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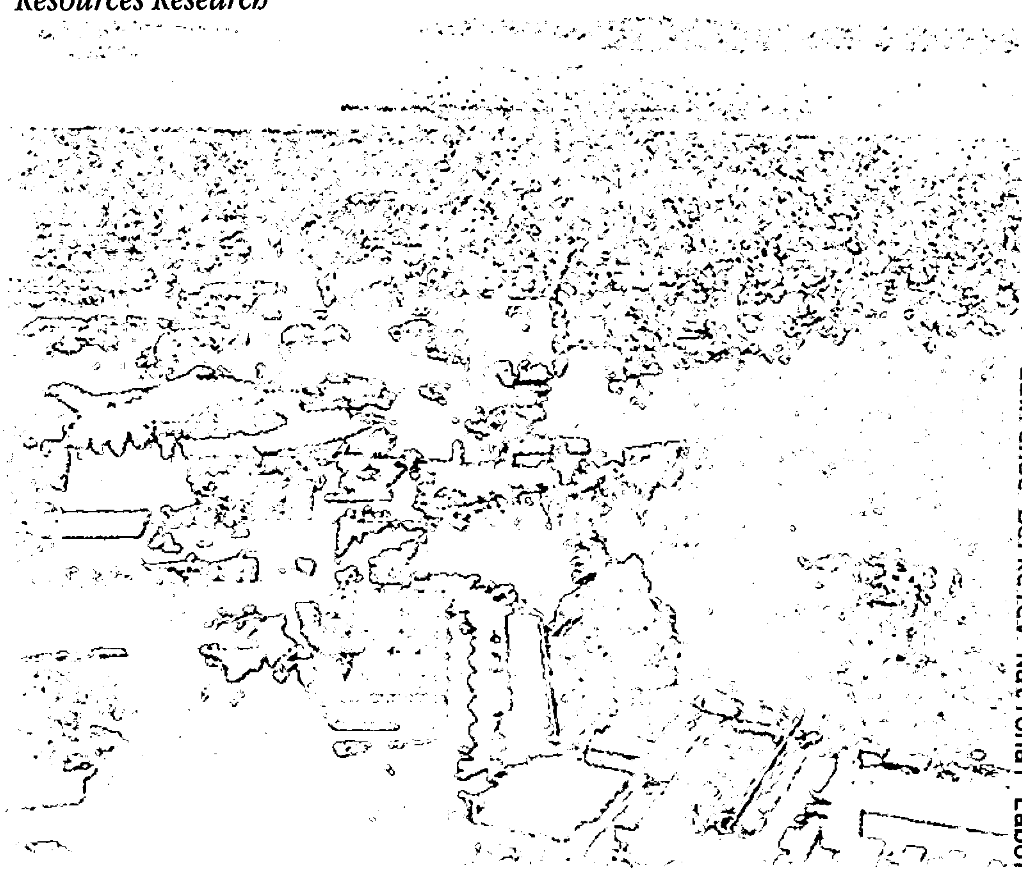

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Multi-Continuum Description of Flow and Transport and Splitting the Fields in Composite Heterogeneous Media

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Abstract

Adequacy of the description and the qualitative prediction of flow and transport processes in subsurface systems essentially depends on how well a model represents the heterogeneity that is intrinsic in real field. One of the simplest models to describe the heterogeneity structure is a so-called composite system. In this model it is assumed that the whole media is composed of homogeneous components that are distributed in space randomly or in a particular periodic manner. Flow and transport simulation in composite systems can usually be reduced to solving partial differential equations with variable discontinuous coefficients and averaging the solutions, which can be accomplished by numerical simulations using the Monte-Carlo approach. A different approach, related to averaging the differential equations of flow and leads to new equations that link averaged fields in composite media. This description is designated as mono-continuum or global description. If the homogeneous components of a composite system, so-called phases, have essentially different hydrodynamic and/or geometric parameters, it is natural to study averaging of the fields on the individual phases of the composite along with the global averaging. This approach reduces to a more detailed description of processes in multi-continua. It takes into consideration the mean fields in the individual continuum phase as well as the cross-flows and cross-forces between continua. However, this description is usually non-closed because the number of equations is less that number of unknown functions (mean fields and exchange terms). To overcome this difficulty, the phenomenological theory of unsteady motion in heterogeneous media (dual-porosity media, fractured porous media) postulates a special interaction mechanism for closing the equations. This paper presents the exact equations of mass-balance and moment-balance for each phase of the composite. The exact physical sense of exchange terms in the multi-continua models is explained. We then demonstrate that joint

consideration of the mono-continual and the multi-continual systems of equations in the case of two-phase random composite leads to a closed description, and from that, we can find the exchange terms. For periodic composite system the same approach leads to a closed description for any number of phases. We successively study the composite systems with a random and periodical structure. The terms describing the interactions between continua (such as the exchange of fluids and momentum between phases) are calculated. Finally, we examine the hypothesis customarily made in the phenomenological models that the cross-flow is proportional to the mean pressure difference. We find the hypothesis as generally unsatisfactory considering its region of applicability (micro and macro isotropic composite medium).

1. INTRODUCTION

The problem of rationally describing flow and transport in real, macroscopically essentially inhomogeneous media is of considerable interest in the theory and its technical applications.

Stochastic approach for flow and transport in heterogeneous random systems (including random composite media) involves the probabilistic treatment of percolation parameters and flow and transport equations, the determination of the functional from the statistical solution or the analysis of equations relating the unknown and given functionals [e.g., *Shvidler,* 1985; *Dagan,* 1989; *Gelhar,* 1993]

Inhomogeneous systems having periodic structure are a convenient model for studying processes in heterogeneous media. The theory of averaging the processes in periodic (as distinct from stochastic) structures is well established, and constructive methods for analyzing many processes in periodic media have been developed [e.g., *Bensoussan et al*, 1978; *Bakhvalov and Panasenko*, 1989; *Jikov at al.*, 1993]

The description in terms of averaged fields represented by the theory of homogenization leads to equations that relate these fields to the effective characteristics of the inhomogeneous medium. Under certain conditions, the averaged equations can be treated as conservation laws, and their system as a mono-continuum model of the process. Obviously, this description must contain and utilize sufficient information on the fields in the individual phases of the periodic or random composite system and the inter-phase transfer processes.

A more detailed description involves the determination of the mean fields in each phase, i.e. the conditionally averaged fields and the equations relating these fields. If it is possible to construct such equations and treat them as the equations of certain process in a phase of the composite system, such a description would be a multi-continua one in accordance with the number of phases.

Irrespective of the method of realizing the multi-continuum description, it is necessary to solve the central problem of closing the systems of equations associated with terms responsible for inter-continuum transfers of mass, momentum, energy, etc.

The phenomenological theory of the unsteady motion of a homogeneous fluid in heterogeneous composite systems (media with dual porosity, fractured porous media), which postulates the special interaction mechanism, is well studied. In this approach, the flow in each phase of the composite is characterized by its own mean pressure or head and mean flow-velocity fields, the relation between which takes the form of Darcy`s law. The rate of fluid transfer among the phases is assumed to be proportional to the difference of the mean head of each phase.

We examine the problem of conditional averaging of a system of flow equations for a weakly compressible fluid in a random and periodic composite medium. The equations of the multi-continua model were developed, and the parameters regulating the interactions between the phase continuum were calculated.

For those cases in which, for one and the same process, a mono-continuum description can be realized and the conservation laws of the multi-continua model can be obtained, it has been shown that the splitting of the globally averaged fields is possible, that the closing transfer terms for the binary random system can be expressed in terms of the characteristic of the mono-continuum and the mean fields in the phase continua , and that their interactions can be calculated. The information thus obtained for some random and periodical systems makes it possible to evaluate and refine the phenomenological closing hypothesis. As an example, we show that when a heterogeneous system is locally isotropic and macro-isotropic, the hypothesis of the proportionality of the cross-flow between the phases to the difference of phase pressures or head can be regarded in some cases as an approximate rule. On the other hand, this relation is generally inadequate in the cases of overall anisotropy.

The article is consisted of 5 sections, with this introduction being section 1. In section 2 we examine the flow in random composite systems. Here, we consider the exact conditional averaged equations of fluid transport and averaged equations of momentum for each phase of the composite random media. These equations make up a non-closed system that is interpreted as a multi-continua model of the process. For a binary composite system, the inclusion of the equations of mono-continua (global) model in this non-closed system of the equations for the same process in the same composite system enables the closed multi-continua description, which makes it possible to directly compute the parameters a multi-continuum models that respond for interaction between continua. We examine a partial but important case of "meso-equilibrium" system and obtain the simple relations for cross-flows between phases, cross-forces, etc.

In section 3 we examine some examples of random composite systems and present the analysis results.

In section 4 the above analysis approach is applied to periodical composite media. Here we also examine the multi-continua model and present conditional averaged equations. In contrast to the stochastic approach, the system that joins the equations of mono-continuum and multi-continua models together with explicit expansion of the local fields with respect to fast and slowly changing variables makes a closed description possible for the composite system with any number of phases.

In section 5 we present some examples of periodic composite systems, for which we can obtain exact expressions for phase heads and their differences. Also examined in detail is the two-phase layered system for which pressure difference and cross-flows are obtained quantitatively. Here we discuss the same method of closing the equations of phenomenological models.

The approach that is presented in this article for random and periodic composite media was briefly published by *Shvidler* [1986a,b; 1988] and *Shvidler and Karasaki* [1994,1999]. In this paper we set forth the problem in more detail and present new theoretical results and some applications.

2. RANDOM COMPOSITE MEDIA

2.1.Mono-continuum (global) description of flow in random medium

Let us consider the unsteady flow of a homogenous, compressible fluid in a heterogeneous, in particular, composite deformable random medium in a three-dimensional domain Ω with boundary $\partial\Omega$. The problem is mathematically described by the equations:

$$div\ v(x,t) + \alpha(x)\frac{\partial u(x,t)}{\partial t} = f(x,t), \tag{2.1}$$

$$\sigma^{-1}(x)\ v(x,t) + \nabla u(x,t) = 0\,, \tag{2.2}$$

$$u(x,0) = u_0(x), \quad u|_{\partial\Omega} = \varphi\ (x,t)\,, \tag{2.3}$$

Here $u(x,t)$ is the head, $v(x,t)$ is the Darcy's velocity vector, $\sigma(x)$ is the symmetric and positive definite conductivity tensor, whose components are a random functions of x , and scalar $\alpha(x)$ is the specific storage of the porous media-fluid system, which is also a positive random function. We assume that both random fields $\alpha(x)$ and $\sigma(x)$ are stochastically homogeneous, that is, all probability density of these random functions are invariant to translation in unbounded space. The source density $f(x,t)$ is a square integrable function. In the present paper we only consider non-random initial and boundary conditions (2.3) for $u(x,t)$. It should be noted that non-random flux condition lead to random boundary condition for $u(x,t)$ and requires special analysis.

We introduce the fields $U(x,t)$ and $V(x,t)$, unconditionally averaged over the ensemble of realizations of the random fields $\sigma(x)$ and $\alpha(x)$:

$$U(x,t) = \langle u(x,t)\rangle,\ V(x,t) = \langle v(x,t)\rangle \tag{2.4}$$

If we assume that δ - so-called micro-scale of the stochastically homogeneous fields $\sigma(x)$ and $\alpha(x)$ - satisfies the condition $\delta << l_\Omega$, where l_Ω is the macro-scale in region Ω, then averaging over the probability measure in (2.4) can be replaced by averaging over the volume of the region ω_Δ, whose meso-scale Δ satisfies the inequalities:

$$\delta << \Delta << l_\Omega \tag{2.5}$$

It is known [e.g., *Bakhvalov and Panasenko*,1989] that unconditional averaging of the system (2.1),(2.2) and (2.3) can be obtained by expanding the fields $u(x,t)$ and $v(x,t)$ in powers of the small parameter $\mu = \delta / l_\Omega$ that is a dimensionless length scale of heterogeneity for the random fields $\alpha(x)$ and $\sigma(x)$. Thus the averaged equations (2.1) and (2.2) can be represented in the form:

$$\operatorname{div} V(x,t) + \alpha^* \frac{\partial U(x,t)}{\partial t} + \mu \frac{\partial}{\partial t} \lambda\left[DU(x,t)\right] = f(x,t) \tag{2.6}$$

$$\left(\sigma^*\right)^{-1} V(x,t) + \nabla U(x,t) = \mu \left(\sigma^*\right)^{-1} \gamma\left[DU(x,t)\right] \tag{2.7}$$

Here the scalar $\alpha^* = \langle\alpha(x)\rangle$ is constant and the tensor $\sigma^* = const$ is so-called the effective conductivity tensor. It should be noted that the non-random constants scalar α^* and tensor σ^* fully define the connection between the vector-field $V(x,t)$ and scalar field $U(x,t)$ in the limiting case of $\mu \to 0$ only. The expressions – the scalar-correlation $\mu\lambda[DU(x,t)] = \langle[\alpha(x) - \langle\alpha(x)\rangle][u(x,t) - U(x,t)]\rangle$ and the vector-correlation $\mu\gamma[DU(x,t)] = \langle[\sigma^* - \langle\sigma(x)\rangle][\nabla u(x,t) - \nabla U(x,t)]\rangle$ are asymptotic series in power of the parameter μ, whose coefficients are linear combinations of the derivatives of the field $U(x,t)$ with respect to x and t. Because both series are infinite and contain the derivatives of any order, the equations (2.6) and (2.7) are non-local.

Obviously, we must add the non-random conditions (2.3) to the equations (2.6) and (2.7) and refer the (2.3) to the function $U(x,t)$, that is: $U(x,t_0) = u_0(x)$, $U(x,t)|_{\partial\Omega} = \varphi(x,t)$. Thus, in terms of $U(x,t)$ and $V(x,t)$ there exists a closed description of the process of non-stationary flow in heterogeneous porous media. The chief difficulties in realizing this description are (a) determining the tensor σ^* and (b) constructing the series $\lambda[DU(x,t)]$ and $\gamma[DU(x,t)]$.

The unconditional averaged system: the equation of mass balance (2.6) and the equation of momentum balance (2.7), describes the mono-continuum model of flow in the medium, in particular, in composite medium, which is a closed description in terms of the mean fields $U(x,t)$ and $V(x,t)$.

