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CORRELATION FUNCTIONS, FUSION RULES, AND THE CLASSICAL YANG-BAXTER EQUATION OF VERTEX OPERATOR ALGEBRAS

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Jianqi Liu

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The Dissertation of Jianqi Liu
is approved:

Professor Chongying Dong, Chair

Professor Haisheng Li

Professor Junecue Suh

Peter Biehl
Vice Provost and Dean of Graduate Studies

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Abstract

Correlation functions, fusion rules, and the classical Yang-Baxter equation of vertex operator algebras

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We introduce the notion of space of correlation functions associated with three modules M^1, M^2, and M^3 over a vertex operator algebra V. By studying the relations between the space of correlation functions with the space of intertwining operators and the bi-modules over Zhu's algebra $A(V)$, we prove a generalized version of the fusion rules theorem for vertex operator algebras. We also give the analog of Rota-Baxter operators for vertex operator algebras as a generalization of the Rota-Baxter operators for Lie algebras. We find some particular types of sub-algebras of the lattice vertex operator algebra V_L to give examples of such operators. Using a general version of Rota-Baxter operators of vertex operator algebra, we find a tensor form of the Yang-Baxter equations for vertex operator algebras that generalizes the classical Yang-Baxter equation for Lie algebras.

To my parents

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Chapter 1

Introduction

This thesis has two primary objectives. The first one is to systematically study the system of correlation functions associated with modules and intertwining operators over vertex operator algebras (see [12, 16, 29, 27]). The second one is to find the analog of Rota-Baxter operators (see [43, 12, 31]) and classical Yang-Baxter equations (see [5, 3, 4]) for vertex operator algebras. Since these are different topics in the field of vertex operator algebras, we separate this thesis into two parts.

In part I of this thesis, after recalling the basics of vertex operator algebras, we study the system of correlation functions associated with three modules M^1, M^2, and M^3 over a vertex operator algebra V. The ultimate goal of part I is to give an alternative version of the fusion rules theorem (see [30, 49]) that allows us to compute the fusion rules of modules over vertex operator algebras by determining certain bi-modules over the algebra $A(V)$ defined Zhu in [73].

In part II of this thesis, we give definitions and examples of Rota-Baxter operators (see [43, 10, 31]) on vertex (operator) algebras and study their basic properties. The ultimate goal of part II is to use a generalization of the Rota-Baxter operators on vertex operator algebras, introduce a notion of Yang-Baxter equations for vertex operator algebras and justify its well-definiteness by relating it with the classical Yang-Baxter equations for Lie algebras.

We will also provide some new results and find some interesting substructures for vertex operator algebras when we achieve these two primary goals. We will give an overview of them in the rest of the introduction. Another central theme of this thesis is the study of Zhu's algebra $A(V)$, which is a fundamental object in the theory of vertex operator algebras and was studied extensively, see for instance [2, 18, 19, 49, 30, 73]. We find that $A(V)$ is noetherian for a strongly finitely generated vertex operator algebra V, and we will give a concrete description of $A(V)$ for a sub-algebra of the lattice vertex operator algebra [29]. In the rest of this thesis, we will abbreviate the term vertex operator algebra by "VOA" for simplicity.

1.1 Correlation functions and fusion rules of vertex operator algebras

The space of intertwining operators (see [27]) of VOAs and its dimension, the so-called fusion rule in the physics literature [61, 65, 66], plays an essential role in studying the tensor product of modules over VOAs, see [41, 53]. In the semi-simple case, the fusion rule is the multiplicity of an irreducible module in a tensor product. For the affine Lie algebras or the associated affine VOAs [30], the fusion rules in case $\widehat{sl_2(\mathbb{C})}$ were computed in [65], and a general version was stated in [66] without proof. In [30], Frenkel and Zhu proposed a formula (Theorem 1.5.2 in [30]) to compute the fusion rules for arbitrary vertex operator algebras by using Zhu's algebra $A(V)$ defined in [73] and some of its (bi)modules. Given irreducible modules M^1, M^2 and M^3 over a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$, Frenkel and Zhu's fusion rules theorem claimed that the space of intertwining operators $I\binom{M^3}{M^1\ M^2}$ could be identified with the vector space $(M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))^*$, where $A(M^1)$ is a bimodule over the Zhu's algebra $A(V)$, and $M^2(0)$ and $M^3(0)$ are the bottom levels of the V-modules M^2 and M^3, which are modules over $A(V)$, see Section 1 in [30] for more details.

Since $A(V)$ is an essential object in the fusion rules theorem, and our objective is to give a general version of this theorem, we will first study $A(V)$ for irrational VOAs in more detail in Chapter 2. Note that if V is rational, then by the main result in [18] Zhu's algebra $A(V)$ is finite-dimensional semi-simple over $\mathbb{C}$. In particular, $A(V)$ is (left) noetherian as a ring. In fact, the noetherian property of $A(V)$ also holds for some irrational VOAs. For example, if $V = M_{\widehat{\mathfrak{h}}}(1, 0)$ be the level one Heisenberg VOA (cf.[29]), then $A(M_{\widehat{\mathfrak{h}}}(1, 0))$ is isomorphic to $S(\mathfrak{h})$ the polynomial ring over $\mathfrak{h}$, hence $A(M_{\widehat{h}}(1, 0))$ is noetherian. More generally, if $V = V_{\widehat{\mathfrak{g}}}(k, 0)$ the level $k \in \mathbb{Z}_{>0}$ vacuum module VOA associated to a finite-dimensional simple Lie algebra $\mathfrak{g}$, then $A(V_{\widehat{\mathfrak{g}}}(k, 0)) \cong U(\mathfrak{g})$ (cf. [30]) which is noetherian as well. And if $V = \bar{V}(c, 0)$ is the Virasoro VOA of central charge c, then $A(\bar{V}(c, 0)) \cong \mathbb{C}[x]$ (cf. [68]). To explain this phenomenon, we rediscovered the epimorphism between the C_2-algebra $R(V)$ and the graded algebra $\mathrm{gr}A(V)$ in

[2], and we use this epimorphism and prove the following (see Theorem 2.2.5):

Theorem 1. *Let V be a CFT-type VOA. If V is strongly finitely generated, or equivalently, C_1-cofinite, then $A(V)$ is (left) noetherian as an algebra.*

The notion of correlation functions on the Riemann surface arose in the conformal field theory and quantum field theory, see [61, 66, 65]. It was first interpreted by the language of VOAs by Frenkel, Lepowsky, and Meurman in [29]. By finding the explicit expression of the correlation functions on $\mathbb{P}^1(\mathbb{C})$ associated to the lattice model, Frenkel, Lepowsky, and Meurman proved the Jacobi identity for the lattice VOAs V_L and the moonshine module VOA $V^\natural$ in [29]. Later, Frenkel and Zhu constructed the affine VOAs (WZNW model) and the Virasoro VOAs (minimal model) by constructing correlation functions on $\mathbb{P}^1(\mathbb{C})$ associated to the highest weight representations of the affine Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t, t^{-1}]\oplus\mathbb{C}K$ and the Virasoro Lie algebra $\mathcal{L} = \bigoplus_{n\in\mathbb{Z}}\mathbb{C}L_n \oplus \mathbb{C}C$, see [30]. Therefore, correlation functions play a central role in constructing these fundamental examples of VOAs. Furthermore, the correlation functions on a torus $\mathbb{C}/\Gamma$ were given by the trace functions associated to a module: $\mathrm{tr}|_M Y_M(a_1, z_1)\dots Y_M(a_n, z_n)q^{L(0)-c/24}$. The recursive formulas for trace functions lead to the modular invariance of characters of the strongly rational VOAs. See [73, 74] for more details.

The correlation functions associated with a module M over a VOA V is given by $\langle v', Y_M(a_1, z_1)\dots Y(a_n, z_n)v\rangle$, where $v' \in M'$, $v \in M$, and $a_1, \dots, a_n \in V$. It is closely related to Zhu's algebra $A(V)$. Zhu used a recursive formula of such correlation functions restricted onto the bottom level $M(0)$ and constructed an irreducible V-module from an irreducible $A(V)$-module, see Theorem 2.2.1 in [73]. Frenkel and Zhu also claimed in [74, 30] that a similar method can be applied to the proof of the fusion rules theorem (Theorem 1.5.2 in [30]), and the details of the proof were omitted. However, it was later realized by Li (see [49]) that some additional conditions are needed in Frenkel and Zhu's fusion rules theorem. Li gave a counter-example in [49] in the case of the universal Virasoro vertex operator algebra that shows that $I\binom{M^3}{M^1\, M^2}$ is not isomorphic to $(M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))^*$ in general. Li also proved in [49] that the fusion rules theorem is true when M^2 and M^3 are the so-called generalized Verma modules constructed in [18]. In particular, it is true for the rational vertex operator algebras (see Section 2 in [49] for more detailed discussions and the counter-example).

In Chapter 3 and Chapter 4, we use a correlation function method as Frenkel and Zhu claimed in [30] and give an alternative proof (of an alternative version) of the fusion rules theorem. The correlation functions we will be focusing on are based on the following prototype $(n+3)$-point correlation functions on $\mathbb{P}^1(\mathbb{C})$:

$$(v_3', Y_{M^3}(a_1, z_1)\dots Y_{M^3}(a_k, z_k) I(v, w) Y_{M^2}(a_{k+1}, z_{k+1})\dots Y_{M^2}(a_n, z_n) v_2), \tag{1.1.1}$$

where $v_3' \in M^3(0)^*$, $v \in M^1$, $v_2 \in M^2$, $a_1, \dots, a_n \in V$, $z_1, \dots, z_n, w$ are complex variables, and I is an intertwining operator of type $\binom{M^3}{M^1\ M^2}$.

Our approach to the fusion rules theorem can be broken down into three steps. First, we introduce a notion of space of correlation functions associated with V-modules M^1, M^2, and M^3, denoted by $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$. The axioms we impose on an element in $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$ were essentially the properties satisfied by the limit rational function of (1.1.1). In fact, $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$ is essentially a quotient of the vector space of three-point genus zero conformal blocks, the dual space to a certain quotient of the tensor product of 3 admissible V-modules (see [66, 72]). Then we prove the following (see Theorem 3.1.5 and Corollary 3.1.6):

Theorem 2. *Let V be a VOA, and M^1, M^2, and M^3 are V-modules. Then $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$ is isomorphic to $I\binom{M^3}{M^1\ M^2}$ as vector spaces.*

In order to relate $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$ with the modules over $A(V)$, we introduce an auxiliary notion of the space of correlation functions associated with M^1, $M^2(0)$, and $M^3(0)$, denoted by $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. This space can be viewed as the space $A(V)$-conformal blocks on the 3-pointed rational curve $\mathbb{P}^1_{\mathbb{C}}$ defined from the representations of Zhu's algebra $A(V)$. The axioms we impose on this space are based on the restriction of the correlation functions (1.1.1) onto the bottom levels $M^2(0)$ and $M^3(0)^*$. In particular, we require a system of correlation functions S in $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ to satisfy two recursive formulas obtained from the expansion of (1.1.1) with respect to the left-most term $Y_{M^3}(a_1, z_1)$ and the right-most term $Y_{M^2}(a_n, z_n)$, one of which is similar to (2.2.1) in [73]. Then our second step is to prove the following isomorphism (see Corollary 3.3.8 and Theorem 3.3.9):

Theorem 3. *Let M^1 be a V-module, and let $M^2(0)$ and $M^3(0)$ be irreducible $A(V)$-modules, then we have the following isomorphism of vector spaces:*

$$\mathrm{Cor}\begin{pmatrix} M^3(0) \\ M^1\ \ M^2(0)\end{pmatrix} \cong \mathrm{Cor}\begin{pmatrix} \bar{M}^3 \\ M^1\ \ \bar{M}^2\end{pmatrix} \cong \mathrm{Cor}\begin{pmatrix} \bar{M}(M^3(0)^*)' \\ M^1\ \ \bar{M}(M^2(0))\end{pmatrix}, \tag{1.1.2}$$

where $\bar{M}^2 = \bar{M}/\mathrm{Rad}(\bar{M})$ and $\bar{M}^{3'} = \tilde{M}/\mathrm{Rad}\tilde{M}$ are quotient modules of the generalized Verma modules $\bar{M}(M^2(0))$ and $\bar{M}(M^3(0)^*)$ defined in [18], respectively.

The method we use to prove Theorem 3 is similar to the proof of Theorem 2.2.1 in [73]. However, unlike building V-modules from $A(V)$-modules based on the ordinary correlation functions $(v', Y(a_1, z_1) \dots Y(a_n, z_n)v)$, in our case, due to the appearance of intertwining operator $I(v, w)$ in (1.1.1), the modules $\bar{M}^2$ and $\bar{M}^3$ constructed by (1.1.1) are not necessarily irreducible. This issue was first observed by Li in [49]. The V-modules $\bar{M}^2$ and $\bar{M}^3$ are quotient modules of certain generalized Verma modules. They can be proved to be irreducible if a technical condition depends only on the (bi)modules over $A(V)$ is satisfied. In particular, $\bar{M}^2$ and $\bar{M}^3$ are irreducible when V is rational. So we have $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)} \cong \mathrm{Cor}\binom{M^3}{M^1\ M^2}$ for irreducible modules M^2 and M^3 over a rational VOA V. Chapter 3 is dedicated to the proof of Theorem 3.

In the third step, we prove that $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ is isomorphic to the following vector space: $(M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$, where $B_h(M^1)$ is a new $A(V)$-bimodule that is given by $B_h(M^1) = M^1/\mathrm{span}\{a \circ u,\ L(-1)v + (L(0) + h_2 - h_3)v : a \in V,\ u, v \in M^1\}$. We will show that $B_h(M^1)$ is a quotient module of $A(M^1)$, and we will give examples to show that the vector spaces $(M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$ and $(M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))^*$ are not isomorphic in general. But they can be proved to be isomorphic when V is rational, see Proposition 4.1.12. We need to mod out the additional terms $L(-1)v+(L(0)v+h_2-h_3)v$ in $A(M^1)$ because otherwise, the $L(-1)$-derivation property of the intertwining operators cannot be correctly reflected.

Chapter 4 is dedicated to the proof of the isomorphism $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)} \cong (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$. Given a linear function f on $M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0)$, we shall use the recursive formulas satisfied by elements in $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ and reconstruct a system of correlation functions in $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. There is one recursive formula of the correlation functions $(v', Y(a_1, z_1) \dots Y(a_n, z_n)v)$, where $v \in M(0)$ and $v' \in M(0)^*$, obtained by expanding the left-most term $Y(a_1, z_1)$ (see (2.2.1) in [73]). However, in our case, this formula alone is insufficient to rebuild the correlation functions from f. The reason is again the appearance of $I(v, w)$ in the correlation functions, which makes expanding the left-most term (v, w) in $S(v_3', (v, w)(a_1, z_1) \dots (a_n, z_n)v_2)$ unreasonable, as the action $v(n)a_i = \mathrm{Res}_z w^{n+h} I(v, w)a_i$ is not yet defined. This explains why we have to introduce an additional recursive formula for the correlation functions (1.1.1) obtained by expanding the right-most term $Y(a_n, z_n)$ in $(v_3', I(v, w)Y(a_1, z_1) \dots Y(a_n, z_n)v_2)$, where $v_3' \in M^3(0)^*$ and $v_2 \in M^2(0)$. We will use both recursive formulas to reconstruct the correlation functions from f. Then by Theorem 2 and 3, we have isomorphisms $I\binom{\bar{M}^3}{M^1\ \bar{M}^2} \cong \mathrm{Cor}\binom{\bar{M}^3}{M^1\ \bar{M}^2} \cong \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)} \cong (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$.

This leads to our alternative version of the fusion rules theorem for general vertex operator algebras:

Theorem 4. *Let V be a CFT-type vertex operator algebra, and let M^1, M^2, and M^3 be V-modules with conformal weights h_1, h_2, and h_3, respectively. Assume $M^2(0)$ and $M^3(0)$ are irreducible $A(V)$-modules, then we have the following isomorphism of vector spaces:*

$$I\binom{\bar{M}(M^3(0)^*)'}{M^1\ \bar{M}(M^2(0))} \cong I\binom{\bar{M}^3}{M^1\ \bar{M}^2} \cong \left(M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0)\right)^*, \tag{1.1.3}$$

where $h = h_1 + h_2 - h_3$. Moreover, if V is rational, then we have an isomorphism:

$$I\binom{M^3}{M^1\ M^2} \cong \left(M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0)\right)^*. \tag{1.1.4}$$

In particular, Frenkel and Zhu's fusion rules theorem (1.1.4) holds for rational VOAs. We call (1.1.3) the generalized fusion rules theorem. Finally, we will use (1.1.3) to determine the fusion rules of the universal Virasoro VOA and the rank-one Heisenberg VOA. The Virasoro VOA case shows that the counter-example given by Li in [49] does not contradict Theorem 4.

1.2 Rota-Baxter operators and analog of classical Yang-Baxter equations for vertex operator algebras

The Rota-Baxter identity was discovered independently by G.-C. Rota in [43] and G. Baxter in [10]. This identity is given by the following equation on an algebraic structure F:

$$P(a) \cdot P(b) = P(P(a) \cdot b) + P(a \cdot P(b)) + \lambda P(a \cdot b), \quad \forall a, b \in F, \tag{1.2.1}$$

where P is a linear operator, $\cdot$ is a product on the algebra F, and λ is a fixed element in the ground field K. If $F = A$ is an associative algebra, then P satisfying (1.2.1) is called a Rota-Baxter operator on A, and (A, P) is called a Rota-Baxter algebra (RBA). The Rota-Baxter identity arises naturally in many fields of mathematics, and examples of Rota-Baxter operators P are also quite abundant, see [31] for more details. A similar identity of this type could also give rise to the operator form of the classical Yang-Baxter equation (CYBE) [64]. Because of these vast appearances of the Rota-Baxter type identities, the Rota-Baxter operators have been vigorously studied on various algebraic structures. See, for instance, [3, 4, 8, 32].

Since vertex (operator) algebras are not quite the same as usual algebraic structures, it is natural to ask if one can introduce a notion of the Rota-Baxter operator for vertex algebras, which could share certain similarities with the usual Rota-Baxter algebras. Chapter 6 is our first attempt at this problem. We define an (ordinary) Rota-Baxter operator of weight $\lambda \in \mathbb{C}$ on a vertex algebra $(V, Y, \mathbf{1})$ to be a linear map $P : V \to V$, satisfying the following relation:

$$Y(P(a), z)P(b) = P(Y(P(a), z)b) + P(Y(a, z)P(b)) + \lambda P(Y(a, z)b), \quad \forall a, b \in V. \tag{1.2.2}$$

In order to give natural examples of Rota-Baxter operators that satisfy (1.2.2), we first study certain sub-algebras of the lattice VOA V_L in Chapter 5. Since the definition and construction of these sub-algebras are similar to the Borel sub-algebra of a simple Lie algebra, we will call them the Borel-type sub-algebras of the lattice VOA. We observe that any additive abelian sub-monoid $M \leq L$ corresponds to a sub-algebra $V_M := \bigoplus_{\alpha \in M} M_{\widehat{\mathfrak{h}}}(1, \alpha)$ of the lattice VOA L. In particular, a Borel-type sub-algebra of a lattice VOA V_L associated to rank r positive definite even lattice L is defined by

$$V_B = \bigoplus_{\alpha \in B} M_{\widehat{\mathfrak{h}}}(1, \alpha), \quad \text{with} \quad B = \mathbb{Z}_{\geq 0}\alpha_1 \oplus \ldots \oplus \mathbb{Z}_{\geq 0}\alpha_r,$$

where $\{\alpha_1, \ldots, \alpha_r\}$ is a basis of L. By definition, V_B is a sub-algebra of the lattice VOA V_L with the same Virasoro element ω as V_L. Moreover, we will prove that a Borel-type sub-algebra satisfies the following properties (see Theorem 5.1.5 and Proposition 5.1.6):

Theorem 5. *Let $B = \mathbb{Z}_{\geq 0}\alpha_1 \oplus \ldots \oplus \mathbb{Z}_{\geq 0}\alpha_r \leq L$, and V_B be the associated Borel-type sub-algebra of V_L. Then V_B is irrational. Moreover, if $(\alpha_i|\alpha_j) \geq 0$ for all $1 \leq i \neq j \leq r$, then V_B is C_1-cofinite.*

We give a more thorough study of the rank-one Borel type sub-algebra $V_{\mathbb{Z}_{\geq 0}\alpha}$ in Section 5.2. In this case, we have a decomposition $V_{\mathbb{Z}\alpha} = V_{\mathbb{Z}\alpha_{\geq 0}} \oplus V_{\mathbb{Z}\alpha_{<0}}$ into vertex Leibniz sub-algebras (see [56]), and we will show that the projection map $P : V_{\mathbb{Z}\alpha} \to V_{\mathbb{Z}_{\geq 0}\alpha}$ along $V_{\mathbb{Z}\alpha_{<0}}$ is a Rota-Baxter operator of the lattice VOA $V_{\mathbb{Z}\alpha}$ of weight -1, and P satisfies $PL(-1) = L(-1)P$. Therefore, our definition (1.2.2) of the Rota-Baxter operators indeed has some natural examples. We also found the presentation of Zhu's algebra $A(V_{\mathbb{Z}_{\geq 0}\alpha})$ (see Proposition 5.2.1 and Theorem 5.2.8):

Theorem 6. *Let $L = \mathbb{Z}\alpha$, with $(\alpha|\alpha) = 2N$ for some fixed positive integer N, and let $V_{\mathbb{Z}_{\geq 0}\alpha} = \bigoplus_{m \in \mathbb{N}} M_{\widehat{\mathfrak{h}}}(1, m\alpha)$ be the Borel-type sub-algebra. Then $A(V_{\mathbb{Z}_{\geq 0}\alpha}) \cong \mathbb{C}[x] \oplus \mathbb{C}y$, where $\mathbb{C}[x]$ is the polynomial ring, $y^2 = 0$, $xy = Ny$, and $yx = -Ny$.*

In the classical theory of Rota-Baxter algebras, the Rota-Baxter identity (1.2.1) has an intimate relationship with the so-called dendriform relations given by Loday in [59], see [31, 8, 4] for more details. And the dendriform relations are the axioms satisfied by a pair of (underlying) operators $(\prec, \succ)$ that form the associative product $a \cdot b = a \prec b + a \succ b$. Note that a vertex algebra $(V, Y, \mathbf{1}, D)$ can be equivalently defined as a vector space V, equipped with a vertex operator Y, a linear map $D : V \to V$, and a vacuum element $\mathbf{1}$, satisfying the truncation property, the vacuum and creation properties, the D-(bracket) derivative property, and the weak associativity (see [9, 37, 54]). We may also view the vertex operator Y as the "product" on V. Therefore, we can introduce a notion of dendriform vertex algebra $(V, \prec_z, \succ_z, D)$ that generalizes the usual dendriform (associative) algebra, see Definition 6.2.3. Then it relates to the Rota-Baxter operators on VOAs as follows (see Theorem 6.2.5, Theorem 6.2.6, and Proposition 6.2.15):

Theorem 7. *Let $(V, Y, \mathbf{1})$ be a vertex algebra, and let $P : V \to V$ be a Rota-Baxter operator on V of weight 0 such that $PD = DP$. Define:*

$$a \prec_z b := Y(a, z)P(b), \quad a \succ_z b := Y(P(a), z)b, \quad \forall a, b \in V. \tag{1.2.3}$$

Then $(V, \prec_z, \succ_z, D)$ is a dendriform vertex algebra.

On the other hand, let $(V, \prec_z, \succ_z, D)$ be a dendriform vertex algebra. Define:

$$\tilde{Y}(a, z)b := a \prec_z b + a \succ_z b, \quad \forall a, b \in V, \tag{1.2.4}$$

then $(V, \tilde{Y}, D)$ is a vertex algebra without vacuum. Moreover, let

$$Y_W : V \to \mathrm{End}(V)[[z, z^{-1}]], \; Y_W(a, z)b := a \succ_z b, \quad \forall a, b, \in V.$$

Then Y_W defines a representation of $(V, \tilde{Y}, D)$ on V itself, where $\tilde{Y}$ is given by (1.2.4).

The Rota-Baxter operators are also closely related to the classical Yang-Baxter equations for both Lie algebras [64, 45] and associative algebras [8]. Chapter 7 is dedicated to exploring such relations in the VOA context. The (parameter independent) classical Yang-Baxter equation (CYBE) is an algebraic equation satisfied by a skew-symmetric two tensor $r = \sum_i a_i \otimes b_i \in \mathfrak{g}^{\otimes 2}$ of a Lie algebra $\mathfrak{g}$:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{1.2.5}$$

where $r_{12} = \sum_i a_i \otimes b_i \otimes 1$, $r_{13} = \sum_i a_i \otimes 1 \otimes b_i$, and $r_{23} = \sum_i 1 \otimes a_i \otimes b_i$ are elements in $U(\mathfrak{g})^{\otimes 3}$. The CYBE (1.2.5) was obtained by taking the semi-classical limit of the quantum Yang-Baxter equation: $R_{12}(u_1, u_2)R_{13}(u_1, u_2)R_{23}(u_1, u_2) = R_{23}(u_1, u_2)R_{13}(u_1, u_2)R_{12}(u_1, u_2)$ discovered independently by C. N. Yang in [71] and R. J. Baxter in [11]. The parameter independent CYBE (1.2.5) was first studied by Belavin and Drinfeld in [5]. They proved that the skew-symmetric solutions to (1.2.5) can give rise to a notion of Lie bialgebras. On the other hand, by using the natural isomorphism of vector spaces:

$$\mathfrak{g} \otimes \mathfrak{g} \cong \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g}),\ r = \sum_i a_i \otimes b_i \mapsto R, \quad \text{where} \quad R(f) = \sum_i a_i \langle f, b_i \rangle,\ \forall f \in \mathfrak{g}^*, \tag{1.2.6}$$

Semenov-Tian-Shansky gave an operator form of the CYBE in [64]. Under this isomorphism, (1.2.5) translates to a relation:

$$[R(f), R(g)] = R(\mathrm{ad}^*(R(f))(g)) - R(\mathrm{ad}^*(R(g))(f)), \quad \forall f, g \in \mathfrak{g}^*, \tag{1.2.7}$$

where $\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$ is the coadjoint representation. In particular, if $\mathfrak{g} \cong \mathfrak{g}^*$ as $\mathfrak{g}$-modules, then (1.2.7) is equivalent to the following equation:

$$[R(f), R(g)] = R([R(f), g]) - R([R(g), f]), \quad \forall f, g \in \mathfrak{g}. \tag{1.2.8}$$

Semenov-Tian-Shansky called $R : \mathfrak{g} \to \mathfrak{g}$ satisfying (1.2.8) an R-matrix and the equation (1.2.8) the operator form CYBE. Observe that R satisfying (1.2.8) is precisely a Rota-Baxter operator of the Lie algebra $\mathfrak{g}$ of weight 0, see [4, 31]. The operator form CYBE was first generalized to the VOA case by Xu in [69] as we mentioned in Section 6.1. However, there was only the operator form generalization of the R-matrix for Lie algebras given in [70], and it is the same as an RBO for VOAs of weight 0 given by (1.2.2)

Semenov-Tian-Shansky's approach was later generalized to arbitrary representations of the Lie algebra $\mathfrak{g}$ by Kupershmidt in [45], wherein he introduced a notion of -operators (relative Rota-Baxter operators) for Lie algebras associated to a module V over $\mathfrak{g}$, which is a linear map $T : V \to \mathfrak{g}$ such that

$$[T(u), T(v)] = T(T(u).v) - T(T(v).u), \quad \forall u, v \in V. \tag{1.2.9}$$

He also proved that skew-symmetric solutions r to the CYBE (1.2.5) are in one-to-one correspondence with the -operators $T_r : \mathfrak{g}^* \to \mathfrak{g}$. On the other hand, Bai proved in [4] that an arbitrary

-operator $T : V \to \mathfrak{g}$ can also give rise to solutions to the CYBE (1.2.5) in the semi-direct product Lie algebra $\mathfrak{g} \rtimes V^*$.

In this Chapter 7, we will generalize these relations between O-operators and skew-symmetric solutions to the CYBE to the VOA case. In particular, by using a generalized version of the RBO for VOA we call relative RBO for VOA, we will give a tensor form analog of the Classical Yang-Baxter equation for VOAs. In particular, for each index m in the vertex operator $Y(a,z) = \sum_{m\in\mathbb{Z}} a_m z^{-m-1}$, we have an equation:

$$r_{12} \cdot_m r_{13} - r_{23} \cdot'_m r_{12} + r_{13} \cdot'^{\mathrm{op}}_m r_{23} = 0. \tag{1.2.10}$$

In this equation, r is in the completion $V \widehat{\otimes} V$ of the tensor product $V \otimes V$ with respect to the natural filtration of $V \otimes V$, and the three products $\cdot_m$, $\cdot'_m$, and $\cdot'^{\mathrm{op}}_m$ are constructed from the vertex operator Y. We call this equation the indexed m-vertex operator Yang-Baxter equation (m-VOYBE). Note that (1.2.6) can be generalized to the complete tensor case:

$$\prod_{t=0}^{\infty} V_t \otimes V_t \cong \prod_{t=0}^{\infty} \mathrm{Hom}(V_t^*, V_t) \cong \mathrm{Hom}_{\mathrm{LP}}(V', V), \quad r \mapsto T_r, \tag{1.2.11}$$

where $\mathrm{Hom}_{\mathrm{LP}}(V', V)$ is the space of level-preserving linear maps $f : V' \to V$, with $f(V_t^*) \subseteq V_t$, for all $t \in \mathbb{N}$. Then we have the following theorem that relates Rota-Baxter operators on VOAs with the vertex-operator Yang-Baxter equation (see Theorem 7.1.10):

Theorem 8. *For a given $m \in \mathbb{Z}$, r is an skew-symmetric solution to the m-VOYBE if and only if the corresponding $T_r : V' \to V$ given by* (1.2.11) *is a level-preserving m-relative RBO, that is, for any $f, g \in V'$, the following equation holds:*

$$T_r(f)_m T_r(g) = T_r(T_r(f)_m g) + T_r(f(m)T_r(g)), \tag{1.2.12}$$

where $T_r(f)_m g = \mathrm{Res}_z z^m Y_{V'}(T_r(f), z) g$ and $f(m)T_r(g) = \mathrm{Res}_z z^m Y_{V'V}^{V'}(f, z) T_r(g)$.

This theorem is a generalization of the Semenov-Tian-Shansky and Kupershmidt's results for the correspondence between RBOs and skew-solutions to the CYBEs of Lie algebras [64, 45]. On the other hand, in order to solve the VOYBE, we can use relative RBOs of VOAs and find solutions in the semi-direct product of a VOA V with the contragredient module W'. However, unlike the Lie algebra case, we need a relative RBO $T : V \to W$ to satisfy some compatibility conditions with the intertwining operators formed by contragredient modules. We call such relative RBOs the strong relative RBOs on VOAs. Then we have the following (see Theorem 7.2.5):

Theorem 9. *Let V be a VOA, W be an (ordinary) V-module of conformal weight* 0*, $U = V \rtimes W'$ be the semi-direct product VOA, $T \in \mathrm{Hom}_{\mathrm{LP}}(W, V)$ be a level-preserving linear operator, and r be $T - T^{21} \in U^{\widehat{\otimes 2}}$, where $T^{21} = \sigma(T)$. Define r_{12}, r_{13}, and r_{23} as follows:*

$$r_{12} := \sum_{t=0}^{\infty} \sum_{i=1}^{p_t} T(v_i^t) \otimes (v_i^t)^* \otimes I - (v_i^t)^* \otimes T(v_i^t) \otimes I,$$

$$r_{13} := \sum_{s=0}^{\infty} \sum_{k=1}^{p_s} T(v_k^s) \otimes I \otimes (v_k^s)^* - (v_k^s)^* \otimes I \otimes T(v_k^s),$$

$$r_{23} := \sum_{r=0}^{\infty} \sum_{l=1}^{p_r} I \otimes T(v_l^r) \otimes (v_l^r)^* - I \otimes (v_l^r)^* \otimes T(v_l^r).$$

Let $m \in \mathbb{Z}$. Then r is a skew-symmetric solution to the m-VOYBE in the VOA $U = V \rtimes W'$ if and only if $T : W \to V$ is a strong m-relative RBO.

This theorem is a generalization of Bai's result for solving the CYBE with -operators of Lie algebras [4]. Finally, by restricting everything to the first-level Lie algebra, we also show that the results of Kupershmidt in [45] and Bai in [4] can be recovered by our general result Theorem 8 and Theorem 9.

We fix some conventions that will be in force throughout this thesis:

(1) The symbols $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ represent the set of integers, rational numbers, real numbers, and complex numbers, respectively.

(2) All vector spaces are defined over the complex number field $\mathbb{C}$, unless we state otherwise.

(3) $\mathbb{N}$ represents the set of natural numbers including zero: $\mathbb{N} = \{0, 1, 2, 3 \dots\}$.

(4) The power series expansion and Laurent series expansion of a complex valued function are both called the "power series expansion".

(5) When we use the integral sign $\int_C f(z)dz$, where C is a simple closed contour of z, it means $\frac{1}{2\pi i} \int_C f(z)dz$.

Part I

Correlation functions and fusion rules of vertex operator algebras

Chapter 2

Basics of vertex operator algebras

This Chapter is about the basics of vertex operator algebras (VOAs). We will recall some previous known definitions and related constructions of VOAs, and present some new discoveries about the basic concepts. We will also give some reinterpretations and alternative proofs of the known results along the way.

Section 2.1 recalls the notions of vertex operator algebras (VOAs), modules over VOAs, the intertwining operators, and some related definitions and properties. In Section 2.2, we recall the definition of Zhu's algebra $A(V)$ and the C_2-algebra $R(V)$, and then present some new discoveries about the $A(V)$ for strongly finitely generated VOA V. We will also study the relations between $R(V)$ and the graded algebra $\mathrm{gr}A(V)$ for some classical examples of VOAs. In Section 2.3, we first recall the definitions of derivation and automorphism of VOAs, and then prove that all derivations of some classical examples of rational VOAs are inner derivations. We will use the closed subgroups of the full automorphism groups and define the generalized orbifold and commutant sub-VOAs. We will also give a generalized notion for the derivations, called the λ-differentials for VOAs. Finally, in Section 2.4, we recall the definition of the correlation functions on the Riemann sphere associated with a module over VOAs as a preparation for the next Chapter. Then we will give a reinterpretation of the equivalency between the Jacobi identity of VOAs and the locality and associativity axioms of correlation functions. We will also generalize the system of correlation functions associated with ordinary modules over VOAs to twisted modules and prove some basic properties.

2.1 Preliminaries

Borcherds gave the notion of vertex algebra in [12], later it was enhanced to the notion of vertex operator algebra by Frenkel, Lepowsky, and Meurman in [29], wherein the key axiom, Jacobi identity, was interpreted by formal variables approach. We skip the recap of formal calculus and refer to [27, 73] for more details. Most of the definitions and related results in this subsection can be found in [12, 16, 29, 27, 55, 73].

Definition 2.1.1. A **vertex algebra (VA)** is a triple $(V, Y, \mathbf{1})$ consisting of a vector space V, a linear map Y called the **vertex operator map** or the state-field correspondence:

$$\begin{aligned} Y : &V \to (\mathrm{End}\, V)[[z, z^{-1}]], \\ &a \mapsto Y(a, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \quad (a_n \in \mathrm{End}\, V), \end{aligned}$$

and a distinguished element $\mathbf{1} \in V$ called the **vacuum vector**, satisfying the following axioms:

(1) (Truncation property) For any $a, b \in V$, $a_n b = 0$ when $n \gg 0$.

(2) (Vacuum property) $Y(\mathbf{1}, z) = \mathrm{Id}_V$.

(3) (Creation property) For any $a \in V$, $Y(a, z)\mathbf{1} \in V[[z]]$ and $\lim_{z\to 0} Y(a, z)\mathbf{1} = a$.

(4) (The Jacobi identity) For any $a, b \in V$,

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(a, z_1)Y(b, z_2) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) Y(b, z_2)Y(a, z_1) \\ &= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(a, z_0)b, z_2). \end{aligned} \tag{2.1.1}$$

A vertex algebra V is said to be generated by a subset $S \subset V$ if

$$V = \mathrm{span}\{a^1_{n_1} a^2_{n_2} \dots a^r_{n_r} b : r \geq 0, n_1, \dots n_r \in \mathbb{Z},\ a^1, \dots, a^r, b \in S\}.$$

V is called **finitely generated** if there exists a finite set S that generates V.

The Jacobi identity (2.1.1) has a component form (cf.[12]):

$$\sum_{i=0}^{\infty} (-1)^i \binom{l}{i} a_{m+l-i} b_{n+i} - \sum_{i=0}^{\infty} (-1)^{l+i} \binom{l}{i} b_{n+l-i} a_{m+i} = \sum_{i=0}^{\infty} \binom{m}{i} (a_{l+i} b)_{m+n-i}, \tag{2.1.2}$$

for all $a, b \in V$, and $m, n, l \in \mathbb{Z}$, which can be obtained by multiplying (2.1.1) with $(z_1 - z_2)^l z_1^m z_2^n$, then take the residue $\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}$. The Jacobi identity (2.1.1) has the following equivalent characterization, see [16, 55, 27]:

Theorem 2.1.2. *Let $(V, Y, \mathbf{1})$ be a vertex algebra, then it satisfies the following properties:*

*(1) (**locality**) For any $a, b \in V$, there exists some integer $k \in \mathbb{N}$ such that*

$$(z_1 - z_2)^k Y(a, z_1)Y(b, z_2) = (z_1 - z_2)^k Y(b, z_2)Y(a, z_1). \tag{2.1.3}$$

*(2) (**weak associativity**) For any $a, b, c \in V$, there exists some integer $k \in \mathbb{N}$ (depending on a and c) such that*

$$(z_0 + z_2)^k Y(Y(a, z_0)b, z_2)c = (z_0 + z_2)^k Y(a, z_0 + z_2)Y(b, z_2)c. \tag{2.1.4}$$

Moreover, if $Y : V \to \mathrm{End}V[[z, z^{-1}]]$ is a linear map that satisfies the truncation property, then the Jacobi identity of Y in the definition of vertex algebra is equivalent to the locality and weak associativity.

Let $(V, Y, \mathbf{1})$ be a vertex algebra. Define a linear operator $D : V \to V$ by letting $Da := a_{-2}\mathbf{1}$, for all $a \in V$. Then $(V, Y, D, \mathbf{1})$ satisfies the ***D*-derivative property**:

$$Y(Da, z) = \frac{d}{dz}Y(a, z), \tag{2.1.5}$$

the ***D*-bracket derivative property**:

$$[D, Y(a, z)] = \frac{d}{dz}Y(a, z), \tag{2.1.6}$$

and the **skew-symmetry**:

$$Y(a, z)b = e^{zD}Y(b, -z)a, \tag{2.1.7}$$

where $a, b \in V$. (2.1.5) and (2.1.6) together are called the ***D*-translation invariance property**. On the other hand, a vertex algebra also has the following equivalent definition, see [55, 44]:

Theorem 2.1.3. *A vertex algebra $(V, Y, D, \mathbf{1})$ is a vector space V, equipped with a linear map $Y : V \to \mathrm{End}(V)[[z, z^{-1}]]$, a distinguished vector $\mathbf{1}$, and a linear map $D : V \to V$, satisfying the truncation property, the vacuum and creation property, the D-bracket derivative property* (2.1.6)*, and the locality* (2.1.3).

Definition 2.1.4. A **vertex operator algebra (VOA)** is a quadruple $(V, Y, \mathbf{1}, \omega)$, where $(V, Y(\cdot, z), \mathbf{1})$ is a $\mathbb{Z}$-graded vertex algebra: $V = \bigoplus_{n\in\mathbb{Z}} V_n$, such that $\mathbf{1} \in V_0$, $\dim V_n < \infty$ for each $n \in \mathbb{Z}$, and $V_n = 0$ for n sufficiently small. $\omega \in V_2$ is another distinguished element called the **Virasoro element**. When we write $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$, that is, $L(n) = \omega_{n+1}$ for each $n \in \mathbb{Z}$, the following additional axioms are satisfied:

(5) (The Virasoro relation)

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c,$$

where $c \in \mathbb{C}$ is called the central charge (or rank) of V.

(6) ($L(-1)$-derivation property) $D = L(-1)$, and

$$\frac{d}{dz}Y(a, z) = Y(L(-1)a, z) = [L(-1), Y(a, z)].$$

(7) ($L(0)$-eigenspace property) $L(0)a = na$, for all $a \in V_n$ and $n \in \mathbb{Z}$. We call n the weight of a homogeneous element $a \in V_n$, and write $\mathrm{wt} a = n$.

A VOA V is said to be of the **CFT-type**, if $V = V_0 \oplus V_+$, where $V_0 = \mathbb{C}\mathbf{1}$ and $V_+ = \bigoplus_{n=1}^{\infty} V_n$. In the rest of this thesis, we will sometimes denote a VOA $(V, Y, \mathbf{1}, \omega)$ by V for simplicity when no confusions occur.

Definition 2.1.5. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra, a **weak V-module** (W, Y_W) is a vector space W, equipped with a linear map

$$\begin{aligned} Y_W : V &\to \mathrm{End}(W)[[z, z^{-1}]], \\ a &\mapsto Y_W(a, z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1}, \end{aligned}$$

satisfying the following axioms:

(1) (Truncation property) For any $a \in V$ and $u \in W$, we have $a(n)u = 0$ for $n \gg 0$.

(2) (Vacuum property) $Y_W(\mathbf{1}, z) = \mathrm{Id}_W$.

(3) (The Jacobi identity) For any $a, b \in V$, and $u \in W$

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_W(a, z_1)Y_W(b, z_2)u - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)Y_W(b, z_2)Y_W(a, z_1)u \\ &= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y_W(Y(a, z_0)b, z_2)u. \end{aligned} \tag{2.1.8}$$

A weak V-module W is said to be generated by a subset $S \subset W$ if

$$W = \mathrm{span}\{a^1(n_1)a^2(n_2)\ldots a^r(n_r)u : r \geq 0, n_1, \ldots n_r \in \mathbb{Z},\ a^1, \ldots, a^r \in V, u \in S\}.$$

A weak V-module W is called **admissible (or $\mathbb{N}$-gradable)** if $W = \bigoplus_{n\in\mathbb{N}} W(n)$, and $a_m W(n) \subset W(\mathrm{wt}a - m - 1 + n)$ for all homogeneous $a \in V$, $m \in \mathbb{Z}$, and $n \in \mathbb{N}$. For $v \in W(n)$, we call n the **admissible degree** of v, and write $\deg v = n$.

An **ordinary V-module** is an admissible V-module W such that $\dim W(n) < \infty$ for each $n \in \mathbb{N}$, and each $W(n)$ is an eigenspace of $L(0) = \mathrm{Res}z Y_W(\omega, z)$ of eigenvalue $\lambda + n$, where $\lambda \in \mathbb{Q}$ is a fixed number called the **conformal weight of** W. We denote $W(n)$ by $W_{\lambda+n}$ for all n, and write $W = \bigoplus_{n\in\mathbb{N}} W_{\lambda+n}$. In particular, (V, Y) itself is an ordinary V-module called the **adjoint module**. By convention, when we say that M is a V-module, it means that M is an ordinary V-module.

Remark 2.1.6. Let $(V, Y, \mathbf{1}, \omega)$ be a VOA, and (W, Y_W) be a V-module. In the rest of this thesis, when no ambiguities occur, we sometimes also write $Y(a, z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1}$. For the module vertex operators, sometimes we write $Y_W(a, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$. The notations of these vertex operators depend on our contexts and references.

The Jacobi identity (2.1.8) also has a component form that is similar to (2.1.2):

$$\begin{aligned}&\sum_{i=0}^{\infty}(-1)^i\binom{l}{i}a(m+l-i)b(n+i)u - \sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}b(n+l-i)a(m+i)u\\ &= \sum_{i=0}^{\infty}\binom{m}{i}(a(l+i)b)(m+n-i)u,\end{aligned} \tag{2.1.9}$$

where $a, b \in V$, $u \in W$, and $m, n, l \in \mathbb{Z}$. Let (W, Y_W) be a weak module over a VOA V. If we write $Y_W(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$, it is proved in [20] that Y_W also satisfies the $L(-1)$-derivative property and the $L(-1)$-bracket derivative property: $Y_W(L(-1)a, z) = \frac{d}{dz}Y_W(a, z) = [L(-1), Y_W(a, z)]$.

Zhu gave the notion of rational VOA in [73], and the concept was later simplified by Dong, Li, and Mason in [18].

Definition 2.1.7. A VOA V is called rational if the admissible V-module category is semisimple. i.e., any admissible V-module M is a direct sum of irreducible admissible V-modules.

The notion of intertwining operator for VOA were defined in Section 5.4 in [27]. Let $V\{z\} := \{\sum_{n\in\mathbb{Q}} a_n z^n : a_n \in V, \forall n \in \mathbb{Q}\}$ be the set of V-valued series with rational powers.

Definition 2.1.8. Let V be a VOA, and $(M^1, Y_{M^1}), (M^2, Y_{M^2})$, and (M^3, Y_{M^3}) be V-modules. An **intertwining operator of type** $\binom{M^3}{M^1\ M^2}$ is a linear map

$$I(\cdot, w) : M^1 \to \mathrm{Hom}(M^2, M^3)\{w\}, \quad I(v, w) = \sum_{n\in\mathbb{Q}} v_n w^{-n-1}\ (v \in M^1,\ v_n \in \mathrm{Hom}(M^2, M^3)),$$

satisfying the following properties:

(1) (Truncation property) For any $u \in M^2$, $v_n u = 0$ when $n \gg 0$.

(2) ($L(-1)$-derivative property)

$$I(L(-1)v, w) = \frac{d}{dw} I(v, w), \quad \forall v \in M^1, \tag{2.1.10}$$

(3) (Jacobi identity) For any $a \in V$, $v \in M^1$, and $u \in M^2$,

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_{M^3}(a, z_1) I(v, z_2) u - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) I(v, z_2) Y_{M^2}(a, z_1) u \\ &= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) I(Y_{M^1}(a, z_0)v, z_2) u. \end{aligned} \tag{2.1.11}$$

The vector space of intertwining operators of type $\binom{M^3}{M^1\ M^2}$ is denoted by $I\binom{M^3}{M^1\ M^2}$, and its dimension is denoted by

$$N_{M^1 M^2}^{M^3} = \dim I\binom{M^3}{M^1\ M^2}. \tag{2.1.12}$$

The numbers $N_{M^1 M^2}^{M^3}$ are called the **fusion rules** associated with the VOA V and modules.

Let M^1, M^2, and M^3 be V-modules, with conformal weights h_1, h_2, and h_3, respectively, and let $I \in I\binom{M^3}{M^1\ M^2}$ be an intertwining operator. Recall that $I(v, w) = \sum_{n\in\mathbb{Z}} v(n) w^{-n-1} \cdot w^{-h}$, where $h = h_1 + h_2 - h_3$, and $v(n) = \mathrm{Res}_w I(v, w) w^{n+h}$. Moreover, $v(n) M^2(m) \subseteq M^3(\deg v - n - 1 + m)$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{N}$, see Proposition 1.5.1 in [30] for more details.

We conclude this section by recalling the definition of contragredient modules of modules over VOAs. Let V be a VOA, and (M, Y_M) be an ordinary V-module. Let M' be the graded dual space $M' := \bigoplus_{n=0}^{\infty} M(n)^*$, and let $Y_{M'} : V \to \mathrm{End}(M')[[z, z^{-1}]]$ be given by

$$\langle Y_{M'}(a, z) v', v\rangle = \langle v', Y_M(e^{zL(1)}(-z^{-2})^{L(0)} a, z^{-1}) v\rangle, \tag{2.1.13}$$

for any $v' \in M'$, $v \in M$, and $a \in V$. Then $(M', Y_{M'})$ is a V-module of the same conformal weight with M, and it is called the **contragredient modules of** M. See Section 5.3 of [27] for more details. We note that (2.1.13) also has a component form:

$$\langle a_n v', v \rangle = \sum_{j\geq 0} \frac{(-1)^{\mathrm{wt} a}}{j!} \langle v', (L(1)^j a)_{2\mathrm{wt} a - n - j - 2} v \rangle, \tag{2.1.14}$$

for $n \in \mathbb{Z}$, $a \in V$ homogeneous, $v \in M$, and $v' \in M'$.

2.2 Zhu's algebra $A(V)$ and the C_2-algebra $R(V)$

The notion of Zhu's algebra $A(V)$ and the C_2-algebra $R(V) = V/C_2(V)$ were both defined by Zhu in [73]. Similar to $C_2(V)$, there is the notion of $C_1(V)$ for CFT-type VOAs given by Li in [51], which is closely related to the strongly finitely generation property of a VOA. In this subsection, we will first recall these concepts and related results, then prove a noetherian property of $A(V)$ for certain finitely generated VOAs. The main content of this section can also be found in [57].

First, we recall the definition of $A(V)$, see [30, 73] for more details. Let V be a VOA, for homogeneous elements $a, b \in V$, define:

$$a \circ b := \mathrm{Res}_z Y(a,z) b \frac{(1+z)^{\mathrm{wt} a}}{z^2} = \sum_{j\geq 0} \binom{\mathrm{wt} a}{j} a_{j-2} b, \tag{2.2.1}$$

$$a * b := \mathrm{Res}_z Y(a,z) b \frac{(1+z)^{\mathrm{wt} a}}{z} = \sum_{j\geq 0} \binom{\mathrm{wt} a}{j} a_{j-1} b. \tag{2.2.2}$$

Let $O(V) = \mathrm{span}\{a \circ b : a, b \in V\}$, and let $A(V) = V/O(V)$. For $a \in V$, we denote $a + O(V) \in A(V)$ by $[a]$ as in [73]. By Theorem 2.1.1 in [73], $O(V)$ is a two-sided ideal with respect to $*$:

$$a * O(V) \subset O(V), \quad \text{and} \quad O(V) * a \subset O(V), \tag{2.2.3}$$

for any $a \in V$. And $A(V)$ is an associative algebra with respect to $*$, with the unit element $[\mathbf{1}]$. By Lemma 2.1.3 in [73], we have the following formulas:

$$a * b \equiv \mathrm{Res}_z Y(b,z) a \frac{(1+z)^{\mathrm{wt} b - 1}}{z} \pmod{O(V)}, \tag{2.2.4}$$

$$a * b - b * a \equiv \mathrm{Res}_z Y(a,z) b (1+z)^{\mathrm{wt} a - 1} \pmod{O(V)}, \tag{2.2.5}$$

for any homogeneous $a, b \in V$. Furthermore, if $m \geq n \geq 0$, we have:

$$\mathrm{Res}_z Y(a,z)b\frac{(1+z)^{\mathrm{wt}a+n}}{z^{2+m}} \equiv 0 \pmod{O(V)}. \tag{2.2.6}$$

By Theorem 2.1.2 in [73], for any admissible V-module $M = \bigoplus_{n=0}^{\infty} M(n)$, the bottom level $M(0)$ is a module over $A(V)$ under the action:

$$A(V) \to \mathrm{End}(M), \quad [a] \mapsto o(a) = a_{\mathrm{wt}a-1}.$$

Furthermore, by Theorem 2.2.1 in [73], given an irreducible $A(V)$-module U, there exists an admissible V-module W such that $W(0) = U$, and there is an one-to-one correspondence between irreducible V-modules and irreducible $A(V)$-modules.

Let V be a CFT-type VOA, then $A(V)$ has a filtration: $A(V)_0 \subseteq A(V)_1 \subseteq A(V)_2 \subseteq \dots$, where $A(V)_n$ is the image of $\bigoplus_{i=0}^{n} V_i \subset V$ in $A(V)$. We call this filtration the **level filtration** of $A(V)$. It is clear that $[\mathbf{1}] \in A(V)_0$, $[\omega] \in A(V)_2$, and by (2.2.2), we have $[a] * [b] \in A(V)_{m+n}$, for $[a] \in A(V)_m$ and $[b] \in A(V)_n$.

These properties indicate that $A(V)$ is a filtered algebra (one can find more details about the filtered algebras in [60, 62]), and we have the associated graded algebra:

$$\mathrm{gr}A(V) = \bigoplus_{n=0}^{\infty} A(V)_n/A(V)_{n-1} = \bigoplus_{n=0}^{\infty} (\mathrm{gr}A(V))_n, \tag{2.2.7}$$

where $(\mathrm{gr}A(V))_n = A(V)_n/A(V)_{n-1}$ for each $n \geq 0$, and $A(V)_{-1} = 0$. For $[a] \in A(V)_n$, we denote the image $[a] + A(V)_{n-1}$ in $\mathrm{gr}A(V)$ by $\overline{[a]}$. For $\overline{[a]} \in A(V)_m/A(V)_{m-1}$ and $\overline{[b]} \in A(V)_n/A(V)_{n-1}$, their product is given by

$$\overline{[a]} * \overline{[b]} = \overline{[a] * [b]} \in A(V)_{m+n}/A(V)_{m+n-1}. \tag{2.2.8}$$

2.2.1 The notherianess of $A(V)$ for C_1-cofinite VOAs

Lemma 2.2.1. $\mathrm{gr}A(V)$ *is a commutative Poisson algebra with the product and the Lie bracket given by:*

$$\overline{[a]} * \overline{[b]} = [a_{-1}b] + A(V)_{m+n-1} \in (\mathrm{gr}A(V))_{m+n}, \tag{2.2.9}$$

$$\{\overline{[a]}, \overline{[b]}\} = [a_0 b] + A(V)_{m+n-2} \in (\mathrm{gr}A(V))_{m+n-1}, \tag{2.2.10}$$

for all $\overline{[a]} \in A(V)_m/A(V)_{m-1}$ *and* $\overline{[b]} \in A(V)_n/A(V)_{n-1}$, *where* $m, n \in \mathbb{N}$.

