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TABLE OF CONTENTS

FOREWORD

This report is submitted in partial fulfillment of Contract No. UCX 2231 with the Lawrence Radiation Laboratory, Livermore, California.

The Investigation was conducted by J. F. Brotchie, Graduate Research Engineer, under the general supervision and technical responsibility of J. Penzien, Associate Professor of Civil Engineering, and E. P. Popov, Professor of Civil Engineering, Department of Civil Engineering, College of Engineering, University of California, Berkeley, California.

SYNOPSIS

The flexural behavior of thin rotational shells in the elastic and post-elastic range of stresses is considered. Attention is restricted to the case of a shell of a linear strain-hardening material, under small deflections, and loaded in such a way that significant stresses are limited to a specific zone of the shell.

A spherical shell is considered in particular, and for the axi-symmetrical case, both the accurate and approximate equations governing displacement are developed. For the general case of a rotational shell subjected to unsymmetrical bending, the approximate equations only are presented.

In the case of axi-symmetrical loading, the following four possible behavior zones are considered: (1) an elastic zone, (2) a zone of circumferential yielding, (3) a zone of meridional yielding, and (4) a zone in which yielding occurs in each direction of principal stress. For unsymmetrical bending, only an elastic zone and a zone of yielding in both directions of principal stress are included. Solutions are presented, where possible, in terms of functions already tabulated, and simplifications are introduced for ready use in design.

The equations presented are further generalized in the axi-symmetrical case to include the effects of a variation in thickness and a variation in the elastic modulus of the shell material.

NOTATION

M	Moment per linear unit of middle surface.
N	Direct force per linear unit of middle surface.
Q	Shear force per linear unit of middle surface.
σ	Direct stress.
ϵ	Direct strain.
χ	Curvature increment.
u, v, w	Circumferential, meridional and radial displacements respectively.
F	Auxiliary function.
R, ϕ, θ	Polar coordinates, Fig. 3.
x, ϕ_e, θ	Conical coordinates, Figs. 4 or 5
Φ	$\equiv w + \lambda F$, a complex potential.
λ	Imaginary constant.
r_1, r_2	Meridional and circumferential radii.
R	Radius of spherical shell.
t	Thickness of shell.
E	Elastic modulus.
E_p	Plastic modulus.
a	$\equiv \frac{E_p}{E}$
ν	= Poisson's ratio.

Other symbols, used less frequently than these, are defined as they appear.

THEORY

SCOPE:

In the analysis which follows, consideration is restricted to thin shells of revolution in which displacements under loading are of smaller order than the thickness of the shell. Each shell is considered to be composed of a material having a bi-linear relationship between uniaxial stress and strain, as shown in Fig. 1. The portion of the curve with slope E_p represents the post-elastic range of the material, which commences at the proportional limit σ_p and extends to the ultimate stress σ_u. This range, σ_p to σ_u, is here referred to as the strain-hardening range.

In the strain-hardening range, attention is restricted to the case where the major component of the inelastic strain is produced by bending, so that the neutral axis is located within the thickness of the shell and the distribution of stress may be idealized as in Fig. 2. The behavior of the shell both in the elastic and strain-hardening ranges will be considered, and structural failure will be assumed to occur when the maximum stress in the shell increases to the ultimate stress σ_u.

The case of a spherical shell under axi-symmetrical bending is considered in particular, and the equations governing elastic and inelastic behavior are developed. In the case where significant deformations are restricted to a specific zone in the shell, eg. under loading conditions associated with stress concentrations in the shell, these equations are simplified, thus enabling practical solutions to be obtained. These simplifications are extended to the case of unsymmetrical bending, and to the corresponding loading conditions in a non-spherical shell of revolution.

I. SPHERICAL SHELLS

Axi-symmetrical bending - accurate equations

Where bending is symmetrical with respect to a central axis of the shell, strain-hardening will occur in concentric or co-axial zones. Clearly, in reality, there will be a gradual transition from one zone to another, but for the purpose of analysis the material in each zone is considered to be entirely in one range of stress in each direction.

Thus, four different zones of behavior are possible. In one, the material is entirely elastic; in another, strain-hardening occurs in the circumferential direction only; in the third, strain-hardening occurs only in the meridional direction, and in the fourth, strain-hardening occurs in both directions simultaneously.

Each zone may be considered separately and may be assumed to have definite boundaries. Within the elastic zone, the stress-strain relationship of the material is assumed to be linear, while in the strain-hardening zone, the stress-strain relationship is assumed to be as shown in Fig. 2.

The differential equations which govern the behavior of the shell are derived from three basic sets of relationships: (1) the equations of equilibrium, (2) the stress-strain equations, and (3) the equations relating strain and displacement. Of these, only the stress-strain relationships differ in each zone. In the limiting case of pure plasticity, however, the whole range of plastic strain corresponds to the stress σ_p and an appropriate criterion* for yield must be substituted for the stress-strain relationship.

The equilibrium equations for axi-symmetrical bending in a spherical

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* eg. that deduced by Tresca, Reference 15.

shell, radially loaded are:

$$\frac{d}{d\phi}\left(N_{\phi} \sin \phi\right) - N_{\theta} \cos \phi - \sin \phi \, Q_{\phi} = 0 \tag{1a}$$

$$N_{\phi} \sin \phi + N_{\theta} \sin \phi + \frac{d}{d\phi}\left(Q_{\phi} \sin \phi\right) + q \, R \sin \phi = 0 \tag{1b}$$

$$\frac{d}{d\phi}\left(M_{\phi} \sin \phi\right) - M_{\theta} \cos \phi - Q_{\phi} R \sin \phi = 0 \tag{1c}$$

in which the notation is conventional, and is listed in a separate section. The spherical coordinates R, ϕ and θ are shown in Fig. 3.

The strain-displacement relationships, at the middle surface, are:

$$\epsilon_{\phi} = \frac{1}{R}\left(\frac{dv}{d\phi} - w\right), \tag{2a}$$

$$\epsilon_{\theta} = \frac{1}{R}\left(v \cot \phi - w\right) \quad ; \tag{2b}$$

and curvatures are related to displacements by

$$\chi_{\phi} = \frac{1}{R^2} \quad \frac{d}{d\phi}\left(v + \frac{dw}{d\phi}\right) \tag{3a}$$

$$\chi_{\theta} = \frac{1}{R^2}\left(v + \frac{dw}{d\phi}\right) \cot \phi \tag{3b}$$

The above relationships, Eqns. 1, 2, and 3, are applicable in each zone, elastic or inelastic, of the shell. The stress-strain relationships, however, are different for each. In the elastic range, the relationship between stress and strain (from Hooke's law) for biaxial stress is

$$\sigma_i = \frac{E}{1-\nu^2}\left[\epsilon_i + \nu \epsilon_j\right] \tag{4}$$

where i and j are the directions of principal stress (eg. i, j = ϕ, θ).

Corresponding relationships may be obtained in the strain-hardening range.

The total strain in the post-elastic range will be assumed to be composed of an elastic component and a purely-plastic component; of which the elastic component is directly proportional to stress. The case of first quadrant bending only will be considered, in which bending stresses are of the same sign, and in which the plastic strains are assumed to occur along planes passing through the center of the shell. In the limiting case of purely plastic yield, the Tresca yield criterion is thus obeyed.* Hence where strain-hardening occurs in one direction only, for example direction j ($j = \phi$ or θ), the relationships between stress and strain (eg. Fig. 1) may be written (for biaxial stress states) as

$$\sigma_i = \frac{E}{1-a\nu^2}\left[\,\epsilon_i + \nu a\,\epsilon_j\,\right] + \nu\sigma_c \qquad (5a)$$

and

$$\sigma_j = \frac{aE}{1-a\nu^2}\left[\,\epsilon_j + \nu\,\epsilon_i\,\right] + \sigma_c \qquad (5b)$$

in which i and j are the directions of principal stress, as before; ν is assumed for simplicity to be everywhere the elastic value of Poisson's ratio; and $a = \frac{E_p}{E}$, the ratio of the inelastic and elastic moduli of elasticity.

Similarly, when strain hardening occurs simultaneously in both directions of principal stress,

$$\varepsilon_i = \frac{\sigma_i - \sigma_c}{aE} - \nu\,\frac{\sigma_j - \sigma_c}{aE}$$

$$\sigma_i = \frac{aE}{1-\nu^2}\left[\,\epsilon_i + \nu\,\epsilon_j\,\right] + \sigma_c \qquad \varepsilon_j = \qquad (6)$$

Thus for thin shells and axi-symmetrical bending in the elastic range, the stress-strain equations lead to the relationships

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* see Appendix I.

$$N_i = \int_{-t/2}^{t/2} \sigma_i \, dz = \frac{Et}{1-\nu^2} (\epsilon_i + \nu \epsilon_j) \tag{7a}$$

$$M_i = \int_{-t/2}^{t/2} \sigma_i \; z \, dz = \frac{-Et^3}{12(1-\nu^2)} \; (\chi_i + \nu \chi_j) \tag{7b}$$

in which ϵ_i, ϵ_j, χ_i, and χ_j are here the elongations and curvatures of the middle surface of the shell.

With strain-hardening in direction j only, corresponding relationships are obtained by dividing the stress distribution into its elastic and purely plastic components as shown in Fig. 2, to give

$$N_i = \int_{-t/2}^{t/2} \sigma_i \, dz = \frac{Et}{1-a\nu^2} \; (\epsilon_i + a\nu \, \epsilon_j) + \frac{2\nu \, \epsilon_j}{\chi_j} \cdot \sigma_c \tag{8a}$$

$$N_j = \int_{-t/2}^{t/2} \sigma_j \, dz = \frac{aEt}{1-a\nu^2} \; (\epsilon_j + \nu \, \epsilon_i) + \frac{2 \, \epsilon_j}{\chi_j} \, \sigma_c \tag{8b}$$

$$M_i = \int_{-t/2}^{t/2} \sigma_i \, z \, dz = \frac{-Et^3}{12(1-a\nu^2)} \; (\chi_i + a\nu \chi_j) + \nu \left[1 - \left\{ \frac{2\epsilon_j}{t\chi_j} \right\}^2 \right] \frac{\sigma_c t^2}{4} \tag{8c}$$

$$M_j = \int_{-t/2}^{t/2} \sigma_j \cdot z \, \partial z = \frac{-aEt^3}{12(1-a\nu^2)} (\chi_j + \nu\chi_i) + \left[1 - \left\{\frac{2\epsilon_j}{t\chi_j}\right\}^2\right] \frac{\sigma_c t^2}{4} \tag{8d}$$

Similarly, for strain-hardening in each direction,

$$N_i = \frac{aEt}{1-\nu^2} (\epsilon_i + \nu\epsilon_j) + \frac{2\epsilon_i}{\chi_i} \sigma_c \tag{9a}$$

$$M_i = \frac{-aEt^3}{12(1-\nu^2)} (\chi_i + \nu\chi_j) + \left[1 - \left\{\frac{2\epsilon_i}{t\chi_i}\right\}^2\right] \frac{\sigma_c t^2}{4} \tag{9b}$$

Equations 8 and 9 are based on the restrictive assumption that the ✓ component of strain due to bending in the extreme fibres is greater than that due to the direct force. Hence substituting Eqns. 2 and 3 into Eqns. 7, membrane forces and moments in the elastic range may be expressed in terms of displacements viz.

$$N_\phi = \frac{Et}{(1-\nu^2)} \frac{1}{R} \left[\frac{dv}{d\phi} - w + \nu (v \cot \phi - w)\right] \tag{10a}$$

$$N_\theta = \frac{Et}{(1-\nu^2)} \frac{1}{R} \left[v \cot \phi - w + \nu \left(\frac{dv}{d\phi} - w\right)\right] \tag{10b}$$

$$M_\phi = \frac{-D}{R^2} \left[\frac{d}{d\phi} \left(v + \frac{dw}{d\phi}\right) + \nu\left(v + \frac{dw}{d\phi}\right) \cot \phi\right] \tag{10c}$$

$$M_\theta = \frac{-D}{R^2} \left[\left(v + \frac{dw}{d\phi}\right) \cot \phi + \nu \frac{d}{d\phi} \left(v + \frac{dw}{d\phi}\right)\right] \tag{10d}$$

See Eq. 17a, b

Substituting Eqns. 2 and 3 into Eqns. 8 and letting $i \equiv \phi$, $j \equiv \theta$, gives the corresponding relationships for the case of circumferential strain-hardening (only):

$$N_\phi = \frac{Et}{(1-a\nu^2)} \frac{1}{R}\left[\frac{dv}{d\phi} - w + a\nu (v \cot \phi - w)\right] + \frac{2\nu R(v \cot \phi - w)\sigma_c}{(v + \frac{dw}{d\phi}) \cot \phi} \qquad (11a)$$

$$N_\theta = \frac{aEt}{(1-a\nu^2)} \frac{1}{R}\left[v \cot \phi - w + \nu(\frac{dv}{d\phi} - w)\right] + \frac{2R(v \cot \phi - w)\sigma_c}{(v + \frac{dw}{d\phi}) \cot \phi} \qquad (11b)$$

$$M_\phi = -\frac{D_1}{R^2}\left[\frac{d}{d\phi}(v + \frac{dw}{d\phi}) + a\nu(v + \frac{dw}{d\phi}) \cot \phi\right] + \nu\left[1 - \left\{\frac{2R(v \cot \phi - w)}{t(v + \frac{dw}{d\phi}) \cot \phi}\right\}^2\right]\frac{\sigma_c t^2}{4} \qquad (11c)$$

$$M_\theta = -\frac{aD_1}{R^2}\left[(v + \frac{dw}{d\phi}) \cot \phi + \nu \frac{d}{d\phi}(v + \frac{dw}{d\phi})\right] + \left[1 - \left\{\frac{2R(v \cot \phi - w)}{t(v + \frac{dw}{d\phi}) \cot \phi}\right\}^2\right]\frac{\sigma_c t^2}{4} \qquad (11d)$$

where $D_1 = \frac{Et^3}{12(1-a\nu^2)}$.

Similarly setting $i \equiv \theta$ and $j \equiv \phi$ in Eqns. 8, expressions for the case of meridional strain-hardening (only) are obtained:

$$N_\phi = \frac{aEt}{1-a\nu^2} \frac{1}{R}\left[\frac{dv}{d\phi} - w + \nu(v \cot \phi - w)\right] + \frac{2R(\frac{dv}{d\phi} - w)\sigma_c}{\frac{d}{d\phi}(v + \frac{dw}{d\phi})} \qquad (12a)$$

$$N_\theta = \frac{Et}{1-a\nu^2} \frac{1}{R}\left[v \cot \phi - w + a\nu(\frac{dv}{d\phi} - w)\right] + \frac{2\nu R(\frac{dv}{d\phi} - w)\sigma_c}{\frac{d}{d\phi}(v + \frac{dw}{d\phi})} \qquad (12b)$$

$$M_\phi = -\frac{aD_1}{R^2}\left[\frac{d}{d\phi}(v+\frac{dw}{d\phi}) + \nu(v+\frac{dw}{d\phi}) \cot \phi\right] + \left[1 - \left\{\frac{2R(\frac{dv}{d\phi} - w)}{t\frac{d}{d\phi}(v+\frac{dw}{d\phi})}\right\}^2\right]\frac{\sigma_c t^2}{4} \qquad (12c)$$

$$M_\theta = -\frac{D_1}{R^2}\left[(v+\frac{dw}{d\phi}) \cot \phi + a\nu \frac{d}{d\phi}(v+\frac{dw}{d\phi})\right] + \nu\left[1 - \left\{\frac{2R(\frac{dv}{d\phi} - w)}{t\cdot\frac{d}{d\phi}(d+\frac{dw}{d\phi})}\right\}^2\right]\frac{\sigma_c t^2}{4} \qquad (12d)$$

For strain-hardening in each direction, substituting Eqns. 2 and 3 into Eqns. 9 gives

$$N_\phi = \frac{aEt}{(1-\nu^2)} \frac{1}{R} \left[\frac{dv}{d\phi} - w + \nu (v \cot \phi - w) \right] + \frac{2R(\frac{dv}{d\phi} - w)\, \sigma_c}{\frac{d}{d\phi}(v + \frac{dw}{d\phi})} \tag{13a}$$

$$N_\theta = \frac{aEt}{(1-\nu^2)} \frac{1}{R} \left[v \cot \phi - w + \nu (\frac{dv}{d\phi} - w) \right] + \frac{2R(v \cot \phi - w)\, \sigma_c}{(v + \frac{dw}{d\phi}) \cot \phi} \tag{13b}$$

$$M_\phi = - \frac{aD}{R^2} \left[\frac{d}{d\phi}(v + \frac{dw}{d\phi}) + \nu (v + \frac{dw}{d\phi}) \cot \phi \right] + \left[1 - \left\{ \frac{2R(\frac{dv}{d\phi} - w)}{t \cdot \frac{d}{d\phi}(v + \frac{dw}{d\phi})} \right\}^2 \right] \frac{\sigma_c t^2}{4} \tag{13c}$$

$$M_\theta = - \frac{aD}{R^2} \left[(v + \frac{dw}{d\phi}) \cot \phi + \nu \frac{d}{d\phi}(v + \frac{dw}{d\phi}) \right] + \left[1 - \left\{ \frac{2R(v \cot \phi - w)}{t \cdot (v + \frac{dw}{d\phi}) \cot \phi} \right\}^2 \right] \frac{\sigma_c t^2}{4} \tag{13d}$$

It then remains to eliminate the shear term Q from Eqns. 1, so that Eqns. 10, 11, 12, and 13 may each be substituted in turn, to give (in each case) two simultaneous differential equations in w and v.

