Title
An Equivariant Main Conjecture in Iwasawa Theory and Applications

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Author
Gambheera, Rusiru

Publication Date
2023

UNIVERSITY OF CALIFORNIA SAN DIEGO

An Equivariant Main Conjecture in Iwasawa Theory and Applications

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Rusiru Gambheera

Committee in charge:

Professor Cristian Popescu, Chair
Professor Russell Impagliazzo
Professor Kiran Kedlaya
Professor Aaron Pollack
Professor Claus Sorenson

2023

The dissertation of Rusiru Gambheera is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

2023

DEDICATION

This is dedicated to my father, who taught me the baby steps of both life and Number Theory.

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ACKNOWLEDGEMENTS

First and foremost, I must express my deepest gratitude to my advisor Cristian Popescu for his unwavering support, invaluable insights, and endless guidance over the past four years. Without his advising, none of my accomplishments would have been possible.

Additionally, I am grateful to my friend Nandagopal Ramachandran for the helpful discussions we shared during our time working together.

I also extend my appreciation to Kiran Kedlaya, Claus Sorensen and Aaron Pollack for the knowledge they imparted during my time in graduate school and also for serving on my thesis committee. I also thank Russel Impagliazzo for serving on my thesis committee.

I am humbled by the hardworking citizens of Sri Lanka who selflessly financed my education, and for that, I am eternally grateful.

Furthermore, I am grateful to have made many wonderful friends during my stay in San Diego, who have made my graduate school experience a truly memorable one. While I cannot thank each of them individually, I express my sincere appreciation to all of them for their invaluable support and camaraderie.

However, I cannot overstate the profound impact that my parents and my brother have had on my life. Their unrelenting support and encouragement have been the bedrock of my success. I owe them a debt of gratitude that can never be repaid.

Lastly, but certainly not least, I thank my beloved wife, Thejani, for being my constant companion and for her love and support over the past five years, as well as

for the joy and happiness that is yet to come.

VITA

2018	B. Sc. in Mathematics , University of Colombo, Sri Lanka
2018-2023	Graduate Teaching Assistant, University of California San Diego
2023	Ph. D. in Mathematics, University of California San Diego

PUBLICATIONS

Rusiru Gambheera, Cristian Popescu, *An Equivariant Main Conjecture in Iwasawa Theory and Applications*, submitted

ABSTRACT OF THE DISSERTATION

An Equivariant Main Conjecture in Iwasawa Theory and Applications

by

Rusiru Gambheera
Doctor of Philosophy in Mathematics

University of California San Diego, 2023

Professor Cristian Popescu, Chair

In this dissertation we prove a new equivariant main conjecture in Iwasawa theory associated to the cyclotomic $\mathbb{Z}_p$–extension of a CM number field over a totally real number field. Our object of interest $\nabla_S^T(H_\infty)_p^-$ is the projective limit of certain p–adic Ritter-Weiss modules which is class field theoretically significant and has nice cohomological properties. Our main result is a number field analogue of the recent results of Bley and Popescu [14] on a certain Drinfeld modular Iwasawa tower of function fields. As an application, we compute the 0-th Fitting ideal of a naturally arising Iwasawa module over the relevant equivariant Iwasawa algebra.

Chapter 1

Introduction

1.1 Why Iwasawa theory ?

One of the main goals of number theory is the study of various invariants of number fields such as their ring of integers, group of units, class group etc. However, most of the time studying individual number fields is hard and not very insightful. So, in **Iwasawa theory**, which is my field of speciality in number theory, we study infinite towers of number fields as a whole. That sometimes gives us valuable information about each individual number field and the growth of its invariants when we go up in the tower.

For instance, in classical Iwasawa theory, for a given number field H, we consider the $\mathbb{Z}_p$–extensions for a fixed prime p. It is a certain class of series of Galois field extensions,

$$H \subseteq H_1 \subseteq H_2 \subseteq \ \dots \ H_\infty := \bigcup_{n=1}^{\infty} H_n$$

where for each n, we have $\Gamma_n := Gal(H_n/H) = \mathbb{Z}/p^n\mathbb{Z}$. Then, $\Gamma = Gal(H_\infty/H) \cong \mathbb{Z}_p$. Now, let $A_n := Cl(H_n) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ where $Cl(H_n)$ is the ideal class group of K_n. For each n, A_n is a $\mathbb{Z}_p[\Gamma_n]$–module. By taking projective limits with respect to norm maps and

Galois restriction maps respectively, we can view $X := \varprojlim_n A_n$ as a module over the classical profinite Iwasawa algebra, $\mathbb{Z}_p[[\Gamma]] := \varprojlim_n \mathbb{Z}_p[\Gamma_n] \cong \mathbb{Z}_p[[T]]$. Now, studying this module structure is interesting in its own right. However, once this is achieved, one can also use Iwasawa co-descent to obtain useful information about the finite layers of the Iwasawa tower.

As of today, Iwasawa theory has been extended to many other settings such as function fields, abelian varieties, modular forms, motives and so on. And also, interesting connections between relevant modules and related $p-$adic L-functions has been established.

In equivariant Iwasawa theory, rather than looking at a single base field H, we look at a Galois field extension H/F. For instance, if H_n is the nth layer of a fixed $\mathbb{Z}_p-$extension of H and if $G_n = Gal(H_n/F)$, then $\mathcal{G} := Gal(H_\infty/F) = \varprojlim_n G_n$. So, using the notaion above, X is a module over the Iwasawa algebra of $\mathcal{G}$, namely $\mathbb{Z}_p[[\mathcal{G}]] := \varprojlim_n \mathbb{Z}_p[G_n]$. Now, just like in the classical setting, we can study this rich and interesting $\mathcal{G}-$equivariant behaviour of X. By doing that, we can also obtain information about the G_n-equivariant behaviour of A_n, for all $n \gg 0$.

1.2 Main conjectures in Iwasawa theory

The classical Iwasawa main conjecture is a fundamental conjecture in algebraic number theory that relates the arithmetic of number fields to the behavior of their associated p-adic L-functions. It was first formulated by Kenkichi Iwasawa and Ralph Greenberg in the 1970s, and is one of the most important and influential conjectures in the field of algebraic number theory. This was proved by Mazur and Wiles [19] in 1984, if the base field is $\mathbb{Q}$ and by Wiles [20] in 1990 in full generality.

More precisely, the classical Iwasawa main conjecture in its simplest form, relates the algebraically defined module X, to an associated $p-$adic L-function. Equivariant main conjectures in Iwasawa theory relates the $\mathbb{Z}_p[[\mathcal{G}]]-$module X (or other arithmetically significant modules) to equivariant $p-$adic L-functions.

In this dissertation we prove a new equivariant main conjecture in the following setting.

Let H/F be a finite abelian CM extension of a totally real number field F. We let S, T be two nonempty disjoint sets of places in F, satisfying some mild conditions, which will be made precise in later chapters. When there is no risk of confusion, we denote the sets of places in H above places in S and T, also by S and T, respectively.

Now, let H_∞ be the cyclotomic $\mathbb{Z}_p-$extension of H (in the case $\zeta_p \in H$ this is obtained by adjoining ζ_{p^n} for each n, to H, where ζ_{p^n} is a primitive n-th root of unity) and $\mathcal{G} := Gal(H_\infty/F)$. For this data, we define a $p-$adic Iwasawa Ritter-Weiss module, $\nabla_S^T(H_\infty)_p$ at the top of the cyclotomic tower. This $\mathbb{Z}_p[[\mathcal{G}]]-$ module is obtained by taking projective limit under the "norm maps" of $p-$adic Ritter-Weiss modules, $\nabla_S^T(H_n) \otimes \mathbb{Z}_p$ as considered in [3], at each finite layer H_n. These modules at finite levels fit into the following short exact sequence.

$$0 \longrightarrow Cl_S^T(H_n) \longrightarrow \nabla_S^T(H_n) \longrightarrow X_{S,H_n} \longrightarrow 0$$

Here $Cl_S^T(H_n)$ is a generalized class group associated to the pair (S,T) and X_{S,H_n} is the submodule of the $\mathbb{Z}-$module $\bigoplus_{v \in S} \mathbb{Z} \cdot v$ whose formal degree is zero. The extension class is coming from class field theory as discussed in Chapter 3. From the above short exact sequence, it is clear that the Ritter-Weiss module contains interesting arithmetic data.

Our equivariant main conjecture, the main result of this dissertation, relates $\nabla_S^T(H_\infty)_p^-$, the (-1)–eigenspace of $\nabla_S^T(H_\infty)_p \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$ under the action of unique complex conjugation automorphism in $\mathcal{G}$, to an equivariant p–adic L-function and shows that the module has desirable homological algebraic properties. The main theorem proved in this dissertation is the following.

Theorem 1.2.1. *Let p be an odd prime, S_∞ be the set of infinite places in F and S_{ram} be the set of primes in F that ramify in the extension H_∞/F. Suppose that S and T satisfy the following properties.*

$$S_\infty \subseteq S, \quad S_{ram} \subseteq S \cup T, \qquad T \nsubseteq S_{ram}$$

Then, we have,

$$Fitt_{\mathbb{Z}_p[[\mathcal{G}]]^-}(\nabla_S^T(H_\infty)_p^-) = (\Theta_S^T(H_\infty/F))$$

Moreover, we have a short exact sequence,

$$0 \to (\mathbb{Z}_p[[\mathcal{G}]]^-)^k \to (\mathbb{Z}_p[[\mathcal{G}]]^-)^k \to \nabla_S^T(H_\infty)_p^- \to 0$$

and $pd_{\mathbb{Z}_p[[\mathcal{G}]]^-}(\nabla_S^T(H_\infty)_p^-) = 1$

Here $Fitt$ is the 0-th Fitting ideal, which is a algebraic object which defined and discussed extensively in Chapter 2. We denote by pd, the projective dimension and $\Theta_S^T(H_\infty/F)$ is the associated equivariant p–adic L-function, which is an element of $\mathbb{Z}_p[[\mathcal{G}]]^-$. All of these objects are defined and discussed in Chapters 3 and 5.

1.3 The structure of this dissertation

In Chapter 2 we define and discuss the properties of *Fitting ideals*, an important algebraic gadget that serves as a bridge, relating the algebraic side (Ritter- Weiss module), to the analytic side (equivariant p–adic L-function). Then, we also discuss, the relatively recent notion of *shifted Fitting ideals*, which is a tool we use for the main application of our main result. The related shifted Fitting ideal computations are done in the Appendix.

In Capter 3, we introduce in detail the Ritter-Weiss module and discuss its arithmetic significance and how it is related to other well known objects. We also discuss how these modules relate to each other in field extensions.

In Chapter 4, we give some motivation coming from function fields. More precisely, we discuss the equivariant main conjecture proved recently by Bley and Popescu [14]. Our result is a direct number field analogue of the result of Bley and Popescu.

In Chaper 5, we will define our set up and prove the main theorem, using the material developed in Chapter 3. Then, we use it to give our main application with the help of computations done in the Appendix. At the end of Chapter 5, we also discuss some possible future directions of our results. Some of them address answering direct follow up questions and others address some vaguely formulated analogies.

Chapter 2

Fitting Ideals

2.1 Basic Properties

Definition 2.1.1. Let R be a commutative ring and M be a finitely generated R–module. Consider the following presentation of M.

$$\bigoplus_{j \in J} R \xrightarrow{\phi} R^n \to M \to 0$$

The i–th Fitting ideal of M over R, denoted by $Fitt_R^i(M)$, is defined as the ideal in R generated by the determinants of all the $(n-i) \times (n-i)$ minors of the (possibly infinite) matrix associated to the map ϕ.

It is implicit in the above definition that the $Fitt_R^i(M)$ does not depend on the choice of presentation.

Definition 2.1.2. In the above definition, if J is finite, M is said to be finitely presented. If J has n elements, M is said to be quadratically presented.

In this document, we are only concerned about the 0–th Fitting ideal, which is also known as the principal Fitting ideal (or just "the Fitting ideal"). We denote this by $Fitt_R(M)$.

It is helpful to think of $Fitt_R(M)$ as the "R–size" of the module M. For instance, if M is a finite abelian group (viewed as a $\mathbb{Z}$–module), then it is easy to see that $Fitt_{\mathbb{Z}}(M) = |M| \cdot \mathbb{Z}$ where $|M|$ is the cardinality of M.

Now, we summarize without proofs, some of the important properties of Fitting ideals. For more details on general properties of Fitting ideals, the reader can consult the Appendix of [19].

Proposition 2.1.3. *Let M, M', M'' finitely generated modules over the commutative ring R.*

- *If I is an ideal of R, we have $Fitt_R(R/I) = I$*
- $Ann_R(M) \subseteq Fitt_R(M)$
- *If we have a surjective R–module morphism $M \to M'$, we have*

  $$Fitt_R(M) \subseteq Fitt_R(M')$$

- *If we have a short exact sequence,*

  $$0 \to M \to M' \to M'' \to 0$$

  of R–modules, then we have

  $$Fitt_R(M) \cdot Fitt_R(M'') \subseteq Fitt_R(M')$$

  We have the equality if the above short exact sequence splits or if M'' is quadratically presented.

- *If M is torsion with $\mathrm{pd}_R(M) \leq 1$ then $Fitt_R(M)$ is an invertible ideal.*

The following theorem shows that Fitting ideals behave well with respect to base change.

Theorem 2.1.4. *Let $\phi : R \to R'$ is a ring morphism and M is an $R-$module, then we have the following.*

$$Fitt_{R'}(R' \otimes_R M) = (\phi(Fitt_R(M)))$$

2.2 Fitting ideals and projective limits

In this section, we prove a slight generalization of Theorem 2.1 in [4] due to by Greither and Kurihara.

Let H/F be an extension of number fields and let H_∞/H be a $\mathbb{Z}_p$-extension for some prime p. Suppose that $G_n := Gal(H_n/F)$ are abelian for all n, where H_n is the nth layer of the $\mathbb{Z}_p$-tower. Therefore, for each n, $G_n \cong G' \times G_{p,n}$ where $G_{p,n}$ is the Sylow p subgroup of G_n and G' is the non-p part. Note that G' does not depend on n and it is isomorphic to the non-p part of $Gal(H/F)$.

Let $\mathcal{G} := Gal(H_\infty/F) = \varprojlim_n G_n$. Then, $\mathcal{G} \cong G' \times G_{p,\infty}$ where $G_{p,\infty} = \varprojlim_n G_{p,n}$. Let G_p be the torsion part of $G_{p,\infty}$. Then, $\mathcal{G} \cong G' \times G_p \times \Gamma$, where $\Gamma \cong \mathbb{Z}_p$. Hence, there is also a subfield H' of H_∞ such that $Gal(H_\infty/H') \cong \Gamma$. Check [10] for details.