Proof. By (2.2.1) and (2.2.8), we have:

$$\overline{[a]} * \overline{[b]} = \overline{[a_{-1}b]} + \sum_{j=1}^{\mathrm{wt}a} \binom{\mathrm{wt}a}{j} \overline{[a_{j-1}b]} = \overline{[a_{-1}b]}$$

since $\mathrm{wt}(a_{j-1}b) = \mathrm{wt}a + \mathrm{wt}b - j \leq m + n - 1$ for any $j \geq 1$. Thus $[a_{j-1}b] \in A(V)_{m+n-1}$ and $\overline{[a_{j-1}b]} = 0$ in $A(V)_{m+n}/A(V)_{m+n-1}$. Moreover, by (2.2.5) and (2.2.8), we have:

$$\overline{[a]} * \overline{[b]} - \overline{[b]} * \overline{[a]} = \sum_{j=0}^{\mathrm{wt}a-1} \binom{\mathrm{wt}a - 1}{j} \overline{[a_j b]} = 0$$

since $\mathrm{wt}(a_j b) = \mathrm{wt}a + \mathrm{wt}b - j - 1 \leq m + n - 1$ for all $j \geq 0$. This shows that $\mathrm{gr}A(V)$ is a commutative algebra over $\mathbb{C}$. It follows from a standard fact of filtered rings (cf. [62]) that $\mathrm{gr}A(V)$ is a Poisson algebra with respect to the bracket

$$\{\overline{[a]}, \overline{[b]}\} := [a] * [b] - [b] * [a] + A(V)_{m+n-2} \in (\mathrm{gr}A(V))_{m+n-1}.$$

Since we have $[a] * [b] - [b] * [a] \equiv [a_0 b] \pmod{A(V)_{m+n-2}}$, it follows that $\mathrm{gr}A(V)$ is a commutative Poisson algebra with respect to the bracket given in (2.2.10). □

The notion of a strongly generated vertex operator algebra is defined by Kac in [44]:

Definition 2.2.2. Let V be a VOA, and let $U \subseteq V$ be a subset. V is said to be **strongly generated by** U if V is spanned by elements of the form:

$$a^1_{-n_1} \ldots a^r_{-n_r} u,$$

where $a^1, \ldots, a^r, u \in U$, and $n_i \geq 1$ for all i. If V is strongly generated by a finite-dimensional subspace U, then V is called **strongly finitely generated**.

Let V be a VOA, the following notions were given by Zhu in [73]:

$$C_2(V) = \mathrm{span}\{a_{-2}b : a, b \in V\}, \quad \text{and} \quad R(V) := V/C_2(V). \tag{2.2.11}$$

V is called **C_2-cofinite** if $R(V)$ is a finite-dimensional vector space. It is also proved by Zhu in [73] that $(R(V), \cdot, \{\cdot, \cdot\})$ is a commutative Poisson algebra, where

$$(a + C_2(V)) \cdot (b + C_2(V)) : = a_{-1}b + C_2(V), \tag{2.2.12}$$

$$\{a + C_2(V), b + C_2(V)\} : = a_0 b + C_2(V), \tag{2.2.13}$$

Let $V = V_0 \oplus V_+$ be a CFT-type VOA, the subspace $C_1(V)$ was defined by Li in [51]:

$$C_1(V) := \text{span}\left(\{a_{-1}b : a, b \in V_+\} \cup \{a_{-2}\mathbf{1} : a \in V\}\right). \tag{2.2.14}$$

The following result was essentially proved by Li, see Theorem 4.11 in [50]:

Proposition 2.2.3. *Let V be a CFT-type VOA, and let $U \subseteq V_+$ be a graded subspace. The following conditions are equivalent:*

(1) V is strongly generated by U.

(2) $V_+ = U + C_1(V)$, where $C_1(V) = span(\{u_{-1}v : u, v \in V_+\} \cup \{L(-1)u : u \in V\})$.

(3) $(U + C_2(V))/C_2(V)$ generates $V/C_2(V)$ as commutative algebra.

The following well-known fact about filtered rings can be found in [60]:

Proposition 2.2.4. *Let R be a filtered ring such that* gr*R is left noetherian, then R is left noetherian.*

Theorem 2.2.5. *Let V be a CFT-type VOA. If V is strongly finitely generated, or equivalently, C_1-cofinite, then $A(V)$ is (left) noetherian as an algebra.*

Proof. First, we show that there is a well-defined epimorphism of commutative Poisson algebras (a similar epimorphism was discovered by Arakawa, Lam, and Yamada in [2]):

$$\begin{aligned} \phi : R(V) = V/C_2(V) \to \text{gr}A(V) = \bigoplus_{n=0}^{\infty} A(V)_n/A(V)_{n-1}, \\ a + C_2(V) \mapsto \overline{[a]} \in A(V)_n/A(V)_{n-1} \quad \text{for } a \in V_n. \end{aligned} \tag{2.2.15}$$

To prove (2.2.15), first define $\phi : V = \bigoplus_{n=0}^{\infty} V_n \to \text{gr}A(V) : \phi(x_1 + \cdots + x_r) = \overline{x_1} + \cdots + \overline{x_r}$, where $x_i \in V_{n_i}$ and $\overline{x_i} \in A(V)_{n_i}/A(V)_{n_i-1}$ for all i. It is clear that ϕ is linear. We claim that $\phi(C_2(V)) = 0$. Indeed, let $a_{-2}b$ be a spanning element in $C_2(V)$, with $a \in V_m$ and $b \in V_n$, where $m \geq 1$ and $n \geq 0$. Then $a_{-2}b \in V_{m+n+1}$ and $\phi(a_{-2}b) = \overline{[a_{-2}b]} \in A(V)_{m+n+1}/A(V)_{m+n}$. Recall that $L(-1)a + L(0)a \equiv 0 \pmod{O(V)}$. Thus $a_{-2}b = (L(-1)a)_{-1}b \equiv (-L(0)a)_{-1}b = -ma_{-1}b$ (mod $O(V)$), with $\text{wt}(a_{-1}b) = m + n$. Hence $\overline{[a_{-2}b]} = \overline{-m[a_{-1}b]} = \overline{[0]}$ in $A(V)_{m+n+1}/A(V)_{m+n}$. Thus, $\phi(C_2(V)) = 0$ and ϕ in (2.2.15) is well-defined.

Since $A(V)_n$ is the image of $\bigoplus_{i=0}^{n} V_n$ in $A(V)$, it is clear that ϕ is surjective. Moreover, by (2.2.12), (2.2.13), and Lemma 2.2.1, we have:

$$\phi((a + C_2(V)) \cdot (b + C_2(V))) = \phi(a_{-1}b + C_2(V)) = [a_{-1}b] + A(V)_{\mathrm{wt}a+\mathrm{wt}b}$$
$$= \phi(a + C_2(V)) * \phi(b + C_2(V)),$$
$$\phi(\{a + C_2(V), b + C_2(V)\}) = \phi(a_0 b + C_2(V)) = [a_0 b] + A(V)_{\mathrm{wt}a+\mathrm{wt}b-1}$$
$$= \{\phi(a + C_2(V)), \phi(b + C_2(V))\}.$$

for all homogeneous $a, b \in V$. Therefore, ϕ given in (2.2.15) is an epimorphism of commutative Poisson algebras. Now let $U = \mathrm{span}\{x^1, \ldots, x^n\}$ be a subspace that strongly generates V. By proposition 2.2.3, $V/C_2(V)$ is generated by $\{x^1 + C_2(V), \ldots, x^n + C_2(V)\}$ as an algebra. In particular, $V/C_2(V)$ is finitely generated, and so its image $\mathrm{gr}A(V)$ under the epimorphism ϕ is also finitely generated. Thus $\mathrm{gr}A(V)$ is noetherian, and so $A(V)$ is also (left) noetherian by Proposition 2.2.4. □

Proposition 2.2.6. *The epimorphism ϕ in* (2.2.15) *is an isomorphism if and only if the following condition holds: For any $a = a_1 + \cdots + a_r \in O(V)$, with $a_i \in V_{n_i}$ for each i and $n_1 < n_2 < \cdots < n_r$, the highest weight summand a_r of a belongs to $C_2(V)$.*

Proof. By the proof of Theorem 2.2.5, we already have $C_2(V) \subseteq \ker\phi$. Then ϕ is an isomorphism if and only if $C_2(V) = \ker\phi$. Also, note that ϕ in (2.2.15) is grading preserving. Assume the condition for $O(V)$ is true, let $x + C_2(V) \in \ker\phi$ with $x \in V_n$, we have $x + O(V) \in A(V)_{n-1}$, and so there exists $y \in \bigoplus_{i=0}^{n-1} V_i$ such that

$$x - y = a = a_1 + \cdots + a_r \in O(V),$$

with $a_i \in V_{n_i}$ for each i and $n_1 < n_2 < \cdots < n_r$. By comparing the highest-weight elements on both sides of this equation, we have $x = a_r \in C_2(V)$. Hence $C_2(V) = \ker\phi$, and ϕ is an isomorphism. Conversely, assume $C_2(V) = \ker\phi$. Let $a = a_1 + \cdots + a_r \in O(V)$, with $a_i \in V_{n_i}$ for each i and $n_1 < n_2 < ... < n_r$, we have:

$$a_r + O(V) = -a_1 - a_2 - \cdots - a_{r-1} + O(V)$$

in $A(V)_{n_r}$. But the right hand side lies in $A(V)_{n_r-1}$ as $n_1 < n_2 < \cdots < n_{r-1} \leq n_r - 1$. Hence $\phi(a_r) = \overline{a_r} = \bar{0} \in A(V)_{n_r}/A(V)_{n_r-1}$, and $a_r \in \ker\phi = C_2(V)$. □

2.2.2 The graded algebra $\mathrm{gr}A(V)$ and $R(V)$

Although the condition for $O(V)$ in Proposition 2.2.6 is obvious for the spanning elements of $O(V)$ since $a \circ b = a_{-2}b + \sum_{j\geq 1}\binom{\mathrm{wt}a}{j}a_{j-2}b$ by definition, and $\mathrm{wt}(a_{j-2}b) < \mathrm{wt}(a_{-2}b)$ for all $j \geq 0$, it is not true for a general element $\sum_{i=1}^{r} u^i \circ v^i$ in $O(V)$ since the highest weight components $u^i_{-2}v^i$ may cancel with each other. But for certain examples of VOAs, especially the VOAs that are also universal highest weight modules over infinite dimensional Lie algebras, we do have the isomorphism $R(V) \cong \mathrm{gr}A(V)$ as commutative Poisson algebras, and it can be proved in different ways.

Proposition 2.2.7. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, equipped with a non-degenerated symmetric invariant bilinear form. Let V be the vacuum module VOA $V_{\widehat{\mathfrak{g}}}(k,0)$ of level $k \in \mathbb{C}$ in [30], then we have: $R(V_{\widehat{\mathfrak{g}}}(k,0)) \cong \mathrm{gr}A(V_{\widehat{\mathfrak{g}}}(k,0))$.*

Proof. By Proposition 5.16 in [21], we have:

$$R(V_{\widehat{\mathfrak{g}}}(k,0)) \cong S(\mathfrak{g}), \quad \text{with} \quad a^1(-1)\dots a^r(-1)\mathbf{1} + C_2(V) \mapsto a^1a^2\dots a^r,$$

for $a^1,\dots,a^r \in \mathfrak{g}$. On the other hand, we have the following identification of the Zhu's algebra, see Section 3 in [30]: $A(V_{\widehat{\mathfrak{g}}}(k,0)) \cong U(\mathfrak{g})$, with $[a^1(-1)\dots a^r(-1)\mathbf{1}] \mapsto a^r\dots a^1$. Then $A(V_{\widehat{\mathfrak{g}}}(k,0))_n = \mathrm{span}\{[a^1(-1)\dots a^r(-1)\mathbf{1}] : a^i \in \mathfrak{g},\ 0 \leq r \leq n\} \cong \mathrm{span}\{a^r\dots a^1 : a^i \in \mathfrak{g},\ 0 \leq r \leq n\} = U(\mathfrak{g})_n$, for each $n \in \mathbb{N}$, where $\{U(\mathfrak{g})_n\}_{n=0}^{\infty}$ is the standard filtration of $U(\mathfrak{g})$. Hence

$$\begin{gathered}\mathrm{gr}A(V_{\widehat{\mathfrak{g}}}(k,0))(\cong \mathrm{gr}U(\mathfrak{g})) \cong S(\mathfrak{g}),\\ [a^1(-1)\dots a^r(-1)\mathbf{1}] + A(V)_{r-1} \mapsto a^r\dots a^1 = a^1\dots a^r.\end{gathered}$$

It follows immediately that we have an isomorphism:

$$\begin{gathered}R(V_{\widehat{\mathfrak{g}}}(k,0)) \cong \mathrm{gr}A(V_{\widehat{\mathfrak{g}}}(k,0)),\\ a^1(-1)\dots a^r(-1)\mathbf{1} + C_2(V) \mapsto [a^1(-1)\dots a^r(-1)\mathbf{1}] + A(V)_{r-1},\end{gathered} \tag{2.2.16}$$

and the morphism (2.2.16) is exactly the linear map ϕ in (2.2.15). □

By adopting a similar method, it is easy to show that $R(M_{\widehat{\mathfrak{h}}}(k,0)) \cong \mathrm{gr}A(M_{\widehat{\mathfrak{h}}}(k,0))$, where $M_{\widehat{\mathfrak{h}}}(k,0)$ is the Heisenberg VOA of level k, and the isomorphism if given by ϕ in (2.2.15).

Proposition 2.2.8. *Let $V = \bar{V}(c,0) = V(c,0)/\langle L_{-1}\mathbf{1}\rangle$ be the Virasoro VOA associated with the (universal) highest weight module $\bar{V}(c,0)$ (cf. [30]). Then*

$$R(\bar{V}(c,0)) \cong \mathrm{gr}A(\bar{V}(c,0)).$$

Proof. Recall that $\bar{V}(c,0) = \text{span}\{L_{-n_1}\dots L_{-n_k}\mathbf{1} : k \geq 0, n_1 \geq n_2 \geq \cdots \geq n_k \geq 2\}$, and the spanning elements are linearly independent. Thus, we have a linear isomorphism:

$$R(\bar{V}(c,0)) \cong \text{span}\{(L_{-2})^n\mathbf{1} + C_2(V) : n \geq 0\} \cong \mathbb{C}[y], \qquad (L_{-2})^n\mathbf{1} + C_2(V) \mapsto y^n,\ \forall n \in \mathbb{N}.$$

On the other hand, it is proved by Wang in [68] that there is an isomorphism of algebras: $A(\bar{V}(c,0)) \cong \mathbb{C}[x]$, $[\omega]^n \mapsto x^n$, for all $n \in \mathbb{N}$. Moreover, we have the following facts in [68]:

$$L_{-n} \equiv (-1)^n((n-1)(L_{-2} + L_{-1}) + L_0) \pmod{O(\bar{V}(c,0))},$$

and $[b] * [\omega] = [(L_{-2} + L_{-1})b]$ for any $b \in \bar{V}(c,0)$. Thus $[L_{-n_1}\dots L_{-n_k}\mathbf{1}] = P([\omega])$ in $A(\bar{V}(c,0))$, where $P(x) \in \mathbb{C}[x]$, with $\deg P \leq k$. So the level filtration of $A(V)$ satisfies:

$$\begin{aligned} A(\bar{V}(c,0))_n &= \text{span}\{[L_{-n_1}\dots L_{-n_k}\mathbf{1}] : k \geq 0, n_1 + \cdots + n_k = n, n_i \geq 2, \forall i\} \\ &= \text{span}\{P([\omega]) : \deg P \leq k \leq \lfloor n/2 \rfloor\} \\ &= \text{span}\{[\mathbf{1}], [\omega], [\omega]^2, \dots, [\omega]^r : r \leq \lfloor n/2 \rfloor\}, \end{aligned}$$

for all $n \in \mathbb{N}$. In particular, $A(\bar{V}(c,0))_{2p} = A(\bar{V}(c,0))_{2p+1} = \text{span}\{[\mathbf{1}], [\omega], \dots, [\omega]^p\}$, for all $p \in \mathbb{N}$. But we also have a filtration $\{F_p\mathbb{C}[x]\}_{p\in\mathbb{N}}$ of $\mathbb{C}[x]$, where $F_{2p}\mathbb{C}[x] = F_{2p+1}\mathbb{C}[x] = \text{span}\{1, x, x^2, \dots, x^p\}$. We have an isomorphism under this filtration:

$$\text{gr}^F\mathbb{C}[x] = \bigoplus_{p=0}^{\infty} F_{2p}\mathbb{C}[x]/F_{2p-1}\mathbb{C}[x] \cong \mathbb{C}[y], \quad x^p + F_{2p-1}\mathbb{C}[x] \mapsto y^p,\ \forall p \in \mathbb{N}.$$

Moreover, we observe the following fact in $A(\bar{V}(c,0))_{2p}/A(\bar{V}(c,0))_{2p-1}$:

$$[\omega]^p + A(\bar{V}(c,0))_{2p-1} = [(L_{-2} + L_{-1})^p\mathbf{1}] + A(\bar{V}(c,0))_{2p-1} = [(L_{-2})^p\mathbf{1}] + A(\bar{V}(c,0))_{2p-1}.$$

It follows that we have an isomorphism:

$$\begin{aligned} &P_2(\bar{V}(c,0))(\cong \mathbb{C}[y]) \cong \text{gr}A(\bar{V}(c,0)), \\ &\quad (L_{-2})^p\mathbf{1} + C_2(V) \mapsto [\omega]^p + A(\bar{V}(c,0))_{2p-1} = [(L_{-2})^p\mathbf{1}] + A(\bar{V}(c,0)),\ \forall p \in \mathbb{N}. \end{aligned} \tag{2.2.17}$$

The morphism (2.2.17) is the same as ϕ in (2.2.15). □

Now we give a counterexample showing that $R(V)$ is not generally isomorphic to $\text{gr}A(V)$. Let L be a positive definite even lattice. The lattice VOA V_L was defined by Frenkel, Lepowsky, and Meurman in [29]. For certain lattice L, $R(V_L)$ is not isomorphic to $\text{gr}A(V_L)$.

Example 2.2.9. Let $L = E_8$ be the root lattice of type E_8. It is well-known that this lattice is unimodular. Dong proved in [13] that V_{E_8} is rational, and its adjoint module is the only irreducible module. The bottom level of this module is $\mathbb{C}\mathbf{1}$. By Theorem 2.2.1 in [73], $\dim A(V_{E_8}) = 1 = \dim \mathrm{gr}A(V_{E_8})$.

On the other hand, for any CFT-type VOA V, we have $V_1 \cap C_2(V) = 0$ since $\mathrm{wt}(a_{-2}b) = \mathrm{wt}a + \mathrm{wt}b + 1 \geq 2$, for any nonzero $a_{-2}b$. Hence $\dim P_2(V_{E_8}) \geq \dim(V_{E_8})_1 \geq \mathrm{rank}E_8 = 8 > \dim \mathrm{gr}A(V_{E_8})$. In fact, a similar argument also shows that $\dim R(V_L) > \dim \mathrm{gr}A(V_L)$, for any unimodular lattice L.

Proposition 2.2.10. *Let $L = \mathbb{Z}\alpha$ be the positive definite lattice of rank 1, with $(\alpha|\alpha) = 2k$, where $k \in \mathbb{Z}_{>0}$. Then we have $R(V_{\mathbb{Z}_\alpha}) \cong \mathrm{gr}A(V_{\mathbb{Z}\alpha})$.*

Proof. Since $\phi : R(V_{\mathbb{Z}\alpha}) \to \mathrm{gr}A(V_{\mathbb{Z}\alpha})$ in (2.2.15) is already an epimorphism, we only have to show that $\dim R(V_{\mathbb{Z}\alpha}) = \dim \mathrm{gr}A(V_{\mathbb{Z}\alpha})$.

Consider the dual lattice $L^\circ = \bigsqcup_{n=-k+1}^{k} L + \frac{n}{2k}\alpha$. By Theorem 3.1 in [13], the irreducible $V_{\mathbb{Z}\alpha}$ modules are $V_{L+\frac{n}{2k}\alpha}$, for $-k+1 \leq n \leq k$. When $n = k$, the bottom level of $V_{L+\frac{1}{2}\alpha}$ is $\mathbb{C}e^{\alpha/2} \oplus \mathbb{C}e^{-\alpha/2}$. When $|n| < k$, we have $(m\alpha + \frac{n}{2k}\alpha | m\alpha + \frac{n}{2k}\alpha) = (m + \frac{n}{2k})^2(\alpha|\alpha) > \left(\frac{n}{2k}\right)^2(\alpha|\alpha)$, for all $m \in \mathbb{Z}\backslash\{0\}$, since $m^2 + \frac{n}{k}m > 0$. So the bottom level of $V_{L+\frac{n}{2k}\alpha}$ is one-dimenional for every $-k+1 \leq n < k$. Thus, by Theorem 2.2.1 in [73],

$$\dim \mathrm{gr}A(V_{\mathbb{Z}\alpha}) = \dim A(V_{\mathbb{Z}\alpha}) = 2^2 + (2k-1)\cdot 1^2 = 2k+3.$$

On the other hand, by Proposition 5.19 in [21], $R(V_{\mathbb{Z}\alpha})$ is a quotient of the polynomial algebra $\mathbb{C}[X, Y, Z]$, modulo the relations: $X^2 = Y^2 = XZ = YZ = 0, XY = \frac{1}{(2k)!}Z^{2k}$. In particular, $R(V_L)$ has a basis $\bar{1}, \bar{X}, \bar{Y}, \bar{Z}, \ldots, \bar{Z}^{2k-1}, \bar{Z}^{2k} = \overline{(2k)!XY}$. So $\dim R(V_{\mathbb{Z}\alpha}) = 2k+3 = \dim \mathrm{gr}A(V_{\mathbb{Z}\alpha})$. $\square$

Remark 2.2.11. By Example 2.2.9 and Proposition 2.2.10, for an affine VOA $L_{\hat{\mathfrak{g}}}(k, 0)$ with positive integer level k, the C_2 algebra $R(L_{\hat{\mathfrak{g}}}(k, 0))$ may or may not be isomorphic to $\mathrm{gr}A(L_{\hat{\mathfrak{g}}}(k, 0))$.

Indeed, since $L = E_8$ is a simple laced root lattice, the lattice VOA V_{E_8} is isomorphic to the affine VOA $L_{\widehat{\mathfrak{g}_{E_8}}}(1, 0)$ (see [30, 29]), where $\mathfrak{g}_{E_8}$ is the simple Lie algebra whose root system is of the type E_8. By Example 2.2.9, $R(L_{\widehat{\mathfrak{g}_{E_8}}}(1, 0)) \ncong \mathrm{gr}A(L_{\widehat{\mathfrak{g}_{E_8}}}(1, 0))$.

On the other hand, $L = \mathbb{Z}\alpha$ with $(\alpha|\alpha) = 2$ is the root lattice of type A_1. Hence $V_L \cong L_{\widehat{sl_2}}(1, 0)$ as VOAs. Then by Proposition 2.2.10, we have $R(L_{\widehat{sl_2}}(1, 0)) \cong \mathrm{gr}A(L_{\widehat{sl_2}}(1, 0))$.

The notherianess of $A(V)$ has the following application. Let V be a VOA. By Definition 2.1.5, a grading subspace $M(n)$ of an admissible V-module M needs not be finite-dimensional. For instance, let U be an infinite-dimensional module over a simple Lie algebra $\mathfrak{g}$, by the construction in [30], the induced module $V_{\widehat{\mathfrak{g}}}(k, U) = U(\widehat{g}) \otimes_{U(\widehat{\mathfrak{g}}_{\geq 0})} U$ is an admissible module over the VOA $V_{\widehat{\mathfrak{g}}}(k, 0)$, and $V_{\widehat{\mathfrak{g}}}(k, U)(0) = U$ is not finite-dimensional.

The bottom level $M(0)$ of any admissible module M is an $A(V)$-module, with the action given by $[a].w = a_{\mathrm{wt}a-1}w$, for all $[a] \in A(V)$ and $w \in M(0)$, see [73, 18] for more details.

Proposition 2.2.12. *Let V be a CFT-type VOA that is C_1-cofinite. Assume M is an admissible V-module such that M is generated by finitely many elements in $M(0)$. Then M must have a maximal submodule.*

Proof. By our assumption, there exists a finite set $S \subset M(0)$ such that

$$M = \mathrm{span}\{a^1_{n_1} \dots a^k_{n_k} w : a^i \in V, k \geq 0,\ n_1, \dots n_k \in \mathbb{Z}, w \in S\}.$$

Given a spanning element $x = a^1_{n_1} \dots a^k_{n_k} w$ of M, if $\mathrm{wt}a^i - n_i - 1 < 0$ for some i, then $a^i_{n_i} w = 0$, and x can be written as a sum of elements of shorter length. So it follows from an easy induction that the bottom level $M(0)$ of M is spanned by elements of the form:

$$a^1_{\mathrm{wt}a^1-1} \dots a^k_{\mathrm{wt}a^k-1} w,$$

for $a^1, \dots, a^k \in V$ homogeneous, and $w \in S$. i.e., $M(0)$ is a finitely generated $A(V)$-module. Since $A(V)$ is noetherian by Theorem 2.2.5, $M(0)$ is a noetherian module, and so $M(0)$ has a maximal submodule U. Let $W \leq M$ be the V-submodule generated by U. Then the bottom level of the quotient module M/W is an irreducible $A(V)$-module $M(0)/U$, and M/W is generated by its bottom level. Hence M/W is a quotient of the generalized Verma module $\bar{M}(M(0)/U)$ constructed in [18]. By Theorem 6.3 in [18], M/W has a maximal submodule $\tilde{W}$, with the property that $\tilde{W} \cap (M(0)/U) = 0$. But then $(M/W)/\tilde{W} \cong L(M(0)/U)$, which is an irreducible V-module since $M(0)/U$ is an irreducible $A(V)$-module. Thus, $\pi^{-1}(\tilde{W}) + W \leq M$ is a maximal submodule, where $\pi : M \to M/W$ is the quotient map. □

We conclude this subsection by recalling the $A(V)$-bimodule $A(M)$ associated with an ordinary V-module M. See Section 1.5 in [30] for more details. For $a \in V$ and $v \in M$, define:

$$a * v := \mathrm{Res}_z Y_M(a, z) v \frac{(1+z)^{\mathrm{wt}a}}{z}, \tag{2.2.18}$$

$$v * a := \mathrm{Res}_z Y_M(a,z)v\frac{(1+z)^{\mathrm{wt}a-1}}{z}, \tag{2.2.19}$$

$$a \circ v := \mathrm{Res}_z Y_M(a,z)v\frac{(1+z)^{\mathrm{wt}a}}{z^2}. \tag{2.2.20}$$

Let $O(M) := \mathrm{span}\{a \circ v : a \in V, v \in M\}$, and let $A(M) := M/O(M)$.

It is proved in [30] that $a * O(M) \subset O(M)$, $O(M) * a \subset O(M)$, and $A(M)$ is a bimodule of $A(V)$, with respect to the left and right action given by (2.2.18) and (2.2.19), respectively. See Theorem 1.5.1 in [30] for more details. Moreover, let M^1, M^2, and M^3 be V-modules, with conformal weights h_1, h_2, and h_3, respectively, and let $I \in I\binom{M^3}{M^1\ M^2}$ be an intertwining operator. Write $I(v,w) = \sum_{n\in\mathbb{Z}} v(n)w^{-n-1} \cdot w^{-h}$, and denote $v(\deg v - 1)$ by $o(v)$. Then we have a linear map $o : M^1 \to \mathrm{Hom}(M^2(0), M^3(0))$, $v \mapsto o(v)$, and we have:

$$o(a * v) = o(a)o(v), \quad o(v * a) = o(v)o(a), \quad o(a \circ v) = 0, \tag{2.2.21}$$

for all $a \in V$ and $v \in M^1$, see Lemma 1.5.2 in [30] for more details.

In Chapter 4, we will construct a new $A(V)$-bimodule $B_\lambda(M)$ associated with M, which can correctly capture the fusion rules.

2.3 Derivations and automorphisms of VOAs

In this section, we first recall the definition of derivations and inner derivations of VOAs. Then we give a concrete description of the derivation algebra for some classical examples of VOAs. Moreover, we prove that the derivations on lattice VOAs are all inner. We then recall the definition of automorphism of VOAs. We will use the closed automorphism groups and their Lie algebras to give a new way to construct new VOAs, as a uniform generalization of the orbifold and commutant construction.

Finally, we generalize the notion of derivations to λ-differentials, which is closely related to the Rota-Baxter operators of VOAs in part II. We will show that the set 1-differentials are in one-to-one correspondence with the automorphism group for a simple VOA. We also propose a way to construct the λ-differentials.

2.3.1 The derivation algebras of the classical examples of VOAs

Let V be a VOA. The notion of derivations of VOAs can be found in [14, 35].

Definition 2.3.1. A linear map $f : V \to V$ is called a derivation if $f(\mathbf{1}) = 0$, $f(\omega) = 0$, and

$$f(Y(a,z)b) = Y(f(a),z)b + Y(a,z)f(b), \quad \forall a, b \in V. \tag{2.3.1}$$

The Lie algebra of all derivations on V is denoted by $\mathrm{Der}(V)$.

Recall that the first level V_1 forms a Lie algebra with respect to the bracket: $[a,b] = a_0 b$ for $a, b \in V_1$. By (2.3.1), any derivation $f \in \mathrm{Der}(V)$ satisfies $f(a_0 b) = f(a)_0 b + a_0 f(b)$. Thus, $f|_{V_1}$ is a derivation on the Lie algebra V_1. Recall the following fact in [14], we write out the proof for completeness:

Lemma 2.3.2. *Let V be a CFT-type VOA. Then for any $a \in V_1$, $f = a_0 = \mathrm{Res}_z Y(a,z) : V \to V$ is a derivation on V.*

Proof. Clearly, $a_0 \mathbf{1} = 0$. Note that $\omega_j a = 0$ for $j \geq 3$, then we have

$$\begin{aligned} a_0 \omega = \omega_{-1} a_0 \mathbf{1} - [\omega_{-1}, a_0] \mathbf{1} &= -\sum_{j \geq 0} \binom{-1}{j} (\omega_j a)_{-1-j} \mathbf{1} \\ &= -(\omega_0 a)_{-1} \mathbf{1} + (\omega_1 a)_{-2} \mathbf{1} - (\omega_2 a)_{-3} \mathbf{1} = -a_{-2} \mathbf{1} + a_{-2} \mathbf{1} + \mu \mathbf{1}_{-3} \mathbf{1} = 0, \end{aligned}$$

where we used the fact that $\omega_2 a \in V_0 = \mathbb{C}\mathbf{1}$. Finally, for any $b \in V$ and $n \in \mathbb{Z}$, we have:

$$[a_0, b_n] = \sum_{j \geq 0} \binom{0}{j} (a_j b)_{n-j} = (a_0 b)_n.$$

It follows that $a_0 Y(b,z) - Y(b,z) a_0 = Y(a_0 b, z)$. i.e., $f = a_0 \in \mathrm{Der}(V)$. □

Definition 2.3.3. [14] Let V be a VOA. Assume that V is CFT-type, or V satisfies $L(1)V_1 = 0$. Then a derivation $f : V \to V$ is called an inner derivation if $f = a_0$ for some $a \in V_1$. The subspace of inner derivations of V is denoted by $\mathrm{Inn}(V)$.

A derivation can be uniquely determined by its actions on the generators:

Lemma 2.3.4. *Assume that V is generated by a subspace $U \subseteq V$. Let $f : V \to V$ be a derivation on V. If there exists some $a \in V_1$ such that $f(b) = a_0 b$ for all $b \in U$, then $f = a_0$ on V.*

Proof. By (2.3.1), for any spanning element $a^1_{n_1} \dots a^r_{n_r} \mathbf{1} \in V$, with $r \geq 0$, $a^i \in U$, and $n_i \in \mathbb{Z}$ for $i = 1, 2, \dots, r$, we have:

$$f(a^1_{n_1} \dots a^r_{n_r} \mathbf{1}) = \sum_{j=1}^{r} a^1_{n_1} \dots f(a^j)_{n_j} \dots a^r_{n_r} \mathbf{1} = \sum_{j=1}^{r} a^1_{n_1} \dots (a_0 a^j)_{n_j} \dots a^r_{n_r} \mathbf{1} = a_0(a^1_{n_1} \dots a^r_{n_r} \mathbf{1}).$$

Thus, $f = a_0$ on V. □

Proposition 2.3.5. *Let $V = \bar{V}(c,0)$ or $L(c,0)$, the (universal) Virasoro VOA. Then* $\mathrm{Der}(V) = 0 = \mathrm{Inn}(V)$.

Proof. Since $V = \bar{V}(c,0)$ and $V = L(c,0)$ are both generated by ω, and any derivation f satisfies $f(\omega) = 0$, then by Lemma 2.3.4, we have $f = 0$ on V. Thus, we have $\mathrm{Inn}(V) = 0 = \mathrm{Der}(V)$. □

Proposition 2.3.6. *Let V be the vacuum module VOA $V_{\widehat{\mathfrak{g}}}(k,0)$, or the affine VOA $L_{\widehat{\mathfrak{g}}}(k,0)$. Then* $\mathrm{Der}(V) = \mathrm{Inn}(V)$.

Proof. Recall that $V_{\widehat{\mathfrak{g}}}(k,0)$ and $L_{\widehat{\mathfrak{g}}}(k,0)$, V are both generated by their first levels, which are the simple Lie algebra $\mathfrak{g}$ via the map $V_1 = \mathrm{span}\{a(-1)\mathbf{1} : a \in \mathfrak{g}\} \to \mathfrak{g} : a(-1)\mathbf{1} \mapsto a,\ \forall a \in \mathfrak{g}$. Let $f \in \mathrm{Der}(V)$. Since $f|_{V_1}$ is a derivation of the Lie algebra $V_1 \cong \mathfrak{g}$, and any derivation of $\mathfrak{g}$ is an inner derivation, then there exists some $a \in V_1$ such that $f|_{V_1} = a_0 : V_1 \to V_1$. But V is generated by V_1, then by Lemma 2.3.4, we have $f = a_0$ on V. i.e., $f \in \mathrm{Inn}(V)$. □

Proposition 2.3.7. *Let V be the Heisenberg VOA $M_{\widehat{\mathfrak{h}}}(1,0)$ associated with a n-dimentional inner product space $\mathfrak{h}$. Then* $\mathrm{Der}V \cong \mathfrak{o}(n,\mathbb{C})$*, the Lie algebra of $n \times n$ skew-symmetric matrices.*

Proof. Similar to the affine VOA case, a derivation $f \in \mathrm{Der}(V)$ is completely determined by its restriction onto the first level $\mathfrak{h}$, i.e., $\mathrm{Der}(V) \to \mathrm{gl}(\mathfrak{h},\mathbb{C}) : f \mapsto f|_{\mathfrak{h}}$ is an embedding of Lie algebras. Since $\mathfrak{h}$ is abelian, each $g \in \mathrm{gl}(\mathfrak{h},\mathbb{C})$ satisfies $g([a,b]) = [g(a),b] + [a,g(b)],\ \forall a,b \in \mathfrak{h}$. Let $\{a^1,\dots,a^n\}$ be an orthonormal basis of $\mathfrak{h}$ with respect to the inner product $(\cdot|\cdot)$. Recall that $\omega = \frac{1}{2}\sum_{i=1}^{n} a^i(-1)^2\mathbf{1}$. For any $f \in \mathrm{Der}(V)$, assume that $f|_{\mathfrak{h}}(a^i) = \sum_{j=1}^{n} c_{ij}a^j$, as $f(\omega) = 0$, we have:

$$
\begin{aligned}
0 &= \frac{1}{2}\sum_{i=1}^{n}(f(a^i)(-1)a^i(-1)\mathbf{1} + a^i(-1)f(a^i)(-1)\mathbf{1}) = \sum_{i=1}^{n} f(a^i)(-1)a^i(-1)\mathbf{1} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}a^j(-1)a^i(-1)\mathbf{1}.
\end{aligned}
$$

Thus, $c_{ij} = -c_{ji}$ for all $1 \le i,j \le n$, and so the image of $\mathrm{Der}(V) \to \mathrm{gl}_{\mathbb{C}}(\mathfrak{h})$ is $\mathfrak{o}(n,\mathbb{C})$. □

Note that $\mathrm{Inn}(M_{\widehat{\mathfrak{h}}}(1,0)) = 0$. Indeed, for any $a, a_1, \dots, a_r \in \mathfrak{h}$, and $n_1,\dots,n_r \ge 1$, we have $a(0)(a_1(-n_1)\dots a_r(-n_r)\mathbf{1}) = 0$. In particular, if $\dim\mathfrak{h} = 1$, then $\mathrm{Der}(M_{\widehat{\mathfrak{h}}}(1,0)) = 0 = \mathrm{Inn}(M_{\widehat{\mathfrak{h}}}(1,0))$; while f $\dim\mathfrak{h} \ge 2$, we have $\mathrm{Der}(M_{\widehat{\mathfrak{h}}}(1,0)) \ne \mathrm{Inn}(M_{\widehat{\mathfrak{h}}}(1,0))$.

For the lattice VOA V_L, a concrete description of the automorphism group $\mathrm{Aut}(V_L)$ is given by Dong and Nagatomo, see Theorem 2.1 in [15]. On the other hand, we will prove that all derivations of the lattice VOA are inner. First, we recall the notion of strongly rational VOAs:

Definition 2.3.8. Let V be a VOA. V is called **strongly rational** if it is of CFT-type, rational, C_2-cofinite, and satisfies $L(1)V_1 = 0$.

According to [50], a stongly rational V carries a nondegenerate symmetric invariant bilinear form $(\cdot|\cdot) : V \times V \to \mathbb{C}$. By definition, the bilinear form is invariant means that

$$(Y(a,z)u|v) = (u|Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})v), \tag{2.3.2}$$

for all $a, u, v \in V$ (cf. (2.1.13)). By rescaling the bilinear form with a nonzero scalar, we may assume that $(\mathbf{1}|\mathbf{1}) = 1$. The following Theorem in [14] describes the derivation algebra for strongly rational VOAs:

Theorem 2.3.9. *Let V be a strongly rational VOA. Assume that $\sum_{m=0}^{n} V_m$ generates V for some $n > 0$. Then $\mathrm{Der}(V)$ is a direct sum of ideals $o(\mathfrak{g})$ and $\mathfrak{g}^\perp$, where $\mathfrak{g}$ is the Lie algebra V_1, and $\mathfrak{g}^\perp$ consists of $d \in \mathrm{Der}(V)$ such that $\mathrm{tr}_{V_n} o(u)d = 0$ for all $u \in V_1$.*

Remark 2.3.10. The proof of Theorem 2.3.9 in [14] also shows that $d|_{V_1} = 0$, for any $d \in \mathfrak{g}^\perp$.

Lemma 2.3.11. *Let $d \in \mathrm{Der}(V)$, and $u \in \mathrm{Ker}\, L(1)$. We have $(du|v) = -(u|dv)$ for all $v \in V$.*

Proof. Since $dV_n \subseteq V_n$, and $(V_n|V_m) = 0$ if $m \neq n$, we may assume that $v \in V_n$ and $u \in V_n \cap \ker L(1)$. Then $u_{2n-1}v \in V_0 = \mathbb{C}\mathbf{1}$, and so $u_{2n-1}v = \lambda\mathbf{1}$ for some $\lambda \in \mathbb{C}$. By (2.3.2), and the fact that $L(1)u = 0$, we have:

$$\begin{aligned}(u|v) = (u_{-1}\mathbf{1}|v) &= (\mathbf{1}|\sum_{j\geq 0} \frac{(-1)^n}{j!}(L(1)^j u)_{2n+1-j-2}v) \\ &= (-1)^n(\mathbf{1}|u_{2n-1}v) = (-1)^n\lambda.\end{aligned}$$

Thus, $u_{2n-1}v = (-1)^n(u|v)\mathbf{1}$. Furthermore, since $d(\mathbf{1}) = 0$ and $dL(1) = L(1)d$, we have $du \in \mathrm{Ker}\, L(1)$, and so $0 = d(u_{2n-1}v) = d(u)_{2n-1}v + u_{2n-1}d(v) = (-1)^n(d(u)|v)\mathbf{1} + (-1)^n(u|d(v))\mathbf{1}$. Therefore, $(du|v) = -(u|dv)$. □

Theorem 2.3.12. *Let L be a positive definite even lattice. Then* $\mathrm{Der}(V_L) = \mathrm{Inn}(V_L)$

Proof. Let $d \in \mathfrak{g}^{\perp}$, by Theorem 2.3.9, we just need to show $d \in \mathrm{Inn}(V_L)$.

We use induction on the level n of V_L to show that each $e^{\alpha} \in (V_L)_n$ is an eigenvector of d. Indeed, when $n = 1$ we have $d = 0$ on $(V_L)_1$, so all e^{α} in the first level are eigenvectors of d of eigenvalue 0. Assume that $de^{\alpha} = \lambda_{\alpha} e^{\alpha}$ for all $\alpha \in L$ with $(\alpha|\alpha) < 2n$. Since $dh = 0$ for all $h \in \mathfrak{h}$, we have $du = \lambda_{\alpha} u$ for all $u \in M(1, \alpha)$, and $\alpha \in L$ s.t. $(\alpha|\alpha) < 2n$. Let

$$U := \bigoplus_{\alpha \in L, (\alpha|\alpha) < 2n} M(1, \alpha) \cap (V_L)_n \leq (V_L)_n.$$

Then we have $dU \subseteq U$. Moreover, since a basis element in $(V_L)_n$ is of the form

$$x = \alpha^1(-n_1) \ldots \alpha^k(-n_k) e^{\beta},$$

where $k \geq 0$, $\alpha^i \in \mathfrak{h}$ for $i = 1, 2, ..., k$, $n_1 \geq \cdots \geq n_k \geq 1$, and $\beta \in L$, while x has weight equal to $n_1 + \cdots + n_k + \frac{(\beta|\beta)}{2} = n$, it follows that $(V_L)_n = U \oplus W$, where $W = \mathrm{span}\{e^{\alpha} : (\alpha|\alpha) = 2n\}$, and $U = span\{\alpha^1(-n_1) \ldots \alpha^k(-n_k) e^{\beta} : k \geq 1, n_1 \geq \cdots \geq n_k \geq 1, (\beta|\beta) < 2n\}$. Let $(\cdot|\cdot)$ be the standard symmetric invariant bilinear form on V_L, then for any spanning element $\alpha^1(-n_1) \ldots \alpha^k(-n_k) e^{\beta} \in U$ and $e^{\alpha} \in W$, we have:

$$(\alpha^1(-n_1) \ldots \alpha^k(-n_k) e^{\beta} | e^{\alpha}) = (\alpha^2(-n_2) \ldots \alpha^k(-n_k) e^{\beta} | \alpha^1(n_1) e^{\alpha}) = 0,$$

hence $U \perp W$. But $(\cdot|\cdot)$ is nondegenerate on $(V_L)_n$, it follows that $(\cdot|\cdot)$ is also nondegenerate on both U and W, and $W = U^{\perp}$. Now for any $u \in U$ and $w \in W$, since $w \in \ker L(1)$, then by Lemma 2.3.11, we have: $(dw|u) = -(w|du) = 0$. Hence $dw \in U^{\perp} = W$. On the other hand, it is easy to see that $W = \bigoplus_{\alpha \in L, (\alpha|\alpha) = 2n} \mathbb{C} e^{\alpha}$, and each subspace $\mathbb{C} e^{\alpha}$ is a common eigenspace of $\mathfrak{h}(0)$ with eigenfunction $(\alpha|\cdot)$. Since d commutes with $\mathfrak{h}(0)$, we have $de^{\alpha} = \lambda_{\alpha} e^{\alpha}$ for all $\alpha \in L$ s.t. $(\alpha|\alpha) = 2n$. This finishes the proof of our claim that each e^{α} is an eigenvector of d.

Now let $L = \mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_r$, and let $de^{\alpha_i} = \lambda_i e^{\alpha_i}$ for $i = 1, 2, \ldots, r$. Since $\mathfrak{h} = \mathbb{C}\alpha_1 \oplus \ldots \oplus \mathbb{C}\alpha_r$, there must exist some $h \in \mathfrak{h}$ s.t. $(h|\alpha_i) = \lambda_i$, for $i = 1, 2, \ldots, r$. Hence $d = h(0)$ on $e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_r}$, which is a set of generators of V_L. This shows that $d = h(0)$ on V_L, and so d is an inner derivation. □

2.3.2 Automorphism groups and λ-differentials of VOAs

The definition of automorphisms of VOAs can be found in [29, 27].

Definition 2.3.13. Let $(V, Y, \mathbf{1}, \omega)$ be a VOA. An automorphism on V is a linear automorphism $\phi \in \mathrm{GL}(V)$, such that $\phi(\mathbf{1}) = \mathbf{1}$, $\phi(\omega) = \omega$, and

$$\phi(Y(a,z)b) = Y(\phi(a),z)\phi(b)\,(\iff \phi(a_n b) = \phi(a)_n\phi(b),\ \forall n \in \mathbb{Z}), \quad \forall a, b \in V. \tag{2.3.3}$$

The subgroup of $\mathrm{GL}(V)$ consisting of all automorphisms of V is denoted by $\mathrm{Aut}\,(V)$.

Lemma 2.3.14. *Let $d \in \mathrm{Der}(V)$, then $e^d = \sum_{n=0}^{\infty} d^n/n! \in \mathrm{Aut}\,(V)$. (This is not true if the VOA V is defined over a ground field $\mathbb{F}$ that is not complete, e.g., $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$.)*

Proof. Since $d(\mathbf{1}) = 0$ and $d(\omega) = 0$, it is clear that $e^d(\mathbf{1}) = \mathbf{1}$ and $e^d(\omega) = \omega$. For any $n \in \mathbb{N}$ and $a \in V_n$, if we view $d : V_n \to V_n$ as a matrix, then by the triangle inequality, we have:

$$||\sum_{n=0}^{N} \frac{d^n a}{n!}|| \leq \sum_{n=0}^{N} \frac{||d^n a||}{n!} \leq \sum_{n=0}^{N} \frac{||d||^n ||a||}{n!} < e^{||d||}||a||,$$

where $||a||$ is given by the standard norm on the finite-dimensional $\mathbb{C}$-vector space V_n. Thus the series $\sum_{n=0}^{\infty}(d^n a)/n!$ is convergent in V_n, and we let the limit be $e^d a \in V_n$. This shows that e^d is well-defined. Moreover, by induction, it is easy to check that $d^m(a_n b) = \sum_{j=0}^{m}(d^{m-j}a)_n(d^j b)$, for all $a, b \in V$, $n \in Z$, and $m \in \mathbb{N}$. It follows immediately that $e^d(a_n b) = (e^d a)_n(e^d b)$, for all $a, b \in V$, and $n \in \mathbb{Z}$. Hence $e^d \in \mathrm{Aut}\,(V)$. □

By Theorem 2.1 in [14], if V is finitely generated, then $\mathrm{Aut}\,(V)$ is a linear algebraic group. By Lemma 2.3.14, for any $a_0 \in \mathrm{Inn}(V)$, we have $e^{a_0} \in \mathrm{Aut}\,(V)$. Let $G \leq \mathrm{Aut}\,(V)$ be the closed subgroup generated by $\{e^{a_0} : a \in V_1\}$. Then it is clear that the Lie algebra $\mathfrak{g}$ of G is precisely the Lie algebra V_1. Moreover, by the main Theorem in [17], if V is strongly rational, $\mathfrak{g}$ is a reductive Lie algebra, then G is a reductive algebraic group.

Remark 2.3.15. It is expected that $\mathrm{Der}(V)$ is isomorphic to the Lie algebra of the linear algebraic group $\mathrm{Aut}\,(V)$. In particular, if $\mathrm{Der}(V) = \mathrm{Inn}(V)$, then $\dim G = \dim \mathrm{Aut}\,(V)$, and so $G = \mathrm{Aut}\,(V)^{\circ}$. We will prove $\mathrm{Der}(V) \cong \mathrm{Lie}(\mathrm{Aut}\,(V))$ in the future.

Since $G \leq \mathrm{Aut}\,(V)$ is an algebraic group, we can discuss the fixed point sub-VOA of a closed subgroup of G. This leads to the following definition:

Definition 2.3.16. Let $H \leq G$ be a closed connected subgroup, with Lie algebra $\mathfrak{h} \leq \mathfrak{g}$. The **generalized orbifold** V^H is defined to be the fixed points sub-VOA of V:

$$V^H := \{v \in V : x(v) = v,\ \forall x \in H\}. \tag{2.3.4}$$

$V^H \leq V$ is a sub-VOA that share the same Virasoro element as V. We have the following classical result from algebraic groups. See Theorem (13.2) in [33].

Theorem 2.3.17. *Let H be a closed subgroup of a linear algebraic group G. If $H \leq G_v = \{g \in G : g(v) = v\}$, then $\mathfrak{h} \subseteq \mathfrak{g}_v = \{X \in \mathfrak{g} : X(v) = 0\}$.*

Proposition 2.3.18. *The generalized orbifold V^H is equal to $V^{\mathfrak{h}} = \{v \in V : a_0 v = 0, \ \forall a \in \mathfrak{h}\}$.*

Proof. For any $v \in V^H$, we clearly have $H \leq G_v$, hence $\mathfrak{h} \subseteq \mathfrak{g}_v$ by Theorem 2.3.17 i.e., for any $a \in \mathfrak{h}$, we have $a_0 v = 0$. This shows $V^H \subseteq V^{\mathfrak{h}}$. Conversely, for any $v \in V^{\mathfrak{h}}$, since $e^X v = v$ for any $X \in \mathfrak{h}$, and $H = \langle e^X : X \in \mathfrak{h}\rangle$, we have $x(v) = v$ for all $x \in H$. i.e., $v \in V^H$. Hence $V^{\mathfrak{h}} \subseteq V^H$. □

Thus, the generalized orbifold V^H is also a **generalized commutant** (see the last Section in [30] for the definition of commutant sub-VOAs):

$$V^H = \{v \in V : a_0 v = 0, \ \forall a \in \mathfrak{h}\}. \tag{2.3.5}$$

We consider the following easy examples:

Example 2.3.19. Let $V = V_L$, $L = \mathbb{Z}\alpha$ the rank one lattice VOA, with $(\alpha|\alpha) = 4$. Then $\mathfrak{g} = \mathbb{C}\alpha(-1)\mathbf{1}$. Let $H = G$ then $V^H = \{v \in V : \alpha(0)v = 0\} = M_{\widehat{\mathfrak{h}}}(1, 0)$.

Example 2.3.20. Let $V = V_{\widehat{\mathfrak{g}}}(k, 0)$ be the level k vacuume module VOA associated to the Lie algebra $\mathfrak{g} = \mathrm{sl}(2, \mathbb{C}) = \mathbb{C}e + \mathbb{C}h + \mathbb{C}f$.

(1) Let H be the Cartan subgroup $D(2, \mathbb{C}) \cap \mathrm{SL}(2, \mathbb{C})$ then $\mathfrak{h} = \mathbb{C}h$. $V^H = \{v \in V : h(0)v = 0\}$ is the eigenspace of $h(0)$ of eigenvalue 0. In particular, by the PBW theorem,

$$V^H = \mathrm{span}\{e^m(-r)f^m(-s)\mathbf{1} : m, r, s \in \mathbb{N}\} + M_{\widehat{\mathfrak{h}}}(k, 0).$$

In fact, a general case of such sub-VOAs denoted by $V(k, 0)(0)$ was also studied by Dong and Wang in [25]. They found a set of generators of this sub-VOA as an intermediate result towards the structure theory of parafermion VOAs. See Theorem 2.1 in [25].

(2) Consider the Borel subgroup $B = T(2, \mathbb{C}) \cap \mathrm{SL}(2, \mathbb{C})$ then its Lie algebra is the Borel subalgebra $\mathfrak{b} = \mathbb{C}e + \mathbb{C}h$. Then the first level of V^B is 0, since the centralizer of $\mathfrak{b}$ in $\mathfrak{g}$ is 0. It is easy to check that the second level is 1-dimensional: $(V^B)_2 = \mathbb{C}\omega$, while the third level is nonzero. e.g., it is easy to see that

$$x = 2e(-2)f(-1)\mathbf{1} + 2e(-1)f(-2)\mathbf{1} - 2h(-3)\mathbf{1} + h(-2)h(-1)\mathbf{1} \in (V^B)_3.$$

So the Virasoro sub-VOA $L(c, 0) \leq V^B$, where c is the central charge of V, but $L(c) \neq V^B$.

We will further study the generalized orbifold and commutant in the future. More precisely, we will see if the generalized orbifold and commutant can give us rational VOAs.

The automorphism group also gives rise to a generalized notion of derivations with weight for VOAs, which is closely related to the Rota-Baxter operators on VOAs we will study in this thesis's second part.

Definition 2.3.21. Let $(V, Y, \mathbf{1})$ be a vertex algebra, and $\lambda \in \mathbb{C}$ be a fixed complex number. A linear map $d : V \to V$ is called a **weak λ-differential** of V if it satisfies

$$d(Y(a,z)b) = Y(da,z)b + Y(a,z)db + \lambda Y(da,z)db, \tag{2.3.6}$$

for all $a, b \in V$. i.e., $d(a_m b) = (da)_m b + a_m(db) + \lambda(da)_m(db)$, for all $a, b \in V$, and $m \in \mathbb{Z}$.

Let $(V, Y, \mathbf{1}, \omega)$ be a VOA. A **λ-differential** on V is a weak λ-differential $d : V \to V$ such that $d\mathbf{1} = 0$ and $d\omega = 0$. The space of λ-differentials on V is denoted by $\mathrm{Diff}_\lambda(V)$.

By Definition 2.3.21, it is easy to see that a 0-differential operator is just a derivation on V. i.e., $\mathrm{Diff}_0(V) = \mathrm{Der}(V)$. The 1-differentials have a nice correspondence with the automorphisms on V. Recall that an endomorphism of V is a linear map $\phi : V \to V$ such that

$$\phi(Y(a,z)b) = Y(\phi(a), z)\phi(b), \tag{2.3.7}$$

$\phi(\mathbf{1}) = \mathbf{1}$, and $\phi(\omega) = \omega$ (cf. [27]). The space of endomorphisms on V is denoted by $\mathrm{End}_V(V)$.