Eliminating Q_ϕ from Eqns. 1a and 1b gives

$$\frac{1}{R \sin \phi} \frac{d^2}{d\phi^2} (N_\phi \sin \phi) - \frac{1}{R \sin \phi} \frac{d}{d\phi} (N_\theta \cos \phi) + \frac{1}{R} (N_\phi + N_\theta) + q = 0 \tag{14a}$$

and eliminating Q_ϕ from Eqns. 1b and 1c gives

$$\frac{1}{R^2 \sin \phi} \frac{d^2}{d\phi^2} (M_\phi \sin \phi) - \frac{1}{R^2 \sin \phi} \frac{d}{d\phi} (M_\theta \cos \phi) + \frac{1}{R} (N_\phi + N_\theta) + q = 0 \tag{14b}$$

Hence substituting Eqns. 10 into Eqns. 14 gives the two equations in v and w, which govern the behavior of the shell in the elastic range. Alternatively, if Eqns. 11 are substituted into Eqns. 14, the equations which govern the behavior in the circumferential strain hardening range result. Similarly if Eqns. 12 and 13 are in turn substituted into Eqns. 14, the corresponding equations for the cases of meridional strain-hardening and strain-hardening in both directions, are obtained.

Firstly, however, Eqns. 10, 11, 12, and 13 may be written in the generalized form

$$N_\phi = b(v' - w) + d\left(\frac{v' - w}{v' + w''}\right) + \nu\left[c(v \cot\phi - w) + e\left(\frac{v - w\tan\phi}{v + w'}\right)\right] \tag{15a}$$

$$N_\theta = f(v\cot\phi - w) + g\left(\frac{v - w\tan\phi}{v + w'}\right) + \nu\left[h(v' - w) + i\left(\frac{v' - w}{v' + w''}\right)\right] \tag{15b}$$

$$M_\phi = k(v' + w'') + \ell\left[1 - \left\{\frac{2R(v' - w)}{t(v' + w'')}\right\}^2\right] + \nu\left[m(v + w')\cot\phi + r\left\{1 - \left\{\frac{2R(v\cot\phi - w)}{t(v + w')\cot\phi}\right\}^2\right\}\right] \tag{15c}$$

$$M_\theta = n(v + w')\cot\phi + p\left[1 - \left\{\frac{2R(v\cot\phi - w)}{t(v + w')\cot\phi}\right\}^2\right] + \nu\left[q(v' + w'') + s\left\{1 - \left\{\frac{2R(v' - w)}{t(v' + w'')}\right\}^2\right\}\right] \tag{15d}$$

in which primes denote differentiation with respect to ϕ.

Substituting Eqns. 15 into Eqns. 14, and omitting, for simplicity, any remaining terms with coefficient ν, there results:

$$b(v''' - w'') + 2\cot\phi\, b(v'' - w') - \cot\phi . f(v'\cot\phi - v\,\mathrm{cosec}^2\phi - w')$$

$$+ 2f(v \cot \phi - w) + d\frac{(v'+w'')(v'''w+v'''w''-v'w^{iv}-w''w''+ww^{iv}+v'w'')}{(v' + w'')^3}$$

$$- \frac{d2(v''+w''')(v''w + v''w''-v'w'-v'w'''-w'w'' + ww''')}{(v' + w'')^3}$$

$$+ 2d \cot \phi \frac{(v''w + v''w'' - v'w' - v'w''' - w'w'' + ww''')}{(v' + w'')^2}$$

$$- g \cot \phi \frac{(v + w')(v'-w' \tan \phi - w \sec^2 \phi) - (v'+w'')(v - w \tan \phi)}{(v + w')^2}$$

$$+ 2g \frac{v - w \tan \phi}{v + w'} = -qR \qquad (16a)$$

$$k(v''' + w^{iv}) + 2k \cot \phi (v'' + w''') - k(v' + w'')$$

$$- n \cot \phi (v' \cot \phi - v \operatorname{cosec}^2 \phi + w'' \cot \phi - w' \operatorname{cosec}^2 \phi)$$

$$+ n \cot \phi (v + w') + bR(v' - w) + fR (v \cot \phi - w)$$

$$+ \ell \Bigg[- \frac{8R^2}{t^2} \Bigg\{ \Bigg\{ \frac{(v'' - w')}{(v' + w'')} - \frac{(v' - w)(v'' + w''')}{(v' + w'')^2} \Bigg\}^2$$

$$- \frac{v' - w}{(v' + w'')} \Bigg\{ \frac{(v'''-w'')}{(v' + w'')} - \frac{2(v''-w')(v'' + w''')}{(v' + w'')^2}$$

$$- \frac{(v' - w)(v''' + w^{iv})}{(v' + w'')^2} + \frac{2(v' - w)(v'' + w''')^2}{(v' + w'')^3} \Bigg\} \Bigg\}$$

$$- \frac{16R^2}{t^2} \cot \phi \cdot \frac{(v' - w)}{(v' + w'')} \Bigg\{ \frac{(v'' - w')}{(v' + w'')} - \frac{(v'- w)(v'' + w''')}{(v' + w'')^2} \Bigg\}$$

$$- 1 + \Bigg\{ \frac{2R(v' - w)}{t(v' + w'')} \Bigg\}^2 \Bigg]$$

$$+ p\left[1-\left\{\frac{2R(v - w\tan\phi)}{t(v + w')}\right\}^2\right.$$

$$- \frac{8R^2}{t^2}\cot\phi\left\{\frac{(v - w\tan\phi)}{(v + w')}\left\{\frac{(v' - w'\tan\phi - w\sec^2\phi)}{(v + w')}\right.\right.$$

$$\left.\left.\left. - \frac{(v - w\tan\phi)(v' + w'')}{(v + w')^2}\right\}\right\}\right] + dR\left[\frac{v' - w}{v' + w''}\right]$$

$$+ gR\left[\frac{v - w\tan\phi}{v + w'}\right] = - qR^2$$

The remaining constants b, d, f, g, k, ℓ , n, and p have different values in each zone of the shell:

In the elastic zone,

$$b = f = \frac{Et}{(1 - \nu^2)R} \quad , \quad d = g = 0$$

$$k = n = -\frac{D}{R^2} \quad , \quad \ell = p = 0 \quad ;$$

for strain-hardening in the circumferential direction only,

$$b = \frac{Et}{(1 - a\nu^2)R} \quad , \quad d = 0$$

$$f = \frac{aEt}{(1-a\nu^2)R} \quad , \quad g = 2R\,\sigma_c$$

$$k = -\frac{D_1}{R^2} \quad , \quad \ell = 0$$

$$n = -\frac{aD_1}{R^2} \quad , \quad p = \frac{\sigma_c t^2}{4}$$

for strain-hardening in the meridional direction only,

$$b = \frac{aEt}{(1-a\nu^2)R} \quad , \quad d = 2R\,\sigma_c$$

$$f = \frac{Et}{(1-a\nu^2)R} \quad , \quad g = 0$$

$$k = -\frac{aD_1}{R^2} \quad , \quad \ell = \frac{\sigma_c t^2}{4}$$

$$n = -\frac{D_1}{R^2} \quad , \quad p = 0$$

and for strain-hardening in both directions simultaneously,

$$b = f = \frac{aEt}{(1-\nu^2)R} \quad , \quad d = g = 2R\,\sigma_c$$

$$k = n = -\frac{aD}{R^2} \quad , \quad \ell = p = \frac{\sigma_c t^2}{4} \quad .$$

However where the equations thus obtained for the elastic case are linear, the equations for the three strain-hardening cases are non-linear and their solutions are virtually intractable.

Axi-symmetrical bending - approximate equations

(a) Spherical coordinates.

Even in the elastic case, it is convenient in practice to introduce approximations in order to reduce the general equations to a usable form. Some degree of simplification (of these equations) was introduced by H. Reissner (3) and Meissner (4) but not sufficient to allow their use for design calculations. Approximate solutions for shallow shells ($\phi < \frac{\pi}{6}$), in the elastic range, has been developed by E. Reissner (5) and Geckeler (6), and apply with increasing accuracy as $\phi \rightarrow 0$. Other solutions by Geckeler (7) and Hetenyi (8), and others (9, 10, 12, 13, 14) increase in accuracy as $\phi \rightarrow \frac{\pi}{2}$.

For present purposes, a solution of a slightly different type is desirable. The class of problem to be considered in particular is that of a loading which produces stress concentrations in the shell but for which significant deformations are limited to a specific zone. The case of a loaded insert or fitting in the shell is included herein.

In this class of problem, the effect on curvature of meridional displacement v, and its derivatives, is of smaller order than the effect of the derivatives of radial displacement w. Thus moments may be expressed in terms of w and ϕ only, with little loss in accuracy. ✓

Hence in the elastic range, we may write

$$M_\phi = -\frac{D}{R^2}\left[\frac{d^2w}{d\phi^2} + \nu \cot\phi \frac{dw}{d\phi}\right] \qquad (17a)$$

$$M_\theta = -\frac{D}{R^2}\left[\cot\phi \frac{dw}{d\phi} + \nu \frac{d^2w}{d\phi^2}\right] \qquad (17b)$$

Similarly if the term containing Q_ϕ in Eqn. 1a is of smaller order than the terms containing the direct forces N_ϕ and N_θ, it is convenient

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to introduce a stress function F such that

$$N_{\phi} \equiv -\frac{L^2}{R^2} \cot \phi \frac{dF}{d\phi} \tag{18a}$$

$$N_{\theta} \equiv -\frac{L^2}{R^2} \frac{d^2F}{d\phi^2} \tag{18b}$$

and

$$L \equiv \frac{R^{(1/2)} t^{(1/2)}}{\left[12(1-\nu^2)\right]^{(1/4)}}$$

The function F may be shown to approximately satisfy Eqn. 1a. Its significance will be more apparent in the case of unsymmetrical bending where there are additional expressions to be satisfied. The parameter L is sometimes referred to as the radius of relative stiffness of the shell and serves to make the operators on F dimensionless.

Substituting Eqns. 17 and 18 into Eqn. 14b, and modifying the coefficients of the first and second derivatives of w, as suggested by Mushtari and Vlasov (24), gives

$$\nabla^4 w + \frac{R}{Et} \nabla^2 F = \frac{qR^2}{Et} \tag{19a}$$

or

$$\nabla^4 w - \frac{R}{Et} (N_{\phi} + N_{\theta}) = \frac{qR^2}{Et} \tag{19b}$$

where $\nabla^4 \equiv \nabla^2 \nabla^2$; and $\nabla^2 \equiv \frac{L^2}{R^2} (\frac{d^2}{d\phi^2} + \cot \phi \frac{d}{d\phi})$, which is Laplace's operation in (dimensionless) spherical coordinates.

A second equation in w and F is obtained from the relevant equation of compatibility which, from Eqns. 2, may be shown (Reference 14) to be of the form

$$\frac{1}{R} \nabla^2 w + \frac{L^2}{R^2 \sin \phi} \left\{ \frac{\partial}{\partial \phi} \left[\sin \phi \frac{\partial \epsilon_\theta}{\partial \phi} + \frac{\partial}{\partial \phi} (\sin \phi)(\epsilon_\theta - \epsilon_\phi) \right] \right\} = 0 \qquad (20)$$

Combining Eqns. 7, 18, and 20 gives the result:

$$\nabla^4 F - \frac{Et}{R} \nabla^2 w = 0 \qquad (21a)$$

or

$$\nabla^2 (N_\phi + N_\theta) + \frac{Et}{R} \nabla^2 w = 0 \qquad (21b)$$

Following the example of Reissner (5) for a shallow shell, if Eqn. 19a is added to λ times Eqn. 21a where

$$\lambda = i \frac{R}{Et}$$

there results

$$\nabla^4 \Phi - i \nabla^2 \Phi = \frac{qR^2}{Et} \qquad (22)$$

where $\Phi = w + \lambda F$.

Alternatively, Eqns. 19a and 21a might have been differently combined to give separate equations in w and F, viz:

$$\nabla^6 w + \nabla^2 w = \frac{R^2}{Et} \nabla^2 q \qquad (23a)$$

$$\nabla^6 F + \nabla^2 F = qR \qquad (23b)$$

A further useful relationship may be obtained by returning to Eqns. 21. Where loads and significant radial displacements do not extend to the outer boundary of the shell, i.e. w and its derivatives with respect to ϕ approach zero as ϕ increases, the sum of the direct forces will also approach zero (from Eqn. 1b), and it may be demonstrated that Eqn. 24 therefore requires that

$$\nabla^2 F - \frac{Et}{R} w = 0 \tag{24a}$$

or

$$N_\phi + N_\theta + \frac{Et}{R} w = 0 \tag{24b}$$

Substituting Eqn. 24a into Eqn. 19a gives

$$\nabla^4 w + w = \frac{qR^2}{Et} \tag{25}$$

For the class of problem to be considered, namely that of a system of loads producing a stress concentration in the shell, and in which significant deformations are limited to a specific zone, Eqns. 19 through 25 are reasonably accurate for any range of latitude in the shell. However, in these equations the operator ∇^2 is expressed in dimensionless polar coordinates, i.e. in the coordinates of the surface of the shell, (Fig. 3), and for practical analysis these (spherical coordinates) are not so readily manipulated. Thus a simpler coordinate surface is desirable.

(b) Conical coordinates

Where stresses are significant only in a specific zone of the shell, it is convenient to introduce a simpler system of coordinates which correspond as closely as possible to the spherical coordinates over the zone considered.

A conical coordinate surface appears to be most suited to this purpose, (Fig. 4). The angle of the cone is chosen to suit the zone to be analyzed, and greatest accuracy is likely to be obtained if the cone is tangent to the sphere at or near the ordinate ϕ_e of the stress concentration. Points on the surface of the sphere within the zone may then be considered to be projected onto the cone.

The conical coordinates are x, measured along the generator of the cone (Fig. 4) and θ, as for the sphere. If points on the surface of the sphere are projected linearly onto the cone so that $x_e - x_1 = \frac{R}{L}(\phi_e - \phi_1)$, Fig. 4, then x is given by

$$x = \frac{R}{L}(\tan\phi_e + \phi - \phi_e) \tag{26}$$

This projection may apparently be used for all values of ϕ_e.

