original context is neither obvious nor trivial but that can turn out to be both once we subject the context to an appropriate transformation, in our case restating the theorem in the context of the type I dual graphs of the interlocking networks. Even though the proof is trivial in that it is merely an identity statement, the result is noteworthy not only by itself but also in light of the following comments:

1. Inasmuch as the geometric lengths of ridge links in a given homogeneous area follow a unimodal frequency distribution similar to the length distribution of channel links [Werner, 1973], theorem 1 states that the mean lengths of ridge and channel boundary paths are approximately equal.

2. The statement of theorem 1 is independent of the topology of the two networks: changing the topology of one while keeping the topology of the other invariant changes the mean number of boundary links not only in one but in both networks and does so by the same amount.

3. It is also interesting that the stated equality of the first moments of the frequency distributions of the link numbers of boundary paths in the two networks does not extend to the second moments which are (generally) different. Both this and the following comment can easily be verified by choosing a suitable example.

4. Likewise, while in any given pair of interlocking networks the mean link numbers of ridge and channel boundary paths are equal, the same does not hold for the mean magnitudes (and therefore the mean number of links) of the subnetworks they enclose.

Theorem 2. The number of boundary paths passing through a given link is equal to the number of links of the boundary path associated with that link.

The proof follows from consideration of all edges of one triangulated polygon intersecting a particular edge of the other and is again based on repeated application of the lemma.

Theorem 3. There exists a one-to-one mapping between the inner nodes of interlocking ridge and channel networks. Specifically, the mapping relates the node X of one network to that node Y of the other network which forms the intersection of the three boundary paths associated with the three links interconnected in X . The same mapping operation relates Y to X , that is, X and Y are (type II) dual nodes.

For a proof see the appendix. In the substantive context of a natural drainage system the theorem relates the node, where a network splits up into two subnetworks, to that node of the network's drainage area boundary that separates

Fig. 5. A pair of corresponding nodes in interlocking networks. The node P joins three links between which three outer links of the other network are positioned. These outer links define three paths which intersect in the node Q. With regard to this relation the nodes P and Q are interchangeable: starting out with Q rather than P and proceeding as above will lead to P as the corresponding node in the other network.

the part of the boundary bordering one subnetwork from the part bordering the other. Moreover, it recognizes the nodes as dual elements: exchanging the concepts of ridge and channel will reverse the relationship between the two nodes (see Figure 5). It is not clear, however, what if any morphological "meaning" can be attributed to the peculiar one-to-one mapping between the inner nodes of C and R as described by the theorem.

As the following theorems will further demonstrate, even though the type II duality between interlocking ridge and channel networks excludes their internal connectivity, their structural interdependence is nevertheless considerable.

Theorem 4. Excluding the boundary paths from the sets of all paths of the interlocking networks C and R , there exists a one-to-one mapping between the network paths of C and R . In particular, if one of the paths is an outer network path, then the other path is an inner path and vice versa, and if one is an intermediate path, then the other will be intermediate too.

For a proof of the theorem see the appendix. We will call any two paths of C and R corresponding if they are mapped onto each other as stated in theorem 4.

The next theorem describes a somewhat surprising property of pairs of corresponding network paths. The description requires the following two definitions. First, two paths in a pair of interlocking networks are said to be quasi-connected whenever (1) one is a channel path and the other is a ridge path, (2) at least one end node U of one path is an inner node and at least one end node V of the other path is an outer node, and (3) U is located on the boundary path of the outer link of which V is the end node.

Furthermore, we say that a sequence of paths in which consecutive paths are quasi-connected defines a quasi-closed curve if (1) the sequence is cyclical, that is, the first

and the last path are quasi-connected and (2) any network node belongs to at most one path of the sequence.

Morphologically speaking, areas delineated by quasi-closed curves have an important and specific meaning: they represent all and only the areas within a fluvially eroded terrain that are closed with respect to nonchannel surface flow. For example, the maximum area that might be affected by the downslope migration of surface pollutants is delineated by the minimum quasi-closed curve containing the pollutant sources.

Theorem 5. Within interlocking networks each pair of corresponding paths defines a quasi-closed curve.

The proof has been omitted, as its reasoning proceeds in a fashion closely parallel to the proof of theorem 4. In particular, the theorem follows from the examination of the first and last triangle and the first and last edge intersecting that chord in (C^*, R^*) which corresponds to the pair of corresponding paths in question.

