The Transformation of Second-Order Linear Systems into Independent Equations

by

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Abstract

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Linear second-order ordinary differential equations arise from Newton's second law combined with Hooke's law and are ubiquitous in mechanical and civil engineering. Perhaps the most prominent example is a mathematical model for small oscillations of particles around their equilibrium positions. However, second-order systems also find applications in such diverse areas as chemical engineering, structural dynamics, linear systems theory or even economics. Very large second-order systems appear, for example, in mathematical modeling of complex structures by finite-element methods.

In general, any system of second-order equations is coupled. Each equation is linked to at least one of its neighbors and the solution of one of the equations requires the solution of all equations. The "classical decoupling problem" is concerned with the elimination of coordinate coupling in linear dynamical systems. The decoupling transforms the system of equations into a collection of mutually independent equations so that each equation can be solved without solving any other equation. In "The Theory of Sound" in 1894, Lord Rayleigh already expounded on the significance of system decoupling. Since then, the problem has attracted the attention of many researchers.

Mathematically, the system of differential equations is defined by three coefficient matrices. The equations are coupled unless all three matrices are diagonal. The "classical decoupling problem" is thus equivalent to the problem of simultaneous conversion of the coefficient matrices into diagonal forms. Current theory emphasizes simultaneous diagonalization of the coefficient matrices by equivalence or similarity transformations. However, it has been shown that no time-invariant linear transformations will decouple every second-order system. Even partial decoupling, i.e. simultaneous conversion of the coefficient matrices into upper triangular forms, is not ensured with time-invariant linear transformations.

The purpose of this work is to present a general method and algorithm to decouple any second-order linear system (possessing symmetric and non-symmetric coefficients). The theory exploits the parameter "time," characteristic of a dynamical

system. The decoupling is achieved by a real, invertible, but generally nonlinear mapping. This mapping simplifies to a real, linear time-invariant transformation when the coefficient matrices can be simultaneously diagonalized by a similarity transformation. A state-space reformulation of the mapping is also derived. In homogeneous systems the configuration-space decoupling transformation is real, linear and time-invariant when cast in state space. In non-homogeneous systems, both the configuration and associated state transformations are nonlinear and depend continuously on the excitation. The theory is illustrated by several numerical examples. Two applications in earthquake engineering demonstrate the utility of the decoupling approach.

I would like to thank my family for all of their love and support.

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Chapter 1

Introduction

We consider the second-order linear dynamical system

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = f(t), \tag{1.1}$$

where all quantities are real and where dots denote derivatives with respect to the independent variable $t \geq 0$. The coefficients M, C, K are $n \times n$ matrices; $q(t)$ and $f(t)$ are n-dimensional column vectors. The initial conditions $q(0)$, $\dot{q}(0)$ as well as $f(t)$ are given. For simplicity, we assume $f(t)$ is continuously differentiable and restrict our attention to invertible M, thus avoiding differential algebraic equations. Equation (1.1) is a cornerstone in vibration theory and, for example, models small oscillations of particles [8, 31, 39, 44, 48]. In vibration terminology, equation (1.1) determines a non-conservative linear system.

Two symmetric positive definite (SPD) matrices M and K can be simultaneously diagonalized by a congruence transformation [25, 46]. The same congruence transformation that diagonalizes M and K also diagonalizes a symmetric C if and only if (see [7])

$$CM^{-1}K = KM^{-1}C. \tag{1.2}$$

Similar restrictive conditions on simultaneous diagonalization apply if M, C and K are arbitrary square matrices (see [32, 35] for an application-based discussion). It follows that system (1.1) generally cannot be uncoupled (decoupled in modern terminology) into a set of mutually independent, real, scalar, second-order equations by a linear mapping $q(t) \to \mathcal{L}q(t)$, with $\mathcal{L}$ independent of t.

Nonetheless, the decoupling of equation (1.1) is desirable from a practical as well as a theoretical perspective. We present a new approach to this old problem and consider a time-dependent, nonlinear mapping $q(t) \to \mathcal{N}(q,t)$ to obtain a set of mutually independent, real, second-order equations. We then compute the solution $q(t)$ of the coupled equation from the solutions of the independent equations. With apologies for using a terminology that may cause confusion, any methodology that uses a linear or nonlinear mapping to obtain a diagonal second-order system is herein referred to as *decoupling*.

The possibility of decoupling any system posessing SPD coefficients by a nonlinear mapping was recently discovered [36, 37]. The decoupling procedure, termed phase synchronization, can be briefly described as follows. First, the quadratic eigenvalue problem associated with (1.1) is solved. A real and diagonal (i.e. decoupled) system is then constructed using the eigenvalues. Let the solution of the decoupled system be $p(t)$. If system solution is an objective, we can recover $q(t)$ by evaluating each component of $p(t)$ with a different time-lag. It is here that we exploit the special parameter t which is characteristic of equation (1.1).

This work extends the developments in [36,37] to systems possessing nonsymmetric coefficients. We also clarify the decoupling procedure under real eigenvalues and provide a broader perspective of the decoupling operations. The few restrictions placed on M, C and K are an indication of the extensive scope of our method. We emphasize the theory of decoupling, rather than its numerous applications such as model reduction, stability analysis, optimal control, earthquake or rotor design. Nevertheless, two examples from earthquake engineering are employed to demonstrate the utility of the decoupling method. Further exposition of applications will be given elsewhere in future papers.

Based upon the notion of structure-preserving transformations (SPT), Garvey and others [9, 10, 18–20] introduced an alternative method for decoupling homogeneous systems. There, algorithms employing linear coordinate transformations in a higher dimensional space (the state space) are utilized to compute a real and second-order diagonal system. If available, this diagonal system is identical to the system we obtain by phase synchronization without recourse to state space or SPTs (modulo a normalization $M_d \rightarrow I$, see (4.3) and (5.14)).

The organization of this work is as follows. Chapter 2 reviews the traditional theory of coordinate coupling in viscously damped linear dynamical systems. This survey sets up the terminology and notation used throughout this work. In Chapter 3, the concept of damped modes is introduced. When vibrating in a mode, each system component performs exponentially decaying or growing harmonic motion with the same frequency and the same exponential decay. However, there is a constant phase difference between any two components. The key, but radical idea is to synchronize all modes by evaluating each component at a different, but fixed time-lag. We call this process phase synchronization. It is shown in Chapter 4 that phase synchronization generates a nonlinear mapping that decouples any linear system. Three numerical examples are provided to illustrate the theory. In Chapter 5, we discuss the decoupling in configuration and state spaces and give an example of a system that can be decoupled in configuration space, but not in state space. We also explain how phase synchronization relates to SPT's. Applications of the decoupling by phase synchronization in structural dynamics are discussed in Chapter 6. We conclude with a summary of major findings in Chapter 7.

1.1 Notational Conventions

We try to reserve capital letters for matrices, lower case Roman letters for column vectors, and lower case Greek letters for scalars. The main exceptions are t (time), e (Euler's number) and $i = \sqrt{-1}$, for obvious reasons. We also reserve n for the number of degrees of freedom (the length of the vector q), and the letters j and k for indexing scalars or vectors. Thus, v_j, $j = 1, \ldots, n$, denotes the sequence of vectors $v_1, v_2, \ldots, v_n$. We sometimes use v_j^k to denote the kth entry of v_j. In addition, we make use of the Hadamard (or Schur, or pointwise) product of two vectors of the same dimension, i.e. $(v \circ w)^k = v^k w^k$. In this context, the vector $u = (1, \ldots, 1)^T$ is useful, where T denotes the transpose (u stands for *unit* and we would have preferred to use e here, as is more common, but decided to reserve e for Euler's number). We apply scalar functions like $\sin(\cdot)$ or $\cos(\cdot)$ to vectors in a component-wise fashion. Thus, $\exp(v)$ is a vector with k^{th} component given by $(\exp(v))^k = \exp(v^k)$. Likewise, $\Re(v)$ and $\Im(v)$ denote respectively the real and imaginary parts of the vector v. We construct diagonal matrices using the notation $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ so that $\Lambda \in C^{n \times n}$ is diagonal with elements $\lambda_1, \ldots, \lambda_n$. Given a sequence of vectors v_j, $j = 1, \ldots, n$, we construct a matrix $V = (v_1, \ldots, v_n) \in C^{n \times n}$, whose columns are the vectors v_j. The identity matrix is denoted by I, and a square matrix with all elements equal zero is denoted by 0. Based upon compatibility, the dimensions of I and 0 can be inferred from the context. Lastly, we express complex vectors in a quasi–polar form, so that $v = r \circ \exp(-if)$ with $f = (\varphi_1, \ldots, \varphi_n)^T$. In strictly polar form, $r^j \geq 0$ and $|\varphi_j| \leq \pi$ for $j = 1, \ldots, n$. For our purposes it appears more sensible to allow r^j to be negative (if necessary) so that $|\varphi_j| \leq \pi/2$. There is no advantage in the problem we consider to have r nonnegative. The choice of sign in $v = r \circ \exp(-if)$ is for convenience only. We call this representation the quasi–polar form.

Chapter 2

Coordinate Coupling in Viscously Damped Linear Systems

The equation of motion of a viscously damped linear system is given by (1.1) with M, C and K being SPD. These characteristics are not arbitrary, but in fact have solid footing in the theory of Lagrangian dynamics. For example, symmetry of M is based upon the fact that the quadratic form of the kinetic energy can always be defined in terms of a symmetric matrix. In addition, K is SPD if the rigid-body modes are eliminated, which is not an essential restriction at all. In general, the equation of motion (1.1) is coupled so that the ith component equation involves not only q_i and its derivatives but also other coordinates and their derivatives as well. The system is decoupled if and only if M, C and K are diagonal matrices. Coordinate coupling is thus not an inherent property of a system but depends on the generalized coordinates used.

The traditional theory of coordinate coupling in viscously damped linear dynamical systems emphasizes simultaneous diagonalization of the coefficient matrices M, C and K by congruence transformations. This theory is concisely surveyed in the present chapter to set up the terminology and notation used throughout this dissertation. To be sure, we assume definiteness of the coefficients M, C and K only in this chapter. We will drop this assumption in Chapter 3 and develop a general methodology for decoupling *any* linear system.

2.1 Decoupling by Classical Modal Analysis

The process of decoupling the equation of motion of an undamped dynamical system ($C = 0$ in (1.1)) is a time-honored procedure termed modal analysis. We present only a brief summary and refer to standard textbooks for complete detail (e.g. [39]).

Associated with the undamped system is the symmetric eigenvalue problem [39,46]

$$Ku = Mu\omega^2. \tag{2.1}$$

Owing to the positive definiteness of the matrices M and K, all n eigenvalues ω_i^2 are real and positive, and the corresponding natural modes u_i are real and orthogonal with respect to either M or K. We define the modal and spectral matrices, respectively, by

$$
\begin{aligned}
U &= [u_1, \ldots, u_n], \tag{2.2} \\
\tilde{K} &= \operatorname{diag}(\omega_1^2, \ldots, \omega_n^2). \tag{2.3}
\end{aligned}
$$

Upon normalization of the natural modes with respect to the mass matrix, the generalized orthogonality of the modes can be expressed in a compact form:

$$
\begin{aligned}
U^T M U &= I, \tag{2.4} \\
U^T K U &= \tilde{K}. \tag{2.5}
\end{aligned}
$$

Define a modal transformation by

$$q(t) = U\tilde{q}(t). \tag{2.6}$$

In terms of the principal coordinate $\tilde{q}$, the equation of motion takes the canonical form

$$\ddot{\tilde{q}}(t) + \tilde{C}\dot{\tilde{q}}(t) + \tilde{K}\tilde{q}(t) = U^T f(t), \tag{2.7}$$

where the SPD matrix

$$\tilde{C} = U^T C U,$$

is referred to as the modal damping matrix. Note that the mass matrix M and the stiffness matrix K have been diagonalized by modal transformation. Thus, an undamped system can always be decoupled by modal analysis. Any coupling in a linear system occurs ultimately through viscous damping.

2.2 Inadequacy of Classical Modal Analysis

A system is classically damped if it can be decoupled by classical modal analysis, whereby the modal damping matrix $\tilde{C}$ in (2.7) is diagonal. In Section 97 of "The Theory of Sound" in 1894, Rayleigh [48] asserted that a system is classically damped if

$$C = \alpha M + \beta K \tag{2.8}$$

for some scalar constants α and β. This requirement, referred to as proportional damping, is sufficient but not necessary for classical damping. In 1965, Caughey and O'Kelly [7] established that a necessary and sufficient condition under which a

system is classically damped is that the commutativity condition $CM^{-1}K = KM^{-1}C$ is satisfied.

Practically speaking, classical damping means that energy dissipation is almost uniformly distributed throughout the system. This assumption is violated for systems consisting of two or more parts with significantly different levels of damping. Examples of such systems include soil-structure systems [12], base-isolated structures [27, 55, 56], and systems in which coupled vibrations of structures and fluids occur. Increasing use of special energy-dissipating devices in control constitutes another important example. In fact, experimental modal testing suggests that no physical system is strictly classically damped [50].