2.2. Multi-continua description of flow in random composite medium

For a more detailed description for composite medium we go over to conditional averaging of the fields $u(x,t)$ and $v(x,t)$ over the composite phases and introduce the random indicator function

$$z_i(x) = \{1 \ , \ if \ \ x \in \Omega_i \ , \ and \ \ 0 \ , \ if \ \ x \notin \Omega_i\} \tag{2.8}$$

where Ω_i is the portion of the domain Ω occupied by the i –th phase, i =1,…,m.

For any x the indicator functions satisfy the relations:

$$\sum_i^m z_i(x) = 1 \ , \ \langle z_i(x)\rangle = \theta_i , \tag{2.9}$$

where θ_i is the volume fraction of the i-th phase in the composite, and for stochastically homogeneous medium $\theta_i = const$. Then from (2.9) we have

$$\sum_i^m \theta_i = 1 \tag{2.10}$$

If the phases are homogeneous ,we can write $\alpha(x) = \sum_i^m \alpha_i z_i(x)$ and $\sigma(x) = \sum_i^m \sigma_i z_i(x)$,where for each i-th phase $\alpha_i = const$ and $\sigma_i = const$, and they are non-random scalars and tensors respectively. If the fields $\alpha(x)$ and $\sigma(x)$ are stochastically homogeneous , after averaging these equations we have $\langle \alpha(x)\rangle = \sum_i^m \alpha_i \theta_i$ and $\langle \sigma(x)\rangle = \sum_i^m \sigma_i \theta_i$.

For describing the conditional averaging of the any random field $y(x)$ we use the following relation:

$$\langle y(x)\rangle_i = \langle y(x)\rangle, \ \ if \, x \in \Omega_i \tag{2.11}$$

and for any random field we can write

$$\langle y(x,t)\rangle_i = \langle z_i(x)\ \ y(x,t)\rangle\ /\ \theta_i \tag{2.12}$$

Thus, for the conditional averaging of $y(x,t)$, it is sufficient to unconditionally average $z_i(x)\ y(x,t)$ and renormalize the result by dividing by θ_i. (It should be noted that here in (2.12) and elsewhere we do not assume summation on repeating indices!)

Taking (2.12) into consideration , we introduce the phase parameter – conditionally averaged head in the i-th phase:

$$U_i(x,t) = \langle\ u(x,t)\ \rangle_i \tag{2.13}$$

It is obvious that unconditionally and conditionally averaged heads are bound by the relation :

$$U(x,t) = \sum_i^m \theta_i U_i(x,t) \tag{2.14}$$

For conditionally averaged flow velocity in the i-th phase:

$$V_i(x,t) = \langle\ v(x,t)\ \rangle_i \tag{2.15}$$

and we have

$$V(x,t) = \sum_i^m \theta_i V_i(x,t) \tag{2.16}$$

We introduce the continuum i-th phase flow velocity which is analogous to Darcy`s velocity, that is the mean velocity of liquid in pores distributed (spread) in all space.

$$V_i^*(x,t) = \theta_i V_i(x,t) \tag{2.17}$$

And from equation (2.16) we have

$$V(x,t) = \sum_{i}^{m} V_i^*(x,t) \tag{2.18}$$

It is easy to see that

$$\langle z_i(x)\, div v(x,t)\rangle = div V_i^*(x,t) + Q_i(x,t) \tag{2.19}$$

$$Q_i(x,t) = -\langle v(x,t)\nabla z_i(x)\rangle \tag{2.20}$$

Then

$$\langle z_i(x) \quad \nabla u(x,t)\rangle = \nabla[\theta_i U_i(x,t)] \; + \; \mathrm{P}_i(x,t) \tag{2.21}$$

$$\mathrm{P}_i(x,t) = -\langle u(x,t)\nabla z_i(x)\rangle \tag{2.22}$$

According to the definition (2.8) the vector $\nabla z_i(x)$ is non-zero only on the boundary $\partial\Omega_i$ that separates the i-th phase from the different phases. To study the behavior of the vector $\nabla z_i(x)$ on the almost everywhere smooth surface $\partial\Omega_i$ we introduce at an arbitrary point $A_i \in \partial\Omega_i$ a local orthogonal coordinate system where the axis ς_A is orthogonal to $\partial\Omega_i$ at point A_i and directed inside Ω_i, and the axes η_A and ζ_A are tangential to $\partial\Omega_i$.

If the equation $\varsigma_A = f_i(\eta_A,\zeta_A)$ describes the face $\partial\Omega_i$, we can write $z_i = H\left[\varsigma_A - f_i(\eta_A,\zeta_A)\right]$, where H[ ] is Heaviside`s step-function. In vicinity of origin we have expansion

$f_i(\eta_A,\zeta_A) = \left[\partial f(0,0)/\partial\eta_A\right]\eta_A + \left[\partial f_i(0,0)/\partial\zeta_A\right]\zeta_A$. Because the axes η_A and ζ_A are tangential, the above derivatives are zero and near point A_i the indicator-function

$z_i(\varsigma_A,\eta_A,\zeta_A) = H(\varsigma_A)$ .Therefore $\nabla z_i\big|_{A_i} = \delta(\varsigma_A)\vec{e}_{\varsigma A}$ where $\delta(\varsigma_A)$ is the Dirac`s δ-function, and $\vec{e}_{\varsigma A}$ is a unit-vector on the axis ς_A .

The scalar correlation $Q_i(x,t)$ and vector correlation $\mathrm{P}_i(x,t)$ have a clear physical significance. Let ω be an arbitrary subdomain of the domain Ω. For each realization inside the subdomain ω the surface S_i^ω separates i-th phase from other phases that are distributed in ω and ,generally speaking, S_i^ω is multiply connected. We consider the expression $Q_i^\omega = |\omega|^{-1} \int_\omega Q_i(x,t)\,d\omega = -|\omega|^{-1} \int_\omega \langle v(x,t)\nabla z_i(x)\rangle d\omega = -|\omega|^{-1} \left\langle \int_\omega v(x,t)\nabla z_i(x)\,d\omega \right\rangle$ and after taking into account that $\nabla z_i(x)$ is zero everywhere excepting the points of surface S_i^ω , we can write $Q_i^\omega = -|\omega|^{-1} \left\langle \int_{S_i^\omega} v_n(x,t)\big|_{S_i^\omega} dS_i^\omega \right\rangle$. Here $v_n(x,t)\big|_{S_i^\omega}$ is the continuos projection of the vector $v(x,t)\big|_{S_i^\omega}$ on the normal $n_i(x)$ to surface S_i^ω ,that is directed inside Ω_i.

Thus, the covariance $Q_i(x,t)$ is the specific mean cross-flow of fluid from the i-th continuum phase to the rest. Because the vector $\nabla z_i(x)$ in point $x \in S_i^\omega$ is perpendicular to S_i^ω and directed inside Ω_i, the positive cross-flow $Q_i(x,t)$ denotes that mean flow from Ω_i is more than the flow into Ω_i.

Similarly we consider the expression $\mathrm{P}_i^\omega = |\omega|^{-1} \int_\omega \mathrm{P}_i(x,t)\,d\omega = -|\omega|^{-1} \int_\omega \langle u(x,t)\nabla z_i(x)\rangle d\omega$

$= |\omega|^{-1} \left\langle \int_\omega u(x,t)\nabla z_i(x)\,d\omega \right\rangle$ and again taking into account the characteristics of the vector $\nabla z_i(x)$, we can write $\mathrm{P}_i^\omega = -|\omega|^{-1} \left\langle \int_{S_i^\omega} u(x,t)\big|_{S_i^\omega} n_i(x)\,dS_i^\omega \right\rangle$.

Thus the vector $\mathrm{P}_i(x,t)$ is the mean specific cross-force from the i-th phase acting on the surface that separates the other phases from the i-th phase.

And obviously, because $\sum_i z_i(x) = 1$ we have from (2.20) and (2.22) the conditions of compatibility:

$$\sum_i^m Q_i(x,t) = 0\,,\ \sum_i^m \mathrm{P}_i(x,t) = 0 \qquad (2.23)$$

Now, multiply the equations (2.1) and (2.2) by $z_i(x)$ and taking into account the relations (2.13), (2.15), (2.17), (2.19), (2.20), (2.21) and (2.22), after averaging we have for the i-th phase

$$div V_i^{*}(x,t) + \alpha_i \theta_i \frac{\partial U_i(x,t)}{\partial t} + Q_i(x,t) = \theta_i f(x,t) \tag{2.24}$$

$$\sigma_i^{-1} V_i^{*}(x,t) + \nabla[\theta_i U_i(x,t)] + P_i(x,t) = 0 \tag{2.25}$$

Although the conservative system of equations (2.24) and (2.25) is non- closed (because the cross-flows $Q_i(x,t)$ and cross-forces $P_i(x,t)$ have not been evaluated in terms of $U_i(x,t)$ and $V_i(x,t)$), this system can be treated as the exact flow equations in the i-th continuum-phase. For this case in the mass balance condition (2.24) the term $Q_i(x,t)$ determines the rate of mass transfer between the i-th continuum-phase and the other continuum-phases. Equation (2.25) is the modified Darcy's law in the form of momentum balance and the vector $P_i(x,t)$ is the specific cross-force from the i-th continuum-phase to the other continuum phases.

Such interpretation of the system of equations (2.24) and (2.25) for all composite phases together with (2.23) – the conditions of compatibility for cross-flows and cross-forces-provides a possibility of a statement about the multi-continua description for transport of flow in the composite media. In this description the conditions of mass and momentum balances in each continuum-phase are realized, and moreover, the continua exchange the fluid and momentum between them.

The system of equations (2.23), (2.24) and (2.25) looks like the phenomenological equations presented earlier by *Rubinstein* [1948] (who studied the heat transport in heterogeneous media) and *Barenblatt at al.* [1960]. But there exist significant differences. For example, the exact equation of balance of momentum (2.25) contains the vector-functions Ψ_i that represent the force interaction between the i-th phase and the other phases that were ignored by these authors. Moreover, contrary to the phenomenological models, the coefficients of the averaged equations are defined exactly through the parameters of the composite media.

For a physical interpretation of the mono and multi-continua models it is possible to use the averaged equations as the balance conditions for any volume when the volumes are sufficiently small. In examination of one representative realization the averaging is derived on surfaces or volumes. For statistical regularity of the results a different kind of averaging is necessary that applies some conditions.

One of these conditions with respect to scales of hierarchy is presented in the inequality (2.5). This condition is sufficient for the mono-continuum description, but for multi-continuum description some conditions that guarantee the stability of conditional averaging should be added. For example, let the composite system be the matrix with randomly or regularly distributed inclusions (so- called granular media) (Fig.1). It is obvious that a control volume ω_Δ must contain a sufficient number of inclusions, and that the surface of control volume $\partial\omega_\Delta$ must dissect some part of inclusions and the fraction of the dissecting surface must be similar to volume fraction of the inclusions. Only under these conditions for volume or surface the averaging is stable and identical to the ensemble averaging. Similar condition must be met for the control volume and surface in a layered system (Fig.2).

2.3.Alternative multi-continua model

Along with the multi-continua model that represent non-closed system of equations (2.23), (2.24) and (2.25), it is possible to construct an alternative and equivalent multi-continua model.

In this case we introduce a scalar function

$$q_i(x,t) = \langle z_i(x)\, divv(x,t) \rangle \tag{2.26}$$

and vector-function $G_i(x,t)$

$$G_i(x,t) = \langle z_i(x)\, \nabla u(x,t) \rangle \tag{2.27}$$

From equations (2.19) and (2.21) we find that

$$q_i(x,t) = divV_i^*(x,t) + Q_i(x,t) \tag{2.28}$$

$$G_i(x,t) = \nabla\left[\theta_i U_i(x,t)\right] + \mathrm{P}_i(x,t) \tag{2.29}$$

Obviously, the cross-flow $Q_i(x,t)$ and flow $q_i(x,t)$ have different physical meanings. Whereas the $q_i(x,t)$ define the total flow from the i-th phase to the rest, the $Q_i(x,t)$ describes the flow transfer between i-th phase and different phases.

Using the functions $q_i(x,t)$ and $G_i(x,t)$ we can rewrite the system of equations (2.23), (2.24) and (2.25) in the different form

$$\alpha_i\, \theta_i \frac{\partial\ U_i(x,t)}{\partial\ t} + q_i(x,t) = \theta_i f(x,t) \tag{2.30}$$

$$\sigma_i^{-1}\ V_i^*\ (x,t) + G_i(x,t) = 0 \tag{2.31}$$

$$\sum_i^m q_i(x,t) = \sum_i^m divV_i^*(x,t) \tag{2.32}$$

$$\sum_i^m G_i(x,t) = \nabla U(x,t) \tag{2.33}$$

The equation (2.30) is the flow balance and the equation (2.31) is the momentum balance for the i-th phase. The equations (2.32) and (2.33) are the conditions of the compatibility for flows

$q_i(x,t)$ and forces $G_i(x,t)$. We conclude that the multi-continua description (2.23), (2.24) and (2.25) are preferable because it is more convenient for understanding the process and will be used in the subsequent analyses. In some cases we use the above relations in addition to computing the flow $q_i(x,t)$ and $G_i(x,t)$.

2.4. Closure problem

In order to close the conditionally averaged system and to determine the fields $U_i(x,t)$, $V_i(x,t)$, the cross-flows $Q_i(x,t)$ and the cross-forces $\mathrm{P}_i(x,t)$, it is natural to employ the results of unconditional averaging of the system of equations (2.1),(2.2) and (2.3) (i.e. , the global averaged system of equations (2.6) and (2.7) and compare the number of dependent variables and equations for them.

It should be noted that after changing variables $U(x,t)$ and $V(x,t)$ in the global averaged closed system (2.6) and (2.7) according to equations (2.14) and (2.16) the new system is non- closed with respect to the variables $U_i(x,t)$ and $V_i(x,t)$.

In addition we should note that although the global averaged system of equations (2.6) and (2.7) and conditional averaged system (2.23),(2.24) and (2.25) are joint the mono-continuum and multi-continua models of the same composite media and non-steady flow , both systems are independent in the sense that the equations of global averaged system in the form (2.6) and (2.7) are not derivable from the system (2.23),(2.24) and (2.25) .