For any character $\chi : G' \to \overline{\mathbb{Q}_p}^\times$ of G', define $R_n^\chi := \mathbb{Z}_p[\chi][G_{p,n}]$ and

$$R_\infty^\chi := \mathbb{Z}_p[\chi][[G_{p,\infty}]] \cong \mathbb{Z}_p[\chi][G_p][[\Gamma]]$$

We define equivalence classes of characters by the identification $\chi \sim \sigma \circ \chi$, for all χ and all $\sigma \in G_{\mathbb{Q}_p}$, where $G_{\mathbb{Q}_p}$ is the absolute Galois group of $\mathbb{Q}_p$.

Let $[\widehat{G'}]$ be the set of equivalence classes and $[\chi]$ be the class of the character χ. Then, we know that $\mathbb{Z}_p[G_n] \cong \bigoplus_{[\chi]\in[\hat{G}']} R_n^\chi$ and $\mathbb{Z}_p[[\mathcal{G}]] \cong \bigoplus_{[\chi]\in[\hat{G}']} R_\infty^\chi$. If M is a $\mathbb{Z}_p[[\mathcal{G}]]$–module then we let $M^\chi := M \otimes_{\mathbb{Z}_p[[\mathcal{G}]]} R_\infty^\chi$. It is easy to see that we have the following $\mathbb{Z}_p[[\mathcal{G}]]$–module isomorphism where G' acts on M^χ component via χ.

$$M \cong \bigoplus_{[\chi]\in[\hat{G}']} M^\chi$$

The following theorem by Greither and Kurihara asserts a compatibility between projective limits and Fitting ideals.

Theorem 2.2.1. ***(Greither-Kurihara)*** *Let $\Lambda := \mathcal{O}[[T]]$ where $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$. Let $R := \Lambda[G]$ where G is a finite p–group and let $R_n := R/((1+T)^{p^n} - 1)R$. Assume that $(A_n)_n$ is a projective system of modules over the projective system of rings $(R_n)_n$ in the obvious sense, satisfying the following two properties:*

(2) The transition maps of $(A_n)_n$ are surjective from some $n_0 \in \mathbb{N}$ onwards.

(2) $A := \varprojlim_n A_n$ is a finitely generated torsion module over Λ.

Then, we have $Fitt_R(A) = \varprojlim_n Fitt_{R_n} A_n$

Now we state and prove a generalization of the above theorem and then a corollary which will be used in Chapter 5.

Theorem 2.2.2. *Let $(A_n)_n$ be a projective system of modules over the projective system of compact local rings $(R_n)_n$ in the obvious sense, such that*

(1) The transition maps $\pi_n : A_{n+1} \to A_n$ and $\pi_n^* : R_{n+1} \to R_n$ are surjective for all $n \gg 0$.

(2) $A := \varprojlim_n A_n$ *is finitely generated over* $R := \varprojlim_n R_n$.

(3) There exists $f \in R$ *such that* $f \cdot A = 0$ *and* R/fR *is local Noetherian, of Krull dimension at most 1.*

Then, we have $Fitt_R(A) = \varprojlim_n Fitt_{R_n} A_n$

Proof. (Sketch) Greither and Kurihara proved their theorem in 8 steps. Except for step (5), all the other steps works identically for the proof of our theorem. A slight modification is needed for step (5).

Step (5): Using the notaion in [4], Theorem 2.1, we need to prove that there exists $r \in \mathbb{N}$ such that B_n is generated by r elements over R_n, for all n.

Since $f \cdot A = 0$, we have $f \cdot R_n^m \subset B_n \subset R_n^m$. So, if we find some r_0 such that all $B_n' := B_n/(f \cdot R_n^m)$ are r_0-generated we will be done, with $r := r_0 + m$. Now, for $n >> 0$, B_n' is a module over R/fR and a submodule of $(R_n/fR_n)^m$. Let B_n'' be the preimage of B_n' in $(R/fR)^m$. It suffices to show that B_n'' is r_0- generated over R/fR for some r_0. But, by (2), R/fR is local noetherian of Krull dimension at most 1. This implies (see the introduction of [5]) the existence of a constant d such that all ideals of R/fR can be generated by d elements. By an easy argument, all submodules of $(R/fR)^m$ can be generated by $r_0 := md$ elements. This completes the proof of step (5) and hence, we have the desired result. □

Observe that when G is a p–group, $\Lambda[G]$ is a local ring with the unique maximal ideal $(\pi, T, g-1; g \in G)$ where π is a uniformizer of $\mathcal{O}$. Since Λ has Krull dimension 2, if $f \in \Lambda$ is nonzero, the Krull dimension of (Λ/f) is 1, hence so is that of $(\Lambda/f)[G] \cong \Lambda[G]/(f)$. Therefore, Theorem 2.2.1 follows from the above theorem. Moreover, the following corollary does not follow from Theorem 2.2.1. For that reason, we need the above nontrivial generalization.

Corollary 2.2.3. *Let $(A_n)_n$ be a projective system of modules over the projective system $(\mathbb{Z}_p[G_n])_n$ in the obvious sense such that*

(1) The transition maps $\pi_n : A_{n+1} \to A_n$ are surjective for $n \gg 0$.

(2) $A := \varprojlim_n A_n$ is finitely generated and torsion over $\Lambda := \mathbb{Z}_p[[\Gamma]]$.

Then, we have $Fitt_{\mathbb{Z}_p[[\mathcal{G}]]}(A) = \varprojlim_n Fitt_{\mathbb{Z}_p[[G_n]]}A_n$

Proof. We are using Theorem 2.2.2. By the description before Theorem 2.2.1, we can split the problem into character components. We fix $\chi \in \hat{G}'$. Suppose $\mathcal{O} = \mathbb{Z}_p[\chi]$.

The group rings $\mathcal{O}[G_{p,n}]$ are compact local. Since their transition maps are induced by Galois restriction, they are surjective. Since A is finitely generated over Λ, clearly it is so over $\Lambda[\chi][G_p] \cong \varprojlim_n \mathcal{O}[G_{p,n}]$. Since A is Λ-torsion, pick $f \in \Lambda \setminus \{0\}$ such that $f \cdot A = 0$. Now, $\Lambda[\chi][G_p]/(f) \cong (\Lambda[\chi]/(f))[G_p]$. We know that $(\Lambda[\chi]/(f))[G_p]$ has Krull dimension 1. And also, clearly, $(\Lambda[\chi]/(f))[G_p]$ is local and Noetherian. Then, by Theorem 2.2.2 we have the result. □

2.3 Shifted Fitting ideals

In this section we introduce the concept of shifted Fitting ideals introduced recently by Kataoka [15] and studied further by Greither, Kataoka and Kurihara [7]. The relevant computations we need for the main application are done in the Appendix.

The following is Theorem 1.7 in [7], defines the $i-$th shifted Fitting ideal and shows that it is well defined.

Theorem 2.3.1. ***(Kataoka)*** *Let M be a finitely generated torsion module over the* *commutative ring R. Take a resolution,*

$$0 \to N \to P_1 \to \ldots \to P_n \to M \to 0$$

where all the modules are finitely generated torsion over R and $pd_R(P_i) \leq 1$, *for all* i. *Define the fractional ideal*

$$Fitt_R^{[n]}(M) := (\prod_{i=1}^{n} Fitt_R(P_i)^{(-1)^i})Fitt_R(N)$$

Then, the above definition is independent from the choice of the resolution. So, $Fitt_R^{[n]}(M)$ *is well defined.*

In Theorem 2.1.3, we saw that Fitting ideals behave nicely with respect to short exact sequences when these are split or the third module is quadratically presented. Now, the following corollary gives a relationship between (shifted) Fitting ideals of the modules involved in short exact sequences. This is a direct consequence of the above theorem.

Corollary 2.3.2. *Consider the following short exact sequence of finitely generated torsion moduels over the commutative ring R.*

$$0 \to M \to M' \to M'' \to 0$$

If $pd_R(M') \leq 1$, *then we have,*

$$Fitt_R(M) = Fitt_R(M') \cdot Fitt_R^{[1]}(M'')$$

Chapter 3

The Ritter-Weiss module

3.1 Definitions and Main Properties

In this section we define the Ritter-Weiss module via its (local) quadratic presentation.

First we recall some notations and results from [3]. Let H/F is an abelian extension of number fields with Galois group G. Let S_∞ be the set of infinite places in F and S_p be the set above primes in F above p. We denote the set of all ramified places in H/F by $S_{ram}(H/F)$ (or simply by S_{ram}). For two disjoint sets of places S, T of F such that $S_\infty \subseteq S$, define,

$$\mathcal{O}^\times_{H,S,T} := \{x \in H^\times; \mathrm{ord}_w(x) = 0, \text{ for all } w \notin S_H, \quad \mathrm{ord}_w(x-1) > 0, \text{ for all } w \in T_H\}.$$

where for any $x \in H^\times$ and a prime ideal w, $ord_w(x)$ is the largest integer k such that $x \in w^k$. Here S_H and T_H be the sets of places in H sitting above places in S and T, respectively. When there is no risk of confusion, we also denote them by S and T. We also define,

$$H_T^\times := \{x \in H^\times ; \mathrm{ord}_w(x-1) > 0, \text{ for all } w \in T_H\}.$$

Now, we define the (S,T)–ray class group by the following exact sequence.

$$0 \longrightarrow \mathcal{O}_{H,S,T}^\times \longrightarrow H_T^\times \xrightarrow{\mathrm{d}iv_{\overline{S\cup T}}} Y_{\overline{S\cup T}}(H) \longrightarrow Cl_S^T(H) \longrightarrow 0.$$

Here $Y_{\overline{S\cup T}}(H) := \bigoplus_{w\notin S\cup T} \mathbb{Z}\cdot w$ is the free $\mathbb{Z}$–module of divisors supported at the places of H outside $S\cup T$ and the map $\mathrm{d}iv_{\overline{S\cup T}}(*) := \sum_{w\notin S_H\cup T_H} \mathrm{ord}_w(*)\cdot w$ is the usual ($S\cup T$–depleted) divisor map and the right–most non–zero map is the divisor–class map. Observe that all the modules in the above sequence hava a natural G–action, which makes the above sequence exact as $\mathbb{Z}[G]$–modules.

Let S, S' and T be a set of places in F satisfying the following properties.

- $S_\infty \subseteq S$. Here S_∞ is the set of all infinite places in F.
- $S \subset S'$ and $S' \cap T = \phi$
- $S_{ram}(H/F) \subseteq S' \cup T$. .
- $Cl_{S'}^T(H) = 1$
- $\bigcup_{w\in S'_H} G_w = G$. Here S'_H is the set of primes in H above the S primes in F and G_w is the decomposition group of the place w.

We fix a place v of F and a place w of H which is above v. Following Ritter and Weiss [6], we define a G_w- module V_w by giving its extension class in the following short exact sequence of G_w- modules. Here, ΔG_w is the augmentation ideal of $\mathbb{Z}[G_w]$.

$$0 \to H_w^\times \to V_w \to \Delta G_w \to 0 \tag{1}$$

Recall that the above sequence is obtained by applying the functor t in [6], Proposition 1 to the following sequence.

$$0 \to W(H_w^{ab}/H_w) \cong H_w^\times \to W(H_w^{ab}/F_v) \to G_w \to 0 \tag{2}$$

Here, W denotes the Weil group.

Now let O_w be the ring of integers in H_w. Define the G_w-module W_w via the following diagram whose rows are short exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & O_w^\times & \longrightarrow & V_w & \longrightarrow & W_w & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow j & & \\ 0 & \longrightarrow & H_w^\times & \longrightarrow & V_w & \longrightarrow & \Delta G_w & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & \mathbb{Z} & & & & & & \end{array}$$

The left vertical map is induced by the inclusion $O_w^\times \subseteq H_w^\times$. Moreover, by the snake lemma, we have the following.

$$0 \to \mathbb{Z} \xrightarrow{i} W_w \xrightarrow{j} \Delta G_w \to 0 \tag{3}$$

Now, we also have the following global short exact sequence,

$$0 \to C_H \to D \to \Delta G \to 0$$

where C_H is the idele class group of H and the extension class

$$\alpha \in Ext_G^1(\Delta G, C_H) = H^1(G, Hom(\Delta G, C_H))$$

such that $\delta'(\alpha) = u_{H/F}$ where $u_{H/F}$ is the global fundamental class and δ' is the connecting homomorphism (isomorphism)

$$\delta' : H^1(G, Hom(\Delta G, C_H)) \to H^2(G, C_H)$$

If M_w is a G_w- module, define,

$$\tilde{\prod_v} M_w := \prod_v Ind_{G_w}^G M_w$$

We define the following modules. (Here U_w are the 1-units of $O_w^\times$)

$$J := \tilde{\prod_{v\notin S\cup T}} O_w^\times \tilde{\prod_{v\in S}} H_w^\times \tilde{\prod_{v\in T}} U_w$$

$$J' := \tilde{\prod_{v\notin S'\cup T}} O_w^\times \tilde{\prod_{v\in S'}} H_w^\times \tilde{\prod_{v\in T}} U_w$$

$$V := \tilde{\prod_{v\notin S'\cup T}} O_w^\times \tilde{\prod_{v\in S'}} V_w \tilde{\prod_{v\in T}} U_w$$

$$W := \tilde{\prod_{v\in S'\setminus S}} W_w \tilde{\prod_{v\in S}} \Delta G_w$$

$$W' := \tilde{\prod_{v\in S'}} \Delta G_w$$

So we have the short exact sequences,

$$0 \to J \to V \to W \to 0$$

$$0 \to J' \to V \to W' \to 0 \tag{4}$$

and by Theorem 1 in [6] we have the following commutative diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & O^\times_{H,S,T} & & V^\theta & & W^\theta & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta_J} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta_w} & & \\
0 & \longrightarrow & C_H & \longrightarrow & D & \longrightarrow & \Delta G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & Cl_S^T(H) & & 0 & & 0 & &
\end{array} \tag{5}$$

By snake the lemma, we have a short exact sequence of $\mathbb{Z}[G]$–modules.

$$0 \to O^{\times}_{H,S,T} \to V^{\theta} \to W^{\theta} \to Cl^{T}_{S}(H) \to 0 \tag{6}$$

Similarly, using J' and W' instead of J and W, we have the following short exact sequence of $\mathbb{Z}[G]$–modules.

$$0 \to O^{\times}_{H,S',T} \to V^{\theta} \to W'^{\theta} \to 0 \tag{7}$$

We recall some more definitions.

$$S'_{ram} := S_{ram} \smallsetminus (S \cup T)$$

$$B := \prod_{v \in S' - S'_{ram}} \mathbb{Z}[G] \prod_{v \in S'_{ram}} \mathbb{Z}[G]^2$$

$$Z := \tilde{\prod_{v \in S}} \mathbb{Z} \tilde{\prod_{v \in S'_{ram}}} Hom(W_w, \mathbb{Z})$$

These modules fits into the following commutative diagram of G- modules.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & W^{\theta} & & B^{\theta} & & Z^{\theta} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W & \xrightarrow{\gamma} & B & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta_W} & & \downarrow{\scriptstyle \theta_B} & & \downarrow{\scriptstyle \theta_Z} & & \\
0 & \longrightarrow & \Delta G & \longrightarrow & \mathbb{Z}[G] & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{8}$$

So, the snake lemma gives the following short exact sequence.

$$0 \to W^{\theta} \to B^{\theta} \to Z^{\theta} \to 0 \tag{9}$$

Define the Ritter-Weiss module,

$$\nabla^{T}_{S}(H) := coker(V^{\theta} \to W^{\theta} \to B^{\theta}) \tag{10}$$

It turns out that the above definition is independant of the choice of S'. We also get the following short exact sequence.

$$0 \to O^{\times}_{H,S,T} \to V^{\theta} \to B^{\theta} \to \nabla^T_S(H) \to 0 \tag{11}$$

The following theorem from [3] shows that, under certain conditions, the above definition can be viewed as a (local) quadratic presentation of the Ritter-Weiss module.