From a mathematical perspective, almost all viscously damped linear systems are non-classically damped. To show this, we wish to determine the dimension of the space of all viscously damped linear systems. This space is defined by three real symmetric square matrices. The symmetry imposes constraints so that only $n(n + 1)/2$ of the n^2 elements of each coefficient matrix are independent. The space of all viscously damped linear systems thus forms an $3n(n + 1)/2$ dimensional linear space over the field of the real numbers. All classically damped linear systems are defined by two real symmetric matrices and n parameters defining viscous damping matrix by the Caughey series [7, 8]

$$C = M \sum_{k=0}^{n-1} \alpha_k (M^{-1}K)^k. \tag{2.9}$$

Classically damped systems thus generate an $n^2 + 2n$ dimensional manifold in the space of all viscously damped linear systems. Viscous damping is thus non-classical with probability one. We conclude that a linear dynamical system is generally non-classically damped and, thus, can not be decoupled by classical modal analysis.

2.3   Inadequacy of State-Space Approach

Classical modal analysis utilizes a real transformation (2.6). Foss and others [17, 58, 59] extended classical modal analysis to a process of complex modal analysis in the state space to treat non-classically damped systems. However, the state-space approach has never appealed to practicing engineers. There are several reasons for this situation. A common excuse is that the state-space approach is computationally more involved because the dimension of the state space is twice the number of degrees of freedom. Another reason is that complex modal analysis still cannot decouple all non-classically damped systems. A condition of non-defective eigenvectors in the state space must be satisfied in order for complex modal analysis to achieve complete decoupling (see Chapter 5 for an example of a system that can be decoupled in configuration space, but not in state space). More importantly, there is little physical insight associated with different elements of complex modal analysis, whereas classical

modal analysis is amenable to physical interpretation. For example, each normal mode u_i represents a physical profile of vibration. Even the eigenvalue problem (2.1) can be interpreted geometrically as the problem of finding the principal axes of an n-dimensional ellipsoid.

2.4   The Classical Decoupling Problem

The "classical decoupling problem" is concerned with the elimination of coordinate coupling in damped linear dynamical systems. It is a well-trodden problem that has attracted the attention of many researchers in the past century. In "The Theory of Sound" in 1894, Rayleigh [48] already expounded on the significance of system decoupling and introduced the concept of proportional damping. Over the years, various types of decoupling approximation were employed in the analysis of damped systems [3, 8, 11, 13, 16, 21, 26, 29, 51, 57]. Different indices of coupling were also introduced to quantify coordinate coupling [2, 33, 34, 43, 45, 47, 54, 62]. However, a solution to the "classical decoupling problem" has not been reported in the open literature. Mathematically, the "classical decoupling problem" is equivalent to the problem of simultaneous conversion of M, C, and K into diagonal forms. Ma and Caughey [35] showed that no time-invariant linear transformations in the configuration space will decouple every damped linear system. Even partial decoupling, i.e. simultaneous conversion of M, C, and K into upper triangular forms, is not ensured with time-invariant linear transformations [6]. As a consequence, any universal decoupling transformation in the configuration space, if it exists, must be at least time-varying or even nonlinear.

Chapter 3

Phase Synchronization

We drop the SPD assumption on M, C, and K to develop a general methodology for the decoupling of (1.1). Associated with (1.1) is the regular quadratic eigenvalue problem [22, 30, 31, 53]

$$(M\lambda^2 + C\lambda + K)v = 0, \tag{3.1}$$

where λ is termed an eigenvalue and v is the corresponding eigenvector. There are $2n$ eigenpairs (λ_j, v_j), $j = 1, \ldots, 2n$, complex in general. All eigenvalues are finite because M is invertible. Since M, C and K are real, complex eigenvalues and eigenvectors occur in complex conjugate pairs. Throughout this paper, we will associate real eigenvectors with real eigenvalues. For simplicity, we assume that the quadratic eigenvalue problem is non-defective. This assumption is neither unduly restrictive (*almost all* systems are non-defective) nor essential. Examples of decoupling defectives systems with SPD coefficients can be found in [36, 37] and complete details are provided in [28].

The solution of (1.1) with $f(t) = 0$ can be written as

$$q(t) = \sum_{j=1}^{2n} v_j e^{\lambda_j t} \gamma_j, \tag{3.2}$$

where $\gamma_j, j = 1, \ldots, 2n$, are constants determined by the initial conditions [22, 31]. Real initial conditions force those constants γ_j, associated with complex conjugate eigensolutions, to likewise occur in complex conjugate pairs. If all eigenvalues are complex, the solution becomes

$$q(t) = \sum_{j=1}^{n} \left(v_j e^{\lambda_j t} \gamma_j + \bar{v}_j e^{\bar{\lambda}_j t} \bar{\gamma}_j \right). \tag{3.3}$$

Every summand is real and represents a vibration pattern, observable in experiments. Following [36, 37], we refer to a summand

$$s_j(t) = v e^{\lambda_j t} \gamma_j + \bar{v}_j e^{\bar{\lambda}_j t} \bar{\gamma}_j = 2\Re(v_j e^{\lambda_j t} \gamma_j) \tag{3.4}$$

as a *mode*. We hope this terminology will not cause additional confusion, since the word mode has been used in different contexts before. For example $v_j e^{\lambda_j t}$ is sometimes called a complex mode.

The real modes $s_j(t)$ play a key role in decoupling (1.1). When vibrating in a mode, each component performs exponentially decaying or growing harmonic motion with the same frequency and the same exponential decay. However, there is a constant phase difference between any two components. The key, but radical idea is to synchronize all modes $s_j(t)$ by evaluating each component $s_j^k(t)$ at a different, but fixed time-lag. We call this process phase synchronization. Phase synchronization generates a nonlinear mapping to decouple the system (the precise meaning of *nonlinear* is clarified in Section 3.3 of this Chapter and in Chapter 4.3). The main attraction in phase synchronization is thus not the formalism itself, but the decoupling algorithm we will extract from it.

Our terminology is not to be confused with the phenomenon of synchronization in nonlinear systems (we specifically mention the work of Blekhman, see e. g. [4] and references therein). The main idea in phase synchronization is to enforce a synchronization by appropriate manipulation of the modes given by (3.4). Because the idea of phase synchronization is new and somewhat unconventional, the derivations of principal formulas will be presented in sufficient detail. We drop the index j on $s(t)$ temporarily because all modes evolve independently of each other.

3.1 Phase Synchronization of Complex Conjugate Eigensolutions

A pair of complex conjugate eigenvalues

$$\lambda, \bar{\lambda} = \alpha \pm i\omega, \quad \omega > 0, \tag{3.5}$$

generates a real mode $s(t)$ given by (3.4). We wish to physically interpret $s(t)$ and, for this reason, we express the eigenvector v and the scalar γ in quasi-polar form

$$v = r \circ \exp(-il), \quad l = (\eta_1, \ldots, \eta_n)^T, \quad |\eta_j| \le \pi/2, \tag{3.6}$$

$$\gamma = \frac{1}{2}\rho e^{-i\theta}, \quad |\theta| \le \pi/2. \tag{3.7}$$

Combining (3.5), (3.6) and (3.7) gives the intermediate result

$$ve^{\lambda t}\gamma = (r \circ \exp(-il))\exp(\alpha t + i\omega t)\frac{1}{2}\rho\exp(-i\theta). \tag{3.8}$$

Since all parameters on the right-hand side of the above equation are real, we can rewrite (3.4) with the aid of the vector $u = (1, \ldots, 1)^T$ in the desired form

$$s(t) = r \circ \cos(u(\omega t - \theta) - l)\rho e^{\alpha t}. \tag{3.9}$$

The constant phase difference between any two system components is now obvious. Phase synchronization eliminates these phase differences by evaluating each component $s^k(t)$ at a different, but fixed time-lag. Upon phase synchronization, we obtain a synchronized vector $y(t)$ with components

$$
\begin{aligned}
y^k(t) &= s^k(t + \eta_k/\omega) \\
&= r_k e^{\alpha \eta_k/\omega} \rho e^{\alpha t} \cos(\omega(t + \eta_k/\omega) - \theta + \eta_k) \\
&= r_k e^{\alpha \eta_k/\omega} \rho e^{\alpha t} \cos(\omega t - \theta).
\end{aligned}
\tag{3.10}
$$

More simply, we write

$$
y(t) = z\psi(t),
\tag{3.11}
$$

with

$$
\psi(t) = \rho e^{\alpha t} \cos(\omega t - \theta), \quad z = \exp(\alpha \omega^{-1} l) \circ r.
\tag{3.12}
$$

The above two equations highlight the synchronization. The argument of the cosine in $y(t)$ is scalar, so that all components perform exponentially decaying (or growing) harmonic motion with the same frequency, passing through their equilibrium positions at the same time. In other words, the mode $y(t)$ physically represents synchronous motion of all system components.

To invert the synchronization, we apply the time-shifting operation

$$
s^k(t) = y^k(t - \eta_j/\omega).
\tag{3.13}
$$

Combination of (3.13) and (3.12) gives a formula for the mode $s(t)$ in terms of $\psi(t)$

$$
s(t) = (\psi(t - \eta_1/\omega), \dots, \psi(t - \eta_n/\omega))^T \circ z.
\tag{3.14}
$$

The above equation plays an important role in the decoupling of system (1.1).

3.2 Phase Synchronization of Two Real and Distinct Eigensolutions

Before we can proceed to decouple (1.1), we must consider modes generated by real eigenvalues. It may appear natural to think of a real eigensolution $ve^{\lambda t}$ as a mode. However, to decouple (1.1) into a set of independent second-order equations, two linearly independent eigensolutions have to be paired up to generate a mode of the form

$$
s(t) = v_a e^{\lambda_a t} \gamma_a + v_b e^{\lambda_b t} \gamma_b, \quad \lambda_a \neq \lambda_b.
\tag{3.15}
$$

The goal is to derive a synchronized vector $y(t)$ of the functional form (3.11). The phase shifts necessary to enforce the synchronization are, however, not obvious. A series of algebraic manipulations, summarized below, permits the necessary phase shifts to manifest themselves. The results in equations (3.21)-(3.24) should be noted.

For the generic case (all n components of v_a, v_b being nonzero), consider the kth component of (3.15),

$$s^k(t) = \exp\left(\frac{\lambda_a + \lambda_b}{2}t\right)$$

$$\times \left(\left(v_a^k \exp\left(\frac{\lambda_a - \lambda_b}{2}t\right)\gamma_a + v_b^k \exp\left(-\frac{\lambda_a - \lambda_b}{2}t\right)\gamma_b\right) \right). \qquad (3.16)$$

Whenever v_a^k or v_b^k equals zero, the corresponding component $s^k(t)$ is either $v_a^k e^{\lambda_a t}\gamma_a$ or $v_b^k e^{\lambda_b t}\gamma_b$ and no extra work is required. In analogy to Section 2.1, we define a vector $y(t)$ by evaluating each component of $s(t)$ with a constant, but yet unkown time-lag τ_k so that

$$y^k(t) = s^k(t + \tau_k) = \exp\left(\frac{\lambda_a+\lambda_b}{2}(t+\tau_k)\right)\left(v_a^k \exp\left(\frac{\lambda_a-\lambda_b}{2}\tau_k\right)\exp\left(\frac{\lambda_a-\lambda_b}{2}t\right)\gamma_a\right.$$

$$\left. +v_b^k \exp\left(-\frac{\lambda_a-\lambda_b}{2}\tau_k\right)\exp\left(-\frac{\lambda_a-\lambda_b}{2}t\right)\gamma_b\right).$$

To achieve a synchronizing effect in $y(t)$ without disturbing the coefficients γ_a and γ_b in equation (3.15), we require that

$$v_a^k \exp\left(\frac{\lambda_a - \lambda_b}{2}\tau_k\right) = v_b^k \exp\left(-\frac{\lambda_a - \lambda_b}{2}\tau_k\right). \qquad (3.17)$$

We solve the above equation for τ_k using the principal value of the logarithm of a complex number, $\ln z = \ln|z| + i\mathrm{Arg}(z)$, and obtain

$$\tau_k = \frac{\ln\left|v_b^k/v_a^k\right|}{\lambda_a - \lambda_b} \times \begin{cases} 1, & \text{for } v_a^k v_b^k > 0, \\ i\pi, & \text{for } v_a^k v_b^k < 0. \end{cases} \qquad (3.18)$$

The above τ_k yields the intermediate results

$$v_a^k \exp\left(\frac{\lambda_a-\lambda_b}{2}\tau_k\right) = \mathrm{sign}(v_a^k)\sqrt{\left|v_a^k v_b^k\right|} \times \begin{cases} 1, & \text{for } v_a^k v_b^k > 0, \\ i, & \text{for } v_a^k v_b^k < 0, \end{cases} \qquad (3.19)$$

and

$$\exp\left(\frac{\lambda_a+\lambda_b}{2}\tau_k\right) = \sqrt{\left|\frac{v_b^k}{v_a^k}\right|^{\left(\frac{\lambda_a+\lambda_b}{\lambda_a-\lambda_b}\right)}}$$
$$\times \begin{cases} 1, & \text{for } v_a^k v_b^k > 0, \\ \exp\left(\frac{1}{2}i\pi\frac{\lambda_a+\lambda_b}{\lambda_a-\lambda_b}\right), & \text{for } v_a^k v_b^k < 0. \end{cases}$$