Alternatively if ϕ_e (but not necessarily ϕ) is restricted to a shallow zone, say $\phi_e < \frac{\pi}{6}$, points on the sphere may be projected vertically onto the cone (Fig. 5) and in this case x is given by

$$x = \frac{R}{L}\,\frac{\sin\phi}{\cos\phi_e} \tag{27}$$

The operator ∇^2 in each case is

$$\nabla^2 = \frac{d^2}{dx^2} + \frac{1}{x}\,\frac{d}{dx}$$

and moments may be expressed in the form:

$$M_\phi = -\frac{D}{L^2}\left[\frac{d^2w}{dx^2} + \nu\frac{1}{x}\,\frac{dw}{dx}\right] \tag{28a}$$

$$M_\theta = -\frac{D}{L^2}\left[\frac{1}{x}\,\frac{dw}{dx} + \nu\frac{d^2w}{dx^2}\right] \tag{28b}$$

In the case where x is given by Eqn. 27, the approximation is made in Eqns. 28 that $\frac{\cos^2\phi}{\cos^2\phi_e} \rightarrow 1$, and this provides a restriction on the

✓

range of ϕ and ϕ_e considered.

Further, direct forces are related to the stress function F by

$$N_\phi = -\frac{1}{x}\frac{dF}{dx} \tag{29a}$$

$$N_\theta = -\frac{d^2F}{dx^2} \tag{29b}$$

and if compatibility is here expressed by the equation

$$-\frac{1}{x}\frac{d\epsilon_\phi}{dx} + \frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\epsilon_\theta}{dx}\right) + \frac{1}{R}\nabla^2 w = 0 \tag{30}$$

then equations 19 through 25 result as before. Deduction of Eqn. 30 (and similarly Eqn. 20) was facilitated by the fact that in the case $R = \infty$, it reduces to the corresponding relationship for the plane stress problem, and in the case $\phi_e = 0$ it reduces to the relationship obtained by Reissner (5) for a shallow shell.

The homogeneous parts of Eqns. 23, in conical coordinates with x given by Eqn. 27, were also obtained in a previous report, (1), using a different method of derivation.

For the special case of $\phi_e = 0$, the conical coordinate surface reduces to a plane. In this case Eqn. 27 reduces to $x \equiv \frac{R \sin \phi}{L}$ and Eqns. 22 and 23 reduce to those of Reissner for a shallow shell. Similarly for $\phi_e = 0$, Eqn. 26 reduces to $x \equiv \frac{R\phi}{L}$ and Eqns. 22 and 23 are equivalent to those derived by Geckeler (6) for a shallow shell. For the case $\phi_e = \frac{\pi}{2}$ the cone becomes a cylinder. Here $\tan \phi_e = \infty$ and it is convenient to introduce a new variable y such that

$$y \equiv x - \frac{R}{L}\tan \phi_e \equiv \frac{R}{L}(\phi - \phi_e)$$

In this case terms containing $\frac{1}{x}$ or powers of $\frac{1}{x}$ vanish, and the equations reduce to the approximate solution proposed by Geckeler (7) for a deeper shell.

Bijlaard (13) shows that Reissner's equation for deflection in a shallow shell may also be reduced to the form of Eqn. 25. In discussing Reissner's solution, Bijlaard compares the effect of direct forces in the shallow shell to the effect of an elastic foundation on a plate. The component of the membrane forces in the z direction is $\frac{1}{R}(N_\phi + N_\theta)$ and from Eqn. 24b

$$\frac{1}{R}(N_\phi + N_\theta) = \frac{-Etw}{R^2} = kw$$

in which the constant k may be considered as the equivalent modulus of the elastic foundation. Thus from Eqns. 24 and 25, it is evident that the above analogy may be extended to the larger values of ϕ considered herein, provided only a specific zone of the shell is deformed. This analogy of the elastic foundation is a useful one in understanding the post-elastic behavior of the shell.

In the inelastic range of stresses, further approximations are required if solutions in a usable form are to be obtained. Attention here will be restricted to the case where, in the zone considered, the strains due to direct forces are small in comparison with the strains produced in the extreme fibres by bending stresses. In this restricted case, the effect of meridional displacement v and middle surface strain ϵ_ϕ on the moments may be neglected.

Considering firstly the case of strain hardening in the circumferential direction only, M_ϕ and M_θ thus reduce to

Since in-plane forces N_ϕ & N_θ are neglected.

$$M_\phi = -\frac{D_1}{L_1^2}\left[\frac{d^2w}{dx_1^2} + a\nu\frac{1}{x_1}\frac{dw}{dx_1}\right] + \nu M_c \qquad (31a)$$

Define ν aE: see p.7

$$M_\theta = -\frac{aD_1}{L_1^{\,2}}\left[\frac{1}{x_1}\frac{dw}{dx_1} + \nu\frac{d^2w}{dx_1^{\,2}}\right] + M_c \tag{31b}$$

in which $x_1 \equiv \frac{R}{L_1}(\tan\phi_e + \phi - \phi_e)$ or $\frac{R\sin\phi}{L_1\cos\phi_e}$; $L_1 \equiv \frac{R^{1/2}t^{1/2}}{\left[12(1-a\nu^2)\right]^{1/4}}$;

and $M_c = \frac{\sigma_c t^2}{4}$.

Similarly in the case of strain-hardening in the meridional direction only, moments are given by

uses

$$M_\phi = \frac{aD_1}{L_1^{\,2}}\left[\frac{d^2w}{dx_1^2} + \frac{1}{x_1}\nu\frac{dw}{dx_1}\right] + M_c \tag{32a}$$

$$M_\theta = \frac{D_1}{L_1^{\,2}}\left[\frac{1}{x_1}\frac{dw}{dx_1} + a\nu\frac{d^2w}{dx_1^2}\right] + \nu M_c \tag{32b}$$

For strain-hardening in each direction simultaneously with $x \equiv \frac{R}{L}(\tan\phi_e + \phi - \phi_e)$ or $\frac{R\sin\phi}{L\cos\phi_e}$ as in the elastic zone, there results

$$M_\phi = \frac{aD}{L^2}\left[\frac{d^2w}{dx^2} + \frac{1}{x}\nu\frac{dw}{dx}\right] + M_c \tag{33a}$$

$$M_\theta = \frac{aD}{L^2}\left[\frac{1}{x}\frac{dw}{dx} + \nu\frac{d^2w}{dx^2}\right] + M_c \tag{33b}$$

Thus in the case of circumferential strain-hardening, Eqns. 14b and 31 may be combined to give

Details

$$w^{iv} + \frac{2}{x_1}w''' - \frac{a}{x_1^{\,2}}w'' + \frac{a}{x_1^{\,3}}w' - \frac{a}{x_1^{\,3}}w'\tan^2\phi - \frac{R}{ET}(N_\phi + N_\theta) = \frac{qR^2}{Et} \tag{34}$$

where primes here, and from here on, denote differentiation with respect

only this term differs.

to x (or x_1). Eqn. 34 is derived using the value of x defined in Eqn. 27 assuming $\frac{\cos^2 \phi}{\cos^2 \phi_e} \rightarrow 1$. For the value of x given in Eqn. 26, slight differences in the terms containing the lower order derivatives of w will occur.

Contradicts ??

Similarly for meridional strain-hardening, substituting Eqn. 32 into Eqn. 14b gives

$$w^{iv} + \frac{2}{x_1} w''' - \frac{1}{a x_1^2} w'' + \frac{1}{a x_1^3} w' - \frac{1}{a x_1^3} w' \tan^2 \phi - \frac{R}{aEt} (N_\phi + N_\theta) = \frac{qR^2}{aEt} \tag{35}$$

For strain-hardening in each direction simultaneously, the corresponding equation, from Eqns. 14b and 33 is

$$w^{iv} + \frac{2}{x} w''' - \frac{1}{x^2} w'' + \frac{1}{x^3} w' - \frac{1}{x^3} w' \tan^2 \phi - \frac{R}{aEt} (N_\phi + N_\theta) = \frac{qR^2}{aEt} \tag{36}$$

For thin shells, where concentrations of stress occur and significant deformations are restricted to a specific zone--or shallow segment, the effect of the various derivatives of w decreases with their order, and the approximation $\frac{1}{x^3} w' \tan^2 \phi \rightarrow 0$, (or that similar terms approach zero depending on the definition of x) may be introduced. The same approximation was introduced in the elastic range.

Quite an approx

Thus for strain-hardening in the circumferential direction only, Eqn. 34 reduces to

$$\nabla_c^4 w - \frac{R}{Et} (N_\phi + N_\theta) = \frac{qR^2}{Et} \tag{37}$$

in which $\nabla_c^4 w = w^{iv} + \frac{2}{x_1} w''' - \frac{a}{x_1^2} w'' + \frac{a}{x_1^3} w'$.

Similarly, for meridional strain-hardening, Eqn. 35 reduces to

$$a \nabla_m^4 w - \frac{R}{Et} (N_\phi + N_\theta) = \frac{qR^2}{Et} \tag{38}$$

where $\nabla_m^4 w \equiv w^{iv} + \frac{2}{x_1} w''' - \frac{1}{a x_1^2} w'' + \frac{1}{a x_1^3} w'$

For strain-hardening in both directions, Eqn. 36 reduces to

$$a \nabla^4 w - \frac{R}{Et} (N_\phi + N_\theta) = \frac{qR^2}{Et} \tag{39}$$

where

$\nabla^4 w \equiv w^{iv} + \frac{2}{x} w''' - \frac{1}{x^2} w'' + \frac{1}{x^3} w'$, as in the elastic range.

It will be noted that the stress function F, as defined in Eqns. 29, is still applicable as it was determined from the equilibrium equations only. Substituting Eqns. 29 into Eqns. 37, 38, and 39 in turn gives: for circumferential strain-hardening,

Elastic case?

$$\nabla_c^4 w + \frac{R}{Et} \nabla^2 F = q \frac{R^2}{Et} \tag{40}$$

for meridional strain-hardening,

$$a \nabla_m^4 w + \frac{R}{Et} \nabla^2 F = q \frac{R^2}{Et} \tag{41}$$

and for strain-hardening in both directions,

$$a \nabla^4 w + \frac{R}{Et} \nabla^2 F = q \frac{R^2}{Et} \tag{42}$$

A second relationship between F and w in each zone is obtained from the compatibility Eqn. 30, as in the elastic range. For this purpose a simplified relationship between N_i and ϵ_i (i, j = ϕ, θ) is desirable, and the two following alternative relationships will be considered: (1) The relationships expressed in Eqns. 8 and 9 for the respective zones will be assumed, except that the term $\frac{\epsilon_j}{\chi_j}$ (or $\frac{\epsilon_i}{\chi_i}$) in each, will be taken as constant. Since this term is thus eliminated from the final differential equation, the assumption will give best results when the ratio $\frac{\epsilon_j}{\chi_j}$ is either nearly constant or is negligible in the range of ϕ considered; (2) As an alternative relationship, Eqn. 7 will be assumed to be applicable to each zone of the shell, i.e., the direct forces will be assumed to remain proportional to the total strains.

These two relationships will generally provide an upper and lower bound to the actual behavior. (In an exceptional case where they do not act as bounds, they will at least indicate the degree of approximation involved.)

Case (1). Thus for circumferential strain-hardening, combining Eqns. 29 and 30 with the modified Eqns. 8, where in this case i = ϕ, j = θ, results in

$$\nabla_c^4 F - \frac{aEt}{R} \nabla^2 w = 0 \tag{43}$$

For meridional strain-hardening, combining Eqns. 29 and 30 with the modified Eqns. 8 where i = θ, j = ϕ, gives

$$\nabla_m^4 F - \frac{Et}{R} \nabla^2 w = 0 \tag{44}$$

For strain-hardening in each direction, combining Eqns. 29 and 30 with the modified Eqns. 9 gives

$$\nabla^4 F - \frac{aEt}{R} \nabla^2 w = 0 \tag{45}$$

Each of Eqns. 43, 44, and 45 may now be combined respectively with Eqns. 40, 41, and 42 in different ways to provide relationships in the inelastic range, corresponding to the relationships of Eqns. 22 and 23 for the elastic range.

Thus for circumferential strain-hardening, different combinations of Eqns. 40 and 43 lead to

$$\nabla_c^4 \nabla_c^4 w + a \nabla^2 \nabla^2 w = \frac{R^2}{Et} \nabla_c^4 q \tag{46a}$$

$$\nabla_c^4 \nabla_c^4 F + a \nabla^2 \nabla^2 F = R a \nabla^2 q \tag{46b}$$

or

$$\nabla_c^4 \Phi_1 - ia^{+1/2} \nabla^2 \Phi_1 = \frac{qR^2}{Et} \tag{46c}$$

where $\Phi_1 \equiv w + \lambda_1 F$ and $\lambda_1 \equiv i \frac{R}{Eta^{1/2}}$.

For meridional strain-hardening, Eqns. 41 and 44 may be combined to give

$$a \nabla_m^4 \nabla_m^4 w + \nabla^2 \nabla^2 w = \frac{R^2}{Et} \nabla_m^4 q \tag{47a}$$

$$a \nabla_m^4 \nabla_m^4 F + \nabla^2 \nabla^2 F = R \nabla^2 q \tag{47b}$$

or

$$\nabla_m^4 \Phi_1 - ia^{-1/2} \nabla^2 \Phi_1 = \frac{qR^2}{aEt} \tag{47c}$$

where $\Phi_1 \equiv w + \lambda_1 F$ and $\lambda_1 \equiv i \frac{R}{Eta^{1/2}}$.

For strain-hardening in both directions simultaneously, Eqns. 42 and 45 combine to give

$$\nabla^6 w + \nabla^2 w = \frac{R^2}{aEt} \nabla^2 q \tag{48a}$$

$$\nabla^6 F + \nabla^2 F = Rq \tag{48b}$$

or

$$\nabla^4 \Phi_2 - i \nabla^2 \Phi_2 = \frac{qR^2}{aEt} \tag{48c}$$

where $\Phi_2 \equiv w + \lambda_2 F$ and $\lambda_2 \equiv i \frac{R}{aEt}$.

The above equations, operators ∇^2, ∇_c^4, and so on can again be derived in <u>spherical</u> instead of <u>conical</u> coordinates.

Case (II) If, alternatively, the direct forces are assumed to remain proportional to the total strains, Eqns. 21 are applicable to each zone of the shell. Combining Eqn. 21a with Eqns. 40, 41, and 42 in turn, enables three more groups of equations, corresponding to Eqns. 46a and b, 47a and b, and 48a and b, to be obtained:

i.e. for circumferential strain-hardening,

$$\nabla^2 \nabla_c^4 w + \nabla^2 w = \frac{R^2}{Et} \nabla^2 q \tag{49a}$$

$$\nabla^4 \nabla_c^4 F + \nabla^2 \nabla^2 F = R \nabla^2 q \tag{49b}$$

for meridional strain-hardening

$$a \nabla^2 \nabla_m^4 w + \nabla^2 w = \frac{R^2}{Et} \nabla^2 q \tag{50a}$$

$$a \nabla^4 \nabla_m^4 F + \nabla^4 F = R \nabla^2 q \tag{50b}$$

and for strain-hardening in each direction

$$a \nabla^6 w + \nabla^2 w = \frac{R^2}{Et} \nabla^2 q \tag{51a}$$

$$a \nabla^6 F + \nabla^2 F = Rq \tag{51b}$$

There is no corresponding complex relationship in this case.

Further, if significant radial displacements do not extend to the outer boundary of the shell, Eqn. 21b again leads to Eqns. 24. Hence for circumferential strain-hardening, combining Eqns. 24b and 40 gives

$$\nabla_c^4 w + w = \frac{qR^2}{Et} \tag{52a}$$

$$\nabla^2 \nabla_c^4 F + \nabla^2 F = Rq \tag{52b}$$

For meridional strain-hardening, from Eqns. 24b and 41,

$$a \nabla_m^4 w + w = \frac{qR^2}{Et} \tag{53a}$$

$$a \nabla^2 \nabla_m^4 F + \nabla^2 F = Rq \tag{53b}$$

and for strain-hardening in each direction, Eqns. 24b and 42 combine to give

$$a \nabla^4 w + w = \frac{qR^2}{Et} \tag{54a}$$

$$a \nabla^6 F + \nabla^2 F = Rq \tag{54b}$$

The behavior of the shell in the elastic and post-elastic ranges is conveniently considered by comparing Eqns. 25, 52a, 53a, and 54a. From inspection these equations, which govern deflection in the various zones of the shell, are seen to differ only in the differential operator contained in the first term of each. A similar difference is noted between Eqns. 22, 46, 47, and 48. In considering the effect of this, it is convenient to recall the analogy between the behavior of an elastic shell and an elastic plate on an elastic foundation.