In physical terms the quasi-closed curve defined by two corresponding paths has its morphological expression in the fact that the end nodes of the channel path are positioned next to and downslope from the end nodes of the ridge path. The terrain delineated by the two paths constitutes a drainage area in that no slope surface runoff will cross its boundary. The area is drained by all and only the networks that enter the channel path on the side facing the drainage area (part C of Figure 6). Thus the concept of a drainage area generalizes the common notion of a drainage basin, that is, an area drained by one and only one channel network.

At least intuitively, the strict separation between boundary paths and all other paths of a network seems implausible, and it is indeed easy to see that boundary paths can be interpreted as being merely a special subset of the set of all network paths, by the following corollary.

Corollary. A boundary path is a network path whose corresponding path in the other network is the null path. As a pair of corresponding paths they form a quasi-closed curve, as defined previously, with the following modification: let b_e be the boundary path associated with the network link e , then e is a link of the two boundary paths associated with the end links of b_e .

Since the path corresponding to the boundary path b_e is the null path, the quasi-closed curve coincides with the boundary path. Morphologically, the term "quasi-closed" refers to the fact that the two end links of b_e are separated only by the outlet link e of the network enclosed by b_e , thus delineating an area that is closed to slope runoff. In the event that the boundary path under consideration is a ridge path, the drainage area delineated by the path coincides with the familiar concept of a drainage basin. Figure 6, parts A and B, shows examples of both types of drainage areas delineated by a boundary path, that is, a channel subnetwork (quasi-) enclosed by a ridge path and a ridge network (quasi-) enclosed by a channel path.

CONCLUDING REMARKS

1. It would not be correct to assume that all axiomatic assumptions postulated in this investigation are required for all of the results. For example, the assumption of network trivalency has no effect on theorem 2, and it is not needed for theorem 1 if we replace the mean link number of boundary paths with the sum of the link numbers of all boundary paths.

Fig. 6. A and B are channel and ridge subnetworks with their respective boundary paths. C represents a pair of corresponding paths forming a quasi-closed curve. The area defined by the curve is closed with respect to slope runoff. If the link number of the channel boundary path is zero, the ridge boundary path delineates the entire area which then coincides with the familiar concept of a drainage basin.

Hence these theorems can be generalized so that they apply to interlocking networks with any number of nodal degrees. In particular, they allow for network nodes of degree 2 which do not occur in nature (unless we change their operational definition) but can be introduced into the research as network links are eliminated to isolate particular subnetworks for study, while keeping the nodes in which they were connected to the subnetwork.

2. It seems likely that the demonstration of other topological interdependencies requires additional axioms. While the internal connectivities of interlocking networks are irrelevant to the theorems of this paper, they are by no means independent of each other. In particular, empirical data confirm what is suggested by the fairly uniform density of natural ridge and channel lines we observe in erosional terrains: the magnitude of any subnetwork of a given network is closely related to the number of links of the associated boundary path which is, of course, a path of the interlocking network [Werner, 1982].

3. The last point has a geometric counterpart: the length of a boundary path and, likewise, the length of a quasi-closed curve must satisfy a lower bound which depends on the area enclosed and is reached when the shape of the area is a circle. Note, however, that the previous statement describes an observed regularity, while this statement is unrelated to the empirical world and is simply the logical consequence of the Euclidean axioms for the two-dimensional space of the real numbers (R^2).

4. Network properties and parameters can conveniently be grouped into those describing network topology and those describing spatial network features. A third category, parameters describing the environment within which networks are embedded, has been part of many drainage systems studies but their impact on the interdependencies between

interlocking networks has not been examined explicitly [Schumm, 1977]. This paper has dealt only with network topology. It explored several interdependencies between the basic topological elements (nodes, links, paths, and subnetworks) of interlocking networks, leading eventually to the derivation of a set of mathematical duality relations and to the formulation of a new concept, the drainage area, which combines elements from both networks and constitutes both a formal conceptualization and a generalization of the familiar notion of a drainage basin. Future research will have to include a systematic investigation of the spatial interdependencies between interlocking networks if the ultimate goal of a comprehensive theory of the fluvially eroded landscape is to be accomplished.

APPENDIX: PROOFS

Proof of Lemma

We label the vertices defining the edges c^* and r^* with q_i , q_j and p_u^* , p_v^* , respectively.

1. We start with the assumption that c^* and r^* intersect. Within the double sequence $S(C^*, R^*)$ and without loss in generality, c^* intersecting r^* is equivalent to

$$q_i^* < p_u^* < q_j^* \quad \sim \{q_i^* < p_v^* < q_j^*\}$$

Hence the outer channel nodes p_u , p_v are located, respectively, inside and outside the channel subnetwork defined by the channel link c , and the path they define must therefore pass through c , that is, it must include c as a link. As we have seen in the foregoing section, this channel path is the boundary path associated with the ridge link whose image is the edge defined by the vertices p_u^* and p_v^* , that is, r^* .