Substitution of equations (3.19) and (3.20) into (3.17) now provides the sought-after expression for the components

$$y^k(t) = (e^{\lambda_a t}\gamma_a + e^{\lambda_b t}\gamma_b)$$

$$\times \sqrt{\left|\frac{(v_b^k)^{\left(\frac{\lambda_a+\lambda_b}{\lambda_a-\lambda_b}+1\right)}}{(v_a^k)^{\left(\frac{\lambda_a+\lambda_b}{\lambda_a-\lambda_b}-1\right)}}\right|} \times \begin{cases} 1, & \text{for } v_a^k v_b^k > 0, \\ \exp\left(\frac{1}{2}i\pi\frac{\lambda_a+\lambda_b}{\lambda_a-\lambda_b}\right), & \text{for } v_a^k v_b^k < 0, \end{cases} \qquad (3.20)$$

of the synchronized mode. By time-shifting every component, we again obtain equation (3.11) with

$$\psi(t) = \gamma_a e^{\lambda_a t} + \gamma_b e^{\lambda_b t}, \quad z = (\zeta_1, \ldots, \zeta_n)^T, \tag{3.21}$$

where

$$\zeta_k = \text{sign}(v_a^k) \sqrt{\left| \frac{(v_b^k)^{\left(\frac{\lambda_a + \lambda_b}{\lambda_a - \lambda_b} + 1\right)}}{(v_a^k)^{\left(\frac{\lambda_a + \lambda_b}{\lambda_a - \lambda_b} - 1\right)}} \right|} \times \begin{cases} 1 & \text{for } v_a^k v_b^k > 0, \\ \exp\left(\frac{1}{2} i\pi \frac{\lambda_a + \lambda_b}{\lambda_a - \lambda_b}\right) & \text{for } v_a^k v_b^k < 0. \end{cases} \tag{3.22}$$

Straightforward calculation shows that the inverse formulae (3.13) and (3.14) remain valid with the substitutions

$$l = (\eta_1, \ldots, \eta_n)^T, \quad \eta_k = \begin{cases} \ln \frac{v_b^k}{v_a^k} & \text{for } v_a^k v_b^k > 0, \\ \ln \left| \frac{v_b^k}{v_a^k} \right| + i\pi & \text{for } v_a^k v_b^k < 0, \end{cases} \tag{3.23}$$

$$\omega = \lambda_a - \lambda_b \neq 0. \tag{3.24}$$

Finally, we would like to point out that phase synchronization reduces to the identity transformation if two distinct eigenvalues share the same real eigenvector.

3.3 Computation of Homogeneous Solution by Phase Synchronization

We wish to express the homogeneous solution in terms of the modes $s(t)$ we just defined. Suppose $2o$ of the $2n$ eigenvalues of (3.1) are complex and $2d = 2(n - o)$ are real (the unusual indices o and d are used to denote *oscillatory* and non-oscillatory *decaying* solutions). For simplicity, we index the eigenvalues such that $\lambda_1, \ldots, \lambda_o$ are complex with increasing positive imaginary parts; $\lambda_{o+1} \leq \cdots \lambda_n \leq \lambda_{n+o+1} \leq \cdots \lambda_{2n}$ are real and $\lambda_{n+1}, \ldots, \lambda_{n+o}$ are the complex conjugates of $\lambda_1, \ldots, \lambda_o$. The indexing scheme is graphically illustrated in Figure 3.1.

For $j = 1, \ldots, o$, define $s_j(t)$ by substituting λ_j, v_j in (3.4). Expressions for $\rho_j, \theta_j, \alpha_j, \omega_j, r_j$ and

$$l_j = (\eta_j^1, \ldots, \eta_j^n) \tag{3.25}$$

can be inferred from equations (3.5), (3.6) and (3.7) by simply reintroducing the index j. Using this notation, define $\psi_j(t)$ and z_j for $j = 1, \ldots, o$ by (3.12). Similarly, for $j = o + 1, \ldots, n$, define $s_j(t)$ by substituting λ_j, v_j for λ_a, v_a and λ_{j+n}, v_{j+n} for λ_b, v_b in (3.15) . With the same substitution, define ψ_j and z_j by (3.21), l_j by (3.23) and, finally, ω_j by (3.24). Assuming that system (3.1) is non-defective, this pairing of real eigenvalues guarantees that eigensolutions with distinct real eigenvalues generate a mode, as required by its definition. We have thus shown that the homogeneous
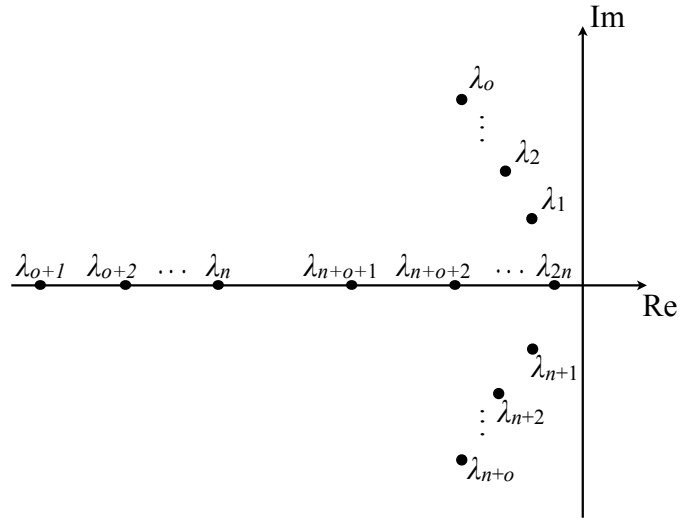
Figure 3.1: Indexing of $2n$ eigenvalues of the quadratic eigenvalue problem (3.1).

solution $q(t)$ is the superposition of n real modes such that

$$q(t) = \sum_{j=1}^{n} s_j(t). \tag{3.26}$$

Note that the above equation is valid for systems with complex and real eigenvalues. By using a similar notation for both real and complex eigensolutions, we are able to write down streamlined formulae for the general case. In particular, with ω in (3.14) representing either the imaginary part of a complex eigenvalue or the difference between two real and distinct eigenvalues (see (3.24)), we obtain

$$q(t) = \sum_{j=1}^{n} (\psi_j(t - \eta_j^1/\omega_j), \ldots, \psi_j(t - \eta_j^n/\omega_j))^T \circ z_j, \tag{3.27}$$

with η_j^k as in (3.25). The above equation represents a mapping from the set of mutually independent functions $\psi_j(t)$ to the homogeneous solution $q(t)$ of equation (1.1) and is linear in the function space spanned by $\{v_j e^{\lambda_j t}\}$, $j = 1, \ldots, 2n$. The equation (3.27) appeared for the first time in [36], however in a more restrictive setting and in different notation.

3.4 Choice of pairing schemes

As indicated above, any two distinct real eigensolutions may be paired up to generate a real mode $s_j(t)$ according to (3.15). We have described only one way. If all

real eigenvalues are distinct, there are in fact $(2d)!/(2^d d!)$ ways to pair the $2d$ distinct eigensolutions to generate d real modes. Each pairing scheme may generate a different set of modes and, consequently, a different set of functions $\psi_j(t)$. Formally, any such set of functions may be used. In applications, the pairing of real eigenvalues warrants further consideration. For example, when the coefficients M, C and K are SPD, the commutativity condition in (1.2) guarantees that the $2n$ eigenvalues occur in n pairs, each pair sharing a common real eigenvector (even if some or all eigenvalues are complex, see [7]). By using this pairing scheme, phase synchronization becomes the identity transformation for all n modes, rendering this specific pairing particularly easy and the scheme of choice in vibration books. A real, invertible, linear mapping (coinciding with classical modal analysis) to decouple the system can be derived from this pairing of eigensolutions (see Chapter 2). In the general case (equation (1.2) not satisfied), phase synchronization is far from being the identity transformation, regardless of the pairing scheme.

It is also reasonable to describe the non-oscillatory part of the solution as a linear combination of $2d$ real eigensolutions $ve^{\lambda t}$, i. e. avoid the somewhat artificial modes $s(t)$ in Section 2.2 altogether. The price for this simplification is to have $o + 2d$ terms in (3.26). With appropriate modifications, $o + 2d$ functions $\psi_j(t)$ can be used to derive an equation similar to (3.27). In other words, phase synchronization may be confined to the oscillatory part of the solution. Following this approach we ultimately generate a set of $o + 2d$ mutually independent equations, o of which are of second order, $2d$ of first order. The second-order structure inherent to (1.1) is thus lost. We prefer to apply phase synchronization to real eigensolutions and preserve the second-order structure.

Chapter 4

The Decoupling of Linear Dynamical Systems

We now proceed to construct a real and diagonal system whose solution generates the solution of the original system (1.1). It is convenient to separate the homogeneous and the inhomogeneous cases.

4.1 Decoupling the Homogeneous Equation

In Section 3.3, we found a mapping from the set of independent functions $\psi_j(t)$ in (3.12) to the homogeneous solution of equation (1.1). Straightforward calculation reveals that the functions $\psi_j(t)$ satisfy the second-order differential equations

$$\ddot{\psi}_j(t) - (\lambda_j + \lambda_{j+n})\dot{\psi}_j(t) + (\lambda_j\lambda_{j+n})\psi_j(t) = 0, \quad j = 1, \ldots, n. \tag{4.1}$$

All coefficients in the above equations are real. To streamline the notation, define

$$\begin{cases} p(t) &= (\psi_1(t), \ldots \psi_n(t))^T, \\ \Lambda_1 &= \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \quad \Lambda_2 = \operatorname{diag}(\lambda_{n+1}, \ldots, \lambda_{2n}), \\ D &= -(\Lambda_1 + \Lambda_2), \quad \Omega = \Lambda_1\Lambda_2, \end{cases} \tag{4.2}$$

and express the n mutually independent, scalar equations (4.1) in a compact matrix form

$$\ddot{p}(t) + D\dot{p}(t) + \Omega p(t) = 0. \tag{4.3}$$

The above represents a decoupled system to which the homogenous equation associated with (1.1) reduces.

The remaining task is to connect the initial conditions of (1.1) and (4.3). The initial conditions render $q(t)$, and hence $p(t)$, unique. By simply evaluating (3.27) and its derivative at $t = 0$, we can only connect q with p at different times. This

difficulty can be avoided by observing that phase synchronization does not disturb the constants γ_j in equation (3.2). Specifically, from (3.2), we have

$$\begin{pmatrix} q(0) \\ \dot{q}(0) \end{pmatrix} = \begin{pmatrix} V_1 & V_2 \\ V_1 L_1 & V_2 L_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{4.4}$$

where the columns of $V_1, V_2 \in C^{n \times n}$ consist of the eigenvectors of (3.1) such that

$$V_1 = (v_1, \ldots, v_n), V_2 = (v_{n+1}, \ldots, v_{2n}), \tag{4.5}$$

and where $c_1^T = (\gamma_1, \ldots, \gamma_n)^T$, $c_2^T = (\gamma_{n+1}, \ldots, \gamma_{2n})^T$. On the other hand, equations (3.12) and (3.21) imply that

$$\begin{pmatrix} p(0) \\ \dot{p}(0) \end{pmatrix} = \begin{pmatrix} I & I \\ \Lambda_1 & \Lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{4.6}$$

Combining (4.6) and (4.4) to eliminate $(c_1^T \; c_2^T)^T$ yields the desired real mapping of initial conditions:

$$\begin{pmatrix} p(0) \\ \dot{p}(0) \end{pmatrix} = \begin{pmatrix} I & I \\ \Lambda_1 & \Lambda_2 \end{pmatrix} \begin{pmatrix} V_1 & V_2 \\ V_1 \Lambda_1 & V_2 \Lambda_2 \end{pmatrix}^{-1} \begin{pmatrix} q(0) \\ \dot{q}(0) \end{pmatrix}. \tag{4.7}$$

The decoupling is now complete. To solve the homogeneous equation associated with (1.1), solve the quadratic eigenvalue problem (3.1), construct and solve the real, diagonal system (4.3) with initial conditions (4.7), and map $p = (\psi_1, \ldots \psi_n)^T$ back to q using (3.27).