Proposition 3.1.1. *Let R be a $\mathbb{Z}[G]-$algebra. Define $V^{\theta}_R := V^{\theta} \otimes_{\mathbb{Z}[G]} R$ and $B^{\theta}_R := B^{\theta} \otimes_{\mathbb{Z}[G]} R$. Suppose $S_{ram}(H/F) \subseteq S \cup T$ and for all primes $v \in T \cap S_{ram}(H/F)$, the rational prime l below v is invertible in R, then V^{θ}_R is projective $R-$module of constant local rank $|S'| - 1$. Consequently, we have the following local quadratic presentation of $\nabla^T_S(H)_R := \nabla^T_S(H) \otimes_{\mathbb{Z}[G]} R$.*

$$V^{\theta}_R \to B^{\theta}_R \to \nabla^T_S(H)_R \to 0$$

Now, combining the sequences (6) and (9), we get the following sequence.

$$0 \to Cl^T_S(H) \to \nabla^T_S(H) \to Z^{\theta} \to 0 \tag{12}$$

3.2 Tate sequences

Let H/F be a Galois extension of number fields with Galois group G. Let S be a set of places of F such that $S_{\infty} \cup S_{ram}(H/F) \subseteq S$. We also assume that S is large enough so that $Cl^{\varnothing}_S = 1$. For this data, in [17], Tate constructed an exact sequence of $\mathbb{Z}[G]-$modules,

$$0 \to \mathcal{O}^{\times}_{H,S} \to A \to B \to \nabla \to 0$$

where $\mathcal{O}^{\times}_{H,S} = \{x \in H^{\times}; \mathrm{ord}_w(x) = 0, \text{ for all } w \notin S_H\}$. Here, the modules A and B are $G-$cohomologically trivial. In this situation, it turns out we can take $\nabla = X_S$, the G

module of divisors of degree 0, supported at $S-$places.

Having a sequence of this type is useful to understand the $G-$module structure of $\mathcal{O}_{H,S}^{\times}$ in terms of cohomology. More precisely, it is easy to see that, for any $r \in \mathbb{Z}$,

$$H^r(G, \mathcal{O}_{H,S}^{\times}) = H^{r+2}(G, \nabla)$$

Threfore, in the above setting considered by Tate in [17], understanding $H^r(G, \mathcal{O}_{H,S}^{\times})$ is reduced to understanding $H^{r+2}(G, X_S)$, which is much easier to tackle with.

Observe that in the previous section, we are looking at a more general situation. After tensoring the short exact sequence (11) by $\mathbb{Z}_p$, we get the following short exact sequence of $\mathbb{Z}_p[G]-$modules,

$$0 \to O_{H,S,T}^{\times} \otimes \mathbb{Z}_p \to V_p^{\theta} \to B_p^{\theta} \to \nabla_S^T(H)_p \to 0 \tag{13}$$

where all the modules have the obvious meaning. Here (S,T) are as in the previous section. So, no largeness condition is imposed on S.

Now, if (S,T) satisfies the conditions of Proposition 3.1.1 for $R = \mathbb{Z}_p[G]$, we have that V_p^{θ} and B_p^{θ} have trivial $G-$cohomology. Therefore, this makes the sequence (13), a Tate type sequence. Therefore, we have for each $r \in \mathbb{Z}$,

$$H^r(G, O_{H,S,T}^{\times} \otimes \mathbb{Z}_p) = H^{r+2}(G, \nabla_S^T(H)_p)$$

And also, observe that if S were large enough such that $Cl_S^T(H) = 1$, then we have $\nabla_S^T(H) \cong X_S$ by the sequence (12), just like in Tate's original sequence.

3.3 Link with the Selmer module

In this section we discuss a link between the Ritter-Weiss module and the Selmer module defined by Burns-Kurihara-Sano [1]

Let H/F be a finite abelian extension of number fields of Galois group G. Let S and T be finite, disjoint sets of places of F, such that $S_\infty \subseteq S$. By taking $\mathbb{Z}$–dual of the map $H_T^\times \xrightarrow{div_{\overline{S\cup T}}} Y_{\overline{S\cup T}}(H)$ we obtain a canonical, injective morphism of $\mathbb{Z}[G]$–modules

$$Y_{\overline{S\cup T}}(H)^* \hookrightarrow (H_T^\times)^*.$$

If one identifies $\prod_{w\notin S_H\cup T_H}\mathbb{Z} \simeq Y_{\overline{S\cup T}}(H)^*$ in the obvious manner, the above injection sends the tuple $(x_w)_w \in \prod_w \mathbb{Z}$ to the homomorphism $* \to \sum_w x_w \mathrm{ord}_w(*)$.

Definition 3.3.1. The Selmer $\mathbb{Z}[G]$–module for the data $(H/F, S, T)$ is given by

$$\mathrm{Sel}_S^T(H) := (H_T^\times)^*/Y_{\overline{S\cup T}}(H)^* \simeq (H_T^\times)^*/\prod_{w\notin S_H\cup T_H}\mathbb{Z},$$

where the isomorphism is given by the identification described above.

Now, we recall the notion of transpose due to Jannsen [16].

Definition 3.3.2. Let M be a module over the commutative ring R with the following presentation by projective modules.

$$P_1 \xrightarrow{\phi} P_2 \to M \to 0$$

For all R–moduels N, let $N^* = Hom_R(N,R)$ under the covariant R–action. Define a transpose of M,

$$M^{tr} = coker(P_2^* \xrightarrow{\phi^*} P_1^*)$$

Observe that in the above definition, M^{tr} depends on the choice of projective presentation. However, under sufficient conditions, we can guarantee that M and M^{tr} have he same Fitting ideal for any choice of the presentation.

Proposition 3.3.3. *Let M be a quadratically presented $R-$module. Then, for any transpose, M^{tr} of M that corresponds to that presentation, we have*

$$Fitt_R(M^{tr}) = Fitt_R(M)$$

Proof. Observe that $Fitt_R(M) = (det(A))$ where A is the square matrix attached to the quadratic presentation of M. But, it is easy to see that the matrix attached to the presentation of M^{tr} is A^T. But, $det(A) = det(A^T)$. This completes the proof. □

It turns out that, with respect to the quadratic presentation in Proposition 3.1.1, Selmer moduel is a transpose of the Ritter-Weiss moduele.

Theorem 3.3.4. *Let $(H/F, G, S, S', T, R)$ be as in Proposition 3.1.1 and $Sel_S^T(H)_R :=$ $Sel_S^T(H) \otimes_{\mathbb{Z}[G]} R$. Then, we have the following presentation,*

$$(B_R^\theta)^* \to (V_R^\theta)^* \to Sel_S^T(H)_R \to 0$$

Therefore, $Sel_S^T(H)_R = \nabla_S^T(H)_R^{tr}$

Proof. Read Appendix A in [3]. □

As a consequence of above two theorems, we have

$$Fitt_R(\nabla_S^T(H)_R) = Fitt_R(Sel_S^T(H)_R)$$

This link will be used in Chapter 5 to compute the Fitting ideals of the relevant Ritter-Weiss moduels.

3.4 Transition maps

Now, our next goal in this chapter is to consider the constructions done in the previous section at two number field extensions of the number field, F (Say K_1 and K_2) such that one contains the other (say $K_1 \subset K_2$)and to construct maps between them. Then, that will allow us to define transition maps between Ritter-Weiss modules, and thereby to give meaningful maps between the sequences (12) at those two levels. In order to do that we start locally. That is, we start by constructing meaningful maps between V_w's at K_1 and K_2.

We first setup the notation. Define $G_1 := Gal(K_1/F)$ and $G_2 := Gal(K_2/F)$. Now we fix a place u of F and then, above that a place v of K_1 and above that a place w of K_2. Now, let F_u, $K_{1,v}$ and $K_{2,w}$ be completions with respect to those places. Then we have the decomposition groups of u in K_1/F and K_2/F are $G_v := Gal(K_{1,v}/F_u)$ and $G_w := Gal(K_{2,w}/F_u)$.

Now, let $\pi : \mathbb{Z}[G_w] \to \mathbb{Z}[G_v]$ be the G_w–module morphism induced by the Galois restriction. Here, $\mathbb{Z}[G_v]$ is viewed as a $\mathbb{Z}[G_w]$-module via Galois restriction. This will also induce a map (which we also call π) between the augmentation ideals, $\pi : \Delta G_w \to \Delta G_v$.

Proposition 3.4.1. *There exist a $\mathbb{Z}[G_w]$-module morphism f_w such that the following diagram commutes. Here the left vertical map is induced by the local norm map.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{2,w}^{\times} & \longrightarrow & V_w^2 & \longrightarrow & \Delta G_w & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle Nm} & & \big\downarrow{\scriptstyle f_w} & & \big\downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & K_{1,v}^{\times} & \longrightarrow & V_v^1 & \longrightarrow & \Delta G_v & \longrightarrow & 0 \end{array}$$

Proof. Observe that we have the following commutative diagram between the sequences (2) at K_1 and K_2.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W(K_{2,w}^{ab}/K_{2,w}) \cong K_{2,w}^{\times} & \longrightarrow & W(K_{2,w}^{ab}/F_u) & \longrightarrow & G_w & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle res} & & \big\downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & W(K_{1,v}^{ab}/K_{1,v}) \cong K_{1,v}^{\times} & \longrightarrow & W(K_{1,v}^{ab}/F_u) & \longrightarrow & G_v & \longrightarrow & 0 \end{array} \tag{14}$$

The middle verical map is Galois restriction. Then, by class field theory, the left vertical map is the local norm map, $Nm : K_{2,w}^{\times} \to K_{1,v}^{\times}$. Now, by applying the functor t of [6], we get our result. □

Now, we need to glue these local commutative diagrams to obtain global diagrams. Let us discuss that general framework.

Suppose, M is a G_w module and N is a G_v module (hence, is also a G_w module via Galois restriction). Let, $f_w : M \to N$ be a G_w- module morphism. Then, we define the global map,

$$f : \mathbb{Z}[G_2] \otimes_{\mathbb{Z}[G_w]} M \to \mathbb{Z}[G_1] \otimes_{\mathbb{Z}[G_v]} N$$

is given by $f(g \otimes m) = \pi(g) \otimes f_w(m)$ for all $g \in G_2$ and $m \in M$.

Observe that, we also have the following commutative diagram where the vertical maps are canonical embeddings.

$$\begin{array}{ccc} \mathbb{Z}[G_2] \otimes_{\mathbb{Z}[G_w]} M & \xrightarrow{f} & \mathbb{Z}[G_1] \otimes_{\mathbb{Z}[G_v]} N \\ \uparrow & & \uparrow \\ M & \xrightarrow{f_w} & N \end{array}$$

Now the following two theorems give an explicit description of f at some important special cases.