Hence in Eqn. 25 the first term $\nabla^4 w$, represents the effect of flexure or bending in the shell or in an elastic plate. The second term represents the effect of membrane action in the shell which corresponds to the effect of the reaction of the elastic foundation on the plate. The third term is a constant times the normal loading intensity ineach case. Similar relationships are found in the inelastic range. Eqn. 52a for circumferential strain-hardening in the shell is again equivalent to the equation for tangential strain-hardening in a plate on an elastic foundation; the first term once more represents the effect of bending in the plate or shell, this time in the inelastic range. Similarly, Equations 53a for meridional strain-hardening is equivalent to the equation for radial strain-hardening in a plate on an elastic foundation. Equation 54a for strain-hardening in each direction is also applicable to an elastically supported plate.

The difference involved in each case between these equations (Case II) for a shell and the corresponding equations for a plate on an elastic foundation is only in the constant parameters involved, and thus the solutions obtained for the latter case are also applicable to the shell. For the case q = 0, these solutions are presented in a previous report (Reference 2).

Thus, the solution for each of Eqns. 25, 52a, 53a, and 54a may be expressed in the form

$$w = w_h + w_p \tag{55}$$

where w_h is the solution to the homogeneous part of the equation and w_p is the particular integral, representing the effect of the loading term.

Thus, for an elastic plate, from Eqn. 25, w_h is given by

$$w_h = \text{Re}\left[A_1 J_o(x\sqrt{i}) + A_2 H_o^{(1)}(x\sqrt{i})\right] \tag{56}$$

where A_1 and A_2 are complex constants and $J_o(\ )$ and $H_o^{(1)}(\ )$ are zero order Bessel functions of the first and third kinds respectively.

The solution may alternatively be expressed in terms of the Bessel-Kelvin functions, ber, bei, ker, and kei, since

$$\text{ber } x = \text{Re } J_o(x\sqrt{i}) \quad , \quad \text{bei } x = -\text{Im } J_o(x\sqrt{i}) \tag{57a, b}$$

$$\text{ker } x = -\frac{\pi}{2} \text{Im } H_o^{(1)}(x\sqrt{i}) \quad , \quad \text{kei } x = -\frac{\pi}{2} \text{Re } H_o^{(1)}(x\sqrt{i}) \tag{57c, d}$$

The functions $J_o(x\sqrt{i})$ and $H_o^{(1)}(x\sqrt{i})$ and their derivatives with respect to x are tabulated in Reference 1, & ($H_o^{(1)}(\ )$) in Ref. 20. Ber x bei x, ker x, and kei x are tabulated in References 19, 21, and 22. Of these, the tables in Reference 19 are the most comprehensive.

For circumferential strain-hardening, from Eqn. 52a,

$$w_h = C_1 R_1 + C_2 R_2 + C_3 R_3 + C_4 R_4 \equiv C_k R_k \tag{58}$$

where C_k are the real constants, and R_k are four independent series solutions, viz:

$$R_1 = 1 + \sum_{n=4,8}^{\infty} [-1]^{n/4} \frac{x^n}{n(n-2)\left[(n-1)^2-a\right].(n-4)(n-6)\left[(n-5)^2-a\right]\ldots 4.2.(3^2-a)}$$

(59a)

$$R_2 = x^2 + \sum_{n=6,10}^{\infty} [-1]^{(n-2)/4} \frac{x^n}{n(n-2)\left[(n-1)^2 - a\right]. \text{-------}.6.4(5^2 - a)}$$

(59b)

$$R_3 = x^{1+a^{1/2}} + \sum_{n=5+a^{1/2},9+a^{1/2}}^{\infty} [-1]^{(n-1-a^{1/2})/4} \frac{x^n}{n(n-2)\left[(n-1)^2-a\right]\ldots(5+a^{1/2})(3+a^{1/2})(4+2a^{1/2})4}$$

(59c)

$$R_4 = x^{1-a^{1/2}} + \sum_{n=5-a^{1/2},9-a^{1/2}}^{\infty} [-1]^{(n-1+a^{1/2})/4} \frac{x^n}{n(n-2)\left[(n-1)^2-a\right]\text{----}(5-a^{1/2})(3-a^{1/2})(4-2a^{1/2})4}$$

(59d)

The above series are independent except where $a = 0, 1, 4$. The special case $a = 1$ is that of a purely elastic plate, and the solution is given in Eqn. 41. Where $a = 0$, the plate is purely plastic (zero rate of hardening) and the solution is given in the previous report (Reference 2).

Similarly, for meridional strain-hardening, from Eqn. 53a,

$$w_h = C_1 S_1 + C_2 S_2 + C_3 S_3 + C_4 S_4 \equiv C_k S_k \qquad (60)$$

where S_k are the four independent series solutions which may be obtained by substituting $\frac{1}{a}$ for a in R_k, Eqns. 59, providing x in this case is given by

$$x_2 \equiv \frac{x_1}{a^{1/4}} \equiv \frac{R \sin \phi}{a^{1/4} L_1 \cos \phi_e} \quad \text{or} \quad \frac{R}{a^{1/4} L_1} (\tan \phi_e + \phi - \phi_e) .$$

The functions R_k and their first three derivatives with respect to x are tabulated in Reference 2. The tables are calculated for $a =$ 1/4, 1/2, 3/4, 2 and 4, for values of x in the range $0 \leq x \leq 10$. The tables, therefore, also give S_k for the reciprocal values of a, i.e. for $a = 4$, 2, 4/3, 1/2, and 1/4, so that in effect both R_k and S_k are tabulated.

For strain-hardening in each direction, the solution for w_h from Eqn. 54a is again given by Eqn. 56 provided a new value x_3 for x is introduced.

where $x_3 = \frac{x}{a^{1/4}} = \frac{R \sin \phi}{a^{1/4} L \cos \phi_e}$ or $\frac{R}{a^{1/4} L}$ $(\tan \phi_e + \phi - \phi_e)$

The particular integral w_p in each case is

$$w_p = \frac{qR^2}{Et} \qquad \text{for q constant.} \tag{61}$$

The corresponding solution for F in each zone may be written

$$F = F_h + F_p \tag{62}$$

where: from Eqn. 23b

$$F_h = \text{Re}\left[B_1 J_o (x\sqrt{i}\,) + B_2 H_o^{(1)} (x\sqrt{i}\,)\right] + c_1 + c_2 \log x \tag{63a}$$

$$F_p = \frac{qR}{4} x^2 \quad , \text{ for q constant} \tag{63b}$$

in which the constants B_e are complex and c_e are real;
from Eqn. 52b

$$F_h = d_1 R_1 + d_2 R_2 + d_3 R_3 + d_4 R_4 + d_5 + d_6 \log x \equiv d_m R_m \tag{64a}$$

$$F_p = \frac{qR}{4} x^2 \qquad \text{for q constant} \tag{64b}$$

in which R_1 through R_4 are given by Eqns. 59, and the constants d_m are real;
from Eqn. 53b

$$F_h = d_1 S_1 + d_2 S_2 + d_3 S_3 + d_4 S_4 + d_5 + d_6 \log x \equiv d_n S_n \tag{65a}$$

$$F_p = a^{1/2} \frac{qR}{4} x^2 \qquad \text{for q constant} \tag{65b}$$

in which S_1 through S_4 are given by substitution of $\frac{1}{a}$ for a in Eqns. 59,
as before, and from Eqn. 54b

$$F_h = \text{Re} \left[B_1 J_o (x\sqrt{i}) + B_2 H_o^{(1)} (x\sqrt{i}) \right] + c_1 + c_2 \log x \tag{66a}$$

$$F_p = \frac{a^{1/2} qRx^2}{4} \qquad \text{for q constant} \tag{66b}$$

From Eqn. 25, the relationship between the complex constants A_e and B_e in Eqns. 56 and 63, or in the corresponding equations in the strain-hardening range, is

$$\overline{B}_e = \frac{1}{\lambda} \overline{A}_e \tag{67}$$

where $\overline{A}_e$ and $\overline{B}_e$ are the conjugates of A_e and B_e. This will be seen later in the solution of Case (1).

Using the previous expressions derived for moments in terms of deflection w, and the equilibrium Eqn. 1c, moments and shears may now be expressed in terms of the above solutions. The direct forces are similarly determined from Eqns. 29.

In the elastic range, the deflection w_h from Eqn. 56 may be written in the form

$$w_h = \text{Re} \left[Z \right] \tag{68a}$$

where $Z \equiv A_1 J_o (x\sqrt{i}) + A_2 H_o^{(1)} (x\sqrt{i})$. The deflection w_p does not contribute to moments and shears where q is constant. Hence from Eqns. 28 the corresponding moments may be written;

$$M_\phi = -\frac{D}{L^2} \text{Re} \left[Z'' + \nu \frac{1}{x} Z' \right] \tag{68b}$$

$$M_\theta = -\frac{D}{L^2} \text{Re} \left[\frac{1}{x} Z' + \nu Z'' \right] \tag{68c}$$

and from Eqn. 1c the shear is given by

$$Q_\phi = + \frac{D}{L^3} \operatorname{Re} \left[i\, Z' \right] \tag{68d}$$

where primes again denote differentiation with respect to x.

In the case of circumferential strain-hardening only, deflection w_h is given by

$$w_h = C_k\, R_k \qquad k = 1,\ 2,\ 3,\ \text{and } 4 \tag{68}$$

Thus from Eqns. 31, the moments are

$$M_\phi = - \frac{C_k D_1}{L_1^{\,2}} \left[R_k'' + a\nu \frac{1}{x} R_k' \right] + \nu M_c \tag{69a}$$

$$M_\theta = - \frac{a C_k D_1}{L_1^{\,2}} \left[\frac{1}{x} R_k' + \nu R_k'' \right] + M_c \tag{69b}$$

and from Eqn. 1c the shear is

$$Q_\phi = - \frac{C_k D_1}{L_1^{\,3}} \left[R_k''' + \frac{1}{x} R_k'' - \frac{a}{x^2} R_k' \right] - \frac{1-\nu}{L_1 x} M_c \tag{69c}$$

Similarly for meridional strain-hardening, where

$$w_h = C_k\, S_k \qquad k = 1,\ 2,\ 3,\ \text{and } 4. \tag{60}$$

moments and shears are given by

$$M_\phi = - \frac{a C_k D_1}{L_2^{\,2}} \left[S_k'' + \nu \frac{1}{x} S_k' \right] + M_c \tag{70a}$$

$$M_\theta = -\frac{C_k D_1}{L_2^2}\left[\frac{1}{x} S_k' + a\nu S_k''\right] + \nu M_c \qquad (70b)$$

and

$$Q_\phi = -\frac{aC_k D_1}{L_2^3}\left[S_k''' + \frac{1}{x} S_k'' - \frac{1}{ax^2} S_k'\right] + \frac{1-\nu}{L_2 x} M_c \qquad (70c)$$

and for strain-hardening in each direction, with

$$w_h = \text{Re}\left[Z\right] \quad \text{as before, and} \qquad (68a)$$

moments and shears are given by

$$M_\phi = -\frac{aD}{L_3^2}\,\text{Re}\left[Z'' + \nu \frac{1}{x} Z'\right] + M_c \qquad (71a)$$

$$M_\theta = -\frac{aD}{L_3^2}\,\text{Re}\left[\frac{1}{x} Z' + \nu Z''\right] + M_c \qquad (71b)$$

and

$$Q_\phi = -\frac{aD}{L_3^3}\,\text{Re}\left[iZ'\right] \qquad (71c)$$

The direct forces N_ϕ and N_θ in each case, are given by Eqns. 29. Alternatively, they may be determined from equilibrium at the section ϕ = constant, and from Eqn. 24a, i.e.

$$N_\phi = -Q_\phi \cot\phi - \frac{R\int_0^\phi q \sin\phi' \cos\phi' d\phi'}{\sin^2\phi} \qquad (72a)$$

and

$$N_\theta = -\frac{Et}{R} w - N_\phi \qquad (72b)$$

The solutions for inelastic bending were presented in terms of

Case (II) since the functions involved in this case were available in tabulated form.

The solutions for Case (I) may again be expressed in series form but only the solutions for yielding in both directions may be expressed in terms of functions already tabulated.

The solutions for w and F in Case (I) may be obtained by solving the complex equation for Φ, and it is of interest to compare the solutions obtained with those of Eqn. 22 for the elastic case.

From Eqn. 22 we obtain

$$\Phi = \Phi_h + \Phi_p \tag{73a}$$

where

$$\Phi_h = A_1 J_o (xi^{3/2}) + A_2 H_o^{(2)} (xi^{3/2}) + A_3 + A_4 \log x \equiv A_k W_k \tag{73b}$$

and

$$\Phi_p = i \frac{qR^2}{4Et} x^2 \qquad \text{for } q = \text{constant} \tag{73c}$$

$$\text{Hence } w = \text{Re } \Phi = \text{Re } (\Phi_h + \Phi_p) = \text{Re } \left[A_k W_k \right] \tag{74a}$$

and

$$F \equiv \text{Re} \left[\frac{1}{\lambda} \Phi \right] \equiv \text{Re } \frac{1}{\lambda} (\Phi_h + \Phi_p) = \text{Re} \left[\frac{1}{\lambda} A_k W_k \right] + \frac{qR}{4} x^2 \tag{74b}$$

The functions $J_o (xi^{3/2})$ and $H_o^{(2)} (xi^{3/2})$ are the complex conjugate functions of $J_o (x\sqrt{i})$ and $H_o^{(1)} (x\sqrt{i})$ respectively. (See References 20, 21, 22), i.e.

$$\text{Re } J_o (x\sqrt{i}) = \text{Re } J_o (xi^{3/2}); \quad \text{Im } J_o (x\sqrt{i}) = - \text{Im } J_o (xi^{3/2}); \tag{75a, b}$$

$$\mathrm{Re}\, H_o^{(1)}(x\sqrt{i}) = \mathrm{Re}\, H_o^{(2)}(xi^{3/2}); \quad \mathrm{Im}\, H_o^{(1)}(x\sqrt{i}) = -\,\mathrm{Im}\, H_o^{(2)}(xi^{3/2}) \tag{75c, d}$$

Thus the solutions for w and F reduce to Eqns. 56 and 63a.