4.2 Decoupling the Inhomogeneous Equation

The homogeneous part of (1.1) can always be decoupled by phase synchronization as described earlier. Thus the remaining task is to determine how the excitation $f(t)$ transforms under phase synchronization. Put differently, we postulate that the coupled system (1.1) can be decoupled into the form

$$\ddot{p}(t) + D\dot{p}(t) + \Omega p(t) = g(t), \tag{4.8}$$

where D and Ω are given by (4.2) and where $g(t)$ is real. To find $g(t)$, recast (1.1) in a first-order form

$$\begin{pmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix} \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}f(t) \end{pmatrix}. \tag{4.9}$$

Should M be ill-conditioned, other forms of first-order conversion may be used [24,38]. Inspired by (4.7), we define a real and invertible mapping by

$$\begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} = \begin{pmatrix} V_1 & V_2 \\ V_1 \Lambda_1 & V_2 \Lambda_2 \end{pmatrix} \begin{pmatrix} I & I \\ \Lambda_1 & \Lambda_2 \end{pmatrix}^{-1} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}. \tag{4.10}$$

Substitution into (4.9) yields the equations

$$p_2(t) = \dot{p}_1(t) - g_1(t), \tag{4.11}$$

$$\ddot{p}_1(t) + D\dot{p}_1(t) + \Omega p_1(t) = (D + I d/dt)g_1(t) + g_2(t), \tag{4.12}$$

where $g_1(t)$ and $g_2(t)$ are given by

$$g_1(t) = ((V_1\Lambda_1 - V_2\Lambda_2 V_2^{-1}V_1)^{-1} + (V_2\Lambda_2 - V_1\Lambda_1 V_1^{-1}V_2)^{-1})M^{-1}f(t),$$
$$g_2(t) = (\Lambda_1(V_1\Lambda_1 - V_2\Lambda_2 V_2^{-1}V_1)^{-1} + \Lambda_2(V_2\Lambda_2 - V_1\Lambda_1 V_1^{-1}V_2)^{-1})M^{-1}f(t). \tag{4.13}$$

Note that $g_1(t)$ and $g_2(t)$ are real and depend continuously on $f(t)$. Specifically, equation (4.11) defines how $p_2(t)$ is connected to the displacements and velocities of the decoupled system (4.8), while (4.12) represents the dynamics of the decoupled system, i.e. $p_1(t) = p(t)$. We thus obtain (4.8) with

$$g(t) = (D + I d/dt)g_1(t) + g_2(t). \tag{4.14}$$

The mapping from p to q, as inferred from equations (4.10), (4.11) and (4.12), is

$$q(t) = (T_1 + T_2 \; d/dt)p(t) - T_2 g_1(t), \tag{4.15}$$

where the real matrices T_1 and T_2 are given by

$$T_1 = (V_1\Lambda_2 - V_2\Lambda_1)(\Lambda_2 - \Lambda_1)^{-1}, \tag{4.16}$$

$$T_2 = (V_2 - V_1)(\Lambda_2 - \Lambda_1)^{-1}. \tag{4.17}$$

As a mapping between p and q, (4.15) is real, time-dependent and nonlinear. More specifically, the transformation is affine in $p(t)$ because $(T_1 + T_2 \; d/dt)p(t)$ is linear and followed by the shift $-T_2 g_1(t)$.

The initial conditions of (1.1) and (4.8) are connected by

$$\begin{pmatrix} p(0) \\ \dot{p}(0) \end{pmatrix} = \begin{pmatrix} I & I \\ \Lambda_1 & \Lambda_2 \end{pmatrix} \begin{pmatrix} V_1 & V_2 \\ V_1\Lambda_1 & V_2\Lambda_2 \end{pmatrix}^{-1} \begin{pmatrix} q(0) \\ \dot{q}(0) \end{pmatrix} + \begin{pmatrix} 0 \\ g_1(0) \end{pmatrix}, \tag{4.18}$$

thus completing the decoupling of (1.1). The formulas for decoupling presented in this section are direct generalizations of those given in [37], applicable only when M, C and K are SPD.

4.3 Nonlinearity and Non-Uniqueness in Decoupling

Based upon physical intuition, the dependence of the nonlinear mapping (4.15) on the excitation $f(t)$ can be explained as follows. If $f(t) = 0$, viscous damping

and gyroscopic forces induce constant time-shifts between the components of a mode $s(t)$. These time-shifts are accounted for by the linear mapping (3.27). For $f(t) \neq 0$, the mapping must account for additional and not necessarily constant time-shifts caused by the external force $f(t)$. Thus, a mapping leading to the decoupling of the inhomogeneous equation must depend on $f(t)$. If $f(t) = 0$, the nonlinear mapping in (4.15) for the inhomogeneous equation reduces to the linear mapping in (3.27).

In decoupling a homogeneous system, we have observed a degree of non-uniqueness when generating the modes. This non-uniqueness is carried over to the decoupling of (1.1). Recall that two systems with identical eigenvalues and multiplicities are termed strictly isospectral. Since the property of being strictly isospectral is reflexive, transitive and symmetric, strictly isospectral systems generate an equivalence class [20]. It is easy to verify that system (1.1) and the decoupled systems (4.3) are strictly isospectral regardless of the pairing of real eigensolutions during phase synchronization. Indeed, every system within the equivalence class can be generated by suitably pairing the real eigensolutions. Thus, phase synchronization generates the path to *all* real and diagonal systems within the equivalence class of systems strictly isospectral to (1.1).

Finally, it is important to point out that decoupling by phase synchronization reduces to a linear mapping if M, C and K are simultaneously diagonalizable to real diagonal matrices by a real equivalence transformation (because phase synchronization becomes the identity in this case).

4.4   Decoupling Algorithm

The procedure for decoupling the linear system (1.1) by phase synchronization may be summarized as an algorithm.

1. Solve the quadratic eigenvalue problem (3.1) and generate the real and diagonal system (4.3).

2. If $f(t) = 0$, decoupling is complete. The solution p of (4.3), with initial conditions (4.7) can be easily obtained. We may recover the homogeneous solution q of (1.1) from p by simply evaluating (3.27).

3. If $f(t) \neq 0$, the decoupled system is (4.8), which can be obtained from (4.3) by incorporating an excitation $g(t)$ given by (4.14). The solution p of the real and diagonal system (4.8), with initial conditions (4.18), can be readily computed. We may recover the solution q of (1.1) from p by using (4.15).

The decoupling algorithm is illustrated in Figure 4.1. Although complex quantities appear in the algorithmic developement, the entire process can be implemented in real arithmetic. Computing the solution by decoupling is particularly attractive if (1.1) is very stiff (eigenvalues vary over many orders of magnitude). In this case,
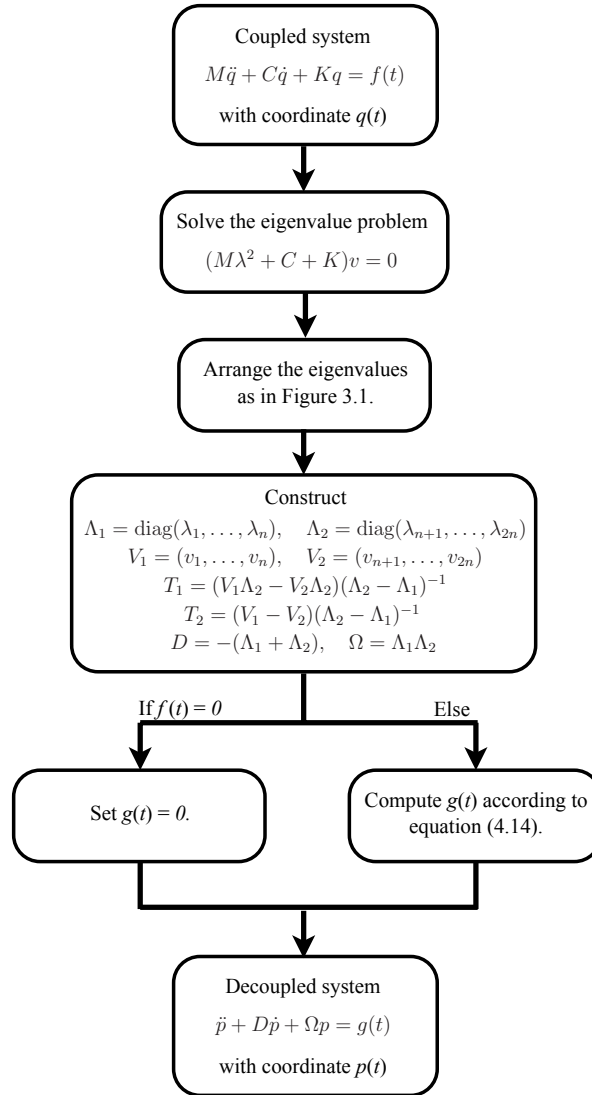
Figure 4.1: Flowchart for decoupling a second-order linear system. All required parameters are obtained through solution of a quadratic eigenvalue problem.

different time stepping methods can be applied to each decoupled equation resulting in substantial improvement in algorithms for response computation. In addition, low-energy components of $p(t)$ may be neglected to obtain powerful model reduction schemes (see Section 6). Further applications of the algorithm will be taken up in future papers.

4.5   Examples

Three examples will be given to illustrate the concept of modes of vibration as well as the process of decoupling by phase synchronization. Complete details are given in the Example 1 to provide physical insight and to reinforce the mathematical concepts expounded earlier.

4.5.1   Example 1: Oscillatory Free Vibration

Consider a homogeneous mass-spring-damper system governed by an equation of the type (1.1), with

$$M = m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{pmatrix}, \quad K = k \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad (4.19)$$

$f(t) = 0$ and initial conditions

$$q(0) = (1, \quad 2)^T, \quad \dot{q}(0) = (-1, \quad 1)^T. \tag{4.20}$$

The system is shown in Figure 4.2. For convenience, let $m = 1$. Three different cases will be examined.
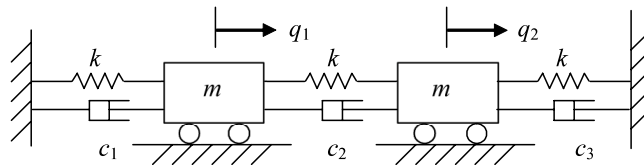
Figure 4.2: The mass-spring-damper system of Example 1.

(a) System is undamped: $c_1 = c_2 = c_3 = 0$. Solution of the symmetric eigenvalue problem (3.1) yields, upon normalization with respect to the mass matrix,

$$\omega = \mathrm{diag}(1, 3), \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4.21}$$

The general solution is a superposition of two modes such that

$$q(t) = \sum_{j=1}^{2} s_j(t) = C_1 \cos(t - \theta_1)u_1 + C_2 \cos(\sqrt{3}t - \theta_2)u_2. \qquad (4.22)$$

The constants C_1, θ_2, C_2 and θ_2 are determined by the initial conditions. As shown in Figure 4.3, the system components in each mode are either in phase or out of phase so that vibration appears truly synchronous. This system can be decoupled by classical modal analysis as explained in Chapter 2.

Figure 4.3: Modes and undamped free response of Example 1a shown in three parts. (a) First mode $s_1(t)$ with first element $s_1^1(t)$ (solid line) and second element $s_1^2(t)$ (dashed line). (b) Second mode $s_2(t)$ with first element $s_2^2(t)$ (solid line) and second element $s_2^2(t)$ (dashed line). (c) Free Response $q(t)$ with first element $q^1(t)$ (solid line) and second element $q^2(t)$ (dashed line).

(b) Classically damped system: $c_1 = c_2 = c_3 = 0.1$. Since $C = 0.1K$, the system is proportionally damped. The general solution is given by

$$q(t) = \sum_{j=1}^{2} s_j(t) = C_1 e^{-0.05t} \cos(1.00t - \theta_1)u_1 + C_2 e^{-0.15t} \cos(1.73t - \theta_2)u_2. \qquad (4.23)$$
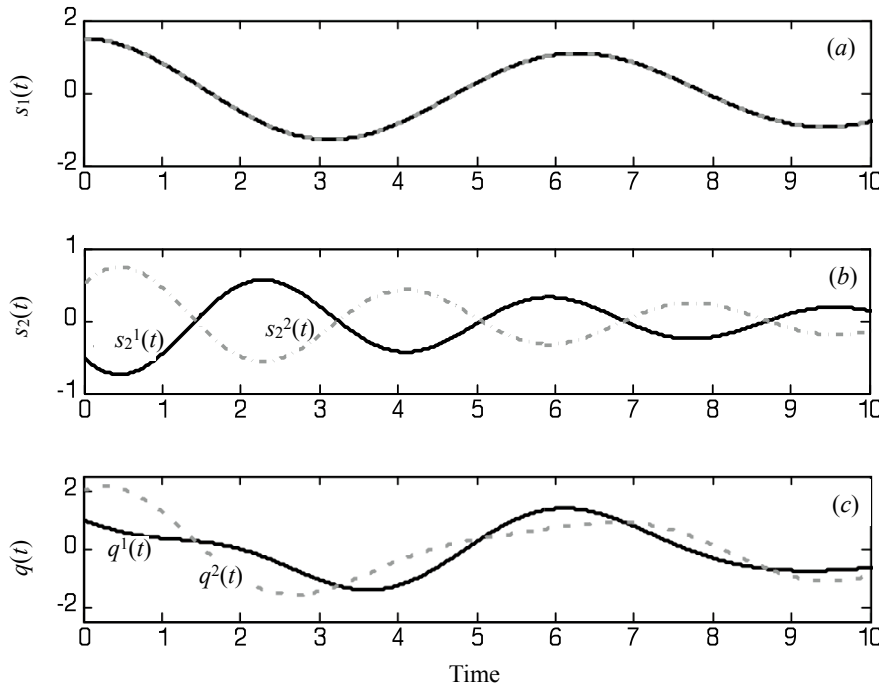
Figure 4.4: Modes and free response of Example 1b shown in three parts. (a) First mode $= s_1(t)$ with first element $s_1^1(t)$ (solid line) and second element $s_1^2(t)$ (dashed line). (b) Second mode $s_2(t)$ with first element $s_2^2(t)$ (solid line) and second element $s_2^2(t)$ (dashed line). (c) Free Response $q(t)$ with first element $q^1(t)$ (solid line) and second element $q^2(t)$ (dashed line).