Proposition 3.4.2. *If $M = \mathbb{Z}[G_w]$, $N = \mathbb{Z}[G_v]$ and $f_w = \pi$ is the local Galois restriction, then f is the global Galois restriction (which we also call π).*

Proof. We know that, $\mathbb{Z}[G_2] \otimes_{\mathbb{Z}[G_w]} \mathbb{Z}[G_w] \cong \mathbb{Z}[G_2]$ and $\mathbb{Z}[G_1] \otimes_{\mathbb{Z}[G_v]} \mathbb{Z}[G_v] \cong \mathbb{Z}[G_1]$. Now, under these isomorphisms, observe that, for all $g \in G_2$, we have $f(g) = f(g \otimes 1) = \pi(g) \otimes \pi(1) = \pi(g) \otimes 1 = \pi(g)$. This completes the proof. □

Proposition 3.4.3. *If* $M = K_{2,w}^{\times}$, $N = K_{1,v}^{\times}$ *and* $f_w = Nm$ *is the local norm map, then* f *induces the global norm map. That is, the following diagram commutes.*

$$\begin{array}{ccc} \mathbb{Z}[G_2] \otimes_{\mathbb{Z}[G_w]} K_{2,w}^{\times} & \xrightarrow{f} & \mathbb{Z}[G_1] \otimes_{\mathbb{Z}[G_v]} K_{1,v}^{\times} \\ \uparrow i_2 & & \uparrow i_1 \\ K_2^{\times} & \xrightarrow{Norm} & K_1^{\times} \end{array}$$

where i_1 *and* i_2 *are the canonical diagonal embeddings.*

Proof. Let $\tilde{\rho}_i$'s are representatives of G_2/G_w in G_2 and let ρ_j's be representatives of G_1/G_v of G_1 such that for each j , $\rho_j = \pi(\tilde{\rho}_i)$ for some i. Now, let $x \in K_2^{\times}$. Then,

$$\begin{aligned} f(i_2(x)) &= f(\sum_i \tilde{\rho}_i \otimes \tilde{\rho}_i^{-1} x) \\ &= \sum_i \pi(\tilde{\rho}_i) \otimes Nm(\tilde{\rho}_i^{-1} x) \\ &= \sum_j \sum_{\pi(\tilde{\rho}_i)=\rho_j} (\rho_j \otimes Nm(\tilde{\rho}_i^{-1} x)) \\ &= \sum_j (\rho_j \otimes \prod_{\pi(\tilde{\rho}_i)\rho_j} Nm(\tilde{\rho}_i^{-1} x)) \end{aligned}$$

Let $\alpha = \prod_{\pi(\tilde{\rho}_i)=\rho_j} Nm(\tilde{\rho}_i^{-1} x)$. Now, let $\theta \in G_2$ such that $\pi(\theta) = \rho_j$. Observe that WLOG we can always choose $\tilde{\rho}_i$'s and ρ_j's such that $\{\tilde{\rho}_i ; \pi(\tilde{\rho}_i) = \rho_j\} = \theta \cdot \{\tilde{\rho}_i ; \pi(\tilde{\rho}_i) = 1\}$. Then, we have,

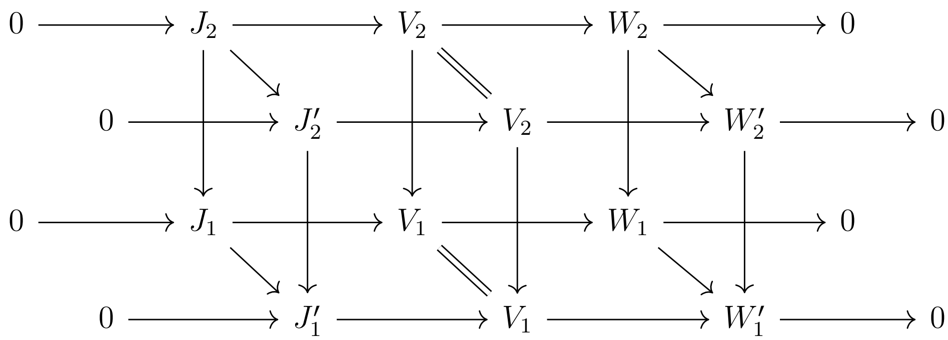
$$\alpha = \prod_{\pi(\tilde{\rho}_i)=1} Nm(\tilde{\rho}_i^{-1} \theta^{-1} x) = \prod_{\pi(\tilde{\rho}_i)=1} \prod_{g \in ker(\pi|_{G_w})} g \tilde{\rho}_i^{-1} (\theta^{-1} x) = Norm(\theta^{-1} x) = \rho_j^{-1} Norm(x)$$

Therefore,

$$f(i_2(x)) = \sum_j (\rho_j \otimes \rho_j^{-1} Norm(x)) = i_1(Norm(x))$$

as desired. □

Now, by gluing the local data from Proposition 3.4.1 according to the above machinery, we get the following diagram.



(15)

The maps between J, J' and W , W' are the obvious ones. The left vertical maps (which we also call "Norm") are induced by the norm maps. The middle vertical maps, f is induced by the local map f_w in Proposition 3.4.1. The right vertical maps (which we also call "π") are induced by π and f_w.

Now, we prove the global analog of Proposition 3.4.1.

Proposition 3.4.4. *There exist a G_2-module morphism d such that the following diagrams commute.*

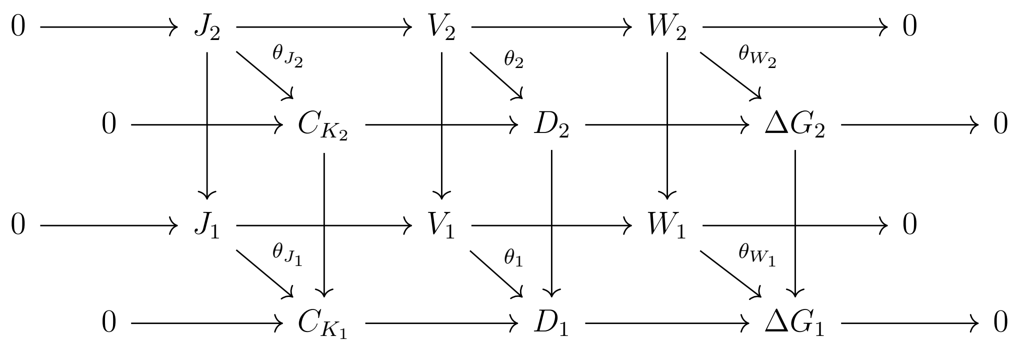
$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_{K_2} & \longrightarrow & D_2 & \longrightarrow & \Delta G_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Norm} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & C_{K_1} & \longrightarrow & D_1 & \longrightarrow & \Delta G_1 & \longrightarrow & 0 \end{array}$$

Proof. Observe that we have the following commutative diagram. This gives a map between the sequences (i) in page 168 of [6] at K_1 and K_2.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W(K_2^{ab}/K_2) \cong C_{K_2} & \longrightarrow & W(K_2^{ab}/F) & \longrightarrow & G_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle res} & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & W(K_1^{ab}/K_1) \cong C_{K_1} & \longrightarrow & W(K_1^{ab}/F) & \longrightarrow & G_1 & \longrightarrow & 0 \end{array} \qquad (16)$$

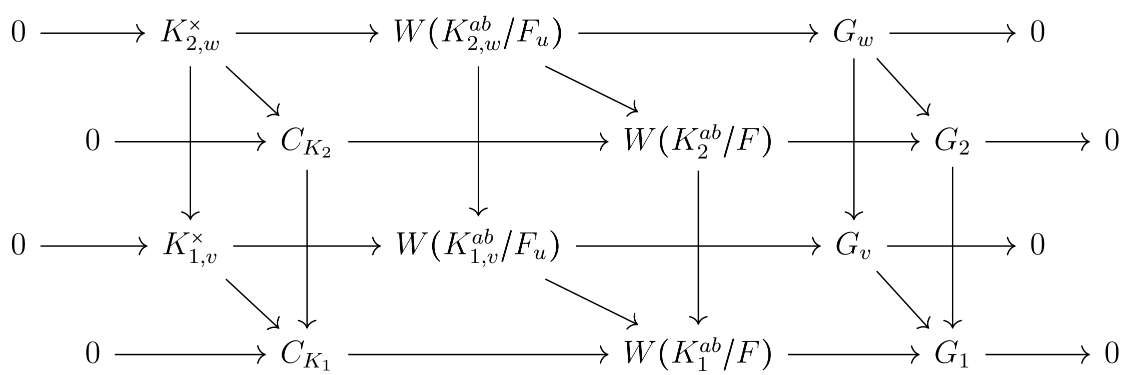
Here, $W(K_*^{ab}/F) \coloneqq W(F)/W(K_*)^c$ where $W(*)$ is the absolute Weil group and res is the natural projection. Then, by class field theory (see [12]), the left vertical map is the norm map, $Norm : C_{K_2} \to C_{K_1}$. Now, by applying the functor t of [6], we get our result. $\square$

Proposition 3.4.5. *The following diagram of G_2-modules commutes.*

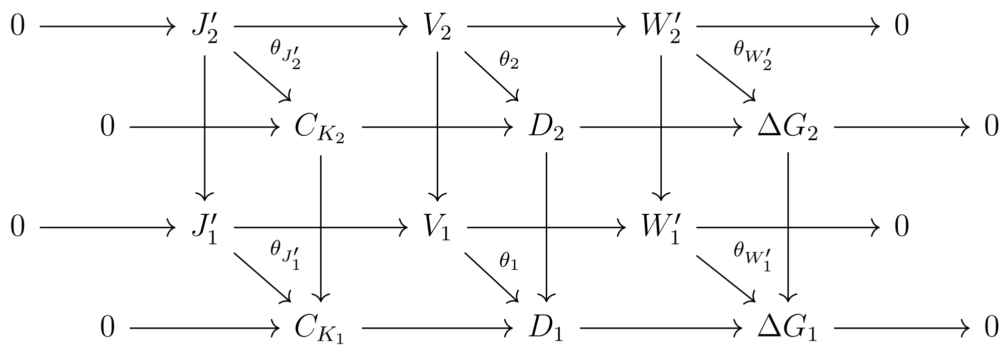


Again, here the left vertical arrows are induced by norm maps, and the right vertical arrows are induced by Galois restriction. Upper and lower levels of the diagram are same as the diagram (5).

Proof. Observe that we have the following commutative diagram which connects diagrams (14) and (16) for a fixed prime w of K_2.



Here, the top and bottom faces are the top face of the diagram in the bottom of page 168 in [6], at levels K_1 and K_2. Now, by applying the translator functor, t of [6] to the above diagram, we get a diagram that connects the diagrams in Proposition 3.4.1 and Proposition 3.4.4. Then, by gluing the local diagrams (inner faces) appropriately using the machinery we defined, we get the following diagram.



Now, by connecting above diagram with diagram (15), we get the desired result. □

Now, the snake lemma yields the following diagram, which is a morphism between diagrams (138) in [3] at levels K_1 and K_2.

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbf{O}^{\times}_{K_2,S,T} & \longrightarrow & V_2^{\theta} & \longrightarrow & W_2^{\theta} & \longrightarrow & Cl_S^T(K_2) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Norm} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle Norm} & & \\ 0 & \longrightarrow & \mathbf{O}^{\times}_{K_1,S,T} & \longrightarrow & V_1^{\theta} & \longrightarrow & W_1^{\theta} & \longrightarrow & Cl_S^T(K_1) & \longrightarrow & 0 \end{array} \tag{17}$$

Now, we look at maps $\gamma : W \to B$ as in diagram (8) at the levels K_1 and K_2, and then construct maps between levels. In order to do that, let us understand the modules W more explicitly. Here we are using the description in [9]. We start by focusing on one level (say K_1).

Let $\overline{G_v} = \langle F \rangle$ be the Galois group of the residue field extension of $K_{1,v}/F_u$ where F is the Frobenius. Then,

$$W_v = \{(x, y) \in \Delta G_v \bigoplus \mathbb{Z}[\overline{G_v}]; \bar{x} = (F - 1)y\}$$

where $\bar{x}$ is the image of x in $\mathbb{Z}[\overline{G_v}]$. Clearly, this is a free $\mathbb{Z}$- module. A $\mathbb{Z}$- basis is given by

$$\{w_g = (g-1, \sum_{i=0}^{a(g)-1} F^i); g \in G_v\}$$

where $a(g)$ defined such that for each $g \in G_v, \bar{g} = F^{a(g)}$ and $0 < a(g) \leq f_1 := |\overline{G_v}|$. Under the notation of the short exact sequence (3), $i(1) = w_1$ and $j(w_g) = g-1$. Now, the G_v action on these basis elements is given by $g \cdot w_h = w_{gh} - w_g + a_{g,h} w_1$ for each $g, h \in G_v$ where $a_{g,h}$ is defined by $a(g) + a(h) = a(gh) + f_1 a_{g,h}$. Observe that $a_{1,h} = 1$ for each $h \in G_v$.

Now, we recall the following technical lemma from [13] about the above quantities at two levels.

Lemma 3.4.6. *Let $\tilde{h} \in G_w$ is a lift of $h \in G_v$ and $g \in G_v$. Let e be the ramification index of the extension $K_{2,w}/K_{1,v}$ and f, f_1 and f_2 be the residue class degrees of the extensions , $K_{2,w}/K_{1,v}$, $K_{1,v}/F_u$ and $K_{2,w}/F_u$. Then, the followings are true.*

(1) $a(\tilde{h}) = a(h) + k_{\tilde{h}} f_1$ for some $k_{\tilde{h}} \in \{0, 1, 2, \dots f-1\}$.
(2)$\sum_{\tilde{g} \to g} a_{\tilde{g},\tilde{h}} = e(a_{g,h} + k_{\tilde{h}})$

Observe that the G_w-module map f_w in Proposition 3.4.1 induces a map f'_w : $V_w^2/\mathbf{O}_w^\times = W_w^2 \to V_v^1/\mathbf{O}_v^\times = W_v^1$. (And also, this map is a local component of the right vertical map in the inner face of diagram (15) at $S' \setminus S$ primes). Now, we give an explicit description of this map in terms of the above mentioned $\mathbb{Z}$-basis elements.

Proposition 3.4.7. *$f'_w : W_w^2 \to W_v^1$ is given by $f'_w(w_{\tilde{g}}) = w_g + k_{\tilde{g}} \cdot w_1$ for all $\tilde{g} \in G_w$. Here, $g = \pi(\tilde{g})$.*

Proof. Proposition 3.4.1 induces maps between the short exact sequences (3) at K_1

and K_2 as below. Here f is the residue class degree of the extension $K_{2,w}/K_{1,v}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{i_2} & W_w^2 & \xrightarrow{j_2} & \Delta G_w & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \times f} & & \downarrow{\scriptstyle f'_w} & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{i_1} & W_v^1 & \xrightarrow{j_1} & \Delta G_v & \longrightarrow & 0 \end{array} \tag{18}$$

Observe that for any $g \in G_v$, by the commutativity of the right square, we have,

$$j_1(f'_w(w_{\tilde{g}})) = \pi(j_2(w_{\tilde{g}})) = \pi(\tilde{g} - 1) = g - 1$$

On the other hand, we know that $j_1(w_g) = g - 1$. Therefore, by the exactness of the upper row, we have $f'_w(w_{\tilde{g}}) = w_g + e_{\tilde{g}} \cdot w_1$ for some $e_{\tilde{g}} \in \mathbb{Z}$. Moreover, from the left commutative square, we have $f'_w(w_{\tilde{1}}) = f \cdot w_1$. Here $\tilde{1}$ is the identity of G_w.