Let us consider the three-dimensional flow process in a composite medium with m-phases. In this case to describe flow in one phase of multiphase media we use two scalar functions $U_i(x,t)$ and $Q_i(x,t)$ and two vector functions $V_i(x,t)$ and $\mathrm{P}_i(x,t)$. That results in a total of 8

(1+1+3+3=8) dependent variables for each phase, and 8m unknown functions for the m-phase composite. On the other hand, for the description of the process in each phase we can use one scalar equation of conservation of mass and one vector equation of conservation of momentum, that is 1+3=4 equations and for m-phase system –4 x m equations. Furthermore, for the m-phase case we have one scalar equation of the compatibility of the cross-flows and one vector equation of the compatibility of cross-forces, that is altogether 4m+4 equations.

The globally averaged system contains one scalar function $U(x,t)$ and one vector function $V(x,t)$, i.e. 4 unknown functions and two equations: one scalar equation (2.6) for conservation of mass and for the composite system as a whole we require the averaged vector equation (2.7) - the condition of conservation of momentum. This globally averaged system is closed and can be solved separately with respect to mean head $U(x,t)$ and mean velocity $V(x,t)$.

So far we have 4m + 4 + 4 = 4m+8 equations. We can add some more equations: the scalar condition (2.14) –the relation between $U(x,t)$ and $U_i(x,t)$ and the vector condition (2.16) – the relation between $V(x,t)$ and $V_i(x,t)$.

Thus, for unsteady flow we finally have 8m + 4 unknown functions for 4m+12 independent equations. It is obvious that for binary composite media, that is for m=2, we have 20 independent equations with 20 unknown functions. The system is closed and, after solving it, we can express all the unknown functions in terms of $U(x,t)$.

By summing the equations (2.24) over all i, we obtain the mass balance equation for the entire composite system of the multi-continua model:

$$div\ V(x,t) + \sum_i \alpha_i \theta_i \frac{\partial\ U_i(x,t)}{\partial\ t} = f(x,t) \tag{2.34}$$

which contains the conditionally and unconditionally averaged fields $U_i(x,t)$ and $V(x,t)$.

Similarly, we can obtain the equation of momentum balance for the whole composite system in terms of $U(x,t)$ and $V_i^*(x,t)$ as

$$\sum_i \sigma_i^{-1} V_i^*(x,t) + \nabla U(x,t) = 0 \tag{2.35}$$

Then after comparing the equations (2.6) and (2.34), we can write one equation for $U_i(x,t)$:

$$\sum_i^m \alpha_i \theta_i \frac{\partial U_i(x,t)}{\partial t} = \alpha^* \frac{\partial U(x,t)}{\partial t} + \mu \frac{\partial}{\partial t} \lambda(DU(x,t)) \tag{2.36}$$

And after differentiating the equation (2.14) we have

$$\sum_i^m \theta_i \frac{\partial U_i(x,t)}{\partial t} = \frac{\partial U(x,t)}{\partial t} \tag{2.37}$$

For two phases i and j and $\alpha_i \neq \alpha_j$ the system of equations (2.36) and (2.37) have unique solutions of $\partial U_i(x,t)/\partial t$ and $\partial U_j(x,t)/\partial t$. Integrating them with respect to time and using the initial conditions $U_i(x,t_0)=u_0(x)$ and $\lambda\left[DU(x,t_0)\right]=0$, we obtain

$$U_i(x,t) = U(x,t) + \mu\theta_i^{-1}\left(\alpha_i - \alpha_j\right)^{-1} \lambda(DU(x,t)), \tag{2.38}$$

$$U_j(x,t) = U(x,t) + \mu\theta_j\left(\alpha_j - \alpha_i\right)^{-1} \lambda(DU(x,t))$$

Combining the globally averaged equation (2.7) with equation (2.23) and the second equation from (2.23) and taking into account the solution (2.38), we find the vectors: $\mathrm{P}_i(x,t)$ and $\mathrm{P}_j(x,t)$

$$\mathrm{P}_i(x,t) = \left(\sigma_j - \sigma_i\right)^{-1}\left[\left(\langle\sigma\rangle - \sigma^*\right)\nabla U(x,t) + \mu\gamma(DU(x,t))\right] + \mu\left(\alpha_i - \alpha_j\right)^{-1} \nabla\lambda(DU(x,t)) \tag{2.39}$$

$$\mathrm{P}_j(x,t) = \left(\sigma_i - \sigma_j\right)^{-1}\left[\left(\langle\sigma\rangle - \sigma^*\right)\nabla U(x,t) + \mu\gamma(DU(x,t))\right] + \mu\left(\alpha_j - \alpha_i\right)^{-1} \nabla\lambda(DU(x,t)) \tag{2.40}$$

We can derive the relations (2.38),(2.39) and (2.40) somewhat differently after computing $\mu\lambda\left[DU(x,t)\right]=\langle\alpha'(x)u'(x,t)\rangle$, where $u'(x,t)=u(x,t)-U(x,t)$ and $\alpha'(x)=\alpha(x)-\langle\alpha(x)\rangle$. For a two-phase composite, we have $\alpha(x)=\alpha_i z_i(x)+\alpha_j z_j(x)$ and after computing $\langle\alpha'(x)u'(x,t)\rangle$, we find $\mu\lambda(DU(x,t))=\theta_i\theta_j(\alpha_i-\alpha_j)\left[U_i(x,t)-U_j(x,t)\right]$ and the relation (2.38). It is appropriate to note that for each component of tensor $\sigma^{lm}(x)$ of two-phase composite medium the correlation moment between $\sigma^{ml}(x)$ and fluctuation of head $u'(x,t)$ can be written as $\langle\sigma^{ml}(x)u'(x,t)\rangle=\theta_i\theta_j\left(\sigma_i^{ml}-\sigma_j^{ml}\right)\left[U_i(x,t)-U_j(x,t)\right]$ and it is proportional to $\langle\alpha'(x)u'(x,t)\rangle$. This results from the fact that for two-phase composite the coefficient of correlation between any component $\sigma^{lm}(x)$ and $\alpha(x)$ is +1 if $K^{lm}=\left(\sigma_i^{lm}-\sigma_j^{lm}\right)\left(\alpha_i-\alpha_j\right)$ is positive, and –1 if K^{lm} is negative. Similarly we find $\langle\sigma'(x)\ \nabla u'(x,t)\rangle=\left(\sigma_i-\sigma_j\right)\left\{\theta_i\theta_j\ \left[\nabla U_i(x,t)-\nabla U_j(x,t)\right]+\mathrm{P}_i(x,t)\right\}$. Using the relationship $\langle\sigma'(x)\nabla u'(x,t)\rangle=\left(\sigma^*-\langle\sigma\rangle\right)\nabla U(x,t)-\mu\gamma(DU)$ which derived from (2.7), we can directly obtain the relations (2.39) and (2.40).

The mean phase velocities are

$$V_i^*(x,t)=-\ \sigma_i\ G_i(x,t)\ ,\ V_i^*(x,t)=-\sigma_j G_i(x,t), \tag{2.41}$$

where the mean phase forces are

$$G_i(x,t)=\left(\sigma_j-\sigma_i\right)^{-1}\left[\left(\sigma_j-\sigma^*\right)\nabla U(x,t)+\mu\gamma(DU(x,t))\right]+2\mu\left(\alpha_i-\alpha_j\right)^{-1}\nabla\lambda(DU(x,t))$$

(2.42)

$$G_j(x,t)=(\sigma_i-\sigma_j)^{-1}\left[(\sigma_i-\sigma^*)\nabla U(x,t)+\mu\gamma(DU(x,t))\right]+2\mu(\alpha_j-\alpha_i)^{-1}\nabla\lambda(DU(x,t))$$

Substituting (2.38) in (2.24), it is possible to determine the cross-flows $Q_i(x,t)$ as

$$Q_i(x,t)=\theta_i\left\{f(x,t)-divV_i(x,t)-\alpha_i\left[\frac{\partial U(x,t)}{\partial t}+\frac{\mu}{\theta_i(\alpha_i-\alpha_j)}\frac{\partial\lambda(DU(x,t))}{\partial\ t}\right]\right\} \tag{2.43}$$

So, let the mono-continua description of flow in a random composite system is realized, e.g. are known the tensor of effective conductivity for all system σ^*, the expressions : the scalar $\mu\lambda\left[DU(x,t)\right]$ and vector $\mu\gamma\left[DU(x,t)\right]$.

We have shown here that in this case for two-phase i and j composite random medium we can Find these fields: the phase mean heads $U_i(x,t)$ and $U_j(x,t)$, the phase mean Darcy's velocity $V_i^*(x,t)$ and $V_j^*(x,t)$, the cross-flows $Q_i(x,t)$ and $Q_j(x,t)$, cross-forces $\mathrm{P}_i(x,t)$ and $\mathrm{P}_j(x,t)$, phase flows $q_i(x,t)$ and $q_j(x,t)$, phase forces $G_i(x,t)$ and $G_j(x,t)$.

2.5. Steady-state flow

Let the source density $f(x,t)$ and boundary function $\varphi(x,t)$ for large $t>>t_0$ weakly depends on t. In this case the flow tend to steady-state and we can use for steady-state stage the above results . All one has to do is to set all derivatives with respect to time t to zero in equations (2.6) and (2.24) and in the expansions $\lambda\left[DU(x,t)\right]$ and $\gamma\left[DU(x,t)\right]$.

It is significant to note, that because the correlations $\langle\alpha'(x)u'(x,t)\rangle$ and $\langle\sigma^{lm}(x)u'(x,t)\rangle$ are proportional, these correlations for $t\to\infty$ have finite limits. Generally speaking, these correlatons are different from zero at these points x, where the conductivity components $\sigma^{lm}(x)$

are correlated with field $u(x)=\lim_{t\to\infty}u(x,t)$. It is obvious that at these points x the mean heads $U_i(x)\neq U_j(x)$.

2.6. Meso-equilibrium approximation

It is a common knowledge that for basic processes in a natural heterogeneous system flow velocities are typically small. When a some perturbation is applied into the flow, the relatively short transition stage in the system creates a slowly changing process in time. Naturally, this stage of the process has been the main interest for application.

Bearing in mind that by applying sufficiently small μ on the meso-scale the system tends fast to a local equilibrium of mean phase heads. This state can be called as meso-equilibrium state. However, for a finite μ this is not to say that the mean phase heads are locally equal or are constant in space and time. As we shoved above, even when the flow is in steady-state, for finite μ in some cases the mean phase heads can be different. And only for $\mu\to 0$ the mean head difference tends to zero.

In this limiting case in all presented equations all terms containing small parameter μ and its positive powers can be neglected. Under these conditions, when a very strong heterogeneity exists, the terms to be neglected can contain large parameters and possibly impose some restriction on the small parameter μ, such that accuracy of the averaged equations is sufficient at least outside the temporal border layer [e.g.,*Bakhvalov and Panasenko,* 1989].

Let us consider the process of flow in a heterogeneous medium in which the scale of heterogeneity μ is so small that in the averaged equations (2.6) and (2.7) it is possible to retain only the dominant terms, i.e. $\mu\to 0$. Then, from (2.38) it follows that

$$U_i(x,t)=U_j(x,t)=U(x,t) \tag{2.44}$$

and from (2.39) and (2.40) the cross-force vectors $P_i(x,t)$ and $P_j(x,t)$ take the forms

$$P_i(x,t) = (\sigma_j - \sigma_i)^{-1}(\langle\sigma\rangle - \sigma^*)\nabla U(x,t) \ , \quad P_j(x,t) = (\sigma_i - \sigma_j)^{-1}(\langle\sigma\rangle - \sigma^*)\nabla U(x,t) \quad (2.45)$$

It follows from (2.45) that in a medium with heterogeneous conductivity, the cross-force $P_i(x,t)$ is zero only in a layered system, provided that $\nabla U(x,t)$ is directed along the layers.

For the mean phase velocities $V_i(x,t)$ we have

$$V_i(x,t) = -\sigma_i^* \nabla U(x,t) \quad (2.46)$$

where the tensor σ_i^* is

$$\sigma_i^* = \theta_i^{-1}\sigma_i(\sigma_j - \sigma_i)^{-1}(\sigma_j - \sigma^*) \quad (2.47)$$

which can be called the phase conductivity. It satisfies the relations

$$\sum_i \theta_i \sigma_i^{-1}\sigma_i^* = I \quad , \quad \sum_i \theta_i \sigma_i^* = \sigma^* \quad (2.48)$$

For the cross-flows in the meso-equilibrium approximation we have from (2.43)

$$Q_i(x,t) = \theta_i \left\{ f(x,t) + div[\sigma_i^* \nabla U(x,t)] - \alpha_i \frac{\partial U(x,t)}{\partial t} \right\} \quad (2.49)$$

or after replacing the $\partial U(x,t)/\partial t$ from global averaged system (2.6) and (2.7) by setting $\mu = 0$ and substituting into (2.49) we have the cross-flow in another form

$$Q_i(x,t) = \frac{\theta_i}{\langle\alpha\rangle} \{(\langle\alpha\rangle - \alpha_i) f(x,t) + div\,[(\langle\alpha\rangle\sigma_i^* - \alpha_i\sigma^*)\nabla U(x,t)] \quad (2.50)$$

If for large time the flow is steady-state, the dependence of the cross-flows $Q_i(x)$ in (2.50) on the parameters α_i is only by appearance because in this case $div\ (\sigma^*\ \nabla U(x)) + f(x) = 0$ then we have from (2.50):

$$Q_i(x) = \theta_i \left[f(x) + div \left(\sigma_i^* \ \nabla U(x) \right) \right] \tag{2.51}$$

For meso-equilibrium stage, the general equation of phase flow $q_i(x,t)$ is

$$q_i(x,t) = \frac{\theta_i}{\langle \alpha \rangle} \left[\left(\langle \alpha \rangle - \alpha_i \right) f(x,t) - \alpha_i \, div \left(\sigma^* \ \nabla U(x,t) \right) \right] \tag{2.52}$$

If the composite system is micro and macro isotropic, and the tensors σ_i and σ^* are isotropic, then for $f(x,t) = 0$ we have

$$Q_i(x,t) = \theta_i \frac{\langle \alpha \rangle \sigma_i^* - \alpha_i \sigma *}{\langle \alpha \rangle} \nabla^2 U(x,t) \tag{2.53}$$

$$q_i(x,t) = -\theta_i \frac{\alpha_i \sigma^*}{\langle \alpha \rangle} \nabla^2 U(x,t) \tag{2.54}$$

or, in another form,

$$Q_i(x,t) = \theta_i \frac{\langle \alpha \rangle \sigma_i^* - \alpha_i \sigma^*}{\sigma^*} \frac{\partial U(x,t)}{\partial t} \tag{2.55}$$

$$q_i(x,t) = -\theta_i \alpha_i \frac{\partial U(x,t)}{\partial t} \tag{2.56}$$

It is obvious that in a fully isotropic medium the phase cross-flows $Q_i(x) = 0$ and phase flows $q_i(x) = 0$ when the flow is steady-state.