As shown in Figure 4.4, the system components in each mode are again either in phase or out of phase but, in contrast to case (a), they decay exponentially. The system can still be decoupled by classical modal analysis.

(c) Non-classically damped system: $c_1 = 0.6$, $c_2 = c_3 = 0.1$. Since condition (1.2) is not satisfied, the system cannot be decoupled by classical modal analysis. Solution of the quadratic eigenvalue problem (3.1) yields

$$\lambda_1 = \bar{\lambda}_3 = -0.18 + 1.00i \quad , \quad v_1 = \bar{v}_3 = \left(0.74e^{-i7.38°}, \quad -0.72e^{-i172.51°} \right)^T \quad (4.24)$$

$$\lambda_2 = \bar{\lambda}_4 = -0.27 + 1.68i \quad , \quad v_2 = \bar{v}_4 = \left(-0.73e^{-i167.13°}, \quad -0.73e^{-i12.68°} \right)^T (4.25)$$

where, for convenience, the eigenvectors are normalized in accordance with

$$2\lambda_j v_j^T M v_j + v_j^T C v_j = 2i\omega_j. \tag{4.26}$$

From (3.9), the two modes are given by

$$s_1(t) = C_1 e^{-0.18t} \begin{pmatrix} 0.74 \cos(1.00t - \theta_1 - 7.38°) \\ -0.72 \cos(1.00t - \theta_1 - 172.51°) \end{pmatrix}, \tag{4.27}$$

$$s_2(t) = C_2 e^{-0.27t} \begin{pmatrix} -0.73 \cos(1.68t - \theta_2 - 167.13°) \\ -0.73 \cos(1.68t - \theta_2 - 12.68°) \end{pmatrix}. \tag{4.28}$$

The general solution is a superposition of these two modes. As can be easily seen in Figure 4.5, there is a constant phase difference between the two components in each mode. Upon decoupling, the equation of motion becomes $\ddot{p}(t) + D\dot{p}(t) + \Omega p(t) = 0$,
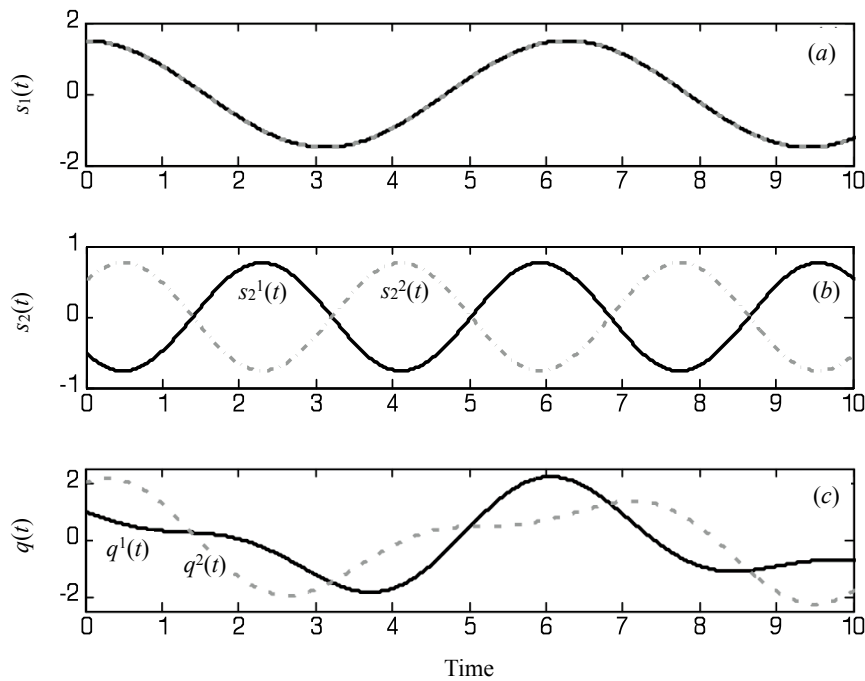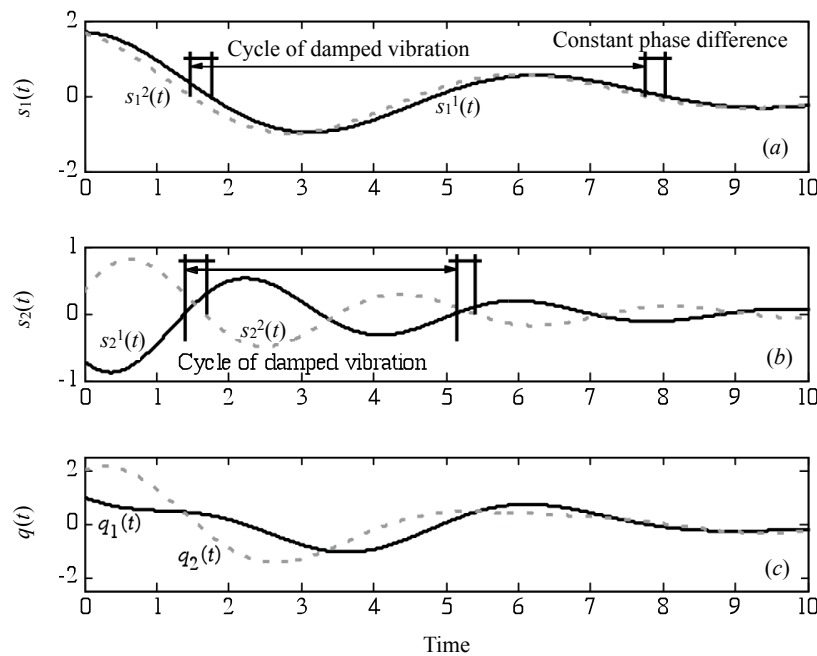

Figure 4.5: Modes and free response of Example 1c shown in three parts. (a) First mode = $s_1(t)$ with first element $s_1^1(t)$ (solid line) and second element $s_1^2(t)$ (dashed line). (b) Second mode $s_2(t)$ with first element $s_2^2(t)$ (solid line) and second element $s_2^2(t)$ (dashed line). (c) Free Response $q(t)$ with first element $q^1(t)$ (solid line) and second element $q^2(t)$ (dashed line).

with

$$D = \text{diag}(0.36, \quad 0.54), \quad \Omega = \text{diag}(1.03, \quad 2.90) \tag{4.29}$$

The initial conditions of the decoupled system are $p(0) = (2.32, \quad -0.71)^T$, $\dot{p}(t) = (-0.50, \quad 2.09)^T$. The solution $q(t)$ of the original system can be readily recovered from solution $p(t)$ of the decoupled system by (3.27). It can be checked that, whether

generated by decoupling or by direct numerical solution of the original equation of motion, is the same.

4.5.2 Example 2: Non-oscillatory Free Vibration

Consider a homogeneous mass-spring-damper system governed by an equation of the type (1.1), with

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & -1 \\ -1 & 8 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \tag{4.30}$$

and initial conditions

$$q(0) = (1, \quad -1)^T, \quad \dot{q}(0) = (1, \quad 1)^T. \tag{4.31}$$

This system does not satisfy (1.2) and, thus, cannot be decoupled by congruence transformations (classical modal analysis, see Chapter 2). Solution of the quadratic eigenvalue problem yields the modes

$$
\begin{aligned}
s_1(t) &= -0.25ie^{-4.16t} \\
&\quad \times \begin{pmatrix} 0.39\cos(i3.58t + \pi/2 + 1.59i + 0.45i) \\ 1.00i\cos(i3.58t + \pi/2 + 1.59i + \pi/2 + 0.04i) \end{pmatrix}, \\
s_2(t) &= 1.72ie^{-1.84t} \\
&\quad \times \begin{pmatrix} 0.94\cos(i1.58t + \pi/2 + 0.81i - 0.04i) \\ -0.73\cos(i1.58t + \pi/2 + 0.81i - +\pi/2 - 0.45i) \end{pmatrix},
\end{aligned}
$$

(4.32)

(4.33)

and the decoupled system $\ddot{p}(t) + D\dot{p}(t) + \Omega p(t) = 0$, with

$$D = \operatorname{diag}(8.33, \quad 3.67), \quad \Omega = \operatorname{diag}(4.54, \quad 0.88), \tag{4.34}$$

and initial conditions $p(0) = (-0.59, \quad 1.55)^T$, $\dot{p}(t) = (0.16, \quad 0.81)^T$. Upon solution of the decoupled equations, we can recover $q(t)$ from $p(t)$ by (3.27).

4.5.3 Example 3: Forced Vibration

Consider the mass-spring-damper system of Example 1 (c) and under the excitation

$$f(t) = (\cos(t), \quad \sin(2t))^T. \tag{4.35}$$

By using (4.16) and (4.17), we compute

$$T_1 = \begin{pmatrix} 0.72 & 0.74 \\ 0.73 & -0.69 \end{pmatrix}, \quad T_2 = \begin{pmatrix} -0.09 & 0.10 \\ 0.09 & 0.10 \end{pmatrix}, \tag{4.36}$$

and the excitation

$$g(t) = \begin{pmatrix} 0.72\cos(t) + 0.18\cos(2t) + 0.09\sin(t) + 0.73\sin(2t) \\ 0.74\cos(t) + 0.20\cos(2t) - 0.10\sin(t) - 0.69\sin(2t) \end{pmatrix}. \tag{4.37}$$

The initial conditions, $p(0) = (2.32, \quad -0.71)^T$, $\dot{p}(t) = (-0.43, \quad -1.61)^T$, of the decoupled system are computed using (4.18). The decoupled system can be readily solved and solution of the original system can be recovered from $p(t)$ by (4.15). Steady-state behaviors of $g(t)$, $p(t)$ and $q(t)$ are shown in Figure 4.6. It can be checked that the response, whether generated by decoupling or by direct numerical integration, is the same.
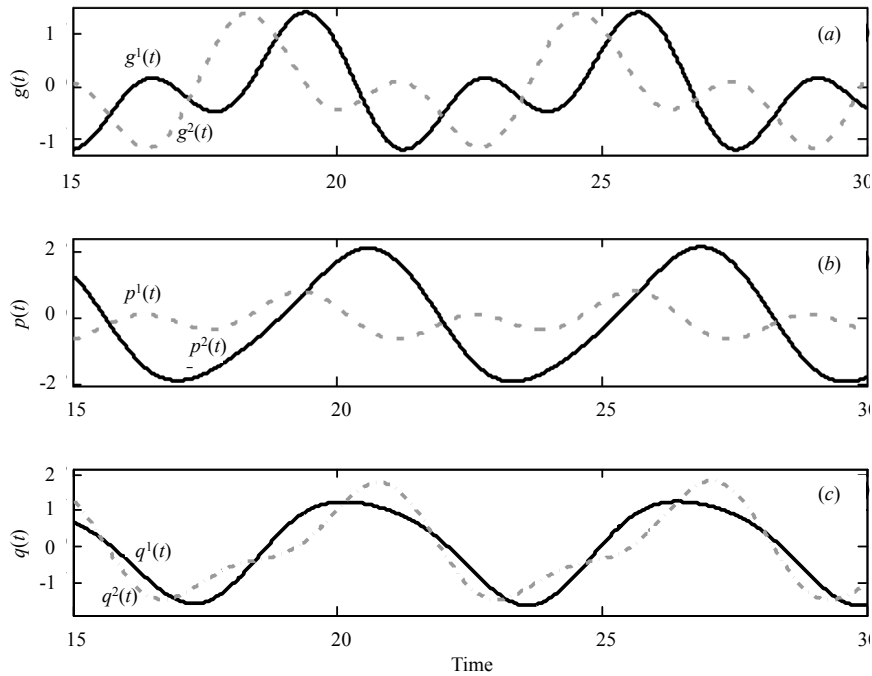
Figure 4.6: Forced response and excitation of Example 3 with a smooth excitation as defined in (4.35). (a) Excitation $g(t)$ of the decoupled system with first element $g^1(t)$ (solid line) and second element $g^2(t)$ (dashed line). (b) Steady-state response of the decoupled system with first element $p^1(t)$ (solid line) and second element $p^2(t)$ (dashed line). (c) Steady-state response $q(t)$ of the original system with first element $q^1(t)$ (solid line) and second element $q^2(t)$ (dashed line).

An implicit assumption in (4.14) is that $f(t)$ be differentiable. However, this assumption can be easily relaxed if all derivatives are interpreted as distributional derivatives [49, 52].

4.6 Efficiency of Solution by Decoupling

Although system solution is probably not the most important reason for decoupling, it may still be instructive to compare the efficiency of solution of (1.1) by direct numerical integration and by decoupling. One measure of the performance of an algorithm is the number of floating point operations (flops) required to evaluate the response at m time points within a given time window.