Now, we are left to prove that $e_{\tilde{g}} = k_{\tilde{g}}$ for all $\tilde{g} \in G_w$. Since we also know that f'_w is G_w- equivariant, for all $\tilde{g}, \tilde{h} \in G_w$ above $g, h \in G_v$ we have, $h \cdot (f'_w(w_{\tilde{g}})) = f'_w(\tilde{h} \cdot w_{\tilde{g}})$. Observe that,

$$\begin{aligned} h \cdot (f'_w(w_{\tilde{g}})) &= h \cdot (w_g + e_{\tilde{g}} \cdot w_1) \\ &= w_{gh} - w_h + a_{a,h} w_1 + e_{\tilde{g}} \cdot w_1 \\ &= w_{gh} - w_h + (a_{g,h} + e_{\tilde{g}}) \cdot w_1 \end{aligned}$$

On the other hand,

$$\begin{aligned} f'_w(\tilde{h} \cdot w_{\tilde{g}}) &= f'_w(w_{\tilde{g}\tilde{h}} - w_{\tilde{h}} + a_{\tilde{g},\tilde{h}} \cdot w_{\tilde{1}}) \\ &= w_{gh} + e_{\tilde{g}\tilde{h}} \cdot w_1 - w_h - e_{\tilde{h}} \cdot w_1 + a_{\tilde{g},\tilde{h}} f \cdot w_1 \\ &= w_{gh} - w_h + (a_{\tilde{g},\tilde{h}} f + e_{\tilde{g}\tilde{h}} - e_{\tilde{h}}) w_1 \end{aligned}$$

Hence, for all $\tilde{g}, \tilde{h} \in G_w$ above $g, h \in G_v$ we have,

$$a_{g,h} + e_{\tilde{g}} = e_{\tilde{g}\tilde{h}} - e_{\tilde{h}} + a_{\tilde{g},\tilde{h}} f$$

Now, by taking the summation of above equation when $\tilde{h} \in G_w$ varies when $\pi(\tilde{h}) = h$ and applying Lemma 3.4.6 (b), we get,

$$ef(a_{g,h} + e_{\tilde{g}}) = \sum_{\pi(\tilde{h})=h} (e_{\tilde{g}\tilde{h}} - e_{\tilde{h}}) + ef(a_{g,h} + k_{\tilde{g}})$$

Now, by taking the summation when h varies through all the elements in G_w, the first term of the right hand side vanishes. Then, we easily get $e_{\tilde{g}} = k_{\tilde{g}}$ as desired. □

Now, we recall the map $\gamma : W \to B$ in the commutative diagram (8) from [3]. Let us look at its component-wise definition at K_1.

- For $v \in S$, γ_v is induced by the inclusion $\Delta G_v \subset \mathbb{Z}[G_v]$
- For $v \in S'_{ram} := S_{ram} \setminus (S \cup T)$, γ_v is induced by (j, s) where $j(w_g) = g - 1$ and

  $$s(w_g) = \sum_{h \in G_v} (r(g) + 1 - a_{g^{-1},h})h$$

  for all $g \in G_v$. Here,

  $$r(g) = \begin{cases} 1 & if \quad g \in I_v \\ 0 & if \quad g \notin I_v \end{cases}$$

  where I_v is the ramification group. This description is from [9].
- For $v \in S' \setminus (S \cup S'_{ram})$, γ_v is induced by s.

Now, we prove the following technical lemma.

Lemma 3.4.8. *For all $\tilde{g} \in G_w$ we have $k_{\tilde{g}^{-1}} + k_{\tilde{g}} = (r(\tilde{g}) + 1)f - (r(g) + 1)$. Here $g = \pi(\tilde{g})$.*

Proof. We split the proof into three cases.

- Case (i) :- $g \notin I_{1,v}$ where $I_{1,v}$ is the ramification group of the extension $K_{1,v}/F_u$

In this case we also have $\tilde{g} \notin I_{2,w}$. Here, $I_{2,w}$ is the ramification group of the extension $K_{2,w}/F_u$. Let, f_1 and f_2 are the residue class degrees of the extensions $K_{1,v}/F_u$ and $K_{2,w}/F_u$ respectively. Therefore,

$$k_{\tilde{g}^{-1}} + k_{\tilde{g}} = \frac{1}{f_1}(a(\tilde{g}) + a(\tilde{g}^{-1}) - a(g) - a(g^{-1})) = \frac{f_2 - f_1}{f_1} = f - 1$$

- Case (ii) :- $g \in I_{1,v}$ and $\tilde{g} \notin I_{2,w}$

$$k_{\tilde{g}^{-1}} + k_{\tilde{g}} = \frac{f_2 - 2f_1}{f_1} = f - 2$$

- Case (iii) :- $\tilde{g} \in I_{2,w}$

In this case we also have $g \in I_{1,v}$

$$k_{\tilde{g}^{-1}} + k_{\tilde{g}} = \frac{2f_2 - 2f_1}{f_1} = 2f - 2$$

In all three cases we have the right hand side as desired. □

From the right square of the commutative diagram (16), the map f'_w is compatible with the maps j's and π. Now we prove a simillar result for the maps s at $K_{1,v}$ and $K_{2,w}$.

Proposition 3.4.9. *Suppose w is unramified in $K_{2,w}/K_{1,v}$. Then, the following diagram of G_w-modules commutes.*

$$\begin{array}{ccc} W_w^2 & \xrightarrow{s_2} & \mathbb{Z}[G_w] \\ \downarrow{\scriptstyle f'_w} & & \downarrow{\scriptstyle \pi} \\ W_v^1 & \xrightarrow{s_1} & \mathbb{Z}[G_v] \end{array}$$

Proof. Observe that, for all $\tilde{g} \in G_w$,

$$s_1(f'_w(w_{\tilde{g}})) = s_1(w_g + k_{\tilde{g}} \cdot w_1) = \sum_{h \in G_v} (r(g) + 1 - a_{g^{-1},h} + k_{\tilde{g}})h$$

where $g = \pi(\tilde{g})$. On the other hand using Lemma 3.4.6 (b),

$$\pi(s_2(w_{\tilde{g}})) = \pi(\sum_{\tilde{h} \in G_w} (r(\tilde{g}) + 1 - a_{\tilde{g}^{-1},\tilde{h}})\tilde{h}) = \sum_{h \in G_v} (f(r(\tilde{g}) + 1) - (a_{g^{-1},h} + k_{\tilde{g}^{-1}}))h$$

Now, by applying Lemma 3.4.8 we have that $s_1(f'_w(w_{\tilde{g}})) = \pi(s_2(w_{\tilde{g}}))$. This completes the proof. $\square$

As a consequence, we have the following global result.

Proposition 3.4.10. *Suppose $S_{ram}(K_2/F) \subseteq S \cup T$. Then, the following diagram of G_2-modules commutes.*

$$\begin{array}{ccc} W_2 & \xrightarrow{\gamma_2} & B_2 \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \pi} \\ W_1 & \xrightarrow{\gamma_1} & B_1 \end{array}$$

Proof. This is obtained by gluing the local diagrams at each prime. At S-primes the local diagram is obvious. Since $S' \smallsetminus S$ primes are unramified the previous theorem gives the local diagram. $\square$

We recall the map θ_B from the diagram (8) from [3]. It's defined component-wise as follows.

- For $v \in S$, θ_B is the identity.
- For $v \in S'_{ram} := S_{ram} \smallsetminus (S \cup T)$, θ_B is the projection on to the first component.
- For $v \in S' \smallsetminus (S \cup S'_{ram})$, $\theta_B(x) = (\sigma_v - 1)x$ where σ_v is the Frobenius. (Observe that in this case v is unramified.)

Now, we prove the following theorem on the compatibility between θ_B maps and Galois restriction.

Proposition 3.4.11. *Suppose $S_{ram}(K_2/F) \subseteq S \cup T$. Then, the following commutes as G_2-modules.*

$$\begin{array}{ccc} B_2 & \xrightarrow{\pi} & B_1 \\ \downarrow{\scriptstyle \theta_{B_2}} & & \downarrow{\scriptstyle \theta_{B_1}} \\ \mathbb{Z}[G_2] & \xrightarrow{\pi} & \mathbb{Z}[G_1] \end{array}$$

Proof. We prove this component-wise. It is obvious for S primes. Now, suppose w be a prime in K_2 above the $S' \setminus S$ prime, v in K_1. Let $\sigma_w \in G_2$ and $\sigma_v \in G_1$ be corresponding Frobenii. This is well defined as these primes are unramified by the assumption. Now, we do the following calculation for a $\tilde{g} \in G_2$ in a w-component of B_2. Here $g = \pi(\tilde{g})$.

$$\theta_{B_1}(\pi(\tilde{g})) = \theta_{B_1}(g) = (\sigma_v - 1)g$$

On the other hand,

$$\pi(\theta_{B_2}(\tilde{g})) = \pi((\sigma_w - 1)\tilde{g}) = (\sigma_v - 1)g$$

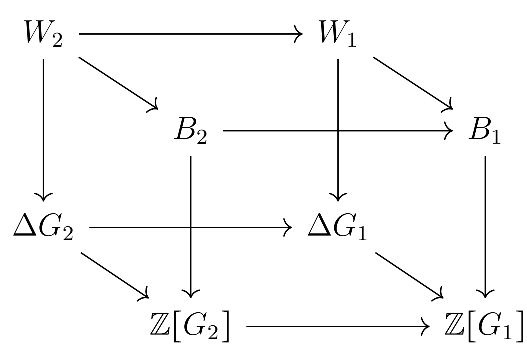
This completes the proof. □

We need one more compatibility result before defining the transition maps we mentioned in the beginning of this section.

Proposition 3.4.12. *Suppose $S_{ram}(K_2/F) \subseteq S \cup T$. Then, the following diagram of G_2-modules commutes.*

$$\begin{array}{ccc} W_2^\theta & \xrightarrow{\gamma_2} & B_2^\theta \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \pi} \\ W_1^\theta & \xrightarrow{\gamma_1} & B_1^\theta \end{array} \tag{19}$$

Proof. Observe that we have the following diagram.



Here, the upper face is by Proposition 3.4.10. Lower face is obvious. Left and right faces are the left squares of diagram (8) at levels K_1 and K_2. Inner face is the right face in the diagram in Proposition 3.4.5. Now, by taking kernels of the vertical maps we get our result. □

Now, we are ready to define the transition maps between Ritter-Weiss modules at K_2 and K_1.

Definition 3.4.13. The G_2-module morphism $\lambda : \nabla_S^T(K_2) \to \nabla_S^T(K_1)$ is defined by the following diagram.

$$\begin{array}{ccccc} V_2^\theta & \longrightarrow & B_2^\theta & \longrightarrow & \nabla_S^T(K_2) \\ \downarrow f & & \downarrow \pi & & \downarrow \lambda \\ V_1^\theta & \longrightarrow & B_1^\theta & \longrightarrow & \nabla_S^T(K_1) \end{array}$$

Here, the left square is obtained by connecting diagram (19) and the middle square of diagram (17).

From [3] we know that B_*^θ are free $\mathbb{Z}[G_*]$ modules. Hence, the middle vertical map is induced by Galois restriction, and hence surjective. Therefore, so is λ.

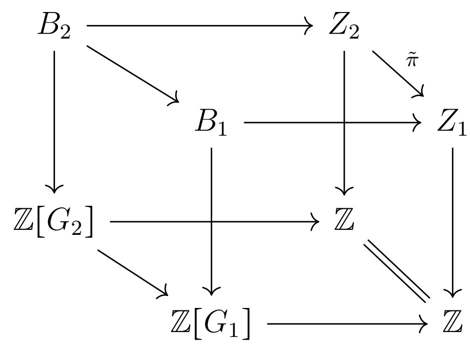
Now, as the very last result of this section we give a map between the sequences (11) at levels K_1 and K_2.

Theorem 3.4.14. *Suppose $S_{ram}(K_2/F) \subseteq S \cup T$. Let X_{S,K_*} be zero divisors of K_* supported at S primes. Then, we have the following diagram.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Cl_S^T(K_2) & \longrightarrow & \nabla_S^T(K_2) & \longrightarrow & X_{S,K_2} & \longrightarrow & 0 \\ & & \downarrow Norm & & \downarrow \lambda & & \downarrow \tilde{\pi} & & \\ 0 & \longrightarrow & Cl_S^T(K_1) & \longrightarrow & \nabla_S^T(K_1) & \longrightarrow & X_{S,K_1} & \longrightarrow & 0 \end{array}$$

Here, $\tilde{\pi}$ is induced by the map (which we also call $\tilde{\pi}$) $(x_w)_w \mapsto (\sum_{w|v} x_w)_v$ on S-divisors.

Proof. By our assumption, $S'_{ram} = \varnothing$. Therefore, $Z_* = Div_S(K_*)$. Hence, we have the following commutative diagram.



Here, the vertical maps of the right face are $(x_w)_w \mapsto \sum_w x_w$. Left face is from the Proposition 3.4.11. Inner and outer faces are the right square in the diagram (8) at levels K_1 and K_2. The lower face is obvious and the upper face is also very easy to prove. Now, by taking kernels we get the following digram.

$$\begin{array}{ccc} B_2^\theta & \longrightarrow & Z_2^\theta \\ \downarrow & & \downarrow \\ B_1^\theta & \longrightarrow & Z_1^\theta \end{array}$$

This together with Proposition 3.4.12 gives us the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_2^\theta & \longrightarrow & B_2^\theta & \longrightarrow & Z_2^\theta & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tilde{\pi}} & & \\ 0 & \longrightarrow & W_1^\theta & \longrightarrow & B_1^\theta & \longrightarrow & Z_1^\theta & \longrightarrow & 0 \end{array} \tag{20}$$

Observe that this gives a map between the sequences (9) at levels K_1 and K_2. Now, by combining the diagrams (17) and (20), noticing that under the given conditions $Z_*^\theta = X_{S,K_*}$, we have the desired result. □

Chapter 4

Motivation from function fields

Recently, Bley and Popescu [14] proved an equivariant main conjecture along any rank one, sign-normalized Drinfeld modular (geometric) Iwasawa tower of a general function field of characteristic p. Let us discuss their result and how it inspires our main conjecture.

We start by recalling their set up.

Set up :- Let k be any function field of characteristic p and ν_∞ is a fixed place of k, which we call the infinite place. Let $A \subset k$ be the set of elements integral away from ν_∞. Also fix an ideal $\mathfrak{f}$ on A and a maximal ideal $\mathfrak{p}$ such that $\mathfrak{p} \nmid \mathfrak{f}$.

Now, for each nonnegative n, define L_n to be the ray-class field of k of conductor $\mathfrak{f}\mathfrak{p}^{n+1}$ in which ν_∞ splits completely. The extension L_n/L_0 is essentially generated by the p^{n+1}–torsion points of a certain type of rank 1, sign-normalized Drinfeld module defined on A. So, we obtain a geometric (Drinfeld modular) Iwasawa tower L_∞/k. Let $G_n = Gal(L_n/k)$, $G_\infty = Gal(L_\infty/k)$ and $\mathbb{Z}_p[[G_\infty]] = \varprojlim \mathbb{Z}_p[[G_n]]$ where the transition maps are induced by Galois restriction.