We can rewrite equation (2.55) as

$$Q_i(x,t) = \theta_i \alpha_i \frac{\kappa_i^* - \kappa^*}{\kappa^*} \frac{\partial U(x,t)}{\partial t} \tag{2.57}$$

where $\kappa^* = \sigma^* / \langle \alpha \rangle$ is effective diffusivity for all system, and $\kappa_i^* = \sigma_i^* / \alpha_i$ is the effective phase diffusivity, which is obviously different from $\kappa_i = \sigma_i / \alpha_i$, the phase local diffusivity.

As can be seen from equation (2.57) the sign of cross-flow $Q_i(x,t)$ is defined by the relation between parameters κ_i^* , κ^* that are dependent on the quantities $\sigma_i\ / \sigma_j$ *and* $\alpha_i\ /\ \alpha_j$ as well as on the geometry of heterogeneity.

Let the conductivity of the composite system be homogeneous, that is $\sigma_i = \sigma_j = \sigma$. For any $f(x,t)$ we have from (2.55)

$$Q_i(x,t) = \theta_i\, \theta_j \left(\alpha_j - \alpha_i\right) \frac{\partial U(x,t)}{\partial t} \qquad (2.58)$$

If $\alpha_i < \alpha_j$ the cross-flow $Q_i(x,t)$ has the same sign as $\partial U(x,t)/\partial t$, that is when the mean pressure increases in time, the i-th phase deliver flow to the j-th phase. Conversely, when the mean pressure falls, the i-th phase obtain flow from the j-th phase. If $\alpha_i > \alpha_j$ the signs are opposite and , when the mean pressure increases, the i-th phase obtains flow, and when mean pressure falls , the i-th phase delivers flow to the j-th phase. In the case where $\alpha_i \to a,\ \alpha_j \to \alpha$, but $\sigma_i \neq \sigma_j$, we have

$$Q_i = \theta_i\, \alpha\, \frac{\sigma_i^* - \sigma^*}{\sigma^*} \frac{\partial U(x,t)}{\partial t} \qquad (2.59)$$

It is easy to show that $\sigma_i^* < \sigma^*$ when $\sigma_i < \sigma_j$ and for $\sigma_i > \sigma_j$ the inequality is $\sigma_i^* > \sigma^*$. So, when $\sigma_i < \sigma_j$ the signs of the cross-flow $Q_i(x,t)$ and the derivative $\partial U(x,t)/\partial t$ are opposite and for $\sigma_i > \sigma_j$ the signs are identical. It is interesting to examine the case $\sigma_i << \sigma_j$ and $\theta_i >> \theta_j$ that in some sense can be related to the i-th porosity system with the j-th fracture system. Neglecting some terms in general system of equations we obtain

$$\alpha_i\theta_i \frac{\partial U(x,t)}{\partial t} + Q_i(x,t) = \theta_i f(x,t) \quad , \quad \theta_j \, div\, V_j(x,t) - Q_i(x,t) = \theta_j f(x,t) \qquad (2.60)$$

$$V_i(x,t) \approx 0 \qquad , \qquad V_j(x,t) = -\theta_j^{-1}\, \sigma^*\, \nabla U(x,t) \qquad (2.61)$$

The system of equation (2.60) and (2.61) bears similarities to the phenomenological equations of flow in fissured porous media derived by *Barenblatt at al.* [1960]. The difference lies in the fact that the system of (2.60) and (2.61) is closed, and all it parameters are completely defined.

Eliminating the phase cross-flow $Q_i(x,t)$ leads to the global averaged equation of pressure $U(x,t)$

$$\alpha_i\, \theta_i \frac{\partial U(x,t)}{\partial t} = \sigma^* \nabla^2 U(x,t) + f(x,t) \qquad (2.62)$$

After determining $U(x,t)$ from (2.62) under appropriate initial and boundary conditions we can determine the i-th phase cross-flow

$$Q_i(x,t) = \theta_i \left[f(x,t) - \alpha_i \frac{\partial U(x,t)}{\partial t} \right] \qquad (2.63)$$

and from (2.61)-the j-th phase flow velocity $V_j(x,t)$, the phase flows $q_i(x,t) = Q_i(x,t)$, $q_j \approx 0$.

We now examine the steady-state flow when $f(x) = 0$ in an binary composite system that is anisotropic for mean phase flow. There can be three variants in this case.

1. Either one or both of the tensors σ_i and σ_j are anisotropic, and the tensor σ^* is anisotropic.

2. Either one or both of the tensors σ_i and σ_j are anisotropic, but the tensor σ^* is isotropic.

3. Tensors σ_i and σ_j are isotropic but the tensor σ^* is anisotropic.

In all of three cases the tensors σ_i^* and σ_j^* are anisotropic and the globally averaged equation is $div\ \left(\sigma^*\ \nabla U(x)\right) = 0$, where the tensor σ^* as noted in case 2 can be isotropic. Analysis of the expression for $Q_i(x)$ in (2.51) shows that for steady-state flow with $f(x) = 0$,

$$Q_i(x) = \theta_i div\left(\sigma_i^* \nabla U(x)\right) \qquad (2.64)$$

Because in the case of phase flow anisotropy the tensors σ_i^* and σ^* are non-similar (i.e., the components of these tensors are non-proportional) and if $\nabla\ U(x) \neq const$, the phase cross-flow $Q_i(x)$ is non-zero.

This result is paradoxical at first sight, but besides the demonstrated calculation, the detailed qualitative analysis explains this effect.

Let the composite medium be a granular medium, that matrix conductivity be σ_j, and the conductivity of the inclusions be σ_i (Fig.1). It is obvious that for steady-state flow, the cross-flow from each i-th phase inclusions in the j-th phase matrix is zero. On the other hand, here we argue that the mean cross-flow from the i-th continuum into the j-th continuum is finite and differ from zero under these conditions.

This contradiction stems from the expanded incorrect transfer of the mechanism of cross-flow from individual inclusion to the aggregate of many inclusions that are contained in the representative control volume. As indicated above, for those inclusions the basic part is completely confined in the control volume and the cross-flow from these inclusions into the matrix is zero when flow is steady-state. But there exists the cross-flow inside the control volume at the surface of those inclusions that are dissected by the control volume surface. If the composite medium is anisotropic for mean phase flow and $\nabla \; U(x) \neq const$, the cross-flows on the cut are not compensated for meso-scale control volume.

We now study a more clear example of layered system that contains homogeneous layers with conductivity σ_i and σ_j (Fig.2). Such a system is micro or macro anisotropic. Each layer intersects the border of control volume at least twice and since it is assumed that the gradient of the mean pressure is not constant, the flow from layer outside the border of the control volume is statistically non-compensated. This means that the cross-flow from i-th layers to j-th layers is statistically non-compensated.

The discussions above show that the multi-continuum description has non-trivial exceptions. For example, the mean cross-flow in granular composite system with isolated inclusions under some conditions is non-zero, whereas for each inclusions the cross-flow is zero, is a peculiar kind of "payment" for continual description of flow in inclusions that do not compose a connected space.

3. EXAMPLES

Let us now study some cases where the meso-equilibrium globally averaged systems can be easily constructed, thus the computation of simple closing relations for the phase cross-flows $Q_i(x,t)$ and the phase cross-forces $\mathrm{P}_i(x,t)$ are possible.

3.1Case1

Let a two-dimensional infinite random heterogeneous system be composed of two subdomains with isotropic conductivities σ_i and σ_j that are statistically equivalently distributed in the plane (for example, like an unbounded chess board with randomly distributed "white" and "black" squares). In this case the mean concentration of the phases are equal and $\theta_i = \theta_j = 1/2$.

It is well known (e.g., *Shvidler, 1985*) the effective conductivity for such a systems is isotropic and $\sigma^* = \sqrt{\sigma_i \, \sigma_j}$, then from (2.46) and (2.47) :

$$\sigma_i^* = \frac{2\sigma_i\sqrt{\sigma_j}}{\sqrt{\sigma_i}+\sqrt{\sigma_j}} \quad , \quad \sigma_j^* = \frac{2\sigma_j\sqrt{\sigma_i}}{\sqrt{\sigma_i}+\sqrt{\sigma_j}} \tag{3.1}$$

$$V_i(x,t) = -\frac{2\sigma_i\sqrt{\sigma_j}}{\sqrt{\sigma_i}+\sqrt{\sigma_j}}\nabla U(x,t) \ , \ V_j(x,t) = -\frac{2\sigma_j\sqrt{\sigma_i}}{\sqrt{\sigma_i}+\sqrt{\sigma_j}}\nabla U(x,t) \tag{3.2}$$

$$\mathrm{P}_i(x,t) = \frac{1}{2}\frac{\sqrt{\sigma_j}-\sqrt{\sigma_i}}{\sqrt{\sigma_i}+\sqrt{\sigma_j}}\nabla U(x,t) \ , \ \mathrm{P}_j(x,t) = -\mathrm{P}_i(x,t) \tag{3.3}$$

Further, by setting $f(x,t) = 0$,

$$Q_i(x,t) = \frac{\sqrt{\sigma_i\sigma_j}\left(\alpha_j\sqrt{\sigma_i}-\alpha_i\sqrt{\sigma_j}\right)}{2\langle\alpha\rangle\left(\sqrt{\sigma_i}+\sqrt{\sigma_j}\right)}\nabla^2 U(x,t) \quad , \quad Q_j(x,t) = -Q_i(x,t) \tag{3.4}$$

$$q_i(x,t) = -\frac{\alpha_i\sqrt{\sigma_i\,\sigma_j}}{\alpha_i+\alpha_j}\nabla^2 U(x,t) \quad , \quad q_j(x,t) = -\frac{\alpha_j\sqrt{\sigma_i\sigma_j}}{\alpha_i+\alpha_j}\nabla^2 U(x,t) \tag{3.5}$$

It is easy to see that when $\sigma_i/\sigma_j = \alpha_i^2/\alpha_j^2$ the mean cross-flow $Q_i(x,t) = 0$ at any time t when flow is transient. The mean cross-force $\mathrm{P}_i(x,t) = 0$ only if $\sigma_i = \sigma_j$. Because the system is micro and macro- isotropic, the phase mean cross-flow $Q_i(x) = 0$ for steady-state flow.

3.2.Case2

The second case is different from the previous one only in the sense that conductivity of subdomains are anisotropic

$$\sigma_i = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \ \sigma_j = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \tag{3.6}$$

This system is globally isotropic and $\sigma^* = \sqrt{ab}$ (Shvidler 1985). After simple manipulation we obtain

$$\sigma_i^* = \frac{2}{\sqrt{a}+\sqrt{b}}\begin{pmatrix}\sqrt{b} & 0\\ 0 & \sqrt{a}\end{pmatrix}, \quad \sigma_j^* = \frac{2}{\sqrt{a}+\sqrt{b}}\begin{pmatrix}\sqrt{a} & 0\\ 0 & \sqrt{b}\end{pmatrix} \tag{3.7}$$

$$V_i(x,t) = -\frac{2\sqrt{ab}}{\sqrt{a}+\sqrt{b}}\begin{pmatrix}\sqrt{a} & 0\\ 0 & \sqrt{b}\end{pmatrix}\nabla U(x,t)\ , \ V_j(x,t) = -\frac{2\sqrt{ab}}{\sqrt{a}+\sqrt{b}}\begin{pmatrix}\sqrt{b} & 0\\ 0 & \sqrt{a}\end{pmatrix}\nabla U(x,t) \tag{3.8}$$

$$\mathrm{P}_i(x,t) = \frac{\sqrt{b}-\sqrt{a}}{2\left(\sqrt{a}+\sqrt{b}\right)}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\nabla U(x,t) \tag{3.9}$$

$$Q_i(x,t) = \frac{1}{2}\left\{ f(x,t) + \frac{2\sqrt{ab}}{\sqrt{a}+\sqrt{b}}\nabla\left[\begin{pmatrix}\sqrt{a} & 0\\ 0 & \sqrt{b}\end{pmatrix}\nabla U(x,t)\right] - \alpha_i\frac{\partial U(x,t)}{\partial t}\right\} \tag{3.10}$$

$$q_i(x,t) = \frac{1}{2}\left[f(x,t) - \alpha_i\frac{\partial U(x,t)}{\partial t}\right] \tag{3.11}$$

It is obvious that when $f(x) = 0$ and under steady-state flow the globally averaged equation is $\nabla^2 U(x) = 0$ and the phase cross-flow $Q_i(x) = 0$ only if $a = b$ or $\nabla U(x) = const$. The phase cross-force $\mathrm{P}_i(x) = 0$ only when $a = b$.