4.6.1 Efficiency of Decoupling the Homogeneous Equation

The flops associated with three procedures will be compared. (a) In direct numerical integration, a standard procedure is to rewrite (1.1) in first-order form using the state companion matrix

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix}. \tag{4.38}$$

The state equation

$$\begin{pmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix} = A \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}, \tag{4.39}$$

is then discretized, and the resulting system of $2n$ coupled difference equations is solved by matrix computations [5]. This procedure involves one-time computation of the exponential matrix $\exp(A\Delta t)$, where Δt is the sampling time, and one matrix-vector multiplication at each step. The estimate of flops for response calculation at m instants is [1, 14, 23, 40]

$$N_1 = 160n^3 + 8mn^2, \tag{4.40}$$

where n is the number of degrees of freedom and $m > n$ in general.

In solving (1.1) by decoupling, two alternative procedures may be used. (b) It is possible to evaluate the n responses by directly invoking (3.27) and the exact responses

$$p^j(t) = e^{\alpha_j t} \left(p^j(0) \cos(\omega_j t) + \frac{\dot{p}^j(0) - \alpha_j p^j(0)}{\omega_j} \sin(\omega_j t) \right). \tag{4.41}$$

This procedure involves one-time solution of (3.1) and (4.7), plus evaluation of (3.27) and (4.41) at each step. The estimate of flops for this procedure is [1, 14, 23, 53]

$$N_2 = 213n^3 + 2mn^2. \tag{4.42}$$

(c) Another method of solution is to decouple (1.1) through solution of the quadratic eigenvalue problem (3.1) followed by direct numerical integration of each decoupled equation (this procedure is also used in forced vibration). If each decoupled equation is solved numerically with the same algorithm used in procedure (a) for direct integration of (1.1), the estimate of flops is

$$N_3 = 213n^3 + 2mn^2 + 8mn + 1280n. \tag{4.43}$$
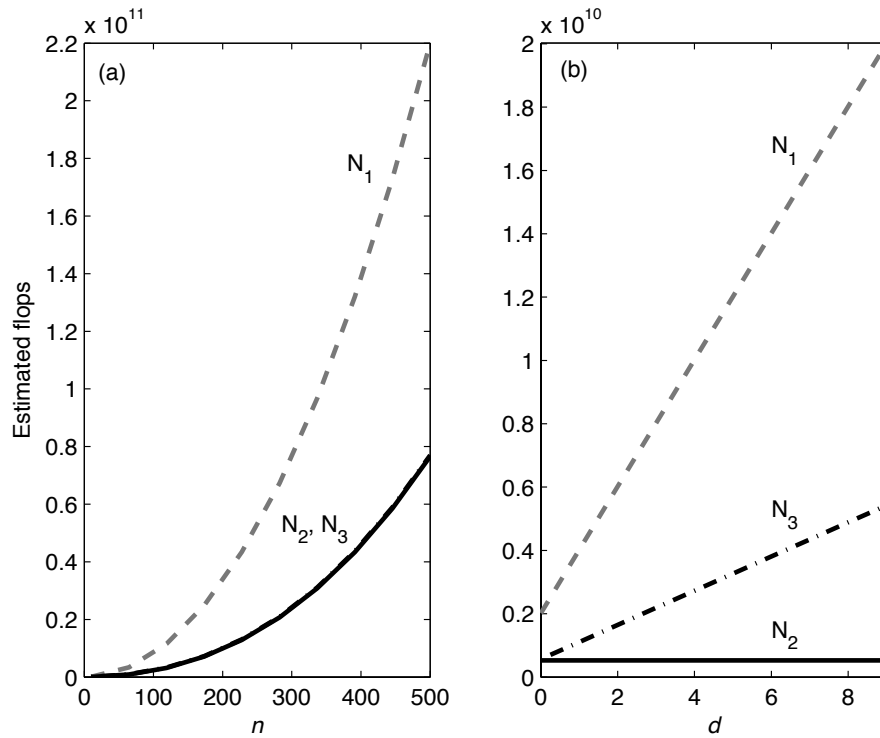
Figure 4.7: Comparison of efficiency of three methods of solution: direct numerical integration (dashed gray line); decoupling by invoking Eqs. (3.27) and (4.41) (solid black line); and decoupling followed by numerical integration of the decoupled equations (dashed black line). (a) Estimated flops to evaluate the response at m instants vs. degree of freedom n. The curves associated with N_2 and N_3 agree within the line thickness. (b) Estimated flops vs. starting time d of window, where the initial conditions are prescribed at $t = 0$, and $m = 10^5$, $n = 50$.

 The variations of N_1, N_2, and N_3 with n are illustrated in Figure 4.7a for a window containing $m = 10^5$ instants. It is observed that the curves associated with N_2, N_3 agree within the line thickness and that procedures (b) and (c) are more efficient than (a). In fact, the estimate of flops shown in Figure 4.7a is very conservative for two reasons. First, N_3 has been estimated by using the same sampling time in the integration of all decoupled equations. If an optimal sampling time is individually chosen for each decoupled equation, N_3 may decrease substantially. Second, Figure 4.7a is generated by using a window of $m = 10^5$ points that begins from $t = 0$, the time at which initial values are prescribed. For any window that begins from a time $d \gg 0$, numerical integration must still start from the initial time $t = 0$. A large number of iterations may be required over the interval $0 < t < d$ before the window of interest is reached. Thus for $d \gg 0$, N_1 and N_3 increase appreciably while N_2

remains constant. This situation is depicted in Figure 4.7b, in which N_1, N_2, and N_3 are plotted against d. It is observed that N_1 increases more rapidly than N_3. Based upon Figure 4.7, it may be stated that solution by decoupling generally reduces the flops and economizes on both core memory and computing time.

4.6.2   Efficiency of Decoupling the Inhomogeneous Equation

The flops associated with two procedures are compared. (a) For direct numerical integration, we recast the second-order equation (1.1) in state space as a first-order system of dimension $2n$ by using the state companion matrix A in (4.38). The state equation

$$\left(\begin{array}{c} \dot{q}(t) \\ \ddot{q}(t) \end{array} \right) = A \left(\begin{array}{c} q(t) \\ \dot{q}(t) \end{array} \right) + \left(\begin{array}{c} 0 \\ M^{-1}f(t) \end{array} \right), \qquad (4.44)$$

is then discretized, and the resulting difference equation is solved by matrix computations [5]. The flops for this standard procedure for response calculation at m instants in forced vibration is [1, 14, 23, 40]

$$N_4 = 160n^3 + 16mn^2, \qquad (4.45)$$

where n is the number of degrees of freedom and $m > n$ in general.

(b) In solving (1.1) by decoupling, the decoupled system (4.8) is obtained through solution of the quadratic eigenvalue problem (3.1) and evaluation of (4.18). Each independent decoupled system in (4.8) is then solved numerically at m instants with the same algorithm used in procedure (a). Subsequently, Eq. (4.15) is employed to compute the response $q(t)$. The estimate of flops is [1, 14, 23, 53]

$$N_5 = 10mn^2 + 16mn + 213n^3 + 4n^2. \qquad (4.46)$$

The variations of N_4 and N_5 with n are illustrated in Figure 4.8 for a window containing $m = 10^6$ instants. It is observed that response calculation by decoupling generally reduces the flops and economizes on both core memory and computing time. In fact, Figure 4.8 is rather conservative because N_5 has been estimated by using the same sampling time in the integration of all decoupled equations. If an optimal sampling time can be individually chosen for each decoupled equation, N_5 may decrease substantially. Moreover, each decoupled equation may be solved exactly in many applications in terms of elementary functions (rather than convolution integrals). On the other hand, the efficiency of response calculation by decoupling depends on the size of the time window. In addition, validity of the above flop estimates requires that the excitation $f(t)$ and response $q(t)$ be sufficiently smooth. Distributional excitation such as an impulse and weak solutions (less than twice differentiable) are excluded [49, 52]. Thus Figure 4.8 should be interpreted as indicative rather than absolute in the comparison of efficiency.
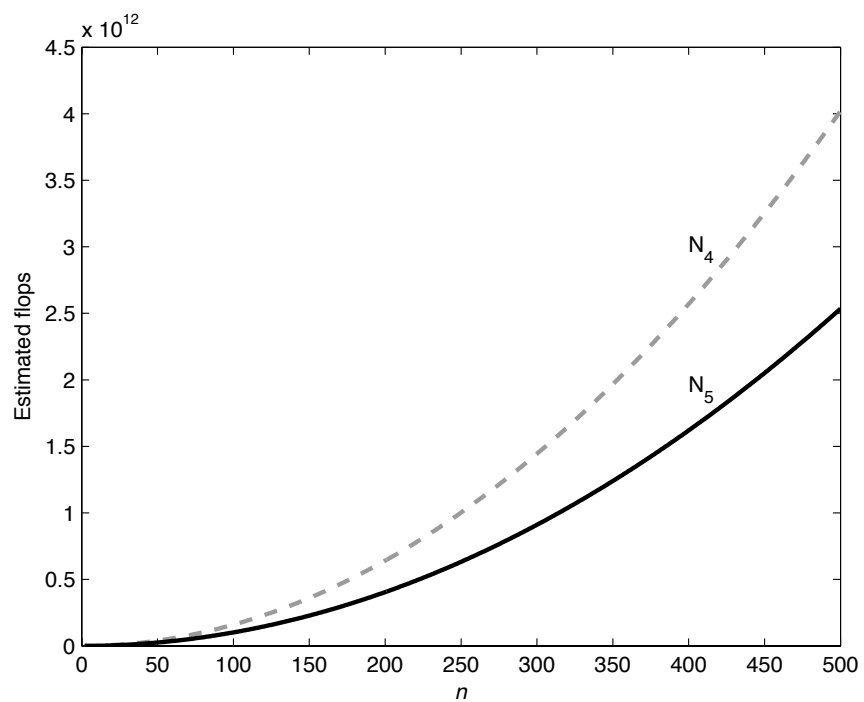
Figure 4.8: Comparison of efficiency in response calculation under forced vibration by direct numerical integration (dashed gray line) and by decoupling (solid black line). Estimated flops to evaluate the response at $m = 10^6$ instants are plotted against the degree of freedom n.

Chapter 5

Decoupling in Configuration and State Spaces

It was shown in Chapter 4 that any linear system can be decoupled in the configuration space by a real, nonlinear, time-dependent transformation. We will show in this chapter that the time-dependent configuration-space decoupling transformation is real, linear and time-invariant when cast in state space. In non-homogeneous systems, both the configuration and associated state transformations are nonlinear (in the displacements $p(t)$ respectively the state $\left(p(t)^T,\ \dot{p}(t)^T\right)^T$) and depend continuously on the excitation. An example is given of a linear system that can be decoupled in configuration but not in state space.

5.1   Simplifying Assumptions

Although the decoupling by phase synchronization can be extended to defective systems [28, 36, 37] so that (1.1) can be decoupled without restrictions, this type of generality will be suppressed in the present chapter. Unless otherwise stated, it will be assumed that (a) all eigenvalues of (3.1) are complex (with non-zero imaginary parts) and distinct, (b) $f(t) = 0$ and (c) M, C, and K are SPD. These assumptions are made to streamline the presentation and, as explained later on, they can be readily relaxed.

5.2   State-Space Formulation of Phase Synchronization

What is the state-space version of the time-dependent decoupling transformation (4.15)? Since physical insight is diminished due to (complex) state transformations, it would be laborious to recast and interpret in state space every equation associated with phase synchronization. This is however not necessary. An efficient

reformulation is provided if a trial state-space version of Eq. (4.15) is first surmised through intuition. The trial version is then rigorously validated.

5.2.1   Derivation of Reformulated Transformation

The free response of the homogeneous system (1.1) is

$$q(t) = V_1 e^{\Lambda_1 t} a_1 + V_2 e^{\Lambda_2 t} a_2, \tag{5.1}$$

where Λ_1 and Λ_2 are defined in (4.2), and V_1 and V_2 are given in (4.5). The n-dimensional columnvectors a_1 and a_2 contain $2n$ constants depending on the initial conditions. Because of the simplifying assumptions in Section 5.1, $\Lambda_2 = \bar{\Lambda}_1$, $V_2 = \bar{V}_1$ and $a_2 = \bar{a}_1$. The state (displacements and velocities) of the homogeneous system (1.1) is given by

$$\left(\begin{array}{c} q(t) \\ \dot{q}(t) \end{array} \right) = \left(\begin{array}{cc} V_1 & V_2 \\ V_1 \Lambda_1 & V_2 \Lambda_2 \end{array} \right) \left(\begin{array}{c} a_1 \\ a_2 \end{array} \right). \tag{5.2}$$

We use the fact that phase synchronization does not disturb the constants a_1, a_2 (see Chapter 4), to write down the state of the homogeneous decoupled system

$$\left(\begin{array}{c} p(t) \\ \dot{p}(t) \end{array} \right) = \left(\begin{array}{cc} I & I \\ \Lambda_1 & \Lambda_2 \end{array} \right) \left(\begin{array}{c} a_1 \\ a_2 \end{array} \right). \tag{5.3}$$

Equations (5.2) and (5.3) can be combined to yield the state transformation

$$\left(\begin{array}{c} q(t) \\ \dot{q}(t) \end{array} \right) = \left(\begin{array}{cc} V_1 & V_2 \\ V_1 \Lambda_1 & V_2 \Lambda_2 \end{array} \right) \left(\begin{array}{cc} I & I \\ \Lambda_1 & \Lambda_2 \end{array} \right)^{-1} \left(\begin{array}{c} p(t) \\ \dot{p}(t) \end{array} \right) = T \left(\begin{array}{c} p(t) \\ \dot{p}(t) \end{array} \right). \tag{5.4}$$

It can be checked that the transformation matrix T is real and nonsingular. Thus the time-dependent configuration-space transformation (3.27) becomes a linear time-invariant transformation (5.4) when cast in state space.