Now, consider the two sets of places of k, S and Σ,

$$S := \{\mathfrak{p}\} \cup \{\nu; \nu - \text{prime in } A \text{ and } \nu | \mathfrak{f}\}$$

and Σ is nonempty and disjoint with S. Associated to the data $(L_n/k, S, \Sigma)$, in [14], the authors defined a Ritter-Weiss type $\mathbb{Z}_p[[G_n]]$–module, $\nabla_S^{(n)}$. This module sits in a short exact sequence,

$$0 \to Pic_S^0(L_n) \otimes \mathbb{Z}_p \to \nabla_S^{(n)} \to \tilde{X}_S^{(n)} \to 0$$

Here

$$Pic_S^0(L_n) := \frac{Div^0(L)}{Div_S^0(L) + div(L^\times)}$$

where $Div^0(L)$ (and $Div_S^0(L)$) are the divisors (supported at places above S–places) of degree zero. $div(*)$ is the usual divisor map and the module $\tilde{X}_S^{(n)}$ is a certain variant (and defined using) the zero divisors supported at places above S.

The reader should view the above short exact sequence as a function field analogue of the sequence (12). And also, observe that, although Σ is needed for the construction of the modules $\nabla_S^{(n)}$ (and relevant other modules), it turns are they are independent of the choice of Σ.

Now, for the data (L_∞, k, S, Σ) define the $\mathbb{Z}_p[[G_\infty]]$–module,

$$\nabla_S^{(\infty)} := \varprojlim \nabla_S^{(n)}$$

where the transition maps are induced by the norm maps between levels L_n. One can also define an equivariant p–adic L-function, $\Theta_{S,\Sigma}^{(\infty)}(u) \in \mathbb{Z}_p[[G_\infty]][[u]]$. In [14], the authors prove the following equivariant main conjecture.

Theorem 4.1. ***(Bley-Popescu)*** *For the data* (k, Σ, S) *the* $\mathbb{Z}_p[[G_\infty]]-$*module* $\nabla_S^{(\infty)}$ *finitely generated, torsion and*

- $pd_{\mathbb{Z}_p[[G_\infty]]}(\nabla_S^{(\infty)}) = 1$
- $Fitt_{\mathbb{Z}_p[[G_\infty]]}(\nabla_S^{(\infty)}) = (\Theta_{S,\Sigma}^{(\infty)}(1))$

The goal of this dissertation is to prove an analogue of the above theorem in the number field setting. More specifically, we consider the cyclotomic $\mathbb{Z}_p-$extension of a CM number field and define an Iwasawa Ritter-Weiss module at the infinite level. Then, we compute its Fitting ideal in terms of an equivariant $p-$adic L-function. We also prove that our module has projective dimension 1 over the equivariant Iwasawa algebra.

Chapter 5

Equivariant Main Conjecture

5.1 Basic definitions and set up

Let H be a number field. Now, we will define a special $\mathbb{Z}_p$−extension of H, which is called the cyclotomic $\mathbb{Z}_p$−extension.

Definition 5.1.1. Let p be an odd prime and ζ_{p^n} be a primitive p^nth root of unity. Let B_n be the unique subfield of $\mathbb{Q}(\zeta_{p^{n+1}})$ such that $[B_n : \mathbb{Q}] = p^n$ and $B_\infty = \bigcup_n B_n$. Then, define the cylotomic $\mathbb{Z}_p$−extension of the number field H to be $H_\infty := HB_\infty$

Observe that for any number field H, H_∞ is indeed a $\mathbb{Z}_p$−extension. That is, H_∞/H is a Galois extension and we have $Gal(H_\infty/H) = \mathbb{Z}_p$. Moreover, if H/F is an abelian extension of number fields, then $H_\infty = HF_\infty$. Therefore, H_∞/F is an abelian extension as well.

Now, we define two important classes of number fields which shows up in our main theorem.

Definition 5.1.2. A number field F is called totally real if for each embedding of F into $\mathbb{C}$, the image lies inside $\mathbb{R}$.

Definition 5.1.3. A number field is called totally imaginary, if none of its embeddings to $\mathbb{C}$, lie inside $\mathbb{R}$. A number field H is called a CM field, if it is totally imaginary and a quadratic extension of a totally real number field.

One of the important properties of CM fields is that it has a complex conjugation automorphism (usually denoted by j) which is independent of its embedding into $\mathbb{C}$.

Now, we are ready to describe the set up in which our main results and applications are proved.

Set up:- Let p be an odd prime and H/F be an abelian CM extension of a totally real number field with $Gal(H/F) = G$. Let H_∞ be the cyclotomic $\mathbb{Z}_p$–extension of H and $\mathcal{G} := Gal(H_\infty/F)$. Let S and T be two nonempty disjoint sets of places in F such that, $S_\infty \subseteq S$, $T \notin S_{ram}(H_\infty/F)$ and $S_{ram}(H_\infty/F) \subseteq S \cup T$.

Observe that since H is a CM field there is a unique complex conjugation automorphism $j \in G$. Throughout this chapter, for any G–module M, we define,

$$M^- := \frac{1}{2}(1-j) \cdot (M \otimes_{\mathbb{Z}} \mathbb{Z}[1/2])$$

Observe that this is a $\mathbb{Z}[G]^-$–module in the obvious sense.

As discussed in the introduction, main conjectures in Iwasawa theory relates algebraic objects to the analytic objects. Now, we define the analytic object, S–depleted, T–smoothed equivariant Artin L-function, which appears in our equivariant main conjecture.

For a place v of F, we let G_v and I_v denote its decomposition and inertia groups in G, respectively and fix $\sigma_v \in G_v$ a Frobenius element for v. The idempotent associated to the trivial character of I_v in $\mathbb{Q}[I_v]$ is given by

$$e_v := \frac{1}{|I_v|} N_{I_v} := \frac{1}{|I_v|} \sum_{\sigma \in I_v} \sigma$$

Then $e_v \sigma_v^{-1} \in \mathbb{Q}[G]$ is independent of the choice of σ_v. As in [3], the $\mathbb{C}[G]$–valued (G–equivariant) L–function associated to $(H/F, S, T)$ of complex variable s is given by the meromorphic continuation to the entire complex plane of the following holomorphic function

$$\Theta_{S,H/F}^T(s) := \prod_{v \notin S} (1 - e_v \sigma_v^{-1} \cdot Nv^{-s})^{-1} \cdot \prod_{v \in T} (1 - e_v \sigma_v^{-1} \cdot Nv^{1-s}), \qquad \mathrm{Re}(s) > 0.$$

The resulting meromorphic continuation (also denoted by $\Theta_{S,H/F}^T(s)$) is holomorphic on $\mathbb{C} \setminus \{1\}$. We are interested in its special value at 0, denoted by $\Theta_S^T(H/F) := \Theta_{S,H/F}^T(0)$. It turns out that, under the conditions given in our setup, we have $\Theta_S^T(H/F) \in \mathbb{Z}_p[G]^-$.

Now, if H'/F is an abelian extension such that $H \subseteq H'$ and $G' = Gal(H'/F)$, we have $\pi(\Theta_S^T(H'/F)) = \Theta_S^T(H/F)$ where $\pi : \mathbb{Z}_p[G'] \to \mathbb{Z}_p[G]$ is induced by the Galois restriction. This is essentially due to the inflation property of Artin L-functions.

Now, let H_n be the nth layer of the cyclotomic $\mathbb{Z}_p$-tower of H and let $G_n := Gal(H_n/F)$. Define $\mathbb{Z}_p[[\mathcal{G}]] := \varprojlim_n \mathbb{Z}_p[G_n]$ where the transition maps are induced by Galois restriction. The property above allow us to define the (S,T)–modified equivariant p–adic L-function,

$$\Theta_S^T(H_\infty/F) := (\Theta_S^T(H_n/F))_n \in \mathbb{Z}_p[[\mathcal{G}]]^-$$

5.2 Main Results

In this section we define a p-adic Ritter-Weiss module at the infinite level of a cyclotomic Iwasawa tower and then, prove an equivariant main conjecture on that module. This can be viewed as a number field analogue of Corollary 3.11 in [14]. We start by recalling our setup.

Fix an odd prime p. Let H/F be an abelian extension of a CM number field over a totally real number field. Let H_∞/H be the cyclotomic $\mathbb{Z}_p$-extension and H_n be the nth layer. We also define $G_n := Gal(H_n/F)$ and $\mathcal{G} := Gal(H_\infty/F)$. Let S and T be two disjoint sets of places in F. When there's no danger of confusion we use the same symbols to denote the places above S and T primes. Throughout this section we assume $S_\infty \subseteq S$, $T \notin S_{ram}(H_\infty/F)$ and $S_{ram}(H_\infty/F) \subseteq S \cup T$.

Definition 5.2.1. Define $\nabla_S^T(H_n)_p := \nabla_S^T(H_n) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\nabla_S^T(H_\infty)_p := \varprojlim_n \nabla_S^T(H_n)_p$ where the transition maps are induced by the λ maps defined in the previous section.

Definition 5.2.2. Define $A_S^T(H_n) := Cl_S^T(H_n) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $X_S^T := \varprojlim_n A_S^T(H_n)$ where the transition maps are induced by the norm maps.

Proposition 5.2.3. *We have the following diagram of* $\mathbb{Z}_p[[\mathcal{G}]]^-$ *modules.*

$$0 \to X_S^{T,-} \to \nabla_S^T(H_\infty)_p^- \to Div_S(H_\infty)^- \otimes_{\mathbb{Z}} \mathbb{Z}_p \to 0$$

Proof. Tensoring the diagram in Theorem 3.4.14 in this context by the flat $\mathbb{Z}$- module $\mathbb{Z}_p$ and taking the minus part, we get for each n,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_S^T(H_{n+1})^- & \longrightarrow & \nabla_S^T(H_{n+1})_p^- & \longrightarrow & X_{S,H_{n+1}}^- \otimes_{\mathbb{Z}} \mathbb{Z}_p & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Norm} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \tilde{\pi}} & & \\ 0 & \longrightarrow & A_S^T(H_n)^- & \longrightarrow & \nabla_S^T(H_n)_p^- & \longrightarrow & X_{S,H_n}^- \otimes_{\mathbb{Z}} \mathbb{Z}_p & \longrightarrow & 0 \end{array}$$

Observe that $X_{S,H_n}^- \otimes_{\mathbb{Z}} \mathbb{Z}_p = Div_S(H_n)^- \otimes_{\mathbb{Z}} \mathbb{Z}_p$. And also, as $Div_S(H_n)^- \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a free $\mathbb{Z}_p$-module, we have

$$\nabla_S^T(H_n)_p^- \cong A_S^T(H_n)^- \bigoplus Div_S(H_n)^- \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

as $\mathbb{Z}_p$-modules. We know that $A_S^T(H_n)^-$ is finite. Therefore, we can topologize $\nabla_S^T(H_n)_p^-$ with discrete topology on $A_S^T(H_n)^-$ and p-adic topology on $Div_S(H_n)^- \otimes_{\mathbb{Z}} \mathbb{Z}_p$, so that the above diagram is of compact modules. Therefore, by taking the projective limit we get the desired result. □

Let $\Lambda := \mathbb{Z}_p[[\Gamma]]$ where $\Gamma = Gal(H_\infty/H')$. Here, H' is as defined in the beginning of Section 2.2. Observe that in the current setting, H_∞ is the cyclotomic $\mathbb{Z}_p$–extension of H'.

Proposition 5.2.4. *$\nabla_S^T(H_\infty)_p^-$ is a finitely generated torsion Λ- module.*

Proof. Observe that for all n we have the following exact sequence (see sequence (11) in [10]).

$$\Delta_{H_n,T} \otimes \mathbb{Z}_p \to A^T(H_n) \to A_{H_n} \to 0$$

Here, $\Delta_{H_n,T} := \bigoplus_{v\in T} \kappa(v)^\times$ where $\kappa(v)$ denotes the residue field associated to the prime v. Here each module is finite, hence compact over the discrete topology. Then, by taking the projective limit over the norm maps, we have,

$$\mathcal{D}_T \to X^T \to X \to 0$$

where $\mathcal{D}_T = \varprojlim_n \Delta_{H_n,T}$. Now, as $\kappa(v)^\times$ is cyclic and T_{H_∞} is finite, $\mathcal{D}_T$ is a finitely generated $\mathbb{Z}_p$-module. Hence, it is a finitely generated torsion Λ module. It is well known that so is X. Therefore, from the above sequence, so is X^T.

We know that for each n we have the surjection, $A^T(H_n) \to A_S^T(H_n)$ of finite groups. So, they are compact with respect to the discrete topology. By taking the projective limit over the norm maps we have the surjection, $X_\varnothing^T \to X_S^T$. Therefore, $X_S^{T,-}$ is a finitely generated torsion Λ-module. Now, since $Div_S(H_\infty)^- \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a finitely generated $\mathbb{Z}_p$-module, it is a finitely generated torsion Λ-module. Now, the result follows from the previous theorem. □

Now, we compute the Fitting ideals of the Ritter-Weiss modules at the finite levels.

Proposition 5.2.5. *For each n and (S,T) as in the setup,*

$$Fitt_{\mathbb{Z}_p[[G_n]]^-}(\nabla_S^T(H_n)_p^-) = (\Theta_S^T(H_n/F))$$

.

Proof. This is a consequence of Lemma 6.1 and Lemma A.8 of [3] and Theorem 6.4 of [13]. □

Now we are ready to state our main result.

Theorem 5.2.6. *Suppose $(H/F,S,T,p)$ are as in the setup. Then, the following hold.*

1. *We have an equality of $\mathbb{Z}_p[[\mathcal{G}]]^-$–ideals*

$$\mathrm{Fitt}_{\mathbb{Z}_p[[\mathcal{G}]]^-}(\nabla_S^T(H_\infty)_p^-) = (\Theta_S^T(H_\infty/F)).$$

2. *The $\mathbb{Z}_p[[\mathcal{G}]]^-$–module $Sel_S^T(H_\infty)_p^-$ sits in a short exact sequence*

$$0 \longrightarrow (\mathbb{Z}_p[[\mathcal{G}]]^-)^k \longrightarrow (\mathbb{Z}_p[[\mathcal{G}]]^-)^k \longrightarrow \nabla_S^T(H_\infty)_p^- \longrightarrow 0,$$

   for some $k > 0$. In particular, $\mathrm{pd}_{\mathbb{Z}_p[[\mathcal{G}]]^-}(\nabla_S^T(H_\infty)_p^-) = 1$.

Proof. Part (1) is obtained by applying Proposition 5.2.5 together with Corollary 2.2.3. Part (2) is a consequence of Proposition 4.9 in [13] and the fact that $\Theta_S^T(H_\infty/F)$ is a nonzero divisor in $\mathbb{Z}_p[[\mathcal{G}]]^-$ as proved in [13]. □

Now, we can use above results to compute the Fitting ideal of the (S,T)– modified Iwasawa module.