Lemma 2.3.22. *If $(V, Y, \mathbf{1}, \omega)$ is a simple VOA, then $\mathrm{End}_V(V)$ is a division algebra over $\mathbb{C}$, with the unit group $\mathrm{Aut}\,(V)$.*

Proof. Let $\phi \in \mathrm{End}_V(V) - \{0\}$, it suffices to show that ϕ is an automorphism. First we note that $\ker\phi$ is an ideal of V: for any $u \in \ker\phi$, $a \in V$, and $m \in \mathbb{Z}$, we have $\phi(a_m u) = \phi(a)_m \phi(u) = 0$ by (2.3.7). Since $\phi \neq 0$, we have $\ker\phi = 0$, and ϕ is injective. Moreover, for any $a \in V_n$, since $\phi(\omega) = \omega$, we have $L(0)\phi(a) = \phi(\omega)_1\phi(a) = \phi(L(0)a) = n\phi(a)$. It follows that $\phi(V_n) \subseteq V_n$ for every $n \in \mathbb{N}$, and since $\dim V_n < \infty$ for each n, we have $\phi|_{V_n} : V_n \to V_n$ is a linear isomorphism for every $n \in \mathbb{N}$. Thus, ϕ is an automorphism. □

Proposition 2.3.23. *Let $(V, Y, \mathbf{1}, \omega)$ be a simple VOA. Then the map $\alpha : \mathrm{Diff}_1(V) \to \mathrm{Aut}\,(V), d \mapsto d + \mathrm{Id}_V$ is a bijection.*

Proof. Since $d\mathbf{1} = 0$ and $d\omega = 0$, we have $(d + \mathrm{Id}_V)(\mathbf{1}) = \mathbf{1}$ and $(d + \mathrm{Id}_V)(\omega) = \omega$. Moreover,

$$\begin{aligned}(d + \mathrm{Id})(Y(a, z)b) &= Y(da, z)b + Y(a, z)db + Y(da, z)db + Y(a, z)b \\ &= Y(da + a, z)b + Y(a + da, z)db \\ &= Y((d + \mathrm{Id})(a), z)(d + \mathrm{Id})(b),\end{aligned}$$

thus $d + \mathrm{Id}_V \in \mathrm{End}_V(V)$. But $d + \mathrm{Id}_V \neq 0$, since otherwise $d = -\mathrm{Id}_V$ does not satsify $d\mathbf{1} = 0$. Hence $\alpha(d) = d + \mathrm{Id}_V \in (\mathrm{End}_V(V))^\times = \mathrm{Aut}\,(V)$, in view of Lemma 2.3.22. On the other hand, for $\phi \in \mathrm{Aut}\,(V)$, we have $(\phi - \mathrm{Id}_V)(\mathbf{1}) = 0$ and $(\phi - \mathrm{Id}_V)(\omega) = 0$, and

$$\begin{aligned}&Y((\phi - \mathrm{Id}_V)(a), z)b + Y(a, z)(\phi - \mathrm{Id}_V)(b) + Y((\phi - \mathrm{Id}_V)(a), z)(\phi - \mathrm{Id}_V)(b) \\ &= Y(\phi(a), z)b - Y(a, z)b + Y(a, z)\phi(b) - Y(a, z)b + Y(\phi(a), z)\phi(b) - Y(a, z)\phi(b) \\ &\quad - Y(\phi(a), z)b + Y(a, z)b \\ &= Y(\phi(a), z)\phi(b) - Y(a, z)b = (\phi - \mathrm{Id}_V)(Y(a, z)b).\end{aligned}$$

Thus, $\phi - \mathrm{Id}_V \in \mathrm{Diff}_1(V)$. Clearly, $\phi \mapsto \phi - Id_V$ is an inverse of α, hence α is a bijection. □

Remark 2.3.24. By a similar argument, we can show that $\beta : \mathrm{Diff}_{-1}(V) \to \mathrm{Aut}\,(V), d \mapsto \mathrm{Id}_V - d$ is a bijection, whose inverse is given by $\phi \mapsto \mathrm{Id}_V - \phi$. Thus, the 1-differentials and -1-differentials on simple VOAs can be completely determined. The natural question is to construct a λ-differential on some specific VOAs for a given $\lambda \in \mathbb{C}$. We will study this problem in the future.

2.4 Correlation functions associated with a module

In this section, we will discuss the system of correlation functions associated with a module M over a VOA $(V, Y, \mathbf{1}, \omega)$. Some results in this section are well-known, and we will write out the proof for part of them as a reference for the later chapters. In particular, we will prove that the locality and associativity of the correlation functions can give rise to the component form Jacobi identity with the assistance of Cauchy's integral Theorem. Such a fact was first pointed out for the lattice VOA in Appendix A. of [29], and later it was reformulated in a formal variable language in [27]. Hence the rationality of products, locality(commutativity), and associativity of the vertex operators given in [27] are the essential axioms of VOAs.

The outline of this section is the following: We first recall Cauchy's integral Theorem and configuration of contours in the Appendix of [29]. Then we introduce the notion of a system of correlation functions associated with a V-module and prove some basic properties. Finally, we will construct the system of correlation functions associated with a twisted module over VOAs [18, 70], and give a component form of the twisted Jacobi identity. We will show that the component form of the twisted Jacobi identity also follows from the locality and associativity of the twisted correlation functions.

2.4.1 The Cauchy–Jacobi identity

We first recall the embedding ι in [29, 27, 73]: $\iota_{z,w}, \iota_{w,z}$, and $\iota_{w,z-w}$ are embeddings of the Laurent polynomial ring $\mathbb{C}[z^{\pm 1}, w^{\pm 1}, (z-w)^{\pm 1}]$ into the fields of power series $\mathbb{C}((z))((w))$, $\mathbb{C}((w))((z))$, and $\mathbb{C}((w))((z-w))$, respectively, they are given as follows:

$$\iota_{z,w}(z^m w^n (z-w)^l) := \sum_{j\geq 0} \binom{l}{j} (-1)^j z^{m+l-j} w^{n+j}, \tag{2.4.1}$$

$$\iota_{w,z}(z^m w^n (z-w)^l) := \sum_{j\geq 0} \binom{l}{j} (-1)^{l-j} w^{n+l-j} z^{m+j}, \tag{2.4.2}$$

$$\iota_{w,z-w}(z^m w^n (z-w)^l) = \sum_{j\geq 0} \binom{m}{j} w^{n+m-j} (z-w)^{l+j}. \tag{2.4.3}$$

for all $m, n, l \in \mathbb{Z}$. i.e., $\iota_{z,w}(f(z,w))$ is the series expansion of a rational function $f(z,w) \in \mathbb{C}[z^{\pm 1}, w^{\pm 1}, (z-w)^{\pm 1}]$ in the domain $0 < |w| < |z|$, $\iota_{w,z}(f(z,w))$ is the series expansion in the domain $0 < |z| < |w|$, and $\iota_{w,z-w}(f(z,w))$ is the series expansion in $0 < |z-w| < |w|$.

We observe the following fact about $\iota_{z,w} f(z,w)$:

$$\begin{aligned} \mathrm{Res}_z \mathrm{Res}_w \left(\iota_{z,w} f(z,w)\right) &= \mathrm{Res}_z \mathrm{Res}_w \left(\sum_{p,q\in\mathbb{Z}} a_{p,q} z^{-p-1} w^{-q-1} \right) = \mathrm{Res}_z \left(\sum_{p\in\mathbb{Z}} a_{p,0} z^{-p-1} \right) \\ &= a_{0,0} = \mathrm{Res}_z \left(\sum_{q\in\mathbb{Z}} a_{0,q} w^{-q-1} \right) = \mathrm{Res}_w \mathrm{Res}_z \left(\sum_{p,q\in\mathbb{Z}} a_{p,q} z^{-p-1} w^{-q-1} \right) \\ &= \mathrm{Res}_w \mathrm{Res}_z \left(\iota_{z,w} f(z,w)\right). \end{aligned} \tag{2.4.4}$$

That is, once we fix the domain $0 < |w < |z|$ of expansion for the rational function $f(z,w)$, the order of taking residues or contour integrals for $f(z,w)$ is interchangeable. Similarly, we have:

$$\mathrm{Res}_z \mathrm{Res}_w \left(\iota_{w,z} f(z,w)\right) = \mathrm{Res}_w \mathrm{Res}_z \left(\iota_{w,z} f(z,w)\right), \tag{2.4.5}$$

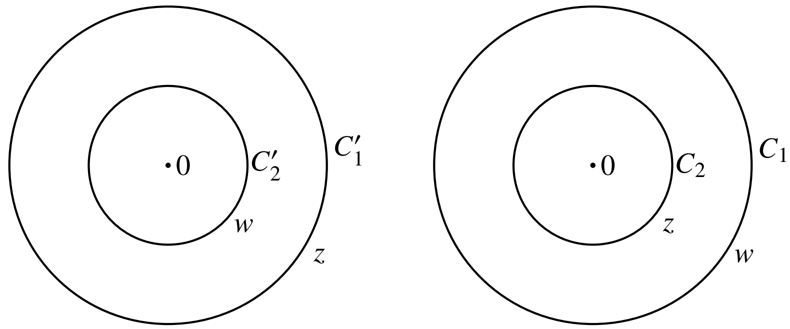

Figure 2.1: Figure of contours C_1, C_2, C_1', C_2'

$$\mathrm{Res}_w \mathrm{Res}_{z-w} \left(\iota_{w,z-w} f(z,w)\right) = \mathrm{Res}_{z-w} \mathrm{Res}_w \left(\iota_{w,z-w} f(z,w)\right). \tag{2.4.6}$$

The following Theorem is a consequence of Cauchy's integral Theorem (or Cauchy's residue Theorem) in complex analysis, see the Appendix A. of [29]:

Theorem 2.4.1. *Let $f(z,w) \in \mathbb{C}[z^{\pm 1}, w^{\pm 1}, (z-w)^{\pm 1}]$, we have:*

$$\mathrm{Res}_z \mathrm{Res}_w \left(\iota_{z,w} f(z,w)\right) - \mathrm{Res}_w \mathrm{Res}_z \left(\iota_{w,z} f(z,w)\right) = \mathrm{Res}_w \mathrm{Res}_{z-w} \left(\iota_{w,z-w} f(z,w)\right). \tag{2.4.7}$$

Proof. First, we observe that $f(z,w)$ has the following form:

$$f(z,w) = \frac{h(z,w)}{z^r w^s (z-w)^t} \in \mathbb{C}[z^{\pm 1}, w^{\pm 1}, (z-w)^{\pm 1}], \quad \text{where } r,s,t \in \mathbb{N},$$

where $h(z,w) \in \mathbb{C}[z,w]$ is a polynomial. Then the only possible poles of $f(z,w)$ are at $z = 0, w = 0$, and $z = w$. We use the contour integration interpretations for the residues in (2.4.7), and we adopt the notations in Proposition A.2.8 in [29]. Let C_1, C_2' be contours of w, and let C_1', C_2 be contours of z. The configuration of these contours is given by Figure 2.1. Then $f(z,w)$ is holomorphic in w inside of C_2' except at the pole $w = 0$, and $f(z,w)$ is holomorphic in z inside of C_2 except at the pole $z = 0$. Then by (2.4.4),

$$\begin{aligned}
&\mathrm{Res}_z \mathrm{Res}_w \left(\iota_{z,w} f(z,w)\right) - \mathrm{Res}_w \mathrm{Res}_z \left(\iota_{w,z} f(z,w)\right) \\
&= \mathrm{Res}_w \mathrm{Res}_z \left(\iota_{z,w} f(z,w)\right) - \mathrm{Res}_w \mathrm{Res}_z \left(\iota_{w,z} f(z,w)\right) \\
&= \int_{C_2'} \int_{C_1'} f(z,w) dz dw - \int_{C_1} \int_{C_2} f(z,w) dz dw. \\
&= \int_{C_1} \int_{C_1'} f(z,w) dz dw - \int_{C_1} \int_{C_2} f(z,w) dz dw,
\end{aligned}$$

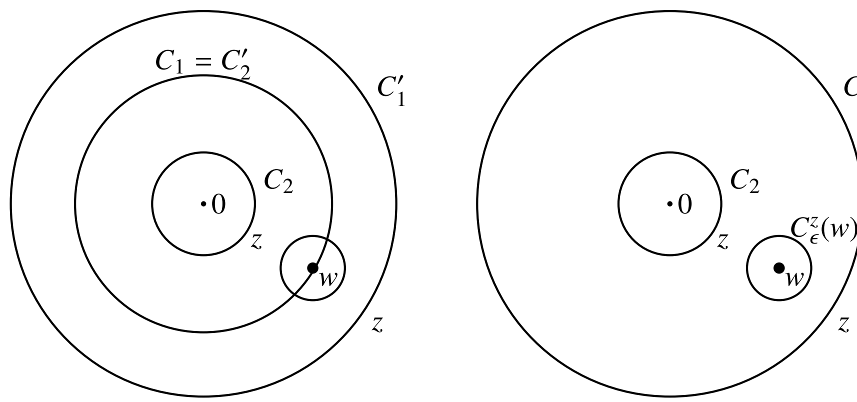

Figure 2.2: Figure of mixed contours

where we choose the contours properly so that $C_2' = C_1$, similar to Proposition A.2.8 in [29]. Then the contours in the integrals above are given by the left diagram in Figure 2.2.

Consider the integral $\int_{C_2'}\int_{C_1'} f(z,w)dzdw$. Since the only possible poles of $f(z,w)$ inside of the contour C_1' are at $z=0$ and $z=w$, by the Cauchy's integral Theorem, $\int_{C_1'} f(z,w)dz$ is equal to the sum of contour integrals of $f(z,w)$ around $z=0$ and $z=w$. Let $C_\epsilon^z(w)$ be a small circle surrounding w, with radius $\epsilon < |w|$, see the diagram on the right in Figure 2.2. Then

$$\int_{C_1'} f(z,w)dz = \int_{C_2} f(z,w)dz + \int_{C_\epsilon^z(w)} f(z,w)dz, \tag{2.4.8}$$

and it follows that

$$\begin{aligned}\mathrm{Res}_z\mathrm{Res}_w\left(\iota_{z,w}f(z,w)\right) - \mathrm{Res}_w\mathrm{Res}_z\left(\iota_{w,z}f(z,w)\right) &= \int_{C_1}\int_{C_1'} f(z,w)dzdw - \int_{C_1}\int_{C_2} f(z,w)dzdw \\ = \int_{C_1}\int_{C_\epsilon^z(w)} f(z,w)dzdw = \mathrm{Res}_w\mathrm{Res}_{z-w}\left(\iota_{w,z-w}f(z,w)\right).\end{aligned}$$

This proves (2.4.7). □

In fact, from the proof of Theorem 2.4.1 and formula (2.4.8), we also have a stronger form of the formula (2.4.7), which is precisely (1.1.4) in [73]:

$$\mathrm{Res}_z\left(\iota_{z,w}f(z,w)\right) - \mathrm{Res}_z\left(\iota_{w,z}f(z,w)\right) = \mathrm{Res}_{z-w}\left(\iota_{w,z-w}f(z,w)\right). \tag{2.4.9}$$

Theorem 2.4.1 can give us an alternative (analytic) proof of Theorem 2.1.2 (the original proof of this Theorem in [27] uses the formal variable approach, see Proposition 3.1.1 in [27]):

Corollary 2.4.2. *Let V be a vector space and $Y : V \to \mathrm{End}(V)[[z, z^{-1}]]$ be a vertex operator satisfying the truncation property, the locality* (2.1.3)*, and the weak associativity* (2.1.4)*. Then Y satisfies the Jacobi identity* (2.1.2)*.*

Proof. Indeed, the locality and weak associativity of Y together indicate that there exists some $f(z, w) \in \mathbb{C}[z^{\pm 1}, w^{\pm 1}, (z-w)^{\pm 1}]$ such that

$$\langle v', Y(a, z)Y(b, w)v\rangle z^m w^n (z-w)^l = \iota_{z,w} f(z, w), \tag{2.4.10}$$

$$\langle v', Y(b, w)Y(a, z)v\rangle z^m w^n (z-w)^l = \iota_{w,z} f(z, w), \tag{2.4.11}$$

$$\langle v', Y(Y(a, z-w)b, w)v\rangle z^m w^n (z-w)^l = \iota_{w,z-w} f(z, w), \tag{2.4.12}$$

for fixed $a, b, v \in V$, $v' \in V^*$, and $m, n, l \in \mathbb{Z}$, see Sections 3.2 and 3.3 in [27] for more details. By substituting (2.4.10)-(2.4.12) into (2.4.7), we have:

$$\begin{aligned}
&\sum_{i=0}^{\infty}(-1)^i\binom{l}{i}\langle v', a_{m+l-i}b_{n+i}v\rangle - \sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}\langle v', b_{n+l-i}a_{m+i}v\rangle \\
&= \mathrm{Res}_z\mathrm{Res}_w\langle v', Y(a, z)Y(b, w)v\rangle z^m w^n (z-w)^l - \mathrm{Res}_w\mathrm{Res}_z\langle v', Y(b, w)Y(a, z)v\rangle z^m w^n (z-w)^l \\
&= \mathrm{Res}_w\mathrm{Res}_{z-w}\langle v', Y(Y(a, z-w)b, w)v\rangle (w + z - w)^m w^n (z-w)^l \\
&= \sum_{i=0}^{\infty}\binom{m}{i}\langle v', (a_{l+i}b)_{m+n-i}v\rangle.
\end{aligned}$$

This shows the Jacobi identity in the component form (2.1.2) since $u \in V$ is equal to 0 if and only if $\langle v', u\rangle = 0$, for all $v' \in V^*$. □

Remark 2.4.3. We believe that Cauchy's Theorem 2.4.1 should have an analog in p-adic analysis, and the axioms (2.4.10)-(2.4.12) should lead to a proper Jacobi identity for the so-called "p-adic VOA". We will study this problem in the future.

2.4.2 Axioms of correlation functions associated with a module

Let $(V, Y, \mathbf{1}, \omega)$ be a VOA, and let V' be the graded dual space of V: $V' = \bigoplus_{n=0}^{\infty} V_n^*$. By definition, a n-points correlation function on $\mathbb{P}^1(\mathbb{C})$ associated with V is the limit of the power series $\langle v', Y(a_1, z_1)Y(a_2, z_2)\dots Y(a_n, z_n)v\rangle$ on the domain $|z_1| > |z_2| > \cdots > |z_n| > 0$, where $a_1, a_2, \dots, a_n, v \in V$, and $v \in V'$, see Proposition 3.5.1 in [27] for more details.

We consider a slightly generalized notion of correlation functions associated with a V-module. Let $M = \bigoplus_{n=0}^{\infty} M(n)$ be an ordinary V-module of conformal weight $\lambda \in \mathbb{C}$, and let

$M' = \bigoplus_{n=0}^{\infty} M(n)^*$ be its contragredient module, see Definition 2.1.5 and (2.1.13). Consider the power series:

$$\langle v', Y_M(a_1, z_1)Y_M(a_2, z_2)\dots Y_M(a_n, z_n)v\rangle \tag{2.4.13}$$

in n complex variables $z_1, \dots, z_n$ with integer powers, where $a_1, \dots, a_n \in V$, $v \in M$, and $v' \in M'$. Recall that the power series (2.4.13) converges in the domain $\mathbb{D} = \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n : |z_1| > |z_2| > \cdots > |z_n| > 0\}$ to a rational function in $z_1, z_2, \dots, z_n$, and $z_i - z_j$, where $1 \leq i \neq j \leq n$ (cf. [27, 72]). We adopt the notations in [72] and denote this rational function by:

$$(v', Y_M(a_1, z_1)Y_M(a_2, z_2)\dots Y_M(a_n, z_n)v). \tag{2.4.14}$$

The rational function (2.4.14) is called a ***n*-point correlation function (on $\mathbb{P}^1(\mathbb{C})$) associated with** M, where $a_1, a_2 \dots, a_n$ can be viewed as n-distinct points on $\mathbb{P}^1(\mathbb{C})$, and $z_1, z_2 \dots, z_n$ are local coordinates around these points. Also recall that the only possible poles of (2.4.14) are at $z_i = 0$ and $z_i = z_j$, for $1 \leq i \neq j \leq n$, see [27, 73] for more details. We use the symbol S_M(or simply S) as in [73] to denote (2.4.14):

$$S_M(v', (a_1, z_1)(a_2, z_2)\dots(a_n, z_n)v) := (v', Y_M(a_1, z_1)Y_M(a_2, z_2)\dots Y_M(a_n, z_n)v). \tag{2.4.15}$$

Then we have a system of multi-linear maps $S_M = \{S_M^n\}_{n=0}^{\infty}$:

$$\begin{aligned} S_M^n : M' \times V \times \cdots \times V \times M &\to \mathcal{F}(z_1, z_2, \dots, z_n), \\ (v', a_1, a_2, \dots, a_n, v) &\mapsto S_M(v', (a_1, z_1)(a_2, z_2)\dots(a_n, z_n)v), \end{aligned} \tag{2.4.16}$$

where $\mathcal{F}(z_1, z_2 \dots, z_n)$ is the vector space of rational functions in n variables $z_1, z_2, \dots, z_n$, with only possible poles at $z_i = 0$, and $z_i = z_j$, for some $1 \leq i \neq j \leq n$.

S_M is called **a system of correlation functions (on $\mathbb{P}^1(\mathbb{C})$) associated with** M. This system of correlation functions S_M satisfies the following properties. See Theorem 2.1 in [72] and Section 4.1 in [73] for more details. A similar result can also be found in [66].

Theorem 2.4.4. *For any $a_1, a_2, \dots, a_n \in V$, $v \in M$, and $v' \in M'$, the system of correlation functions S_M given by* (2.4.16) *satisfies the following* **genus-zero** *properties:*

(1) (Truncation property) For fixed $a \in V$ and $v \in M$, the series expansion of $S_M(v', (a, z)v)$ around $z = 0$ has a uniform lower bound for z independent of $v' \in M'$. i.e., $S(v', (a, z)v) = \sum_{n \leq N} a_n z^{-n-1}$, for all $v' \in M'$.

(2) (Vacuum property)

$$S_M(v', (\mathbf{1}, z)(a_1, z_1) \dots (a_n, z_n)v) = S_M(v', (a_1, z_1) \dots (a_n, z_n)v). \tag{2.4.17}$$

(3) ($L(-1)$-derivation property)

$$S_M(v', (L(-1)a_1, z_1)(a_2, z_2) \dots (a_n, z_n)v) = \frac{d}{dz_1} S_M(v', (a_1, z_1) \dots (a_n, z_n)v). \tag{2.4.18}$$

(4) (Locality) The terms $(a_1, z_1), (a_2, z_2), \dots, (a_n, z_n)$ can be permuted arbitrarily. i.e.,

$$S_M(v', (a_1, z_1)(a_2, z_2) \dots (a_n, z_n)v) = S_M(v', (a_{i_1}, z_{i_1})(a_{i_2}, z_{i_2}) \dots (a_{i_n}, z_{i_n})v). \tag{2.4.19}$$

(5) (Associativity) For any $k \in \mathbb{Z}$, we have:

$$\int_C S_M(v', (a_1, z_1)(a_2, z_2) \dots (a_n, z_n)v)(z_1 - z_2)^k dz_1 = S_M(v', ((a_1)_k a_2, z_2) \dots (a_n, z_n)v), \tag{2.4.20}$$

where C is a contour of z_1 surrounding z_2, with $z_3, \dots, z_n$ lying outside of C.

(6) (The Virasoro relation) Let $\omega \in V$ be the Virasoro element, and let $x, x_1, \dots, x_m$ be complex variables, denote the rational function

$$S_M(v', (\omega, x_1) \dots (\omega, x_m)(a_1, z_1) \dots (a_n, z_n)v)$$

by S for simplicity. Assume that $v', v, a_1, \dots, a_n$ are the highest-weight vectors for the Virasoro algebra, then we have:

$$\begin{aligned} &S_M(v', (\omega, x)(\omega, x_1) \dots (\omega, x_m)(a_1, z_1) \dots (a_n, z_n)v) \\ &= \sum_{k=1}^{n} \frac{x^{-1} z_k}{x - z_k} \frac{d}{dz_k} S + \sum_{k=1}^{n} \frac{\mathrm{wt} a_k}{(x - z_k)^2} S + \frac{\mathrm{wt} v}{(x - w)^2} S \\ &\quad + \frac{\mathrm{wt} v}{x^2} S + \sum_{k=1}^{m} \frac{x^{-1} w_k}{x - x_k} \frac{d}{dx_k} S + \sum_{k=1}^{m} \frac{2}{(x - x_k)^2} S \\ &\quad + \frac{c}{2} \sum_{k=1}^{m} \frac{1}{(x - x_k)^4} S_M(v', (\omega, x_1) \dots \widehat{(\omega, x_k)} \dots (\omega, x_m)(a_1, z_1) \dots (a_n, z_n)v) \end{aligned} \tag{2.4.21}$$

(7) (The generating property for M) For any $a \in V$ and $m \in \mathbb{Z}$, we have:

$$S_M(v', (a_1, z_1) \dots (a_n, z_n) a(m) v) = \int_C S_M(v', (a_1, z_1) \dots (a_n, z_n)(a, z)v) z^m dz, \tag{2.4.22}$$

where $C = C_R(0)$ is a contour of z surrounding 0, with $z_1, \dots, z_n$ lying outside.

Proof. (1), (2), and (3) are clear. For the proof of (2.4.19) and (2.4.20), see Theorem 2.1 in [72]. To prove (6), we note that the left-hand side of (2.4.21) is the limit of the power series:

$$\langle v', Y_M(\omega, x)Y_M(\omega, x_1)\dots Y_M(\omega, x_m)Y_M(a_1, z_1)\dots Y_M(a_n, z_n)v\rangle$$

on the domain $|x| > |x_1| > \cdots > |x_m| > |z_1| > \cdots > |z_n|$. Since $L(n)v = 0$ for $n > 0$, $L(0)v = \mathrm{wt}v \cdot v$, $L(-n)v' = 0$ for $n < 0$, and $\langle v', L(n)v\rangle = \langle L(-n)v', v\rangle$, we can write the power series as follows:

$$\begin{aligned}
&\langle v', Y_M(\omega, x)Y_M(\omega, x_1)\dots Y_M(\omega, x_m)Y_M(a_1, z_1)\dots Y_M(a_n, z_n)v\rangle\\
&= \langle v', \sum_{n\geq 0} L(n)x^{-n-2}Y_M(\omega_1, x_1)\dots Y_M(a_n, z_n)v\rangle + \langle v', \sum_{n<0} L(n)Y_M(\omega_1, x_1)\dots Y_M(a_n, z_n)v\rangle\\
&= \sum_{n\geq 0}\langle v', [L(n), Y_M(\omega_1, x_1)\dots Y_M(a_n, z_n)]v\rangle x^{-n-2} + \langle v', Y_M(\omega_1, x_1)\dots Y(a_n, z_n)v\rangle \frac{\mathrm{wt}v_2}{x^2}.
\end{aligned}$$

For the highest-weight vector a of the Virasoro algebra, it is easy to show that

$$\sum_{n\geq -1}[L(n), Y_M(a, z)]x^{-n-2} = \frac{x^{-1}z}{x-z}\frac{d}{dz}Y_M(a, z) + \frac{\mathrm{wt}a}{(x-z)^2}Y_M(a, z).$$

Furthermore, by the Virasoro relation, we have:

$$\begin{aligned}
&\sum_{n\geq 0}[L(n), Y_M(\omega, x_k)]x^{-n-2}\\
&= \sum_{n\geq 0}\left(Y_M(\omega_0\omega, x_k)x_k^{n+1} + (n+1)Y_M(\omega_1\omega, x_k)x_k^n + \binom{n+1}{3}Y_M(\frac{c}{2}\mathbf{1}, x_k)x_k^{n-2}\right)x^{-n-2}\\
&= \frac{x^{-1}x_k}{x-x_k}\frac{d}{dx_k}Y_M(\omega, x_k) + \frac{2}{(x-x_k)^2}Y_M(\omega, x_k) + \frac{c}{2}\frac{1}{(x-x_k)^4}.
\end{aligned}$$

This shows (2.4.21) by taking the limit of the resulting power series. In the next chapter, we give a more general version of this Theorem, wherein the correlation functions are given by intertwining operators, and we will prove a more general version of the generating property (7). So we omit the proof for them in this Theorem. □

In fact, a particular converse of this Theorem is also true, and it was claimed in [72] without proof. In other words, the genus-zero properties (1)–(7) in Theorem 2.4.4 are good enough to characterize a V-module.

Let $(V, Y, \mathbf{1}, \omega)$ be a VOA, and let $M = \bigoplus_{n=0}^{\infty} M(n)$ be a graded vector space, equipped with a linear operator $Y_M : V \to \mathrm{End}(M)[[z, z^{-1}]],\ a \mapsto Y_M(a, z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1}$ (we do not require (M, Y_M) to satisfy any axiom). Let $M' = \bigoplus_{n=0}^{\infty} M(n)^*$ be the graded dual space of M. Then we have the following:

Theorem 2.4.5. *With the settings as above, suppose there exists a system of correlation functions $S_M = \{S_M^n\}_{n=0}^{\infty}$:*

$$S_M^n : M' \times V \times \cdots \times V \times M \to \mathcal{F}(z_1, z_2, \ldots, z_n), \qquad (v', a_1, a_2, \ldots, a_n, v) \mapsto S_M(v', (a_1, z_1)(a_2, z_2) \ldots (a_n, z_n)v), \tag{2.4.23}$$

satisfying the genus-zero properties (1)-(7) in Theorem 2.4.4, where the term $a(m)v$ in (2.4.22) *is defined by* $\mathrm{Res}_z z^m Y_M(a, z)v$*, and the* 0*-point function in* (2.4.23) *is given by $S_M(v', v) = \langle v', v\rangle$, for any $v \in M$ and $v' \in M'$. Then (M, Y_M) is an admissible V-module.*

Proof. By the generating property (2.4.22) and the assumption that $S(v', v) = \langle v', v\rangle$, we have:

$$\langle v', Y_M(a, z)v\rangle = \sum_{m\in\mathbb{Z}} \langle v', a(m)v\rangle z^{-m-1} = \sum_{m\in\mathbb{Z}} \left(\int_C S_M(v', (a, z)v) z^m dz \right) z^{-m-1},$$

where the right-hand side is the Laurent series expansion of the rational function $S(v', (a, z)v)$. Then we have the following equality of rational functions:

$$S_M(v'(a, z)v) = (v', Y_M(a, z)v), \tag{2.4.24}$$

in view of (2.4.14), and by the Truncation property (1), we have $a(m)v = 0$ for $m \gg 0$. i.e., Y_M also satisfies the truncation property. By (2.4.24), (2.4.17), and (2.4.18), we have:

$$Y_M(\mathbf{1}, z)v = v, \quad \text{and} \quad Y_M(L(-1)a, z) = \frac{d}{dz} Y_M(a, z),$$

for all $a \in V$ and $v \in M$. It remains to prove the Jacobi identity (2.1.9) of Y_M. Indeed, by using the notations in Theorem 2.4.1, we have:

$$\begin{aligned}
\langle v', \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} a(m + l - i)b(n + i)v\rangle &= \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} S(v', a(m + l - i)b(n + i)v) \\
&= \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} \int_{C_1'} S(v', (a, z)b(n + i)v) z^{m+l-i} dz \\
&= \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} \int_{C_1'} \int_{C_2'} S(v', (a, z)(b, w)v) z^{m+l-i} w^{n+i} dw dz \\
&= \int_{C_1'} \int_{C_2'} S(v', (a, z)(b, w)v) z^m w^n (z - w)^l dw dz
\end{aligned} \tag{2.4.25}$$

$$= \mathrm{Res}_z \mathrm{Res}_w \left(\iota_{z,w} S(v', (a, z)(b, w)v) z^m w^n (z - w)^l \right).$$

where the contours C_1' and C_2' are contours centered at 0 of w and z in Figure 2.1, and we denote S_M by S for simplicity. Similarly, we have:

$$\begin{aligned}
&\langle v', \sum_{i=0}^{\infty} (-1)^{l+i} \binom{l}{i} b(n + l - i) a(m + i) v \rangle = \sum_{i=0}^{\infty} (-1)^{l+i} \binom{l}{i} S(v', b(n + l - i) a(m + i) v) \\
&= \sum_{i=0}^{\infty} (-1)^{l+i} \binom{l}{i} \int_{C_1} S(v', (b, w) a(m + i) v) w^{n+l-i} dw \\
&= \sum_{i=0}^{\infty} (-1)^{l+i} \binom{l}{i} \int_{C_1} \int_{C_2} S(v', (b, w)(a, z) v) z^{m+i} w^{n+l-i} dz dw \\
&= \int_{C_1} \int_{C_2} S(v', (b, w)(a, z) v) z^m w^n (z - w)^l dz dw \\
&= \mathrm{Res}_w \mathrm{Res}_z \left(\iota_{w,z} S(v', (b, w)(a, z) v) z^m w^n (z - w)^l \right),
\end{aligned} \tag{2.4.26}$$

where C_1, C_2 are contours of w, z centered at 0 in Figure 2.1. Finally, by (2.4.24) and the associativity (2.4.20), we have:

$$\begin{aligned}
&\sum_{i \geq 0} \binom{m}{i} \langle v', (a_{l+i} b)(m + n - i) v \rangle = \sum_{i \geq 0} \binom{m}{i} \int_{C_1} S(v', (a_{l+i} b, w) v) w^{m+n-i} \\
&= \sum_{i \geq 0} \binom{m}{i} \int_{C_1} \int_{C_\epsilon^z(w)} S(v', (a, z)(b, w) v)(z - w)^{l+i} w^{n+m-i} dz dw \\
&= \int_{C_1} \int_{C_\epsilon^z(w)} S(v', (a, z)(b, w) v)(z - w)^l \iota_{w,z-w} (w + (z - w))^m w^n dz dw \\
&= \mathrm{Res}_w \mathrm{Res}_{z-w} \left(\iota_{w,z-w} S(v', (a, z)(b, w) v) z^m w^n (z - w)^l \right),
\end{aligned} \tag{2.4.27}$$

where C_2 and $C^z(w)$ are the contours in Figure 2.2. By the locality (2.4.19), we have an equality $S(v', (a, z)(b, w)v) = S(v', (b, w)(a, z)v)$ in the Laurent polynomial ring $\mathbb{C}[z^{\pm 1}, w^{\pm 1}, (z - w)^{\pm 1}]$. Then it follows from Theorem 2.4.1 that

$$\begin{aligned}
&\mathrm{Res}_z \mathrm{Res}_w \left(\iota_{z,w} S(v', (a, z)(b, w) v) z^m w^n (z - w)^l \right) - \mathrm{Res}_w \mathrm{Res}_z \left(\iota_{w,z} S(v', (b, w)(a, z) v) z^m w^n (z - w)^l \right) \\
&= \mathrm{Res}_w \mathrm{Res}_{z-w} \left(\iota_{w,z-w} S(v', (a, z)(b, w) v) z^m w^n (z - w)^l \right).
\end{aligned}$$

Since $v' \in M'$ is chosen arbitrarily, the component form Jacobi identity (2.1.9) holds, in view of (2.4.25), (2.4.26), and (2.4.27). Thus, (M, Y_M) is a V-module. □

Remark 2.4.6. In the proof of the Jacobi identity for Y_M in Theorem 2.4.5, we have only used (2.4.24), which follows from the truncation property (1), the locality (2.4.19), associativity

(2.4.20), and the generating property for M (2.4.22). Therefore, we only need (1), (2), (4), (5), and (7) in Theorem 2.4.5 hold to obtain a V-module structure on (M, Y_M). In the next chapter, we will give a more explicit description of a generalization of the correspondence give in Theorem 2.4.5, wherein the correlation functions are defined by both vertex operators and intertwining operators.

Remark 2.4.7. A system of correlation functions $S_M = \{S^n_M\}_{n=0}^{\infty}$ can be built from a module U over $A(V)$, by using the recursive formula satisfied by the series (2.4.13). This is the essential idea of constructing a V-module from an $A(V)$-module. See Theorem 2.2.1 in [73].

We can also consider the case when $M = V$. Then the axioms of the VOA V itself correspond to the genus-zero axioms (1)–(7) in Theorem 2.4.4, with M replaced by V. More precisely, let $V = \bigoplus_{n=0}^{\infty} V_n$ be a graded vector space, with $\dim V_n < \infty$ for all $n \in \mathbb{N}$, and $V' = \bigoplus_{n=0}^{\infty} V_n^*$ be its graded dual space. Let $Y : V \to \mathrm{End}(V)[[z, z^{-1}]],\ a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$ be a linear map. Again, we do not require Y to satisfy any axiom at this point. By adopting a similar proof as Theorem 2.4.5, we have the following:

Corollary 2.4.8. *With the settings as above, suppose there exists a system of correlation functions $S = \{S^n\}_{n=0}^{\infty}$, where for each $n \in \mathbb{N}$,*

$$S^n : V' \times V \times \cdots \times V \times V \to \mathcal{F}(z_1, z_2, \dots, z_n), \tag{2.4.28}$$
$$(v', a_1, a_2, \dots, a_n, v) \mapsto S(v', (a_1, z_1)(a_2, z_2) \dots (a_n, z_n) v),$$

satisfying the truncation property (1), the locality (4), the associativity (5), and the generating property (7) in Theorem 2.4.4, where the term $a_1(k)a_2$ in (2.4.20) *is defined by* $\mathrm{Res}_z z^k Y(a_1, z) a_2$*, and the term $a(m)v$ in* (2.4.22) *is defined by* $\mathrm{Res}_z z^m Y(a, z) v$*, and the* 0*-point function in* (2.4.28) *is given by $S(v', v) = \langle v', v \rangle$, for any $a_1, \dots, a_n, v \in$, and $v' \in V'$. Then (V, Y) satisfies the truncation property and the Jacobi identity* (2.1.2)*.*

Corollary 2.4.8 (or a similar format) was used in the construction of the lattice VOA V_L in Section A.3 of [29], and the affine and Virasoro VOAs in [30]. We believe that Corollary 2.4.8, together with the $A(V)$-theory, should also be useful in studying the simple current extensions of VOAs.

2.4.3 Twisted correlation functions

Now we consider the correlation functions associated with a twisted module over a VOA V. Let $(V, Y, \mathbf{1}, \omega)$ be a VOA, and let $g \in \mathrm{Aut}(V)$ be an order $T \in \mathbb{N}$ automorphism of V. Then $V = \bigoplus_{r=0}^{T-1} V^r$, where $V^r = \{a \in V : T(a) = e^{2\pi ir/T} a\}$ is the eigenspace of T of eigenvalue $e^{2\pi ir/T}$. Then it is easy to see that V^0 is a sub-VOA of V, and each V^r is a module over V^0, with respect to the same vertex operator Y (cf. [16, 18, 29]). Recall the following definition in [18, 70]:

Definition 2.4.9. With the settings as above, a **weak g-twisted V-module** is a vector space M, equipped with a linear map

$$\begin{aligned} Y_M : V &\to \mathrm{End}(M)\{z\}, \\ b &\mapsto Y_M(b, z) = \sum_{n\in\mathbb{Q}} b(n)z^{-n-1}, \end{aligned}$$

satisfying the following axioms for all $0 \leq r \leq T-1$, $a \in V^r$, $b \in V$, and $u \in M$:

(1) (Index property) $Y_M(a, z) = \sum_{n\in \frac{r}{T}+\mathbb{Z}} a(n)z^{-n-1}$.

(2) (Truncation property) $a(n)u = 0$ for $n \gg 0$.

(3) (Vacuum property) $Y_M(\mathbf{1}, z) = \mathrm{Id}_M$.

(4) (Twisted Jacobi identity)

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(a, z_1)Y_M(b, z_2)u - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) Y_M(b, z_2)Y_M(a, z_1)u \\ &= z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)^{-r/T} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(a, z_0)b, z_2)u. \end{aligned} \tag{2.4.29}$$

A weak g-twisted V-module is called **admissible** if $M = \bigoplus_{n\in\frac{1}{T}\mathbb{Z}_+} M(n)$, such that $a(m)M(n) \subset M(\mathrm{wt} a - m - 1 + n)$, for all $m \in \mathbb{Z}$, $n \in \mathbb{Z}_+/T$, and $a \in V$ homogeneous.

A weak g-twisted V-module is called a **g-twisted V-module** if $M = \coprod_{\lambda\in\mathbb{C}} M_\lambda$, with $\dim M_\lambda < \infty$, and $M_\lambda = \{u \in M : L(0)u = \lambda u\}$, for each $\lambda \in \mathbb{C}$. Moreover, for fixed $\lambda \in \mathbb{C}$, we have $M_{\lambda+\frac{n}{T}} = 0$ for $n \in \mathbb{Z}$ small enough.

We want to derive a component form of the twisted Jacobi identity (2.4.29). Since an extra term appears on the right-hand side of (2.4.29), we need more subtle discussions for the formal delta functions. Recall that by convention, $(z_1+z_2)^\alpha = \sum_{j\geq 0}\binom{\alpha}{j} z_1^{\alpha-j} z_2^j$, for any $\alpha\in\mathbb{C}$. We observe the following fact for this expansion:

Lemma 2.4.10. *Let z_0, z_1, z_2 be formal variables, and $\alpha\in\mathbb{C}$, then we have:*

$$(z_0+(z_2+z_1))^\alpha = (z_0+(z_1+z_2))^\alpha = ((z_0+z_1)+z_2)^\alpha \tag{2.4.30}$$

Proof. The first equality is clear, and we only prove the second equality:

$$\begin{aligned}(z_0+(z_1+z_2))^\alpha &= \sum_{m\geq 0}\sum_{j\geq 0}\binom{\alpha}{m}\binom{m}{j} z_0^{\alpha-m} z_1^{m-j} z_2^j = \sum_{m,j\geq 0, m-j\geq 0}\frac{\alpha(\alpha-1)\ldots(\alpha-m+1)}{(m-j)!j!} z_0^{\alpha-m} z_1^{m-j} z_2^j\\ &= \sum_{i\geq 0, j\geq 0}\frac{\alpha(\alpha-1)\ldots(\alpha-i-j+1)}{i!j!} z_0^{\alpha-i-j} z_1^i z_2^j = \sum_{j\geq 0}\sum_{i\geq 0}\binom{\alpha}{j}\binom{\alpha-j}{i} z_0^{\alpha-i-j} z_1^i z_2^j\\ &= ((z_0+z_1)+z_2)^\alpha,\end{aligned}$$

where we changed the variable $m\mapsto i+j$ in the third equality. □

The following formula was used in Section 5 of [18] without proof. We will write out the proof for it for completeness.

Lemma 2.4.11.

$$z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-r/T}\delta\left(\frac{z_1-z_0}{z_2}\right) = z_1^{-1}\left(\frac{z_2+z_0}{z_1}\right)^{r/T}\delta\left(\frac{z_2+z_0}{z_1}\right) \tag{2.4.31}$$

Proof. Recall that the binomial coefficients satisfy $\binom{\alpha}{j} = \binom{-\alpha-1+j}{j}(-1)^j$, for all $\lambda\in\mathbb{C}$. Then

$$\begin{aligned} z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-r/T}\delta\left(\frac{z_1-z_0}{z_2}\right) &= \sum_{n\in\mathbb{Z}}\sum_{j\geq 0}\binom{n-\frac{r}{T}}{j}(-1)^j z_1^{n-\frac{r}{T}-j} z_2^{-n-1+\frac{r}{T}} z_0^j\\ &= \sum_{n\in\mathbb{Z}}\sum_{j\geq 0}\binom{-n+\frac{r}{T}-1+j}{j} z_1^{n-\frac{r}{T}-j} z_2^{-n-1+\frac{r}{T}} z_0^j = \sum_{m\in\mathbb{Z}}\sum_{j\geq 0}\binom{m+\frac{r}{T}}{j} z_1^{-m-1-\frac{r}{T}} z_2^{m-j+\frac{r}{T}} z_0^j\\ &= \sum_{m\in\mathbb{Z}} z_1^{-m-1-\frac{r}{T}}(z_2+z_0)^{m+\frac{r}{T}} = z_1^{-1}\left(\frac{z_2+z_2}{z_1}\right)^{r/T}\delta\left(\frac{z_2+z_2}{z_1}\right).\end{aligned}$$

This shows (2.4.31). □

By the proof of (2.4.31), together with (2.4.30), it is easy to derive the following equality, and the proof is also similar to the usual case:

$$z_1^{-1}\left(\frac{z_2+z_0}{z_1}\right)^{r/T}\delta\left(\frac{z_2+z_0}{z_1}\right)(z_1-z_2)^l = z_1^{-1}\left(\frac{z_2+z_0}{z_1}\right)^{r/T}\delta\left(\frac{z_2+z_0}{z_1}\right) z_0^l. \tag{2.4.32}$$

Proposition 2.4.12. *Let M be a g-twisted V-module. The twisted Jacobi identity* (2.4.29) *has the following component form: Let $a \in V^r$, $b \in V^s$, for some $0 \le r, s \le T-1$, $u \in M$, and $m, n, l \in \mathbb{Z}$. We have:*

$$\sum_{i=0}^{\infty}\binom{l}{i}(-1)^i a(\frac{r}{T}+m+l-i)b(\frac{s}{T}+n+i)-\sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}b(\frac{s}{T}+n+l-i)a(m+\frac{r}{T}+i)$$
$$=\sum_{j=0}^{\infty}\binom{m+\frac{r}{T}}{j}(a_{j+l}b)(m+n+\frac{r+s}{T}-j). \tag{2.4.33}$$

Proof. First, we observe that $a_p b \in V^{r+s}$, for all $p \in \mathbb{Z}$. Multiply (2.4.29) with $z_1^{m+\frac{r}{T}} z_2^{n+\frac{s}{T}}(z_1 - z_2)^l$, then apply $\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}$, we have:

$$\text{L.H.S. of } \mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_1^{m+\frac{r}{T}} z_2^{n+\frac{s}{T}}(z_1-z_2)^l \cdot (2.4.29)$$
$$=\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\sum_{p,q\in\mathbb{Z}} a(p+\frac{r}{T})b(q+\frac{s}{T})z_1^{-p-1+m}z_2^{-q-1+n}(z_1-z_2)^l$$
$$-\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\sum_{p,q\in\mathbb{Z}} b(q+\frac{s}{T})a(p+\frac{r}{T})z_1^{-p-1+m}z_2^{-q-1+n}(-z_2+z_1)^l$$
$$=\sum_{i=0}^{\infty}\binom{l}{i}(-1)^i a(\frac{r}{T}+m+l-i)b(\frac{s}{T}+n+i)-\sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}b(\frac{s}{T}+n+l-i)a(m+\frac{r}{T}+i).$$

On the other hand, by the proof of (2.4.30), together with (2.4.32), we have:

$$\text{R.H.S. of } \mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_1^{m+\frac{r}{T}} z_2^{n+\frac{s}{T}}(z_1-z_2)^l \cdot (2.4.29)$$
$$=\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2} z_1^{-1}\left(\frac{z_2+z_0}{z_1}\right)^{r/T}\delta\left(\frac{z_2+z_0}{z_1}\right)Y_M(Y(a,z_0)b,z_2)z_1^{m+\frac{r}{T}}z_2^{n+\frac{s}{T}}z_0^l$$
$$=\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}\sum_{t\in\mathbb{Z}}\sum_{j\ge 0}\binom{t+\frac{r}{T}}{j}z_1^{-t-1-\frac{r}{T}}z_2^{t-j+\frac{r}{T}}z_0^j\sum_{p,q\in\mathbb{Z}}(a_pb)(q+\frac{r+s}{T})$$
$$\cdot z_0^{-p-1}z_2^{-q-1-\frac{r}{T}-\frac{s}{T}}z_1^{m+\frac{r}{T}}z_2^{n+\frac{s}{T}}z_0^l$$
$$=\mathrm{Res}_{z_0}\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}\sum_{t\in\mathbb{Z}}\sum_{j\ge 0}\binom{t+\frac{r}{T}}{j}\sum_{p,q\in\mathbb{Z}}z_1^{-t-1+m}z_2^{t-j-q-1+n}z_0^{j-p+l-1}(a_pb)(q+\frac{r+s}{T})$$
$$=\sum_{j\ge 0}\binom{m+\frac{r}{T}}{j}(a_{j+l}b)(m-j+n+\frac{r+s}{T})\quad(\text{with } p=j+l,\ t=m,\ q=t-j+n).$$

This proves (2.4.33) □

By the proof of Proposition 2.4.12, together with (3.5) in [18], we can easily derive a twisted version of the rationality of product, locality, and weak associativity:

Proposition 2.4.13. *Let (M, Y_M) be a g-twisted V-module, and let $a \in V^r$, $b \in V^s$, for some $0 \leq r, s \leq T-1$, $u \in M$, $u' \in M'$, and $m, n, l \in \mathbb{Z}$. Then there exists $f(z_1, z_2) \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{\pm 1}]$, such that*

$$\langle u', Y_M(a, z_1)Y_M(b, z_2)u\rangle z_1^{m+\frac{r}{T}} z_2^{n+\frac{s}{T}} (z_1 - z_2)^l = \iota_{z_1,z_2} f(z_1, z_2), \tag{2.4.34}$$

$$\langle u', Y_M(b, z_2)Y_M(a, z_1)u\rangle z_1^{m+\frac{r}{T}} z_2^{n+\frac{s}{T}} (z_1 - z_2)^l = \iota_{z_2,z_1} f(z_1, z_2), \tag{2.4.35}$$

$$\langle u', Y_M(Y(a, z_1 - z_2)b, z_2)u\rangle z_1^{m+\frac{r}{T}} z_2^{n+\frac{s}{T}} (z_1 - z_2)^l = \iota_{z_2,z_1-z_2} f(z_1, z_2). \tag{2.4.36}$$

Furthermore, the component form of the twisted Jacobi identity (2.4.33) *follows from* (2.4.34)-(2.4.36)*, together with Theorem 2.4.1.*

Now we consider the correlation functions associated with a g-twisted V-module (M, Y_M). For simplicity, we let $M = \bigoplus_{n\in\frac{1}{T}\mathbb{Z}_+} M_{\lambda+n} = \bigoplus_{n\in\frac{1}{T}\mathbb{Z}_+} M(n)$. Recall that M has the contragredient module $(M', Y_{M'})$(cf. [18, 70]), where $M' = \bigoplus_{n\in\frac{1}{T}\mathbb{Z}_+} M(n)^*$, and

$$\langle Y_{M'}(a, z)f, u\rangle = \langle f, Y_M(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})u\rangle, \tag{2.4.37}$$

for all $a \in V$, $f \in M'$, and $u \in M$. It was proved in [70] that $(M', Y_{M'})$ is a g^{-1}-twisted V-module.

Let $a_1, a_2, \ldots, a_n \in V$ be homogeneous with respect to the T-grading, where $a_1 \in V^{r_1}, a_2 \in V^{r_2}, \ldots, a_n \in V^{r_n}$, for some $0 \leq r_1, r_2, \ldots, r_n \leq T-1$. Consider the series

$$\langle u', Y_M(a_1, z_1)Y_M(a_2, z_2)\ldots Y_M(a_n, z_n)u\rangle z_1^{r_1/T} z_2^{r_2/T} \ldots z_n^{r_n/T}, \tag{2.4.38}$$

where $u \in M$ and $u' \in M'$. Similar to the beginning of subsection 2.4.3, by (2.4.34)-(2.4.36), together with the Definition of twisted modules, the series (2.4.38) only has integral powers of $z_1, z_2, \ldots, z_n$, and it is convergent in the domain $\mathbb{D} = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_1| > |z_2| > \cdots > |z_n| > 0\}$ to a rational function in $z_1, z_2, \ldots, z_n$, and $z_i - z_j$, where $1 \leq i \neq j \leq n$. We denote the limit rational function of (2.4.38) by:

$$S_M(u', (a_1, z_1)(a_2, z_2)\ldots(a_n, z_n)u). \tag{2.4.39}$$

The rational function (2.4.39) is called a **n-point twisted correlation function associated with M**. S_M in (2.4.39) also give rise to a sequence of functions

$$\begin{aligned} S_M^n : M' \times V \times \cdots \times V \times M &\to \mathcal{F}(z_1, z_2, \ldots, z_n), \\ (v', a_1, a_2, \ldots, a_n, v) &\mapsto S_M(v', (a_1, z_1)(a_2, z_2)\ldots(a_n, z_n)v). \end{aligned} \tag{2.4.40}$$

We call $S_M = \{S_M^n\}_{n=0}^{\infty}$ a **system of twisted correlation functions associated with** M. Moreover, we have the following properties of the system of twisted rational functions (2.4.40) that are similar to the genus-zero properties in Theorem 2.4.4:

Theorem 2.4.14. *For any $a_1 \in V^{r_1}, \dots, a_n \in V^{r_n}$, where $0 \le r_1, \dots r_n \le T-1$, $u \in M$, and $u' \in M'$, the system of correlation functions $S_M = \{S_M^n\}_{n=0}^{\infty}$ defined by* (2.4.36) *satisfies the following* **twisted genus-zero** *properties:*

(1) (Truncation property) For fixed $a \in V^r$ and $u \in M$, the series expansion of $S_M(u', (a, z)u)$ around $z = 0$ has a uniform lower bound for z independent of $u' \in M'$. i.e., there exists $N \in \mathbb{N}$ such that $S_M(u', (a, z)u) = \sum_{n \le \frac{r}{T}+N} a_n z^{-n+\frac{r}{T}-1}$, for all $u' \in M'$.

(2) (Vacuum property)

$$S_M(u', (\mathbf{1}, z)(a_1, z_1) \dots (a_n, z_n)u) = S_M(u', (a_1, z_1) \dots (a_n, z_n)u). \tag{2.4.41}$$

(3) ($L(-1)$-derivation property)

$$S_M(u', (L(-1)a_1, z_1) \dots (a_n, z_n)v) = \frac{d}{dz_1}\left(S_M(u', (a_1, z_1) \dots (a_n, z_n)u) z_1^{-r/T}\right) z_1^{r/T}. \tag{2.4.42}$$

(4) (Locality) The terms $(a_1, z_1), (a_2, z_2), \dots, (a_n, z_n)$ can be permuted arbitrarily. i.e.,

$$S_M(u', (a_1, z_1)(a_2, z_2) \dots (a_n, z_n)u) = S_M(u', (a_{i_1}, z_{i_1})(a_{i_2}, z_{i_2}) \dots (a_{i_n}, z_{i_n})u). \tag{2.4.43}$$

(5) (Associativity) For any $k \in \mathbb{Z}$, we have:

$$\int_C S_M(u', (a_1, z_1)(a_2, z_2) \dots (a_n, z_n)u)(z_1 - z_2)^k dz_1 = S_M(u', ((a_1)_k a_2, z_2) \dots u) z_1^{r_1/T} z_2^{-r_1/T}, \tag{2.4.44}$$

where C is a contour of z_1 surrounding z_2, with $z_3, \dots, z_n$ lying outside of C.