This surprising result, surmised through intuition, can be readily validated. In free vibration, the state-space versions of (1.1) and (4.8) are given, respectively, by

$$\left(\begin{array}{c} \dot{q}(t) \\ \ddot{q}(t) \end{array} \right) = \left(\begin{array}{cc} 0 & I \\ -M^{-1}K & -M^{-1}C \end{array} \right) \left(\begin{array}{c} q(t) \\ \dot{q}(t) \end{array} \right) = A \left(\begin{array}{c} q(t) \\ \dot{q}(t) \end{array} \right), \tag{5.5}$$

$$\left(\begin{array}{c} \dot{p}(t) \\ \ddot{p}(t) \end{array} \right) = \left(\begin{array}{cc} 0 & I \\ -\Omega & -D \end{array} \right) \left(\begin{array}{c} p(t) \\ \dot{p}(t) \end{array} \right) = B \left(\begin{array}{c} p(t) \\ \dot{p}(t) \end{array} \right), \tag{5.6}$$

where D, Ω are defined in (4.2). Observe that the quadratic eigenvalue problems associated with (1.1) and (4.8) have the same eigenvalues with the same multiplicities. In addition, the quadratic eigenvalue problem (3.1) and the matrix A in (5.5) have identical eigenvalues, and the same is true for the quadratic eigenvalue problem associated with the decoupled system (4.8) and the matrix B in (5.6). As a consequence,

A, B have the same eigenvalues, i.e., they are isospectral, and each is diagonalizable because if the eigenvalues are distinct. From linear algebra, two diagonalizable matrices are isospectral if and only if they are similar. It can be checked by direct manipulations that

$$T^{-1}AT = B \tag{5.7}$$

where T is defined in (5.4). Thus the state transformation (5.4) converts (5.5) into (5.6) through a similarity transformation. The state-space version of the decoupling transformation (3.27) is indeed given by (5.4). While (3.27) decouples the homogeneous equation (1.1) in configuration space, the state transformation (5.4) does not decouple the state-space version of system (1.1) because B is not diagonal. Rather, Eq. (5.4) operates in such a way that (5.5) is converted into (5.6), from which the homogeneous decoupled system is extracted.

5.2.2 Relaxation of Assumptions

Subject to the simplifying assumptions of Section 5.1, the time-dependent decoupling transformation (3.27) becomes a linear time-invariant transformation in state space. It can be shown that the same observation is true for free vibration under real, complex, or repeated eigenvalues, as long as (3.1) is non-defective. If there exist $2r \leq 2n$ distinct real eigenvalues, there is an equivalence class of $(2r)!/2^r r!$ different forms of D, Ω associated with the real eigenvalues [37]. For each member of this equivalence class, the corresponding time-dependent configuration-space decoupling transformation is equivalent to a linear time-invariant state transformation. When (3.1) is defective, Jordan sub-matrices appear in many formulas associated with decoupling [28, 36, 37]. As a result, both the configuration-space decoupling transformation and its state-space version are time-dependent.

If $f(t) \neq 0$, the nonlinear configuration-space decoupling transformation (4.8) depends continuously on the excitation $f(t)$. Consequently, its reformulated state-space version also involves $f(t)$. From (4.10), it can be shown that

$$\begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} = \begin{pmatrix} V_1 & V_2 \\ V_1\Lambda_1 & V_2\Lambda_2 \end{pmatrix} \begin{pmatrix} I & I \\ \Lambda_1 & \Lambda_2 \end{pmatrix}^{-1} \begin{pmatrix} p(t) \\ \dot{p}(t) \end{pmatrix} - \begin{pmatrix} 0 \\ g_2(t) \end{pmatrix}, \tag{5.8}$$

is the state-space version of the decoupling transformation for forced vibration (4.15). Phase synchronization can be used to decouple systems with symmetric or non-symmetric coefficients, provided that M is nonsingular. The observations in this section thus remain valid when M, C and K are not symmetric.

5.3 Phase Synchronization and Structure Preserving Transformations

Garvey et al. [18, 19] defined structure preserving transformations (SPT) and applied them to decouple certain linear dynamical systems of second order. In our notation, an equivalence transformation $\{U_L, U_R\}$ is structure-preserving if and only if

$$U_L^T \begin{pmatrix} 0 & K \\ K & C \end{pmatrix} U_R = \begin{pmatrix} 0 & K_D \\ K_D & C_D \end{pmatrix}, \tag{5.9}$$

$$U_L^T \begin{pmatrix} K & 0 \\ 0 & -M \end{pmatrix} U_R = \begin{pmatrix} K_D & 0 \\ 0 & -M_D \end{pmatrix}, \tag{5.10}$$

$$U_L^T \begin{pmatrix} C & M \\ M & 0 \end{pmatrix} U_R = \begin{pmatrix} C_D & M_D \\ M_D & 0 \end{pmatrix}, \tag{5.11}$$

where U_L, U_R are real, invertible $2n \times 2n$ matrices and M_D, C_D, K_D are real $n \times n$ matrices. It is easy to check that the eigenvalues (and their multiplicities) of equation (3.1) remain the same if M, C and K are replaced respectively by M_D, C_D and K_D. Thus, SPTs are strictly isospectral. An SPT is termed diagonalizing if M_D, C_D, K_D are diagonal. To illustrate how a diagonalizing SPT decouples (1.1) for $f(t) = 0$, consider the first-order formulation

$$\begin{pmatrix} C & M \\ M & 0 \end{pmatrix} \begin{pmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix} + \begin{pmatrix} K & 0 \\ 0 & -M \end{pmatrix} \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} = 0. \tag{5.12}$$

A diagonalizing SPT leads to

$$\begin{pmatrix} C_D & M_D \\ M_D & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_D(t) \\ \ddot{q}_D(t) \end{pmatrix} + \begin{pmatrix} K_D & 0 \\ 0 & -M_D \end{pmatrix} \begin{pmatrix} q_D(t) \\ \dot{q}_D(t) \end{pmatrix} = 0, \tag{5.13}$$

from which the decoupled second-order equation

$$M_D \ddot{q}_D(t) + C_D \dot{q}_D(t) + K_D q_D(t) = 0, \tag{5.14}$$

can be extracted. It was pointed out in [10] that current algorithms for constructing diagonalizing SPTs can be quite restrictive.

We have argued that the decoupled system (4.3) is the unique real and diagonal system, isospectral to (1.1). Thus, a diagonalizing SPT must generate the same decoupled system as phase synchronization. It is indeed straightforward to show that $C_D = D$ and $K_D = \Omega$, provided we choose $M_D = I$, which can be done without loss of generality. Thus, a diagonalizing SPT leads to the same decoupled system as phase synchronization.

This somewhat surprising observation can be explained as follows. A diagonalizing SPT lives in the state space, i.e. the space of dimension $2n$ spanned by displacements

and velocities. Phase synchronization, on the other hand, is a method in the configuration space of dimension n spanned only by the displacements. Any procedure in configuration space can of course be executed in state space (but not vice versa). We thus may interpret diagonalizing SPTs to be the state-space version of phase synchronization. Starting from phase synchronization, a diagonalizing SPT can be constructed, if it is available. The mapping in equation (4.7), for example, defines a diagonalizing structure-preserving congruence transformation for symmetric and positive definite M, C, K and a clever normalization of eigenvectors. On the other hand, it is generally not possible to obtain phase synchronization from diagonalizing SPTs (see [42]).

5.4   Illustrative Examples

Two examples illustrate the theoretical developments of this chapter. The first examples shows that there are systems that can be decoupled in configuration space, but not in state space. The second emphasizes SPTs.

5.4.1   Example 1: Coordinate Coupling in State Space

A two-degree-of-freedom system in free vibration is defined by $M = I$,

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \tag{5.15}$$

This damped system is already in a decoupled form, and both classical modal analysis (see Chapter 2) and phase synchronization (see Chapter 4) reduce to identity transformation in configuration space. The eigenvalues of the state companion matrix

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -4 & 0 & -4 \end{pmatrix} \tag{5.16}$$

are $\lambda_1 = -1$, $\lambda_2 = -2$ and each is repeated. However, there is only one eigenvector $(1, 0, -1, 0)^T$ associated with λ_1 and also only one eigenvector $(0, 1, 0, -2)^T$ associated with λ_2. Therefore, the matrix A is defective and cannot be diagonalized. As a result, the system in this example cannot be decoupled by complex modal analysis in state space. A generalization is obvious: a classically damped multi-degree-of-freedom system cannot be decoupled by complex modal analysis in state space if one or more degrees are critically damped.

There should not be any confusion about the role played by structure-preserving transformations: they are state-space transformations aiming at decoupling systems in the configuration space. It is easy to show that a diagonalizing SPT for this example is given by $U_L = U_R = I$.

5.4.2 Example 2: Diagonalizing Structure-Preserving Transformation

Consider a non-classically damped system governed by (1.1) with $M = I$,

$$C = \begin{pmatrix} 0.5 & -0.1 \\ -0.1 & 1 \end{pmatrix}, K = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \tag{5.17}$$

and $f(t) = 0$. Solution of the quadratic eigenvalue problem (3.1) yields

$$\lambda_1 = \bar{\lambda}_3 == 0.25 + 0.97i \quad, \quad v_1 = \bar{v}_3 = \begin{pmatrix} 1.00e^{-i0.0002}, & -0.03e^{-i1.49} \end{pmatrix}^T \tag{5.18}$$

$$\lambda_2 = \bar{\lambda}_4 == 0.50 + 1.93i \quad, \quad v_2 = \bar{v}_4 = \begin{pmatrix} 0.07e^{-i1.66}, & -1.00e^{-i3.14} \end{pmatrix}^T \tag{5.19}$$

Phase synchronization converts (1.1) into (4.8), for which

$$D = \text{diag}(0.50, \quad 1.00), \quad \Omega = \text{diag}(1.00, \quad 3.99), \tag{5.20}$$

and $g(t) = 0$. The configuration-space decoupling transformation (3.27) becomes

$$\begin{pmatrix} q^1(t) \\ q^2(t) \end{pmatrix} = \begin{pmatrix} 1.00p_1(t - 0.002) + 0.04p_2(t - 0.86) \\ -0.02p_1(t - 1.53) - 0.44p_2(t - 1.63) \end{pmatrix} \tag{5.21}$$

From (5.4), the state-space version of (3.27) is given by

$$\begin{pmatrix} q^1(t) \\ q^2(t) \\ \dot{q}^1(t) \\ \dot{q}^2(t) \end{pmatrix} = \begin{pmatrix} 1.00 & -0.02 & -0.00 & -0.04 \\ 0.01 & 1.00 & 0.04 & 0.00 \\ 0.00 & 0.14 & 1.00 & 0.01 \\ -0.03 & -0.00 & -0.01 & 1.00 \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \\ \dot{p}_1(t) \\ \dot{p}_2(t) \end{pmatrix} = T \begin{pmatrix} p_1(t) \\ p_2(t) \\ \dot{p}_1(t) \\ \dot{p}_2(t) \end{pmatrix} \tag{5.22}$$

The matrix T above defines a diagonalizing SPT $\{U_L, U_R\}$ with $U_L = U_R = T$. This transformation satisfies equations (5.9)-(5.11) above, for which $M_D = I$, $C_D = D$, and $K_D = \Omega$.

Chapter 6

Applications in Earthquake Engineering

The response of a building to earthquake excitation is of great importance in structural engineering. Although the dynamic properties are nonlinear, experimental studies using shaking tables have shown that the system response can often be simulated to a satisfactory degree of accuracy by a linear viscoelastic model. The equation of motion (1.1) is thus widely used in earthquake engineering to model the dynamic behavior of, for example, multi-story buildings, nuclear power-plants or base-isolated structures [8, 12, 27, 55, 56, 60, 61]). In such applications, M and K represent respectively the inertia and elastic properties of the structure under investigation and C describes the energy dissipation. All three system matrices are SPD. Practically speaking, the commutativity condition (1.2) is satisfied if energy dissipation is almost uniformly distributed throughout the system. This condition is violated for systems consisting of two or more subsystems with significantly different levels of damping. We shall consider two examples in the analysis of (1.1) when (1.2) is not satisfied.

6.1 Response of Light Equipment in a Base-Isolated Structure

The use of base-isolation is known to attenuate not only the response of the building (termed the primary structure), but also the response of a secondary system mounted on the primary structure, for example internal equipment. The isolation system, primary structure and secondary system are usually made of different materials with significantly different energy dissipation characteristics. Hence the commutativity condition (1.2) is not satisfied [55, 56].