Similarly for Eqn. 46e,

$$\phi_1 = \phi_{1h} + \phi_{1p} \tag{76a}$$

where

$$\phi_{1h} = B_1 X_1 + B_2 X_2 + B_3 X_3 + B_4 X_4 = B_k X_k \tag{76b}$$

in which B_k are the complex constants and X_k are given by

$$X_1 = 1 \tag{76c}$$

$$X_2 = \sum_{n=2,4,6}^{\infty} i^{(n-2)} \frac{(n-a)^2 \cdot (n-4)^2 \cdot \text{-----------} 4^2 . 2^2 . (i^{3/2}x)^n}{n(n-2)\left[(n-1)^2-a\right] \cdot (n-2)(n-4)\left[(n-3)^2-a\right] \cdot \text{-----} 4.2.(3^2-a).} \tag{76d}$$

$$X_3 = \sum_{n=1+a^{1/2},\,3+a^{1/2}}^{\infty} i^{(n-1+a^{1/2})} \frac{(n-2)^2. \text{-----------}(1+a^{1/2})^2.(i^{3/2}x)^n}{n(n-2)\left[(n-1)^2-a\right] \text{---}(3+a^{1/2})(1+a^{1/2})(2+2a^{1/2})2.} \tag{76e}$$

$$X_4 = \sum_{n=1-a^{1/2},\,3-a^{1/2}}^{\infty} i^{(n-1+a^{1/2})} \frac{(n-2)^2. \text{-----------}(1-a^{1/2})^2.(i^{3/2}x)^n}{n(n-2)\left[(n-1)^2-a\right] \text{---}(3-a^{1/2})(1-a^{1/2})(2-2a^{1/2})2.} \tag{76f}$$

and $x \equiv x_2 \equiv x_1 a^{1/4} \equiv \dfrac{a^{1/4} R \sin\phi}{L_1 \cos\phi_e}$ or $\dfrac{a^{1/4}R}{L_1}(\tan\phi_e + \phi - \phi_e)$ (76g)

The particular solution in this case is

$$\phi_{1p} = i \frac{qR^2}{4aEt} x^2 \tag{76h}$$

hence

$$w \equiv \text{Re}\ \Phi_1 \equiv \text{Re}\left[B_k\ X_k\right] \tag{76j}$$

and

$$F \equiv \text{Re}\ \frac{1}{\lambda_1}\ \Phi_1 \equiv \text{Re}\left[\frac{1}{\lambda_1} B_k\ X_k\right] + \frac{qR}{4a^{1/2}}\ x^2 \tag{76k}$$

In the same manner, the solution of Eqn. 47c may be expressed in the form

$$\Phi_1 = \Phi_{1h} + \Phi_{1p} \tag{77a}$$

where

$$\Phi_{1h} = C_1\ Y_1 + C_2\ Y_2 + C_3\ Y_3 + C_4\ Y_4 \pm C_k\ Y_k \tag{77b}$$

in which Y_k may be obtained by substituting $\frac{1}{a}$ for a in X_k, and

$$x \equiv x_3 \equiv \frac{x_1}{a^{1/4}} \equiv \frac{R \sin \phi}{a^{1/4} L_1 \cos \phi_e} \quad \text{or} \quad \frac{R}{a^{1/4} L_1} \left(\tan \phi_e + \phi - \phi_e\right) \tag{77c}$$

and

$$\Phi_{1p} = i\ \frac{qR^2}{4Et}\ x^2 \tag{77d}$$

Thus

$$w = \text{Re}\ \Phi_1 = \text{Re}\left[C_k\ Y_k\right] \tag{77e}$$

and

$$F = \text{Re}\ \frac{1}{\lambda_1}\ \Phi_1 = \text{Re}\left[\frac{1}{\lambda_1}\ C_k\ Y_k\right] + \frac{qRa^{1/2}}{4}\ x^2 \tag{77f}$$

Similarly, the solution for Eqn. 48c may be expressed as

$$\Phi_2 = \Phi_{2h} + \Phi_{2p} \tag{78a}$$

where

$$\Phi_{2h} = A_k W_k \tag{78b}$$

in which $x \equiv \frac{R \sin \phi}{L \cos \phi_e}$ or $\frac{R}{L} (\tan \phi + \phi - \phi_e)$ and W_k is (78c)

given by Eqn. 73b, and

$$\Phi_{2p} = i \frac{qR^2}{4aEt} x^2 \tag{78d}$$

Thus

$$w = \text{Re}\ \Phi_2 = \text{Re} \left[A_k W_k \right] \tag{78e}$$

and

$$F = \text{Re}\ \frac{1}{\lambda_2} \Phi_2 = \text{Re} \left[\frac{1}{\lambda_2} A_k W_k \right] + \frac{qR}{4} x^2 \tag{78f}$$

Moments, shears, and direct forces are then obtained by substituting for w and F in Eqns. 68, 69, 70, 71, and 29.

Unsymmetrical bending

In the case of unsymmetrical bending past the elastic range of stress, the zones of inelastic behavior are no longer coaxial. Further, since the directions of yield are the directions of principal stress and are not necessarily circumferential or meridional, we need only distinguish between three zones: an elastic zone, a zone in which strain-hardening occurs in one direction only, and a zone in which strain-hardening occurs in both directions of the principal stress simultaneously.

Of the three, only the cases of elastic behavior and of strain-hardening in both directions simultaneously are readily handled. Where strain-hardening occurs in one direction only, the material properties vary, in effect, with the unknown directions of principal stress. The problem is analogous to the case of an orthotropic plate or shell in which the principal directions of orthotropy vary with the directions of principal stress, and a general solution appears to be intractable.

The equations of equilibrium required in the general case increase to five, viz:

$$\frac{\partial}{\partial \phi}(N_\phi \sin \phi) + \frac{\partial}{\partial \theta} N_{\phi\theta} - N_\theta \cos \phi - \sin \phi \, Q_\phi = 0 \tag{79a}$$

$$\frac{\partial}{\partial \phi}(N_{\phi\theta} \sin \phi) + \frac{\partial N_\theta'}{\partial \theta} + N_{\phi\theta} \cos \phi - \sin \phi \, Q_\theta = 0 \tag{79b}$$

$$+ \frac{\partial}{\partial \phi}(Q_\phi \sin \phi) + \frac{\partial Q_\theta}{\partial \theta} + N_\phi \sin \phi + N_\theta \sin \phi + q R \sin \phi = 0 \tag{79c}$$

$$\frac{\partial}{\partial \phi}(M_\phi \sin \phi) + \frac{\partial}{\partial \theta} M_{\phi\theta} - M_\theta \cos \phi - Q_\phi R \sin \phi = 0 \tag{79d}$$

$$\frac{\partial}{\partial \phi}(M_{\phi\theta} \sin \phi) + \frac{\partial M_\theta}{\partial \theta} + M_{\phi\theta} \cos \phi - Q_\theta R \sin \phi = 0 \tag{79e}$$

The strains and curvatures may be expressed in terms of circumferential, meridional, and radial displacements u, v, and w, viz.

$$\epsilon_\phi = \frac{1}{R}\left(\frac{dv}{d\phi} - w\right) \tag{80a}$$

$$\epsilon_\theta = \frac{1}{R}\left(v \cot \phi - w + \frac{\partial u}{\partial \theta} \operatorname{cosec} \phi\right) \tag{80b}$$

$$\gamma_{\phi\theta} = \frac{1}{R}\left(\frac{\partial v}{\partial \theta} \operatorname{cosec} \phi + \sin \phi \frac{\partial}{\partial \phi}(u \operatorname{cosec} \phi)\right) \tag{80c}$$

$$\chi_\phi = \frac{1}{R^2}\left[\frac{d}{d\phi}\left\{\left(\frac{dw}{d\phi}\right) + v\right\}\right] \tag{80d}$$

$$\chi_\theta = \frac{1}{R^2}\left[\left(\frac{dw}{d\phi}\right) \cot \phi + \operatorname{cosec}^2 \phi \frac{\partial^2 w}{\partial \theta^2} + \frac{\partial u}{\partial \theta} + v \cot \phi\right] \tag{80e}$$

$$\chi_{\phi\theta} = \frac{1}{R^2}\left[\frac{\partial}{\partial\phi}\left(\operatorname{cosec}\phi\frac{\partial w}{\partial\theta} + u\right)\right] + \operatorname{cosec}\phi\left(\frac{\partial v}{\partial\theta} - v\cos\phi\right) \tag{80f}$$

u,v and their derivatives are neglected when compared to w and its derivatives in further calculations.

The relationships between stress and strain again are different for the different zones of the shell. In the elastic zone,

$$N_\phi = \frac{Et}{1-\nu^2}\left(\epsilon_\phi + \nu\epsilon_\theta\right) \tag{81a}$$

$$N_\theta = \frac{Et}{1-\nu^2}\left(\epsilon_\theta + \nu\epsilon_\phi\right) \tag{81b}$$

$$N_{\phi\theta} = \frac{Et}{2(1+\nu)}\gamma_{\phi\theta} - \frac{M_{\phi\theta}}{R} \tag{81c}$$

$$M_\phi = -D\left(\chi_\phi + \nu\chi_\theta\right) \tag{81d}$$

$$M_\theta = -D\left(\chi_\theta + \nu\chi_\phi\right) \tag{81e}$$

$$M_{\phi\theta} = -(1-\nu)D\chi_{\phi\theta} \tag{81f}$$

It is again convenient to introduce a stress function F which in the general case is defined in conical coordinates by

$$N_\phi = -\frac{1}{x}\frac{\partial F}{\partial x} - \frac{1}{\cos^2\phi_e\, x^2}\frac{\partial^2 F}{\partial\theta^2} \tag{82a}$$

$$N_\theta = -\frac{\partial^2 F}{\partial x^2} \tag{82b}$$

$$N_{\phi\theta} = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial F}{\partial\theta}\right)\sec\phi_e \tag{82c}$$

and is seen to satisfy the equilibrium equations 79a and 79b provided shears are of smaller order than direct forces and may therefore be neglected.

Hence proceeding in the same way as for the symmetrical case, we again derive the equations

$$\nabla^4 w + \frac{R}{Et} \nabla^2 F = \frac{qR^2}{Et} \qquad \text{See 19a} \tag{83}$$

$$\nabla^4 F - \frac{Et}{R} \nabla^2 w = 0 \qquad \text{Eq. 21a} \tag{84}$$

where $\nabla^4 \equiv \nabla^2 \cdot \nabla^2$ and ∇^2 in this case is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{x} \frac{\partial}{\partial x} + \frac{1}{\cos^2 \phi_e \, x^2} \frac{\partial^2}{\partial \theta^2}$$

and $x \equiv \frac{R \sin \phi}{L \cos \phi_e}$ or $\frac{R}{L} (\tan \phi_e + \phi - \phi_e)$ as before.

Equations 83 and 84 may again be combined to give

$$\nabla^6 F + \nabla^2 F = qR \tag{85}$$

$$\nabla^6 w + \nabla^2 w = \frac{R^2}{Et} \nabla^2 q \tag{86}$$

or alternatively,

$$\nabla^4 \Phi - i \nabla^2 \Phi = q \frac{R^2}{Et} \tag{87}$$

where $\Phi \equiv w + \lambda F$, and $\lambda \equiv i \frac{R}{Et}$,

as in the symmetrical case.

Where radial displacements tend to approach zero at the external boundary of the shell, Equation 79c again requires that

$$N_\phi + N_\theta + \frac{Etw}{R} = 0 \tag{28}$$

and from Eqn. 83

$$\nabla^4 w + w = q \frac{R^2}{Et} \quad \text{, as before.} \tag{88}$$

The ordinate ϕ_e, in the case of unsymmetrical bending, tends to have less significance since the concentration of stress considered need not occur at a constant ϕ ordinate around the shell. However, choosing a value for ϕ_e in the region of maximum stress again allows the bending term ($\nabla^4 w$) in Eqn. 88 to more nearly approach the corresponding term for a plate and thus leads to greater accuracy than that given by the value $\phi_e = 0$.

In the case of strain-hardening in each direction, the stress-strain relationships give

$$N_\phi = \frac{aEt}{1-\nu^2}(\epsilon_\phi + \nu\epsilon_\theta) + \frac{2\epsilon_\phi}{\chi_\phi}\sigma_c \tag{89a}$$

$$N_\theta = \frac{aEt}{1-\nu^2}(\epsilon_\theta + \nu\epsilon_\phi) + \frac{2\epsilon_\theta}{\chi_\theta}\sigma_c \tag{89b}$$

$$N_{\phi\theta} = \frac{aEt}{2(1-\nu)}\gamma_{\phi\theta} - \frac{M_{\phi\theta}}{R} \tag{89c}$$

$$M_\phi = -aD(\chi_\phi + \nu\chi_\theta) + \left[1 - \left\{\frac{2\epsilon_\phi}{t\chi_\phi}\right\}^2\right] M_c \tag{89d}$$

$$M_\theta = -aD(\chi_\theta + \nu\chi_\phi) + \left[1 - \left\{\frac{2\epsilon_\theta}{t\chi_\theta}\right\}^2\right] M_c \tag{89e}$$

$$M_{\phi\theta} \cong -(1-\nu)\, aD\, \chi_{\phi\theta} \tag{89f}$$

The procedure for derivation of the differential equations is again analogous to the symmetrical case.

Case (I) Assuming the terms $\frac{\epsilon_\phi}{\chi_\phi}$ and $\frac{\epsilon_\theta}{\chi_\theta}$ to be effectively constant, or negligible, we again obtain the equations

$$a \nabla^4 w + \frac{R}{Et} \nabla^2 F = q \frac{R^2}{Et} \tag{90}$$

$$\nabla^4 F - \frac{aEt}{R} \nabla^2 w = 0 \tag{91}$$

which may be combined as before to give

$$\nabla^6 F + \nabla^2 F = qR \tag{92}$$

$$\nabla^6 w + \nabla^2 w = \frac{R^2}{aEt} \nabla^2 q \tag{93}$$

or $$\nabla^4 \Phi - i \nabla^2 \Phi = \frac{qR^2}{aEt} \tag{94}$$

in which $\Phi \equiv w + \lambda_p F$ and $\lambda_p \equiv i \frac{R}{aEt}$.

Where displacements and direct forces tend to approach zero at the external boundary, Eqn. 91 requires that

$$N_\phi + N_\theta - \frac{aEtw}{R} = 0 \tag{95}$$

and from Eqn. 90,

$$\nabla^4 w + w = \frac{qR^2}{aEt} \tag{96}$$

Case (II). Alternatively, using Eqns. 83 and 90 to relate direct forces to stresses, we obtain

$$a \nabla^6 F + \nabla^2 F = qR \tag{97}$$

$$a \nabla^4 w + w = \frac{qR^2}{Et} \tag{98}$$

and $\nabla^4 \Phi - i\, a^{-1/2} \nabla^2 \Phi = \frac{qR^2}{aEt}$ (99)

where $\Phi \equiv w + \lambda_p' F$ and $\lambda_p' \equiv i \frac{R}{a^{1/2} Et}$

Each of the above operators may alternatively be expressed in spherical coordinates as before, where for unsymmetrical bending

$$\nabla^2 \equiv \frac{L^2}{R^2}\left[\frac{\partial^2}{\partial \phi^2} + \frac{1}{\tan \phi}\frac{\partial}{\partial \phi}\right] + \frac{L^2}{R^2 \sin^2 \phi}\frac{\partial^2}{\partial \theta^2}$$

The solutions for w and F in each zone may again be conveniently determined from the solution for Φ.

The solutions for Φ may each be expressed in the form

$$\Phi = \Phi_h + \Phi_p \tag{100}$$

In the elastic range of stresses, Φ_h is given by

$$\Phi_h = \sum_{n=0}^{\infty}\left[A_v J_v(xi^{3/2}) + B_v H_v^{(2)}(xi^{3/2})\right] \begin{matrix}\cos n\theta \\ \sin n\theta\end{matrix}$$

$$+ C_o \log x + D_o + \sum_{n=1}^{\infty}\left[C_v x^{-v} + D_v x^{v}\right] \begin{matrix}\cos n\theta \\ \sin n\theta\end{matrix} \tag{101}$$

where $v = n/\cos \phi_e$, A_v through D_v are complex constants, and $J_v(\ )$ and $H_v^{(2)}(\ )$ are bessel functions of order v. Where v is not an integer $H_v^{(2)}(\ )$ is replaced by $J_{-v}(\ )$ in the solution. The variable x is defined by Eqns. 26 or 27 as before. The particular solution Φ_p is given by

$$\Phi_p = i \frac{qR^2}{4Et} x^2 , \quad \text{for q constant.} \tag{102}$$

Thus from the restrictions previously imposed,

$$w \equiv \text{Re}\,\phi = \text{Re}\sum_{n=0}^{\infty}\left[A_v J_v\,(xi^{3/2}) + B_v H_v^{(2)}(xi^{3/2})\right]\begin{matrix}\cos n\theta\\ \sin n\theta\end{matrix} + \frac{qR^2}{Et}$$

(103)

$$F \equiv \text{Re}\left[\phi\frac{1}{\lambda}\right] = \frac{R}{Et}\,\text{Im}\sum_{n=0}^{\infty}\left[A_v J_v\,(xi^{3/2}) + B_v H_v^{(2)}(xi^{3/2})\right]\begin{matrix}\cos n\theta\\ \sin n\theta\end{matrix}$$

$$+ c_o \log x + d_o + \sum_{n=1}^{\infty}\left[c_v x^{-v} + d_v x^{v}\right]\begin{matrix}\cos n\theta\\ \sin n\theta\end{matrix} + \frac{qR}{4}x^2 \qquad (104)$$

where c_v and d_v are real constants.