(6) (The Virasoro relation) Let $\omega \in V$ be the Virasoro element, and let $x, x_1, \dots, x_m$ be complex variables, denote the rational function

$$S_M(u', (\omega, x_1) \dots (\omega, x_m)(a_1, z_1) \dots (a_n, z_n)u)$$

by S for simplicity. Assume that $u', u, a_1, \dots, a_n$ are the highest-weight vectors for the Virasoro algebra, then we have:

$$S_M(u', (\omega, x)(\omega, x_1) \dots (\omega, x_m)(a_1, z_1) \dots (a_n, z_n)u)$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \frac{x^{-1}z_k}{x-z_k} z_k^{r_k/T} \frac{d}{dz_k}(S \cdot z_k^{-r_k/T}) + \sum_{k=1}^{n} \frac{\mathrm{wt} a_k}{(x-z_k)^2} S + \frac{\mathrm{wt} v}{(x-w)^2} S \\
&+ \frac{\mathrm{wt} v}{x^2} S + \sum_{k=1}^{m} \frac{x^{-1} w_k}{x-x_k} \frac{d}{dx_k} S + \sum_{k=1}^{m} \frac{2}{(x-x_k)^2} S \\
&+ \frac{c}{2} \sum_{k=1}^{m} \frac{1}{(x-x_k)^4} S_M(v', (\omega, x_1) \dots \widehat{(\omega, x_k)} \dots (\omega, x_m)(a_1, z_1) \dots (a_n, z_n) u)
\end{aligned}
\tag{2.4.45}
$$

(7) (The generating property for M) For any $a \in V^r$ and $m \in \mathbb{Z}$, we have:

$$
S_M(u', (a_1, z_1) \dots (a_n, z_n) a(m + \frac{r}{T}) u) = \int_C S_M(u', (a_1, z_1) \dots (a_n, z_n)(a, z) u) z^m dz,
\tag{2.4.46}
$$

where $C = C_R(0)$ is a contour of z surrounding 0, with $z_1, \dots, z_n$ lying outside.

Proof. (1) and (2) follow from the Definition and (2.4.38), (3) follows from $Y_M(L(-1)a, z) = \frac{d}{dz} Y_M(a, z)$, see (3.9) in [18]. (4) and (5) follows from Proposition 2.4.13. We write out some details of the associativity (5). By definition and Proposition 2.4.13, let $|z_1|, |z_2| < |z_3|, \dots, |z_n|$,

$$
\begin{aligned}
&\text{L.H.S. of (2.4.44)} = \int_C \iota_{z_1, z_2} \langle u', Y_M(a_1, z_1) Y_M(a_2, z_2) \dots u \rangle z_1^{r_1/T} z_2^{r_2/T} \dots z_n^{r_n/T} (z_1 - z_2)^k dz_1 \\
&= \int_C \iota_{z_2, z_1 - z_2} \langle u', Y_M(Y(a_1, z_1 - z_2) a_2, z_2) \dots Y(a_n, z_n) u \rangle z_1^{r_1/T} z_2^{r_2/T} \dots z_n^{r_n/T} (z_1 - z_2)^k dz_1 \\
&= \left(\langle u', Y_M((a_1)_k a_2, z_2) \dots Y(a_n, z_n) u \rangle z_2^{(r_1 + r_2)/T} \dots z_n^{r_n/T} \right) z_1^{r_1/T} z_2^{-r_1/T} \\
&= \text{R.H.S. of (2.4.44)},
\end{aligned}
$$

where we used the fact that $(a_1)_k a_2 \in V^{r+s}$. The proof of (6) is similar to the corresponding one in Theorem 2.4.4. We just need to observe that $\omega \in V^0$, and so $Y_M(\omega, x) = \sum_{n \in \frac{0}{T} + \mathbb{Z}} L(n) z^{-n-2}$. (7) follows from (2.4.38) and $Y_M(a, z) = \sum_{m \in \frac{r}{T} + \mathbb{Z}} a(m) z^{-m-1}$. □

Finally, we also have a twisted version of Theorem 2.4.5. Let $(V, Y, \mathbf{1}, \omega)$ be a VOA, and $g \in \mathrm{Aut}(V)$ be of order T. Let $M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)$ be a graded vector space, equipped with a linear operator $Y_M : V \to \mathrm{End}(M)[[z^{\frac{1}{T}}, z^{-\frac{1}{T}}]]$, $a \in V^r \mapsto Y_M(a, z) = \sum_{n \in \frac{r}{T} + \mathbb{Z}} a(n) z^{-n-1}$. Again, we do not require (M, Y_M) to satisfy any axiom. Let $M' = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+}^{\infty} M(n)^*$ be the graded dual space of M. Then we have the following:

Theorem 2.4.15. *With the settings as above, suppose there exists a system of correlation functions $S_M = \{S_M^n\}_{n=0}^{\infty}$:*

$$
S_M^n : M' \times V \times \dots \times V \times M \to \mathcal{F}(z_1, z_2, \dots, z_n),
\tag{2.4.47}
$$

$$(u', a_1, a_2, \dots, a_n, u) \mapsto S_M(u', (a_1, z_1)(a_2, z_2)\dots(a_n, z_n)u),$$

satisfying the twisted genus-zero properties (1)-(7) in Theorem 2.4.14, where the term $a(m+\frac{r}{T})u$ *in* (2.4.46) *is defined by* $\mathrm{Res}_z z^{\frac{r}{T}+m} Y_M(a, z)u$*, and the* 0*-point function in* (2.4.47) *is given by* $S_M(u', u) = \langle u', u\rangle$*, for any* $u \in M$ *and* $u' \in M'$*. Then* (M, Y_M) *is a g-twisted admissible* V*-module.*

Proof. The proof is also similar to Theorem 2.4.5, with some index adjustments. We write out the details for completeness. Let $a \in V^r$, then by (2.4.46) and the assumption, we have:

$$\langle u', Y_M(a, z)u\rangle z^{\frac{r}{T}} = \sum_{m\in\mathbb{Z}} \langle u', a(m + \frac{r}{T})u\rangle z^{-m-1} = \sum_{m\in\mathbb{Z}} \left(\int_C S_M(u', (a, z)u) z^m dz \right) z^{-m-1},$$

which is the power series expansion of $S_M(u', (a, z)u)$. Thus, we have:

$$S_M(u', (a, z)u) = \lim \langle u', Y_M(a, z)u\rangle z^{\frac{r}{T}}. \tag{2.4.48}$$

By the truncation property (1) of S_M, we have $a(m)u = 0$, for $m \gg 0$. By (2.4.41) and (2.4.48), we have $Y_M(\mathbf{1}, z) = \mathrm{Id}_M$. By (2.4.42), (2.4.48), and the uniform convergence of the power series, we have:

$$\begin{aligned}
\lim\langle u', Y_M(L(-1)a, z)u\rangle z^{\frac{r}{T}} &= S_M(u', (L(-1)a, z)u) = \frac{d}{dz}\left(S_M(u', (a, z)u)z^{-r/T}\right) z^{r/T} \\
&= \frac{d}{dz}\left(\lim Y_M(L(-1)a, z)u\rangle z^{\frac{r}{T}} z^{-r/T}\right) z^{r/T} = \lim \frac{d}{dz}\langle u', Y_M(L(-1)a, z)u\rangle z^{r/T}.
\end{aligned}$$

Thus, $Y_M(L(-1)a, z) = \frac{d}{dz} Y_M(a, z)$. Finally, we prove the component form twisted Jacobi identity (2.4.33). Let $a \in V^r$ and $b \in V^s$, by (2.4.46) and (2.4.48), we have:

$$\begin{aligned}
&\langle u', \sum_{i=0}^{\infty} \binom{l}{i} (-1)^i a(\frac{r}{T} + m + l - i) b(\frac{s}{T} + n + i)u\rangle \\
&= \sum_{i=0}^{\infty} \binom{l}{i} (-1)^i S_M(u', a(\frac{r}{T} + m + l - i) b(\frac{s}{T} + n + i)u) \\
&= \sum_{i=0}^{\infty} \binom{l}{i} (-1)^i \int_{C_1'} S_M(u', (a, z_1) b(\frac{s}{T} + n + i)u) z_1^{m+l-i} dz_1 \\
&= \sum_{i=0}^{\infty} \binom{l}{i} (-1)^i \int_{C_1'} \int_{C_2'} S_M(u', (a, z_1)(b, z_2)u) z_1^{m+l-i} z_2^{n+i} dz_1 dz_2 \\
&= \int_{C_1'} \int_{C_2'} S_M(u', (a, z_1)(b, z_2)u) z_1^m z_2^n (z_1 - z_2)^l dz_1 dz_2
\end{aligned}$$

$$= \mathrm{Res}_{z_1}\mathrm{Res}_{z_2}\left(\iota_{z_1,z_2}S_M(u',(a,z_1)(b,z_2)u)z_1^m z_2^n(z_1-z_2)^l\right),$$

where C_1' and C_2' are given by Figure 2.1, with $z = z_1$ and $w = z_2$. Similarly, we have:

$$\langle u', -\sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}b(\frac{s}{T}+n+l-i)a(m+\frac{r}{T}+i)u\rangle$$
$$= -\mathrm{Res}_{z_2}\mathrm{Res}_{z_1}\left(\iota_{z_2,z_1}S_M(u',(b,z_2)(a,z_1)u)z_1^m z_2^n(z_1-z_2)^l\right),$$

Moreover, since $a_{j+l}b \in V^{r+s}$ for all $j \geq 0$, by the associativity (2.4.44) and (2.4.48), we have

$$\langle u', \sum_{j=0}^{\infty}\binom{m+\frac{r}{T}}{j}(a_{j+l}b)(m+n+\frac{r+s}{T}-j)u\rangle = \sum_{j=0}^{\infty}\binom{m+\frac{r}{T}}{j}\int_{C_1}S_M(u',(a_{j+l}b,z_2)u)z_2^{m+n-j}dz_2$$
$$= \sum_{j=0}^{\infty}\binom{m+\frac{r}{T}}{j}\int_{C_1}\int_{C_\epsilon^{z_1}(z_2)}S_M(u',(a,z_1)(b,z_2)u)(z_1-z_2)^l z_2^{m+n-j}z_1^{-r/T}z_2^{r/T}dz_2dz_1$$
$$= \int_{C_1}\int_{C_\epsilon^{z_1}(z_2)}S_M(u',(a,z_1)(b,z_2)u)(z_1-z_2)^l(z_2+z_1-z_2)^{m+\frac{r}{T}}z_2^n z_1^{-r/T}dz_2dz_1$$
$$= \int_{C_1}\int_{C_\epsilon^{z_1}(z_2)}S_M(u',(a,z_1)(b,z_2)u)(z_1-z_2)^l z_1^m z_2^n$$
$$= \mathrm{Res}_{z_2}\mathrm{Res}_{z_1-z_2}\left(\iota_{z_2,z_1-z_2}S_M(u',(a,z_1)(b,z_2)u)(z_1-z_2)^l z_1^m z_2^n\right).$$

where C_1 and $C_\epsilon^{z_1}(z_2)$ are given by Figure 2.2. Now the component form Jacobi identity follows from the locality (2.4.43), together with Theorem 2.4.1. □

Remark 2.4.16. Since the twisted correlation functions satisfy the similar properties as the usual correlation functions, we believe that Zhu's method of constructing an irreducible V-module from an irreducible $A(V)$-module (Theorem 2.1.2 in [73]) by using the recursive formula of correlation functions can be easily generalized to the twisted case. This would result in a one-to-one correspondence between irreducible twisted V-modules and irreducible $A_g(V)$-modules, where $A_g(V)$ is the twisted Zhu's algebra defined by Dong, Li, and Mason in [18]. Such a correspondence for twisted modules was first obtained by a purely algebraic approach in [18].

Remark 2.4.17. Another possible way to generalize the system of correlation functions on $\mathbb{P}^1(\mathbb{C})$ is to define it over a higher dimensional complex manifold, for example, $\mathbb{P}^r(\mathbb{C})$, where $r \geq 2$. Then instead of one coordinate z_i, we need several coordinates $[z_1, \ldots, z_r]$. In other words, we need to define the generalized vertex operator "$Y(\vec{a}, (z_1, \ldots, z_r))$", and find the axioms of such vertex operators that generalize the usual rationality of products, locality, and associativity. A natural candidate for "$Y(\vec{a}, (z_1, \ldots, z_r))$" is obviously the tensor product of vertex operators:

$$Y(a_1, z_1) \otimes Y(a_2, z_2) \otimes \cdots \otimes Y(a_r, z_r),$$

but there are still a lot of details to work out. We will study this problem in the future.

Chapter 3

Space of correlation functions

In Section 2.4 of Chapter 2, we introduced the notion of a system of correlation functions S_M associated with a module M over VOA V. Such a function was built from the module vertex operator: $\langle v', Y_M(a_1, z_1) Y_M(a_2, z_2) \dots Y(a_n, z_n) v \rangle$.

In this Chapter, we will consider a general case of the system of correlation functions that are defined by both the module vertex operators and an intertwining operator $I \in I\binom{M^3}{M^1\ M^2}$, where M^1, M^2, and M^3 are V-modules of conformal weights h_1, h_2, and h_3, respectively. Namely, we will study the rational function defined by the limit of the following series:

$$\langle v_3', Y_{M^3}(a_1, z_1) \dots Y_{M^3}(a_{k-1}, z_{k-1}) I(v, w) Y_{M^2}(a_k, z_k) \dots Y_{M^2}(a_n, z_n) v_2 \rangle w^{h_1+h_2-h_3}, \tag{3.0.1}$$

where $v_3' \in (M^3)'$, $v \in M^1$, $v_2 \in M^2$, and $a_1, \dots, a_n \in V$. Such correlation functions satisfy some similar axioms as the ones in Theorem 2.4.4. Furthermore, instead of one system of correlation functions S_M, we will study the vector space spanned by all systems of correlation functions defined by (3.0.1) in this Chapter, where we let the intertwining operator I vary in $I\binom{M^3}{M^1\ M^2}$. We denote this vector space by $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$, and we will show that $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$ can be naturally identified with the vector space $I\binom{M^3}{M^1\ M^2}$. Hence the fusion rule $N_{M^1 M^2}^{M^3}$ can also be computed by $N_{M^1 M^2}^{M^3} = \dim \mathrm{Cor}\binom{M^3}{M^1\ M^2}$.

The domain $(M^3)' \times V \times \dots \times M^1 \times \dots \times M^2$ of the system of correlation functions defined by (3.0.1) can also be restricted onto the bottom levels: $M^3(0)^* \times V \times \dots \times M^1 \times \dots \times M^2(0)$, and the restricted functions have intimate relations with the Zhu's algebra $A(V)$ and some of its bimodules. We denote the vector space of the restricted correlation functions by $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. Then we use certain generating formulas satisfied by the correlation functions and prove that

$\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ is isomorphic to both $\mathrm{Cor}\binom{\bar{M}^3}{M^1\ \bar{M}^2}$ and $\mathrm{Cor}\binom{\bar{M}(M^3(0)^*)'}{M^1\ \bar{M}(M^2(0))}$ when $M^2(0)$ and $M^3(0)$ are irreducible modules over $A(V)$, where $\bar{M}(M^2(0))$ and $\bar{M}(M^3(0)^*)$ are the generalized Verma modules (see [18]) associated with irreducible $A(V)$-modules $M^2(0)$ and $M^3(0)^*$, respectively. However, unlike building V-modules from $A(V)$-modules (see Theorem 2.2.1 in [73]) based on the ordinary correlation functions $(v', Y_M(a_1, z_1) \dots Y_M(a_n, z_n)v)$, in our case, due to the appearance of the intertwining operator $I(v, w)$ in (3.0.1), the modules $\bar{M}^2$ and $\bar{M}^3$ constructed by (3.0.1) are not necessarily irreducible. This issue was first observed by Li in [49]. The V-modules $\bar{M}^2$ and $\bar{M}^3$ are quotient modules of certain generalized Verma modules. They can be proved to be irreducible if a technical condition depends only on the (bi)modules over $A(V)$ is satisfied. In particular, if the VOA V is rational, then the generalized Verma modules $\bar{M}(M^2(0))$ and $\bar{M}(M^3(0)^*)$ are both irreducible, see Theorem 6.3 in [18], and so the fusion rule of irreducible V-modules M^1, M^2, and M^3 can be computed by $N_{M^1M^2}^{M^3} = \dim \mathrm{Cor}\binom{M^3}{M^1\ M^2} = \dim \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$.

We fix some notations for this Chapter. Let $V = (V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra (VOA) which is of the CFT-type: $V = \bigoplus_{n=0}^{\infty} V_n$, with $V_0 = \mathbb{C}\mathbf{1}$. A module M over V is an ordinary V-module: $M = \bigoplus_{n=0}^{\infty} M_{\lambda+n}$, where each $M_{\lambda+n}$ is an eigenspace of $L(0)$ with eigenvalue $\lambda + n$. Any V-module M is $\mathbb{N}$-gradable (or admissible): $M = \bigoplus_{n=0}^{\infty} M(n)$, with $M(n) = M_{\lambda+n}$ for each n. We write $Y_M(a, z) = \sum_{n\in\mathbb{Z}} a(n) z^{-n-1}$, for all $a \in V$, and we write $Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$. The Definitions and notations of these concepts can be found in Section 2.1. The main content of this Chapter can also be found in [58].

3.1 Space of correlation functions associated with three modules M^1, M^2, and M^3

In this Section, based on the properties of the limit rational function of the series (3.0.1), we will introduce the notion of space of correlation functions associated with V-modules M^1, M^2, and M^3, $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$. We will also prove that $\mathrm{Cor}\binom{M^3}{M^1\ M^2} \cong I\binom{M^3}{M^1\ M^2}$ as vector spaces by using the techniques we developed in Section 2.4.

3.1.1 The $(n+3)$-point correlation functions

Let M^1, M^2, and M^3 be V-modules with conformal weights h_1, h_2, and h_3, respectively, and let $I \in I\binom{M^3}{M^1\ M^2}$ be an intertwining operator. Recall that $I(v,w) = \sum_{n\in\mathbb{Z}} v(n)w^{-n-1} \cdot w^{-h}$, where $h = h_1 + h_2 - h_3$, and $v(n) = \mathrm{Res}_w I(v,w)w^{n+h}$. Moreover, $v(n)M^2(m) \subseteq M^3(\deg v - n - 1 + m)$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{N}$, see Section 2.1. Consider the power series (3.0.1):

$$\langle v_3', Y(a_1,z_1)\dots Y(a_k,z_k)I(v,w)Y(a_{k+1},z_{k+1})\dots Y(a_n,z_n)v_2\rangle w^h \tag{3.1.1}$$

in $n+1$ complex variables $z_1,\dots,z_n,w$ with integer powers, where $a_1,\dots,a_n \in V$, $v \in M^1$, $v_2 \in M^2$, and $v_3' \in (M^3)'$ which is the contragredient module of M^3. We also omit the module notations M^3 and M^2 in Y_{M^3} and Y_{M^2} in (3.1.1) for simplicity. We multiply the term w^h to avoid the appearance of the logarithm when computing the integrations.

Similar to Corollary 2.4.2, the product of an intertwining operator and module vertex operators also satisfies the rationality of product property, locality, and associativity. In other words, with the notations as above, for any $v_3' \in (M^3)'$, $a \in V$, $v_2 \in M^2$, and $m,n,l \in \mathbb{Z}$, there exists some $f(z,w) \in \mathbb{C}[z^{\pm1}, w^{\pm1}, (z-w)^{\pm1}]$, such that

$$\langle v_3', Y_{M^3}(a,z)I(v,w)v_2\rangle z^m w^{n+h}(z-w)^l = \iota_{z,w}f(z,w), \tag{3.1.2}$$

$$\langle v_3', I(v,w)Y_{M^2}(a,z)v_2\rangle z^m w^{n+h}(z-w)^l = \iota_{w,z}f(z,w), \tag{3.1.3}$$

$$\langle v_3', I(Y_{M^1}(a,z-w)v,w)v_2\rangle z^m w^{n+h}(z-w)^l = \iota_{w,z-w}f(z,w), \tag{3.1.4}$$

and the proof is also similar to the corresponding ones in [27], we omit the details of the proof. Then it follows that the power series (3.1.1) converges in the domain $\mathbb{D} = \{(z_1,\dots,z_n,w) \in \mathbb{C}^{n+1} \| z_1| > |z_2| > \dots > |w| > \dots > |z_n| > 0\}$ to a rational function in $z_1,\dots,z_n,w$, $z_i - z_j$ and $z_k - w$, where $1 \le i \ne j \le n$ and $1 \le k \le n$. We denote this rational function by:

$$(v_3', Y(a_1,z_1)\dots Y(a_k,z_k)I(v,w)Y(a_{k+1},z_{k+1})\dots Y(a_n,z_n)v_2), \tag{3.1.5}$$

Note that the only possible poles of (3.1.5) are at $z_i = 0$, $w = 0$, $z_i = z_j$ and $z_k = w$. Moreover, by (3.1.2) and (3.1.3), together with Corollary 2.4.2, the rational function (3.1.5) does not depend on the place k where $I(v,w)$ is placed at. In other words, for any permutation $(i_1, i_2, \dots, i_n)$ of $(1,2,\dots,n)$ and any $k = 0,1,\dots,n$ the power series

$$\langle v_3', Y(a_{i_1},z_{i_1})\dots Y(a_{i_k},z_{i_k})I(v,w)Y(a_{i_{k+1}},z_{i_{k+1}})\dots Y(a_{i_n},z_{i_n})v_2\rangle w^h$$

have the same limit function (3.1.5) on their corresponding domain of convergence. We use the symbol S_I to denote the limit function (3.1.5):

$$S_I(v_3', (a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2) := (v_3', Y(a_1, z_1)\dots I(v, w)\dots Y(a_n, z_n)v_2). \qquad (3.1.6)$$

Then we have a system of linear maps $S_I = \{(S_I)^n_{V\dots M^1\dots V}\}_{n=0}^{\infty}$:

$$\begin{aligned}(S_I)^n_{V\dots M^1\dots V} &: (M^3)' \times V \times \dots \times M^1 \times \dots V \times M^2 \to \mathcal{F}(z_1, \dots, z_n, w),\\ &(v_3', a_1, \dots, v, \dots, a_n, v_2) \mapsto S_I(v_3', (a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2),\end{aligned} \qquad (3.1.7)$$

where $\mathcal{F}(z_1, \dots, z_n, w)$ is the space of rational functions in $n+1$ variables $z_1, z_2, \dots, z_n, w$, with only possible poles at $z_i = 0,\ w = 0,\ z_i = z_j,\ z_k = w$. For a fixed $n \in \mathbb{N}$, we have $(S_I)^n_{M^1 V\dots V} = (S_I)^n_{VM^1\dots V} = \dots = (S_I)^n_{V\dots VM^1}$, since the terms $(a_1, z_1), \dots, (a_n, z_n)$, and (v, w) can be permuted within S_I in (3.1.6). We may view S_I as a $(n+3)$-point correlation function on $\mathbb{P}^1(\mathbb{C})$, where we associated V-modules: $(M^3)'$, $V, \dots M^1, \dots V$, and M^2 to these points.

We introduce the following notion that generalizes both Definition 4.1.1 in [73] and the genus-zero axioms in Theorem 2.4.4:

Definition 3.1.1. A system of linear maps $S = \{S^n_{V\dots M^1\dots V}\}_{n=0}^{\infty}$,

$$\begin{aligned}S^n_{V\dots M^1\dots V} &: (M^3)' \times V \times \dots \times M^1 \times \dots V \times M^2 \to \mathcal{F}(z_1, \dots, z_n, w),\\ &(v_3', a_1, \dots, v, \dots, a_n, v_2) \mapsto S(v_3', (a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2),\end{aligned}$$

is said to satisfy the **genus-zero property associated with M^1, M^2, and M^3** if it satisfies

(1) (Truncation property) For fixed $v \in M^1$ and $v_2 \in M^2$, the Laurent series expansion of $S(v_3', (v, w)v_2)$ around $w = 0$ has a uniform lower bound for w independent of $v_3' \in (M^3)'$. i.e., $S(v_3', (v, w)v_2) = \sum_{n \leq N} a_n w^{-n-1}$ for all $v_3' \in (M^3)'$.

(2) (Locality) The terms $(a_1, z_1), \dots, (a_n, z_n)$, and (v, w) can be permuted arbitrarily within S. i.e., $S^n_{M^1 V\dots V} = S^n_{VM^1\dots V} = \dots = S^n_{V\dots VM^1}$ for any fixed $n \in \mathbb{N}$.

(3) (Vacuum property)

$$S(v_3', (\mathbf{1}, z)(a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2) = S(v_3', (a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2). \quad (3.1.8)$$

(4) ($L(-1)$-derivation property)

$$\begin{aligned} S(v_3', (L(-1)a_1, z_1)\dots(a_n, z_n)(v, w)v_2) &= \frac{d}{dz_1} S(v_3', (a_1, z_1)\dots(a_n, z_n)(v, w)v_2), \\ S(v_3', (L(-1)v, w)(a_1, z_1)\dots v_2)w^{-h} &= \frac{d}{dw}\left(S(v_3', (v, w)(a_1, z_1)\dots v_2)w^{-h}\right). \end{aligned} \tag{3.1.9}$$

(5) (Associativity)

$$\begin{aligned} \int_C S(v_3', (a_1, z_1)(v, w)\dots(a_n, z_n)v_2)(z_1 - w)^k dz_1 &= S(v_3', (a_1(k)v, w)\dots(a_n, z_n)v_2), \\ \int_C S(v_3', (a_1, z_1)(a_2, z_2)\dots(v, w)v_2)(z_1 - z_2)^k dz_1 &= S(v_3', (a_1(k)a_2, z_2)\dots(v, w)v_2), \end{aligned} \tag{3.1.10}$$

where in the first equation of (3.1.10), C is a contour of z_1 surrounding w, with $z_2, \dots, z_n$ outside of C; while in the second equation of (3.1.10), C is a contour of z_1 surrounding z_2, with $z_3, \dots, z_n, w$ outside of C.

(6) (The Virasoro relation) Let $\omega \in V$ be the Virasoro element, and let $x, x_1, \dots, x_m$ be complex variables, denote the rational function

$$S(v_3', (\omega, x_1)\dots(\omega, x_m)(a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2)$$

by S for simplicity. Assume that $v_3', v, v_2, a_1, \dots, a_n$ are the highest-weight vectors for the Virasoro algebra, then we have:

$$\begin{aligned} &S(v_3', (\omega, x)(\omega, x_1)\dots(\omega, x_m)(a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2) \\ &= \sum_{k=1}^{n} \frac{x^{-1}z_k}{x - z_k}\frac{d}{dz_k}S + \sum_{k=1}^{n}\frac{\mathrm{wt}a_k}{(x - z_k)^2}S + \frac{x^{-1}w}{x - w}w^h\frac{d}{dw}(S \cdot w^{-h}) + \frac{\mathrm{wt}v}{(x - w)^2}S \\ &+ \frac{\mathrm{wt}v_2}{x^2}S + \sum_{k=1}^{m}\frac{x^{-1}w_k}{x - x_k}\frac{d}{dx_k}S + \sum_{k=1}^{m}\frac{2}{(x - x_k)^2}S \\ &+ \frac{c}{2}\sum_{k=1}^{m}\frac{1}{(x - x_k)^4}S(v_3', (\omega, x_1)\dots\widehat{(\omega, x_k)}\dots(\omega, x_m)(a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2) \end{aligned} \tag{3.1.11}$$

(7) (The generating property for M^2) For any $a \in V$ and $m \in \mathbb{Z}$, we have:

$$\begin{aligned} &S(v_3', (a_1, z_1)\dots(v, w)\dots(a_n, z_n)a(m)v_2) \\ &= \int_C S(v_3', (a_1, z_1)\dots(v, w)\dots(a_n, z_n)(a, z)v_2)z^m dz, \end{aligned} \tag{3.1.12}$$

where $C = C_R(0)$ is a contour of z surrounding 0 with $z_1, \dots, z_n, w$ lying outside.

(8) (The generating property for $(M^3)'$) Denote $(e^{z^{-1}L(1)}(-z^2)^{L(0)}a, z)$ by $(a, z)'$, then

$$\begin{aligned} &S(a(m)v_3', (a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2) \\ &= \int_{C'} S(v_3', (a, z)'(a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2)z^{-m-2}dz, \end{aligned} \tag{3.1.13}$$

where $C' = C_r(0)$ is a contour of z surrounding 0 with $z_1, \dots, z_n, w$ lying inside.

Definition 3.1.2. The vector space of the system of linear maps $S = \{S^n_{V\dots M^1\dots V}\}_{n=0}^{\infty}$ satisfying the genus-zero property associated with M^1, M^2, and M^3 is called the **space of correlation functions associated with** M^1, M^2**, and** M^3. We denote it by $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$.

Proposition 3.1.3. *The system of functions S_I given by* (3.1.6) *and* (3.1.7) *satisfies the genus-zero property associated with M^1, M^2, and M^3 in Definition 3.1.1. Thus $S_I \in$* $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$.

Proof. The properties (1) - (6) for S_I follow immediately from the Definition of S_I in (3.1.6), (3.1.7), together with (3.1.2)–(3.1.4) and the expansion formula of the vertex operator Y. See Section 5.6 in [27] and the proof of Theorem 2.4.4 for more details.

To prove (3.1.12), we note that the Laurent series expansion of the rational function (3.1.6) on the domain $|z| < |z_i|, |w|$ for all i is $\sum_{m\in\mathbb{Z}}(v_3', Y(a_1, z_1)\dots I(v, w)\dots a(m)v_2)z^{-m-1}$. The coefficient of z^{-m-1} in the Laurent series is also

$$\int_C (v_3', Y(a_1, z_1)\dots I(v, w)\dots Y(a_n, z_n)Y(a, z)v_2)z^m dz,$$

where $C = C_R(0)$ is a contour of z surrounding 0 with $z_1, \dots, z_n$ and w lying outside. This proves (3.1.12). To prove (3.1.13), we denote the term $\sum_{j\geq 0}\frac{1}{j!}(-1)^{\mathrm{wt}a}(L(1)a^j)(2\mathrm{wt}a - m - j - 2)$ by $a'(m)$, then by the definition of contragredient module (see (5.2.4) in [27]), the series

$$\begin{aligned} &\sum_{m\in\mathbb{Z}}(a(m)v_3', Y(a_1, z_1)\dots I(v, w)\dots Y(a_n, z_n)Y(a, z)v_2)z^{-m-1} \\ &= \sum_{m\in\mathbb{Z}}(v_3', a'(m)Y(a_1, z_1)\dots I(v, w)\dots Y(a_n, z_n)v_2)z^{-m-1} \end{aligned}$$

is the expansion of $(v_3', Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})Y(a_1, z_1)\dots I(v, w)\dots Y(a_n, z_n)v_2)$ on the domain $|z^{-1}| > |z_i|, |w|$, or equivalently, $|z| < 1/|z_i|, 1/|w|$, for $i = 1, \dots, n$. By comparing the Laurent coefficient of z^{-m-1}, we have:

$$(a(m)v_3', Y(a_1, z_1)\dots I(v, w)\dots Y(a_n, z_n)Y(a, z)v_2)$$

$$= \int_{C_R(0)} (v_3', Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})Y(a_1, z_1)\ldots I(v, w)\ldots Y(a_n, z_n)v_2)z^m dz, \tag{3.1.14}$$

where R is small enough such that $R < 1/|z_i|, 1/|w|$, for $i = 1, \ldots, n$. Change the variable $z \to 1/z$ in the integral (3.1.14). Note that the parametrization of $1/z$ is $(1/R)e^{-i\theta}$, which gives us a clockwise orientation, and $d(1/z) = -(1/z^2)dz$. Let $C' = C_r(0)$, with radius $r = 1/R > |z_i|, |w|$ for $i = 1, \ldots, n$, equipped with the counterclockwise orientation. Then $z_1, \ldots, z_n, w$ are inside of C', and

$$\begin{aligned}(3.1.14) &= -\int_{C'} (v_3', Y(e^{z^{-1}L(1)}(-z^2)^{L(0)}a, z)Y(a_1, z_1)\ldots I(v, w)\ldots Y(a_n, z_n)v_2)z^{-m}(-z^{-2})dz\\ &= \int_{C'} (v_3', Y(e^{z^{-1}L(1)}(-z^2)^{L(0)}a, z)Y(a_1, z_1)\ldots I(v, w)\ldots Y(a_n, z_n)v_2)z^{-m-2}dz\\ &= \int_{C'} S_I(v_3', (a, z)'(a_1, z_1)\ldots(v, w)\ldots(a_n, z_n)v_2)z^{-m-2}dz.\end{aligned}$$

This proves (3.1.13). □

Remark 3.1.4. Let $S \in \mathrm{Cor}\binom{M^3}{M^1\ M^2}$. With the notations of Proposition 3.1.3, we have:

$$\begin{aligned}&S(a'(m)v_3', (a_1, z_1)\ldots(v, w)\ldots(a_n, z_n)v_2)\\ &= \sum_{j\geq 0} \frac{1}{j!}(-1)^{\mathrm{wt}a} \int_{C'} S(v_3', (e^{z^{-1}L(1)}(-z^2)^{L(0)}(L(1)^j a), z)(a_1, z_1)\ldots(a_n, z_n)v_2)z^{-2\mathrm{wt}a+m+j}dz\\ &= \int_{C'} S(v_3', (e^{z^{-1}L(1)}(-z^2)^{L(0)}e^{zL(1)}(-z^{-2})^{L(0)}a, z)(a_1, z_1)\ldots(a_n, z_n)v_2)z^m dz\\ &= \int_{C'} S(v_3', (e^{z^{-1}L(1)}e^{-z^{-1}L(1)}a, z)(a_1, z_1)\ldots(a_n, z_n)v_2)z^m dz\\ &= \int_{C'} S(v_3', (a, z)(a_1, z_1)\ldots(v, w)\ldots(a_n, z_n)v_2)z^m dz.\end{aligned}$$

Hence the generating property for $(M^3)'$ (8) in Definition 3.1.1 is equivalent to:

$$\begin{aligned}&S(a'(m)v_3', (a_1, z_1)\ldots(v, w)\ldots(a_n, z_n)v_2)\\ &= \int_{C'} S(v_3', (a, z)(a_1, z_1)\ldots(v, w)\ldots(a_n, z_n)v_2)z^m dz,\end{aligned} \tag{3.1.15}$$

where $a'(m) = \sum_{j\geq 0} \frac{1}{j!}(-1)^{\mathrm{wt}a}(L(1)^j a)(2\mathrm{wt}a - m - j - 2)$ and $C' = C_r(0)$ as in (8).

As a consequence of Proposition 3.1.3, we have a well-defined linear map:

$$\alpha : I\binom{M^3}{M^1\ M^2} \to \mathrm{Cor}\binom{M^3}{M^1\ M^2}, \qquad I \mapsto S_I, \tag{3.1.16}$$

where S_I is given by (3.1.6) and (3.1.7).

3.1.2 The space of correlation functions and the space of intertwining operators

Although the genus-zero property associated with three V-modules in Definition 3.1.1 seems long and intrinsic, it is good enough to characterize an intertwining operator. In other words, we can construct an inverse of the map α in (3.1.16).

Fix a system of correlation functions S in $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$, we construct an intertwining operator $I_S \in I\binom{M^3}{M^1\ M^2}$ in the following way:

Let $v \in M^1$, define a linear map $v(n) : M^2 \to M^3$ by the formula:

$$\langle v_3', v(n)v_2\rangle := \int_C S(v_3', (v, w)v_2)w^n dw, \tag{3.1.17}$$

where C is a contour of w surrounding 0. Note that an element $u \in M^3$ is uniquely determined by the value $\langle v_3', u\rangle$ for $v_3' \in (M^3)'$, so we have a well-defined element $v(n)v_2$ in M^3. Then we define $I_S(v, w)$ as the following power series:

$$I_S(v, w) := \sum_{n\in\mathbb{Z}} v(n)w^{-n-1} \cdot w^{-h}, \tag{3.1.18}$$

where $h = h_1 + h_2 - h_3$. It is clear that $I(v, w) \in \mathrm{Hom}(M^2, M^3)\{z\}$.

Theorem 3.1.5. *The series $I_S(v, w)$ defined by* (3.1.17) *and* (3.1.18) *is an intertwining operator of type* $\binom{M^3}{M^1\ M^2}$.

Proof. By Definition 3.1.1, $S(v_3', (v, w)v_2)$ is a rational function in w with the only possible pole at $w = 0$, and the term (3.1.17) is the Laurent coefficient of $S(v_3', (v, w)v_2)$. Thus the series $\langle x_3', I_S(v, w)x_2\rangle w^h$ is the Laurent series expansion of $S(x_3', (v, w)x_2)$ around $w = 0$ by (3.1.18). In particular, if we denote the limit of the Laurent series $\langle v_3', I(v, w)v_2\rangle w^h$ by $(v_3', I(v, w)v_2)$, then we have the following equality of rational functions:

$$(v_3', I_S(v, w)v_2) = S(v_3', (v, w)v_2) \tag{3.1.19}$$

Since S satisfies the property (1) in Definition 3.1.1, for $v \in M^1$ and $v_2 \in M^2$, there exists $N \in \mathbb{Z}$ such that $\langle v_3', I_S(v, w)v_2\rangle w^h = \sum_{n\leq N} \left(\int_C S(v_3', (v, w)v_2)w^n dw\right) w^{-n-1}$, for all $v_3' \in (M^3)'$. Hence we have $v(n)v_2 = 0$ for $n \gg 0$. By the $L(-1)$-derivative property of S, together with (3.1.18), we have:

$$\langle v_3', I_S(L(-1)v, w)v_2\rangle = \frac{d}{dw}(S(v_3', (v, w)v_2)w^{-h}) = \frac{d}{dw}\langle v_3', I_S(v, w)v_2\rangle.$$

Hence $I_S(L(-1)v, w) = \frac{d}{dw} I_S(v, w)$. Moreover, we claim that the following equation holds:

$$\sum_{i=0}^{\infty} \binom{m}{i} (a(l+i)v)(m+n-i)v_2 \tag{3.1.20}$$
$$= \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} a(m+l-i)v(n+i)v_2 - \sum_{i=0}^{\infty} (-1)^{l+i} \binom{l}{i} v(n+l-i)a(m+i)v_2,$$

for all $m, n, l \in \mathbb{Z}$, $a \in V$, $v \in M^1$, and $v_2 \in M^2$. Note that (3.1.20) is the component form of the Jacobi identity for the intertwining operator I_S.

Indeed, the proof is similar to the proof of Theorem 2.4.5 and 2.4.14, with a different order of integration and choice of radii . By (3.1.17) and the generating property of $(M^3)'$ of S (3.1.15), we have:

$$\langle v_3', \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} a(m+l-i)v(n+i)v_2 \rangle = \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} \int_{C_1'} S(a'(m+l-i)v_3', (v, w)v_2) w^{n+i} dw$$
$$= \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} \int_{C_1'} \int_{C_2'} S(v_3', (a, z)(v, w)v_2) z^{m+l-i} w^{n+i} dz dw \tag{3.1.21}$$
$$= \int_{C_1'} \int_{C_2'} S(v_3', (a, z)(v, w)v_2) z^m w^n (z-w)^l dz dw$$
$$= \mathrm{Res}_z \mathrm{Res}_w \left(\iota_{z,w} S(v_3', (a, z)(v, w)v_2) z^m w^n (z-w)^l \right).$$

where C_1' is a contour of w centered at 0, and C_2' is a contour of z centered at 0 such that w is lying inside, and the last equality follows from (2.4.4). So C_1' and C_2' can be given in Figure 2.1. On the other hand, by (3.1.17) and the generating property (3.1.12) of S, we have:

$$\langle v_3', \sum_{i=0}^{\infty} (-1)^{l+i} \binom{l}{i} v(n+l-i)a(m+i)v_2 \rangle = \sum_{i=0}^{\infty} (-1)^{l+i} \binom{l}{i} \int_{C_1} S(v_3', (v, w)a(m+i)v_2) w^{n+l-i} dw$$
$$= \sum_{i=0}^{\infty} (-1)^{l+i} \binom{l}{i} \int_{C_1} \int_{C_2} S(v_3', (v, w)(a, z)v_2) z^{m+i} w^{n+l-i} dw dz \tag{3.1.22}$$
$$= \int_{C_1} \int_{C_2} S(v_3', (v, w)(a, z)v_2) z^m w^n (z-w)^l dw dz$$
$$= \mathrm{Res}_w \mathrm{Res}_z \left(\iota_{w,z} S(v_3', (v, w)(a, z)v_2) z^m w^n (z-w)^l \right),$$

where C_1 is a contours in w centered at 0, and C_2 is a contour of z centered at 0, with w lying outside. Thus, C_1 and C_2 can also be given by Figure 2.1. Then by the Definition formulas (3.1.17), (3.1.21), and (3.1.22), together with (2) and (5) in Definition 3.1.1, we have:

$$\langle v_3', \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} a(m+l-i)v(n+i)v_2 - \sum_{i=0}^{\infty} (-1)^{l+i} \binom{l}{i} v(n+l-i)a(m+i)v_2 \rangle$$

$$\begin{aligned}
&= \mathrm{Res}_z\mathrm{Res}_w \left(\iota_{z,w} S(v_3', (a,z)(v,w)v_2) z^m w^n (z-w)^l\right)\\
&\quad - \mathrm{Res}_w\mathrm{Res}_z \left(\iota_{w,z} S(v_3', (v,w)(a,z)v_2) z^m w^n (z-w)^l\right),\\
&= \mathrm{Res}_w\mathrm{Res}_{z-w} \left(\iota_{w,z-w} S(v_3', (a,z)(v,w)v_2) z^m w^n (z-w)^l\right),\\
&= \int_{C_2}\int_{C_\epsilon^z(w)} S(v_3', (a,z)(v,w)v_2)(z-w)^l z^m w^n dz dw \qquad (3.1.23)\\
&= \int_{C_2}\int_{C_\epsilon^z(w)} S(v_3', (a,z)(v,w)v_2)(z-w)^l \iota_{w,z-w}(w+(z-w))^m w^n dz dw\\
&= \sum_{i\geq 0}\binom{m}{i}\int_{C_2}\int_{C_\epsilon^z(w)} S(v_3', (a,z)(v,w)v_2)(z-w)^{l+i} w^{n+m-i} dz dw\\
&= \sum_{i\geq 0}\binom{m}{i}\int_{C_2} S(v_3', (a(l+i)v,w)v_2) w^{m+n-i}\\
&= \sum_{i\geq 0}\binom{m}{i}\langle v_3', (a(l+i)v)(m+n-i)v_2\rangle,
\end{aligned}$$

where the contours C_2 and $C_\epsilon^z(w)$ are given by Figure 2.2, and we've used Theorem 2.4.1 to obtain the third equality. Since v_3' in (3.1.23) can be choosen arbitraily, the Jacobi identity (3.1.20) follows, and so I_S given by (3.1.18) is an intertwining operator of type $\binom{M^3}{M^1\ M^2}$. □

Corollary 3.1.6. *The vector space of intertwining operators* $I\binom{M^3}{M^1\ M^2}$ *is isomorphic to the vector space* $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$ *in Definition 3.1.2.*

Proof. Theorem 3.1.5 indicates that there exists a well-defined linear map:

$$\beta : \mathrm{Cor}\binom{M^3}{M^1\ M^2} \to I\binom{M^3}{M^1\ M^2}, \quad S \mapsto I_S. \qquad (3.1.24)$$

By (3.1.6) and (3.1.19), it is clear that β is an inverse of the linear map α in (3.1.16). Hence $I\binom{M^3}{M^1\ M^2} \cong \mathrm{Cor}\binom{M^3}{M^1\ M^2}$ as vector spaces. □

Remark 3.1.7. If we consider the case when $M^1 = V$ and $M^3 = M^2 = M$, then an intertwining operator $I \in \binom{M}{V\ M}$ is just a vertex operator $Y_M : V \to \mathrm{End}(M)[[z, z^{-1}]]$, and Corollary 3.1.6 in this case is precisely Theorem 2.4.5 about the correlation function associated with one module.

3.2 The correlation functions defined on the bottom levels

In this Section, we will restricted a system of correlation function $S \in \mathrm{Cor}\binom{M^3}{M^1\ M^2}$ onto the bottom levels $M^3(0)^*$ and $M^2(0)$ of $(M^3)'$ and M^2, respectively, and use the properties of the restricted correlation functions to give an auxiliary notion of the space of correlation functions associated with M^1, $M^2(0)$, and $M^3(0)$, denoted by $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. We will give a lift for any $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ to a system of functions on the domain $M^3(0)^* \times V \times \cdots \times M^1 \times \cdots \times V \times \bar{M}$, where $\bar{M}$ is a free object associated to $M^2(0)$. We will also define the radical of the lifted functions, and prove some properties of the radical.

Recall that the bottom level $M(0)$ of any $\mathbb{N}$-gradable V-module $M = \bigoplus_{n=0}^{\infty} M(n)$ is a module over the Zhu's algebra $A(V)$ under the module action: $[a].v = o(a)v = a(\mathrm{wt} a - 1)v$, for all $[a] \in A(V)$ and $v \in M(0)$, see Section 2.2. For the rest of this Section, we assume that the $A(V)$-modules $M^2(0)$ and $M^3(0)$ are irreducible.

3.2.1 The space of correlation functions associated with M^1, $M^2(0)$, and $M^3(0)$

Let $S \in \mathrm{Cor}\binom{M^3}{M^1\ M^2}$, and let $I \in I\binom{M^3}{M^1\ M^2}$ be its corresponding intertwining operator under the isomorphism β in (3.1.24). For each $n \in \mathbb{N}$, consider the restriction of S onto the bottom levels $M^2(0)$ and $M^3(0)^*$:

$$S|_{M^3(0)^* \times \ldots M^1 \cdots \times M^2(0)} : M^3(0)^* \times V \times \cdots \times M^1 \cdots \times V \times M^2(0) \to \mathcal{F}(z_1, \ldots, z_n, w). \quad (3.2.1)$$

To simplify our notation, we use the same symbol S to denote the restricted function (3.2.1). Clearly, S in (3.2.1) satisfies properties (1)–(6) in Definition 3.1.1, with the elements v_3' and v_2 in these properties belong to $M^3(0)^*$ and $M^2(0)$, respectively. Moreover, since $(v_3', I(v, w)v_2) = S(v_3', (v, w)v_2)$ by (3.1.19), and $v(n)M^2(m) \subseteq M^3(m + \deg v - n - 1)$ for all $v \in M^1$ homogeneous, $n \in \mathbb{Z}$, and $m \in \mathbb{N}$ (see (1.5.4) in [30]), then we have:

$$S(v_3', (v, w)v_2) = \langle v_3', v(\deg v - 1)v_2\rangle w^{-\deg v}. \quad (3.2.2)$$

We introduce the following intermediate notion based on the properties satisfied by the system of restricted correlation functions (3.2.1).

Definition 3.2.1. Let $M^2(0)$ and $M^3(0)$ be irreducible $A(V)$-modules, and let $S = \{S^n_{V \ldots M^1 \ldots V}\}_{n=0}^{\infty}$ be a system of linear maps:

$$S^n_{V \ldots M^1 \ldots V} : M^3(0)^* \times V \times \ldots \times M^1 \times \ldots V \times M^2(0) \to \mathcal{F}(z_1, \ldots, z_n, w),$$
$$(v_3', a_1, \ldots, v, \ldots, a_n, v_2) \mapsto S(v_3', (a_1, z_1) \ldots (v, w) \ldots (a_n, z_n)v_2).$$

Then S is said to satisfy the **genus-zero property associated with M^1, $M^2(0)$, and $M^3(0)$** if the following conditions are satisfied:

(1) S satisfies properties (2) − (6) in Definition 3.1.1, with the elements v_3' and v_2 in these properties belong to $M^3(0)^*$ and $M^2(0)$, respectively.

(2) There exists a linear functional $f : M^1 \to \mathrm{Hom}_{\mathbb{C}}(M^2(0), M^3(0)), v \mapsto f_v$, such that

$$S(v_3', (v, w)v_2) = \langle v_3', f_v(v_2)\rangle w^{-\deg v}, \tag{3.2.3}$$

for all $v_2 \in M^2(0)$ and $v_3' \in M^3(0)^*$.

(3) (The recursive formula for $M^3(0)^*$) For any $v_3' \in M^3(0)^*$, $v \in M^1$, $v_2 \in M^2(0)$, and $a_1, \dots, a_n \in V$,

$$\begin{aligned} &S(v_3', (a, z)(a_1, z_1) \dots (a_n, z_n)(v, w)v_2) = S(v_3' o(a), (a_1, z_1) \dots (a_n, z_n)(v, w)v_2) z^{-\mathrm{wt} a} \\ &+ \sum_{k=1}^{n} \sum_{i \geq 0} F_{\mathrm{wt} a, i}(z, z_k) S(v_3', (a_1, z_1) \dots (a(i)a_k, z_k) \dots (a_n, z_n)(v, w)v_2) \\ &+ \sum_{i \geq 0} F_{\mathrm{wt} a, i}(z, w) S(v_3', (a_1, z_1) \dots (a_n, z_n)(a(i)v, w)v_2), \end{aligned} \tag{3.2.4}$$

where $F_{\mathrm{wt} a, i}(z, w)$ is a rational function in z, w given by:

$$\begin{aligned} \iota_{z,w}(F_{\mathrm{wt} a, i}(z, w)) &= \sum_{j \geq 0} \binom{\mathrm{wt} a + j}{i} z^{-\mathrm{wt} a - j - 1} w^{\mathrm{wt} a + j - i}, \\ F_{m,i}(z, w) &= \frac{z^{-m}}{i!} \left(\frac{d}{dw}\right)^i \frac{w^m}{z - w}, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{3.2.5}$$

and $v_3' o(a)$ is given by the natural right module action on $M^3(0)^*$.

(4) (The recursive formula for $M^2(0)$) For any $v_3' \in M^3(0)^*$, $v \in M^1$, $v_2 \in M^2(0)$, and $a_1, \dots, a_n \in V$, we have:

$$\begin{aligned} &S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)(a, z)v_2) = S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w) o(a) v_2) z^{-\mathrm{wt} a} \\ &+ \sum_{k=1}^{n} \sum_{i \geq 0} G_{\mathrm{wt} a, i}(z, w) S(v_3', (a_1, z_1) \dots (a(i)a_k, z_k) \dots (a_n, z_n)(v, w)v_2) \\ &+ \sum_{i \geq 0} G_{\mathrm{wt} a, i}(z, w) S(v_3', (a_1, z_1) \dots (a_n, z_n)(a(i)v, w)(a, z)v_2), \end{aligned} \tag{3.2.6}$$

where $G_{\mathrm{wt} a, i}(z, w)$ is a rational function defined by

$$\begin{aligned} \iota_{w,z}(G_{\mathrm{wt} a, i}(z, w)) &= -\sum_{j \geq 0} \binom{\mathrm{wt} a - 2 - j}{i} w^{\mathrm{wt} a - j - 2 - i} z^{-\mathrm{wt} a + 1 + j}, \\ G_{m,i}(z, w) &= \frac{z^{-m+1}}{i!} \left(\frac{d}{dw}\right)^i \left(\frac{w^{m-1}}{z - w}\right), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.2.7}$$

The vector space of the system of functions satisfying the genus-zero property associated with M^1, $M^2(0)$, and $M^3(0)$ is denoted by $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$.

We observe that the rational functions F and G given by (3.2.5) and (3.2.7) satisfy the following relation:

$$F_{m,i}(z,w) - G_{m,i}(z,w) = \frac{z^{-m}}{i!}\left(\frac{d}{dw}\right)^i \left(\frac{w^m}{z-w} - \frac{zw^{m-1}}{z-w}\right) = -\binom{m-1}{i} z^{-m} w^{m-1-i},$$

for all $m \in \mathbb{N}$. In particular, we have:

$$F_{\mathrm{wt}a,i}(z,w) - G_{\mathrm{wt}a,i}(z,w) = -\binom{\mathrm{wt}a-1}{i} z^{-\mathrm{wt}a} w^{\mathrm{wt}a-1-i}. \tag{3.2.8}$$

The equation (3.2.8) will be used multiple times in Chapter 4 when we build a system of correlation functions S from a linear map on a tensor product of $A(V)$-modules.

Proposition 3.2.2. *Let $S \in \mathrm{Cor}\binom{M^3}{M^1\ M^2}$. Then the system of restricted functions S in* (3.2.1) *satisfies the genus-zero property associated with M^1, $M^2(0)$, and $M^3(0)$.*

Proof. By our discussion in the begining of this subsection, S in (3.2.1) satisfies (1) and (2) in Definition 3.2.1, where the f_v in (3.2.3) is given by $f_v = v(\deg v - 1)$, for all $v \in M^1$. The proof of (3.2.4) is similar to the proof of Lemma 2.2.1 in [73]. We omit the details. To prove (3.2.6), we only consider the case when $n = 0$ (the general case follows from a similar argument.) Note that $a(n)v_2 = 0$ if $\mathrm{wt}a - n - 1 < 0$, it follows that $\langle v_3', I(v,w)Y(a,z)v_2\rangle = \langle v_3', I(v,w)o(a)v_2\rangle z^{-\mathrm{wt}a} + \sum_{\mathrm{wt}a-n-1>0}\langle v_3', I(v,w)a(n)v_2\rangle z^{-n-1}$. By the definition of contragredient modules, we have $\langle v_3', a(n)u\rangle = \sum_{i\geq 0}\frac{1}{i!}(-1)^i\langle (L(i)a)(2\mathrm{wt}a - n - i - 2)v_3', u\rangle$, for any $n \in \mathbb{Z}$. But $(L(i)a)(2\mathrm{wt}a - n - i - 2)v_3' \in (M^3)'(-\mathrm{wt}a + n + 1) = 0$ when $\mathrm{wt}a - n - 1 > 0$. Thus

$$\begin{aligned}
&\sum_{wta-n-1>0}\langle v_3', I(v,w)a(n)v_2\rangle z^{-n-1} = -\sum_{\mathrm{wt}a-n-1>0}\langle v_3', [a(n), I(v,w)]v_2\rangle z^{-n-1}\\
&= -\sum_{\mathrm{wt}a-n-1>0}\sum_{i\geq 0}\binom{n}{i}\langle v_3', I(a(i)v,w)v_2\rangle z^{-n-1}w^{n-i}\\
&= -\sum_{j\geq 0}\sum_{i\geq 0}\binom{\mathrm{wt}a-j-2}{i} z^{-\mathrm{wt}a+j+2-1} w^{\mathrm{wt}a-j-2-i}\langle v_3', I(a(i)v,w)v_2\rangle\\
&= \sum_{i\geq 0}\iota_{w,z}(G_{\mathrm{wt}a,i}(z,w))\langle v_3', I(a(i)v,w)v_2\rangle,
\end{aligned}$$

where the last equality follows from (3.2.7). Hence we have:

$$\langle v_3', I(v,w)Y(a,z)\rangle = \langle v_3', I(v,w)o(a)v_2\rangle z^{-\mathrm{wt}a} + \sum_{i\geq 0}\iota_{w,z}(G_{\mathrm{wt}a,i}(z,w))\langle v_3', I(a(i)v,w)v_2\rangle$$

as power series. By taking the limit of this series, we obtain (3.2.6) for $n = 0$. □

As a consequence of Proposition 3.2.2, we have a well-defined restriction map:

$$\varphi : \mathrm{Cor}\binom{M^3}{M^1\ M^2} \to \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}, \quad S \mapsto S|_{M^3(0)^* \times \ldots M^1 \cdots \times M^2(0)}, \tag{3.2.9}$$

where M^2 and M^3 are any V-modules, with bottom levels $M^2(0)$ and $M^3(0)$, respectively.