The classical engineering approach to the analysis of base-isolated systems rests upon classical modal analysis (see Chapter 2). That is, we solve the symmetric eigenvalue problem (2.1) to obtain n real eigenvectors u_j and n natural frequencies ω_j,

$j = 1, \ldots, n$. The eigenvectors are orthogonal with respect to M or K. Normalize the eigenvectors with respect to M and use the normalized eigenvectors as columns to construct the modal matrix U in (2.2). Upon modal transformation $q(t) = U\tilde{q}(t)$ in (2.6), the equation of motion (1.1) becomes (2.7) with modal damping matrix $\tilde{C} = U^T C U$ and spectral matrix $\tilde{K} = U^T K U$ (see (2.2)). The modal damping matrix $\tilde{C}$ is diagonal if and only if (1.2) is satisfied [7]. When $\tilde{C}$ is diagonal, (2.7) is decoupled and methods for analysis and design are readily available (see [8]). When $\tilde{C}$ is not diagonal, (2.7) is often decoupled by simply neglecting the off-diagonal elements of $\tilde{C}$. This procedure, termed the *decoupling approximation*, is relatively routine in structural engineering. The decoupling approximation appears intuitive if the off-diagonal elements in $\tilde{C}$ are small in magnitude when compared to the diagonal elements, i. e. $\tilde{C}$ is diagonally dominant. However, even when $\tilde{C}$ is diagonally dominant, the errors can still be large and exhibit rather surprising behaviors [21, 29, 41].

Tsai and Kelly [55] investigated the validity of the decoupling approximation in computation of the seismic response of attached equipment in a base-isolated building. In [55], they derived a linear viscoelastic model for a five-story, base-isolated building with internal equipment. The model has three degrees of freedom, representing the displacement of the base, the primary structure and the equipment respectively (see [55] for the details of the modeling process). The response of the equipment to the 1940 El Centro earthquake is studied in detail. Here, we utilize their formulation and apply the decoupling algorithm described in Section 4.4 to this problem.

Given M, C and K as in [55], we first solve the quadratic eigenvalue problem (3.1). Using the eigen-data, we construct the matrices Λ_1 and Λ_2 according to (4.2) and V_1, V_2 according to (4.5). With an excitation $g(t)$ derived from (4.13) and (4.14), the decoupled equation (4.8) is obtained, where

$$D = \mathrm{diag}(0.1910, \ 0.1644, \ 0.8178)$$

and

$$\Omega = \mathrm{diag}(0.3505, \ 33.0195, \ 33.7899).$$

We apply an explicit Runge-Kutta (4,5) formula [15] (implemented in the Matlab function *ode45*) to solve the decoupled equations (4.8), i.e. compute $p(t)$. The time step is adaptively chosen for each independent decoupled coordinate and can be as large as 0.08. Equation (4.15) is used to map $p(t)$ back to $q(t)$. The upper part of Figure 6.1 shows the response of the attached equipment as computed by the method of phase synchronization. The lower part shows the results obtained by employing the decoupling approximation. As already observed by Tsai and Kelly [55], the results obtained by the decoupling approximation underestimate the response of the equipment because the coupling in (2.7) is significant. In contrast, the method of phase synchronization generates the decoupled equation (4.8) by capturing all coupling effects. Using (4.8) instead of decoupling approximation, an accurate system response is computed. Furthermore, system solution by phase synchronization can be
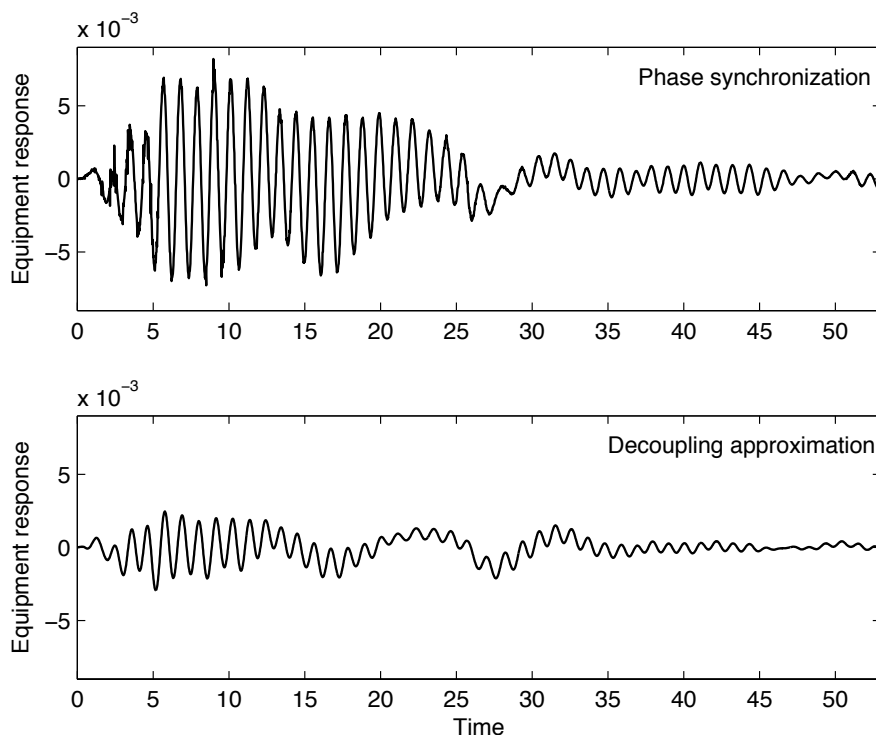
Figure 6.1: Response of attached equipment to the 1940 El Centro earthquake. Top: phase synchronization. Bottom: decoupling approximation.

substantially more efficient than direct simulation because a different time step may be chosen for each independent decoupled equation (see Section 4.6.2). For example, the time step in the present example could be chosen twice as big for the decoupled equations. However, the main advantage of phase synchronization lies beyond response calculation: all streamlined design and analysis methods applicable only to independent single-degree-of-freedom equations can be used once (4.8) is obtained.

6.2 A Simplified Linear Viscoelastic Model of a Nuclear Power Plant

We now consider the seismic response of a nuclear power plant. We use the 8 degree-of-freedom linear viscoelastic model presented in [27] and shown in Figure 6.2 for convenience. The model consists of four interconnected rigid structures representing the core, prestressed concrete pressure vessel (PCPV), basement and building respectively. Each structure has two degrees of freedom representing the sway (U)

Figure 6.2: An 8 degree-of-freedom model of a nuclear power plant (taken from [27]).

and rocking angle (Φ). The coefficient matrices M, C and K are given in [27]. We are interested in the response of the core (U_1) when the structure is subject to the 1940 El Centro earthquake.

We compare three approaches to this problem. First, we simulate the response of the system using the Runge-Kutta formula already mentioned in the previous section. The results serve as the benchmark solution and are shown in the second uppermost part of Figure 6.3. Next, we apply the method of phase synchronization and decouple the system to obtain (4.8). Upon solution of the decoupled equations using the same Runge-Kutta formula as above, we compute the energy

$$E_j = \frac{1}{2} \int_0^T \dot{p}_j(t)^2 + \Omega_j^2 p_j(t)^2 dt \tag{6.1}$$

in each decoupled coordinate $p_j(t)$. In the above equation, $T = 10$, and Ω_j is the jth diagonal entry of Ω in (4.2). The results are shown in the top part of Figure 6.3 and it is clear that the coordinates $p_1(t)$ and $p_5(t)$ carry most of the energy within the system. As a result, an approximation to the response can be obtained by solving only the first five decoupled equations (which are associated with the lowest frequencies), followed by transformation to $q(t)$ using (4.15). The results of this approximation, shown in the third uppermost part of Figure 6.3, are in very good agreement with the core response obtained by direct numerical simulation.
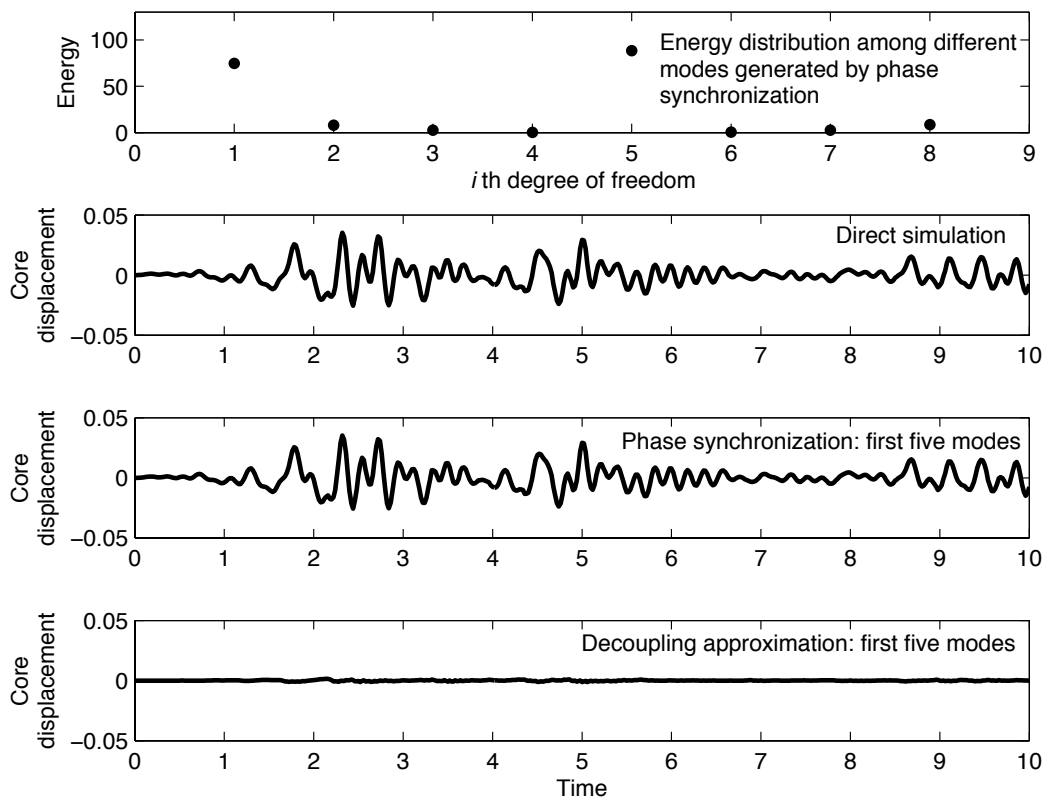
Figure 6.3: Comparison of three approaches for response calculation.

Finally, we can obtain another decoupled system using the decoupling approximation. When the first five coordinates of this decoupled system are used to generate the system response, the resulting approximation dramatically underestimates the core response, as is shown in the bottom part of Figure 6.3. Perhaps this is not surprising. The decoupling approximation produces a decoupled system by neglecting the coupling caused by non-zero off-diagonal elements of the modal damping matrix $\tilde{C}$. In contrast, the method of phase synchronization generates a decoupled equation by incorporating all coupling effects properly.

To be sure, direct numerical simulation certainly yields the correct core response, but the method of phase synchronization offers a lot more than just efficient evaluation of response. Incidentally, we have outlined in this example a new method of model reduction, i.e. solving fewer differential equations while achieving good accuracy in response computations. The decoupling approximation, although in popular use, is not feasible for model reduction unless coupling is negligible.

Chapter 7

Conclusions

This dissertation builds upon previously published work. Phase synchronization and the decoupling of all non-defective, purely oscillatory linear systems with symmetric coefficients in free vibration was first presented in [36]. The procedure was extended in [37] to apply to non-defective, non-oscillatory systems with symmetric coefficients in free or forced vibration. Currently, an extension of the theory to defective systems with symmetric coefficients in under review [28]. Chapter 5 is based upon the extended discussion on decoupling in configuration and state space published in [42].

Here, we have presented a theory and an algorithm to decouple all non-defective second-order linear dynamical systems (with symmetric and non-symmetric coefficients). Previous work on decoupling emphasizes simultaneous diagonalization of the three system matrices, however this theory exploits the parameter t (time), characteristic of a dynamical system. The decoupling methodology developed herein possesses ample physical insight and it also lends itself to numerical computations. Major findings are summarized in the following statements.

1. The decoupling is achieved by a real, invertible, but nonlinear mapping. This mapping simplifies to a real, linear time-invariant transformation when the three system matrices can be simultaneously diagonalized.

2. All parameters required for the decoupling of a second-order system are obtained from the solution of a quadratic eigenvalue problem.

3. In homogeneous systems, the real, time-dependent, configuration-space decoupling transformation is real, linear and time-invariant when cast in state space. In addition, the configuration-space decoupling transformation generates a diagonalizing structure-preserving transformation.

4. In non-homogeneous systems, both the configuration-space decoupling transformation and associated state transformation are nonlinear and depend continuously on the excitation.

5. There are damped linear systems that can be decoupled by modal analysis or phase synchronization in configuration space but not by complex modal analysis in state space.

System decoupling plays a fundamental role in such diverse areas as linear vibration, quantum mechanics, mathematical economics, and computational science. It not only provides an efficient means of evaluating the system response but also greatly facilitates qualitative analysis. Unlike any methods of decoupling approximation, phase synchronization is an exact decoupling technique that accounts for all effects of coupling in full. As demonstrated by examples, the main advantage of decoupling lies beyond system solution. It is the possibility, for example, of model reduction, damping characterization, or stability design that would make system decoupling worthwhile. The study of other issues, such as energy distribution among the independent decoupled coordinates or numerical algorithms for decoupling are also worthwhile in a subsequent course of investigation.

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