For rather obvious reasons of symmetry a parallel line of reasoning will show that the ridge path defined by the outer ridge nodes q_i , q_j consists of links located both within and without the subnetwork that has the ridge link r as its root; this path must therefore contain r as a link. Since q_i^* and q_j^* are the vertices of (C^*, R^*) that define the dual image c^* of c , the path (q_i, q_j) is the boundary path associated with link c . This proves the "if" portion of the lemma.

2. Let c be a link of the channel boundary path b_r associated with the ridge link r , and let r be a link of the ridge boundary path b_c associated with the channel link c . Within the sequence $S(C)$ let c_{i+1} , c_j be the first and last outer link of the channel subnetwork C_c defined by c , and let c_u and c_v be the outer links of b_r . Since c is a link of the path b_r and since, in graph theoretical terms, C_c is a tree, it follows that (without loss of generality) c_u is a link of C_c while c_v is not:

$$c_{i+1} \leq c_u \leq c_j \quad \sim \{c_{i+1} \leq c_v \leq c_j\}$$

As was demonstrated in a previous section, the boundary path associated with the channel link c is the ridge path whose outer links are positioned directly before and after the first and last outer channel link of the subnetwork C_c within the double sequence of outer ridge and channel links $S(C, R)$. Hence within that sequence it is $r_i < c_u < r_j$ and $\sim \{r_i < c_v < r_j\}$. Substituting these links by their associated outer nodes and the nodes by the corresponding vertices of the double triangulation (C^*, R^*) produces the inequalities

$$q_i^* < p_u^* < q_j^* \quad \sim \{q_i^* < p_v^* < q_j^*\}$$

Since q_i , q_j are the outer nodes of the boundary path b_c of the link c , their dual images, the vertices q_i^* , q_j^* , define the edge c^* in (C^*, R^*) ; similarly, the vertices p_u^* , p_v^* define the edge r^* in (C^*, R^*) . Thus the above inequalities simply state the intersection of c^* and r^* in (C^*, R^*) , which completes the proof of the lemma.

Proof of Theorem 2

The proof will be carried out in the terminology of the type I dual graphs of the interlocking networks C , R . As will become clear from the following considerations, the type I dual version of theorem 2 states that the triangles of C^* and R^* intersect in such a way that each triangle of one intersects with exactly one triangle of the other, and it does so in such a manner that each of its edges intersects with exactly two edges of the other, thus forming a hexagon as the intersection of the two triangles.

Let A , B , C be the vertices of a triangle of C^* with $A < B < C$. We define $\{e_i | i = 1, 2, \dots\}$ as the set of all edges in R^* that intersect both edges (A, B) and (A, C) . If X , Y are the end points of such an edge, then the definition of the set implies, without loss in generality, the inequalities $A < X < B$ and $\sim \{A < Y < C\}$. Let U , V be the end points of the edge in the set $\{e_i\}$ that is minimal in the particular sense that for every edge $e_x \in \{e_i\}$ with end points P , Q and $P < Q$ it is $A < P \leq U$; furthermore, it is $A > Q \geq V$ if $A > V$ and $\sim \{A < Q < V\}$ otherwise. The set $\{e_i\}$ cannot be empty because, by construction of (C^*, R^*) , the edges (A, B) and (A, C) are certainly intersected by the outer edge connecting the two vertices of R^* that are positioned adjacent to A within the double sequence $S(C^*, R^*)$. Moreover, the ordering of the edges $\{e_i\}$ as described is feasible because they are edges of a triangulated polygon and have therefore at most one point in common. Simply put, the minimal edge among all edges intersecting (A, B) and (A, C) is the one "furthest away" from A .

For parallel reasons there exist two additional edges (E, F) and (G, H) in R^* which intersect the pairs of edges (B, C) and (B, A) and (C, A) and (C, B) and meet equivalent definitions of minimality. The three edges (U, V) , (E, F) , and (G, H) must form a triangle because if they did not, there would exist additional edges within the subpolygon U, E, F, G, H, V because R^* , being the dual graph of the trivalent tree R , is composed of triangular regions only. Thus $U = E$, $F = G$, and $H = V$.

The type I dual images of the intersecting triangles A, B, C of C^* and U, F, H of R^* are two inner nodes X, Y of the interlocking networks C and R . Since the triangles are related by a one-to-one correspondence, the same relation holds for the nodes X and Y . We recall that the edges of a triangle are the type I dual images of the links joint in that network node which is the type I dual image of the triangle. Furthermore, according to the lemma proved earlier, intersecting edges in a double triangulation are the images of ridge and channel network links that are part of each other's boundary paths. Thus X is the channel network node in which the three boundary paths of the ridge links connected in Y intersect, and Y is the ridge network node in which the three boundary paths of the channel links connected in X intersect, which completes the proof.