Theorem 5.2.7. *Suppose $(H/F, S, T, p)$ are as in the setup. Then, the following hold.*

$$Fitt_{\mathbb{Z}_p[[\mathcal{G}]]^-}(X_S^{T,-}) = (\Theta_T^{S\cap S_{ram}}(H_\infty/F)) \prod_{v\in(S\cap S_{ram})\smallsetminus S_p} \left(1, \frac{N(I_v)}{\sigma_v - 1}\right) \prod_{v\in S_p} Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]^-}(\mathbb{Z}_p[\mathcal{G}/\mathcal{G}_v]^-)$$

Here, $\mathcal{G}_v$ and I_v are the decomposition and inetria groups of the prime v in the extension H_∞/F. Here, $N(I_v) = \sum_{g\in I_v} g$ and σ_v is any choice of Frobenius.

Proof. By Propositions 5.2.6, 5.2.3, 5.2.4 and 2.3.2, we have the following.

$$Fitt_{\mathbb{Z}_p[[\mathcal{G}]]^-}(X_S^{T,-}) = (\Theta_T^S(H_\infty/F)) \prod_{v\in S} Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]^-}(\mathbb{Z}_p[\mathcal{G}/\mathcal{G}_v]^-)$$

Now, by projecting into each character of G_n we know that for each n, we have,

$$\Theta_T^S(H_n/F) = \Theta_T^{S\cap S_{ram}}(H_n/F) \prod_{v\in S\smallsetminus S_{ram}} (1 - \sigma_v^{-1})$$

Now, by taking the projective limit we have,

$$\Theta_T^S(H_\infty/F) = \Theta_T^{S\cap S_{ram}}(H_\infty/F) \prod_{v\in S\smallsetminus S_{ram}} (1 - \sigma_v^{-1})$$

We know that for unramified primes

$$Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]^-}(\mathbb{Z}_p[\mathcal{G}/\mathcal{G}_v]^-) = \left(\frac{1}{1 - \sigma_v^{-1}}\right)$$

Moreover, by Proposition 1.8 of [7] , for ramified, non-p primes,

$$Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]^-}(\mathbb{Z}_p[\mathcal{G}/\mathcal{G}_v]^-) = \left(1, \frac{N(I_v)}{1 - \sigma_v}\right)$$

This completes the proof. □

As a consequence of the theorem above and the explicit computations done in the Appendix, we have the following main theorem.

Theorem 5.2.8. *Under the notation defined earlier, we have the following.*

$$Fitt_{\mathbb{Z}_p[[\mathcal{G}]]^-}(X_S^{T,-}) = (\Theta_T^{S\cap S_{ram}}(H_\infty/F)) \prod_{v\in(S\cap S_{ram})} (N(A)\Delta B^{r_B-2} \text{ ; } \mathcal{G}_v = A\times B, A\text{–}torsion)$$

Proof. This is an easy consequence of the above theorem and Theorem A.6. □

Observe that the right hand side of the above equality does not depend on unramified primes in S. This is not an accident since it is an easy exercise in class field theory to show that, the module X_S^T itself is independent of unramified S–primes.

5.3 Future directions

There are several directions and plans to continue our research extending from the theorems and techniques we have developed so far. In this section we briefly discuss some of them.

1) In [14] Bley and Popescu proved that in the function field setting, the Ritter-Weiss type module is isomorphic to the Γ-coinvariance of the p–adic realization of Picard 1-motives, where Γ is the Galois group of the arithmetic Iwasawa tower. We believe that an analogous result must exist in the number field setting as well. That is, there must be a link (probably an isomorphism) between $(Sel_S^T(H_\infty)_p^{-,*})_\Gamma$ and $\nabla_S^T(H)_p^-$ where $Sel_S^T(H_\infty)_p$ is a Selmer module defined in [13] at the infinite level of the cyclotomic Iwasawa tower.

2) Let Y/X be a Galois cover of finite connected graphs and if $Y_1 \subseteq Y_2 \subseteq ...$ is an Iwasawa tower of graphs above Y. We believe that an equivariant main conjecture exist for the module $\varprojlim_n Pic(Y_n)$.

3) The Theorem A.6 is true when $\mathcal{G}$ is an abelian p–adic Lie group with $\mathbb{Z}_p$-rank one. We believe that there's an analogous result for higher rank groups as well. We also believe, that will have a lot of applications in number fields, function fields and graph theoretic settings.

4) All of our theorems are proved in the commutative setting. That is, H_∞/H is the **cyclotomic** $\mathbb{Z}_p$ extension, H/F is abelian and therefore so is H_∞/F. We would like to extend our theory to noncommutative settings such as $\mathbb{Z}_p$–extensions which are not cyclotomic. For that we will have to use the theory of Fitting invariants introduced by Nickel [18] rather than Fitting ideals as the latter only works for commutative rings.

5) A very big improvement of Theorem 5.2.8 would be computing $Fitt_{\mathbb{Z}_p[[\mathcal{G}]]^-}(X^-)$. That is, removing S and T from the theorem. However, removing the T–part would still be an interesting and challenging project. We would also like to prove simillar results at the finite level, such as computing $Fitt_{\mathbb{Z}_p[G]^-}(A_S^T(H))$.

Appendix A

Shifted Fitting Ideal Computations

In this section, we are computing shifted Fitting ideals of the divisors of $S-$primes (or any finite set of primes) in the extension H_∞/F. For this it is enough to find $Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]}(\mathbb{Z}_p[\mathcal{G}/\mathcal{G}_v])$ for some fixed prime v.

Let us start with our set up. Let H' be as defined in the beginning of Section 2.2 so that H_∞ is the cyclotomic $\mathbb{Z}_p-$extension of H'. Define, $\mathcal{G} := Gal(H_\infty/F)$ and $\Gamma := Gal(H_\infty/H') = \overline{<\gamma>}$ where γ is a topological generator of Γ. For each $n \in \mathbb{N}$, let H_n be the $n-$th layer of the cyclotomic $\mathbb{Z}_p-$tower and $G_n := Gal(H_n/F)$. Now, we fix a large n such that no $S-$prime splits above H_n in the cyclotomic tower. Then we have, $\mathcal{G}/\mathcal{G}_v \cong G_n/G_{n,v}$ where $\mathcal{G}_v$ and $G_{n,v}$ are the decomposition groups of v in the extensions H_∞/F and H_n/F respectively. Observe that the isomorphism is given by Galois restriction.

Now, by [7], Proposition 4.2, in order to compute $Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]}(\mathbb{Z}_p[\mathcal{G}/\mathcal{G}_v])$, it is enough to find a resolution,

$$R^{t_3} \xrightarrow{A} R^{t_2} \to R^{t_1} \to Y_v \to 0$$

Here, $Y_v = \mathbb{Z}_p[\mathcal{G}/\mathcal{G}_v]$, $R = \mathbb{Z}_p[G_n]$ and $A \in M_{t_2 \times t_3}(R)$. Then, we have,

$$Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]}(Y_v) = (\gamma^{p^n} - 1)^{t_2 - t_1} \sum_{e=0}^{t_2} (\gamma^{p^n} - 1)^{-e} Min_e(\tilde{A}) \tag{1}$$

Here, $\tilde{A} \in M_{t_2 \times t_3}(\mathbb{Z}_p[[\mathcal{G}]])$ is a lift of A and $Min_e(\tilde{A})$ is the ideal generated by the e−minors of $\tilde{A}$. Now, our first goal in this section is to find an explicit resolution for Y_v.

Let a cyclic decomposition of $G_{n,v}$ be,

$$G_{n,v} = \langle g_1 \rangle \times \langle g_2 \rangle \times \dots \ \langle g_r \rangle$$

Let $\beta_i = g_i - 1$ and $\alpha_i = N(\langle g_i \rangle) := \sum_{k=1}^{ord(g_i)} g_i^k$ for all $1 \le i \le r$. Then, it is easy to see that $Y_v \cong \mathbb{Z}_p[G_n/G_{n,v}] \cong R/(\beta_1, \beta_2, \dots \ \beta_r)$. Now, observe that we have the following short exact sequence.

$$0 \to K \to R^r \xrightarrow{f} R \to Y_v \to 0$$

where f is defined as $f(e_i) = \beta_i$ for each i, where e_i's are the standard basis elements. Here, $K = ker(f)$. Therefore, $K = \{(y_1, y_2, \dots \ , y_r) \in R^r ; \sum_{i=1}^r y_i \beta_i = 0\}$. Now, we prove the following theorem which gives an explicit description of K.

Proposition A.1. *Define the sequence of matrices $Q_k \in M_{k \times k(k-1)/2}(R)$ inductively as follows.*

$$\textit{Let } Q_1 \textit{ be the empty matrix and } Q_k = \begin{pmatrix} Q_{k-1} & & -\beta_k I_{k-1} & & \\ 0 & \beta_1 & \beta_2 & \dots & \beta_{k-1} \end{pmatrix}$$

for each $2 \le k \le r$. Now, define A_k as $A_k = \begin{pmatrix} diag(\alpha_i)_{i=1}^k & Q_k \end{pmatrix}$. Then, $K = Image(A_r)$.

Proof. We prove a slightly stronger result. Define, $K_k := \{(y_1, y_2, \dots \ , y_k) \in R^k ; \sum_{i=1}^k y_i \beta_i = 0\}$. Let us use induction on k to prove that $K_k = Image(A_k)$.

We claim that $y_1\beta_1 = 0$ iff $y_1 = \alpha_1\alpha'$ for some $\alpha' \in R$. Observe that the backward implication is due to the fact that $\alpha_1\beta_1 = 0$. Now, for the forward implication, let u_i be a set of representatives of $G_n/\langle g_1\rangle$ in G_n. Then, as a set $G_n = \{u_i \cdot g_1^j\}$. So, suppose that $y_1 = \sum_i \sum_{j=1}^{ord_{G_n}(g_1)} c_{i,j}(u_i \cdot g_1^j)$ for some $c_{i,j} \in \mathbb{Z}_p$. Now, by setting $c_{i,0} = c_{i,ord_{G_n}(g_1)}$ we have,

$$0 = y_1\beta_1 = \sum_i \sum_{j=1}^{ord_{G_n}(g_1)} c_{i,j}(u_i \cdot (g_1^j - g_1^{j+1})) = \sum_i \sum_{j=1}^{ord_{G_n}(g_1)} (c_{i,j} - c_{i,j-1})u_i \cdot g_1^j$$

Therefore, for each i, we have

$$c_{i,0} = c_{i,1} = \ \dots \ = c_{i,ord_{G_n}(g_1)-1} =: c_i$$

Hence, $y_1 = \sum_i \sum_{j=1}^{ord_{G_n}(g_1)} c_i(u_i \cdot g_1^j) = (\sum_i c_i u_i) \sum_{j=1}^{ord_{G_n}(g_1)} g_1^j = \alpha_1 \cdot \alpha'$ by setting $\alpha' = \sum_i c_i u_i$. This completes the proof of the claim and thereby the $k = 1$ case because, $A_1 = (\alpha_1)$.

Now, we assume the result for all integers less than k. Suppose $(y_i)_{i=1}^k \in K_k$. That is,

$$\sum_{i=1}^{k} y_i\beta_i = 0 \tag{2}$$

Let us view this equation in the group ring $R' = R/(\beta_1, \beta_2, \dots \ \beta_{k-1}) \cong \mathbb{Z}_p[G_n/\langle g_1\rangle \times \langle g_2\rangle \times \dots \ \langle g_{k-1}\rangle]$. Let $\bar{x}$ be the image of $x \in R$ in R'. Then, we have $\overline{y_k} \cdot \overline{\beta_k} = 0$. Observe that $ord_{G_n}(g_k) = ord_{G_n/\langle g_1\rangle\times\langle g_2\rangle\times\dots\ \langle g_{k-1}\rangle}(\overline{g_k})$. Again, by an easy group ring calculation(similar to what we did for $k = 1$ case), we have, $\overline{y_k} = \overline{\alpha_k}\cdot\overline{\alpha'}$ for some $\alpha' \in R$. Therefore, we have $y_k \in \alpha_k \cdot \alpha' + (\beta_1, \beta_2, \dots \ \beta_{k-1})$ and hence, $y_k = \alpha_k \cdot \alpha' + \sum_{i=1}^{k-1} \beta_i\theta_i$ for some $\theta_i \in R$. This together with the fact that $\alpha_k\beta_k = 0$ and (2) implies,

$$\sum_{i=1}^{k-1} (y_i + \theta_i\beta_k)\beta_i = 0$$

Now, by induction hypothesis $(y_i+\theta_i\beta_k)_{i=1}^{k-1} \in Image(A_{k-1})$ which implies that $(y_i)_{i=1}^{k-1} \in Image(\begin{pmatrix} A_{k-1} & | & -\beta_k I_{k-1} \end{pmatrix})$. But, we also have $y_k \in Image(\begin{pmatrix} \beta_1 & \beta_2 & \dots & \beta_{k-1} & \alpha_k \end{pmatrix})$. Combining these results, we have

$$(y_i)_{i=1}^{k} \in Image(\begin{pmatrix} A_{k-1} & & -\beta_k I_{k-1} & & & 0 \\ 0 & \beta_1 & \beta_2 & \dots & \beta_{k-1} & \alpha_k \end{pmatrix})$$

Now, by rearranging the columns of the matrix we have, $(y_i)_{i=1}^{k} \in Image(A_k)$. So, we have proved that $K_k \subset Image(A_k)$.

Now, in order to prove the other inclusion, observe the following partial description of the matrices Q_k.

$$Q_k \doteq \begin{pmatrix} & \{i,j\} & \\ & & \\ & \beta_j & \\ & 0 & \\ & -\beta_i & \\ & & \end{pmatrix} \begin{matrix} \\ \\ i \\ \\ j \\ \\ \end{matrix} \tag{3}$$

Here, columns are indexed by the pairs $\{i,j\}$ where $i \neq j$ and $1 \leq i,j \leq k$ and $\doteq$ means the equality up to multiplying columns by -1 and rearranging. But, when we are considering images, the signs and the order of columns don't matter. Now, by using the fact that $\alpha_i\beta_i = 0$ for all i, and an easy calculation shows the other inclusion, completing the inductive step. Now, as $K = K_r$, this completes the proof. □

Now, we have the resolution of Y_v we were looking for.

$$R^{r(r+1)/2} \xrightarrow{A_r} R^r \xrightarrow{f} R \to Y_v \to 0$$

Now, by (1) in order to compute the shifted Fitting ideals explicitly, we need to pick generators g_i and a lift $\tilde{A}_r$ of A_r.