For strain-hardening in each direction, Case (I)

$$w = \text{Re}\sum_{n=0}^{\infty}\left[A_v J_v\,(xi^{3/2}) + B_v H_v^{(2)}(xi^{3/2})\right]\begin{matrix}\cos n\theta\\ \sin n\theta\end{matrix} + \frac{qR^2}{aEt} \qquad (105)$$

$$F = \frac{R}{aEt}\text{Im}\sum_{n=0}^{\infty}\left[A_v J_v\,(xi^{3/2}) + B_v H_v^{(2)}(xi^{3/2})\right]\begin{matrix}\cos n\theta\\ \sin n\theta\end{matrix}$$

$$+ c_o \log x + d_o + \sum_{n=1}^{\infty}\left[c_v x^{-v} + d_v x^{v}\right]\begin{matrix}\cos n\theta\\ \sin n\theta\end{matrix} + \frac{qR}{4}x^2 \qquad (106)$$

Similarly for Case (II), w is again given by Eqn. 105 and F is given by

$$F = \frac{R}{a^{1/2}Et}\,\text{Im}\sum_{n=0}^{\infty}\left[A_v J_v(xi^{3/2}) + B_v H_v^{(2)}(xi^{3/2})\right]\begin{matrix}\cos n\theta\\ \sin n\theta\end{matrix}$$

$$+ c_o \log x + d_o + \sum_{n=1}^{\infty}\left[c_v x^{-v} + d_v x^{v}\right]\begin{matrix}\cos n\theta\\ \sin n\theta\end{matrix} + \frac{qRa^{1/2}x^2}{4} \qquad (107)$$

with $x = \frac{x}{a^{1/4}} = \frac{R\sin\phi}{a^{1/4}\cdot L\cos\phi_e}$ or $\frac{R}{a^{1/4}L}(\tan\phi_e + \phi - \phi_e)$

From Eqns. 80 and 81, moments in the elastic range are given by

$$M_\phi = -\frac{D}{L^2}\left[\frac{\partial^2 w}{\partial x^2} + \nu\left(\frac{1}{x}\frac{\partial w}{\partial x} + \frac{1}{\cos^2\phi_e\, x^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \tag{108a}$$

$$M_\theta = -\frac{D}{L^2}\left[\frac{1}{x}\frac{\partial w}{\partial x} + \frac{1}{\cos^2\phi_e\, x^2}\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial x^2}\right] \tag{108b}$$

$$M_{\phi\theta} = \frac{(1-v)}{\cos\phi_e}\frac{D}{L^2}\left[\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial w}{\partial \theta}\right)\right] \tag{108c}$$

and from Eqns. 79 and 108 a, b, and c shears are given by

$$Q_\phi = -\frac{D}{L^3}\frac{\partial}{\partial x}(\nabla^2 w) \tag{108d}$$

$$Q_\theta = -\frac{D}{\cos\phi_e\, L^3}\frac{1}{x}\frac{\partial}{\partial \theta}(\nabla^2 w) \tag{108e}$$

From Eqns. 80 and 89, moments in the strain-hardening range are given approximately by

$$M_\phi = -\frac{aD}{L^2}\left[\frac{\partial^2 w}{\partial x^2} + \nu\left(\frac{1}{x}\frac{\partial w}{\partial x} + \frac{1}{\cos^2\phi_e\, x^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] + M_c \tag{109a}$$

$$M_\theta = -\frac{aD}{L^2}\left[\frac{1}{x}\frac{\partial w}{\partial x} + \frac{1}{\cos^2\phi_e\, x^2}\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial x^2}\right] + M_c \tag{109b}$$

$$M_{\phi\theta} = -\frac{(1-v)}{\cos\phi_e}\frac{aD}{L^2}\left[\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial w}{\partial \theta}\right)\right] \tag{109c}$$

and from Eqns. 79 and 109a, b, and c, shears are given by

$$Q_\phi = -\frac{aD}{L^3}\frac{\partial}{\partial x}(\nabla^2 w) \tag{109d}$$

$$Q_\theta = -\frac{aD}{\cos\phi_e L^3} \frac{1}{x} \frac{\partial}{\partial\theta}(\nabla^2 w) \tag{109e}$$

Hence for unsymmetrical bending in the elastic range, with

$$w_h = \mathrm{Re}\left[Z_v\right] \cos n\theta^{*} \qquad \text{where } n = 0, 1, 2 \ldots\ldots \infty^{**} \tag{110a}$$

and

$$Z_v = A_v J_v (x\sqrt{i}) + B_v H_v^{(1)}(x\sqrt{i})$$

Eqn. 108 gives

$$M_\phi = -\frac{D}{L^2} \mathrm{Re}\left[Z_v'' + \nu\left(\frac{1}{x} Z_v' - \frac{n^2}{\cos^2\phi_e\, x^2} Z_v\right)\right] \cos n\theta \tag{110b}$$

$$M_\theta = -\frac{D}{L^2} \mathrm{Re}\left[\frac{1}{x} Z_v' - \frac{n^2}{\cos^2\phi_e\, x^2} Z_v + \nu Z_v''\right] \cos n\theta \tag{110c}$$

$$M_{\phi\theta} = +\frac{(1-\nu)}{\cos\phi_e} \frac{D}{L^2} \mathrm{Re}\left[n\left(\frac{1}{x} Z_v' - \frac{1}{x^2} Z_v\right)\right] \sin n\theta \tag{110d}$$

$$Q_\phi = \frac{D}{L^3} \mathrm{Re}\left[i Z_v'\right] \cos n\theta \tag{110e}$$

$$Q_\theta = \frac{-D}{\cos\phi_e L^3} \frac{1}{x} \mathrm{Re}\left[i\, n Z_v\right] \sin n\theta \tag{110f}$$

The corresponding equations in the strain-hardening range are

---

*Where symmetry occurs about a plane through the poles ($\theta = 0, \pi$) only the cosine series is required.

**Summation from $n = 0$ to $n = \infty$ is here intended.

$$W_h = \text{Re}\left[Z_v \right] \cos n\,\theta^{*} \qquad n = 0, 1, 2, \ldots \infty^{**} \tag{111a}$$

as before, and from Eqns. 109

$$M_\phi = M_c - \frac{aD}{L^2}\,\text{Re}\left[Z_v'' + \nu\left(\frac{1}{x} Z_v' - \frac{n^2}{\cos^2 \phi_e\, x^2} Z_v\right)\right] \cos n\,\theta \tag{111b}$$

$$M_\theta = M_c - \frac{aD}{L^2}\,\text{Re}\left[\frac{1}{x} Z_v' - \frac{n^2}{\cos^2 \phi_e\, x^2} Z_v + \nu Z_v'' \right] \cos n\,\theta \tag{111c}$$

$$M_{\phi\theta} = (1-\nu)\,\frac{aD}{\cos \phi_e\, L^2}\,\text{Re}\left[n\left(\frac{1}{x} Z_v' - \frac{1}{x^2} Z_v\right)\right] \sin n\,\theta \tag{111d}$$

$$Q_\phi = \frac{aD}{L^3}\,\text{Re}\left[i\, Z_v' \right] \cos n\,\theta \tag{111e}$$

$$Q_\theta = \frac{-aD}{\cos \phi_e\, L^3}\,\frac{1}{x}\,\text{Re}\left[i\, n\, Z_v \right] \sin n\,\theta \tag{111f}$$

From Eqns. 82 the direct forces in each case (for $q = 0$) are

$$N_\phi = \frac{R}{Et}\,\text{Im}\left[\frac{1}{x} Z_v' - \frac{v^2}{x^2} Z_v \right] \cos n\,\theta + \frac{c_o}{x^2} + \sum_{n=1}^{\infty} \left[c_v\left(-v - \frac{n^2}{\cos^2 \phi_e}\right) x^{-v-2} + d_v\left(v - \frac{n^2}{\cos^2 \phi_e}\right) x^{v-2} \right] \cos n\,\theta \tag{112a}$$

$$N_\theta = \frac{R}{Et}\,\text{Im}\left[Z_v'' \right] \cos n\,\theta + \frac{-c_o}{x^2} + \sum_{n=1}^{\infty} \left[c_v\, v(v+1)\, x^{-v-2} + d_v\, v(v-1)\, x^{v-2} \right] \cos n\,\theta \tag{112b}$$

$$N_{\phi\theta} = \frac{Rn}{Et} \frac{1}{\cos \phi_e} \operatorname{Im} \left[\frac{1}{x} Z_v' - \frac{1}{x^2} Z_v \right] \sin n \theta + \sum_{n=1}^{\infty} n \left[C_v (-v-1) x^{-v-2} + d_v (v-1) x^{v-2} \right] \frac{\sin n \theta}{\cos \phi_e} \qquad (112c)$$

Boundary conditions

The boundaries to the various zones of behavior must necessarily be determined by trial, and in the case of unsymmetrical bending this will be the most difficult part of the problem.

At the common boundary of two zones, the conditions of continuity (of deflections, slopes, moments, and shears) are applicable. Other boundary conditions required to determine the integration constants are:

1. at a clamped edge, deflection w and slope w' are zero.

2. at a simply supported edge, deflection w and normal moments M_n are zero, and

3. at a free edge both normal moments M_n and effective normal shears V_n are zero, where

$$V_n = Q_n - \frac{\partial M_{nt}}{\partial s}$$

The degree of restraint imposed on inplane forces at a boundary provides further determinative conditions.

In the special case where $a = 0$, i.e. of purely plastic bending, it is often convenient to consider the zone of yield to be retracted to a line, in which case the condition that slope is continuous is necessarily replaced by the yield condition

$$M_n = M_p$$

or more accurately, by

$$M_n = M_p \left[1 - \left(\frac{N_n}{N_p} \right)^2 \right] \tag{113}$$

where M_p and N_p are respectively the yield capacities for moment and direct force.

II. ROTATIONAL SHELLS

General Case

The equations for a specific zone in a spherical shell may be generalized to the case of a specific zone in a general shell of revolution as follows.

In the elastic range, following Mushtari and Vlasov (Reference 24) there results

$$\nabla^4 \Phi - i \nabla_\gamma^2 \Phi = \frac{qR_o^2}{Et} \tag{114a}$$

where, in conical coordinates,

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{1}{x} \frac{\partial}{\partial x} + \frac{1}{\cos^2 \phi_e \, x^2} \frac{\partial^2}{\partial \theta^2} \tag{114b}$$

$$\nabla_\gamma^2 \equiv \frac{R_o}{r_2} \left[\frac{\partial^2}{\partial x^2} + \gamma \left(\frac{1}{x} \frac{\partial}{\partial x} + \frac{1}{\cos^2 \phi_e \, x^2} \frac{\partial^2}{\partial \theta^2} \right) \right] \tag{114c}$$

$$\gamma \equiv \frac{r_2}{r_1} \tag{114d}$$

$$x \equiv \frac{1}{L_o} \left[r_2 \tan \phi_e + s - s_e \right] \tag{114e}$$

$$\Phi \equiv w + \lambda F \tag{114f}$$

$$\lambda \equiv i \frac{R_o}{Et} \qquad (114g)$$

r_1, r_2 = meridional and circumferential radii of curvature respectively

R_o = radius at $\phi = 0$

$$L_o = \frac{R_o^{1/2} \, t^{1/2}}{\left[12(1-v^2)\right]^{1/4}} \qquad (114h)$$

s = curve length

Similarly, in the inelastic range for Case (I) as before, stress and displacements are governed by

$$\nabla_p^4 \Phi - i\, b\, \nabla_\gamma^2 \Phi = \frac{qR_o^2}{cEt} \qquad \lambda = \frac{iRo}{dEt} \qquad (115a)$$

where, for strain-hardening in the circumferential direction only (axi-symmetrical bending)

$$\nabla_p^4 \equiv \nabla_c^4 \quad , \qquad b \equiv a^{1/2} = d, \qquad c \equiv 1 \qquad (115b)$$

for strain-hardening in the meridional direction only (axi-symmetrical bending)

$$\nabla_p^4 \equiv \nabla_m^4 \quad , \qquad b \equiv a^{-1/2} \quad , \qquad c \equiv a, \quad d = a^{1/2} \qquad (115b)$$

and for strain-hardening in both directions of principal stress (axi-symmetrical or unsymmetrical bending)

$$\nabla_p^4 \equiv \nabla^4 \quad , \qquad b \equiv 1 \quad , \qquad c \equiv a = d \qquad (115d)$$

The form of the Equations 114a and 115a may be inferred from the corresponding equations for the spherical shell. The first term in each represents the plate or inplane effect of bending, and direct forces, and the second the curvature or membrane effect. The third term is the loading term. Thus the effect of plasticity is to modify the ∇^4 operator in Eqn. 29 and the effect of the varying curvatures is to modify the operator ∇^2. The constants b, c and d allow for change in the coefficients due to modifications to the elastic modulus.

Special case

Where the radii r_1 and r_2 are constant over the zone considered, we have $\frac{R_o}{r_2}$ = constant, $\frac{r_2}{r_1}$ = constant, and Eqns. 114a and 115a may be readily solved by separation of variables and the method of Frobenius (25).

Non-uniform thickness

For shells of varying thickness the equations are developed in the same manner as for the uniform shell, except that the terms D, D_1, t, and t_1, must be considered as variables. For an axi-symmetrical shell, i.e. $t = t(\phi)$ under axi-symmetrical loading $(q = q(\phi))$, and with the further generalization that $E = E(\phi)$ the resultant equations are

$$\nabla_{p1}^4 \circ \nabla_{p2}^4 w + b \nabla_\gamma^2 \circ \nabla_\gamma^2 w = \frac{R_o^2}{cE_o t_o} \circ \nabla_{p2}^4 q \qquad (116a)$$

$$\nabla_{p1}^4 \cdot \nabla_{p2}^4 F + b \nabla_\gamma^2 \circ \nabla_\gamma^2 F = R_o \circ b \cdot \nabla_\gamma^2 q \qquad (116b)$$

where,

$$\nabla_{p1}^4 = \frac{1}{D_o} \left[\frac{1}{x} \frac{d^2}{dx^2} (D_1 x \frac{d^2}{dx^2}) - \frac{e}{x} \frac{d}{dx} (D_1 \frac{1}{x} \frac{d}{dx}) \right]$$

$$\nabla_{p2}^{4} \equiv E_o t_o \left[\frac{1}{x} \frac{d^2}{dx^2} \left(\frac{1}{Et} x \frac{d^2}{dx^2} \right) - \frac{e}{x} \frac{d}{dx} \left(\frac{1}{Et} \frac{1}{x} \frac{d}{dx} \right) \right]$$

D_o, E_o, t_o, R_o = $(D, E, t, R)_{\phi = 0}$; and b, c, and e are constants.

In the elastic range,

$$b \equiv c \equiv e \equiv 1$$

for circumferential strain-hardening,

$$b \equiv e \equiv a \quad ; \quad c \equiv 1$$

for meridional strain-hardening,

$$b = \frac{1}{a} \quad , \quad c = a \quad , \quad e = \frac{1}{a}$$

and for strain-hardening in each direction,

$$b \equiv e \equiv 1 \quad , \quad c = a .$$

III. SIMPLIFICATION FOR DESIGN

In the class of problem considered, that of a stress concentration in the shell, yielding is normally limited to a relatively narrow zone. Where the zone considered is narrow, and stresses vary considerably over the zone, the lower order derivatives of F and w in Eqns. 114a and 115a tend to become less significant and for design calculations may be neglected. In this way the whole second term in each equation may be omitted leaving, for Eqn. 114a

$$\nabla^4 \Phi = \frac{qR_o^2}{Et} \tag{117}$$

and for Eqn. 115a

$$\nabla_p^4 \Phi = \frac{qR_o^2}{cEt} \tag{118}$$

where p and c are defined in Eqns. 115.

The real parts of these equations may be inferred from the analogy of the plate on an elastic foundation, which may be extended to the rotational shell. As the width of the zone considered approaches zero, the reaction, and hence the effect, of the elastic foundation, becomes of smaller order than the bending term. Hence in Eqns. 114a and 115a, the second term--describing the effect of the equivalent foundation--may be neglected.