3.3.Case 3

The next example involves three-dimensional, two-phase layered medium composed of homogeneous anisotropic layers. Let the layers be directed perpendicular to the x_3-axis, and the phases conductivity be such that

$$\sigma_i = \begin{pmatrix} a_i & 0 & 0 \\ 0 & b_i & 0 \\ 0 & 0 & c_i \end{pmatrix} \quad , \quad \sigma_j = \begin{pmatrix} a_j & 0 & 0 \\ 0 & b_j & 0 \\ 0 & 0 & c_j \end{pmatrix} \tag{3.12}$$

Then

$$\sigma^* = \begin{pmatrix} \langle a \rangle & 0 & 0 \\ 0 & \langle b \rangle & 0 \\ 0 & 0 & \langle c^{-1} \rangle^{-1} \end{pmatrix}, \sigma_i^* = \begin{pmatrix} a_i & 0 & 0 \\ 0 & b_i & 0 \\ 0 & 0 & \langle c^{-1} \rangle^{-1} \end{pmatrix} \tag{3.13}$$

Where

$$\langle a \rangle = a_i\, \theta_i + a_j \theta_j \quad , \quad \langle b \rangle = b_i \theta_i + b_j \theta_j \quad , \quad \langle c^{-1} \rangle^{-1} = c_i\, c_j \left(c_i \theta_j + c_j \theta_i \right)^{-1} \tag{3.14}$$

and the global averaged flow equation is

$$\langle \alpha \rangle \frac{\partial U(x,t)}{\partial t} = \langle a \rangle \frac{\partial^2 U(x,t)}{\partial x_1^2} + \langle b \rangle \frac{\partial^2 U(x,t)}{\partial x_2^2} + \langle c^{-1} \rangle^{-1} \frac{\partial^2 U(x,t)}{\partial^2 x_3^2} + f(x,t) \tag{3.15}$$

Then, from (2.46) and (2.45), we compute the components of the i-th phase flow velocity and cross-force as

$$V_{i1}(x,t) = -\, a_i \frac{\partial U(x,t)}{\partial x_1} \quad , \quad V_{i2}(x,t) = -b_i \frac{\partial U(x,t)}{\partial x_2} \quad , \quad V_{i3}(x,t) = -\langle c^{-1} \rangle^{-1} \frac{\partial U(x,t)}{\partial x_3} \tag{3.16}$$

$$\mathrm{P}_{i1}(x,t) = 0 \;,\; \mathrm{P}_{i2}(x,t) = 0 \;,\; \mathrm{P}_{i3}(x,t) = \theta_i^{-1} \frac{\langle c \rangle - \langle c^{-1} \rangle^{-1}}{c_j - c_i} \frac{\partial U(x,t)}{\partial x_3} \tag{3.17}$$

Notice that the longitudinal components of the phase cross-force are zero. This is partly because in our example the longitudinal main axes of the tensors σ_i and σ_j are aligned with the layers.

For the mean phase cross-flow and phase flow we can write

$$Q_i(x,t) = \theta_i \left[f(x,t) + a_i \frac{\partial^2 U(x,t)}{\partial x_1^2} + b_i \frac{\partial^2 U(x,t)}{\partial x_2^2} + \langle c^{-1} \rangle^{-1} \frac{\partial^2 U(x,t)}{\partial x_3^2} - \alpha_i \frac{\partial U(x,t)}{\partial t} \right] \tag{3.18}$$

$$q_i(x,t) = -\alpha_i\,\theta_i \frac{\partial U(x,t)}{\partial t} \tag{3.19}$$

If the averaged flow is one-dimensional and perpendicular to the layers, the cross-flow is

$$Q_i(x,t) = \theta_i\left(\langle\alpha\rangle - \alpha_i\right)\frac{\partial U(x,t)}{\partial t} \tag{3.20}$$

and is proportional to $q_i(x,t)$.

Let the averaged three-dimensional flow be a steady-state. Then the averaged equation is

$$\langle a\rangle \frac{\partial^2 U(x)}{\partial x_1^2} + \langle b\rangle \frac{\partial^2 U(x)}{\partial x_2^2} + \left\langle c^{-1}\right\rangle^{-1} \frac{\partial^2 U(x)}{\partial\ x_3^2} + f(x) = 0 \tag{3.21}$$

and for the cross-flow $Q_i(x)$ we have

$$Q_i(x) = \theta_i\left[f(x) + a_i \frac{\partial^2 U(x)}{\partial x_1^2} + b_i \frac{\partial^2 U(x)}{\partial x_2^2} + \left\langle c^{-1}\right\rangle^{-1} \frac{\partial^2 U(x)}{\partial x_3^2}\right] \tag{3.22}$$

It is obvious that for $f(x) = 0$, if the longitudinal components of tensors σ_i ,σ_j are not equal and when $\nabla U(x,t) \neq const$, the phase cross-flow $Q_i(x) \neq 0$.

3.4.Case 4

We consider a model that imitates some porous space with system of fractures. Let an unbounded porous media – a matrix with isotropic conductivity σ_m and diffusivity α_m be randomly and statistically uniformly dissected by three infinite and mutually orthogonal systems of plates with parameters σ_f and α_f that simulate the infinite fractures along each Cartesian axis. In this case the Cartesian axis are principal axis for the global effective conductivity tensor.

We assume that the matrix conductivity σ_m is significantly smaller than the fracture conductivity σ_f and the concentration of each parallel system of the fractures c_k (orthogonal to the k-Cartesian axis) is significantly less than unity.

Under these assumptions the σ_k^* - mean component of the effective conductivity tensor that are associated with k-axis is defined mainly by the matrix and the fractures that are parallel to k-axis. The contribution in the k-component of effective conductivity from the fractures that are orthogonal to k-axis is significantly less if the $\sigma_m << \sigma_f$, and if the fracture concentration $c = c_1 + c_2 + c_3 << 1$.

Thus effective conductivity tensor σ^* is approximated by

$$\sigma^* = \langle \sigma \rangle I - \sigma_f C \qquad (3.23)$$

where $\langle \sigma \rangle = \sigma_m (1-c) + \sigma_f c$, and the tensor C is

$$C = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \qquad (3.24)$$

Then the global averaged flow equation in a fractured medium with porous parallelepiped blocks has the form

$$\nabla[\sigma^* \nabla U(x,t)] + f(x,t) = \langle \alpha \rangle \frac{\partial U(x,t)}{\partial t} \qquad (3.25)$$

After using for σ^* the expression (3.23) we have

$$\sigma_m^* = \sigma_m \left[I + \frac{\sigma_f}{(1-c)(\sigma_f - \sigma_m)} C \right], \quad \sigma_f^* = \sigma_f \left[I - \frac{\sigma_f}{c(\sigma_f - \sigma_m)} C \right]$$

(3.26)

$$V_m(x,t) = -\sigma_m \left[I + \frac{\sigma_f}{(1-c)\ (\sigma_f - \sigma_m)} C \right] \nabla U(x,t)\ ,\ V_f(x,t) = -\sigma_f \left[I - \frac{\sigma_f}{c\ (\sigma_f - \sigma_m)} C \right] \nabla U(x,t)$$

The mean m-th and f-th phase cross-forces are

$$\mathrm{P}_m(x,t) = \frac{\sigma_f}{\sigma_f - \sigma_m} C\, \nabla U(x,t)\ ,\ \mathrm{P}_f(x,t) = \frac{\sigma_f}{\sigma_m - \sigma_f} C \nabla U(x,t) \tag{3.27}$$

Then for mean phase cross-flow

$$Q_m(x,t) = (1-c)\, f(x,t) + \sigma_m (1-c) \nabla \left\{ \left[I + \frac{\sigma_f}{(1-c)(\sigma_f - \sigma_m)} C \right] \nabla U(x,t) \right\} - \alpha_m (1-c) \frac{\partial U(x,t)}{\partial t} \tag{3.28}$$

It is obvious that if the effective conductivity tensor is non-isotropic (for this it is sufficient that not all concentrations c_k are equal), the source density $f(x)$ is zero under steady-state flow, and if $\nabla U(x) \neq const$, then the mean phase cross-flow $Q_m(x) \neq 0$.

If the porous medium with fractures is macro-isotropic ($c_1 = c_2 = c_3 = c/3$), we have

$$\sigma^* = \left[\sigma_m (1-c) + \sigma_f (2c/3) \right] I \tag{3.30}$$

$$\sigma_m^* = \sigma_m \left[1 + \frac{\sigma_f\, c}{3\,(1-c)(\sigma_f - \sigma_m)} \right] I \quad , \quad \sigma_f^* = \sigma_f \left[1 - \frac{\sigma_f}{3\,(\sigma_f - \sigma_m)} \right] I \tag{3.31}$$

$$V_m(x,t) = -\sigma_m \left[1 + \frac{\sigma_f c}{3(1-c)(\sigma_f - \sigma_m)} \right] \nabla U(x,t)\ ,\ V_f(x,t) = -\sigma_f \left[1 - \frac{\sigma_f}{3\ (\sigma_f - \sigma_m)} \right] \nabla U(x,t) \tag{3.32}$$

$$\mathrm{P}_m(x,t) = \frac{c\ \sigma_f}{3\ (\sigma_f - \sigma_m)} \nabla U(x,t) \tag{3.33}$$

$$Q_m(x,t) = (1-c)\left\{f(x,t)+\sigma_m\left[1+\frac{c\sigma_f}{3(1-c)(\sigma_f-\sigma_m)}\right]\nabla U(x,t)-\alpha_m\frac{\partial U(x,t)}{\partial t}\right\} \tag{3.34}$$

In this case $Q_m(x) = Q_f(x) = 0$ when $f(x)=0$ and the flow is steady-state.

We have now completed the analysis of the mesoscale equilibrium approximation for multi-continuum models in stochastic media. It should be noted again that the more exact description that takes into account the deviation from the mesoscale equilibrium resulted in a better representation of global averaging.

4. PERIODIC COMPOSITE SYSTEMS

4.1. Problem formulation. Mono-continuum description

Let us now introduce a positive length-dimension parameter ε and determine the functions $\sigma^{\varepsilon}(x) = \sigma(x/\varepsilon)$ and $\alpha^{\varepsilon}(x) = \alpha(x/\varepsilon)$, for which Y-periodic functions $\sigma(y)$ *and* $\alpha(y)$ are εY - periodic in the variable x. As $\varepsilon \to 0$ the edges of the period of these functions tend to zero and, consequently, $\sigma^{\varepsilon}(x)$ and $\alpha^{\varepsilon}(x)$ are a model of the system with small-scale periodic heterogeneity.

Following *Bakhvalov and Panasenko* [1989], we use the standard method of solving the equations (2.1), (2.2) and (2.3) in which $\alpha(x)$ and $\sigma(x)$ are periodic functions. The solution is found in the form of two-scale expansion in the fast $y= x/\varepsilon$ and slow x variables asymptotic with respect to the parameter ε

$$u^{\varepsilon}(x,t)=\sum_{n=0}^{\infty}\varepsilon^n u_n(x,y,t) \quad , \quad v^{\varepsilon}(x,t)=\sum_{n=0}^{\infty}\varepsilon^n v_n(x,y,t) \tag{4.1}$$

where the functions $u_n(x,y,t)$ and $v_n(x,y,t)$ are Y-periodic in the fast variable y.

Substituting (4.1) in the system (2.1), (2.2) and (2.3) that describes the flow in periodic media also, if $\alpha(x)$ and $\sigma(x)$ are periodic functions, and expanding the operators in the powers of ε, we can obtain a set of equations for $u_n\ (x,y,t)$, $v_n\ (x,y,t)$ whose solutions satisfy the expansion (4.1). Averaging these equations over the representative volume of the region ω_Δ (whose meso-scale Δ satisfies the inequalities $\varepsilon << \Delta << l_\Omega$) is equivalent to averaging over the domainY-cell period by means of the operator $\langle f(x,y,t)\rangle = |Y|^{-1}\int_y f(x,y,t)dy$, where $|Y|$ is volume of domain Y.

The averaged system for the mean functions $U(x,t)=\langle u^\varepsilon(x,t)\rangle$, $V(x,t)=\langle v^\varepsilon(x,t)\rangle$ is

$$divV+\alpha^*\frac{\partial\ U}{\partial\ t}+\varepsilon\frac{\partial\bar{\lambda}(x,t)}{\partial t}(x,t)=f\ ,\ \bar{\lambda}(x,t)\ =\langle\alpha(y)(u_1+\varepsilon u_2+...)\rangle \tag{4.2}$$

$$\left(\sigma^*\right)^{-1}V(x,t)+\nabla U(x,t)\ =\ \varepsilon\bar{\gamma}(x,t)\ ,\bar{\gamma}(x,t)=-\ \sigma(y)\left(\nabla_x u_1+\nabla_y u_2+\varepsilon\nabla_x u_2+....\right)\ \rangle \tag{4.3}$$

The effective storage capacity $\alpha^*=const$ and conductivity tensor $\sigma^*=const$ can be written in the form:

$$\alpha^*=\langle\alpha(y)\rangle\ ,\ \ \sigma^*_{ij}=\langle\sigma_{ij}(y)\rangle+\left\langle\sum_k\sigma_{ik}(y)\frac{\partial W^j}{\partial y_k}\right\rangle \tag{4.4}$$

where $W^l(y)$ is the Y-periodic generalized solution of the problem

$$\sum_{i,j}\frac{\partial}{\partial y_i}\left[\sigma_{ij}(y)\frac{\partial W^l}{\partial y_j}\right]=-\sum_i\frac{\partial}{\partial y_i}\sigma_{ij}(y)\ \ ,\ \ \langle W^l(y)\rangle=0 \tag{4.5}$$

The expansion (4.1) for $u^\varepsilon(x,y,t)$ can be written in the form:

$$u^\varepsilon(x,t)=u_0(x,t)+\varepsilon u_1(x,y,t)+\varepsilon^2u_2(x,y,t)+... \tag{4.6}$$

where

$$u_0(x,t) = U(x,t) \quad , \quad u_1(x,y,t) \;=\; \sum_s W^s(y)\frac{\partial U(x,t)}{\partial x_s} \tag{4.7}$$

$$u_2(x,y,t) \;=\; W(y)\frac{\partial U(x,t)}{\partial t} + \sum_{r,p} W^{rp}(y)\frac{\partial^2 U(x,t)}{\partial x_r \partial x_p} \tag{4.8}$$

Here $W(y)$ and $W^{rp}(y)$ are the Y-periodic solutions of the following equations

$$\sum_{k,j}\frac{\partial}{\partial y_k}\left[\sigma_{kj}(y)\,\frac{\partial W(y)}{\partial y_j}\right] \;=\; \alpha(y) - \langle \alpha(y) \rangle \tag{4.9}$$

$$\sum_{k,j}\frac{\partial}{\partial y_k}\left[\sigma_{kj}(y)\frac{\partial W^{rp}(y)}{\partial y_j}\right] = \sigma^*_{rp} - \sigma_{rp}(y) - \sum_k \frac{\partial}{\partial y_k}\left[\sigma_{kr}(y)W^p(y)\right] - \sigma_{rj}(y)\frac{\partial W^p(y)}{\partial y_j} \tag{4.10}$$

that satisfy the conditions

$$\langle W(y) \rangle = 0\,, \langle W^{rp}(y) \rangle = 0 \tag{4.11}$$

Then for $v^{\varepsilon}(x,t)$ we have

$$v^{\varepsilon}(x,t) = -\sigma(y)\left\{\nabla_x u_0(x,t) + \nabla_y u_1(x,y,t) + \varepsilon\left[\nabla_x u_1(x,y,t) + \nabla_y u_2(x,y,t)\right] + \varepsilon^2 \nabla_x u_2(x,y,t) + \ldots\right\} \tag{4.12}$$

The averaged system in equations (4.2) and (4.3) has the same form as the globally averaged stochastic system of equation (2.6) and (2.7). The difference is that, in the periodic case, an explicit procedure for calculating the tensor σ^* and the parameters of the averaged equations scalar $\bar{\lambda}(x,t)$ and vector $\bar{\gamma}(x,t)$ are shown. Furthermore it is especially significant that the explicit expansions (4.6) and (4.12) for fields $u^{\varepsilon}(x,t)$ and $v^{\varepsilon}(x,t)$ are shown here. As demonstrated by Sanchez-Palencia (1980), the $U(x,t)$ and $V(x,t)$ fields, the means over a small representative volume ω_{Δ} in the space of the slow variable x containing sufficiently number of cell-periods, are macroscopic fields. The usual integral conservation conditions for arbitrary macroscopic domains can be written in terms of $U(x,t)$ and $V(x,t)$. The identity of the volume and surface averaging of the velocity field is ensured by satisfying the condition $div_y\ v_0(x,y,t)=0$, which follows from equations (2.1), (2.2), (4.7) and (4.12)

Thus the closed system of equations (4.2) and (4.3) together with the equations from (4.4) to (4.12) describes the mono-continual model of flow in a periodical composite medium in terms of mean head $U(x,t)$ and mean flow velocity $V(x,t)$.