Proof of Theorem 4

From a graph theoretical standpoint the concept of a double triangulation (C^*, R^*) has merit only as an illustration of a more abstract concept which consists of a cyclically ordered set $S = S(C^*, R^*)$ of vertices and a subset $\{e^*\}$ of its cartesian product $\{S \times S\}$. This subset is, of course, the set of edges of the double triangulation. We divide the remaining elements in $S \times S$ into two subsets, the subset $\{f^*\}$ consisting of all pairs of adjacent vertices and the subset $\{g^*\}$ consisting of all remaining vertex pairs: $\{g^*\} = \{\{S \times S\} - \{e^*\} - \{f^*\}\}$. We call the elements of $\{g^*\}$ the chords of (C^*, R^*) .

Similarly, we subdivide the set of all paths in C , and the set of all paths in R , into those that are boundary paths as defined earlier, labeled $\{b_c\}$ and $\{b_r\}$, respectively, and all other paths which we label $\{p_c\}$ and $\{p_r\}$, respectively. The principal idea of the proof is to demonstrate that both sets $\{p_c\}$ and $\{p_r\}$ are related to the set $\{g^*\}$ by one-to-one correspondences. In turn, the transitivity of these correspondence relationships establishes the one-to-one correspondence between $\{p_c\}$ and $\{p_r\}$, as stated in the theorem.

Let g^* be an element of the set $\{g^*\}$, that is, a chord connecting two nonadjacent vertices in (C^*, R^*) . Then g^* intersects edges of the two triangulations in such a way that it passes through a sequence of adjacent triangles of each triangulation, thus defining one path in each of the two type I primal networks C and R .

We will show (1) for each path p_c in C there exists exactly one chord in (C^*, R^*) intersecting the edges that are the dual images of the links of p_c ; note that for reasons of symmetry an equivalent statement must then hold for ridge paths: for each path p_r in R there exists exactly one chord in (C^*, R^*) that intersects the edges that are the dual images of the links of p_r , and (2) the mappings thus described are one-to-one correspondences between each of the two sets $\{p_c\}$, $\{p_r\}$ of network paths in C , R and the set $\{g^*\}$ of all chords in (C^*, R^*) .

Proof of First Assertion. Let p be a path in (C, R) with x and y as its end links. Without loss in generality we can assume that p is a channel path and that the magnitudes of the links of p , starting with x , increase monotonically (that can always be accomplished by temporarily defining, if need be, another outer link as the root of C such that the links of p have the same orientation relative to the root). Successive links of p define subnetworks, each of which is a proper subnetwork of the following one. Thus if we label the links of p with $e(i)$, where $i = 1, \dots, t$, $x = e(1)$, and $y = e(t)$ and the vertices of $e(i)^*$ in (C^*, R^*) with u_i^* and v_i^* , then $u_i^* \geq u_{i+1}^*$ and $v_i^* \leq v_{i+1}^*$ in $S(C^*, R^*)$ for all i .

We now choose a vertex P of (C^*, R^*) as follows: if $e(1)$ is an outer network link, then its dual edge $e(1)^*$ is an outer edge and its end points u_1^* , v_1^* are neighboring vertices of C^* . Within the double sequence $S(C^*, R^*)$ there is exactly one vertex of R^* positioned between u_1^* and v_1^* ; this vertex we will choose as P . Should $e(1)$ be an inner link, then $e(1)^*$ is an inner edge of the polygon C^* and as such a link of two triangles of C^* . We choose as P the third vertex of the triangle whose third vertex, within the ordered sequence $S(C^*, R^*)$, is positioned between u_1^* and v_1^* : $u_1^* < P < v_1^*$.

In like fashion we select a second vertex denoted by Q : if the end link $e(t)$ of the path p is an outer link, then Q is the vertex positioned between u_t^* and v_t^* within the sequence

$S(C^*, R^*)$ (in this case, to maintain simplicity of the inequalities, we temporarily define $e(t)$ as the outlet link of C). If, on the other hand, $e(t)$ is an inner link, then we choose Q as the third vertex of the triangle that has $e(t)^*$ as a side and whose third vertex lies outside the interval defined by the vertices of $e(t)^*$: $\sim\{u_t^* < Q < v_t^*\}$. By virtue of the set of inequalities

$$P > u_1^* \geq u_t^* > Q$$

$$P < v_1^* \leq v_t^* < Q$$

for all i the two vertices of each of the edges $e(i)^*$ are positioned on opposite sides of the chord PQ , and all edges $e(i)^*$ do therefore intersect PQ .