Thus in the case q = 0, over the zone considered, there results for axi-symmetry,

In a narrow elastic zone

$$w = c_1 + c_2 x^2 + c_3 \log x + c_4 x^2 \log x \tag{119a}$$

$$M_\phi = -\frac{D}{L_o^2}\left[2(1+\nu)c_2 - (1-\nu)c_3 \frac{1}{x^2} + c_4\left\{2 \log x + 3 + \nu(2 \log x + 1\right\}\right] \tag{119b}$$

$$M_\theta = -\frac{D}{L_o^2}\left[2(1+\nu)c_2 + (1-\nu)c_3 \frac{1}{x^2} + c_4\left\{2 \log x + 1 + \nu(2 \log x + 3\right\}\right] \tag{119c}$$

$$Q_\phi = -\frac{D}{L_o^3}\left[4 c_4 \frac{1}{x}\right] \tag{119d}$$

$$N_\phi = C_5 + C_6 \frac{1}{x^2} + C_7 (2 \log x + 1) \tag{119e}$$

$$N_\theta = C_5 - C_6 \frac{1}{x^2} + C_7 (2 \log x + 3) \tag{119f}$$

For circumferential strain-hardening over a narrow zone

$$w = C_1 + C_2 x^2 + C_3 x^{1+a^{1/2}} + C_4 x^{1-a^{1/2}} \tag{120a}$$

$$M_\phi = -\frac{D_1}{L_o^2}\left[2C_2(1+a\nu) + C_3(1+a^{1/2})(a^{1/2}+\nu a)x^{-1+a^{1/2}} + C_4(1-a^{1/2})(-a^{1/2}+\nu a)x^{-1-a^{1/2}}\right] + \nu M_c \tag{120b}$$

$$M_\theta = -\frac{aD_1}{L_o^2}\left[2C_2(1+\nu) + C_3(1+a^{1/2})(1+a^{1/2}\nu)x^{-1+a^{1/2}} + C_4(1-a^{1/2})(1-a^{1/2}\nu)x^{-1-a^{1/2}}\right] + M_c \tag{120c}$$

$$Q_\phi = -\frac{D_1}{L_o^3}\left[\frac{2C_2}{x}\left(1+\frac{1}{a}\right)\right] - \left(\frac{1-\nu}{x}\right)\frac{1}{L_o} M_c \tag{120d}$$

$$N_\phi = C_5 + C_6 (1+a^{1/2})\, x^{-1+a^{1/2}} + C_7 (1-a^{1/2})\, x^{-1-a^{1/2}} \tag{120e}$$

$$N_\theta = C_5 + C_6 (1+a^{1/2})\, a^{1/2}\, x^{-1+a^{1/2}} - C_7 (1-a^{1/2})\, a^{1/2}\, x^{-1-a^{1/2}} \tag{120f}$$

For meridional strain-hardening over a narrow zone

$$w = C_1 + C_2 x^2 + C_3 x^{1+a^{-1/2}} + C_4 x^{1-a^{-1/2}} \tag{121a}$$

$$M_\phi = -\frac{aD_1}{L_o^2}\left[2C_2(1+\nu) + C_3(1+a^{-1/2})(a^{-1/2}+\nu)\, x^{-1+a^{-1/2}} + C_4(1-a^{-1/2})(-a^{-1/2}+\nu)\, x^{-1-a^{-1/2}}\right] + M_c \tag{121b}$$

$$M_\theta = -\frac{D_1}{L_o^2}\left[2C_2\left(\frac{1}{a}+\nu\right) + C_3(1+a^{-1/2})\left(\frac{1}{a}+\nu a^{-1/2}\right) x^{-1+a^{-1/2}} + C_4(1-a^{-1/2})\left(\frac{1}{a}-\nu a^{-1/2}\right) x^{-1-a^{-1/2}}\right] + \nu M_c \tag{121c}$$

$$Q_\phi = -\frac{aD_1}{L_o^3}\left[2\frac{C_2}{x}\left(1-\frac{1}{a}\right)\right] + \frac{1-\nu}{L_o x} M_c \qquad (121d)$$

$$N_\phi = C_5 + C_6\,(1+a^{-1/2})\,x^{-1+a^{-1/2}} + C_7\,(1-a^{-1/2})\,x^{-1-a^{-1/2}} \qquad (121e)$$

$$N_\theta = C_5 + C_6(1+a^{-1/2})\,a^{-1/2}\,x^{-1+a^{-1/2}} - C_7(1-a^{-1/2})\,a^{-1/2}x^{-1-a^{-1/2}} \qquad (121f)$$

With strain-hardening in each direction, deflection w is given by Eqn. 119a

$$M_\phi = -\frac{aD}{L_o^2}\left[2(1+\nu)C_2 - (1-\nu)\,C_3\frac{1}{x^2} + C_4\left\{2\log x + 3 + \nu(2\log x + 1)\right\}\right] + M_c \qquad (122a)$$

$$M_\theta = -\frac{aD}{L_o^2}\left[2(1+\nu)C_2 + (1-\nu)C_3\frac{1}{x^2} + C_4\left\{2\log x + 1 + \nu(2\log x + 3)\right\}\right] + M_c \qquad (122b)$$

$$Q_\phi = -\frac{aD}{L_o^3}\left[4\,C_4\frac{1}{x}\right] \qquad (122c)$$

and N_ϕ and N_θ are given by Eqns. 119e and f.

Alternatively direct forces are given by Eqns. 72 in each zone, thus eliminating the constants C_5 through C_7.

For unsymmetrical bending in a narrow elastic zone, with q = 0, from Eqn. 117

$$\underset{(F)}{w} = C_1 + C_2x^2 + C_3\log x + C_4x^2\log x + \left[C_{14}'\,F(v)\,x\log x\right]^{*}\begin{Bmatrix}\cos\theta\\ \sin\theta\end{Bmatrix}$$

* This term is necessary for a fourth independent solution, for the case v = 1, when n = 1 and $\cos\phi_e = 1$, i.e., $\phi_e = 0$.

$$+ \left[C_{n1}x^{v} + C_{n2}\, x^{v+2} + C_{n3}x^{-v} + C_{n4}x^{-v+2} \right] \begin{cases} \cos n\theta \\ \sin n\theta \end{cases}$$

$$n = 1,\ 2,\ 3\ \ldots.\ \infty \qquad (123)$$

where $v = \dfrac{n}{\cos \phi_e}$, as before and $F(v) = \begin{cases} 1, v = 1 \\ 0, v \neq 1 \end{cases}$

and moments, shears, and direct forces are given by Eqns. 108 and 82. In a narrow inelastic zone with strain-hardening in each direction, w is again given by Eqn. 125 and direct forces by Eqns. 82 but moments and shears in this case are given by Eqn. 109.

As a further, or alternative, simplification where ϕ_e is small (for example $\phi_e < \frac{\pi}{6}$) it is convenient to introduce the approximation

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{x^2}\frac{\partial^2}{\partial \theta^2}$$

and hence $v \doteq n$, in the previous equations.

The variable x is still defined by Eqns. 26 or 27, and the coordinate system is thus equivalent in effect to a plane polar system with radial ordinates modified by the inclusion of the terms containing ϕ_e.

An advantage of the above solutions (Eqns. 119 through 122) is that they are in closed form, and are therefore readily evaluated at any point. Where the zone of yield is particularly narrow (as it may be when $a \to 0$) and the direction of yield is essentially meridional, the zone may be considered to contract in width to a line and may be treated as a boundary condition (see boundary conditions, also Reference 23), further simplifying the analysis. This is of special value when $a = 0$, and in the unsymmetrical case.

Where the zone considered is not as narrow as in the cases above, the equations previously presented (Eqns. 21 through 99) are required, and in the case of a spherical shell, their solutions are largely available in tabulated form, e.g. for axi-symmetrical bending in the elastic zone, the functions describing deflections, moments, shears, and direct forces are tabulated in Reference 1 (and References 19 through 22), and for the inelastic zone (Case II) equivalent tables are presented in Reference 2. For the cases of unsymmetrical bending considered, only the lower order Bessel functions in the solutions presented are directly tabulated (References 19 through 22) but recurrence formulae (Reference 22) allow the higher order functions, if required, to be expressed in terms of these. For design calculations the effect of these higher order functions may generally be neglected.

DISCUSSION

Thus equations are presented for the analysis of small displacements in thin rotational shells, under bending in the elastic and linear-strain-hardening ranges. Emphasis is given to the case of stress concentrations in the shell and to the solution, for the purpose of design, of stresses in the region of this concentration. (Outside this region, and for cases other than this, in which stress concentrations do not occur, the simpler membrane solutions are normally sufficient.)

In the special case of a spherical shell of uniform thickness, under axi-symmetrical bending, the differential equations which govern the behavior of the shell are presented at essentially three different orders of accuracy.

The equations are first presented in a form which may be considered exact within the limits of thin shell and small deflection theory. They are then reduced to a simpler form to facilitate solution by neglecting certain lower order terms. This restricts their application to a specific zone of the shell. When this zone is <u>very</u> narrow as it may well be for a zone of increased thickness surrounding an opening or fitting in the shell, the equations may be further reduced by neglecting more terms as described, resulting in closed form solutions in these zones. (An estimate of the order of accuracy of the approximate solutions may be obtained by substitution into the more accurate equations presented.) For the general case of a rotational shell and unsymmetrical bending, the two latter (approximate) forms of solution only are considered.

When the zone of yield is extremely narrow, as it may be for the case of yielding in the meridional or nearly meridional direction only, and particularly when the ratio "a" of Ep to E is small, a further simplification may be introduced by considering the zone of yield to be reduced in width to a line. It may then be considered merely as a boundary condition, as previously noted, which is of special advantage in the unsymmetrical case.

The solution of a typical design problem, using the equations presented, is outlined in Example 1. The use of the basic equations for the direct design (for t) of narrow zone of increased variable thickness (required to resist high stress concentrations at the edge of a fitting or opening) is outlined in Example 2.

Thus from the equations presented, a range of problems involving the bending of shells in the post-elastic range may be readily handled, allowing the design of thinner shells when load capacity is the criterion. The magnitude of the material savings thus involved is dependent on the distribution of loading, the proportions of the shell, and the stress-strain properties of the shell material, and can be quite considerable in the class of problem considered. Thus where weight is a significant factor in design, the utilization of the post-elastic range of the material is well justified, and in some cases, essential.

Examples:

1. Consider the axi-symmetrical problem of a thin spherical shell with a rigid cylindrical insert, axially loaded into the strain-hardening range of the shell. It is required to determine the thickness t of the shell necessary to just carry a load P_u at the center of the insert. The ordinate at the insert edge is ϕ_o and the material properties (of the shell) are denoted by a_1, E_1, and Ep_1.

The required thickness is necessarily determined by trial. Assuming that radial yielding only occurs prior to failure and that the width of the yielded zone is very narrow (in the case considered), the deflection within the yielded zone is given by Eqn. 121a and the corresponding moments and shears, by Eqns. 121b through 121d. In the surrounding, and not so narrow, elastic zone, deflections, moments, and shears are given by Eqns. 68. The integration constants contained in the above equations and the corresponding load P_1' are then determined from the boundary conditions,

e.g. at the insert edge ($\phi = \phi_o$),

$$w' = 0$$

$$Q_\phi = -\frac{P_1}{2\pi R \sin \phi_o}$$

at the edge of yield, which is assumed to be at $\phi = \phi_1$,

$$M_{\phi_e} = M_{\phi_p} = M_p$$

$$Q_{\phi_e} = Q_{\phi_p}$$

$$w_e = w_p$$

$$w_e' = w_p'$$

(in which the subscripts e and p denote the elastic and inelastic zones respectively) and as ϕ becomes large we have the further conditions

$$w \rightarrow 0$$

$$w' \rightarrow 0$$

Thus P_1 and $M_\phi(\phi = \phi_o)$ are determined and compared to P_u and M_u respectively. By successive approximation ϕ_1 and t_1 are adjusted to give the conditions $P_1 = P_u$ and $M_{\phi_o} = M_u$.

Where the thickness is given and the loading capacity is required, the procedure is slightly simpler, and only ϕ, must necessarily be determined by trial,---to satisfy the condition $M_{\phi_o} = M_u$ as before.

2. In the previous example, a shell of uniform thickness was considered, and the shell analysed, using the equations presented to determine the thickness, t, required. It may be of interest, however, to consider an alternative approach.

If the thickness t were allowed to vary with ϕ (i.e. $t = t(\phi)$), then t may be considered as the dependent variable in the problem, and may be solved for directly. This process may be termed as direct design (as distinct from analysis as previously considered), and is of particular advantage in designing to resist high stress concentrations around openings or fittings in the shell.

Thus if, in the example previously considered, a very narrow band immediately surrounding the insert is allowed to vary in thickness, the behavior of the shell and its ultimate loading capacity may be considerably improved. The required thickness is determined by combining with the equations of equilibrium a suitable behavior criterion for the shell.

(a) For example, if $a = \frac{Ep}{E} = 0$, and a condition of pure yield ($M_\phi = M_\theta = M_p(\phi)$) is allowed throughout the narrow zone considered, substituting M_p for M_ϕ and M_θ in Eqn. 1c gives:

$$Q_\phi = \frac{1}{R}\frac{d}{d\phi}(M_p)$$

and from the loading P_1 on the insert (neglecting inplane forces)

$$Q_\phi = -\frac{P_1}{2\pi R \sin\phi}$$

Equating these two expressions for Q_ϕ and integrating with respect to ϕ gives

$$M_p = -\frac{P_1}{2\pi}\log\phi + C_1$$

and hence $t = \left(\frac{4M_p}{\sigma_p}\right)^{1/2} = \left[\frac{4}{\sigma_p}\left(-\frac{P_1}{2\pi}\log\phi + C_1\right)\right]^{1/2}$

in which the constant C_1 may be evaluated to give optimum behavior or economy in the surrounding, uniform shell. Other behavior criteria may be similarly met giving different expression for t

(b) e.g. if the ratio a of E_p to E is finite, and the insert is elastic (e.g. a fitting in the shell) and of such proportions that $\sigma_\phi < \sigma_\theta$ over the narrow zone considered, the design criterion $\sigma_\theta = \sigma_u$ over this zone might be introduced. Thus neglecting the effect of Poisson's ratio for simplicity, circumferential moment may be expressed as

$$M_\theta = M_c - \frac{aD}{L^2}\frac{1}{x}w' = c\,t^2$$

in which

$$c = \frac{1}{6}\left(\sigma_u + \frac{1}{2}\sigma_c\right)$$

and operating with $\frac{d}{dx}(x.....)$ gives

$$M_c - \frac{a}{L^2} Dw'' = c \frac{d}{dx} (x\, t^2)$$

$$= M_\phi \text{ , for the case where } M_\phi \text{ ,}$$

also, is in the strain-hardening range. (This must later be ascertained by substitution.)

Substituting the above expressions for M_ϕ and M_θ , and the previous expression for Q_ϕ , into Eqn. 1c as before, results in

$$c \left[\frac{d}{dx} \left(x \frac{d}{dx} (x\, t^2) \right) - t^2 \right] = - \frac{P_1}{2\pi}$$

and solving (by integration) this differential equation for t gives

$$t = \left[\frac{-P_1}{4\pi c} \log x + C_1 x^{-2} + C_2 \right]^{1/2}$$

? $t = [C_1 \log r + C_2 r^{+} + C_3]^{1/2}$

in which C_1 and C_2 are arbitrary constants of the integration which may be determined (from the boundary conditions) to provide optimum behavior or economy in the remainder of the shell. Other applications of this procedure are considered in Reference 26.