The following Lemma will be used in the next chapter:

Lemma 3.2.3. *Let $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, and let $f : M^1 \to \mathrm{Hom}_{\mathbb{C}}(M^2(0), M^3(0)), v \mapsto f_v$ be the linear functional in Definition 3.2.1. Suppose that $f_v = 0$ for all $v \in M^1$. Then $S = 0$.*

Proof. We use induction on n to show that $S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w)v_2) = 0$ for all $v_3' \in M^3(0)^*$, $v \in M^1$, $v_2 \in M^2(0)$, and $a_1, \ldots, a_n \in V$. When $n = 0$, by the assumption and (3.2.3), we have: $S(v_3', (v, w)v_2) = \langle v_3', f_v(v_2)\rangle w^{-\deg v} = \langle v_3', 0\rangle w^{-\deg w} = 0$, for all $v_3' \in M^3(0)^*$, $v \in M^1$, and $v_2 \in M^2(0)$. For $n > 0$, by the recursive formula (3.2.4), we have

$$\begin{aligned}
&S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w)v_2) = S(v_3' o(a_1), (a_2, z_2) \ldots (a_n, z_n)(v, w))z^{-\mathrm{wt} a_1} \\
&\quad + \sum_{k=2}^{n} \sum_{i \geq 0} F_{\mathrm{wt} a_1, i}(z_1, z_k) S(v_3', (a_2, z_2) \ldots (a_1(i)a_k, z_k) \ldots (a_n, z_n)(v, w)v_2) \\
&\quad + \sum_{i \geq 0} F_{\mathrm{wt} a_1, i}(z_1, w) S(v_3', (a_2, z_2) \ldots (a_n, z_n)(a_1(i)v, w)v_2).
\end{aligned}$$

Since each term on the right-hand side has a smaller length, the right-hand side is equal to 0 by the induction hypothesis, so we have $S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w)v_2) = 0$. □

3.2.2 Generalized Verma modules and the radical of correlation functions

Recall that for any irreducible $A(V)$-module U, Dong, Li, and Mason constructed a generalized Verma module $\bar{M}(U)$ in [18]. By their construction, $\bar{M}(U) = (U(\mathcal{L}(V)) \otimes_{U(\mathcal{L}(V)_{\geq 0})} U)/U(\mathcal{L}(V))W$, where

$$\mathcal{L}(V) = V \otimes \mathbb{C}[t, t^{-1}]/(L(-1) \otimes 1 + 1 \otimes \frac{d}{dt})(V \otimes \mathbb{C}[t, t^{-1}]) \tag{3.2.10}$$

is the Lie algebra associated with the VOA V (cf. [12, 18]).$\mathcal{L}(V)$ is a graded vector space: $\mathcal{L}(V) = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}(V)_n$, where $\deg(\overline{a \otimes t^n}) := \mathrm{wt}a - n - 1$, for all homogeneous $a \in V$ and $n \in \mathbb{Z}$, and $\mathcal{L}(V)_n$ is spanned by elements in $\mathcal{L}(V)$ of degree n. Moreover, recall that $\mathcal{L}(V)_{\geq 0} =$

$\bigoplus_{n\in\mathbb{N}}\mathcal{L}(V)_n$, and $\mathcal{L}(V)_0$ is a Lie subalgebra. There exists an epimorphism of Lie algebras $\mathcal{L}(V)_0 \to A(V)_{\text{Lie}}$, and so U is a modules over $\mathcal{L}(V)_{\geq 0}$. W is the subspace of $U(\mathcal{L}(V))\otimes_{U(\mathcal{L}(V)_{\geq 0})} U$ spanned by the coefficients of the weak associativity equality (2.1.4), see Section 5 in [18] for more details.

$\bar{M}(U)$ is $\mathbb{N}$-gradable: $\bar{M}(U) = \bigoplus_{n=0}^{\infty}\bar{M}(n)$, with the bottom level $\bar{M}(U)(0) = U$. It satisfies a universal property in the sense that any $\mathbb{N}$-gradable V-module with bottom level U is a quotient module of $\bar{M}(U)$ (Theorem 6.2 in [18]). Moreover, $\bar{M}(U)$ admits a unique maximal graded $\mathcal{L}(V)$-submodule J subject to $J\cap U = 0$, and $L(U) = \bar{M}(U)/J$ is an irreducible V-module (Theorem 6.3 in [19]).

In Section 2 of [49], Li gave an alternative definition of the generalized Verma module $\bar{F}(U)$ associated with U, namely, $\bar{F}(U) = (U(\mathcal{L}(V))\otimes_{U(\mathcal{L}(V)_{\geq 0})} U)/J(U)$, where $J(U)$ is the intersection of $\ker\alpha$, where α runs over all $\mathcal{L}(V)$-homomorphisms from $\bar{F}(U)$ to weak V-modules. Clearly, $\bar{M}(U) = \bar{F}(U)$ since they satisfy the same universal property.

Let $\bar{M} := T(\mathcal{L}(V))\otimes_{\mathbb{C}} M^2(0)$, where $T(\mathcal{L}(V))$ is the tensor algebra of $\mathcal{L}(V)$. Given a $S \in \text{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, we want to extend the domain of S to $M^3(0)^* \times V \times \cdots \times M^1 \times \cdots \times V \times \bar{M}$. To simplify our notation, we omit the tensor symbol in an element of $\bar{M}$ and denote an element $\overline{b\otimes t^n}$ in $\mathcal{L}(V)$ by (b, n), then an element in $\bar{M}$ can be written as:

$$x = (b_1, i_1)(b_1, i_2)\dots(b_m, i_m)v_2 \tag{3.2.11}$$

where $b_i \in V,\ i_k \in \mathbb{Z},\ v_2 \in M^2(0)$, and (b, i) linear in b. We extend the last input space of S from $M^2(0)$ to $\bar{M}$ by repeatedly using the generating formula (3.1.12). i.e., we let:

$$\begin{aligned}
&S : M^3(0)^* \times V \times \cdots \times M^1 \times \cdots \times V \times \bar{M} \to \mathcal{F}(z_1, \dots, z_n, w),\\
&S(v_3', (a_1, z_1)\dots(a_n, v_n)(v, w)x) \qquad\qquad (3.2.12)\\
&:= \int_{C_1}\cdots\int_{C_m} S(v_3', (a_1, z_1)\dots(a_n, z_n)(v, w)(b_1, w_1)\dots(b_m, w_m)v_2)w_1^{i_1}\dots w_m^{i_m}dw_1\dots dw_m,
\end{aligned}$$

where C_k is a contour of w_k, C_k contains C_{k+1} for each k, C_m contains 0, and $z_1, \dots, z_n, w$ are lying outside of C_1. We first prove the well-definedness of S in (3.2.12). By (3.2.10), we just need to show that S in (3.2.12) agrees on the elements:

$$(b_1, i_1)\dots(L(-1)b_k, i_k)\dots(b_m, i_m)v_2, \quad \text{and} \quad -i_k(b_1, i_1)\dots(b_k, i_k - 1)\dots(b_m, i_m)v_2.$$

Indeed, by the Definition 3.2.1, S in (3.3.1) satisfies (3.1.9). Thus,

$$S(v_3', (a_1, z_1)\dots(a_n, v_n)(v, w)(b_1, i_1)\dots(L(-1)b_k, i_k)\dots(b_m, i_m)v_2)$$

$$= \int_{C_1} \cdots \int_{C_m} \frac{d}{dw_k} S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w) \dots (b_k, w_k) \dots v_2) w_1^{i_1} \dots w_k^{i_k} \dots w_m^{i_m} dw_1 \dots dw_m$$
$$= -\int_{C_1} \cdots \int_{C_m} S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w) \dots (b_k, w_k) \dots v_2) w_1^{i_1} \dots (i_k) w_k^{i_k - 1} \dots w_m^{i_m} dw_1 \dots dw_m$$
$$= S(v_3', (a_1, z_1) \dots (a_n, v_n)(v, w)(-i_k)(b_1, i_1) \dots (b_k, i_k - 1) \dots (b_m, i_m) v_2).$$

Introduce a natural gradation on $\bar{M}$ by letting

$$\deg((b_1, i_1)(b_1, i_2) \dots (b_m, i_m) v_2) := \sum_{k=1}^{m} (\mathrm{wt} b_k - i_k - 1), \tag{3.2.13}$$

and denote the degree n subspace by $\bar{M}(n)$. Then $\bar{M} = \bigoplus_{n \in \mathbb{Z}} \bar{M}(n)$, with $M^2(0) \subseteq \bar{M}(0)$. Similar to (2.2.30) in [73], we define the radical of S on $\bar{M}$ by

$$\begin{aligned} \mathrm{Rad}(S) := \{ x \in \bar{M} | S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w) x) = 0, \\ \forall n \geq 0,\ a_1, \dots a_n \in V,\ v \in M^1,\ v_3 \in M^3(0)^* \}, \end{aligned} \tag{3.2.14}$$

then let $\mathrm{Rad}(\bar{M}) := \bigcap_S \mathrm{Rad}(S)$, where the intersection is taken over all $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. In fact, we can take the intersection over all nonzero S since $\mathrm{Rad}(S) = \bar{M}$ if $S = 0$.

It is clear that the extended S in (3.2.12) factors through $\bar{M}/\mathrm{Rad}(\bar{M})$. Next, we show some essential properties of $\mathrm{Rad}(\bar{M})$, which will eventually lead to the conclusion that $\bar{M}/\mathrm{Rad}(\bar{M})$ carries a structure of $\mathbb{N}$-gradable V-module whose bottom level is $M^2(0)$.

Lemma 3.2.4. *Let W be the subspace of $\bar{M}$ spanned by the following elements:*

$$\begin{aligned} &\sum_{i=0}^{\infty} \binom{m}{i} (a(l+i)b, m+n-i) x \\ &-\Big(\sum_{i=0}^{\infty} (-1)^i \binom{l}{i} (a, m+l-i)(b, n+i) x - \sum_{i=0}^{\infty} (-1)^{l+i} \binom{l}{i} (b, n+l-i)(a, m+i) x \Big), \end{aligned} \tag{3.2.15}$$

where $a, b \in V$, $m, n, l \in \mathbb{Z}$, and $x \in \bar{M}$. Then we have $W \subset \mathrm{Rad}(\bar{M})$.

Proof. By the formula (3.2.12), for the following element in $\bar{M}$, it is easy to see that

$$x' = (b_1, i_1) \dots (b_m, i_m) x,$$

where $x = (c_1, j_1) \dots (c_n, j_n) v_2$ for some $b_i, c_j \in V$ and $i_k, j_l \in \mathbb{Z}$, we have:

$$S(v_3', (a_1, z_1) \dots (a_n, v_n)(v, w) x') \tag{3.2.16}$$

$$= \int_{C_1} \cdots \int_{C_m} S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)(b_1, w_1) \dots (b_m, w_m)x) w_1^{i_1} \dots w_m^{i_m} dw_1 \dots dw_m,$$

where C_k is a contour of w_k, C_{k+1} is inside of C_k for each k, C_m contains 0, and $z_1, \dots, z_n, w$ are lying outside of C_1. Now we fix a nonzero element $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$.

Denote the element (3.2.15) by y. We adopt the notations in Proposition A.2.8 in [29] again. Let C_R^i be the circle of w_i, $i = 1, 2$, centered at 0 with radius R, and let $C_\epsilon^1(w_2)$ be the circle of w_1 centered at w_2 with radius ϵ. We may choose ϵ small enough so that $|w_1 - w_2| < |w_2|$ for any w_1 lying on $C_\epsilon^1(w_2)$. Choose $R, r, \rho > 0$ so that $R > \rho > r$. By (3.2.16) and the locality (2) in Definition 3.1.1 of S, we have:

$$\begin{aligned}
& S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)y) \\
&= \int_{C_\rho^2} \sum_{i=0}^{\infty} \binom{m}{i} S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)(a(l+i)b, w_2)x) w_2^{m+n-i} dw_2 \\
&\quad - \int_{C_R^1} \int_{C_\rho^2} \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)(a, w_1)(b, w_2)x) w_1^{m+l-i} w_2^{n+i} dw_1 dw_2 \\
&\quad + \int_{C_\rho^2} \int_{C_r^1} \sum_{i=0}^{\infty} (-1)^{l+i} \binom{l}{i} S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)(b, w_2)(a, w_1)x) w_1^{m+i} w_2^{n+l-i} dw_1 dw_2 \\
&= \int_{C_\rho^2} \sum_{i=0}^{\infty} \binom{m}{i} S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)(a(l+i)b, w_2)x) w_2^{m+n-i} dw_2 \\
&\quad - \int_{C_R^1} \int_{C_\rho^2} S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)(a, w_1)(b, w_2)x) \cdot \iota_{w_1, w_2}((w_1 - w_2)^l) w_1^m w_2^n dw_1 dw_2 \\
&\quad + \int_{C_\rho^2} \int_{C_r^1} S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)(b, w_2)(a, w_1)x) \cdot \iota_{w_2, w_1}((-w_2 + w_1)^l) w_1^m w_2^n dw_1 dw_2 \\
&= \int_{C_\rho^2} \sum_{i=0}^{\infty} \binom{m}{i} S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)(a(l+i)b, w_2)x) w_2^{m+n-i} dw_2 \\
&\quad - \int_{C_\rho^2} \int_{C_\epsilon^1(w_2)} S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)(a, w_1)(b, w_2)v_2)(w_1 - w_2)^l w_1^m w_2^n dw_1 dw_2. \\
&= \int_{C_\rho^2} \sum_{i=0}^{\infty} \binom{m}{i} S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)(a(l+i)b, w_2)x) w_2^{m+n-i} dw_2 \\
&\quad - \int_{C_\rho^2} \int_{C_\epsilon^1(w_2)} \sum_{i=1}^{\infty} \binom{m}{i} S(v_3', (a_1, z_1) \dots (v, w)(a, w_1)(b, w_2)v_2)(w_1 - w_2)^{l+i} w_2^{m+n-i} dw_1 dw_2 \\
&= 0,
\end{aligned}$$

for all $v_3' \in M^3(0)^*$, $a_1, \dots a_n \in V$, and $v \in M^1$, where the last equality follows from the associativity (5) in Definition 3.1.1. This shows $y \in \mathrm{Rad}(S)$. But S is chosen arbitrarily. Hence we have $y \in \mathrm{Rad}(\bar{M})$. □

Lemma 3.2.5. *$\bar{M}$ and $\mathrm{Rad}(\bar{M})$ satisfy the following properties:*
(a) If $x \in \mathrm{Rad}(\bar{M})$, then $(b, i)x \in \mathrm{Rad}(\bar{M})$, for any $b \in V$ and $i \in \mathbb{Z}$.
(b) $M^2(0) \cap \mathrm{Rad}(\bar{M}) = 0$.
(c) $\bar{M}(n) \subset \mathrm{Rad}(\bar{M})$ for all $n < 0$.

Proof. Since $\mathrm{Rad}(\bar{M}) = \bigcap_S \mathrm{Rad}(S)$, we just need to show that (a), (b), and (c) hold for $\mathrm{Rad}(S)$, where $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ is nonzero.

(a) Let $x \in \mathrm{Rad}(S)$, by (3.2.12) and the definition (3.2.14) of $\mathrm{Rad}(S)$, we have

$$S(v_3', (a_1, z_1) \dots (v, w)(b, i)x) = \int_C S(v_3', (a_1, z_1) \dots (v, w)(b, w_1)x) w_1^i dw_1 = \int_C 0 \cdot w_1^i dw_1 = 0,$$

where C is a contour of w_1, with $z_1, \dots, z_n, w$ lying outside. Thus $(b, i)x \in \mathrm{Rad}(S)$.

(b) Suppose there exists some $v_2 \neq 0$ in $M^2(0) \cap \mathrm{Rad}(S)$, then by (3.2.3) and the recursive formula (3.2.6), we have

$$\begin{aligned} 0 &= \iota_{w,z}(S(v_3', (a, z)(v, w)v_2)) \\ &= S(v_3', (v, w)o(a)v_2)z^{-\mathrm{wt}a} + \sum_{i\geq 0} \iota_{w,z}(G_{\mathrm{wt}a,i}(z, w))S(v_3', (a(i)v, w)v_2) \qquad (3.2.17) \\ &= \langle v_3', f_v(o(a)v_2)\rangle z^{-\mathrm{wt}a} w^{-\deg w} - \sum_{i,j\geq 0} \binom{\mathrm{wt}a - 2 - j}{i} w^{\deg v - j - 1} z^{-\mathrm{wt}a + 1 + j} \langle v_3', f_{a(i)v}(v_2)\rangle, \end{aligned}$$

for any $a \in V$, $v_3' \in M^3(0)^*$, and $v \in M^1$. By comparing the coefficients of $z^{-\mathrm{wt}a} w^{-\deg w}$ on both sides of (3.2.17), we have $\langle v_3', f_v(o(a)v_2)\rangle = 0$ for all $v_3 \in M^3(0)^*$, $a \in V$, and $v \in M^1$. Then $f_v(M^2(0)) = 0$, since $M^2(0)$ is an irreducible $A(V)$-module, and $M^2(0) = A(V).v_2 = span\{o(a)v_2 | a \in V\}$. It follows that $f_v = 0$ for all $v \in M^1$. By Lemma 3.2.3, we have $S = 0$, which is a contradiction.

(c) Let $x = (b_m, i_m) \dots (b_1, i_1)v_2$, with $\sum_{k=1}^m (\mathrm{wt}b_k - i_k - 1) < 0$. We use induction on the length m of x to show that $x \in \mathrm{Rad}(S)$. For the base case, let $x = (b, t)v_2$ with $\mathrm{wt}b - t - 1 < 0$, then by (3.2.12) and (3.2.6), we have

$$\begin{aligned} S(v_3', (a_1, z_1) \dots (v, w)x) &= \int_C S(v_3', (a_1, z_1) \dots (v, w)(b, z)v_2) z^t dz \\ &= \int_C S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)o(b)v_2) z^{t - \mathrm{wt}b} dz \qquad (3.2.18) \end{aligned}$$

$$+\int_C \sum_{k=1}^{n} \sum_{i\geq 0} G_{\mathrm{wt}b,i}(z,z_k) S(v_3', (a_1,z_1)\dots(b(i)a_k,z_k)\dots(v,w)v_2) z^t dz$$
$$+\int_C \sum_{i\geq 0} G_{\mathrm{wt}b,i}(z,w) S(v_3', (a_1,z_1)\dots(b(i)v,w)v_2) z^t dz,$$

where C is a contour of z surrounding 0, with all other variables lying outside C. In particular, we have $|z| < |z_k|$ for all k, and $|z| < |w|$. Then by (3.2.7),

$$\int_C G_{\mathrm{wt}b,i}(z,z_k) z^t dz = \int_C \frac{z^{-\mathrm{wt}b+1+t}}{i!}\Big(\frac{d}{dz_k}\Big)^i \Big(\frac{z_k^{\mathrm{wt}b-1}}{z-z_k}\Big) dz = 0, \tag{3.2.19}$$

since $-\mathrm{wt}b + 1 + t > 0$, and $1/(z - z_k)$ is a sum of nonnegative powers in z for all z lying on the contour C. We also have $\int_C z^{t-\mathrm{wt}b} dz = 0$, since $t - \mathrm{wt}b > -1$. It follows that all the integrals on the right-hand side of (3.2.18) are equal to 0. This finishes the base case.

Now let $m > 0$, and consider $x = (b_m, i_m)\dots(b_1, i_1)v_2 \in \bar{M}$. We have:

$$\begin{aligned}
&S(v_3', (a_1,z_1)\dots(v,w)x)\\
&= \int_{C_m}\cdots\int_{C_1} S(v_3', (a_1,z_1)\dots(v,w)(b_m,w_m)\dots(b_1,w_1)v_2) w_m^{i_m}\dots w_1^{i_1} dw_1\dots dw_m\\
&= \int_{C_m}\cdots\int_{C_1} S(v_3', (a_1,z_1)\dots(v,w)(b_m,w_m)\dots \underset{(1)}{o(b_1)v_2}) w_m^{i_m}\dots w_1^{-\mathrm{wt}b_1+i_1} dw_1\dots dw_m\\
&\quad+ \int_{C_m}\cdots\int_{C_1} \sum_{k=1}^{n}\sum_{i\geq 0} G_{\mathrm{wt}b_1,i}(w_1,z_k) S(v_3', \dots \underset{(2)}{(b_1(i)a_k,z_k)\dots}(v,w)\dots v_2) w_m^{i_m}\dots w_1^{i_1} dw_1\dots dw_m\\
&\quad+ \int_{C_m}\cdots\int_{C_1} \sum_{i\geq 0} G_{\mathrm{wt}b_1,i}(w_1,w) S(v_3', \dots (b_1(i)v,w)\underset{(3)}{(b_m,w_m)}\dots v_2) w_m^{i_m}\dots w_1^{i_1} dw_1\dots dw_m\\
&\quad+ \int_{C_m}\cdots\int_{C_1} \sum_{l=2}^{m}\sum_{i\geq 0} G_{\mathrm{wt}b_1,i}(w_1,w_l) S(v_3', \dots (v,w)\dots \underset{(4)}{(b_1(i)b_l,w_l)}\dots v_2) w_m^{i_m}\dots w_1^{i_1} dw_1\dots dw_m\\
&= (1) + (2) + (3) + (4),
\end{aligned}$$

where C_1 is a contour of w_1 surrounding 0, with all other variables lying outside. We need to show that the sum of these integrals equals 0. i.e., (1) + (2) + (3) + (4) = 0.

Case 1. $\mathrm{wt}b_1 - i_1 - 1 < 0$.

Similar to (3.2.19), we have $\int_{C_1} G_{\mathrm{wt}b_1,i}(w_1, z) w_1^{i_1} dw_1 = 0$, for $z = z_k$, w or w_l. Thus we have (2) = (3) = (4) = 0. We also have (1) = 0 because $-\mathrm{wt}b_1 + i_1 > -1$.

Case 2. $\mathrm{wt}b_1 - i_1 - 1 > 0$.

Then $-\mathrm{wt}b_1 + i_1 < -1$, which implies (1) = 0. Moreover, by (3.2.7) we have:

$$\int_{C_1} G_{\mathrm{wt}b_1,i}(w_1,z)w_1^{i_1}dw_1 = \mathrm{Res}_{w_1=0}\Big(-\sum_{j\geq 0}\binom{\mathrm{wt}b_1-2-j}{i}z^{\mathrm{wt}b_1-j-2-i}w_1^{-\mathrm{wt}b_1+1+j+i_1}\Big)$$
$$= -\binom{i_1}{i}z^{i_1-i}. \tag{3.2.20}$$

for $z = z_k$, w or w_l. Apply (3.2.20) to (2), (3), and (4), and we have:

$$(2) = -\int_{C_m}\cdots\int_{C_2}\sum_{k=1}^{n}\sum_{i\geq 0}\binom{i_1}{i}z_k^{i_1-i}S(v_3',\ldots(b_1(i)a_k,z_k)\ldots(v,w)(b_m,w_m)\ldots(b_2,w_2)v_2)$$
$$= -\sum_{k=1}^{n}\sum_{i\geq 0}\binom{i_1}{i}z_k^{i_1-i}S(v_3',(a_1,z_1)\ldots(b_1(i)a_k,z_k)\ldots(a_n,z_n)(v,w)y),$$

where $y = (b_m,i_m)\ldots(b_2,i_2)v_2$. Note that $\deg y = \deg x - (\mathrm{wt}b_1 - i_1 - 1) < 0$, and the length of y is $m-1$, then by the induction hypothesis we have (2) = 0. Similarly, (3) = 0.

$$(4) = \int_{C_m}\cdots\int_{C_1}\sum_{l=2}^{m}\sum_{i\geq 0}\binom{i_1}{i}w_l^{i_1-i}S(v_3',\ldots(v,w)\ldots(b_1(i)b_l,w_l)\ldots v_2)w_m^{i_m}\ldots w_1^{i_1}dw_1\ldots dw_m$$
$$= \sum_{l=2}^{m}\sum_{i\geq 0}\binom{i_1}{i}S(v_3',(a_1,z_1)\ldots(a_n,z_n)(v,w)y_l),$$

where $y_l = (b_m,i_m)\ldots(b_1(i)b_l, i_1+i_l-i)\ldots(b_2,i_2)v_2$. Note that

$$\deg(b_1(i)b_l, i_1+i_l-i) = \mathrm{wt}b_1 + \mathrm{wt}b_l - i - 1 - i_1 - i_l + i - 1 = \deg(b_1,i_1) + \deg(b_l,i_l).$$

Thus, $\deg y_l = \sum_{k=1}^{m}\mathrm{wt}(b_k,i_k) = \deg x < 0$, and the length of y_l is $m-1$ for each l. Hence (4) = 0 by the induction hypothesis.

Case 3. $\mathrm{wt}b_1 - i_1 - 1 = 0$.

In this case, we have: $\int_{C_1} G_{\mathrm{wt}b_1,i}(w_1,z)w_1^{i_1}dw_1 = 0$ in view of (3.2.19). Hence (2) = (3) = (4) = 0. Moreover, since $-\mathrm{wt}b_1 + i_1 = -1$, we have:

$$(1) = \int_{C_m}\cdots\int_{C_2} S(v_3',(a_1,z_1)\ldots(v,w)(b_m,w_m)\ldots o(b_1)v_2)w_m^{i_m}\ldots w_2^{i_2}dw_2\ldots dw_m$$
$$= S(v_3',(a_1,z_1)\ldots(a_n,z_n)(v,w)y),$$

where $y = (b_m,i_m)\ldots(b_2,i_2)v_2$. Since $\deg y = \deg x < 0$, and the length of y is $m-1$, we have (1) = 0 by the induction hypothesis. Now the proof of (c) is complete. □

Remark 3.2.6. Lemma 3.2.4 and 3.2.5 indicate that the properties of $\mathrm{Rad}(\bar{M})$ are actually encoded by the recursive formulas in Definition 3.2.1. As we will see in the next Section, the V-modules M^2 and $(M^3)'$ can be characterized by the correlation functions on the bottom levels $M^2(0)$ and $M^3(0)^*$, together with the recursive formulas (3.2.4) and (3.2.6).

3.3 Extension from the bottom levels

Using the properties we proved in the previous Section, we will show that the restriction map φ in (3.2.9) has an inverse for certain V-modules M^2 and M^3, with irreducible bottom levels $M^2(0)$ and $M^3(0)$, respectively. Then for such V-modules M^2 and M^3, we have isomorphisms $I\binom{M^3}{M^1\ M^2} \cong \mathrm{Cor}\binom{M^3}{M^1\ M^2} \cong \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. Moreover, the second isomorphism holds automatically if V is a rational VOA, and M^2 and M^3 are irreducible V-modules. So we can compute the fusion rules of rational VOAs by determining the dimension of spaces of correlation functions $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$ or $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$.

3.3.1 The V-modules constructed from bottom levels and correlation functions

Choose an element S in $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, then S is a system of multi-linear maps:

$$S : M^3(0)^* \times V \times \cdots \times M^1 \times \cdots \times V \times M^2(0) \to \mathcal{F}(z_1, \ldots, z_n, w) \tag{3.3.1}$$

We will extend the first and the last input vector spaces from $M^3(0)^*$ and $M^2(0)$ to some V-modules $\tilde{M}/\mathrm{Rad}\tilde{M}$ and $\bar{M}/\mathrm{Rad}\bar{M}$, which are certain quotient modules of the generalized Verma modules $\bar{M}(M^3(0)^*)$ and $\bar{M}(M^2(0))$, respectively.

We first extend $M^2(0)$, and we will proceed like the proof of Theorem 2.2.1 in [73]. In our case, however, the extended V-module is **not** necessarily irreducible like the extended module in Theorem 2.2.1 [73].

Define a vertex operator $Y_{\bar{M}^2}$ on the quotient space $\bar{M}^2 = \bar{M}/\mathrm{Rad}(\bar{M})$ as follows:

$$Y_{\bar{M}^2}(a, z)(b_1, i_1) \ldots (b_m, i_m)v_2 := \sum_{n\in\mathbb{Z}} (a, n)(b_1, i_1) \ldots (b_m, i_m)v_2 z^{-n-1}, \tag{3.3.2}$$

where $a \in V$, $(b_1, i_1) \ldots (b_m, i_m)v_2 \in \bar{M}^2$, and we use the same notation $(b_1, i_1) \ldots (b_m, i_m)v_2$ for its image in the quotient space $\bar{M}^2$. We can express (3.3.2) in the component form:

$$a(n)(b_1, i_1) \ldots (b_m, i_m)v_2 = (a, n)(b_1, i_1) \ldots (b_m, i_m)v_2, \tag{3.3.3}$$

for all $a \in V$, $n \in \mathbb{Z}$, and $(b_1, i_1)\dots(b_m, i_m)v_2 \in \bar{M}$.

Proposition 3.3.1. *$\bar{M}^2 = \bar{M}/\mathrm{Rad}(\bar{M})$, together with $Y_{\bar{M}^2} : V \to \mathrm{End}(\bar{M}^2)[[z, z^{-1}]]$ given by* (3.3.2) *and* (3.3.3)*, is a weak V-module.*

Proof. By (a) of Lemma 3.2.5, we have $a(n)\mathrm{Rad}(\bar{M}) \subseteq \mathrm{Rad}(\bar{M})$. Hence $Y_{\bar{M}^2}$ is well-defined. Let $x = (b_1, i_1)\dots(b_m, i_m)v_2 \in \bar{M}^2$, we claim that $\mathbf{1}(-1)x = x$ and $\mathbf{1}(n)x = 0$ for any $n \neq -1$. Indeed, for any $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, by the definition formula (3.2.12), the recursive formula (3.2.6), together with the fact that $\mathbf{1}(j)a = 0$ for all $j \geq 0$, $a \in V$, and $\mathbf{1}(j)v = 0$ for all $j \geq 0$, $v \in M^1$, we have:

$$\begin{aligned} &S(v_3', (a_1, z_1)\dots(v, w)\mathbf{1}(n)x) \\ &= \int_{C_0}\int_{C_m}\cdots\int_{C_1} S(v_3', (\mathbf{1}, w_0)(a_1, z_1)\dots(v, w)(b_1, w_1)\dots v_2)w_0^n w_1^{i_1}\dots w_m^{i_m} dw_1\dots dw_m dw_0 \\ &= \int_{C_0}\int_{C_m}\cdots\int_{C_1} S(v_3' o(\mathbf{1}), (a_1, z_1)\dots(v, w)(b_1, w_1)\dots v_2)w_0^n w_1^{i_1}\dots w_m^{i_m} dw_1\dots dw_m dw_0 \\ &= \delta_{n+1,0}\cdot S(v_3', (a_1, z_1)\dots(v, w)x), \end{aligned}$$

where the last equality follows from the fact that $\int_{C_0} w_0^n dw_0 = \delta_{n+1,0}$. Thus, $(\mathbf{1}(n)x - \delta_{n+1,0}x) \in \mathrm{Rad}(\bar{M})$, and so $\mathbf{1}(n)x = \delta_{n+1,0}x$ in $\bar{M}^2$. Moreover, given homogeneous elements $x \in \bar{M}$ and $a \in V$, by (3.2.13) and (3.3.3), $\deg(a(n).x) = \mathrm{wt} a - n - 1 + \deg x < 0$ when $n >> 0$. Then by part (c) of Lemma 3.2.5, we have $a(n)x = 0$ in $\bar{M}^2$ when n is large enough. Finally, by Lemma 3.2.4 and (3.3.3), $(\bar{M}^2, Y_{\bar{M}^2})$ satisfies the Jacobi identity. Hence it is a weak V-module. □

Proposition 3.3.2. *$\bar{M}^2$ has a gradation $\bar{M}^2 = \bigoplus_{n=0}^{\infty} \bar{M}^2(n)$, where $\bar{M}^2(n)$ is an eigenspace of $L(0)$ of eigenvalue $\lambda + n$ for each $n \in \mathbb{N}$, and $\bar{M}^2(0) = M^2(0)$. In particular, $\bar{M}^2$ is an ordinary V-module, and if $M^2(0)$ is the bottom level of some ordinary V-module M^2, with conformal weight h_2, then $\lambda = h_2$.*

Proof. Let $\bar{M}^2(n)$ be the image of $\bar{M}(n)$ under the quotient map $\bar{M} \to \bar{M}^2$. By Lemma 3.2.5, we have $\bar{M}^2 = \sum_{n\geq 0} \bar{M}^2(n)$ and $M^2(0) \subseteq \bar{M}^2(0)$. We claim that

$$a(\mathrm{wt} a - 1)v_2 = o(a)v_2, \tag{3.3.4}$$

for all $v_2 \in M^2(0)$ and homogeneous $a \in V$. Indeed, we only need to show that $(a, \mathrm{wt} a - 1)v_2 - o(a)v_2 \in \mathrm{Rad}(S)$, for all $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. By (3.2.12) and (3.2.6),

$$S(v_3', (a_1, z_1)\dots(a_n, z_n)(v, w)(a, \mathrm{wt} a - 1)v_2)$$

$$= \int_C S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)(a, w_1)v_2) w_1^{\mathrm{wt} a - 1} dw_1$$
$$= \int_C S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w) o(a) v_2) w_1^{-\mathrm{wt} a} w_1^{\mathrm{wt} a - 1} dw_1$$
$$+ \sum_{k=1}^{n} \sum_{i \geq 0} \int_C G_{\mathrm{wt} a, i}(w_1, z_k) S(v_3', (a_1, z_1) \dots (a(i)a_k, z_k) \dots (a_n, z_n)(v, w)v_2) w_1^{\mathrm{wt} a - 1} dw_1$$
$$+ \sum_{i \geq 0} \int_C G_{\mathrm{wt} a, i}(w_1, w) S(v_3', (a_1, z_1) \dots (a_n, z_n)(a(i)v, w)v_2) w_1^{\mathrm{wt} a - 1} dw_1,$$

where C is a contour of w_1 surrounding 0, with all other variables lying outside of C. Since $|z_k|, |w| > |w_1|$ for all k, where w_1 is lying on C, then we have

$$\int_C G_{\mathrm{wt} a, i}(w_1, z) w_1^{\mathrm{wt} a - 1} dw_1 = \int_C w_1^{\mathrm{wt} a - 1} \frac{w_1^{-\mathrm{wt} a + 1}}{i!} \left(\frac{d}{dz}\right)^i \left(\frac{z^{\mathrm{wt} a - 1}}{w_1 - z}\right) dw_1 = 0,$$

for $z = z_k$ or w. Hence $(a, \mathrm{wt} a - 1)v_2 - o(a)v_2 \in \mathrm{Rad}(S)$. This shows (3.3.4).

Since $L(0) = \omega(\mathrm{wt}\omega - 1)$ on $\bar{M}^2$, it follows from (3.3.4) that $L(0)$ preserves $M^2(0)$. On the other hand, we have $[L(0), a(n)] = (\mathrm{wt} a - n - 1)a(n)$ (see (4.2.2) in [27]). Then by (3.3.4) again, we have $[L(0), o(a)]v_2 = [L(0), a(\mathrm{wt} a - 1)]v_2 = 0$. Since $M^2(0)$ is an irreducible $A(V)$-module which is of countable dimension, then by the Schur's Lemma (Lemma 1.2.1 in [73]), there exists $\lambda \in \mathbb{C}$ such that $L(0) = \lambda \cdot \mathrm{Id}$ on $M^2(0)$. If $M^2(0)$ is the bottom level of M^2, with conformal weight h_2, then $L(0) = h_2 \cdot \mathrm{Id}$ on $M^2(0)$, and so $h_2 = \lambda$.

Now for any spanning element $x = (b_1, i_1) \dots (b_m, i_m)v_2 = b_1(i_1) \dots b_m(i_m)v_2$ of $\bar{M}^2(n)$, we have $L(0)x = (\sum_{k=1}^{m}(\mathrm{wt} b_k - i_k - 1) + \lambda)x = (n + \lambda)x$. Therefore, $\bar{M}^2(n)$ is an eigenspace of $L(0)$ of eigenvalue $n + \lambda$ for every $n \in \mathbb{N}$, and $\bar{M}^2 = \bigoplus_{n=0}^{\infty} \bar{M}^2(n)$.

Finally, for any spanning element $x = b_1(i_1) \dots b_m(i_m)v_2$ of $\bar{M}^2(0)$, it follows from (3.3.4) and an easy induction that $x \in M^2(0)$, therefore $\bar{M}^2(0) = M^2(0)$. □

Remark 3.3.3. Unlike the construction of V-modules from the correlation functions in Theorem 2.2.1 in [73], in our case, it is unclear whether $\bar{M}^2 = \bar{M}/\mathrm{Rad}(\bar{M})$ is an irreducible V-module. The reason is the following:

Assume $N \leq \bar{M}^2$ is a submodule, by Proposition 3.3.2 we have $N = \bigoplus_{n=0}^{\infty} N(n)$, with $N(n) = N \cap \bar{M}^2(n)$ for each n. If $N(0) \neq 0$, then clearly $N = \bar{M}^2$. Thus, in order to show that $\bar{M}^2$ is irreducible, we need to show that $N = 0$ when $N(0) = 0$.

This is true for the module $\bar{M}/\mathrm{Rad}(\bar{M})$ constructed in Theorem 2.2.1 in [73], wherein the correlation function $S(v', (a_1, z_1) \dots (a_n, z_n)N)$, with $v' \in M^2(0)$, is essentially the limit function of $\langle v', Y(a_1, z_1) \dots Y(a_n, z_n)N\rangle$. It is zero because $Y(a, z)N \subset N((z))$, and the bottom level of N is 0. Thus, $N \subseteq \mathrm{Rad}(S)$, and so $N = 0$ in $\bar{M}/\mathrm{Rad}(\bar{M})$. However, in our case, $S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)N)$ with $v_3' \in M^3(0)^*$ is essentially the limit function of $\langle v_3', I(v, w)Y(a_1, z_1) \dots Y(a_n, z_n)N\rangle w^h$. Although the components of $Y(a, z)$ still leave N invariant, the intertwining operator $I(v, w)$ could send some element in N to a nonzero element of $M^3(0)$. Hence we cannot conclude that $N \subseteq \mathrm{Rad}(\bar{M})$ in general.

We give a sufficient condition under which $\bar{M}^2$ is irreducible.

Lemma 3.3.4. *Let* $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. *Suppose* S *satisfies the condition that*

$$\sum_{i\geq 0} \binom{n}{i} \langle v_3', f_{b(i)v}(v_2)\rangle = 0, \tag{3.3.5}$$

for all $b \in V$, $n \in \mathbb{Z}$ *such that* $\mathrm{wt} b - n - 1 > 0$, $v \in M^1$, $v_3' \in M^3(0)^*$, *and* $v_2 \in M^2(0)$. *Then* $S(v_3', (v, w)y) = 0$ *for any* $y \in M(m)$ *with* $m \geq 1$, $v_3' \in M^3(0)^*$, *and* $v \in M^1$.

Proof. It follows from an easy induction that y can be written as a sum of the elements of the form $(b_m, n_m) \dots (b_1, n_1)v_2$, where $m \geq 1$, $v_2 \in M^2(0)$, and $\mathrm{wt} b_j - n_j - 1 > 0$ for all j.

Let $y = (b_m, n_m) \dots (b_1, n_1)v_2$. We use induction on m to show that $S(v_3', (v, w)y) = 0$. For the base case $m = 1$ and $y = (b, n)v_2$, with $\mathrm{wt} b - n - 1 > 0$, by (3.2.12), (3.2.3), (3.2.6), (3.2.7), and the assumption (3.3.5), we have:

$$\begin{aligned}
S(v_3', (v, w)y) &= \int_C S(v_3', (v, w)(b, z)v_2)z^n dz \\
&= \int_C S(v_3', (v, w)o(b)v_2)z^{-\mathrm{wt} b+n} dz + \int_C \sum_{i\geq 0} G_{\mathrm{wt} b, i}(z, w)S(v_3', (b(i)v, w)v_2)z^n dz \\
&= 0 + \sum_{i\geq 0} \int_C -\sum_{j\geq 0} \binom{\mathrm{wt} b - 2 - j}{i} w^{\mathrm{wt} b - j - 2 - i} z^{n - \mathrm{wt} b + 1 + j} S(v_3', (b(i)v, w)v_2) dz \\
&= -\sum_{i\geq 0} \binom{n}{i} \langle v_3', f_{b(i)v} v_2\rangle w^{-\mathrm{wt} b - \deg v + 1 + n} = 0.
\end{aligned}$$

Now let $m > 1$. Then by (3.2.12) and (3.2.6), we have

$$\begin{aligned}
S(v_3', (v, w)y) &= \int_{C_m} \cdots \int_{C_1} S(v_3', (v, w)(b_m, z_m) \dots (b_1, z_1)v_2) z_1^{n_1} \dots z_m^{n_m} dz_1 \dots dz_m \\
&= \int_{C_m} \cdots \int_{C_1} S(v_3', (v, w)(b_m, z_m) \dots (b_2, z_2)o(b_1)v_2) z_1^{-\mathrm{wt} b_1 + n_1} \dots z_m^{n_m} dz_1 \dots dz_m
\end{aligned}$$

$$
\begin{aligned}
&+\int_{C_m}\cdots\int_{C_1}\sum_{i\geq 0}G_{\mathrm{wt}b_1,i}(z_1,w)S(v_3',(b_1(i)v,w)(b_m,z_m)\dots(b_2,z_2)v_2)z_1^{n_1}\dots z_m^{n_m}dz_1\dots dz_m\\
&+\int_{C_m}\cdots\int_{C_1}\sum_{k=2}^{m}\sum_{i\geq 0}G_{\mathrm{wt}b_1,i}(z_1,z_k)S(v_3',(v,w)\dots(b_1(i)b_k,z_k)\dots(b_2,z_2)v_2)z_1^{n_1}\dots z_m^{n_m}dz_1\dots dz_m\\
&=0+\int_{C_m}\cdots\int_{C_2}\sum_{i\geq 0}\int_{C_1}-\sum_{j\geq 0}\binom{\mathrm{wt}b_1-2-j}{i}w^{\mathrm{wt}b_1-j-2-i}z_1^{n_1-\mathrm{wt}b_1+1+j}\\
&\qquad\cdot S(v_3',(b_1(i)v,w)(b_m,z_m)\dots(b_2,z_2)v_2)z_2^{n_2}\dots z_m^{n_m}dz_2\dots dz_m\\
&+\int_{C_m}\cdots\int_{C_2}\sum_{k=2}^{m}\sum_{i\geq 0}\int_{C_1}-\sum_{j\geq 0}\binom{\mathrm{wt}b_1-2-j}{i}z_k^{n_k+\mathrm{wt}b_1-j-2-i}z_1^{n_1-\mathrm{wt}b_1+1+j}\\
&\qquad\cdot S(v_3',(v,w)(b_m,z_m)\dots(b_1(i)b_k,z_k)\dots(b_2,z_2)v_2)z_2^{n_2}\dots\widehat{z_k^{n_k}}\dots z_m^{n_m}dz_2\dots dz_m\\
&=-\int_{C_m}\cdots\int_{C_2}\sum_{i\geq 0}\binom{n_1}{i}w^{n_1-i}S(v_3',(b_1(i)v,w)(b_m,z_m)\dots(b_2,z_2)v_2)z_2^{n_2}\dots z_k^{n_k}dz_2\dots dz_m\\
&-\int_{C_m}\cdots\int_{C_2}\sum_{k=2}^{m}\sum_{i\geq 0}\binom{n_1}{i}S(v_3',(v,w)\dots(b_1(i)b_k,z_k)\dots v_2)z_2^{n_2}\dots z_k^{n_1+n_k-i}\dots z_m^{n_m}dz_2\dots dz_m\\
&=-\sum_{i\geq 0}\binom{n_1}{i}w^{n_1-i}S(v_3',(b_1(i)v,w)(b_m,n_m)\dots(b_2,n_2)v_2)\\
&-\sum_{k=2}^{m}\sum_{i\geq 0}\binom{n_1}{i}S(v_3',(v,w)(b_m,n_m)\dots(b_1(i)b_k,n_1+n_k-i)\dots(b_2,n_2)v_2)\\
&=0,
\end{aligned}
$$

where the last equality follows from the induction hypothesis, together with $\deg(b_1(i)b_k, n_1 + n_k - i) = \mathrm{wt}b_1 - n_1 - 1 + \mathrm{wt}b_k - n_k - 1 > 0$, for any $i \geq 0$. □

Corollary 3.3.5. *For any fixed $v \in M^1$ and $y \in \bar{M}^2 = \bar{M}/\mathrm{Rad}(\bar{M})$, let $n \in \mathbb{Z}$ be an integer such that $n > \deg v + \deg y - 1$. Then we have*

$$
\int_C S(v_3', (v, w)y)w^n dw = 0, \tag{3.3.6}
$$

for all $v_3' \in M^3(0)$, where C is a contour of w surrounding 0. In particular, for fixed $v \in M^1$ and $y \in \bar{M}^2$, the power series expansion of $S(v_3', (v, w)y)$ has a uniform lower bound for w independent of $v_3' \in M^3(0)^*$.

Proof. It suffices to show (3.3.6) for $y = (b_m, n_m)\dots(b_1, n_1)v_2$, where $v_2 \in M^2(0)$, $m \geq 0$, and $\mathrm{wt}b_j - n_j - 1 > 0$ for all j. Again, we use induction on m. When $m = 0$, we have $y = v_2$

and $\deg y = 0$. Then by (3.2.3) and $-\deg v + n > -1$, we have: $\int_C S(v_3', (v,w)v_2)w^n dw = \int_C \langle v_3', f_v(v_2)\rangle w^{-\deg v+n} dw = 0$. Now let $m > 0$, and let $n \in \mathbb{Z}$ be such that $n > \deg v + \deg y - 1$. Since $-\mathrm{wt} b_1 + n_1 < -1$, by the calculations in Lemma 3.3.4, we have:

$$\int_C S(v_3', (v,w)y)w^n dw = -\sum_{i\geq 0}\int_C \binom{n_1}{i} w^{n+n_1-i} \underbrace{S(v_3', (b_1(i)v, w)(b_m, n_m)\dots(b_2, n_2)v_2)dw}_{(1)}$$
$$-\sum_{k=2}^{m}\sum_{i\geq 0}\int_C \binom{n_1}{i} w^n \underbrace{S(v_3', (v,w)(b_m, n_m)\dots(b_1(i)b_k, n_1+n_k-i)\dots(b_2,n_2)v_2)dw}_{(2)}$$
$$= (1) + (2).$$

Since $n > \deg v + \deg y - 1$, we have $n + n_1 - i > \deg(b_1(i)v) + \sum_{j=2}^{m}(\mathrm{wt} b_j - n_j - 1) - 1$ for all $i \geq 0$. Then by the induction hypothesis, $(1) = 0$ for all $v_3' \in M^3(0)^*$. On the other hand, since $\deg(b_1(i)b_k, n_1 + n_k - i) = \mathrm{wt} b_1 - n_1 - 1 + \mathrm{wt} b_k - n_k - 1$ for all $i \geq 0$, we have $(2) = 0$ for all $v_3' \in M^3(0)^*$. Thus $\int_C S(v_3', (v,w)y)w^n dw = 0$. □

Proposition 3.3.6. *Suppose every* $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ *satisfies the condition* (3.3.5)*, then* $\bar{M}^2 = \bar{M}/\mathrm{Rad}(\bar{M})$ *is an irreducible* V*-module with bottom level* $M^2(0)$*. In particular,* $\bar{M}^2$ *is isomorphic to* $L(M^2(0))$*, the unique irreducible* V*-module with bottom level* $M^2(0)$*.*

Proof. Note that for any $x \in M$, $S(v_3', (a_1, z_1)\dots(a_n, z_n)(v,w)x)$ is also a rational function in $z_1, \dots, z_n, w$ by (3.2.12) and (3.3.3), and it has Laurent series expansion:

$$S(v_3', (a_1,z_1)\dots(a_n,z_n)(v,w)x) = S(v_3', (v,w)(a_1,z_1)\dots(a_n,z_n)x)$$
$$= \sum_{i_1,\dots,i_n\in\mathbb{Z}}\left(\int_{C_n}\cdots\int_{C_1} S(v_3', (v,w)(a_n,z_n)\dots(a_1,z_1)x)z_1^{i_1}\dots z_n^{i_n}dz_1\dots dz_n\right)z_1^{-i_1-1}\dots z_n^{-i_n-1}$$
$$= \sum_{i_1,\dots,i_n\in\mathbb{Z}} S(v_3', (v,w)a_n(i_n)\dots a_1(i_1)x)z_1^{-i_1-1}\dots z_n^{-i_n-1} \tag{3.3.7}$$

on the domain $\mathbb{D} = \{(z_1, \dots, z_n, w) : |w| > |z_n| > \dots > |z_1| > 0\}$. Let N be a submodule of $\bar{M}^2$ such that $N(0) = 0$, we need to show that $N = 0$. Let $x \in N$, we have $y = a_n(i_n)\dots a_1(i_1)x \in N$, and if $y \neq 0$ then $\deg(y) > 0$. By Lemma 3.3.4, we have $S(v_3', (v,w)y) = 0$. Thus, the rational function $S(v_3', (a_1,z_1)\dots(a_n,z_n)(v,w)x)$ is equal to 0 by (3.3.7). i.e., $x \in \mathrm{Rad}(S)$ for all $S \in \mathrm{Cor}\binom{M^3}{M^1\ M^2}$. Thus $N = 0$. □

In conclusion, given a $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, the extended S in (3.2.12) factors though an $\mathbb{N}$-gradable V-module $\bar{M}^2 = \bar{M}/\mathrm{Rad}(\bar{M})$ whose bottom level is $M^2(0)$. It is irreducible if the condition (3.3.5) is satisfied. Therefore, by (3.2.12) and (3.3.3), we have a well-defined system of $(n+3)$-point correlation functions:

$$\begin{aligned}
&S : M^3(0)^* \times V \times \cdots \times M^1 \times \cdots \times V \times \bar{M}^2 \to \mathcal{F}(z_1, \ldots, z_n, w),\\
&S(v_3', (a_1, z_1)\ldots(a_n, z_n)(v, w)b_1(i_1)\ldots b_m(i_m)v_2) \qquad (3.3.8)\\
&= \int_{C_1}\cdots\int_{C_m} S(v_3', (a_1, z_1)\ldots(a_n, z_n)(v, w)(b_1, w_1)\ldots(b_m, w_m)v_2)w_1^{i_1}\ldots w_m^{i_m}dw_1\ldots dw_m,
\end{aligned}$$

for all $b_1(i_1)\ldots b_m(i_m)v_2 \in \bar{M}^2$, where C_k is a contour of w_k, C_k contains C_{k+1} for all k, C_m contains 0, and $z_1, \ldots, z_n, w$ are outside of C_1.

In particular, S in (3.3.8) satisfies the generating formula (3.1.12) with $M^2 = \bar{M}^2$, since the extended S is defined by this formula. Moreover, by Corollary 3.3.5 and the fact that the orginal S in (3.3.1) belongs to $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, it is easy to see that the S in (3.3.8) also satisfies the properties (1) − (6) in Definition 3.1.1, with $v_3' \in M^3(0)^*$ and $v_2 \in \bar{M}^2$.

We adopt a similar method to extend the first input component of S in (3.3.8) from $M^3(0)^*$ to a V-module by using the other generating formula (3.1.13). First, we let

$$\tilde{M} := T(\mathcal{L}(V)) \otimes_{\mathbb{C}} M^3(0)^*.$$

Then $\tilde{M}$ is spanned by elements of the form: $y = (b_1, i_1)\ldots(b_m, i_m)v_3'$, where $b_j \in V$, $i_j \in \mathbb{Z}$ for $j = 1, \ldots, m$, and $v_3' \in M^3(0)^*$. Next, we extend S in (3.3.8) by iterating the generating formula (3.1.13). i.e., we define:

$$\begin{aligned}
&S : \tilde{M} \times V \times \cdots \times M^1 \times \cdots \times V \times \bar{M}^2 \to \mathcal{F}(z_1, \ldots, z_n, w)\\
&S((b_1, i_1)\ldots(b_m, i_m)v_3', (a_1, z_1)\ldots(a_n, z_n)(v, w)x_2) \qquad (3.3.9)\\
&:= \int_{C_1}\cdots\int_{C_m} S(v_3', (b_m, w_m)'\ldots(b_1, w_1)'(a_1, z_1)\ldots(v, w)x_2)w_1^{-i_1-2}\ldots w_m^{-i_m-2}dw_m\ldots dw_1,
\end{aligned}$$

where $(b, w)' = (e^{w^{-1}L(1)}(-w^2)^{L(0)}b, w)$, C_k is a contour of w_k s.t. C_k contains C_{k-1} for each k, and C_1 contains all the variables $z_1, \ldots, z_n, w$. For S in (3.3.9), we similarly define

$$\mathrm{Rad}(S) := \{y \in \tilde{M} : S(y, (a_1, z_1)\ldots(a_n, z_n)(v, w)x) = 0, \forall a_i \in V, v \in M^1, x \in \bar{M}^2\},$$

and let $\mathrm{Rad}(\tilde{M}) := \bigcap \mathrm{Rad}(S)$, where the intersection is taken over all $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, with the extension given by (3.3.9). Clearly, S factors though $\tilde{M}/\mathrm{Rad}(\tilde{M})$.