Now, let $G = Gal(H'/F)$. Observe that $\mathcal{G} \cong G \times \Gamma$ and $G \cong Tor(\mathcal{G})$. Now, since $rank_{\mathbb{Z}_p}(\mathcal{G}_v) = 1$, we have $\mathcal{G}_v = Tor(\mathcal{G}_v) \times \overline{\langle y \rangle}$ where $y = (g, \gamma^t) \in G \times \Gamma$. Observe that we can choose γ and y such that t is a power of p and g has a $p-$power order. Now, we also choose a bigger n to make sure that $\gamma^{p^n} \in \langle y \rangle$.

Now, we know that, if π_n is the Galois restriction,

$$\pi_n : \mathcal{G} \cong G \times \Gamma \to G_n \cong G \times (\Gamma/\Gamma^{p^n})$$

where under relevant identifications $\pi_n(G) = G$ and $\pi_n(\Gamma) = \Gamma/\Gamma^{p^n}$. Therefore, $G_{n,v} = \pi_n(\mathcal{G}_v) \cong \pi_n(Tor(\mathcal{G}_v)) \times (\overline{\langle y \rangle}/\Gamma^{p^n})$. Observe that, here $\pi_n(Tor(\mathcal{G}_v)) \subseteq G$ is independent of n. Now, choose generators of $\pi_n(Tor(\mathcal{G}_v))$ such that,

$$\pi_n(Tor(\mathcal{G}_v)) = \langle g_1 \rangle \times \langle g_2 \rangle \times \ldots \ \langle g_{r-1} \rangle$$

and let $g_r = \pi_n(y)$. Then we have

$$G_{n,v} = \langle g_1 \rangle \times \langle g_2 \rangle \times \ldots \ \langle g_r \rangle$$

For, these generators we have the corresponding $\alpha_i, \beta_i \in R$. Observe that out of these elements, only α_r, β_r depends on n. So, when we are choosing the lift $\tilde{A}_r$, we can set the lifts of the entries to be, $\tilde{\alpha_i} = \alpha_i$ and $\tilde{\beta_i} = \beta_i$ for $1 \leq i \leq r-1$, $\tilde{\beta_r} = y - 1$ and $\tilde{\alpha_r} = \sum_{i=0}^{ord_{G_n}(g_r)-1} y^i$. Therefore, in $\tilde{A}_r$ only the entry $\tilde{\alpha_r}$ depends on n.

Now, we prove some technical theorems about the matrix $\tilde{A}_r$.

Lemma A.2. *Let* $Q'_k \in M_{k\times k(k-1)/2}(\mathbb{Z}_p[x_1, x_2, \dots, x_k])$ *be defined as*

$$Q'_k = \begin{pmatrix} & \{i,j\} & \\ & & \\ & x_j & \\ & 0 & \\ & -x_i & \\ & & \end{pmatrix} \begin{matrix} \\ \\ i \\ \\ j \\ \\ \end{matrix}$$

Then, we have $Min_k(Q'_k) = (0)$ *for each* $k \le r$.

Proof. Observe that $\mathbb{Z}_p[x_1, x_2, \dots, x_k]$ is an integral domain and,

$$\begin{pmatrix} x_1 & x_2 & \dots & x_k \end{pmatrix} \cdot Q'_k = 0$$

Therefore, all of the $k-$minors of Q'_k must be zero. This completes the proof. □

Lemma A.3. *Let* $A'_k := \begin{pmatrix} diag(y_i)_{i=1}^k & Q'_k \end{pmatrix} \in M_{k\times k(k+1)/2}(\mathbb{Z}_p[x_1, \dots, x_k, y_1, \dots, y_k])$. *Then, any monomial which leads to a* $k-$*minor of* A'_k *satisfies one (or more) of the followings.*

1. *It is equal to* $y_1 y_2 \dots y_k$
2. *It has a term* $x_i y_i$ *for some* i *as a factor.*
3. *It cancels out when the determinant that correspond to the* $k-$*minor is taken and the resultant* $k-$*minor is zero.*

Proof. We use induction on k. When $k = 1$, clearly 1) occurs because, $A'_1 = (y_1)$. Now, let us assume the result for all the integers less than k and prove it for k.

Suppose the monomial of interest has no y_i term as a factor. Then, it is a monomial leading to a $k-$minor of Q'_k. Then by Lemma A.2 it must get canceled out and 3) will occur. Now suppose it has a y_i term (WLOG say y_k). If x_k is also a factor, we are done because, 2) occurs. If not, observe that the monomial is of the following form.

$$y_k \cdot (\text{a monomial which leads to a } (k-1) - \text{minor in } A'_{k-1}) \tag{4}$$

Therefore, the k−minor which correspond to the above monomial is of the form,

$$y_k \cdot (\text{a } (k-1) - \text{minor of } A'_{k-1})$$

Now, by induction hypothesis, 1) , 2) or 3) must be true for the degree $k-1$ monomial in (4), which implies that 1), 2) or 3) must be true respectively for the monomial of interest. This completes the induction. □

Now, we are ready to compute the terms in equation (1).

Proposition A.4.

$$Min_r(\tilde{A}_r) = \Big(\prod_{i=1}^{r} \tilde{\alpha}_i \ , \ \tilde{\alpha}_r \tilde{\beta}_r^{\ t_r} \prod_{i \in L} \tilde{\alpha}_i \prod_{i \in L^c} \tilde{\beta}_i^{\ t_i} \ ; \ L \subset \{1, 2, \ \dots \ , r-1\} \ , \\ \sum t_i = r - |L| - 1 \ , \ t_i \geq 0 \ , \ t_r \geq 1 \Big)$$

Proof. Observe that the minors of $\tilde{A}_r$ are the images of the minors of A'_r (defined in Lemma A.3) under the map $h : \mathbb{Z}_p[x_1, x_2, \dots , x_k, y_1, y_2, \dots , y_k] \to \mathbb{Z}_p[[\mathcal{G}]]$ defined by sending x_i to $\tilde{\beta}_i$ and y_i to $\tilde{\alpha}_i$. Now, by Lemma A.3 and the fact that $\tilde{\alpha}_i \tilde{\beta}_i = 0$ for all $1 \leq i \leq r-1$, the only monomials that generate nonzero r−minors are the ones on the RHS. Therefore, we have $LHS \subseteq RHS$.

Now, we prove the other inclusion. Observe that, $\prod_{i=1}^{r} \tilde{\alpha}_i = det(diag(\tilde{\alpha}_i)_{i=1}^{r}) \in Min_r(\tilde{A}_r)$. Now, let $L \subseteq \{1, 2, \dots , r-1\}$ which correspond to a generator, $u = \tilde{\alpha}_r \tilde{\beta}_r^{\ t_r} \prod_{i \in L} \tilde{\alpha}_i \prod_{i \in L^c} \tilde{\beta}_i^{\ t_i}$ of second type in RHS. Observe that, since $\tilde{A}_r$ is symmetric with respect to the indices from 1 to r, up to multiplying columns by −1, in order to show that $u \in LHS$, WLOG we can assume that $L = \{1, 2, \dots , a\}$ and $t_{r-1} \geq t_{r-2}, \dots$

such that $\sum t_i = r-a-1$. Now, whenever $t_{r-\theta} > 0$, we have $\theta < 1+t_r+t_{r-1}+ \dots +t_{r-(\theta-1)}$. Therefore,

$$r-\theta > (r-1) - (t_r + t_{r-1} + \dots + t_{r-(\theta-1)}) =: a_\theta \tag{4}$$

Now let

$$A = \begin{pmatrix} diag(\alpha_i)_{i=1}^{a} \\ 0_{(r-a)\times a} \end{pmatrix} \quad B = \begin{pmatrix} 0_{(r-1)\times 1} \\ \tilde{\alpha_r} \end{pmatrix}$$

Now, set $a_0 = r-1$ and for all θ such that $t_{r-\theta} > 0$, define,

$$C_{r-\theta} = \begin{pmatrix} & & 0_{a_{\theta+1}\times t_{r-\theta}} & \\ & & \tilde{\beta_{r-\theta}} I_{t_{r-\theta}} & \\ & & 0_{((r-\theta)-a_\theta-1)\times t_{r-\theta}} & \\ -\beta_{a_{\theta+1}+1} & -\beta_{a_{\theta+1}} & \dots & -\beta_{a_\theta} \\ & & 0_{\theta\times t_{r-\theta}} & \end{pmatrix}$$

Observe that, by the inequality (4), the dimensions of the block matrices above are non-negative. Hence, all $C_{r-\theta}$ are well defined. Now, define,

$$E = \begin{pmatrix} A | \dots & C_{r-\theta} & \dots & C_{r-1} | C_r | B \end{pmatrix}$$

Now, if we look closely, up to rearranging columns and multiplying by -1, E is a $r \times r$ sub matrix of $\tilde{A}_r$. Therefore, $det(E) \in LHS$. On the other hand, observe that E is in fact upper triangular and $det(E) = u$. Hence, $u \in LHS$, completing the other inclusion. This completes the proof. $\square$

Now we are ready to give an explicit description of $Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]}(Y_v)$ in terms of the generators of $\mathcal{G}_v$

Theorem A.5. *Under the notation introduced earlier, we have*

$$Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]}(Y_v) = \Bigg(\frac{1}{\tilde{\beta}_r} \prod_{i=1}^{r-1} \tilde{\alpha}_i \ , \ \prod_{i \in L} \tilde{\alpha}_i \prod_{i \in L^c \cup \{r\}} \tilde{\beta}_i^{\,t_i} \\ ; L \subseteq \{1, 2, \ \dots \ , r-1\} \ , \ \sum t_i = r - |L| - 2 \ , \ t_i \geq 0 \Bigg)$$

Proof. From the resolution we obtained, the equation (1) reads as,

$$Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]}(Y_v) = \sum_{e=0}^{r} (\gamma^{p^n} - 1)^{r-1-e} Min_e(\tilde{A}_r)$$

First, we compute the $e = r$ term. Observe that, from the way we chose our n, we have, $\tilde{\alpha_r}\tilde{\beta_r} = (\sum_{i=0}^{ord_{G_n}(g_r)-1} y^i)(y-1) = y^{ord_{G_n}(g_r)} - 1 = \gamma^{p^n} - 1$. Since we know that $\gamma^{p^n} - 1$ is a nonzero divisor in the Iwasawa algebra, so are $\tilde{\alpha_r}$ and $\tilde{\beta_r}$. Then, by Proposition A.4, we have,

$$\frac{1}{\gamma^{p^n} - 1} Min_r(\tilde{A}_r) = RHS =: I$$

Now, we look at the $e = r-1$ term. Observe that, all the nonzero monomials which lead to $(r-1)$–minors are of the form $\tilde{\alpha_r}^b \prod_{i \in L} \tilde{\alpha}_i \prod_{i \in L^c \cup \{r\}} \tilde{\beta}_i^{\,t_i}$ where $\sum t_i = r - |L| - b - 1$ for $b = 0$ or 1. Clearly, each of those monomials are multiples of the generators of the second type in I. Therefore, we have $Min_{r-1}(\tilde{A}_r) \subset I$.

Now, clearly all the $e < r-1$ terms are sub-ideals of $(\gamma^{p^n} - 1)$. Therefore, we have

$$I \subseteq Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]}(Y_v) \subseteq I + (\gamma^{p^n} - 1) \tag{5}$$

Since I has no $\tilde{\alpha_r}$ in it, I is independent of n. Therefore, (5) is true if we replace n with any $m > n$. And also, we know that $(\gamma^{p^n} - 1)I = Min_r(\tilde{A}_r)$ is an ideal of

$\mathbb{Z}_p[[\mathcal{G}]]$. So, we have the following inclusions of (closed under the standard topology) ideals of the Noetherian ring $\mathbb{Z}_p[[\mathcal{G}]]$.

$$(\gamma^{p^n}-1)I \subseteq (\gamma^{p^n}-1)Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]}(Y_v) \subseteq (\gamma^{p^n}-1)I + (\gamma^{p^n}-1)(\gamma^{p^m}-1)$$

We know that under the standard topology $\lim_{m\to\infty}(\gamma^{p^m}-1) = 0$. Therefore, by an easy topological argument, we have,

$$(\gamma^{p^n}-1)I \subseteq (\gamma^{p^n}-1)Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]}(Y_v) \subseteq (\gamma^{p^n}-1)I$$

Now, canceling out the ideal generated by the nonzero divisor $\gamma^{p^n}-1$ yields the desired result. □

As the final result of this section, we give a more intrinsic description, which does not depend on the choice of generators, for $Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]}(Y_v)$.

Theorem A.6.

$$Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]}(Y_v) = (N(A)\Delta B^{r_B-2} \ ;\ \mathcal{G}_v = A\times B, A-torsion)$$

Here A, B runs through all the possibilities such that $\mathcal{G}_v = A\times B$ where A is torsion. $\Delta B = (g-1 \ ;\ g\in B)$ is the augmentation ideal of B and r_B is the minimum number of generators of B.

Proof. Observe that, if $A = \prod_{i\in L}\langle g_i\rangle$, $B = \prod_{i\in L^c}\langle g_i\rangle \times \overline{\langle y\rangle}$ where $L \neq \{1,2,\ \dots\ r-1\}$ then, $N(A) = \prod_{i\in L}\tilde{\alpha}_i$ and $\Delta B^{r_B-2} = (\prod_{i\in L^c\cup\{r\}}\tilde{\beta}_i^{\,t_i}\ ;\ \sum t_i = r-|L|-2\ ,\ t_i\geq 0)$. If $L = \{1,2,\ \dots\ r-1\}$, Then, $N(A)\Delta B^{r_B-2} = (\prod_{i\in L}\tilde{\alpha}_i)(y-1)^{-1} = (\frac{1}{\tilde{\beta}_r}\prod_{i=1}^{r-1}\tilde{\alpha}_i)$. Therefore, by Theorem A.5, $LHS \subseteq RHS$.

But, on the other hand, $Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]}(Y_v)$ must be independent of the choice of generators of $\mathcal{G}_v$. Therefore, for any choice of A, B (and their generators), by Theorem A.5, we should have that

$$N(A)\Delta B^{r_B-2} \subseteq Fitt^{[1]}_{\mathbb{Z}_p[[\mathcal{G}]]}(Y_v)$$

As a consequence, we have $RHS \subseteq LHS$. This completes the proof. □

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