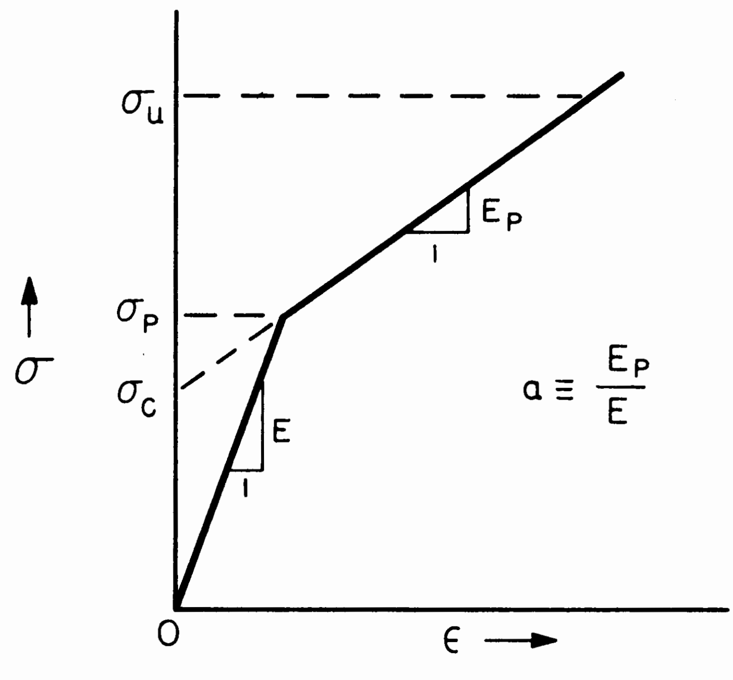


FIG. 1 STRESS-STRAIN RELATIONSHIP

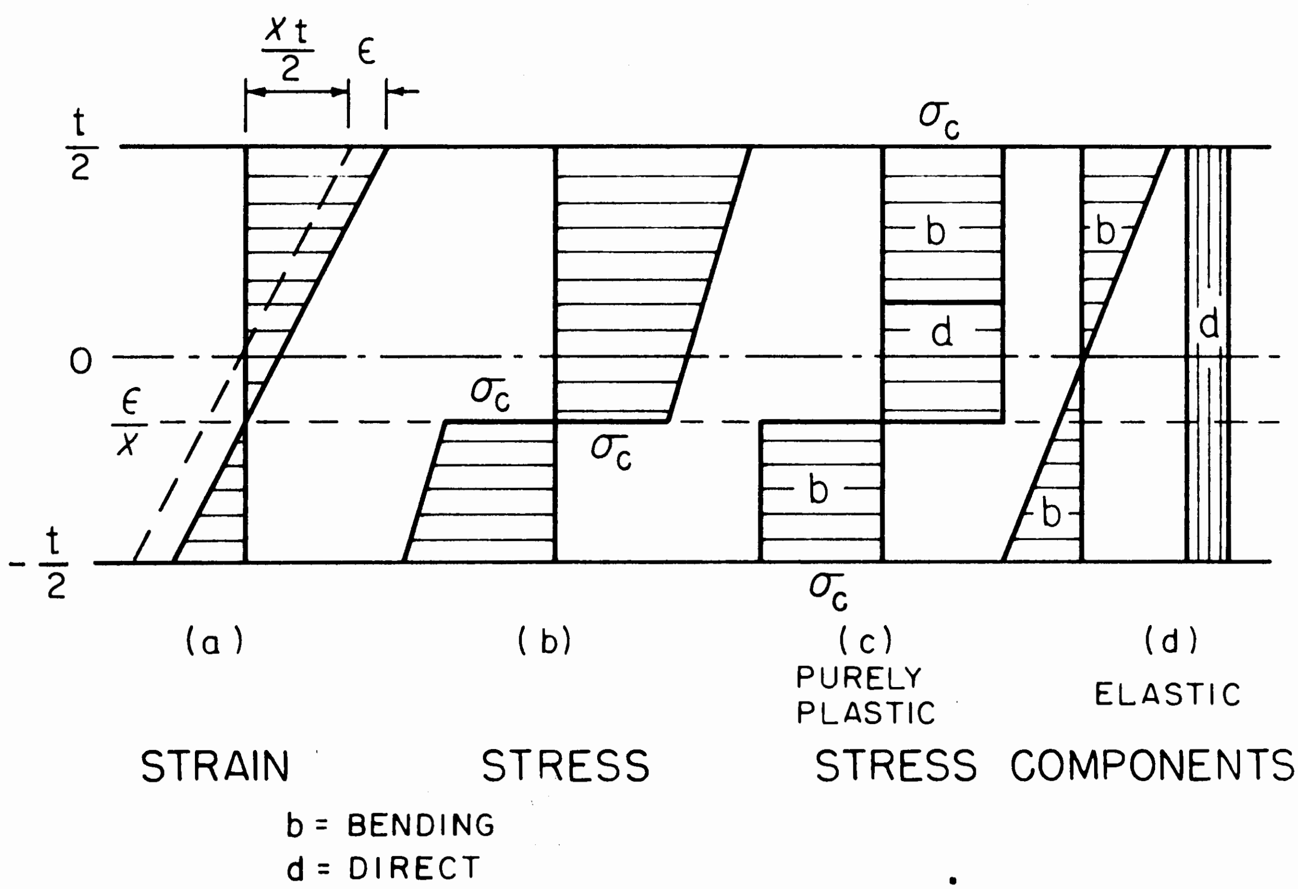


FIG. 2 ASSUMED DISTRIBUTIONS FOR STRESS AND STRAIN IN THE POST-ELASTIC RANGE

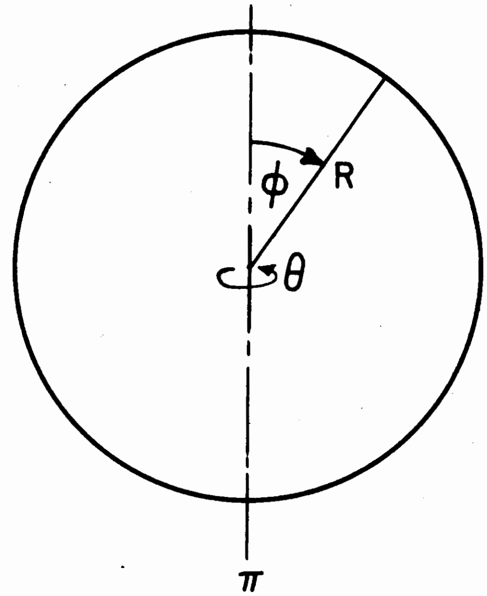
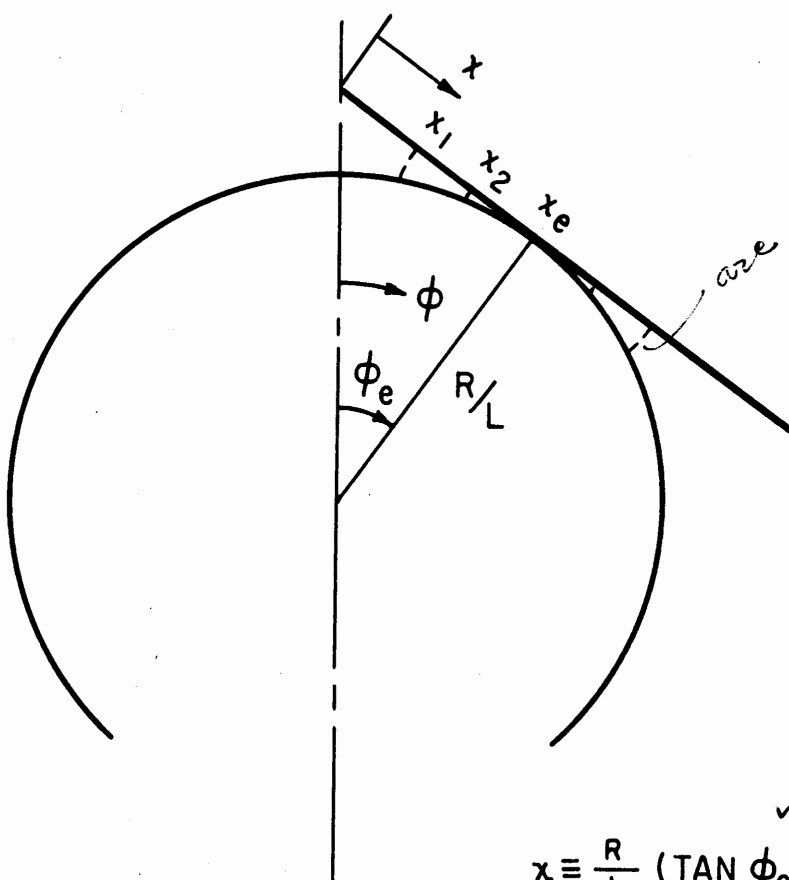


FIG. 3 SPHERICAL COORDINATES



$$x \equiv \frac{R}{L}\,(\mathrm{TAN}\,\phi_e + \phi - \phi_e)$$

FIG. 4 PROJECTION OF ZONE OF SHELL ONTO
CONICAL COORDINATE SURFACE -
LINEAR PROJECTION OF MERIDIONAL ORDINATES.

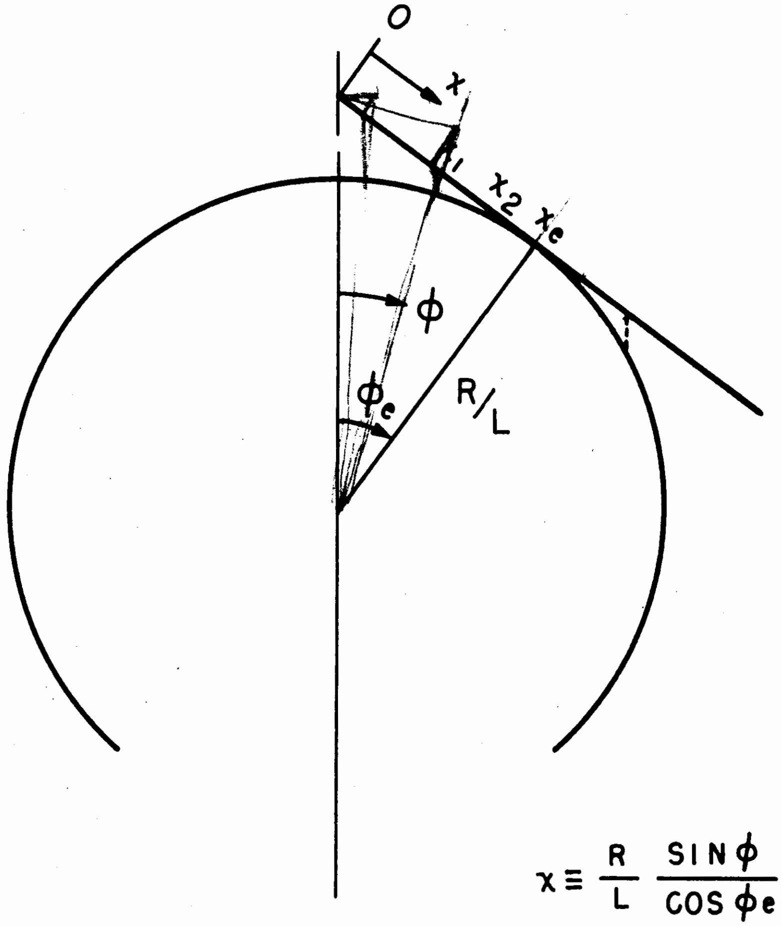


FIG. 5 VERTICAL PROJECTION ONTO CONICAL COORDINATE SURFACE

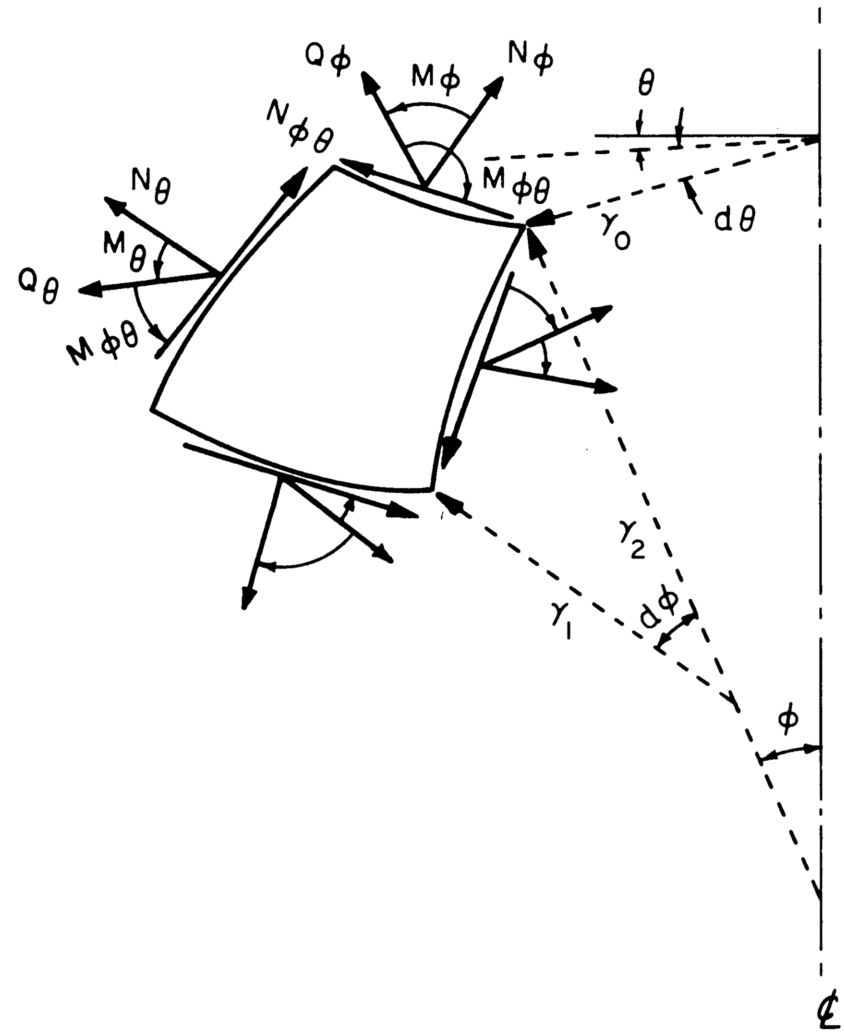


FIG. 6 SIGN CONVENTION AND NOTATION
ROTATIONAL SHELL

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APPENDIX I

With the assumptions stated for the stress-strain relationship in the inelastic range, the total strains in the principal directions can be expressed in terms of stresses as

$$\epsilon_i = \frac{\sigma_i}{E} - \frac{\nu\sigma_j}{E}$$

$$\epsilon_j = \frac{\sigma_j - \sigma_c}{aE} - \frac{\nu\sigma_i}{E}$$

which gives

$$\sigma_i = \frac{E}{1 - a\nu^2}\left[\epsilon_i + a\nu\epsilon_j\right] + \frac{\nu\sigma_c}{1 - a\nu^2} \qquad \text{(A)}$$

$$\sigma_j = \frac{aE}{1 - a\nu^2}\left[\epsilon_j + \nu\epsilon_i\right] + \frac{\sigma_c}{1 - a\nu^2} \qquad \text{(B)}$$

The requirements for the stress-strain relationship in the inelastic range are:

1) Since in the strain hardening range, the material is considered initially rigid (i.e., rigid strain hardening) then for the initial condition $\epsilon_i = 0 = \epsilon_j$, we require

$$\sigma_j = \sigma_c \quad \text{and} \quad \sigma_i = \nu\sigma_c \, .$$

Actually σ_j ∴ $\frac{\sigma_c}{1-a\nu^2}$ if ϵ_i & $\epsilon_j = 0$ in 1st relations. Hence eqs 5a & 5b are better.

2) For the special case $a = 0$, Tresca's yield criterion is to be satisfied and this requires

$$\sigma_j = \sigma_c = \sigma_p \quad \text{and} \quad \sigma_i = E\epsilon_i + \nu\sigma_p \, .$$

3) For the special case $a = 1$, and hence $\sigma_c = 0$, we require Hooke's law to be obeyed.

These requirements are explicitly met by Eqns. (A) and (B) if we modify the last terms in them by letting $(1 - a\nu^2) \simeq 1$, which is quite justifiable as always $a < 1$ and $\nu^2 << 1$. Hence we have

$$\sigma_i = \frac{E}{1 - a\nu^2}\left[\epsilon_i + a\nu\epsilon_j\right] + \nu\sigma_c$$

$$\sigma_j = \frac{aE}{1 - a\nu^2}\left[\epsilon_j + \nu\epsilon_i\right] + \sigma_c \quad .$$