4.2.Multi-continua description

In order to analyze the fields in the phases of a composite system, we introduce the indicator function $z_i(y)$ of the fast variable y defined as

$$z_i(y)=\{1\ \ if\ \ y\in Y_i\ and\ 0\ \ if\ y\in Y\setminus Y_i\ \} \tag{4.13}$$

and the mean value of the function $\varphi(x,y,t)$ of the i- phase, that is local in the space of slow variable x and time t

$$\varphi_i(x,t)=\langle\varphi(x,y,t)\rangle_i=\langle\varphi(x,y,t)\ z_i(y)\rangle\theta_i^{-1}\ ,\quad \theta_i=|Y_i|/|Y| \tag{4,14}$$

where $\theta_i=$ const, which is the volume fraction of the i-th phase in the cell-period.

Now for periodic case we discuss the conditional averaging of the initial system (2.1), (2.2) and (2.3) over the representative volume ω_Δ , taking into account the fact that the conditional averaging operation commutes with differentiation with respect to time and the slow variable x, we obtain the following system of equations for each i-th phase from m-phase composite medium:

$$divV_i^*(x,t)+\alpha_i\theta_i\frac{\partial U_i(x,t)}{\partial\ t}+Q_i(x,t)=\theta_i\ f(x,t) \tag{4.15}$$

$$\sigma_i^{-1} V_i^*(x,t) + \theta_i \nabla U_i(x,t) + \mathrm{P}_i(x,t) = 0 \tag{4.16}$$

Here

$$U_i(x,t) = \langle u^\varepsilon(x,t) \rangle \quad , \quad V_i^*(x,t) = -\sigma_i \theta_i \langle \nabla u^\varepsilon(x,t) \rangle \tag{4.17}$$

$$Q_i(x,t) = -\langle v^\varepsilon(x,t) \nabla z_i(y) \rangle , \mathrm{P}_i(x,t) = -\langle [u^\varepsilon(x,t) - U(x,t)] \nabla z_i(y) \rangle \tag{4.18}$$

Since $\sum_{i=1}^{m} z_i(x) = 1$, we have

$$\sum_{i=1}^{m} Q_i(x,t) = 0 , \sum_{i=1}^{m} \Psi_i(x,t) = 0 \tag{4.19}$$

At this point we need to take into account that any representative volume ω_Δ consists of two parts - ω_Δ^1 and ω_Δ^2 . The first part ω_Δ^1 includes all whole internal cells and the second part ω_Δ^2 includes non-integer cells inside the representative volume ω_Δ and adjoining to the border $\partial\omega_\Delta$. In some of the non-integer cells, the border $\partial\omega_\Delta$ intersects only one phase, in the rest of the cells the border intersects at least two phases. As indicated earlier, in the stochastic case, the cross-flow from dissected inclusions under some conditions (e.g. in steady-state flow) is significantl.

It is easy to see that the conditionally averaged system (4.15), (4.16) and (4.19) is completely identical to the conditionally averaged system (2.23), (2.24) and (2.25), that corresponds to the stochastic composite media. This formal expression is nonrandom because the media with periodical structure are a special case of all the realization of the stochastic field formed by random shift of one periodical structure. In the stochastic problem the treatment of the equation in (2.1) as continuum conservation laws is based on the obvious fact of the multiple and fairly arbitrary dissection by the surface of the representative control volume of various subdomains in the heterogeneous random system . In contrast, in the periodic system, the identity of the conditional means over the macroscopic volume and surface requires the satisfaction of certain additional conditions. It can be shown that for phase flow the equality of the means over the volume and the surface of the cell-period is satisfied by the condition $div_y\, v_0(x,y,t) = 0$ in the absence of sources of the field $z_i(y)v_0(x,y,t)$, which is equivalent to the orthogonality of the velocity $v_o(x,y,t)$ to the phase surface. This is also equivalent to (a) the condition $\sigma_i = 0$, or (b) when $\sigma_i \neq 0$, the system is layered and the flow occurs in the directions of the layers. These conditions considerably limit the class of media in question, which again emphasizes the point that, in the multi-continuum approach the requirement that surface and volume means over each cell be equal is physically unjustified. It should be replaced by the natural condition of equality of the surface and volume means in macroscopic domain containing not only many whole cells but also fractions of cells dissected by the surface of the control volume, which, therefore cannot be arbitrary.

The scalar function $Q_i(x,t)$ and the vector function $\mathrm{P}_i(x,t)$ have a clear physical significance. The $Q_i(x,t)$ is the specific mean cross-flow of fluid from the i-th continuum-phase to the different phases and $\mathrm{P}_i(x,t)$ is the mean specific force from the other phases acting on the surface bounding the i-th phase.

The system (4.15), (4.16) and (4.19) is closed since after substitution of the expressions (4.6) and (4.12) into equations (4.17 and (4.18) the functions $Q_i(x,t)$ and $\mathrm{P}_i(x,t)$ have been evaluated in terms $U_i(x,t)$. The system can be treated as the exact flow equations in the i -th continuum. Equation (4.15) is the mass balance for the continuum-phase and equation (4.16) is the modified Darcy's law – the impulse balance for the phase.

Considering the system (4.15), (4.16) and (4.19) together with the system (4.2), (4.3) in the multiphase case results in a closed set of equations, that link the conditional mean fields

$U_i(x,t)$, $V_i(x,t)$, $Q_i(x,t)$, $\mathrm{P}_i(x,t)$ and $G_i(x,t)=\theta_i\langle\nabla u(x,t)\rangle_i$ with the conditional phase flow $q_i(x,t)$ by the relation

$$q_i(x,t) = Q_i(x,t) + div\ V_i^*(x,t) \tag{4.20}$$

in terms of the averaged field $U(x,t)$ that can be found from global system (4.2) and (4.3).Combining the systems (4.2),(4.3) and (4.15),(4.16) and (4.19) we can write the two-phase composite meso-scale approximation of the phase fields and the interaction parameters as

$$U_i(x,t)=U(x,t)\ ,\ V_i^*(x,t)=-\ \theta_i\sigma_i^*\nabla U(x,t)\ ,\quad \sigma_i^*=\theta_i^{-1}\sigma_i(\sigma_j-\sigma_i)^{-1}(\sigma_j-\sigma^*) \tag{4.21}$$

$$Q_i(x,t)=\theta_i\left[f(x,t)\ +\ div\left(\sigma_i^*\nabla U\right)-\alpha_i\frac{\partial U(x,t)}{\partial t}\right] \tag{4.22}$$

$$\mathrm{P}_i(x,t)=\left(\sigma_j-\sigma_i\right)^{-1}\left(\langle\sigma\rangle-\sigma^*\right)\nabla U(x,t)\ ,\ G_i(x,t)=\theta_i\sigma_i^{-1}\sigma_i^*\nabla U(x,t) \tag{4.23}$$

$$q_i(x,t)=\theta_i f(x,t)-\alpha_i\theta_i\frac{\partial U(x,t)}{\partial t}\quad or\quad q_i(x,t)=\theta_i f(x,t)-\frac{\alpha_i}{\langle\alpha\rangle}\theta_i div\left(\sigma^*\nabla U(x,t)\right) \tag{4.24}$$

The identity of the globally and conditionally averaged equations for periodic and stochastic media leads to the complete formal coincidence of the characteristics presented in section 2 of this paper. The result concerning the finiteness of the cross-flow $Q_i(x,t)$ in systems with anisotropy for steady-state process remains valid. Note if $\sigma^*/\langle\alpha\rangle=\sigma_i^*/\alpha_i$, that is $\kappa^*=\sigma^*/\langle\alpha\rangle$ - the effective diffusivity for composite is equal to i-th phase effective diffusivity $\kappa_i^*=\sigma_i^*/\alpha_i$, the cross-flow $Q_i(x,t)=0$ for any non-steady-state flow.

5.APPLICATION EXAMPLES

As already mentioned above, for closure purposes of the phenomenological theories , the hypothesis concerning the structure of the transfer terms between the phase continua are used . In particular, *Rubinstein* [1948],*Barenblatt, Zheltov and Kochina* [1960], *Khoroshun and Soltanov* [1984] assumed a proportional relationship between the cross flow and the phase pressure difference (the phase temperature difference in similar heat transport problems). In the case of periodic systems this hypothesis can be tested by direct calculation.

Applying the conditional averaging operator $\langle\ \rangle_i$ to the expressions (4.6), (4.7) and (4.8) we obtain the following quadratic (in ε) expression for the mean i-th phase pressure :

$$U_i(x,t) = U(x,t) + \varepsilon \sum_s \beta_i^s \frac{\partial U(x,t)}{\partial x_s} + \varepsilon^2 \left[\beta_i \frac{\partial U(x,t)}{\partial t} + \sum_{r,p} \beta_i^{rp} \frac{\partial^2 U(x,t)}{\partial x_r \partial x_p} \right] \tag{5.1}$$

$$\beta_i^s = \langle W^s(y) \rangle_i \ , \quad \beta_i = \langle W(y) \rangle_i \ , \quad \beta_i^{rp} = \langle W^{rp}(y) \rangle_i \tag{5.2}$$

As can be seen from the system of equations (4.5),(4.9) and (4.10), the signs and the modules of the vector components β_i^s and the tensor-components β_i^{rp} depend on the conductivity field $\sigma(x)$ only , but the scalar β_i depends on both $\sigma(x)$ and $\alpha(x)$ fields.

If the cell-period Y contains two phases i and j , we have

$$\theta_i\, \beta_i^s + \theta_j\, \beta_j^s = 0 \ , \quad \theta_i\, \beta_i + \theta_j \beta_j = 0 \ , \quad \theta_i\, \beta_i^{rp} + \theta_j\, \beta_j^{rp} = 0 \tag{5.3}$$

and the phase head difference $\Delta_{ij}(x,t) = U_i(x,t) - U_j(x,t)$ has the form

$$\Delta_{ij}(x,t) = \theta_j^{-1} \left\{ \varepsilon \sum_s \beta_i^s \frac{\partial U(x,t)}{\partial x_s} + \varepsilon^2 \left[\beta_i \frac{\partial U(x,t)}{\partial t} + \sum_{r,p} \beta_i^{rp} \frac{\partial^2 U(x,t)}{\partial x_r \partial x_p} \right] \right\} \tag{5.4}$$

and in general $\Delta_{ij}(x,t)$ is **the first order** in small parameter ε.

As shown by *Bakhvalov and Panasenko* [1989], if in the cell-period Y the tensor $\sigma(y)$ has certain symmetry properties, then the functions $W^s(y)$ will possess corresponding symmetry. In particular, if the plane $y_h = 0$ is a plane of symmetry of the tensor $\sigma(y)$, then the function $W^h(y)$ will be odd with respect to the variable y_h, and consequently $\beta_i^h = 0$.

We will consider the case in which the tensor $\sigma(y)$ is symmetric about all the coordinate planes, for example, there is a spherical inclusion at the center of the space cell. If $\sigma(y)$ is an isotropic tensor, then this periodic system will be micro-isotropic and macro-isotropic and the effective conductivity tensor is spherical ($\sigma^*_{rp} = \sigma^*_o \delta_{rp}$). Because of symmetry $\beta_i^s = 0$, $\beta_i^{rp} = \beta_i^* \delta_{rp}$ and the parameter $\eta_{ij}(x,t) = \Delta_{ij}(x,t)/\varepsilon^2$ is obtained from (5.4) as

$$\eta_{ij}(x,t) = \theta_j^{-1}\left(\beta_i \frac{\partial U(x,t)}{\partial\ t} + \beta_i^* \nabla^2 U(x,t)\right) \tag{5.5}$$

The general conclusion that the vector $\beta_i^s = 0$ and the tensor β_i^{rp} is proportional to the unit-tensor δ_{rp} is implied from the fact that the completely isotropic field $\sigma(x)$ determines a unique zero –vector and a unit-tensor.