Conversely, if the vertices w^* , z^* of an edge e^* and C^* are positioned on either side of the chord PQ : $P > w^* > Q$ and $P < z^* < Q$, then the network link e corresponding to e^* is a link of the path p . That becomes clear when we consider the subnetwork C_e defined by e . Since C^* is a triangulated polygon, the edge e^* does not intersect any of the other edges of C^* . In particular, e^* does not intersect the edges of the triangles (u_1^*, P, v_1^*) and (u_t^*, Q, v_t^*) . Thus $u_1^* \geq w^* \geq u_t^*$ and $v_1^* \leq z^* \leq v_t^*$. Translating these inequalities from C^* into C means that the end link e_1 of p is located in the subnetwork C_e while the other, e_t , is not; hence the path itself must pass through the root link e of the subnetwork C_e , making e a link of the path p .

For the sake of completeness we should mention the limiting case of p consisting of one inner link e only. In this case it is $e(1) = e(t)$ and therefore $u_1^* = u_t^*$, $v_1^* = v_t^*$, and P and Q are the third vertices of the two triangles in C^* having the edge e^* in common. Hence there can be no edge other than e^* intersecting the chord PQ . If e is an outer link, then P is the vertex positioned between the two vertices defining e^* while Q is the third vertex of the triangle defined by e^* .

Proof of Second Assertion. The first part of the proof established a one-to-one correspondence between all paths $\{p_c\}$ of C and a set of chords in $\{g^*\}$. We still need to demonstrate that for every chord in $\{g^*\}$ there corresponds a path in $\{p_c\}$, that is, the set of chords corresponding to the set of paths $\{p_c\}$ is $\{g^*\}$. To this end we simply show that the set $\{p_c\}$ of (nonboundary) paths in C and the set $\{g^*\}$ of chords in (C^*, R^*) are of equal size. The implication of the proof for ridge paths is straightforward: Inasmuch as the interlocking ridge network R is of the same magnitude as the channel network C , its set $\{p_r\}$ of (nonboundary) paths has the same size as the set $\{p_c\}$ in C and therefore $\{g^*\}$ in (C^*, R^*) .

Given that the magnitude of the network C is n , the number of outer and inner nodes in C , that is, the total number of nodes, is $(n + 1) + (n - 1) = 2n$. Hence the number of different paths in C is $(2n)(2n - 1)(1/2)$ or $n(2n - 1)$, and the number of network paths excluding the $2n - 1$ boundary paths, that is, the size of the set $\{p_c\}$, is $(n - 1)(2n - 1)$.

The number of vertices of (C^*, R^*) is $2(n + 1)$, and the number of all possible vertex pairs in $\{S \times S\}$ is therefore $(2n + 2)(2n + 1)(1/2)$. We subtract the number of elements in $\{e^*\}$, that is, the $2(2n - 1)$ vertex pairs of defining the edges of (C^*, R^*) ; they correspond to the links in C and R and therefore simultaneously to the boundary paths associ-

ated with these links. We also subtract the $2(n + 1)$ pairs of neighboring vertices (i.e., the elements of the set $\{f^*\}$). The result is the size of the set $\{g^*\}$, which is $(n + 1)(2n + 1) - 2(2n - 1) - 2(n + 1) = (n - 1)(2n - 1)$. QED

The last part of the proof is obvious: the relationship derived above between the paths $\{p_c\}$ of C and the chords $\{g^*\}$ in (C^*, R^*) is based on properties of C with respect to which the networks C and R are indistinguishable. Thus the same line of argument applies to the relationship between the paths $\{p_r\}$ of R and the chords $\{g^*\}$ in (C^*, R^*) . As one-to-one relationships between sets are transitive, it follows that there exists a one-to-one correspondence between the paths p_c of C and the paths p_r of R . Specifically, two paths from C and R , respectively, correspond to each other if and only if the images of their links intersect the same chord of (C^*, R^*) .

The second part of the theorem follows readily (1) from the construction of a chord for a given network path as presented in the first part of the proof and (2) from the arrangement of vertices and outer edges in the double triangulation (C^*, R^*) . Clearly, a chord that has a vertex of C^* (R^*) as an end point cannot intersect the two outer edges connected in that vertex but must intersect the outer edge connecting the two vertices of R^* (C^*) that are positioned next to and on either side of the vertex of C^* (R^*). Translating these statements into the type I dual language of C , R will give the desired result.

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