Similar to our previous argument, one can show that $\bar{M}^{3\prime} = \tilde{M}/\mathrm{Rad}(\tilde{M})$ has a natural $\mathbb{N}$-gradable V-module structure $\bar{M}^{3\prime} = \bigoplus_{n=0}^{\infty} \bar{M}^{3\prime}(n)$, with $\bar{M}^{3\prime}(0) = M^3(0)^*$. Moreover, $\bar{M}^{3\prime} =$

$\tilde{M}/\mathrm{Rad}(\tilde{M})$ is irreducible if the condition 3.3.5 is satisfied. Thus we have a well-defined system of correlation functions S:

$$
\begin{aligned}
&S : \bar{M}^{3\prime} \times V \times \cdots \times M^1 \times \cdots \times V \times \bar{M}^2 \to \mathcal{F}(z_1, \ldots, z_n, w), \\
&S(b_1(i_1)\ldots b_m(i_m)v_3', (a_1, z_1)\ldots(a_n, z_n)(v, w)x_2) \qquad (3.3.10)\\
&= \int_{C_1} \cdots \int_{C_m} S(v_3', (b_m, w_m)' \ldots (b_1, w_1)'(a_1, z_1)\ldots(v, w)x_2) w_1^{-i_1-2} \ldots w_m^{-i_m-2} dw_m \ldots dw_1,
\end{aligned}
$$

for all $b_1(i_1)\ldots b_m(i_m)v_3' \in \bar{M}^{3\prime}$ and $x_2 \in \bar{M}^2$. Moreover, by Remark 3.1.4, we also have:

$$
\begin{aligned}
&S(b_1'(i_1)\ldots b_m'(i_m)v_3', (a_1, z_1)\ldots(a_n, z_n)(v, w)x_2) \qquad (3.3.11)\\
&= \int_{C_1} \cdots \int_{C_m} S(v_3', (b_m, w_m)\ldots(b_1, w_1)(a_1, z_1)\ldots(a_n, z_n)(v, w)x_2) w_1^{i_1} \ldots w_m^{i_m} dw_m \ldots dw_1,
\end{aligned}
$$

where $b'(i) = \sum_{j\geq 0} \frac{1}{j!}(-1)^{\mathrm{wt}b}(L(1)^j b)(2\mathrm{wt}b - i - j - 2)$, C_k is a contour of w_k such that C_k contains C_{k-1} for each k, and $z_1, \ldots, z_n, w$ are inside of C_1. Since (3.3.10) and (3.3.11) are given by iterating the generating formula (3.1.13), it is clear that S in (3.3.10) also satisfies (3.1.13) with $M^2 = \bar{M}^2$ and $(M^3)' = \bar{M}^{3\prime}$. Denote the contragredient module of $\bar{M}^{3\prime}$ by $\bar{M}^3$.

3.3.2 The correspondence between the space of correlation functions on the bottom levels and the space of intertwining operators

Theorem 3.3.7. *The system of extended correlation functions S in* (3.3.10) *lies in* $\mathrm{Cor}\binom{\bar{M}^3}{M^1\ \bar{M}^2}$. *Hence we have an isomorphism of vector spaces* $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)} \cong \mathrm{Cor}\binom{\bar{M}^3}{M^1\ \bar{M}^2} \cong I\binom{\bar{M}^3}{M^1\ \bar{M}^2}$.

Proof. We have already proven that S satisfies (7) and (8) in Definition 3.1.1, with $M^2 = \bar{M}^2$ and $(M^3)' = \bar{M}^{3\prime}$. It remains to show that S in (3.3.10) satisfies the properties (1) − (6) in Definition 3.1.1, with $M^2 = \bar{M}^2$ and $M^3 = \bar{M}^3$. In fact, by the definition formulas (3.3.8) and (3.3.11), together with the fact that the orginal S in (3.3.1) lies in $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, the properties (2) − (6) are straightforward. To prove (1), we need an intermediate result first. We introduce the following notation:

$$
\begin{aligned}
S(v_3', b_1(n_1)\ldots b_m(n_m)(v, w)x_2) := \int_{C_m} \cdots \int_{C_1} & S(v_3', (b_1, z_1)\ldots(b_m, z_m)(v, w)x_2) \qquad (3.3.12)\\
& \cdot z_1^{n_1} \ldots z_m^{n_m} dz_1 \ldots dz_m,
\end{aligned}
$$

where $m \geq 0$, $x_2 \in \bar{M}^2$, $b_k \in V$, $n_k \in \mathbb{Z}$, C_k is a contour of z_k s.t. C_k contains C_{k+1} for all k, and w is inside of C_m. Assume $\mathrm{wt} b_1 - n_1 - 1 < 0$. We claim that:

$$\begin{aligned}
&S(v_3', b_1(n_1)\dots b_m(n_m)(v,w)x_2) \\
&= \sum_{l=2}^{m}\sum_{i\geq 0}\binom{n_1}{i} S(v_3', b_2(n_2)\dots(b_1(i)b_l)(n_1+n_l-i)\dots b_m(n_m)(v,w)x_2) \\
&\quad + \sum_{i\geq 0}\binom{n_1}{i} S(v_3', b_2(n_2)\dots b_m(n_m)(b_1(i)v,w)x_2)w^{n_1-i} \\
&\quad + S(v_3', b_2(n_2)\dots b_m(n_m)(v,w)(b_1(n_1)x_2)).
\end{aligned} \tag{3.3.13}$$

Let $x_2 = c_1(i_1)\dots c_r(i_r)v_2$, for some $c_j \in V$, $i_j \in \mathbb{Z}$ for all j, and $v_2 \in M^2(0)$. Note that $b_1(n_1)v_2 = 0$ as $\mathrm{wt} b_1 - n_1 - 1 < 0$. For $|z_1| > |w|$, by (3.2.5) we have:

$$\int_{C_1} F_{\mathrm{wt} b_1, i}(z_1, w) z_1^{n_1} dz_1 = \sum_{j\geq 0}\int_{C_1}\binom{\mathrm{wt} b_1 + j}{i} z_1^{n_1 - \mathrm{wt} b_1 - j - 1} w^{\mathrm{wt} b_1 + j - i + i_t} dz_1 = \binom{n_1}{i} w^{n_1 - i},$$

where C_1 is a contour of z_1, with w lying inside. Then by (3.3.12), (3.3.8), the recursive formula (3.2.4), together with the fact that $-\mathrm{wt} b_1 + n_1 > -1$, we have:

$$\begin{aligned}
&S(v_3', b_1(n_1)\dots b_m(n_m)(v,w)x_2) \\
&= \int_{C_m}\cdots\int_{C_1}\sum_{l=2}^{m}\sum_{i\geq 0} F_{\mathrm{wt} b_1, i}(z_1, z_l) S(v_3', (b_2, z_2)\dots(b_1(i)b_l, z_l)\dots(v,w)x_2) z_1^{n_1}\dots z_m^{n_m} dz_1\dots dz_m \\
&\quad + \int_{C_m}\cdots\int_{C_1}\sum_{i\geq 0} F_{\mathrm{wt} b_1, i}(z_1, w) S(v_3', (b_2, z_2)\dots(b_m, z_m)(b_1(i)v, w)x_2) z_1^{n_1}\dots z_m^{n_m} dz_1\dots dz_m \\
&\quad + \int_{C_m}\cdots\int_{C_1}\Bigg(\int_{C_1'}\cdots\int_{C_r'}\sum_{i\geq 0} F_{\mathrm{wt} b_1, i}(z_1, w_t) S(v_3', (b_2, z_2)\dots(b_m, z_m)(v,w)(c_1, w_1)\dots \\
&\qquad (b_1(i)c_t, w_t)\dots(c_r, w_r)v_2)\cdot w_1^{i_1}\dots w_r^{i_r} dw_r\dots dw_1\Bigg) z_1^{n_1}\dots z_m^{n_m} dz_1\dots dz_m \\
&= \int_{C_m}\cdots\int_{C_2}\sum_{l=2}^{m}\sum_{i\geq 0}\binom{n_1}{i} S(v_3', (b_2, z_2)\dots(b_1(i)b_l, z_l)\dots(b_m, z_m)(v,w)x_2) \\
&\qquad \cdot z_2^{n_2}\dots z_l^{n_1 - i + n_l}\dots z_m^{n_m} dz_2\dots dz_m \\
&\quad + \int_{C_m}\cdots\int_{C_2}\sum_{i\geq 0}\binom{n_1}{i} S(v_3', (b_2, z_2)\dots(b_m, z_m)(b_1(i)v, w)x_2) w^{n_1 - i} z_2^{n_2}\dots z_m^{n_m} dz_2\dots dz_m \\
&\quad + \int_{C_m}\cdots\int_{C_2}\sum_{i\geq 0}\binom{n_1}{i} S(v_3', (b_2, z_2)\dots(v,w)\,(c_1(i_1)\dots(b_1(i)c_t)(n_1 - i + i_t)\dots c_r(i_r)v_2)) \\
&\qquad \cdot z_2^{n_2}\dots z_m^{n_m} dz_2\dots dz_m
\end{aligned}$$

$$= \sum_{l=2}^{m} \sum_{i \geq 0} \binom{n_1}{i} S(v_3', b_2(n_2) \ldots (b_1(i) b_l)(n_1 + n_l - i) \ldots b_m(n_m)(v, w) x_2)$$
$$+ \sum_{i \geq 0} \binom{n_1}{i} S(v_3', b_2(n_2) \ldots b_m(n_m)(b_1(i) v, w) x_2) w^{n_1 - i}$$
$$+ S(v_3', b_2(n_2) \ldots b_m(n_m)(v, w)(b_1(n_1) x_2)).$$

This proves (3.3.13). Now let $x_3' = b_m(n_m) \ldots b_1(n_1) v_3' \in \bar{M}^{3'}$, with $\mathrm{wt} b_i - n_i - 1 > 0$ for all i. We use induction on m to show that

$$\int_C S(b_m(n_m) \ldots b_1(n_1) v_3', (v, w) x_2) w^n dw = 0, \tag{3.3.14}$$

for any fixed $v \in M^1$, $x_2 \in \bar{M}^2$, and $n \in \mathbb{Z}$ such that $n > \deg v + \deg x_2 - 1$. The base case $m = 0$ follows from the Corollary 3.3.5. Let $m > 0$, then by (3.3.10) and (3.3.12), we have:

$$\int_C S(b_m(n_m) \ldots b_1(n_1) v_3', (v, w) x_2) w^n dw$$
$$= \int_C \int_{C_m} \cdots \int_{C_1} S(v_3', (b_1, z_1)' \ldots (b_m, z_m)'(v, w) x_2) z_1^{-n_1 - 2} \ldots z_m^{-n_m - 2} w^n dz_1 \ldots dz_m dw$$
$$= \sum_{j_1 \geq 0, \ldots, j_m \geq 0} \frac{(-1)^{\mathrm{wt} b_1 + \cdots + \mathrm{wt} b_m}}{j_1! \ldots j_m!} \int_C \int_{C_m} \cdots \int_{C_1} S(v_3', (L(1)^{j_1} b_1, z_1) \ldots (L(1)^{j_m} b_m, z_m)(v, w) x_2)$$
$$\cdot z_1^{2\mathrm{wt} b_1 - n_1 - 2 - j_1} \ldots z_m^{2\mathrm{wt} b_m - n_m - 2 - j_m} w^n dz_1 \ldots dz_m dw.$$
$$= \sum_{j_1 \geq 0, \ldots, j_m \geq 0} \frac{(-1)^{\mathrm{wt} b_1 + \cdots + \mathrm{wt} b_m}}{j_1! \ldots j_m!} \int_C S(v_3', (L(1)^{j_1} b_1)(2\mathrm{wt} b_1 - n_1 - 2 - j_1) \ldots \tag{3.3.15}$$
$$\ldots (L(1)^{j_m} b_m)(2\mathrm{wt} b_m - n_m - 2 - j_m)(v, w) x_2).$$

It suffices to show that each summand in (3.3.15) is 0. For simplicity, we denote the term $(L(1)^{j_i} b_i)(2\mathrm{wt} b_i - n_i - 2 - j_i)$ by $c_i(r_i)$ for each i, note that

$$\mathrm{wt} c_1(r_1) = \mathrm{wt}(L(1)^{j_1} b_1)(2\mathrm{wt} b_1 - n_1 - 2 - j_1) = -\mathrm{wt} b_1 + n_1 + 1 < 0.$$

Then by (3.3.13), together with the definition formulas (3.3.12) and (3.3.11), we have:

$$\int_C S(v_3', c_1(r_1) \ldots c_m(r_m)(v, w) x_2) w^n dw$$
$$= \sum_{l=2}^{m} \sum_{i \geq 0} \binom{r_1}{i} \int_C S(v_3', c_2(r_2) \ldots (c_1(i) c_l)(r_1 + r_l - i) \ldots c_m(r_m)(v, w) x_2) w^n dw$$
$$+ \sum_{i \geq 0} \binom{r_1}{i} \int_C S(v_3', c_2(r_2) \ldots c_m(r_m)(c_1(i) v, w) x_2) w^{n + r_1 - i} dw$$

$$
\begin{aligned}
&+ \int_C S(v_3', c_2(r_2)\dots c_m(r_m)(v,w)(c_1(r_1)x_2))w^n dw \\
&= \sum_{l=2}^{m}\sum_{i\geq 0}\binom{r_1}{i}\int_C \underset{(1)}{S(c_m'(r_m)\dots(c_1(i)c_l)'(r_1+r_l-i)\dots c_2'(r_2)v_3',(v,w)x_2)w^n dw} \\
&+ \sum_{i\geq 0}\binom{r_1}{i}\int_C \underset{(2)}{S(c_m'(r_m)\dots c_2'(r_2)v_3',(c_1(i)v,w)x_2)w^{n+r_1-i}dw} \\
&+ \int_C \underset{(3)}{S(c_m'(r_m)\dots c_2'(r_2)v_3',(v,w)(c_1(r_1)x_2))w^n dw} \\
&= (1)+(2)+(3).
\end{aligned}
$$

Since $\mathrm{wt}c_1 - r_1 - 1 < 0$ and $n > \deg v + \deg x_2 - 1$, we have

$$
\begin{aligned}
\deg(c_1(i)v) + \deg x_2 - 1 &= \deg v + \deg x_2 - 1 + \mathrm{wt}c_1 - i - 1 < n + r_1 - i, \\
\deg v + \deg(c_1(r_1)x_2) - 1 &= \deg v + \deg x_2 + \mathrm{wt}c_1 - r_1 - 1 - 1 < n,
\end{aligned}
$$

for all $i \geq 0$. Then by the induction hypothesis, we have $(1) = (2) = (3) = 0$. This finishes the proof of (3.3.14). Hence S in (3.3.10) belongs to $\mathrm{Cor}\binom{\bar{M}^3}{M^1\ \bar{M}^2}$. □

So far in this subsection, by abuse of notations, we used the same symbol S (3.3.10) for the extension of a system of correlation functions S in (3.3.1). We denote the extended S in (3.3.10) by $\psi(S)$ for the rest of this subsection. Then by the Theorem 3.3.7, we have a linear map:

$$
\psi : \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)} \to \mathrm{Cor}\binom{\bar{M}^2}{M^1\ \bar{M}^2}, \quad S \mapsto \psi(S), \tag{3.3.16}
$$

which is an inverse of the restriction map φ in (3.2.9), with $M^2 = \bar{M}^2$ and $M^3 = \bar{M}^3$.

Corollary 3.3.8. *Let* $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. *Then the linear functional* f *in Definition 3.2.1 is given by* $f_v = o(v) = v(\deg v - 1) = \mathrm{Res}_z I(z,w)w^{\deg v - 1 + h}$, *where* $I \in I\binom{\bar{M}^2}{M^1\ \bar{M}^2}$ *is the intertwining operator corresponds to* $\psi(S)$ *in* $\mathrm{Cor}\binom{\bar{M}^2}{M^1\ \bar{M}^2}$.

Proof. By (3.2.3), we have $S(v_3',(v,w)v_2) = \langle v_3', f_v(v_2)\rangle w^{-\deg v}$, for all $v_3' \in M^3(0)^*$, $v_2 \in M^2(0)$, and $v \in M^1$. On the other hand, by (3.1.19),

$$
S(v_3',(v,w)v_2) = \psi(S)(v_3',(v,w)v_2) = (v_3', I(v,w)v_2) = \langle v_3', v(\deg v - 1)v_2\rangle w^{-\deg v},
$$

since $v(m)M^2(0) \subseteq M^3(\deg v - m - 1)$ for any $m \in \mathbb{Z}$. Thus, $f_v = v(\deg v - 1)$. □

We finish this subsection by showing another property of the space of correlation functions associated with three modules. By (3.3.8) and (3.3.10), the $\psi(S)$ in (3.3.16) satisfies:

$$
\begin{aligned}
&\psi(S)(c_1(j_1)\dots c_m(j_m)v_3', (a_1,z_1)\dots(a_p,z_p)(v,w)b_1(i_1)\dots b_n(i_n)v_2)\\
&= \int_{C_1'}\cdots\int_{C_m'}\int_{C_n}\cdots\int_{C_1} S(v_3', (c_m,w_m)'\dots(c_1,w_1)'(a_1,z_1)\dots(v,w)(b_1,x_1)\dots(b_n,x_n)v_2)\\
&\quad\cdot x_1^{i_1}\dots x_n^{i_n}w_1^{-j_1-2}\dots w_m^{-j_m-2}dx_1\dots dx_n dw_m\dots dw_1, \qquad (3.3.17)
\end{aligned}
$$

where $v_3'\in M^3(0)^*$, $v_2\in M^2(0)$, $v\in M^1$, $a_r, b_s, c_t\in V$ for all r,s,t, C_k' is a contour of w_k, C_l is a contour of x_l for all k,l, such that $C_1\subset\cdots\subset C_n\subset C_1'\subset\cdots\subset C_m'$ (we use the subset symbol to indicate one contour is inside of the other), and $z_1,\dots,z_n,w$ are outside of C_1' but inside of C_n.

By Proposition 3.3.2 and Theorem 6.2 in [18], we have an epimorphism of V-modules $\pi:\bar{M}(M^2(0))\to\bar{M}^2$, where $\bar{M}(M^2(0))$ is the generalized Verma module with bottom level $M^2(0)$. Similarly, there is an epimorphism $\pi:\bar{M}(M^3(0)^*)\to\bar{M}^{3\prime}$. More generally, let N^2 and N^3 be any V-modules that are generated by their corresponding bottom levels, and assume that $N^2(0)=M^2(0)$ and $N^3(0)=M^3(0)$. Suppose there exist epimorphisms $\pi:N^2\to\bar{M}^2$ and $\pi:N^{3\prime}\to\bar{M}^{3\prime}$. If we write $\mathrm{Res}_z Y_N(b,z)z^j=b_j$ and $\mathrm{Res}_z Y_{\bar{M}}(b,z)z^j=b(j)$, then

$$
\pi(c^1_{j_1}\dots c^m_{j_m}v_3')=c^1(j_1)\dots c^m(j_m)v_3', \quad \text{and} \quad \pi(b^1_{i_1}\dots b^n_{i_n}v_2)=b^1(i_1)\dots b^n(i_n)v_2,
$$

where $c^k, b^l\in V$, $j_k, i_l\in\mathbb{Z}$ for all k,l, $v_3'\in M^3(0)^*$, and $v_2\in M^2(0)$. Thus, we have a linear map: $\pi^*:\mathrm{Cor}\binom{\bar{M}^3}{M^1\ \bar{M}^2}\to\mathrm{Cor}\binom{N^3}{M^1\ N^2}$ that is given by:

$$
\begin{aligned}
&\pi^*(S)(c^1_{j_1}\dots c^m_{j_m}v_3', (a_1,z_1)\dots(a_n,z_n)(v,w)b^1_{i_1}\dots b^n_{i_n}v_2) \qquad (3.3.18)\\
&= S(c^1(j_1)\dots c^m(j_m)v_3', (a_1,z_1)\dots(a_n,z_n)(v,w)b^1(i_1)\dots b^n(i_n)v_2).
\end{aligned}
$$

Compose ψ and π^*, we have a linear map $\pi^*\psi:\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}\to\mathrm{Cor}\binom{N^3}{M^1\ N^2}$. We claim that $\pi^*\psi$ is the inverse of the restriction map $\varphi:\mathrm{Cor}\binom{N^3}{M^1\ N^2}\to\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$.

Indeed, for $S\in\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, by (3.3.17) and (3.3.18), we have:

$$
\begin{aligned}
\varphi(\pi^*\psi)(S)(v_3', (a_1,z_1)\dots(a_n,z_n)(v,w)v_2) &= \psi(S)(\pi(v_3'), (a_1,z_1)\dots(a_n,z_n)(v,w)\pi(v_2))\\
&= S(v_3', (a_1,z_1)\dots(a_n,z_n)(v,w)v_2),
\end{aligned}
$$

where $v_2\in M^2(0)$ and $v_3'\in M^3(0)^*$. Hence $\varphi(\pi^*\psi)=1$. On the other hand, for $S\in\mathrm{Cor}\binom{N^3}{M^1\ N^2}$, again by (3.3.17) and (3.3.18), together with the fact that S satisfies (3.1.12) and (3.1.13), we have for any $c^1_{j_1}\dots c^m_{j_m}v_3'\in N^{3'}$, $b^1_{i_1}\dots b^n_{i_n}v_2\in N^2$, $a_1,\dots,a_n\in V$, and $v\in M^1$,

$$
\begin{aligned}
&(\pi^*\psi)\varphi(S)(c^1_{j_1}\dots c^m_{j_m}v_3',(a_1,z_1)\dots(a_n,z_n)(v,w)b^1_{i_1}\dots b^n_{i_n}v_2)\\
&=\psi(\varphi(S))(c^1(j_1)\dots c^m(j_m)v_3',(a_1,z_1)\dots(a_n,z_n)(v,w)b^1(i_1)\dots b^n(i_n)v_2)\\
&=\int_{C_1'}\cdots\int_{C_m'}\int_{C_n}\cdots\int_{C_1}\varphi(S)(v_3',(c^m,w_m)'\dots(c^1,w_1)'(a_1,z_1)\dots(v,w)(b^1,x_1)\dots(b^n,x_n)v_2)\\
&\quad\cdot x_1^{i_1}\dots x_n^{i_n}w_1^{-j_1-2}\dots w_m^{-j_m-2}dx_1\dots dx_ndw_m\dots dw_1,\\
&=\int_{C_1'}\cdots\int_{C_m'}\int_{C_n}\cdots\int_{C_1}S(v_3',(c^m,w_m)'\dots(c^1,w_1)'(a_1,z_1)\dots(v,w)(b^1,x_1)\dots(b^n,x_n)v_2)\\
&\quad\cdot x_1^{i_1}\dots x_n^{i_n}w_1^{-j_1-2}\dots w_m^{-j_m-2}dx_1\dots dx_ndw_m\dots dw_1,\\
&=S(c^1_{j_1}\dots c^m_{j_m}v_3',(a_1,z_1)\dots(a_n,z_n)(v,w)b^1_{i_1}\dots b^n_{i_n}v_2).
\end{aligned}
$$

This shows $(\pi^*\psi)\varphi=1$, and so we have $\mathrm{Cor}\binom{N^3}{M^1\ N^2}\cong\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. In particular, choose $N^2=\bar{M}(M^2(0))$ and $N^3=\bar{M}(M^3(0)^*)'$, then we have:

$$
\mathrm{Cor}\binom{\bar{M}(M^3(0)^*)'}{M^1\ \bar{M}(M^2(0))}\cong\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}\cong\mathrm{Cor}\binom{\bar{M}^3}{M^1\ \bar{M}^2}\tag{3.3.19}
$$

Now by (3.3.19), Corollary 3.1.6, and Theorem 3.3.7, we have the following theorem:

Theorem 3.3.9. *Let M^1 be a V-module, and let $M^2(0)$ and $M^3(0)$ be irreducible $A(V)$-modules, then we have the following isomorphism of vector spaces:*

$$
I\binom{\bar{M}(M^3(0)^*)'}{M^1\ \bar{M}(M^2(0))}\cong\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}\cong I\binom{\bar{M}^3}{M^1\ \bar{M}^2}.\tag{3.3.20}
$$

If the VOA V is rational, then the generalized Verma module $\bar{M}(U)$ is an irreducible V-module for any irreducible $A(V)$-module U. Thus, $\bar{M}(M^2(0))=\bar{M}^2=L(M^2(0))$, and $\bar{M}(M^3(0)^*)'=\bar{M}^3=L(M^3(0))$. On the other hand, by Theorem 2.2.2 in [73], if M^2 and M^3 are irreducible V-module, then $M^2(0)$ and $M^3(0)$ are irreducible $A(V)$-module.

Corollary 3.3.10. *Let V be an rational VOA, and let M^1, M^2, and M^3 be V-modules. Suppose M^2 and M^3 are irreducible, then we have* $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}\cong I\binom{M^3}{M^1\ M^2}$.

By the argument above, we also have the following easy observation, which is useful in the computation of fusion rules of general V-modules:

Remark 3.3.11. Let W^2 and W^3 be any $\mathbb{N}$-gradable V-module that are generated by their corresponding bottom levels, and assume that $W^2(0)=M^2(0)$ and $W^3(0)=M^3(0)$. Then there exist epimorphisms: $\pi : \bar{M}(M^2(0)) \to W^2$, and $\pi : \bar{M}(M^3(0)^*) \to W^{3'}$. Similar to (3.3.18), π induces a linear map: $\pi^* : \mathrm{Cor}\binom{W^3}{M^1\ W^2} \hookrightarrow \mathrm{Cor}\binom{\bar{M}(M^3(0)^*)'}{M^1\ \bar{M}(M^2(0))}$, which is injective since π are surjective. Then by Corollary 3.1.6, (3.3.19), and (3.3.20), we have the following estimate for the fusion rule:

$$\dim I\binom{W^3}{M^1\ W^2} \leq \dim \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}. \tag{3.3.21}$$

We will use this estimate in the next Chapter.

Chapter 4

$A(V)$-theory and correlation functions

In the last Chapter, we introduced two different notions of space of correlation functions. The first one was the space of correlation functions associated with V-modules M^1, M^2, and M^3, denoted by $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$ in 3.1. The second one was the space of correlation functions associated with V-module M^1, and $A(V)$-modules $M^2(0)$ and $M^3(0)$, denoted by $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ in 3.2. We proved that $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$ is isomorphic to $I\binom{M^3}{M^1\ M^2}$ as vector spaces, see Corollary 3.1.6, and $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ is isomorphic to $\mathrm{Cor}\binom{\bar{M}(M^3(0)^*)'}{M^1\ \bar{M}(M^2(0))}$, where $\bar{M}(M^2(0))$ and $\bar{M}(M^3(0)^*)$ are the generalized Verma module of V associated with $A(V)$-modules $M^2(0)$ and $M^3(0)^*$, respectively, see Corollary 3.3.8.

Let $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. By Definition 7.1.6, S is a system of $(n+3)$-point functions, with $n \in \mathbb{N}$, and the three-point function of S has the following form:

$$S(v_3', (v, w)v_2) = \langle v_3', f_v(v_2)\rangle w^{-\deg v}, \quad (f_v \in \mathrm{Hom}_{\mathbb{C}}(M^2(0), M^3(0)),\ \forall v \in M^1).$$

Our goal in this Chapter is to show that S can be uniquely determined by the coefficient $\langle v_3', f_v(v_2)\rangle$ of the three-point function of S, which can be viewed as the value of a linear functional f on a tensor product space $M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0)$, where $M^2(0)$ and $M^3(0)$ are left modules over $A(V)$, and $B_h(M^1)$ is an $A(V)$-bimodule, which is a quotient module of $A(M^1)$ in 2.2. Therefore, we can compute the fusion rule of certain modules over VOAs by computing the dimension of dual vector space: $\left(M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0)\right)^*$. This is the so-called "fusion rules Theorem" claimed by Frenkel and Zhu in [30]. In fact, if V is rational, we can show that $M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0)$ is isomorphic to $M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0)$ as vector spaces. Also, recall that any generalized Verma module associated with an irreducible

$A(V)$-module U is irreducible; see 3.2. Thus, the original version of the fusion rules Theorem in [30] is true for the rational VOAs.

In this Chapter, we again assume that $M^2(0)$ and $M^3(0)$ are irreducible $A(V)$-modules. By Proposition 3.3.2, $L(0) = o(\omega) = h_2 \cdot \mathrm{Id}$ on $M^2(0)$, and $L(0) = h_3 \cdot \mathrm{Id}$ on $M^3(0)$, for some $h_2, h_3 \in \mathbb{C}$. Moreover, h_2 and h_3 are the conformal weights of $\bar{M}^2$ and $\bar{M}^3$, respectively.

4.1 $A(V)$-bimodules and construction of correlation functions

We will construct a new $A(V)$-bimodule $B_\lambda(W)$ in this Section, where λ is an arbitrary complex number, and W is a V-module. To construct $B_\lambda(W)$, instead of mod out the terms given by left module actions Y_W as in (2.2.20), we also mod out the terms given by right modules actions Y_{WV}^W. Then the $L(-1)$-derivation property of an intertwining operator $I \in I\binom{M^3}{M^1\, M^2}$ is encoded within $B_h(M^1)$, where we choose $\lambda = h = h_1 + h_2 - h_3$.

Although $A(W)$ is not isomorphic to $B_\lambda(W)$ in general, we will show that $M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0) \cong M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0)$ when $A(V)$ is semisimple, which is guaranteed if V is rational, see [18, 73]. Finally, before proving the general fusion rules Theorem in the next two Sections, we will discuss some easy consequences of the fusion rules Theorem for rational VOAs, which are related to the tensor product of modules over VOAs.

4.1.1 The $A(V)$-bimodule $B_\lambda(W)$

Let W be a V-module with conformal weight h'. A sequence of $A_N(V)$-bimodules $A_N(W)$ was constructed by Huang and Yang in Section 4 of [42]. In particular, the $A_0(V) = A(V)$-bimodule $A_0(W)$ is defined as follows:

$A_0(W) = W/O_0(W)$, where $O_0(W) = \mathrm{span}\{a \circ u, L(-1)u + (L(0) - h')u : \forall a \in V, u \in W\}$. It is proved (see Theorem 4.7 in [42]) that $A_0(W)$ is an $A(V)$-bimodule under the left and right actions: $a *_0 u = \mathrm{Res}_z Y_W(a, z)u \frac{(1+z)^{\mathrm{wt}\, a}}{z}$ and $v *_0 a = \mathrm{Res}_z Y_{WV}^W(u, z)a \frac{(1+z)^{\deg u}}{z}$, where Y_{WV}^W is defined by the skew-symmetry formula (5.1.5) in [27]:

$$Y_{WV}^W(u, z)a = e^{zL(-1)} Y_W(a, -z)u. \tag{4.1.1}$$

Now let $\lambda \in \mathbb{C}$ be a fixed complex number, we construct another $A(V)$-bimodule $B_\lambda(W)$ that is similar to $A_0(W)$ in the following way:

Definition 4.1.1. For homogeneous elements $u \in W$ and $a \in V$, define:

$$u \circ_{\substack{W \\ WV}} a := \mathrm{Res}_z \left(Y_{WV}^{W}(u, z)a \frac{(1+z)^{\deg u + \lambda}}{z^2} \right), \tag{4.1.2}$$

then extend $\circ$ bilinearly to $\circ : W \times V \to W$. Let $O_{WV}^{W}(W)$ be the vector space spanned by elements (4.1.2) for all $a \in V$ and $u \in W$, and let $B_\lambda(W) := W/(O(W) + O_{WV}^{W}(W))$, where $O(W) = \mathrm{span}\{a \circ u = \mathrm{Res}_z \left(Y_W(a, z)u \frac{(1+z)^{\mathrm{wt} a}}{z^2} \right) : \forall a \in V, u \in W\}$.

Lemma 4.1.2. *Let $u \in W$ and $a \in V$ by homogeneous elements, and $m \geq n \geq 0$. Then*

$$\mathrm{Res}_z Y_{WV}^{W}(u, z)a \frac{(1+z)^{\deg u + \lambda + n}}{z^{2+m}} \in O_{WV}^{W}(W). \tag{4.1.3}$$

Proof. Since $Y_{WV}^{W}(L(-1)u, z) = \frac{d}{dz} Y_{WV}^{W}(u, z)$, the proof of (4.1.3) is almost the same as the proof of Lemma 2.1.2 in [73], we omit the details. □

Recall that the module actions of $A(V)$ on its bimodule $A(W)$ are given by:

$$b * v = \mathrm{Res}_z \left(Y_W(b, z)v \frac{(1+z)^{\mathrm{wt} b}}{z} \right), \quad \text{and} \quad v * b = \mathrm{Res}_z \left(Y_W(b, z)v \frac{(1+z)^{\mathrm{wt} b - 1}}{z} \right),$$

where $b \in V$ is homogeneous and $v \in W$ (see Definition 1.5.2 in [30]).

Lemma 4.1.3. *$b * O_{WV}^{W}(W) \subseteq O_{WV}^{W}(W)$ and $O_{WV}^{W}(W) * b \subseteq O_{WV}^{W}(W)$, for all $b \in V$.*

Proof. Let $u \in W$ and $b \in V$ be homogeneous, and let $a \in V$. By Definition 4.1.1, Lemma 4.1.2, and the Jacobi identity of the intertwining operator Y_{WV}^{W}, we have:

$$\begin{aligned}
b * (u \circ_{\substack{W \\ WV}} a) &\equiv \mathrm{Res}_{z_1} Y_W(b, z_1) \frac{(1+z_1)^{\mathrm{wt} b}}{z_1} \mathrm{Res}_{z_2} Y_{WV}^{W}(u, z_2)a \frac{(1+z_2)^{\deg u + \lambda}}{z_2^2} \\
&\quad - \mathrm{Res}_{z_2} Y_{WV}^{W}(u, z_2) \frac{(1+z_2)^{\deg u + \lambda}}{z_2^2} \mathrm{Res}_{z_1} Y_V(u, z_1)a \frac{(1+z_1)^{\mathrm{wt} b}}{z_1} \pmod{O_{WV}^{W}(W)} \\
&= \mathrm{Res}_{z_0} \mathrm{Res}_{z_2} Y_{WV}^{W}(Y_W(b, z_0), z_2)a \frac{(1+z_2+z_0)^{\mathrm{wt} b}}{z_2+z_0} \cdot \frac{(1+z_2)^{\deg u + \lambda}}{z_2^2} \\
&= \mathrm{Res}_{z_0} \mathrm{Res}_{z_2} \sum_{i=0}^{\mathrm{wt} b} \sum_{j \geq 0} Y_{WV}^{W}(Y_W(b, z_0)u, z_2)a \binom{\mathrm{wt} b}{i} (-1)^j z_0^{i+j} \frac{(1+z_2)^{\deg u + \lambda + \mathrm{wt} b - i}}{z_2^{2+j+1}} \\
&= \sum_{i=0}^{\mathrm{wt} b} \sum_{j \geq 0} \binom{\mathrm{wt} b}{i} \mathrm{Res}_{z_2} Y_{WV}^{W}(b_{i+j}u, z_2)a \frac{(1+z_2)^{\deg(b_{i+j}u) + \lambda + (j+1)}}{z_2^{2+(j+1)}} \\
&\equiv 0 \pmod{O_{WV}^{W}(W)},
\end{aligned}$$

where the last congruence follows from Lemma 4.1.2. By a similar computation, we have:

$$(u \circ_{WV}^{W} a) * b \equiv \sum_{i=0}^{\mathrm{wt} b-1} \sum_{j\geq 0} \binom{\mathrm{wt} b-1}{i} \mathrm{Res}_{z_2} Y_{WV}^{W}(b_{i+j}u, z_2) a \frac{(1+z_2)^{\deg(b_{i+j}u)+\lambda+j}}{z_2^{2+(j+1)}}$$
$$\equiv 0 \pmod{O_{WV}^{W}(W)}.$$

Hence we have $b * O_{WV}^{W}(W) \subseteq O_{WV}^{W}(W)$, and $O_{WV}^{W}(W) * b \subseteq O_{WV}^{W}(W)$. □

By Lemma 4.1.3 and Theorem 1.5.1 in [30], $B_\lambda(W) = W/(O(W) + O_{WV}^{W}(W))$ has an $A(V)$-bimodule structure with respect to $b * v$ and $v * b$. Moreover, $B_\lambda(W)$ is a quotient module of $A(W)$. In particular, we have the following formula holds in $B_\lambda(W)$:

$$a * u - u * a \equiv \sum_{j\geq 0} \binom{\mathrm{wt} a - 1}{j} a(j)u \pmod{O_{WV}^{W}(W) + O(W)}, \tag{4.1.4}$$

where $a \in V$ homogeneous, and $u \in W$. Let

$$O_\lambda(W) := \mathrm{span}\{a \circ u,\ L(-1)u + (L(0) - h' + \lambda)u : \forall a \in V, u \in W\} \subset W. \tag{4.1.5}$$

Lemma 4.1.4. *For any $u \in W$, we have $L(-1)u + (L(0) - h' + \lambda)u \in O_{WV}^{W}(W)$.*

Proof. Let $u \in W$ be homogeneous. Since $\deg u = (L(0) - h')u$, we have:

$$u \circ_{WV}^{W} \mathbf{1} = \mathrm{Res}_z e^{zL(-1)} Y_W(\mathbf{1}, -z) u \frac{(1+z)^{\deg u+\lambda}}{z^2} = \mathrm{Res}_z \sum_{j\geq 0} \frac{z^j}{j!} L(-1)^j \sum_{i=0}^{\deg u+\lambda} \binom{\deg u+\lambda}{i} z^{i-2}$$
$$= \binom{\deg u+\lambda}{0} L(-1)u + \binom{\deg u+\lambda}{1} L(-1)^0 u = (L(-1) + L(0) - h' + \lambda)u.$$

Hence $(L(-1) + (L(0) - h' + \lambda))u \in O_{WV}^{W}(W)$. □

Lemma 4.1.5. *We have $O(W) + O_{WV}^{W}(W) = O_\lambda(W)$. In particular, $B_\lambda(W) = W/O_\lambda(W)$.*

Proof. By Lemma 4.1.4, we only need to show that $O_{WV}^{W}(W) \subseteq O_\lambda(W)$. Similar to the proof of Lemma 2.1.3 in [73], for any homogeneous $u \in W$ and $a \in V$, we have: $Y_{WV}^{W}(u, z)a \equiv (1+z)^{-\deg u-\lambda-\mathrm{wt} a} Y_W\left(a, \frac{-z}{1+z}\right) u \pmod{O_\lambda(W)}$. It follows that

$$u \circ_{WV}^{W} a = \mathrm{Res}_z Y_{WV}^{W}(u, z) a \frac{(1+z)^{\deg u+\lambda}}{z^2} \equiv \mathrm{Res}_z Y_W\left(a, \frac{-z}{1+z}\right) u \frac{(1+z)^{-\mathrm{wt} a}}{z^2} \pmod{O_\lambda(W)}$$
$$\equiv -\mathrm{Res}_w Y_W(a, w) u \frac{(1+w)^{\mathrm{wt} a}}{w^2} \pmod{O_\lambda(W)}.$$

Hence $u \circ_{WV}^{W} a \equiv -a \circ u \pmod{O_\lambda(W)}$, and so $O_{WV}^{W}(W) + O(W) = O_\lambda(W)$. □

Now let $W = M^1$, and $\lambda = h = h_1 + h_2 - h_3$. Then by (4.1.5) and Lemma 4.1.5, $B_h(M^1) = M^1/O_h(M^1)$, where $O_h(M^1) = \text{span}\{a \circ u,\ L(-1)u + (L(0)+h_2-h_3)u : \forall a \in V, u \in M^1\}$.

Lemma 4.1.6. *Let $I \in I\binom{\bar{M^3}}{M^1\ \bar{M^2}}$, then the linear map*

$$o : M^1 \to \text{Hom}_{\mathbb{C}}(M^2(0), M^3(0)),\ o(v) = v(\deg v - 1) = \text{Res}_z I(v,z) z^{\deg v - 1 + h}$$

factors through $B_h(M^1) = M^1/O_h(M^1)$.

Proof. By Lemma 4.1.5, we need to show that $o(O_h(M^1)) = 0$. By Lemma 1.5.2 in [30], we already have $o(a \circ u) = 0$ for all $a \in V$ and $u \in M^1$. Furthermore, by the $L(-1)$-derivation property of I, we have:

$$\begin{aligned} o(L(-1)v) &= \text{Res}_z I(L(-1)v, z) z^{\deg v + 1 - 1 + h} = \text{Res}_z \left(\frac{d}{dz} I(v,z) \right) z^{\deg v + h} \\ &= \text{Res}_z I(v,z)(-\deg v - h) z^{\deg v + h - 1} = -((L(0) - h_1 + h)v)(\deg v - 1) \\ &= -o((L(0) + h_2 - h_3)v). \end{aligned}$$

Hence $o(O_h(M^1)) = 0$, and so o factors through $B_h(M^1)$. □

Proposition 4.1.7. *There exists an injective linear map:*

$$\begin{aligned} \nu : \text{Cor}\binom{M^3(0)}{M^1\ M^2(0)} &\to (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^* \qquad (4.1.6) \\ S \mapsto f_S, &\qquad f_S(v_3' \otimes v \otimes v_2) := \langle v_3', f_v(v_2) \rangle, \end{aligned}$$

where we use the same symbol v for its image in $B_h(M^1)$.

Proof. First, we have $f_v = o(v)$ by Corollary 3.3.8, where $o(v) = \text{Res}_w I(v,w) w^{\deg v - 1 + h}$, and $I \in I\binom{\bar{M^3}}{M^1\ \bar{M^2}}$ is the intertwining operator corresponds to $\psi(S)$ in $\text{Cor}\binom{\bar{M^3}}{M^1\ \bar{M^2}}$, see Theorem 3.3.9. Moreover, it follows from Lemma 4.1.6 that $o(O_h(M^1)) = 0$. Hence ν is well-defined. The injectivity of ν follows from Lemma 3.2.3. □

Remark 4.1.8. Although our definition for $B_h(M^1)$ is similar to the $A(V)$-bimodule $A_0(M^1)$ constructed by Huang and Yang in [42], they are not isomorphic as $A(V)$-bimodules. We will give a counter-example in the last Section.

4.1.2 The $A(V)$-bimodules for rational VOAs

We will explore the relations between the $A(V)$-bimodules $A(M^1)$ and $B_h(M^1)$ in this subsection. First, we recall some basic facts about the semisimple associative algebras over V. The following results and definitions can be found in [63]:

Lemma 4.1.9. *Let $A = M_{n_1}(\mathbb{C}) \times M_{n_2}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$ be a semisimple algebra over $\mathbb{C}$. Up to isomorphism, the modules $V_1 = \mathbb{C}^{n_1}, V_2 = \mathbb{C}^{n_2}, \dots, V_n = \mathbb{C}^{n_r}$ on which the matrix elements act as left-multiplications with column vectors are all the irreducible left A modules. The dual spaces $V_1^*, V_2^*, \dots, V_r^*$, with $\langle f.a, v\rangle := \langle f, a.v\rangle$, for any $a \in A$, $f \in V_i^*$, $v \in V_i$, and $1 \le i \le r$, are all the irreducible right A modules.*

Let A be an associative algebra over $\mathbb{C}$. The **enveloping algebra of** A is defined by $A^e := A \otimes_{\mathbb{C}} A^{\mathrm{op}}$. Denote the product on A by $a \cdot b = ab$, then the product on A^e is given by:

$$(a \otimes b) \cdot (a_1 \otimes b_1) := a \cdot a_1 \otimes b \cdot_{\mathrm{op}} b_1 = aa_1 \otimes b_1 b,$$

for any $a, a_1, b, b_1 \in A$. We have the following fact regarding enveloping algebra:

Lemma 4.1.10. *An A-bimodule M is the same as a left A^e module, with the left A^e-module action given by $(a \otimes b).m = (a.m).b$, for any $a, b \in A$, and $m \in M$. If A is semisimple, then A^e is also semisimple, and all the irreducible left modules of A^e are $V_i \otimes_{\mathbb{C}} V_j^*$, $0 \le i, j \le r$.*

In particular, if A is a semisimple algebra, then any A-bimodule M can be written as a direct sum of certain copies of $V_i \otimes_{\mathbb{C}} V_j^*$, $0 \le i, j \le r$.

Corollary 4.1.11. *Let V be a VOA such that $A(V)$ is semisimple, and let $M^1, M^2, \dots, M^p$ be all the irreducible $A(V)$-modules up to isomorphism. Then any $A(V)$-bimodule U is also semisimple, and U is a direct sum of copies of $M^i(0) \otimes_{\mathbb{C}} M^j(0)^*$, $0 \le i, j \le p$.*

Proposition 4.1.12. *If $A(V)$ is a semisimple algebra, then we have the isomorphism*

$$M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0) \cong M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0) \tag{4.1.7}$$

of vector spaces, where M^1, M^2, and M^3 are V-modules of conformal weights h_1, h_2, and h_3, respectively. In particular, (4.1.7) *is true if V is rational.*

Proof. By Lemma 4.1.5 and 4.1.6, $B_h(M^1) = A(M^1)/\mathrm{span}\{[(L(-1)+L(0)+h_2-h_3)u] : u \in M^1\}$. Observe the following fact in $A(M^1)$: $[\omega] * [u] - [u] * [\omega] = \mathrm{Res}_z Y_{M^1}(\omega, z)u(1+z)^{\mathrm{wt}\,\omega - 1} =$

$[L(-1)u + L(0)u]$, for all $u \in M^1$. Hence $[(L(-1) + L(0) + h_2 - h_3)u] = [\omega] * [u] - [u] * [\omega] + (h_2 - h_3)[u]$, for all $u \in M^1$. Let

$$J = \mathrm{span}\{[\omega] * [u] - [u] * [\omega] + (h_2 - h_3)[u] : u \in M^1\} \subset A(M^1).$$

Recall that $[\omega] \in A(V)$ is a central element, and $A(M^1)$ is a bimodule over $A(V)$, see Section 2.2. Then for each spanning element $\alpha = [\omega] * [u] - [u] * [\omega] + (h_2 - h_3)[u]$ of J, we have:

$$[a].\alpha = [\omega] * ([a] * [u]) - ([a] * [u]) * [w] + (h_2 - h_3)([a] * [u]) \in J,$$

$$\alpha.[a] = [\omega] * ([u] * [a]) - ([u] * [a]) * [\omega] + (h_2 - h_3)([u] * [a]) \in J.$$

Thus, $J \leq A(M^1)$ is a sub-bimodule over $A(V)$, and we have $B_h(M^1) = A(M^1)/J$. Since $A(V)$ is semisimple by assumption, then by Corollary 4.1.11, $A(M^1)$ is a semisimple $A(V)$-bimodule, and so $A(M^1) \cong J \oplus (A(M^1)/J) = J \oplus B_h(M^1)$ as $A(V)$-bimodules.

We claim that $M^3(0)^* \otimes_{A(V)} J \otimes_{A(V)} M^2(0) = 0$ in $M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0)$. Indeed, for any $v_3' \in M^3(0)^*$ and $v_2 \in M^2(0)$, we have:

$$\begin{aligned} & v_3' \otimes ([\omega] * [u] - [u] * [\omega] + (h_2 - h_3)[u]) \otimes v_2 \\ & = v_3'(o(\omega) - h_3) \otimes [u] \otimes v_2 - v_3' \otimes [u] \otimes (o(\omega) - h_2)v_2 \\ & = v_3'(L(0) - h_3) \otimes [u] \otimes v_2 - v_3' \otimes [u] \otimes (L(0) - h_2)v_2 \\ & = 0, \end{aligned}$$

since $\langle v_3' L(0), v_3 \rangle = \langle v_3', L(0)v_3 \rangle = h_3 \langle v_3', v_3 \rangle$, for all $v_3 \in M^3(0)$, by the definition of right module actions on $M^3(0)^*$. Then by the decomposition of $A(M^1)$, we have:

$$\begin{aligned} & M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0) \cong M^3(0)^* \otimes_{A(V)} (J \oplus B_h(M^1)) \otimes_{A(V)} M^2(0) \\ & \cong \left(M^3(0)^* \otimes_{A(V)} J \otimes_{A(V)} M^2(0)\right) \oplus \left(M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0)\right) \\ & = 0 \oplus \left(M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0)\right). \end{aligned}$$

This proves (4.1.7). □

Remark 4.1.13. In general, the $A(V)$-bimodule $A(M^1)$ do not have the decomposition $A(M^1) \cong J \oplus (B_h(M^1))$ into sub-bimodules. Therefore, we do not have $M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0)$ isomorphic to $M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0)$ for general VOAs. We will give an example in Section 4.2. Thus, the construction of $B_h(M^1)$ in this Section is necessary for the formulation of the fusion rules Theorem for general VOAs.

As we will show in Section 4.2, if V is a rational VOA, and M^2 and M^3 are irreducible V-modules, then Frenkel and Zhu's original fusion rules Theorem holds true. i.e., there is an isomorphism of vector spaces:

$$I\binom{M^3}{M^1\ M^2} \cong \left(M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0)\right)^*. \tag{4.1.8}$$

By the Hom-tensor duality, we can compute the fusion rules as follows:

$$N_{M^1 M^2}^{M^3} = \dim \mathrm{Hom}_{A(V)}\left(A(M^1) \otimes_{A(V)} M^2(0), M^3(0)\right). \tag{4.1.9}$$

It was observed by Li in [51] that the fusion rule $N_{M^1 M^2}^{M^3}$ is finite if $A(M^1)$ is a finitely generated $A(V)$-bimodule. Later it was proved by Huang that $N_{M^1 M^2}^{M^3}$ is finite if the modules M^1, M^2, and M^3 are C_1-cofinite, see Theorem 3.1 in [36]. We proved in [57] that $N_{M^1 M^2}^{M^3}$ is finite if V is C_1-cofinite as a VOA, and M^1 is C_1-cofinite as a V-module.

Section The construction of correlation functions on bottom levels by recursive formulas By Proposition 4.1.7, $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ can be embedded in the vector space $(M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$ via ν in (4.1.6). Our goal next is to construct an inverse map of ν.

Given a $f \in (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$, we need to construct a corresponding system of correlation functions S in $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. Our strategy is to use the recursive formulas (3.2.4) and (3.2.6) and construct the system of functions S inductively. The key is to show the locality ((2) in Definition 3.1.1) in each step, which can be achieved by the properties of the $A(V)$-bimodule $B_h(M^1)$, together with the formula (3.2.8).

From now on, in this Section, we fix a linear function $f \in (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$. We will construct a system of correlation functions $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ from f.

4.1.3 Construction of 4-point and 5-point functions

Definition 4.1.14. Define $S_M : M^3(0)^* \times M^1 \times M^2(0) \to \mathcal{F}(w)$ by

$$S_M(v_3', (v, w)v_2) := f(v_3' \otimes v \otimes v_2) w^{-\deg v}, \tag{4.1.10}$$

where on the right-hand side we use the same symbol v for its image $v + O(M^1)$ in $B_h(M^1)$.

Define $S_{VM}^L : M^3(0)^* \times V \times M^1 \times M^2(0) \to \mathcal{F}(z, w)$ by

$$\begin{aligned} S_{VM}^L(v_3', (a, z)(v, w)v_2) :&= S_M(v_3' o(a), (v, w)v_2) z^{-\mathrm{wt} a} \\ &+ \sum_{i \geq 0} F_{\mathrm{wt} a, i}(z, w) S_M(v_3', (a(i)v, w)v_2). \end{aligned} \tag{4.1.11}$$

Finally, define $S^R_{MV} : M^3(0)^* \times M^1 \times V \times M^2(0) \to \mathcal{F}(z,w)$ by

$$
\begin{aligned}
S^R_{MV}(v_3', (v,w)(a,z)v_2) :&= S_M(v_3', (v,w)o(a)v_2)z^{-\mathrm{wt}a} \\
&\quad + \sum_{i\geq 0} G_{\mathrm{wt}a,i}(z,w) S_M(v_3', (a(i)v,w)v_2).
\end{aligned}
\tag{4.1.12}
$$

The upper index L (resp.R) in the 4-point functions S indicates that we use the expansion formula for the left (resp. right) most term, namely, (3.2.4) (resp.(3.2.6)) to construct the new S from the 3-point function. We will denote the 3-point function S_M by S.

Proposition 4.1.15. *As rational functions in $\mathcal{F}(z,w)$, we have:*

$$
S^L_{VM}(v_3', (a,z)(v,w)v_2) = S^R_{MV}(v_3', (v,w)(a,z)v_2).
$$

Proof. By Definition 4.1.14, (3.2.8), and the property of $M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0)$,

$$
\begin{aligned}
&S^L_{VM}(v_3', (a,z)(v,w)v_2) - S^R_{MV}(v_3', (v,w)(a,z)v_2) \\
&= f(v_3' o(a) \otimes v \otimes v_2) w^{-\deg v} z^{-\mathrm{wt}a} - f(v_3' \otimes v \otimes o(a)v_2) w^{-\deg v} z^{-\mathrm{wt}a} \\
&\quad + \sum_{i\geq 0} (F_{\mathrm{wt}a,i}(z,w) - G_{\mathrm{wt}a,i}(z,w)) S_M(v_3', (a(i)v,w)v_2) \\
&= f(v_3' \otimes a * v \otimes v_2) w^{-\deg v} z^{-\mathrm{wt}a} - f(v_3' \otimes v * a \otimes v_2) w^{-\deg v} z^{-\mathrm{wt}a} \\
&\quad - \sum_{i\geq 0} \binom{\mathrm{wt}a - 1}{i} f(v_3' \otimes a(i)v \otimes v_2) w^{-\deg v - \mathrm{wt}a + i + 1} z^{-\mathrm{wt}a} w^{\mathrm{wt}a - 1 - i} \\
&= f(v_3' \otimes (a * v - v * a) \otimes v_2) w^{-\deg v} z^{-\mathrm{wt}a} - \sum_{i\geq 0} \binom{\mathrm{wt}a - 1}{i} f(v_3' \otimes a(i)v \otimes v_2) z^{-\mathrm{wt}a} w^{-\deg v}.
\end{aligned}
$$

By (4.1.4), we also have $a * v - v * a = \sum_{i\geq 0} \binom{\mathrm{wt}a-1}{i} a(i)v$ holds in the $A(V)$-bimodule $B_h(M^1)$. Hence $S^L_{VM}(v_3', (a,z)(v,w)v_2) - S^R_{MV}(v_3', (v,w)(a,z)v_2) = 0$. □

By Proposition 4.1.15, the 4-point functions S^L_{VM} and S^R_{MV} in definition 4.1.14 give rise to one single 4-point function S that satisfies

$$
S(v_3', (a,z)(v,w)v_2) = S(v_3', (v,w)(a,z)v_2), \tag{4.1.13}
$$

and this function can be defined either by (4.1.11) or (4.1.12).