Comparing the expression (5.5) with the globally averaged system (4.2) and (4.3), and eliminating $\nabla^2 U(x,t)$ from (5.5), we obtain with the same accuracy :

$$\eta_{ij}(x,t) = \theta_j^{-1}\left(\beta_i + \beta_i^* \frac{\langle\alpha\rangle}{\sigma_0^*}\right)\frac{\partial U(x,t)}{\partial t} \tag{5.6}$$

It is easy to show that the when $f(x,t)$ same approximation for microscopic and macroscopic isotropic systems in (4.22) and (4.24) leads to relations

$$Q_i(x,t) = \theta_i (\sigma_0^*)^{-1}\left[\langle\alpha\rangle\sigma_i^* - \alpha_i\sigma_0^*\right]\frac{\partial U(x,t)}{\partial t} \ , \ q_i(x,t) = -\theta_i\alpha_i \frac{\partial U(x,t)}{\partial t} \tag{5.7}$$

Thus, $\eta_{ij}(x,t)$ with the same accuracy is approximately proportional to $Q_i(x,t)$ or $q_i(x,t)$ and

$$Q_i(x,t) = \frac{\theta_i\theta_j\left[\langle\alpha\rangle\sigma_i^* - \alpha_i\sigma_0^*\right]}{\varepsilon^2\left[\beta_i\sigma_0^* + \beta_i^*\langle\alpha\rangle\right]}\left(U_i(x,t) - U_j(x,t)\right), \tag{5.8}$$

$$q_i(x,t) = -\frac{\theta_i\theta_j\alpha_i\sigma_0^*}{\varepsilon^2\left[\beta_i\sigma_0^* + \beta_i^*\langle\alpha\rangle\right]}\left(U_i(x,t) - U_j(x,t)\right),\ Q_i(x,t) = \left[1 - \frac{\langle\alpha\rangle\sigma_i^*}{\alpha_i\sigma_0^*}\right]q_i(x,t) \tag{5.9}$$

If the symmetry conditions are not satisfied for the tensor $\sigma(y)$, then, in order to estimate the order of the pressure difference with respect to the parameter ε, it is necessary to solve a fairly complex problem for the cell-period.

5.1.Case 1

As an example we consider the problem of two-dimensional cell depicted in Fig.3 and let (a) $\sigma(y) = \sigma\varpi \ \ if \ \ y_1 < y_2 \ \ and\ (b) \ \ \sigma(y) = \sigma \ \ if \ \ y_1 > y_2$ where the arbitrary parameter $\varpi >> 1$. After solving the cell-problem asymptotically with respect to the large parameter ϖ (see Bakhvalov and Panasenko, 1989) we determine that the head difference can be written as

$$\Delta_{21}(x,t) = U_2(x,t) - U_1(x,t) = 0.0386470\,\varepsilon\left(\frac{\partial U(x,t)}{\partial x_1} - \frac{\partial U(x,t)}{\partial x_2}\right) \tag{5.10}$$

Obviously, the quantity $\Delta_{21}(x,t)$ is positive if the vector $\nabla U(x,t)$ is directed from the domain of high conductivity D_2 into the domain D_1; otherwise it is negative. If the vector $\nabla U(x,t)$ is directed along the phase interface in the cell $(y_1 = y_2)$, then the head difference will be zero. We note that for the global averaged steady and spatially homogeneous flow, the expression for $U_2(x) - U_1(x)$ being linear in ε is exact, and the cross-flow $Q_1(x) = Q_2(x) = 0$.

5.2.Case 2

We will consider a problem that can be solved exactly as above. Let, the cell-period have the form depicted in Fig.4. Solving asymptotically the corresponding problem for the cell, we obtain the head difference:

$$\Delta_{21}(x,t) = U_2(x,t) - U_1(x,t) = 0.0338216\,\varepsilon \frac{\partial U(x,t)}{\partial x_1}, \tag{5.11}$$

which is positive for a vector $\nabla U(x,t)$ directed into the right-hand half plane ; otherwise it is negative.

It is obvious that creating the finite quantity with dimension like cross-flow from the head difference is possible after dividing the head difference by ε^2. In our case this operation leads to an unlimited amount of cross-flow when $\varepsilon \to 0$ and therefore, the linear proportional dependence between cross flow and head difference does not exist.

5.3. Case 3

We will now consider the case of an inhomogeneous layered system for which all the computations can be performed in by quadratic approximation in small parameter ε. In the cell-period $(|y_3| \le 1/2)$ let the parameters be as follows: the capacity $\alpha(y) = \alpha_1 \; if \; y_3 > y_0 \quad and \; \alpha(y) = \alpha_2 \; if \; y_3 < y_0$ and the tensor of conductivity $\sigma(y) = \sigma_1 \; if \; y_3 > y_0 \quad and \quad \sigma(y) = \sigma_2 \; if \; y_3 < y_0$.

For example, let $y_0 = 0$, i.e., $\theta_1 = \theta_2 = 1/2$. Solution of the equation (4.5), (4.9), (4.10) and (4.11) leads to the determination of $W^s(y)$, $W(y)$, $W^{rp}(y)$. Because of symmetry $\beta^s = 0$ the expressions $\eta_{12}(x,t) = (U_1(x,t) - U_2(x,t))/\varepsilon^2$, $Q_1(x,t)$, $q_1(x,t)$ take the form

$$\eta_{12}(x,t)=\frac{1}{6\sigma_{33}^{*}}\left[\left(\frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}-\frac{\sigma^2{}_{11}-\sigma_{11}^1}{\sigma_{11}^1+\sigma_{11}^2}\right)\sigma_{11}^{*}\frac{\partial^2 U(x,t)}{\partial x_1^2}+\left(\frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}-\frac{\sigma_{22}^2-\sigma_{22}^1}{\sigma_{22}^1+\sigma_{22}^2}\right)\sigma_{22}^{*}\frac{\partial^2 U(x,t)}{\partial x_2^2}+\right.$$

$$\left.+\left(\frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}+\frac{\sigma_{33}^2-\sigma_{33}^1}{\sigma_{33}^1+\sigma_{33}^2}\right)\sigma_{33}^{*}\frac{\partial^2 U(x,t)}{\partial x_3^2}\right] \tag{5.12}$$

$$Q_1(x,t)=\frac{1}{2\langle\alpha\rangle}\left[\left(\sigma_{11}^1\langle\alpha\rangle-\alpha_1\sigma_{11}^{*}\right)\frac{\partial^2 U}{\partial x_1^2}+\left(\sigma_{22}^1\langle\alpha\rangle-\alpha_2\sigma_{22}^{*}\right)\frac{\partial^2 U}{\partial x_2^2}+\left(\langle\alpha\rangle-\alpha_1\right)\sigma_{33}^{*}\frac{\partial^2 U}{\partial x_3^2}\right] \tag{5.13}$$

$$q_1(x,t)=-\frac{\alpha_1}{2\langle\alpha\rangle}\left(\sigma_{11}^{*}\frac{\partial^2 U(x,t)}{\partial x_1^2}+\sigma_{22}^{*}\frac{\partial^2 U(x,t)}{\partial x_2^2}+\sigma_{33}^{*}\frac{\partial^2 U(x,t)}{\partial x_3^2}\right) \tag{5.14}$$

We note that in these expressions the derivative with respect to time has been eliminated with the aid of the globally averaged equation

$$\langle\alpha\rangle\frac{\partial U(x,t)}{\partial t}=\sum_i \sigma_{ii}^{*}\frac{\partial^2 U(x,t)}{\partial x_i^2}\ ,\ \sigma_{11}^{*}=\langle\sigma_{11}\rangle\ ,\ \sigma_{22}^{*}=\langle\sigma_{22}\rangle\ ,\ \sigma_{33}^{*}=\left\langle(\sigma_{33})^{-1}\right\rangle^{-1} \tag{5.15}$$

It is easy to see from (5.12) and (5.13) that in the general case of unsteady three-dimensional flow for arbitrary layered systems, the head difference $U_1(x,t)-U_2(x,t)$ is not proportional to $Q_1(x,t)$ or $q_1(x,t)$.

However, under certain conditions proportionality may be observed. In concluding the present study we make the following observations.

1. If the transverse component of conductivity of the layers are equal $(\sigma_{33}^1=\sigma_{33}^2=\sigma_{33})$, then for three-dimensional non-steady flow

$$Q_1(x,t)=3\ \sigma_{33}\,\varepsilon^{-2}\left(U_1(x,t)-U_2(x,t)\right) \tag{5.16}$$

2. If the layers are isotropic but inhomogeneous and the globally averaged flow has a transverse component, then the cross flow $Q_i(x,t)$ and the head difference are not proportional.

3. If the global flow is purely longitudinal $(\partial U(x,t)/\partial x_3 = 0)$, the cross flow and the head difference are related trough the expression

$$Q_1(x,t) = \frac{6\sigma_1\sigma_2\,\varepsilon^{-2}}{\sigma_1+\sigma_2}(U_1(x,t)-U_2(x,t)) \tag{5.17}$$

4. For purely one-dimensional unsteady transverse flow$\left(\partial U(x,,t)/\partial x_1 = \partial U(x,t)/\partial x_2 = 0\right)$, the relationship between $Q_1(x,t)$ and $(U_1 - U_2)$ is given by

$$Q_1(x,t) = \frac{3\sigma_2(\alpha_0-1)}{\alpha_0\sigma_0-1}\varepsilon^{-2}\left(U_1(x,t)-U_2(x,t)\right) \tag{5.18}$$

where $\alpha_0 = \alpha_2/\alpha_1$, $\sigma_0 = \sigma_2/\sigma_1$.

In Fig.5, the first quadrant of the plane (α_0, σ_0) is divided by the straight line $\alpha_0 = 1$, along which the cross-flow is zero and the hyperbola $\alpha_0\sigma_0 = 1$, along which the head difference is zero, into four regions, within which the sign of the proportionality factor is constant. Obviously, in region II and IV the sign of cross flow $Q_1(x,t)$ and that of head difference are opposite, which implies that cross flow occurs from the phase with the reduced head into the phase in which the head is higher than mean. We note that in phenomenological constructions using the proportionality hypothesis it is routinely assumed that the signs are identical. It seems likely that an analogous situation exist for fully isotropic systems. Because the denominator as well as the numerator in (5.8) can be positive or negative, the coefficient in front of the head difference may be negative by some combination of the σ and α fields.

5. If in a layered system with isotropic layers the flow is in a steady-state limit, the cross-flow and the head difference are proportional and have the same sign

$$Q_1(x) = \frac{3\sigma_1\sigma_2}{\sigma_1 + \sigma_2}\varepsilon^{-2}\left(U_1(x) - U_2(x)\right) \tag{5.19}$$

6. In all the cases involving the same layered system, when any of the derivatives entering into the globally averaged equation (5.15) vanishes, the cross flow and the head difference will be proportional. However, in accordance with (5.17), (5.18) and (5.19) the proportionality factor will depend to a considerable extent on the process considered. In the case of a purely transverse flow the proportionality factor may change sign depending on the relation between the conductivity and the capacities of the layer. Thus, the proportionality factor depends not only on the geometric and physical parameters of the layered system, but also on the process realized, or more precisely, on the macroscopic boundary conditions.

Comparing the following examples we can see how sensitive this dependence is to the process. Let us compare purely longitudinal flow in quasi-steady period with purely steady-state and quasi-longitudinal flow (expression (5.17) and (5.19)). Obviously, the proportionality factors differ by a factor of two. A similar comparison between purely transverse flow in the quasi-steady period and steady-state, quasi-transverse flow leads to a comparison of expression (5.18) and (5.19). In this case it is possible to observe not only a quantitative but also a qualitative difference in the proportionality factors. We note that in accordance with (5.12),(5.13) and (5.14) the proportionality factor is a ratio of linear combinations of the derivatives entering into the global averaged equation. Obviously, in the neighborhood of zero values of all derivatives the behavior of this ratio will depend on the rate at which each derivative tends to zero and, in principle, may be arbitrary. In other words, the proportionality factor essentially depends on the process presented in the composite system. Similar conclusions follow from an examination of layered systems composed of anisotropic layers. Consequently, we can state that fairly generally such parameters of the multi-continuum description as the cross-flows $Q_i(x,t)$, the head difference $U_i(x,t) - U_j(x,t)$, and moreover, the flows $q_i(x,t)$ are pair-wise independent in the sense that, except for certain special situations, none of these parameters is proportional to each others.

7. Thus, having evaluated the hypothesis of the proportionality of the cross-flow to the mean head difference, we can, considering its region of applicability (micro and macro isotropic composite medium) regard the hypothesis as generally unsatisfactory.

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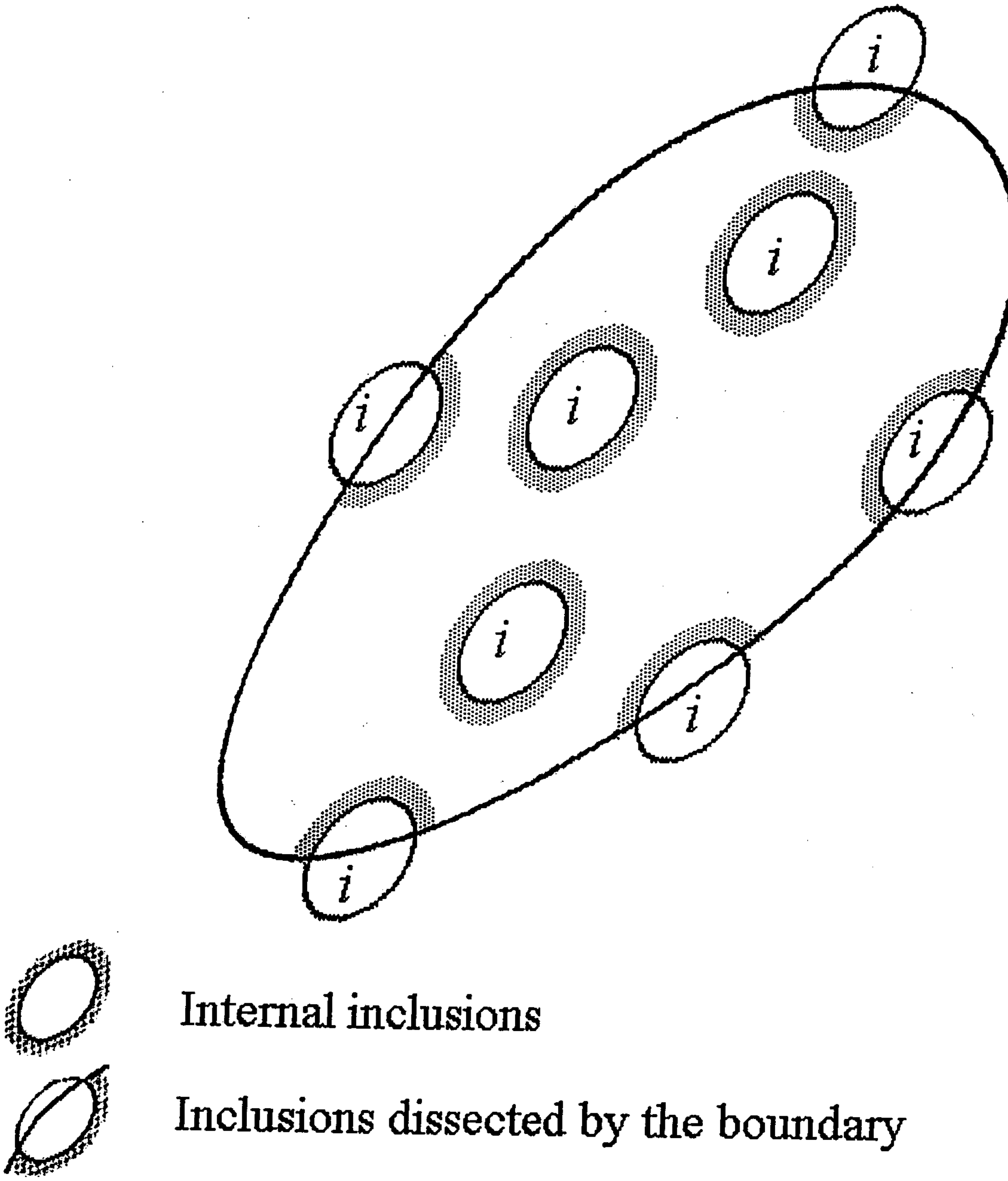
List of Figures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Internal inclusions
Inclusions dissected by the boundary

Figure 1

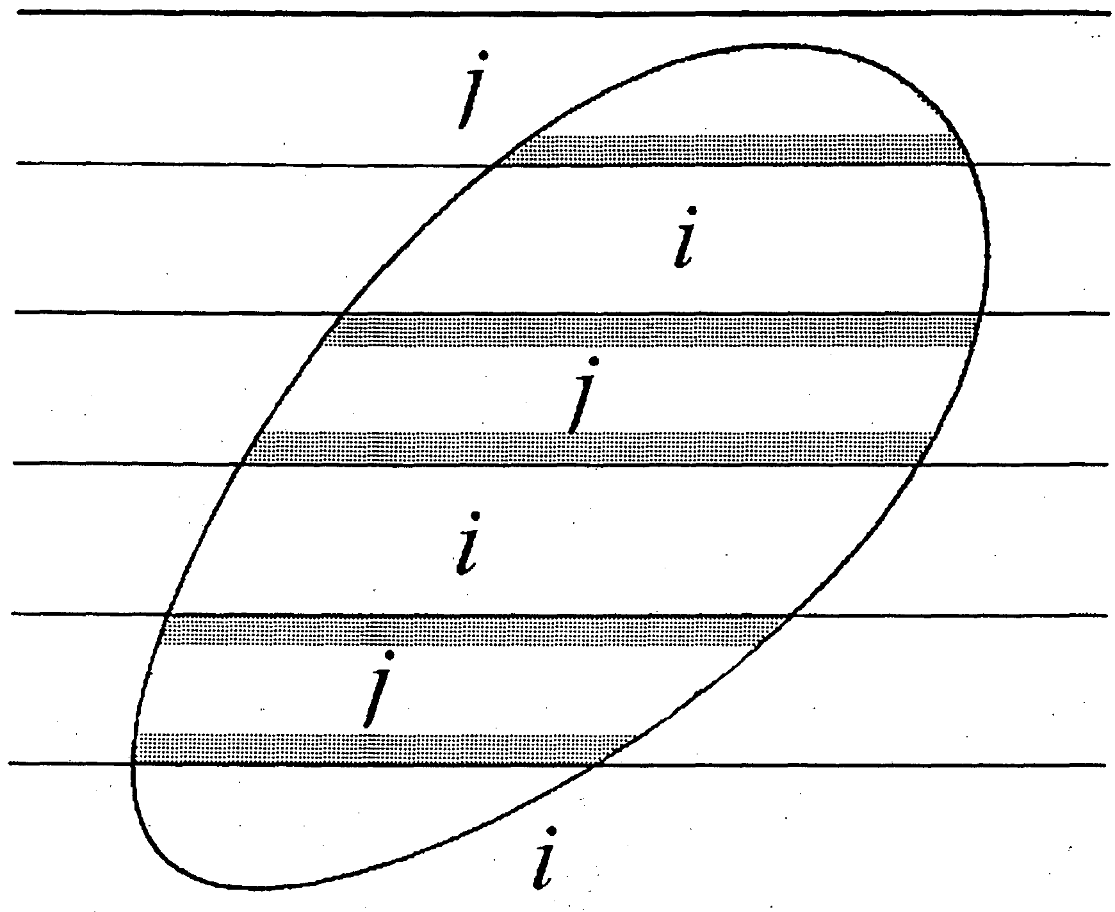

Figure 2

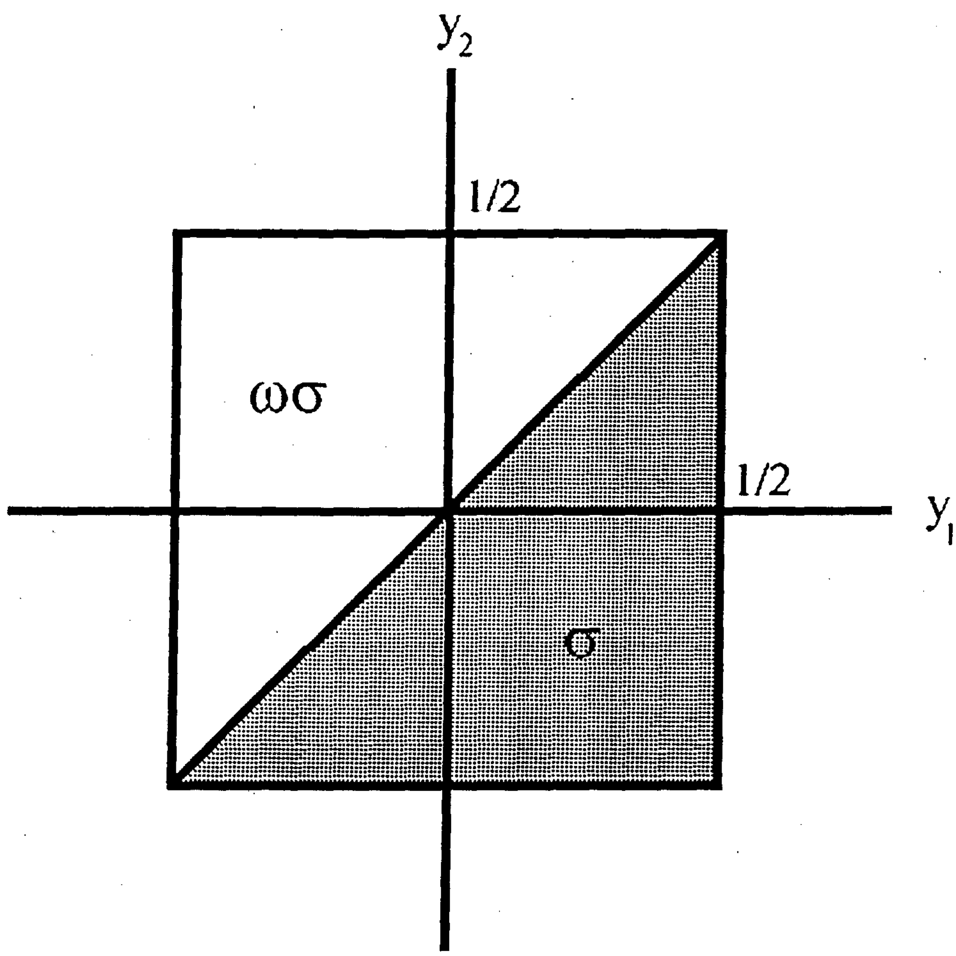

Figure 3

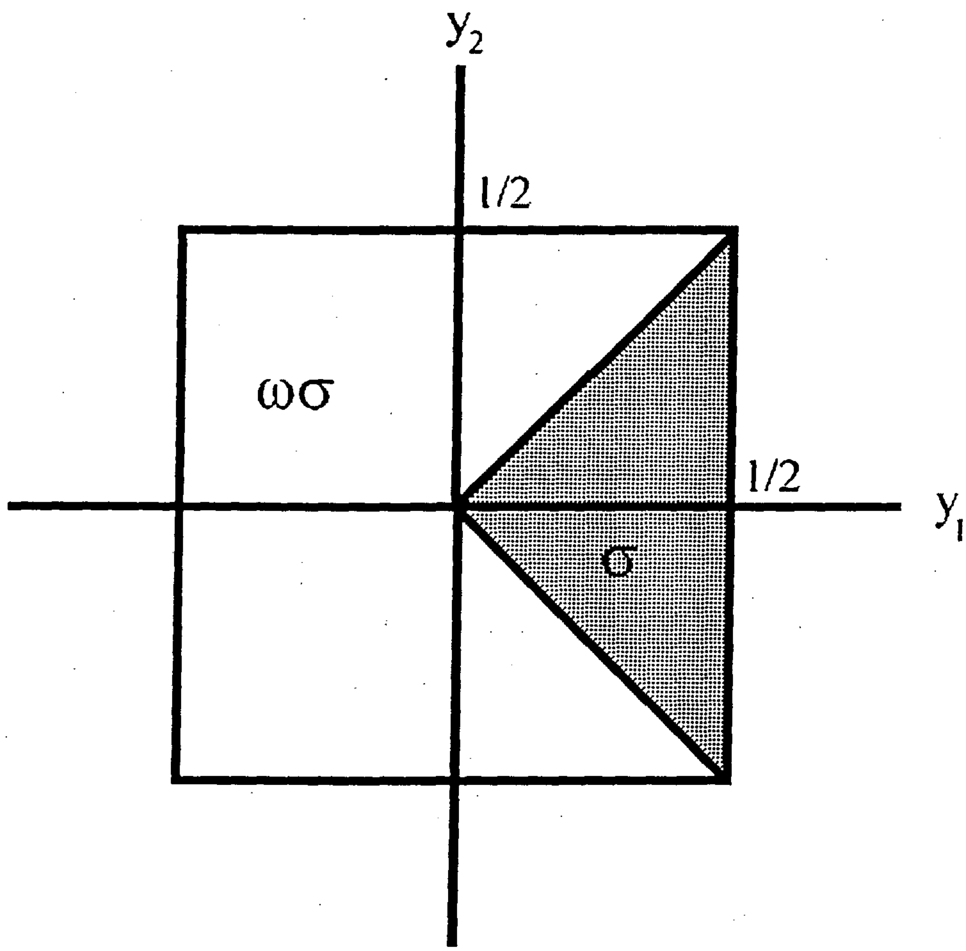

Figure 4

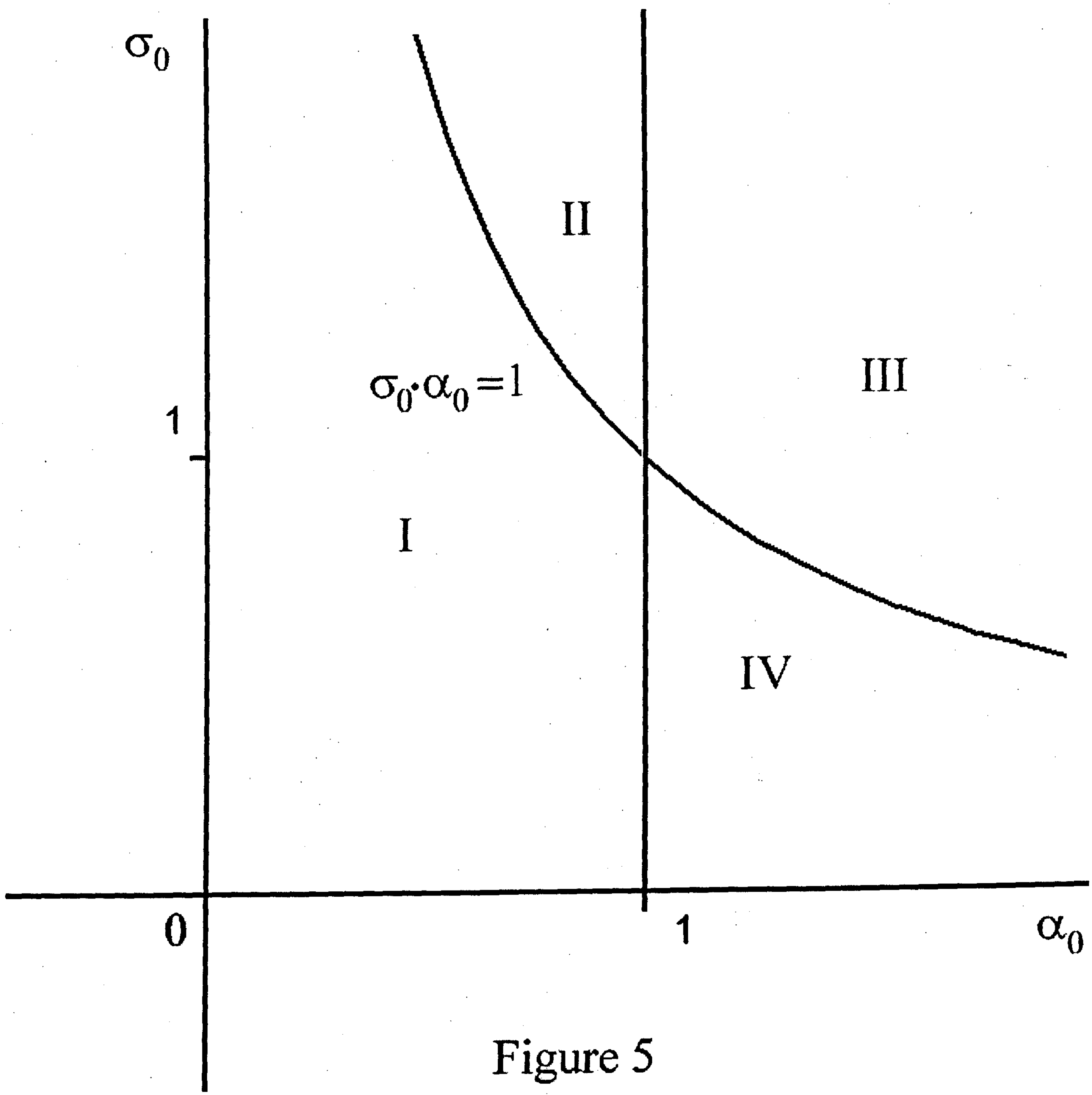

Figure 5

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