We adopt a similar method to construct 5-point functions. As long as the term (v,w) does not appear at the left-most place, we use the formula (3.2.4) to construct S from the 4-point function; if (v,w) appears at the left-most place, we use (3.2.6) to construct S.

Definition 4.1.16. Define the 5-point functions with the upper index L,

$$S^L_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2), \quad \text{and} \quad S^L_{VVM}(v_3', (a_1, z_1)(a_2, z_2)(v, w)v_2),$$

by expanding (a_1, z_1) from the left, which is given by the common formula:

$$\begin{aligned} S(v_3' o(a_1), (v, w)(a_2, z_2)v_2) z_1^{-\mathrm{wt} a_1} &+ \sum_{j\geq 0} F_{\mathrm{wt} a_1, j}(z_1, w) S(v_3', (a_1(j)v, w)(a_2, z_2)v_2) \\ &+ \sum_{j\geq 0} F_{\mathrm{wt} a_1, j}(z_1, z_2) S(v_3', (v, w)(a_1(j)a_2, z_2)v_2). \end{aligned} \tag{4.1.14}$$

Define the 5-point functions with upper index R,

$$S^R_{VMV}(v_3', (a_2, z_2)(v, w)(a_1, z_1)v_2), \quad \text{and} \quad S^R_{MVV}(v_3', (v, w)(a_2, z_2)(a_1, z_1)v_2),$$

by expanding (a_1, z_1) from the right, which is given by the common formula:

$$\begin{aligned} S(v_3', (a_2, z_2)(v, w) o(a_1) v_2) z_1^{-\mathrm{wt} a_1} &+ \sum_{j\geq 0} G_{\mathrm{wt} a_1, j}(z_1, w) S(v_3', (a_2, z_2)(a_1(j)v, w)v_2) \\ &+ \sum_{j\geq 0} G_{\mathrm{wt} a_1, j}(z_1, z_2) S(v_3', (a_1(j)a_2, z_2)(v, w)v_2). \end{aligned} \tag{4.1.15}$$

The function S in (4.1.14) and (4.1.15) is the (common) 4-point function in Definition 4.1.14. By (4.1.13), it makes sense to define S^L_{VMV} and S^L_{VVM} by the same formula, same for S^R_{VMV} and S^R_{MVV}. We will show that all the 5-point functions in Definition 4.1.16 are the same. First, we observe that the term $S_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2)$ has the following two expressions: $S^L_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2)$ and $S^R_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2)$.

Proposition 4.1.17. *If* (4.1.14)=(4.1.15)*, then we have:*

$$S^L_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2) = S^R_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2).$$

Proof. Note that (4.1.14) is a generalization of the function (2.2.6) in [73]. By a similar calculation, it is easy to see that the formula (2.2.11) in [73] also holds for our case. i.e., we can swap the terms (a_1, z_1) and (a_2, z_2) in S^L_{VVM}:

$$S^L_{VVM}(v_3', (a_1, z_1)(a_2, z_2)(v, w)v_2) = S^L_{VVM}(v_3', (a_2, z_2)(a_1, z_1)(v, w)v_2). \tag{4.1.16}$$

By the assumption that (4.1.14)=(4.1.15), Definition 4.1.16, and (4.1.16), we have:

$$S^L_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2) = S^L_{VVM}(v_3', (a_1, z_1)(a_2, z_2)(v, w)v_2)$$

$$= S^L_{VVM}(v_3', (a_2,z_2)(a_1,z_1)(v,w)v_2) = S^L_{VMV}(v_3', (a_2,z_2)(v,w)(a_1,z_1)v_2)$$
$$= S^R_{VMV}(v_3', (a_1,z_1)(v,w)(a_2,z_2)v_2),$$

where the last equality follows from the assumption that (4.1.14)=(4.1.15). □

Next, we show that (4.1.14)=(4.1.15). We use symbols (1), (2), and (3) to denote the difference of the three summands in the term (4.1.14) − (4.1.15):

$$S(v_3' o(a_1), (v,w)(a_2,z_2)v_2) z_1^{-\mathrm{wt} a_1} - S(v_3', (a_2,z_2)(v,w)o(a_1)v_2) z_1^{-\mathrm{wt} a_1}. \tag{1}$$

$$\sum_{j\geq 0} (F_{\mathrm{wt} a_1, j}(z_1,w) - G_{\mathrm{wt} a_1, j}(z_1,w)) S(v_3', (a_1(j)v, w)(a_2,z_2)v_2). \tag{2}$$

$$\sum_{j\geq 0} (F_{\mathrm{wt} a_1, j}(z_1,z_2) - G_{\mathrm{wt} a_1, j}(z_1,z_2)) S(v_3', (v,w)(a_1(j)a_2, z_2)v_2). \tag{3}$$

So we need to show that (1)+(2)+(3)=0.

By (4.1.13), we may use the formula (4.1.11) and expand both terms in (1) with respect to (a_2, z_2) from the left. Then (1) can be expressed as:

$$
\begin{aligned}
& S(v_3' o(a_1), (v,w)(a_2,z_2)v_2) z_1^{-\mathrm{wt} a_1} - S(v_3', (a_2,z_2)(v,w)o(a_1)v_2) z_1^{-\mathrm{wt} a_1} \\
&= S(v_3' o(a_1) o(a_2), (v,w)v_2) z_1^{-\mathrm{wt} a_1} z_2^{-\mathrm{wt} a_2} + \sum_{i\geq 0} F_{\mathrm{wt} a_2, i}(z_2,w) S(v_3' o(a_1), (a_2(i)v, w)v_2) z_1^{-\mathrm{wt} a_1} \\
&\quad - S(v_3' o(a_2), (v,w)o(a_1)v_2) z_1^{-\mathrm{wt} a_1} z_2^{-\mathrm{wt} a_2} + \sum_{i\geq 0} F_{\mathrm{wt} a_2, i}(z_2,w) S(v_3', (a_2(i)v, w)o(a_1)v_2) z_1^{-\mathrm{wt} a_1} \\
&= \underbrace{f(v_3' \otimes a_1 * a_2 * v \otimes v_2) w^{-\deg v} z_1^{-\mathrm{wt} a_1} z_2^{-\mathrm{wt} a_2}}_{(11)} - \underbrace{f(v_3' \otimes a_2 * v * a_1 \otimes v_2) w^{-\deg v} z_1^{-\mathrm{wt} a_1} z_2^{-\mathrm{wt} a_2}}_{(12)} \\
&\quad + \underbrace{\sum_{i\geq 0} F_{\mathrm{wt} a_2, i}(z_2,w) f(v_3' \otimes (a_1 * (a_2(i)v)) - (a_2(i)v) * a_1) \otimes v_2) w^{-\mathrm{wt} a_2 - \deg v + i + 1} z_1^{-\mathrm{wt} a_1}}_{(13)} \\
&= (11) + (12) + (13).
\end{aligned}
$$

For the term (2), we use the formula (4.1.11) agian and expand each summand in (2) with respect to (a_2, z_2) from the left. Then by (3.2.8), (2) can be expressed as:

$$
\begin{aligned}
((2)) &= \sum_{j\geq 0} F_{\mathrm{wt} a_1, j}(z_1,w) S(v_3' o(a_2), (a_1(j)v, w)v_2) z_2^{-\mathrm{wt} a_2} \\
&\quad + \sum_{j\geq 0} \sum_{i\geq 0} F_{\mathrm{wt} a_1, j}(z_1,w) F_{\mathrm{wt} a_2, i}(z_2,w) S(v_3', (a_2(i)a_1(j)v, w)v_2) \\
&\quad - \sum_{j\geq 0} G_{\mathrm{wt} a_1, j}(z_1,w) S(v_3' o(a_2), (a_1(j)v, w)v_2) z_2^{-\mathrm{wt} a_2}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{j\geq 0}\sum_{ß\geq 0} G_{\mathrm{wt}a_1,j}(z_1,w)F_{\mathrm{wt}a_2,i}(z_2,w)S(v_3',(a_2(i)a_1(j)v,w)v_2)\\
&=\sum_{j\geq 0}-\binom{\mathrm{wt}a_1-1}{j}\underbrace{S(v_3'o(a_2),(a_1(j)v,w)v_2)z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2}w^{\mathrm{wt}a_1-1-j}}_{(21)}\\
&\quad+\underbrace{\sum_{j\geq 0}\sum_{i\geq 0}-\binom{\mathrm{wt}a_1-1}{j}z_1^{-\mathrm{wt}a_1}w^{\mathrm{wt}a_1-1-j}F_{\mathrm{wt}a_2,w}(z_2,w)S(v_3',(a_2(i)a_1(j)v,w)v_2)}_{(22)}\\
&=(21)+(22).
\end{aligned}
$$

Finally, for the term (3), we expand each of its summand with respect to $(a_1(j)a_2,z_2)$ from the left, so (3) can be expressed as:

$$
\begin{aligned}
((3))&=\sum_{j\geq 0}F_{\mathrm{wt}a_1,j}(z_1,z_2)S(v_3'o(a_1(j)a_2),(v,w)v_2)z_2^{-\mathrm{wt}a_1-\mathrm{wt}a_2+j+1}\\
&\quad+\sum_{j\geq 0}\sum_{i\geq 0}F_{\mathrm{wt}a_1,j}(z_1,z_2)F_{\mathrm{wt}a_1+\mathrm{wt}a_2-j-1,i}(z_2,w)S(v_3',((a_1(j)a_2)(i)v,w)v_2)\\
&\quad-\sum_{j\geq 0}G_{\mathrm{wt}a_1,j}(z_1,z_2)S(v_3'o(a_1(j)a_2),(v,w)v_2)z_2^{-\mathrm{wt}a_1-\mathrm{wt}a_2+j+1}\\
&\quad+\sum_{j\geq 0}\sum_{i\geq 0}G_{\mathrm{wt}a_1,j}(z_1,z_2)F_{\mathrm{wt}a_1+\mathrm{wt}a_2-j-1,i}(z_2,w)S(v_3',((a_1(j)a_2)(i)v,w)v_2)\\
&=\underbrace{\sum_{j\geq 0}-\binom{\mathrm{wt}a_1-1}{j}z_1^{-\mathrm{wt}a_1}z_2^{\mathrm{wt}a_1-1-j}S(v_3'o(a_1(j)a_2),(v,w)v_2)z_2^{-\mathrm{wt}a_1-\mathrm{wt}a_2+j+1}}_{(31)}\\
&\quad+\underbrace{\sum_{j\geq 0}\sum_{i\geq 0}-\binom{\mathrm{wt}a_1-1}{j}z_1^{-\mathrm{wt}a_1}z_2^{\mathrm{wt}a_1-1-j}F_{\mathrm{wt}a_1+\mathrm{wt}a_2-j-1,i}(z_2,w)S(v_3',(a_1(j)a_2)(i)(v,w)v_2)}_{(32)}\\
&=(31)+(32).
\end{aligned}
$$

We need to show that $(11)+(12)+(13)+(21)+(22)+(31)+(32)=0$. In fact, since $a*v-v*a=\mathrm{Res}_zY(a,z)v(1+z)^{\mathrm{wt}a-1}=\sum_{j\geq 0}\binom{\mathrm{wt}a-1}{j}a(j)v$ in $B_h(M^1)$, see (4.1.4), and $a_1*a_2-a_2*a_1=\sum_{j\geq 0}\binom{\mathrm{wt}a_1-1}{j}a_1(j)a_2$ in $A(V)$, we can rewrite (21) and (31) as:

$$
\begin{aligned}
(21)&=-\sum_{j\geq 0}\binom{\mathrm{wt}a_1-1}{j}w^{-\mathrm{wt}a_1-\deg v+j+1}z_1^{\mathrm{wt}a_1}z_2^{\mathrm{wt}a_2}w^{\mathrm{wt}a_1-j-1}f(v_3'o(a_2)\otimes a_1(j)v\otimes v_2)\\
&=-w^{-\deg v}z_1^{-\mathrm{wt}a_1}z_2^{\mathrm{wt}a_2}f(v_3'\otimes(a_2*a_1*v-a_2*v*a_1)\otimes v_2);\\
(31)&=-\sum_{j\geq 0}\binom{\mathrm{wt}a_1-1}{j}z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2}w^{-\deg v}f(v_3'o(a_1(j)a_2)\otimes v\otimes v_2)\\
&=-z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2}w^{-\deg v}f(v_3'\otimes(a_1*a_2*v-a_2*a_1*v)\otimes v_2).
\end{aligned}
$$

Then by the bimodule property of $B_h(M^1)$, we have:

$$\begin{aligned}
&(11) + (12) + (21) + (31)\\
&= f(v_3' \otimes a_1 * a_2 * v \otimes v_2) w^{-\deg v} z_1^{-\mathrm{wt} a_1} z_2^{-\mathrm{wt} a_2} - f(v_3' \otimes a_2 * v * a_1 \otimes v_2) w^{-\deg v} z_1^{-\mathrm{wt} a_1} z_2^{-\mathrm{wt} a_2}\\
&- w^{-\deg v} z_1^{-\mathrm{wt} a_1} z_2^{\mathrm{wt} a_2} f(v_3' \otimes (a_2 * a_1 * v - a_2 * v * a_1) \otimes v_2)\\
&- z_1^{-\mathrm{wt} a_1} z_2^{-\mathrm{wt} a_2} w^{-\deg v} f(v_3' \otimes (a_1 * a_2 * v - a_2 * a_1 * v) \otimes v_2)\\
&= 0.
\end{aligned}$$

It remains to show that (13) + (22) + (32) = 0.

Lemma 4.1.18. *Let M be a V module, and let $a_1, a_2 \in V$, $v \in M$, and $n \in \mathbb{N}$. We have:*

$$\begin{aligned}
&\sum_{i,j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j} \binom{\mathrm{wt} a_2 + n}{i} (a_1(j) a_2(i) v - a_2(i) a_1(j) v)\\
&= \sum_{i,j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j} \binom{\mathrm{wt} a_1 + \mathrm{wt} a_2 - j - 1 + n}{i} (a_1(j) a_2)(i) v
\end{aligned} \tag{4.1.17}$$

Proof. Choose complex variables z_1, z_2 in the domain $|z_1| < 1,\ |z_2| < 1,\ |z_1 - z_2| < |1 + z_2|$. By the Jacobi identity in the residue form, the left-hand side of (4.1.17) can be written as:

$$\begin{aligned}
&\mathrm{Res}_{z_1,z_2} \sum_{i,j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j} \binom{\mathrm{wt} a_2 + n}{i} z_1^j z_2^i (Y(a_1, z_1) Y(a_2, z_2) v - Y(a_2, z_2) Y(a_1, z_1) v)\\
&= \mathrm{Res}_{z_1,z_2} (1 + z_1)^{\mathrm{wt} a_1 - 1} (1 + z_2)^{\mathrm{wt} a_2 + n} (Y(a_1, z_1) Y(a_2, z_2) v - Y(a_2, z_2) Y(a_1, z_1) v)\\
&= \mathrm{Res}_{z_2} \mathrm{Res}_{z_1 - z_2} (1 + z_2 + (z_1 - z_2))^{\mathrm{wt} a_1 - 1} (1 + z_2)^{\mathrm{wt} a_2 + n} Y(Y(a_1, z_1 - z_2) a_2, z_2) v\\
&= \mathrm{Res}_{z_2} \mathrm{Res}_{z_1 - z_2} \sum_{j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j} (1 + z_2)^{\mathrm{wt} a_1 - 1 - j + \mathrm{wt} a_2 + n} (z_1 - z_2)^j Y(Y(a_1, z_1 - z_2) a_2, z_2) v\\
&= \sum_{i,j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j} \binom{\mathrm{wt} a_1 + \mathrm{wt} a_2 - j - 1 + n}{i} (a_1(j) a_2)(i) v,
\end{aligned}$$

which is the right-hand side of (4.1.17). □

We use the formula (4.1.4) again and rewrite (13) as:

$$(13) = \sum_{i,j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j} z_1^{-\mathrm{wt} a_1} w^{-\mathrm{wt} a_2 - \deg v + i + 1} F_{\mathrm{wt} a_2, i}(z_2, w) f(v_3' \otimes a_1(j) a_2(i) v \otimes v_2).$$

Since the map $\iota_{z_2,w}$ is injective (see Section 3 in [27]), we only need to show that $\iota_{z_2,w}((13) + (22) + (32)) = 0$. By (3.2.5), $\iota_{z_2,w}(F_{\mathrm{wt} a_2, i}(z_2, w))$ can be written as:

$$\iota_{z_2,w}(F_{\mathrm{wt} a_2, i}(z_2, w)) = \sum_{n\geq 0} \binom{\mathrm{wt} a_2 + n}{i} w^{\mathrm{wt} a_2 + n - i} z_2^{-\mathrm{wt} a_2 - n - 1}$$

To simplify our notation, we denote $z_1^{\mathrm{wt} a_1} w^{-\deg v+n+1} z_2^{-\mathrm{wt} a_2-n-1}$ by γ. By Lemma 4.1.18,

$$
\begin{aligned}
&\iota_{z_2,w}(13)+\iota_{z_2,w}(22)\\
&=\sum_{i,j\geq 0}\binom{\mathrm{wt} a_1-1}{j} z_1^{\mathrm{wt} a_1} w^{-\mathrm{wt} a_2-\deg v+i+1}\left(\sum_{n\geq 0}\binom{\mathrm{wt} a_2+n}{i} w^{\mathrm{wt} a_2+n-i} z_2^{-\mathrm{wt} a_2-n-1}\right)\\
&\cdot\left(f(v_3'\otimes a_1(j)a_2(i)v\otimes v_2)-f(v_3'\otimes a_2(i)a_1(j)v\otimes v_2)\right)\\
&=\sum_{i,j,n\geq 0}\binom{\mathrm{wt} a_1-1}{j}\binom{\mathrm{wt} a_2+n}{i}\gamma\cdot f(v_3'\otimes(a_1(j)a_2(i)v-a_2(i)a_1(j)v)\otimes v_2)\\
&=\sum_{i,j,n\geq 0}\binom{\mathrm{wt} a_1-1}{j}\binom{\mathrm{wt} a_1+\mathrm{wt} a_2+n-j-1}{i}\gamma\cdot f(v_3'\otimes(a_1(j)a_2)(i)v\otimes v_2)\\
&=-\iota_{z_2,w}(32).
\end{aligned}
$$

Now the proof of (4.1.14)=(4.1.15) is complete.

Therefore, the 5 point functions in Definition 4.1.16 give rise to one single 5-point function S that satisfies:

$$
\begin{aligned}
&S(v_3',(a_1,z_1)(a_2,z_2)(v,w)v_2)=S(v_3',(a_2,z_2)(a_1,z_1)(v,w)v_2)\\
&=S(v_3',(a_1,z_1)(v,w)(a_2,z_2)v_2)=S(v_3',(a_2,z_2)(v,w)(a_1,z_1)v_2)\\
&=S(v_3',(v,w)(a_1,z_1)(a_2,z_2)v_2)=S(v_3',(v,w)(a_2,z_2)(a_1,z_1)v_2).
\end{aligned}
\tag{4.1.18}
$$

In particular, the 5-point function S satisfies the locality in Definition 3.1.1, with $v_3'\in M^3(0)^*$ and $v_2\in M^2(0)$. Moreover, $S(v_3',(a_1,z_1)(a_2,z_2)(v,w)v_2)$ also satisfies both of the recursive formula (3.2.4) and (3.2.6) by its definition.

4.1.4 Construction of $(n+3)$-point functions

We construct the general $(n+3)$-point function S using induction on n. We have finished the base cases $n=1,2$ in the previous subsection. Now assume that the $(n+2)$-point functions $S : M^3(0)^*\times V\times\cdots\times M^1\times\cdots\times V\times M^2(0)\to\mathcal{F}(z_1,\dots,z_{n-1},w)$ exist and satisfy the following two properties: Let $\{(b_1,w_1),(b_2,w_2),\dots,(b_n,w_n)\}$ be the same set as $\{(a_1,z_1),\dots,(a_{n-1},z_{n-1}),(v,w)\}$. The first property is the locality:

$$S(v_3',(a_1,z_1)(a_2,z_2)\dots(a_{n-1},z_{n-1})(v,w)v_2)=S(v_3',(b_1,w_1)(b_2,w_2)\dots(b_n,w_n)v_2),\tag{I}$$

that is, the terms (a_1, z_1),$(a_2, z_2), \dots, (a_{n-1}, z_{n-1})$, and (v, w) can be permutated arbitrarily within S. Denote by S^L (resp. S^R) the expansion of the $(n+1)$-point function S with respect to the left (resp. right)-most term using (3.2.4) (resp. (3.2.6)). The second property is that

$$\begin{aligned} S(v_3', (b_1, w_1)(b_2, w_2)\dots(b_n, w_n)v_2) &= S^L(v_3', (b_1, w_1)(b_2, w_2)\dots(b_n, w_n)v_2) \\ &= S^R(v_3', (b_1, w_1)(b_2, w_2)\dots(b_n, w_n)v_2), \end{aligned} \tag{II}$$

where (b_1, w_1) in S^L is not (v, w), and (b_n, w_n) in S^R is not (v, w).

Note that properties I and II are satisfied by the 4-point and 5-point functions (see (4.1.13) and (4.1.18).) We construct $(n + 3)$-point functions as follows:

Definition 4.1.19. Assume the number of V in the sub-indices of $S^L_{VV\dots M^1\dots V}$ and $S^R_{V\dots M^1\dots VV}$ are both equal to n, the sub-index M^1 in S^L is not at the first place, and the sub-index M^1 in S^R is not at the last place. Define $S^L_{VV\dots M^1\dots V}$ by

$$\begin{aligned} &S^L_{VV\dots M^1\dots V}(v_3', (a_1, z_1)\dots(v, w)\dots v_2) := S(v_3' o(a_1), (a_2, z_2)\dots(a_n, z_n)(v, w)v_2)z_1^{-\mathrm{wt}a_1} \\ &+ \sum_{k=2}^{n}\sum_{j\geq 0} F_{\mathrm{wt}a_1, j}(z_1, z_k)S(v_3', (a_2, z_2)\dots(a_1(j)a_k, z_k)\dots(a_n, z_n)(v, w)v_2) \\ &+ \sum_{j\geq 0} F_{\mathrm{wt}a_1, j}(z_1, w)S(v_3', (a_2, z_2)\dots(a_n, z_n)(a_1(j)v, w)v_2). \end{aligned} \tag{4.1.19}$$

Define $S^R_{V\dots M^1\dots VV}$ by

$$\begin{aligned} &S^R_{V\dots M^1\dots VV}(v_3', \dots(v, w)\dots(a_1, z_1)v_2) := S(v_3', (a_2, z_2)\dots(a_n, z_n)(v, w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1} \\ &+ \sum_{k=2}^{n}\sum_{j\geq 0} G_{\mathrm{wt}a_1, j}(z_1, z_k)S(v_3', (a_2, z_2)\dots(a_1(j)a_k, z_k)\dots(a_n, z_n)(v, w)v_2) \\ &+ \sum_{j\geq 0} G_{\mathrm{wt}a_1, j}(z_1, w)S(v_3', (a_2, z_2)\dots(a_n, z_n)(a_1(j)v, w)v_2), \end{aligned} \tag{4.1.20}$$

where the S on right-hand sides of (4.1.19) and (4.1.20) is the $(n + 2)$-point function.

The definition above indicates that $S^L_{VMV\dots V} = S^L_{VVM\dots V} = \dots = S^L_{VV\dots VM}$, which is reasonable because the $(n + 2)$-point function S on the right-hand side of (4.1.19) satisfies the locality property (I). For a similar reason, we can also expect that $S^R_{MV\dots VV} = S^R_{VM\dots VV} = \dots = S^R_{V\dots VMV}$. We need to show that

$$\begin{aligned} &S^L_{V\dots M\dots V}(v_3', (a_1, z_1)\dots(v, w)\dots(a_2, z_2)v_2) \\ &= S^R_{V\dots M\dots V}(v_3', (a_1, z_1)\dots(v, w)\dots(a_2, z_2)v_2), \end{aligned} \tag{4.1.21}$$

for all $S^L_{VV\ldots M\ldots V}$ and $S^R_{V\ldots M\ldots VV}$.

Indeed, as we mentioned in Proposition 4.1.15, since (4.1.19) is the generalization of (2.2.6) in [73], by a similar argument as the proof of (2.2.11) in [73], we have:

$$\begin{aligned} &S^L_{VV\ldots M\ldots V}(v_3', (a_1, z_1)(a_2, z_2)\ldots(v, w)\ldots v_2) \\ &= S^L_{VV\ldots M\ldots V}(v_3', (a_2, z_2)(a_1, z_1)\ldots(v, w)\ldots v_2). \end{aligned} \tag{4.1.22}$$

Proposition 4.1.20. *If* $S^L_{VV\ldots M\ldots V}(v_3', (a_1, z_1)\ldots v_2) = S^R_{V\ldots M\ldots VV}(v_3', \ldots (a_1, z_1)v_2)$*. i.e., if the right-hand side of* (4.1.19) *is equal to the right-hand side of* (4.1.20)*, then* (4.1.21) *holds.*

Proof. The proof is similar to the proof of Proposition 4.1.17. By (4.1.22) and the assumption,

$$\begin{aligned} &S^L_{V\ldots M\ldots V}(v_3', (a_1, z_1)\ldots(v, w)\ldots(a_2, z_2)v_2) = S^L_{VV\ldots M\ldots V}(v_3', (a_1, z_1)(a_2, z_2)\ldots(v, w)\ldots v_2) \\ &= S^L_{VV\ldots M\ldots V}(v_3', (a_2, z_2)(a_1, z_1)\ldots(v, w)\ldots v_2) = S^R_{V\ldots M\ldots VV}(v_3', (a_1, z_1)\ldots(v, w)\ldots(a_2, z_2)v_2) \end{aligned}$$

as asserted. □

Now we are left to show that (4.1.19) = (4.1.20). i.e.,

$$S^L_{VV\ldots M\ldots V}(v_3', (a_1, z_1)\ldots(v, w)\ldots v_2) = S^R_{V\ldots M\ldots VV}(v_3', \ldots(v, w)\ldots(a_1, z_1)v_2). \tag{4.1.23}$$

Similar to the previous subsection, we use the symbols $(1), (2), (3)$, and (4) to denote the following summands on the right-hand side of the difference (4.1.19) − (4.1.20):

$$S(v_3' o(a_1), (a_2, z_2)\ldots(v, w)v_2)z^{-\mathrm{wt}a_1} - S(v_3', (a_2, z_2)\ldots(v, w)o(a_1)v_2)z^{-\mathrm{wt}a_1}. \tag{1}$$

$$\sum_{j\geq 0}(F_{\mathrm{wt}a_1, j}(z_1, z_2) - G_{\mathrm{wt}a_1, j}(z_1, z_2))S(v_3', (a_1(j)a_2, z_2)\ldots(a_n, z_n)(v, w)v_2). \tag{2}$$

$$\sum_{k=3}^{n}\sum_{j\geq 0}(F_{\mathrm{wt}a_1, j}(z_1, z_k) - G_{\mathrm{wt}a_1, j}(z_1, z_k))S(v_3', (a_2, z_2)\ldots(a_1(j)a_k, z_k)\ldots(v, w)v_2). \tag{3}$$

$$\sum_{j\geq 0}((F_{\mathrm{wt}a_1, j}(z_1, w) - G_{\mathrm{wt}a_1, j}(z_1, w))S(v_3', (a_2, z_2)\ldots(a_n, z_n)(a_1(j)v, w)v_2). \tag{4}$$

Then we need to show that (1)+(2)+(3)+(4)=0. Our strategy is to apply the expansion formula (3.2.4) and expand each summand of (1) − (4) with respect to the left-most term. Then we add them all up and show that the sum equals 0.

Remark 4.1.21. Since we will be using the recursive formula (3.2.4) twice and the 3-point function cannot be further expanded, the construction of the 5-point function in the previous subsection is necessary for our induction process.

Start with (1), note that $S(v_3'o(a_1),(a_2,z_2)\dots(a_n,z_n)(v,w)v_2)z^{-\mathrm{wt}a_1}$ can be written as:

$$\begin{aligned}
&S(v_3'o(a_1)o(a_2),(a_3,z_3)\dots(a_n,z_n)(v,w)v_2)z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2} \qquad (*)\\
&+\sum_{t=3}^{n}\sum_{i\geq 0}F_{\mathrm{wt}a_2,i}(z_2,z_t)S(v_3'o(a_1),(a_3,z_3)\dots(a_2(i)a_t,z_t)\dots(a_n,z_n)(v,w)v_2)z_1^{-\mathrm{wt}a_1}\\
&+\sum_{i\geq 0}F_{\mathrm{wt}a_2,i}(z_2,w)S(v_3'o(a_1),(a_3,z_3)\dots(a_n,z_n)(a_2(i)v,w)v_2)z_1^{-\mathrm{wt}a_1}.
\end{aligned}$$

$S(v_3',(a_2,z_2)\dots(a_n,z_n)(v,w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}$ can be written as

$$\begin{aligned}
&S(v_3'o(a_2),(a_3,z_3)\dots(a_n,z_n)(v,w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2} \qquad (**)\\
&+\sum_{t=3}^{n}\sum_{i\geq 0}F_{\mathrm{wt}a_2,i}(z_2,z_t)S(v_3',(a_3,z_3)\dots(a_2(i)a_t,z_t)\dots(a_n,z_n)(v,w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}\\
&+\sum_{i\geq 0}F_{\mathrm{wt}a_2,i}(z_2,w)S(v_3',(a_3,z_3)\dots(a_n,z_n)(a_2(i)v,w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}.
\end{aligned}$$

We denote the first, second, and third corresponding terms in $(*)-(**)$ by (11), (12), and (13), respectively. In particular, (11) is

$$\begin{aligned}
&S(v_3'o(a_1)o(a_2),(a_3,z_3)\dots(a_n,z_n)(v,w)v_2)z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2}\\
&-S(v_3'o(a_2),(a_3,z_3)\dots(a_n,z_n)(v,w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2}.
\end{aligned} \qquad (11)$$

Lemma 4.1.22. *As* $(n+1)$*-point function, we have:*

$$\begin{aligned}
&S(v_3'o(a_1),(a_3,z_3)\dots(a_n,z_n)(v,w)v_2)-S(v_3',(a_3,z_3)\dots(a_n,z_n)(v,w)o(a_1)v_2)\\
&=\sum_{k=3}^{n}\sum_{j\geq 0}\binom{\mathrm{wt}a_1-1}{j}z_k^{\mathrm{wt}a_1-j-1}S(v_3',(a_3,z_3)\dots(a_1(j)a_k,z_k)\dots(a_n,z_n)(v,w)v_2) \qquad (4.1.24)\\
&+\sum_{j\geq 0}\sum_{j\geq 0}\binom{\mathrm{wt}a_1-1}{j}w^{\mathrm{wt}a_1-j-1}S(v_3',(a_3,z_3)\dots(a_n,z_n)(a_1(j)v,w)v_2)
\end{aligned}$$

Proof. By the induction hypothesis for the $(n+2)$-point functions and (3.2.8), we have:

$$\begin{aligned}
0&=S(v_3',(a_1,z_1)(a_3,z_3)\dots(a_n,z_n)(v,w)v_2)-S(v_3',(a_3,z_3)\dots(a_n,z_n)(a_1,z_1)(v,w)v_2)\\
&=S(v_3'o(a_1)(a_3,z_3)\dots(a_n,z_n)(v,w)v_2)z_1^{-wta_1}-S(v_3'(a_3,z_3)\dots(a_n,z_n)(v,w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=3}^{n}\sum_{j\geq 0}(F_{\mathrm{wt}a_1,j}(z_1,z_k)-G_{\mathrm{wt}a_1,j}(z_1,z_k))S(v_3',(a_3,z_3)\ldots(a_1(j)a_k,z_k)\ldots(a_n,z_n)(v,w)v_2)\\
&+\sum_{j\geq 0}(F_{\mathrm{wt}a_1,j}(z_1,w)-G_{\mathrm{wt}a_1,j}(z_1,w))S(v_3',(a_3,z_3)\ldots(a_n,z_n)(a_1(j)v,w)v_2)\\
=\;&S(v_3'o(a_1)(a_3,z_3)\ldots(a_n,z_n)(v,w)v_2)z_1^{-\mathrm{wt}a_1}-S(v_3'(a_3,z_3)\ldots(a_n,z_n)(v,w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}\\
&+\sum_{k=3}^{n}\sum_{j\geq 0}-\binom{\mathrm{wt}a_1-1}{j}z_k^{\mathrm{wt}a_1-j-1}S(v_3',(a_3,z_3)\ldots(a_1(j)a_k,z_k)\ldots(a_n,z_n)(v,w)v_2)\\
&+\sum_{j\geq 0}-\binom{\mathrm{wt}a_1-1}{j}w^{\mathrm{wt}a_1-j-1}S(v_3',(a_3,z_3)\ldots(a_n,z_n)(a_1(j)v,w)v_2).
\end{aligned}
$$

This proves (4.1.24). □

It follows from the Lemma 4.1.22 that (12) and (13) can be written as:

$$
\begin{aligned}
(12)=\;&\sum_{t=3}^{n}\sum_{k=3,k\neq t}^{n}\sum_{i,j\geq 0}F_{\mathrm{wt}a_2,i}(z_2,z_t)\binom{\mathrm{wt}a_1-1}{j}z_1^{-\mathrm{wt}a_1}z_k^{\mathrm{wt}a_1-1-j}\\
&\cdot\underbrace{S(v_3',(a_3,z_3)\ldots(a_1(j)a_k,z_k)\ldots(a_2(i)a_t,z_t)\ldots(a_n,z_n)(v,w)v_2)}_{(121)}\\
&+\sum_{t=3}^{n}\sum_{i,j\geq 0}F_{\mathrm{wt}a_2,i}(z_2,z_t)\binom{\mathrm{wt}a_1-1}{j}z_1^{-\mathrm{wt}a_1}z_t^{\mathrm{wt}a_1-1-j}\\
&\cdot\underbrace{S(v_3',(a_3,z_3)\ldots(a_1(j)a_2(i)a_t,z_t)\ldots(a_n,z_n)(v,w)v_2)}_{(122)}\\
&+\sum_{t=3}^{n}\sum_{i,j\geq 0}F_{\mathrm{wt}a_2,i}(z_2,w)\binom{\mathrm{wt}a_1-1}{j}z_1^{-\mathrm{wt}a_1}w^{\mathrm{wt}a_1-1-j}\\
&\cdot\underbrace{S(v_3',(a_3,z_3)\ldots(a_2(i)a_t,z_t)\ldots(a_n,z_n)(a_1(j)v,w)v_2)}_{(123)}\\
=\;&(121)+(122)+(123),\\
(13)=\;&\sum_{k=3}^{n}\sum_{i,j\geq 0}F_{\mathrm{wt}a_2,i}(z_2,z_k)\binom{\mathrm{wt}a_1-1}{j}z_1^{-wta_1}z_k^{\mathrm{wt}a_1-1-j}\\
&\cdot\underbrace{S(v_3',(a_3,z_3)\ldots(a_1(j)a_k,z_k)\ldots(a_n,z_n)(a_2(i)v,w)v_2)}_{(131)}\\
&+\sum_{i,j\geq 0}F_{\mathrm{wt}a_2,i}(z_2,w)\binom{\mathrm{wt}a_1-1}{j}z_1^{-\mathrm{wt}a_1}w^{\mathrm{wt}a_1-1-j}\underbrace{S(v_3',(a_3,z_3)\ldots(a_n,z_n)(a_1(j)a_2(i)v,w)v_2)}_{(132)}\\
=\;&(131)+(132).
\end{aligned}
$$

Then (1)=(11)+(121)+(122)+(123)+(131)+(132).

Now we expand (2), (3), and (4) with respect to their corresponding left-most terms. By (3.2.8), they can be expressed as follows:

$$(2) = \sum_{j\geq 0} -\binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} z_2^{-\mathrm{wt}a_2} \underbrace{S(v_3' o(a_1(j)a_2), (a_3, z_3) \dots (a_n, z_n)(v, w)v_2)}_{(21)}$$

$$+ \sum_{k=3}^{n} \sum_{i,j\geq 0} -\binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} z_2^{\mathrm{wt}a_1 - 1 - j} F_{\mathrm{wt}a_1 + \mathrm{wt}a_2 - j - 1, i}(z_2, z_k)$$

$$\underbrace{\cdot S(v_3', (a_3, z_3) \dots ((a_1(j)a_2)(i)a_k, z_k) \dots (a_n, z_n)(v, w)v_2)}_{(22)}$$

$$+ \sum_{i,j\geq 0} -\binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} z_2^{-\mathrm{wt}a_1 - 1 - j} F_{\mathrm{wt}a_1 + \mathrm{wt}a_2 - j - 1, i}(z_2, w)$$

$$\underbrace{\cdot S(v_3', (a_3, z_3) \dots (a_n, z_n)((a_1(j)a_2)(i)v, w)v_2)}_{(23)}$$

$$= (21) + (22) + (23).$$

$$(3) = \sum_{k=3}^{n} \sum_{j\geq 0} -\binom{wta_1 - 1}{j} z_1^{-\mathrm{wt}a_1} z_k^{\mathrm{wt}a_1 - 1 - j} \underbrace{S(v_3' o(a_2), (a_3, z_3) \dots (a_1(j)a_k, z_k) \dots v_2) z_2^{-\mathrm{wt}a_2}}_{(31)}$$

$$+ \sum_{k=3}^{n} \sum_{j,i\geq 0} -\binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} z_k^{\mathrm{wt}a_1 - 1 - j} F_{\mathrm{wt}a_2, i}(z_2, w)$$

$$\underbrace{\cdot S(v_3', (a_3, z_3) \dots (a_1(j)a_k, z_k) \dots (a_n, z_n)(a_2(i)v, w)v_2)}_{(32)}$$

$$+ \sum_{k=3}^{n} \sum_{t=3, t\neq k}^{n} \sum_{j,i\geq 0} -\binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} z_k^{\mathrm{wt}a_1 - 1 - j} F_{\mathrm{wt}a_2, i}(z_2, z_t)$$

$$\underbrace{\cdot S(v_3', (a_3, z_3) \dots (a_2(i)a_t, z_t) \dots (a_1(j)a_k, z_k) \dots (a_n, z_n)(v, w)v_2)}_{(33)}$$

$$+ \sum_{k=3}^{n} \sum_{j,i\geq 0} -\binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} z_k^{\mathrm{wt}a_1 - 1 - j} F_{\mathrm{wt}a_2, i}(z_2, z_k)$$

$$\underbrace{\cdot S(v_3', (a_3, z_3) \dots (a_2(i)a_1(j)a_k, z_k) \dots (a_n, z_n)(v, w)v_2)}_{(34)}$$

$$= (31) + (32) + (33) + (34).$$

$$(4) = \sum_{j\geq 0} -\binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} w^{\mathrm{wt}a_1 - 1 - j} \underbrace{S(v_3' o(a_2), (a_3, z_3) \dots (a_n, z_n)(a_1(j)v, w)v_2) z_2^{-\mathrm{wt}a_2}}_{(41)}$$

$$+ \sum_{t=3}^{n} \sum_{j,i\geq 0} -\binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} w^{\mathrm{wt}a_1 - 1 - j} F_{\mathrm{wt}a_2, i}(z_2, z_k)$$

$$\underbrace{\cdot S(v_3', (a_3, z_3) \dots (a_2(i)a_k, z_k) \dots (a_n, z_n)(a_1(j)v, w)v_2)}_{(42)}$$

$$+\sum_{j,i\geq 0} -\binom{\mathrm{wt} a_1 - 1}{j} z_1^{-\mathrm{wt} a_1} w^{\mathrm{wt} a_1 - 1 - j} F_{\mathrm{wt} a_2, i}(z_2, w) \underset{(43)}{S(v_3', (a_3, z_3) \dots (a_n, z_n)(a_2(i) a_1(j) v, w) v_2)}$$

$$= (41) + (42) + (43).$$

By Lemma 4.1.18 and the formula (3.2.5) of $\iota_{z_2, z_t} F_{n,i}(z_2, z_t)$, we have:

$$\begin{aligned}
&\sum_{i,j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j} F_{\mathrm{wt} a_2, i}(z_2, z_t) a_1(j) a_2(i) a_t + \sum_{i,j\geq 0} -\binom{\mathrm{wt} a_1 - 1}{j} F_{\mathrm{wt} a_2, i}(z_2, z_t) a_1(j) a_2(i) a_t \\
&+ \sum_{i,j\geq 0} -\binom{\mathrm{wt} a_1 - 1}{j} F_{\mathrm{wt} a_1 + \mathrm{wt} a_2 - j - 1, i}(z_2, z_t)(a_1(j) a_2)(i) a_t \qquad (4.1.25) \\
&= 0,
\end{aligned}$$

and the same equation holds if we replace z_t with w and a_i with v. Using (4.1.25), we have the cancelations (122) + (22) + (34) = 0, and (132) + (23) + (43) = 0. Moreover, it follows directly from the expressions of the terms (123), (42), (121), (33), (131), and (32) that

$$(123) + (42) = 0, \quad (121) + (33) = 0, \ \text{ and } \ (131) + (32) = 0.$$

Now it remains to show (11)+(21)+(31)+(41)=0, or equivalently,

$$\begin{aligned}
&S(v_3' o(a_1) o(a_2), (a_3, z_3) \dots (a_n, z_n)(v, w) v_2) - S(v_3' o(a_2), (a_3, z_3) \dots (a_n, z_n)(v, w) o(a_1) v_2) \\
&= \sum_{j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j} S(v_3' o(a_1(j) a_2), (a_3, z_3) \dots (a_n, z_n)(v, w) v_2) \\
&+ \sum_{k=3}^{n} \sum_{j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j} z_k^{\mathrm{wt} a_1 - 1 - j} S(v_3' o(a_2), (a_3, z_3) \dots (a_1(j) a_k, z_k) \dots (v, w) v_2) \qquad (4.1.26) \\
&+ \sum_{j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j} w^{\mathrm{wt} a_1 - 1 - j} S(v_3' o(a_2), (a_3, z_3) \dots (a_n, z_n)(a_1(j) v, w) v_2),
\end{aligned}$$

but this is a consequence of Lemma 4.1.22. In fact,

$$\begin{aligned}
&\text{L.H.S. of } (4.1.26) \\
&= S(v_3' o(a_1) o(a_2), (a_3, z_3) \dots (a_n, z_n)(v, w) v_2) - S(v_3' o(a_2) o(a_1), (a_3, z_3) \dots (a_n, z_n)(v, w) v_2) \\
&+ S(v_3' o(a_2) o(a_1), (a_3, z_3) \dots (a_n, z_n)(v, w) v_2) - S(v_3' o(a_2), (a_3, z_3) \dots (a_n, z_n)(v, w) o(a_1) v_2).
\end{aligned}$$

Since S is linear in the place $M^3(0)^*$, we have

$$S(v_3' o(a_1) o(a_2), (a_3, z_3) \dots (a_n, z_n)(v, w) v_2) - S(v_3' o(a_2) o(a_1), (a_3, z_3) \dots (a_n, z_n)(v, w) v_2)$$

$$= S(v_3'[o(a_1), o(a_2)], (a_3, z_3)\dots(a_n, z_n)(v, w)v_2)$$
$$= \sum_{j\geq 0} \binom{\mathrm{wt}a_1 - 1}{j} S(v_3' o(a_1(j)a_2), (a_3, z_3)\dots(a_n, z_n)(v, w)v_2),$$

which is the first term on the right-hand side of (4.1.26). Moreover, by Lemma 4.1.22,

$$S(v_3' o(a_2)o(a_1), (a_3, z_3)\dots(a_n, z_n)(v, w)v_2) - S(v_3' o(a_2), (a_3, z_3)\dots(a_n, z_n)(v, w)o(a_1)v_2)$$
$$= \sum_{k=3}^{n} \sum_{j\geq 0} \binom{\mathrm{wt}a_1 - 1}{j} z_k^{\mathrm{wt}a_1 - 1 - j} S(v_3' o(a_2), (a_3, z_3)\dots(a_1(j)a_k, z_k)\dots(v, w)v_2)$$
$$+ \sum_{j\geq 0} \binom{\mathrm{wt}a_1 - 1}{j} w^{\mathrm{wt}a_1 - 1 - j} S(v_3' o(a_2), (a_3, z_3)\dots(a_n, z_n)(a_1(j)v, w)v_2),$$

which gives us the last two summands on the right-hand side of (4.1.26). This proves (4.1.26). Hence (1) + (2) + (3) + (4) = 0, and so (4.1.23) holds.

Then by Proposition 4.1.20, all the $(n+3)$-point functions $S^L_{VV\dots M\dots V}$ and $S^R_{V\dots M\dots VV}$ defined by (4.1.19) and (4.1.20) give rise to one single $(n+3)$-point function:

$$S : M^3(0)^* \times V \times \cdots \times M^1 \times \cdots \times V \times M^2(0) \to \mathcal{F}(z_1, \dots, z_n, w), \tag{4.1.27}$$

where M^1 can be placed anywhere in between the first and the last place of V. Moreover, by Definition 4.1.19 and (4.1.21), S in (4.1.27) satisfies the locality I and the expansion property II, with n replaced by $n+1$. Therefore, the induction step is complete.

4.2 The general fusion rules Theorem

In this Section, we will show that $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ can be identified with the vector space $(M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$. Using this isomorphism, together with the isomorphism given by Theorem 3.3.9, we can prove the fusion rules Theorem for general VOAs.

However, there are counter-examples showing that this identification is false if one replaces $B_h(M^1)$ by the $A(V)$-bimodule $A(M^1)$ constructed in Theorem 1.5.1 in [30] or $A_0(M^1)$ constructed in Section 4 of [42]. The reason is that the correct $L(-1)$-derivation property of the intertwining operators cannot be captured by $A(M^1)$ nor $A_0(M^1)$, and this property cannot be dissolved by taking the tensor product with bottom levels as in Proposition 4.1.12 for general VOAs neither. We will give more details and examples in the last subsection.

4.2.1 The correspondence between correlation functions on the bottom levels and functions on $A(V)$-modules

Theorem 4.2.1. *The system of* $(n+3)$*-point functions* S *we constructed by Definitions 4.1.14, 4.1.16, and 4.1.19 in this subsection lies in* $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$.

Proof. Since S is constructed inductively by the recursive formulas (3.2.4) and (3.2.6) in view of Defintions 4.1.14, 4.1.16, and 4.1.19, it obviously satisfies (3.2.4) and (3.2.6). By (4.1.10), we have $S(v_3', (v,w)v_2) = f(v_3'\otimes v\otimes v_2)w^{-\deg v}$, for any $v_3'\in M^3(0)^*$, $v\in M^1$, and $v_2\in M^2(0)$. By the Hom-tensor duality, we have a well-defined element $f_v\in \mathrm{Hom}_{\mathbb{C}}(M^2(0), M^3(0))$ such that $\langle v_3', f_v(v_2)\rangle = f(v_3'\otimes v\otimes v_2)$ for each $v\in M^1$. Hence S satisfies (3.2.3).

In view of Definition 7.1.6, it remains to show that S satisfies (2) − (6) in Definition 3.1.1 for $v_2\in M^2(0)$ and $v_3'\in M^3(0)^*$. Indeed, the locality follows from (I), and by (4.1.19),

$$
\begin{aligned}
&S(v_3', (\mathbf{1}, z)(a_1, z_1)\dots(a_n, z_n)(v, w)v_2) = S(v_3' o(\mathbf{1}), (a_1, z_1)\dots(a_n, z_n)(v, w)v_2)z^{-\mathrm{wt}\mathbf{1}}\\
&\quad+\sum_{k=1}^{n}\sum_{j\geq 0} F_{\mathrm{wt}\mathbf{1},j}(z, z_j)S(v_3', (a_1, z_1)\dots(\mathbf{1}(j)a_k, z_k)\dots(a_n, z_n)(v, w)v_2)\\
&\quad+\sum_{j\geq 0} F_{\mathrm{wt}\mathbf{1},j}(z, w)S(v_3', (a_1, z_1)\dots(a_n, z_n)(\mathbf{1}(j)v, w)v_2)\\
&= S(v_3', (a_1, z_1)\dots(a_n, z_n)(v, w)v_2),
\end{aligned}
$$

since $\mathbf{1}(j)a_k = \mathbf{1}(j)v = 0$ when $j\geq 0$, and $o(\mathbf{1}) = \mathrm{Id}$.

Again because S in (4.1.27) satisfies (4.1.19), it is easy to verify the following associativity formulas by a similar argument to the proof of (2.2.9) in [73]:

$$
\begin{aligned}
&\int_C S(v_3', (a_1, z_1)(v, w)\dots(a_n, z_n)v_2)(z_1-w)^n dz_1 = S(v_3', (a_1(k)v, w)\dots(a_n, z_n)v_2),\\
&\int_C S(v_3', (a_1, z_1)(a_2, z_2)\dots(v, w)v_2)(z_1-z_2)^n dz_1 = S(v_3', (a_1(k)a_2, z_2)\dots(v, w)v_2),
\end{aligned}
\tag{4.2.1}
$$

where in the first equation of (4.2.1), C is a contour of z_1 surrounding w with $z_2, \dots, z_n$ outside of C; while in the second equation of (4.2.1), C is a contour of z_1 surrounding z_2 with $z_3, \dots, z_n, w$ outside of C. We also have:

$$
\begin{aligned}
&S(v_3', (L(-1)a_1, z_1)\dots(a_n, z_n)(v, w)v_2) = \frac{d}{dz_1}S(v_3', (a_1, z_1)\dots(a_n, z_n)(v, w)v_2),\\
&S(v_3', (L(-1)v, w)(a_1, z_1)\dots(a_n, z_n)v_2)w^{-h} = \frac{d}{dw}(S(v_3', (v, w)(a_1, z_1)\dots v_2)w^{-h}).
\end{aligned}
\tag{4.2.2}
$$

The first equation in (4.2.2) is similar to (2.2.8) in [73]. We omit the details of the proof. To show the second equation in (4.2.2), we use induction on n. When $n = 0$, by (4.1.5) and Lemma 4.1.5, we have: $L(-1)v + (L(0) + h_2 - h_3)v \equiv 0 \mod O_h(M^1)$ for all $v \in M^1$. Then

$$\begin{aligned}
&S(v_3', (L(-1)v, w)v_2)w^{-h} = f(v_3' \otimes L(-1)v \otimes v_2)w^{-\deg v - 1 - h} \\
&= -f(v_3' \otimes (L(0) + h_2 - h_3)v \otimes v_2)w^{-\deg v - 1 - h} = f(v_3' \otimes v \otimes v_2)\frac{d}{dw}(w^{-\deg v - h}) \\
&= \frac{d}{dw}(S(v_3', (v, w)v_2)w^{-h}).
\end{aligned} \tag{4.2.3}$$

Now assume the second equation of (4.2.2) holds for the $(n + 2)$-point function, then by the properties (I) and (II) of S, we have:

$$\begin{aligned}
&S(v_3', (L(-1)v, w)(a_1, z_1) \ldots (a_n, z_n)v_2)w^{-h} = S^L(v_3', (a_1, z_1) \ldots (a_n, z_n)(L(-1)v, w)v_2)w^{-h} \\
&= S(v_3' o(a_1), (a_2, z_2) \ldots (a_n, z_n)(L(-1)v, w)v_2)z_1^{-\mathrm{wt} a_1} w^{-h} \\
&+ \sum_{k=2}^{n} \sum_{j \geq 0} F_{\mathrm{wt} a_1, j}(z_1, z_k) S(v_3', (a_2, z_2) \ldots (a_1(j)a_k, z_k) \ldots (a_n, z_n)(L(-1)v, w)v_2)w^{-h} \\
&+ \sum_{j \geq 0} F_{\mathrm{wt} a_1, j}(z_1, w) S(v_3', (a_2, z_2) \ldots (a_n, z_n)(a_1(j)L(-1)v, w)v_2)w^{-h}.
\end{aligned} \tag{4.2.4}$$

Note that we can apply the induction hypothesis to the first two terms of (4.2.4). Moreover, by the $L(-1)$-bracket formula (4.2.1) in [27], we have:

$$a_1(j)L(-1)v_2 = L(-1)a_1(j)v_2 - [L(-1), a_1(j)]v_2 = L(-1)a_1(j)v_2 + ja_1(j-1)v_2.$$

It follows from the induction hypothesis and (3.2.5) that

$$\begin{aligned}
&\sum_{j \geq 0} F_{\mathrm{wt} a_1, j}(z_1, w) S(v_3', (a_2, z_2) \ldots (a_n, z_n)(a_1(j)L(-1)v, w)v_2)w^{-h} \\
&= \sum_{j \geq 0} F_{\mathrm{wt} a_1, j}(z_1, w) \frac{d}{dw}(S(v_3', (a_2, z_2) \ldots (a_n, z_n)(a_1(j)v, w)v_2)w^{-h}) \\
&+ \sum_{j \geq 1} \frac{z_1^{-\mathrm{wt} a_1}}{(j-1)!} \left(\frac{d}{dw}\right)^j \left(\frac{w^{\mathrm{wt} a_1}}{z_1 - w}\right) S(v_3', (a_2, z_2) \ldots (a_n, z_n)(a_1(j-1)v, w)v_2)w^{-h} \\
&= \frac{d}{dw} \sum_{j \geq 0} F_{\mathrm{wt} a_1, j}(z_1, w) S(v_3', (a_2, z_2) \ldots (a_n, z_n)(a_1(j)v, w)v_2)w^{-h}.
\end{aligned}$$

This proves (4.2.2). Finally, let $v_3' \in M^3(0)^*, v \in M^1, v_2 \in M^2(0)$, and $a_1, \ldots, a_n \in V$ be highest weight vectors of the Virasoro algebra. By property (I) and (4.2.2) of S, we have:

$$S(v_3', (\omega, x)(\omega, x_1) \ldots (\omega, x_m)(a_1, z_1) \ldots (a_n, z_n)(v, w)v_2)$$

$$
\begin{aligned}
&= S(v_{3'}, (\omega, x_1)\dots(a_n, z_n)(v, w)o(\omega)v_2)x^{-2} \\
&+ \sum_{k=1}^{m}\sum_{j\geq 0} G_{2,j}(x, x_k) S(v_3', (\omega, x_1)\dots(\omega_j\omega, x_k)\dots(a_n, z_n)(v, w)v_2) \\
&+ \sum_{k=1}^{n}\sum_{j\geq 0} G_{2,j}(x, z_k) S(v_3', (\omega, x_1)\dots(\omega_j a_k, z_k)\dots(a_n, z_n)(v, w)v_2) \\
&+ \sum_{j\geq 0} G_{2,j}(x, w) S(v_3', (\omega, x_1)\dots(a_n, z_n)(\omega_j v, w)v_2).
\end{aligned}
$$

By the definition formula (3.2.7), it is easy to verify that:

$$
G_{2,0}(x,z) = \frac{x^{-1}z}{x-z}, \quad G_{2,1}(x,z) = \frac{1}{(x-z)^2}, \quad G_{2,3}(x,z) = \frac{1}{(x-z)^4}.
$$

Then by using the properties of the Virasoro element ω (see Section 2.3 in [27]), we have:

$$
\begin{aligned}
&S(v_3', (\omega, x)(\omega, x_1)\dots(\omega, x_m)(a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2) \\
&= \sum_{k=1}^{n} \frac{x^{-1}z_k}{x-z_k}\frac{d}{dz_k}S + \sum_{k=1}^{n}\frac{\mathrm{wt} a_k}{(x-z_k)^2}S + \frac{x^{-1}w}{x-w}w^h\frac{d}{dw}(S\cdot w^{-h}) + \frac{\mathrm{wt} v}{(x-w)^2}S \\
&+ \frac{h_2}{x^2}S + \sum_{k=1}^{m}\frac{x^{-1}wx_k}{x-x_k}\frac{d}{dx_k}S + \sum_{k=1}^{m}\frac{2}{(x-x_k)^2}S \\
&+ \frac{c}{2}\sum_{k=1}^{m}\frac{1}{(x-x_k)^4}S(v_3', (\omega, x_1)\dots\widehat{(\omega, x_k)}\dots(\omega, x_m)(a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2),
\end{aligned}
$$

where $S = S(v_3', (\omega, x_1)\dots(\omega, x_m)(a_1, z_1)\dots(a_n, z_n)(v, w)v_2)$. This shows that the S in (3.3.4) also satisfies (3.1.11), with $v_3' \in M^3(0)^*$ and $v_2 \in M^2(0)$. Therefore, $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. □

Remark 4.2.2. By equation (4.2.3), we see that it is necessary to have the equality $L(-1)v + (L(0)+h_2-h_3)v = 0$ hold in the bimodule $B_h(M^1)$ to show the $L(-1)$-derivation property (4.2.2) of S. However, in general, such equality does not hold in the bimodule $A(M^1)$ in [30] by its construction. This partially explain the reason why $I\binom{M^3}{M^1\ M^2}$ is not isomorphic to $(M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))^*$ in general.

Theorem 4.2.1 indicates that we have a well-defined linear map:

$$
\mu : (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^* \to \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}, \quad f \mapsto S_f, \tag{4.2.5}
$$

where S_f is the S we constructed in this subsection by Defintions 4.1.14, 4.1.16, and 4.1.19.

Since we have $S_f(v_3', (v,w)v_2) = f(v_3' \otimes v \otimes v_2)w^{-\deg v}$ by (4.1.10), and $f_{S_f}(v_3' \otimes v \otimes v_2)w^{-\deg v} = S_f(v_3', (v,w)v_2)$ by (4.1.6) and Definition 7.1.6, then $f_{S_f} = f$. i.e., $\nu\mu = 1$. On the other hand, for $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, again by (4.1.10) and (4.1.6), we have:

$$S_{f_S}(v_3', (v,w)v_2) = f_S(v_3' \otimes v \otimes v_2)w^{-\deg v} = S(v_3', (v,w)v_2).$$

Moreover, S_{f_S} and S satisfy the same recursive formulas by (4.1.19), (4.1.20), (3.2.4), and (3.2.6), then it follows from an easy induction that $S_{f_S} = S$. i.e., $\mu \circ \nu = 1$, and so μ is an isomorphism. Now we have our main result for Chapter 3 and Chapter 4:

Theorem 4.2.3. *Let M^1, M^2, and M^3 be V-modules, with conformal weight h_1, h_2, and h_3, respectively. Assume $M^2(0)$ and $M^3(0)$ are irreducible $A(V)$-modules. Then we have the following isomorphism of vector spaces:*

$$\begin{aligned} I\binom{\bar{M}(M^3(0)^*)'}{M^1\ \bar{M}(M^2(0))} \cong I\binom{\bar{M^3}}{M^1\ \bar{M^2}} &\cong (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*, \\ I \mapsto f_I, \quad f_I(v_3' \otimes v \otimes v_2) &= \langle v_3', o(v)v_2\rangle, \end{aligned} \tag{4.2.6}$$

for all $v_3' \in M^3(0)^*$, $v \in M^1$, and $v_2 \in M^2(0)$, where $h = h_1 + h_2 - h_3$, and $M^2 = \bar{M}/\mathrm{Rad}(\bar{M})$ and $M^3 = (\tilde{M}/\mathrm{Rad}\tilde{M})'$ are quotient modules of the generalized Verma module $\bar{M}(M^2(0))$ and $\bar{M}(M^3(0))$, respectively.

Proof. This is a direct consequence of Corollary 3.1.6, Theorem 3.3.9, and Theorem 4.2.1, together of which give us the isomorphism: $I\binom{\bar{M}(M^3(0)^*)'}{M^1\ \bar{M}(M^2(0))} \cong I\binom{\bar{M^3}}{M^1\ \bar{M^2}} \cong \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)} \cong (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$, such that $I \mapsto f_I$ as in (4.2.6). □

Recall that V-modules $\bar{M^2}$ and $\bar{M^3}'$ are irreducible if condition (3.3.5) is satisfied (see Proposition 3.3.6). By the isomorphism (4.2.5), condition (3.3.5) translates to the following:

For any $f \in (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$, one has:

$$\sum_{i\geq 0} \binom{n}{i} f(v_3' \otimes b(i)v \otimes v_2) = 0, \tag{4.2.7}$$

for all $b \in V$, $n \in \mathbb{Z}$ such that $\mathrm{wt} b - n - 1 > 0$, $v \in M^1$, $v_3' \in M^3(0)^*$, and $v_2 \in M^2(0)$.

Corollary 4.2.4. *Let M^1, M^2, and M^3 be V-modules, with conformal weight h_1, h_2, and h_3, respectively. Suppose M^2 and M^3 are irreducible, and condition (4.2.7) is satisfied, then we have an isomorphism: $I\binom{M^3}{M^1\ M^2} \cong (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$.*

Suppose M^2 and M^3 are V-modules (not necessarily irreducible) that are generated by their corresponding bottom levels $M^2(0)$ and $M^3(0)$, respectively, which are irreducible $A(V)$-modules. Then by (3.3.21) and (4.2.6), we have the following estimate of the fusion rule:

$$\dim I\binom{M^3}{M^1\ M^2} \leq \dim(M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*. \tag{4.2.8}$$

Finally, if V is rational, by Theorem 4.2.3, Corollary 3.3.10, and Proposition 4.1.12, the original version of the fusion rules Theorem in [30] is true:

Corollary 4.2.5. *Let V be a rational VOA, and let M^1, M^2, and M^3 be V modules, with conformal weight h_1, h_2, and h_3, respectively. Suppose M^2 and M^3 are irreducible, then*

$$I\binom{M^3}{M^1\ M^2} \cong (M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))^*. \tag{4.2.9}$$

4.2.2 Examples

In this subsection, we will use (4.2.6) and the estimating formula (4.2.8) and compute the fusion rules for certain modules over the Virasoro VOAs and the Heisenberg VOAs.

Example 4.2.6. A counter-example that shows $I\binom{M^3}{M^1\ M^2}$ is not isomorphic to $(M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))^*$ was presented in Section 2 in [49]. It was given as follows:

Recall that the (universal) Virasoro VOA $M_c = M(c, 0)/\langle L(-1)v_{c,0}\rangle$ defined in [30] has Zhu's algebra $A(M_c) \cong \mathbb{C}[t]$, with $[\omega]^n \mapsto t^n$. Let $M(c, h)$ be the Verma module of highest weight h and central charge c over the Virasoro algebra, then $M(c, h)$ is a module over M_c, and we have the following equalities held in $A(M(c, h))$:

$$[b] * [\omega]^n = [(L(-2) + L(-1))^n b], \qquad [\omega]^n * [b] = [(L(-2) + 2L(-1) + L(0))^n b],$$

for all $b \in M(c, h)$ and $n \in \mathbb{N}$. Hence there is an identification of $\mathbb{C}[t] \cong A(M_c)$-bimodules:

$$\begin{aligned} &\mathbb{C}[t_1, t_2] \cong A(M(c, h)) \\ &f(t_1, t_2) \mapsto f(L(-2) + 2L(-1) + L(0), L(-2) + L(-1))v_{c,h}, \end{aligned} \tag{4.2.10}$$

where $C[t_1, t_2]$ is a bimodule over $\mathbb{C}[t]$ on which the actions are given by:

$$t^n.f(t_1, t_2) = t_1^n f(t_1, t_2), \qquad f(t_1, t_2).t^n = t_2^n f(t_1, t_2).$$

For $h_1, h_2 \in \mathbb{C}$ such that $M(c, h_1)$ and $M(c, h_2)$ are irreducible, it is proved (see (2.37) in [49]) that $I\binom{M(c,h_2)}{M(c,h_1)\ M_c} = 0$, while $\dim(M(c, h_2)(0)^* \otimes_{A(M_c)} A(M(c, h_1)) \otimes_{A(M_c)} M_c(0))^* = 1$.

Although $M^2 = M_c$ is neither a generalized Verma module nor irreducible, we can still use (4.2.6) and (4.2.8) to obtain the correct fusion rules. Indeed, since M_c and $M(c, h_2)$ are both generalized by their bottom levels, by (4.2.8), we have:

$$\dim I\binom{M(c,h_2)}{M(c,h_1)\ M_c} \leq \dim(M(c,h_2)(0)^* \otimes_{A(M_c)} B_h(M(c,h_1)) \otimes_{A(M_c)} M_c(0))^*. \qquad (4.2.11)$$

Moreover, since $h = h_1 + 0 - h_2$, it follows from Lemma 4.1.4 and Lemma 4.1.5 that

$$B_h(M(c,h_1)) = A(M(c,h_1))/\mathrm{span}\{(L(-1) + L(0) - h_2)[b] : b \in M(c,h_1)\}.$$

Then $[L(-1)b] = -[(\deg b + h_1 - h_2)b]$ in $B_h(M(c,h_1))$. It follows from (4.2.10) that

$$B_h(M(c,h_1)) \cong \mathbb{C}[t_0], \quad \text{with} \quad [(L(-2) - L(0) + h_2)^n v_{c,h_1}] \mapsto t_0^n,$$

and $\mathbb{C}[t_0]$ is a $\mathbb{C}[t](\cong A(M_c))$-bimodule on which the actions are given by:

$$f(t_0).t^n = t_0^n f(t_0), \quad \text{and} \quad t.f(t_0) = (t_0 + h_2)^n f(t_0).$$

Hence we have $B_h(M(c,h_1)) \otimes_{A(M_c)} M_c(0) \cong \mathbb{C}[t_0] \otimes_{\mathbb{C}[t]} M_c(0) \cong M_c(0)$, and so

$$(M(c,h_2)(0)^* \otimes_{A(M_c)} B_h(M(c,h_1)) \otimes_{A(M_c)} M_c(0))^* \cong \mathrm{Hom}_{A(M_c)}(M_c(0), M(c,h_2)(0)) = 0,$$

since $o(\omega)v_{c,0} = 0$, $o(\omega)v_{c,h_2} = h_2 v_{c,h_2}$ and $h_2 \neq 0$. Thus, $I\binom{M(c,h_2)}{M(c,h_1)\ M_c} = 0$ by (4.2.11).

We give another example that shows that the bimodule $B_h(M^1)$ in (4.2.6) cannot be replaced by the $A(V)$-bimodule $A_0(M^1)$ defined in [42] either.

Example 4.2.7. Let $V = M_{\widehat{\mathfrak{h}}}(1,0)$ be the Heisenberg VOA of level 1 associated to a one-dimensional vector space $\mathfrak{h} = \mathbb{C}\alpha$ with $(\alpha|\alpha) = 1$. By Theorem 3.1.1 in [30], one has $A(M_{\widehat{\mathfrak{h}}}(1,0)) \cong \mathbb{C}[x]$, with $[\alpha(-i_1 - 1)\dots\alpha(-i_n - 1)\mathbf{1}] \mapsto (-1)^{i_1+\dots+i_n} x^n$.

Let $\lambda \in \mathfrak{h}$, we have a V-module $M_{\widehat{\mathfrak{h}}}(1,\lambda) = M_{\widehat{\mathfrak{h}}}(1,0) \otimes_{\mathbb{C}} \mathbb{C}e^\lambda$, with conformal weight $h = \frac{(\lambda|\lambda)}{2}$. Note that $M_{\widehat{\mathfrak{h}}}(1,\lambda)$ is the Verma module over the Heisenberg Lie algebra $\widehat{\mathfrak{h}}$. Since $M_{\widehat{\mathfrak{h}}}(1,\lambda)$ is irreducible, it is automatically a generalized Verma module associated with its bottom level $\mathbb{C}e^\lambda$. By Theorem 3.2.1 in [30], we have:

$$A(M_{\widehat{\mathfrak{h}}}(1,\lambda)) \cong \mathbb{C}e^\lambda \otimes_{\mathbb{C}} \mathbb{C}[x], \quad \text{with} \quad [\alpha(-i_1 - 1)\dots\alpha(-i_n - 1)e^\lambda] \mapsto (-1)^{i_1+\dots+i_n} e^\lambda \otimes x^n,$$

where the bimodule actions are given by $x.(e^\lambda \otimes x^n) = e^\lambda \otimes x^{n+1} + (\lambda|\alpha)e^\lambda \otimes x^n$, and $(e^\lambda \otimes x^n).x = e^\lambda \otimes x^{n+1}$ for all $n \in \mathbb{N}$. By definition in Section 4 of [42],

$$A_0(M_{\widehat{\mathfrak{h}}}(1,\lambda)) = A(M_{\widehat{\mathfrak{h}}}(1,\lambda))/\mathrm{span}\{[(L(-1) + L(0) - (\lambda|\lambda)/2)b] : b \in M_{\widehat{\mathfrak{h}}}(1,\lambda)\}.$$

Choose $\lambda \in \mathfrak{h}$ such that $(\lambda|\alpha) \neq 0$. Recall that $\omega = \frac{1}{2}\alpha(-1)^2\mathbf{1}$, and so

$$L(-1)e^{\lambda} = \mathrm{Res}_z Y_W(\omega, z)e^{\lambda} = \sum_{i\geq 0} \alpha(-1-i)\alpha(i)e^{\lambda} = (\lambda|\alpha)\alpha(-1)e^{\lambda}.$$

Then we have $[(\lambda|\alpha)\alpha(-1)e^{\lambda}] = [L(-1)e^{\lambda}] = -[(L(0) - (\lambda|\lambda)/2)e^{\lambda}] = 0$ in $A_0(M_{\widehat{\mathfrak{h}}}(1,\lambda))$, and $[\alpha(-1)e^{\lambda}] = 0$ in $A_0(M_{\widehat{\mathfrak{h}}}(1,\lambda))$. For any spanning element $[\alpha(-i_1-1)\dots\alpha(-i_n-1)e^{\lambda}]$ of $A_0(M_{\widehat{\mathfrak{h}}}(1,\lambda))$, we then have $[\alpha(-i_1-1)\dots\alpha(-i_n-1)e^{\lambda}] = (-1)^{i_1+\dots+i_n}[\alpha(-1)^n e^{\lambda}] = 0$ for $n > 0$. Thus, $A_0(M_{\widehat{\mathfrak{h}}}(1,\lambda)) \cong \mathbb{C}[e^{\lambda}]$, with the module actions given by:

$$x.[e^{\lambda}] = (\lambda|\alpha)[e^{\lambda}], \quad \text{and} \quad [e^{\lambda}].x = 0. \tag{4.2.12}$$

Now choose $\mu \in \mathfrak{h}$ such that $(\mu|\alpha) \neq 0$, it is well-known that $\dim I\binom{M_{\widehat{\mathfrak{h}}}(1,\lambda+\mu)}{M_{\widehat{\mathfrak{h}}}(1,\lambda)\ M_{\widehat{\mathfrak{h}}}(1,\mu)} = 1$. But

$$A_0(M_{\widehat{\mathfrak{h}}}(1,\lambda)) \otimes_{A(M_{\widehat{\mathfrak{h}}}(1,0))} M_{\widehat{\mathfrak{h}}}(1,\mu)(0) \cong \mathbb{C}[e^{\lambda}] \otimes_{\mathbb{C}[x]} \mathbb{C}e^{\mu} = 0,$$

since it follows from (4.2.12) that $[e^{\lambda}] \otimes e^{\mu} = \frac{1}{(\mu|\alpha)}[e^{\lambda}] \otimes o(\alpha(-1)\mathbf{1})e^{\mu} = \frac{1}{(\mu|\alpha)}[e^{\lambda}].x \otimes e^{\mu} = 0$ in the tensor product above. Then we have:

$$\dim(M_{\widehat{\mathfrak{h}}}(1,\lambda+\mu)(0)^* \otimes_{A(M_{\widehat{\mathfrak{h}}}(1,0))} A_0(M_{\widehat{\mathfrak{h}}}(1,\lambda)) \otimes_{A(M_{\widehat{\mathfrak{h}}}(1,0))} M_{\widehat{\mathfrak{h}}}(1,\mu)(0))^* = 0 \neq 1.$$

This shows that the isomorphism (4.2.6) is not true if one replaces $B_h(M^1)$ with $A_0(M^1)$.

Now we verify (4.2.6) in this case. Indeed, since $\mathfrak{h} = \mathbb{C}\alpha$, then $(\lambda|\alpha) \neq 0$ and $(\mu|\alpha) \neq 0$ imply that $\lambda = m\alpha$ and $\mu = n\alpha$, with $m \neq 0$ and $n \neq 0$. Hence

$$h = \frac{(\lambda|\lambda)}{2} + \frac{(\mu|\mu)}{2} - \frac{(\lambda+\mu|\lambda+\mu)}{2} = -(\lambda|\mu) = -mn \neq 0.$$

By definition 4.1.1, we have the following equality holds in $B_h(M_{\widehat{\mathfrak{h}}}(1,\lambda))$:

$$[(\lambda|\alpha)\alpha(-1)e^{\lambda}] = [L(-1)e^{\lambda}] = -[(L(0) - \frac{(\lambda|\lambda)}{2} + h)e^{\lambda}] = -(\lambda|\mu)[e^{\lambda}]$$

Then for any spanning element $[\alpha(-i_1-1)\dots\alpha(-i_n-1)e^{\lambda}]$ of $B_h(M_{\widehat{\mathfrak{h}}}(1,\lambda))$, we have:

$$[\alpha(-i_1-1)\dots\alpha(-i_n-1)e^{\lambda}] = (-1)^{i_1+\dots+i_n}[\alpha(-1)^n e^{\lambda}] = (-1)^{i_1+\dots+i_n}\left(\frac{-(\lambda|\mu)}{(\lambda|\alpha)}\right)^n [e^{\lambda}].$$

Thus $B_h(M_{\widehat{\mathfrak{h}}}(1,\lambda)) = \mathbb{C}[e^{\lambda}]$, with the module actions given by

$$[e^{\lambda}].x = \frac{-(\lambda|\mu)}{(\lambda|\alpha)}[e^{\lambda}](\neq 0), \quad \text{and} \quad x.[e^{\lambda}] = \frac{-(\lambda|\mu)}{(\lambda|\alpha)}[e^{\lambda}] + (\lambda|\alpha)[e^{\lambda}]. \tag{4.2.13}$$

Then by (4.2.13), we have $B_h(M_{\widehat{\mathfrak{h}}}(1,\lambda)) \otimes_{A(M_{\widehat{\mathfrak{h}}}(1,0))} M_{\widehat{\mathfrak{h}}}(1,\mu)(0) \cong \mathbb{C}[e^\lambda] \otimes_{\mathbb{C}[x]} \mathbb{C}e^\mu$ is a one-dimensional vector space, with $x.[e^\lambda] \otimes e^\mu = [e^\lambda].x \otimes e^\mu + (\lambda|\alpha)[e^\lambda] \otimes e^\mu = (\lambda + \mu|\alpha)[e^\lambda] \otimes e^\mu$. On the other hand, $x.e^{\lambda+\mu} = (\lambda+\mu|\alpha)e^{\lambda+\mu}$. Thus we have:

$$\dim \mathrm{Hom}_{A(M_{\widehat{\mathfrak{h}}}(1,0))}(B_h(M_{\widehat{\mathfrak{h}}}(1,\lambda)) \otimes_{A(M_{\widehat{\mathfrak{h}}}(1,0))} M_{\widehat{\mathfrak{h}}}(1,\mu)(0), M_{\widehat{\mathfrak{h}}}(1,\lambda+\mu)(0)) = 1.$$

This shows (4.2.6) is true for $M^1 = M_{\widehat{\mathfrak{h}}}(1,\lambda)$, $M^2 = M_{\widehat{\mathfrak{h}}}(1,\mu)$, and $M^3 = M_{\widehat{\mathfrak{h}}}(1,\lambda+\mu)$.

Furthermore, the argument above also shows that $B_h(M_{\widehat{\mathfrak{h}}}(1,\lambda)) \otimes_{A(M_{\widehat{\mathfrak{h}}}(1,0))} M_{\widehat{\mathfrak{h}}}(1,\mu)(0)$ is a one-dimensional vector space spanned by an eigenvector of $\mathfrak{h}$ of eigenfunction $(\lambda+\mu|\cdot)$. Hence we have:

$$\mathrm{Hom}_{A(M_{\widehat{\mathfrak{h}}}(1,0))}(B_h(M_{\widehat{\mathfrak{h}}}(1,\lambda)) \otimes_{A(M_{\widehat{\mathfrak{h}}}(1,0))} M_{\widehat{\mathfrak{h}}}(1,\mu)(0), M_{\widehat{\mathfrak{h}}}(1,\gamma)(0)) = 0,$$

if $\gamma \neq \lambda+\mu$. On the other hand, for $\gamma \neq \lambda+\mu$, it is well-known that $I\binom{M_{\widehat{\mathfrak{h}}}(1,\gamma)}{M_{\widehat{\mathfrak{h}}}(1,\lambda)\ M_{\widehat{\mathfrak{h}}}(1,\mu)} = 0$. Thus, the rank one Heisenberg VOA verifies (4.2.6).

Remark 4.2.8. For irrational VOAs, the previous examples indicate that the vector spaces

$$\left(M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0)\right)^*, \quad \text{and} \quad \left(M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0)\right)^*$$

maybe still be isomorphic to each other. For the universal Virasoro VOA, Example 4.2.6 indicate that these two spaces are **not** isomorphic. For the rank one Heisenberg VOA in Example 4.2.7, it is easy to see that these two spaces are isomorphic. We suspect that this isomorphism is also true for the vacuum module VOA $V_{\widehat{\mathfrak{g}}}(k,0)$ with positive integral level $k \in \mathbb{Z}_+$.

4.2.3 The fusion tensor of modules over rational VOAs

The notion of the tensor product of modules over VOAs is defined by the universal property, which is a generalization of the universal property of the tensor product of modules over compact Lie groups or their Lie algebras, see [38, 53]:

Definition 4.2.9. Let V be a VOA, and M, N be two V-modules. The tensor product of M and N is a pair $(M \boxtimes N, I)$, where $M \boxtimes N$ is a V-module, and I is an intertwining operator of the form $\binom{M\boxtimes N}{M\ N}$, satisfying the following universal property:

For any V-module W and any intertwining operator $\mathcal{Y} \in I\binom{W}{M\ N}$, there exists a unique homomorphism of V-modules $f : M \boxtimes N \to W$ such that $f(I(u,z)v) = \mathcal{Y}(u,z)v$, for any $u \in M$ and $v \in N$. If a tensor product exists, then it must be unique up to isomorphism.

The tensor product of modules over VOAs was constructed by Huang and Lepowsky in [38, 39, 40], and later generalized into the logarithmic tensor product by Huang, Lepowsky, and Zhang in [41]. For any two strongly graded generalized V-modules W_1 and W_2, they proved the existence of tensor product $W_1 \boxtimes_{Q(z)} W_2$ that satisfies the universal property in Definition 4.2.9. They also proved the associativity of their $Q(z)$-tensor product.

Let V be a rational and C_2-cofinite VOA, and $M^1, M^2, \dots, M^p$ be all the irreducible modules over V up to isomorphism. We may denote the $Q(z)$-tensor product $M^i \boxtimes_{Q(z)} M^j$ simply by $M^i \boxtimes M^j$, and denote the fusion rules $N_{M^i M^j}^{M^k}$ simply by N_{ij}^k, for $1 \leq i, j, k \leq p$. It was proved by Abe, Buhl, and Dong in [1] that all the irreducible V-modules are also C_2-cofinite. Thus, all the fusion rules N_{ij}^k are finite. Then the tensor products of irreducible modules have the following decomposition into a direct sum of irreducible V-modules:

$$M^i \boxtimes M^j \cong \bigoplus_{k=0}^{p} N_{ij}^k M^k, \quad \forall 1 \leq i, j \leq p. \tag{4.2.14}$$

For rational and C_2-cofinite VOAs, one can also construct the tensor product $M^i \boxtimes M^j$ by defining it to be the direct sum of modules on the right-hand side of (4.2.14), then this direct sum satisfies the universal property in Definition 4.2.9, and we call $M^i \boxtimes M^j$ the **fusion tensor** of V-modules M^i and M^j, see [53] for more details.

However, if we construct a tensor product in this way, then one can only prove the associativity of the tensor product by showing the following equality of the fusion rules, see [61]:

$$\sum_{k,l=1}^{p} N_{ij}^k N_{ks}^l = \sum_{k,l=1}^{p} N_{ik}^l N_{js}^k, \quad \forall 1 \leq i, j, s \leq p. \tag{4.2.15}$$

Indeed, by (4.2.14), we have:

$$(M^i \boxtimes M^j) \boxtimes M^s = \sum_{k=1}^{p} N_{ij}^k (M^k \boxtimes M^s) = \sum_{k,l=1}^{p} N_{ij}^k N_{ks}^l M^l,$$
$$M^i \boxtimes (M^j \boxtimes M^s) = M^i \boxtimes \sum_{k=1}^{p} N_{js}^k M^k = \sum_{k,l=1}^{p} N_{ik}^l N_{js}^k M^l.$$

It was conjectured by Dong (cf. [24]) that (4.2.15) can be proved by the fusion rules Theorem of rational VOAs (4.1.9), together with an isomorphism of certain $A(V)$-bimodules. Indeed, we can use (4.2.15) and translate the associativity of tensor product into an isomorphism of $A(V)$-bimodules.

From now on, we assume that V is a rational and C_2-cofinite VOA, with irreducible modules $M^1, M^2, \ldots, M^p$ up to isomorphism, and the tensor product of M^i and M^j is given by the fusion tensor (4.2.14).

Lemma 4.2.10. *As left modules over $A(V)$, we have the following isomorphism:*

$$(M^i \boxtimes M^j)(0) \cong A(M^i) \otimes_{A(V)} M^j(0). \tag{4.2.16}$$

Proof. Since $A(M^i) \otimes_{A(V)} M^j(0)$ is a direct sum of irreducible left $A(V)$-modules, then by the definition formula (4.2.14), Lemma 4.1.10, and the Schur's lemma, we have:

$$\begin{aligned}(M^i \boxtimes M^j)(0) = \bigoplus_{k=1}^{p} N_{ij}^{k} M^k(0) &= \bigoplus_{k=1}^{p} \dim \mathrm{Hom}_{A(V)}(A(M^i) \otimes_{A(V)} M^j(0), M^k(0)) M^k(0) \\ &\cong A(M^i) \otimes_{A(V)} M^j(0)\end{aligned}$$

as left $A(V)$-modules, where we have used the fact that the admissible level $(M^i \boxtimes M^j)(n)$ coincides with the admissible level $\bigoplus_{k=0}^{p} N_{ij}^{k} M^k(n)$, for any $1 \leq i, j \leq p$, and $n \in \mathbb{N}$. □

Proposition 4.2.11. *Suppose we have the following isomorphism of $A(V)$-bimodules:*

$$A(M^i) \otimes_{A(V)} A(M^j) \cong A(M^j) \otimes_{A(V)} A(M^i), \quad \forall 1 \leq i, j \leq p, \tag{4.2.17}$$

then the fusion tensor (4.2.14) *satisfies the associativity.*

Proof. Note that (4.2.16) can be generalized to

$$(M \boxtimes N)(0) \cong A(M) \otimes_{A(V)} N(0), \tag{4.2.18}$$

where M, N are finite direct sum of irreducible V-modules. This is clear because the tensor operators $\cdot \boxtimes \cdot$ and $\cdot \otimes \cdot$ preserve finite direct sum (colimit) of modules, and $A(M \oplus N) \cong A(M) \oplus A(N)$, where the second isomorphism follows from $O(M \oplus N) = O(M) \oplus O(N)$ in $M \oplus N$, see Section 2.2. Let W be an irreducible V-module. By (4.2.18) and (4.2.17), we have:

$$\begin{aligned}(M^i \boxtimes (M^j \boxtimes W))(0) &\cong A(M^i) \otimes_{A(V)} (M^j \boxtimes W)(0) \cong A(M^i) \otimes_{A(V)} (A(M^j) \otimes_{A(V)} W(0)) \\ &\cong A(M^j) \otimes_{A(V)} (A(M^i) \otimes_{A(V)} W(0)) \cong (M^j \boxtimes (M^i \boxtimes W))(0).\end{aligned} \tag{4.2.19}$$

On the other hand, by (4.2.19) and $M \boxtimes N \cong N \boxtimes M$, we have:

$$(M^j \boxtimes (M^i \boxtimes W))(0) \cong (M^j \boxtimes (W \boxtimes M^i))(0) \cong (W \boxtimes (M^j \boxtimes M^i))(0)$$

$$\cong ((M^j \boxtimes M^i) \boxtimes W)(0) \cong ((M^i \boxtimes M^j) \boxtimes W)(0).$$

Since irreducible $A(V)$-modules are in one-to-one correspondence with irreducible V-modules (cf. [18, 73]), it follows that two admissible V-modules M and N are isomorphic if they have isomorphic bottom levels. Therefore, $M^j \boxtimes (M^i \boxtimes W) \cong (M^j \boxtimes M^i) \boxtimes W$. □

There is an easy way to show that $A(M) \otimes_{A(V)} A(N)$ is isomorphic to $A(N) \otimes_{A(V)} A(M)$ as vector spaces, where M, N are two irreducible V-modules. Recall that $A(V)$ has an anti-involution $\phi : A(V) \to A(V), [a] \mapsto [e^{L(1)}(-1)^{L(0)}a]$, for all $[a] \in A(V)$, see Section 2 in [73]. Similarly, we can define anti-involutions of $A(M)$ and $A(N)$ as follows:

$$\phi_M : A(M) \to A(M), [u] \mapsto [e^{L(1)}(-1)^{L(0)}u],$$
$$\phi_N : A(N) \to A(N), [v] \mapsto [e^{L(1)}(-1)^{L(0)}v],$$

for all $[u] \in A(M)$ and $[v] \in A(N)$. Similar to the computation in [30] (see also [19]), it is easy to show the following compatibility properties of ϕ, ϕ_M, and ϕ_N:

$$\phi_M([a] * [u]) = \phi_M([u]) * \phi([a]), \qquad \phi_M([u] * [a]) = \phi([a]) * \phi_M([u]),$$
$$\phi_N([a] * [v]) = \phi_N([v]) * \phi([a]), \qquad \phi_N([v] * [a]) = \phi([a]) * \phi_N([v]),$$

for all $[u] \in A(M)$, $[v] \in A(N)$, and $[a] \in A(V)$. Define

$$\tilde{\phi} : A(M) \otimes_{A(V)} A(N) \to A(N) \otimes_{A(V)} A(M),\ [u] \otimes [v] \mapsto \phi_N([v]) \otimes \phi_M([u]). \tag{4.2.20}$$

We observe that $\tilde{\phi}$ is well-defined since for any $[u] * [a] \otimes [v] = [u] \otimes [a] * [v] \in A(M) \otimes_{A(V)} A(N)$,

$$\begin{aligned}\tilde{\phi}([u] * [a] \otimes [v]) &= \phi_N([v]) \otimes \phi_M([u] * [a]) = \phi_N([v]) \otimes \phi([a]) * \phi_M([u]) \\ &= \phi_N([v]) * \phi([a]) \otimes \phi_M([u]) = \phi_N([a] * [v]) \otimes \phi_M([u]) = \tilde{\phi}([u] \otimes [a] * [v]),\end{aligned}$$

and similarly $\tilde{\phi}([u] \otimes [a] * [v]) = \tilde{\phi}([u] * [a] \otimes [v])$. Clearly, $\tilde{\phi}$ satisfies $\tilde{\phi}^2 = \mathrm{Id}$, thus $\tilde{\phi}$ is a linear isomorphism. However, $\tilde{\phi}$ is **not** a homomorphism of $A(V)$-bimodules in general. Indeed,

$$\begin{aligned}\tilde{\phi}([a].([u] \otimes [v])) &= \phi_N([v]) \otimes \phi_M([a] * [u]) = \phi_N([v]) \otimes \phi_M([u]) * \phi([a]) \\ &= \tilde{\phi}([u] \otimes [v]).\phi([a]),\end{aligned}$$

which is not equal to $[a].\tilde{\phi}([u] \otimes [v])$ in general. In fact, by Lemma 4.1.10, we have a decomposition $A(N) \otimes_{A(V)} A(M) = \bigoplus_{i,j=1}^{p} m_{ij} M^i(0) \otimes_{\mathbb{C}} M^j(0)^*$ into irreducible $A(V)$-bimodules. However, the left action of $[a]$ is not necessarily the same as the right action of $\phi([a])$ on each irreducible pieces $M^i(0) \otimes_{\mathbb{C}} M^j(0)^*$.

Therefore, we need to find another way to show the isomorphism between these two bimodules. By studying the filtration on $A(M)$ and the associated graded modules $\mathrm{gr}A(M)$ over $\mathrm{gr}A(V)$, we proved (4.2.17) under an extra condition, see Theorem 3.17 in [57].

There is another sufficient condition for the associativity of the fusion tensor given by certain isomorphism of $A(V)$-bimodules.

Proposition 4.2.12. *The fusion tensor* (4.2.14) *satisfies the associativity if* $A(M^i \boxtimes M^j) \cong A(M^i) \otimes_{A(V)} A(M^j)$ *as* $A(V)$*-bimodules, for any* $1 \leq i, j \leq p$.

Proof. Let W be an irreducible V-module. By (4.2.18), we have the following isomorphisms:

$$((M^i \boxtimes M^j) \boxtimes W)(0) = A(M^i \boxtimes M^j) \otimes_{A(V)} W(0),$$
$$(M^i \boxtimes (M^j \boxtimes W))(0) = A(M^i) \otimes_{A(V)} (M^j \boxtimes W)(0) = A(M^i) \otimes_{A(V)} A(M^j) \otimes_{A(V)} W(0).$$

for any $1 \leq i, j \leq p$. By assumption, $A(M^i \boxtimes M^j) \otimes_{A(V)} W(0)$ is isomorphic to $A(M^i) \otimes_{A(V)} A(M^j) \otimes_{A(V)} W(0)$ as left $A(V)$-modules, hence $(M^i \boxtimes M^j) \boxtimes W \cong M^i \boxtimes (M^j \boxtimes W)$ as V-modules. □

Remark 4.2.13. We suspect that $A(M^i) \otimes_{A(V)} A(M^j)$ might have some further connections with the correlation functions on the bottom levels in Section 3.2. We might be able to construct a more general version of the correlation function from a space like

$$\left(M^k(0)^* \otimes_{A(V)} A(M^i) \otimes_{A(V)} A(M^j) \otimes_{A(V)} M^l(0)\right)^*,$$

and use the properties of such correlation functions to prove the isomorphism (4.2.17). We will study this problem in the future.

Part II

Rota-Baxter operators and the classical Yang-Baxter equations of vertex operator algebras

Chapter 5

Borel-type sub-algebras of the lattice vertex operator algebras

This Chapter begins this thesis's second part, which is dedicated to studying analogs of Rota-Baxter operators (RBOs) and the classical Yang-Baxter equations for vertex operator algebras. We study some special types of sub-algebras V_B of the lattice VOAs V_L constructed in [29] in this Chapter. They are similar to the Borel algebras $\mathfrak{b} = \mathfrak{n}_+ \oplus \mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ that has a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. We call such sub-algebras of lattice VOA the "Borel-type sub-algebras". Similar to the Lie algebra case, we will also introduce a notion of parabolic-type sub-algebras of the lattice VOA based on their relations with the Borel-type sub-algebras. We study this particular type of sub-algebras because they can give rise to natural (projection) Rota-Baxter operators on the lattice VOAs. Finding examples of Rota-Baxter operators on VOAs is a challenging task. It amounts to solving the vertex operator analog of the classical Yang-Baxter equations, which is quite intrinsic. We will see this in the next two chapters. Therefore, a natural example of RBO on VOA could also justify our definition axioms for Rota-Baxter operators on VOAs in the next Chapter.

In the first Section of this Chapter, we will give the definitions of Borel and parabolic type sub-algebras of the lattice VOAs and study their basic properties. In particular, we will show that these VOAs are of CFT-type, irrational, and some special Borel-type sub-algebras are C_1-cofinite. In the second Section of this Chapter, we will be focusing on the easiest nontrivial case of the Borel-type sub-algebra $V_{\mathbb{Z}_{\geq 0}\alpha}$ of the rank one VOA $V_{\mathbb{Z}\alpha}$. We will show that Zhu's algebra $A(V)$ of $V_{\mathbb{Z}_{\geq 0}\alpha}$ is a one-dimensional non-abelian nilpotent extension of the polynomial algebra $\mathbb{C}[x]$. Then we will use this result about $A(V_{\mathbb{Z}_{\geq 0}\alpha})$ to classify the irreducible modules over $V_{\mathbb{Z}_{\geq 0}\alpha}$. We will show that the complete list of irreducible modules over $V_{\mathbb{Z}_{\geq 0}\alpha}$ is the same as the complete list of irreducible modules over the Heisenberg VOA $M_{\widehat{\mathfrak{h}}}(1,0)$. We will also propose a way to extend a module over a Borel-type sub-algebra V_B to a module over V_L.

5.1 The Borel-type and parabolic-type sub-algebras of V_L

In this section, we first review the construction of lattice VOAs in [29] and some related results, then we give the definitions of Borel-type and parabolic type sub-algebras of a lattice VOA V_L, based on the decomposition of V_L as irreducible modules over the Heisenberg sub-VOA. We will prove that these sub-algebras are irrational, and some Borel-type sub-algebras are C_1-cofinite.

5.1.1 Sub-algebras of V_L associated to abelian sub-monoid

Let L be a positive definite even lattice of rank $d \geq 1$, equipped with $\mathbb{Z}$-bilinear form $(\cdot|\cdot) : L\times L \to \mathbb{Z}$. Let $\mathfrak{h} := \mathbb{C}\otimes_{\mathbb{Z}} L$, extend $(\cdot|\cdot)$ to a nondegenerate $\mathbb{C}$-bilinear form $(\cdot|\cdot) : \mathfrak{h}\times\mathfrak{h} \to \mathbb{C}$, and let $M_{\widehat{\mathfrak{h}}}(1,0)$ be the level-one Heisenberg VOA associated to $\widehat{\mathfrak{h}} = \mathfrak{h}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$.

We give a brief recap of the construction of the Heisenberg VOA $M_{\widehat{\mathfrak{h}}}(1,0)$ and its irreducible modules $M_{\widehat{\mathfrak{h}}}(1,\lambda)$, which will be used later in this and the next Chapter. Recall that the Lie bracket on the affine algebra $\widehat{\mathfrak{h}}$ is given by:

$$[h_1(m), h_2(n)] = m\delta_{m+n,0}K, \quad \forall h_1, h_2 \in \mathfrak{h}, \text{ and } m, n \in \mathbb{Z}, \tag{5.1.1}$$

where we denote $h\otimes t^m \in \widehat{\mathfrak{h}}$ by $h(m)$. Then $\widehat{\mathfrak{h}} = (\widehat{\mathfrak{h}})_+ \oplus (\widehat{\mathfrak{h}})_0 \oplus (\widehat{\mathfrak{h}})_-$, where $(\widehat{\mathfrak{h}})_\pm = \bigoplus_{n\in\mathbb{Z}_\pm} \mathfrak{h}\otimes\mathbb{C}t^n$, and $(\widehat{\mathfrak{h}})_0 = \mathfrak{h}\otimes\mathbb{C}1\oplus\mathbb{C}K$. Let $(\widehat{\mathfrak{h}})_{\geq 0} = (\widehat{\mathfrak{h}})_+ \oplus (\widehat{\mathfrak{h}})_0$, which is a Lie sub-algebra of $\widehat{\mathfrak{h}}$.

For each $\lambda \in \mathfrak{h}$, let e^λ be a formal symbol associated to λ. Then $\mathbb{C}e^\lambda$ is a module over $(\widehat{\mathfrak{h}})_{\geq 0}$, with the module actions given by $h(0)e^\lambda = (h|\lambda)e^\lambda$, $K.e^\lambda = e^\lambda$, and $h(n)e^\lambda = 0$, for all $h \in \mathfrak{h}$ and $n > 0$. Then define $M_{\widehat{\mathfrak{h}}}(1,\lambda)$ to be the induced module:

$$M_{\widehat{\mathfrak{h}}}(1,\lambda) := \mathrm{Ind}_{\widehat{\mathfrak{h}}_{\geq 0}}^{\widehat{\mathfrak{h}}} \mathbb{C}e^\lambda = U(\widehat{\mathfrak{h}}) \otimes_{U(\widehat{\mathfrak{h}}_{\geq 0})} \mathbb{C}e^\lambda. \tag{5.1.2}$$

Then we have $M_{\widehat{\mathfrak{h}}}(1,\lambda) \cong U(\widehat{\mathfrak{h}}_{<0}) \otimes_{\mathbb{C}} \mathbb{C}e^\lambda = \mathrm{span}\{h_1(-n_1)\dots h_k(-n_k)e^\lambda : k \geq 0,\ h_1,\dots,h_k \in \mathfrak{h},\ n_1 \geq \cdots \geq n_k \geq 0\}$ as vector spaces. It was proved in [29] that $M_{\widehat{\mathfrak{h}}}(1,0)$ is a VOA called the level-one Heisenberg VOA, and $M_{\widehat{\mathfrak{h}}}(1,\lambda)$, with different $\lambda\in\mathfrak{h}$, are all the irreducible modules over $M_{\widehat{\mathfrak{h}}}(1,0)$ up to isomorphism.

Write $\mathbb{C}^{\epsilon}[L]=\bigoplus_{\alpha\in L}\mathbb{C}e^{\alpha}$, where e^{α} is a formal symbol associated to α for each $\alpha\in L$ (e^{α} is denoted by $\iota(\alpha)$ in [29]), and $\epsilon: L\times L\to\langle\pm1\rangle$ is a 2-cocycle of the abelian group L such that $\epsilon(\alpha,\beta)\epsilon(\beta,\alpha)=(-1)^{(\alpha|\beta)}$, for any $\alpha,\beta\in L$. Let $V_L=M_{\widehat{\mathfrak{h}}}(1,0)\otimes_{\mathbb{C}}\mathbb{C}^{\epsilon}[L]$, then by the discussion above, we have the following spanning set of V_L:

$$V_L=\mathrm{span}\{h_1(-n_1)\dots h_k(-n_k)e^{\alpha}: k\geq 0,\alpha\in L,\ h_1,\dots,h_k\in\mathfrak{h},\ n_1\geq\cdots\geq n_k\geq 0\},$$

where we omit the tensor sign $\otimes$ in the term $h_1(-n_1)\dots h_k(-n_k)e^{\alpha}$. Let the vertex operators $Y: V_L\to\mathrm{End}(V_L)[[z,z^{-1}]]$ be given as follows on the spanning elements:

$$Y(h(-1)\mathbf{1},z):=h(z)=\sum_{n\in\mathbb{Z}}h(n)z^{-n-1}\quad\left(h(n)e^{\alpha}:=0,\ h(0)e^{\lambda}:=(h|\alpha)e^{\alpha}\right),\tag{5.1.3}$$

$$Y(e^{\alpha},z):=E^{-}(-\alpha,z)E^{+}(-\alpha,z)e_{\alpha}z^{\alpha}\quad\left(z^{\alpha}(e^{\beta}):=z^{(\alpha|\beta)}e^{\beta},\ e_{\alpha}(e^{\beta}):=\epsilon(\alpha,\beta)e^{\alpha+\beta}\right),\tag{5.1.4}$$

$$Y(h_1(-n_1-1)\dots h_k(-n_k-1)e^{\alpha},z):={}^{\circ}_{\circ}(\partial_z^{(n_1)}h_1(z))\dots(\partial_z^{(n_k)}h_k(z))Y(e^{\alpha},z){}^{\circ}_{\circ},\tag{5.1.5}$$

for any $k\geq 1$, $n_1\geq\cdots\geq n_k\geq 0$, $h,h_1,\dots h_k\in\mathfrak{h}$, and $\alpha,\beta\in L$, where $E^{\pm}$ and $\partial_z^{(n)}$ in (5.1.3)–(5.1.5) are given as follows:

$$E^{\pm}(-\alpha,z)=\exp\left(\sum_{n\in\mathbb{Z}_{\pm}}\frac{-\alpha(n)}{n}z^{-n}\right),\quad \partial_z^{(n)}=\frac{1}{n!}.$$

The normal ordering in (5.1.5) rearranges the terms in such a way that the right hand side of (5.1.5) is given as follows:

$$\sum_{m_1>0,\dots,m_k>0}\ \sum_{n_1\geq0,\dots,n_k\geq0}c_{m_1,\dots,n_k}h_1(-m_1)\dots h_k(-m_k)E^{-}(-\alpha,z)e_{\alpha}z^{\alpha}E^{+}(-\alpha,z)h_1(n_1)\dots h_k(n_k).\tag{5.1.6}$$

Let $\{\alpha_1,\dots,\alpha_d\}$ be an orthonormal basis of $\mathfrak{h}$, and let $\omega=\frac{1}{2}\sum_{i=1}^{d}\alpha_i(-1)^2\mathbf{1}\in M_{\widehat{\mathfrak{h}}}(1,0)\subset V_L$.

It is proved in the appendix A.2 in [29] that $(V_L,Y,\omega,\mathbf{1})$ is a VOA with $M_{\widehat{\mathfrak{h}}}(1,0)$ a sub-VOA. In particular, V_L has the same Virasoro element ω and the vacuum element $\mathbf{1}$ with the Heisenberg sub-VOA $M_{\widehat{\mathfrak{h}}}(1,0)$. Recall that V_L has the following decomposition as a module over the Heisenberg VOA $M_{\widehat{\mathfrak{h}}}(1,0)$ (cf. [13, 17]):

$$V_L=\bigoplus_{\alpha\in L}M_{\widehat{\mathfrak{h}}}(1,\alpha),\tag{5.1.7}$$

where $M_{\widehat{\mathfrak{h}}}(1,\alpha) = M_{\widehat{\mathfrak{h}}}(1,0) \otimes \mathbb{C}e^{\alpha}$ for each $\alpha \in L$.

Let $L^{\circ} = \{h \in \mathfrak{h} : (h|\alpha) \in \mathbb{Z}, \ \forall \alpha \in L\}$ be the dual lattice of L. For each element $\lambda \in L^{\circ}$. It was proved in [29] that $V_{L+\lambda} = M_{\widehat{\mathfrak{h}}}(1,0) \otimes_{\mathbb{C}} \mathbb{C}^{\epsilon}[L+\lambda]$ is a module over V_L, with the module vertex operator $Y_M : V_L \to \mathrm{End}(V_{L+\lambda})[[z,z^{-1}]]$ given by similar formulas as (5.1.3)–(5.1.5), the only differences are $h(0)e^{\beta+\lambda} := (h|\beta+\lambda)e^{\beta+\lambda}$, $z^{\alpha}(e^{\beta+\lambda}) := z^{(\alpha|\beta+\lambda)}e^{\beta+\lambda}$, and $e_{\alpha}e^{\beta+\lambda} := \epsilon(\alpha,\beta)e^{\alpha+\beta+\lambda}$, for any $h \in \mathfrak{h}$, $\alpha,\beta \in L$ and $\lambda \in L^{\circ}$.

Furthermore, Dong classified the irreducible modules over V_L in [13]. The main result is the following: Let $L^{\circ}/L = \bigsqcup_{i=1}^{p}(L+\lambda_i)$ be the coset decomposition of the subgroup L in L°. Then $\{V_{L+\lambda_1}, \dots, V_{L+\lambda_p}\}$ are all the irreducible module over V_L up to isomorphism. Furthermore, V_L is a rational VOA.

Observe that a lattice L is an abelian monoid, with the commutative associative product given by its addition, and the identity element given by 0. An **abelian sub-monoid** of L is a subset $M \subset L$ such that $0 \in M$, and M is closed under the addition of L.

Proposition 5.1.1. *Let $M \leq L$ be an abelian sub-monoid, with identity element $0 \in L$, and let $V_M := \bigoplus_{\alpha\in M} M_{\widehat{\mathfrak{h}}}(1,\alpha)$. Then $(V_M, Y, \omega, \mathbf{1})$ is a CFT-type sub-VOA of $(V_L, Y, \omega, \mathbf{1})$.*

Proof. By (5.1.3) and (5.1.4), for any $\alpha,\beta \in M$, we have

$$\begin{aligned} Y(e^{\alpha},z)e^{\beta} &= E^{-}(-\alpha,z)E^{+}(-\alpha,z)e_{\alpha}z^{\alpha}(e^{\beta}) = E^{-}(-\alpha,z)E^{+}(-\alpha,z)\epsilon(\alpha,\beta)z^{(\alpha|\beta)}e^{\alpha+\beta}, \\ &= \exp\Big(\sum_{n<0} -\frac{\alpha(n)}{n}z^{-n}\Big)\epsilon(\alpha,\beta)z^{(\alpha|\beta)}e^{\alpha+\beta} \end{aligned}$$

which is contained in $M_{\widehat{\mathfrak{h}}}(1,\alpha+\beta)((z)) \subset V_M((z))$, in view of the decomposition (5.1.7). More generally, for any $h_1(-n_1-1)\dots h_k(-n_k-1)e^{\alpha} \in M_{\widehat{\mathfrak{h}}}(1,\alpha)$ and $h_1'(-m_1-1)\dots h_r'(-m_r-1)e^{\beta} \in M_{\widehat{\mathfrak{h}}}(1,\beta)$, with $\alpha,\beta \in M$, it is easy to see from (5.1.5) and (5.1.6) that

$$Y(h_1(-n_1-1)\dots h_k(-n_k-1)e^{\alpha},z)h_1'(-m_1-1)\dots h_r'(-m_r-1)e^{\beta} \in M_{\widehat{\mathfrak{h}}}(1,\alpha+\beta)((z)).$$

Since M is closed under addition and $M_{\widehat{\mathfrak{h}}}(1,0) \subset V_M$, it follows that V_M is a sub-VOA of V_L. Since V_M has the same Virasoro element as V_L, we have $(V_M)_n \subseteq (V_L)_n$ for each $n \geq 0$, and $(V_M)_0 = (V_L)_0 = \mathbb{C}\mathbf{1}$. Thus V_M is of the CFT-type. □

The proof of Proposition 5.1.1 essentially depends on the fact that

$$Y(a,z)b \in M_{\widehat{\mathfrak{h}}}(1,\alpha+\beta)((z)), \quad \forall a \in M_{\widehat{\mathfrak{h}}}(1,\alpha),\ b \in M_{\widehat{\mathfrak{h}}}(1,\beta),\ \text{and}\ \alpha,\beta \in L, \tag{5.1.8}$$

where Y is the vertex operator of the lattice VOA V_L. We will use this fact multiple times in the next subsection. We call $(V_M, Y, \omega, \mathbf{1})$ in Proposition 5.1.1 the **sub-algebra of V_L associated to M**. We will use Proposition 5.1.1 and give the definition of Borel-type and parabolic-type sub-algebra in the next subsection by choosing suitable abelian sub-monoid of L.

Similarly, by (5.1.8), if $S \leq L$ is a sub-semigroup. i.e., S is only required to be closed under the addition of L, then it is easy to show that $V_S = \bigoplus_{\alpha\in S} M_{\hat{\mathfrak{h}}}(1,\alpha) \subset V_L$ is closed under the vertex operator Y of V_L. Such a sub-structure is called a **vertex Leibniz sub-algebra**, see [56] for the definition of vertex Leibniz algebra.

Remark 5.1.2. When L is a rank one lattice: $L = \mathbb{Z}\alpha$, it is observed by Dong (see Proposition 4.1 in [13]) that $V_{\mathbb{N}\alpha}$ is a sub-VOA of $V_{\mathbb{Z}\alpha}$. Proposition 5.1.1 is a generalization of this result, noting that $\mathbb{N}\alpha$ is an abelian sub-monoid of $\mathbb{Z}\alpha$.

5.1.2 Definition and first properties of Borel-type and parabolic-type sub-algebras

In the Lie theory, recall that a Borel subgroup B of a connected linear algebraic group G is defined to be a closed connected solvable subgroup of G that is maximal subject to these conditions. A parabolic subgroup can be equivalently characterized as a closed subgroup P that includes a Borel subgroup. See Chapter 21 in [33] for more details. The Lie algebra $\mathfrak{b}$ of B is called a Borel sub-algebra of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$, and a parabolic sub-algebra $\mathfrak{p}$ is just a Lie sub-algebra of $\mathfrak{g}$ that includes a Borel sub-algebra $\mathfrak{b}$.

If a Lie algebra $\mathfrak{g}$ is simple, then it has a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, where $\mathfrak{h}$ is a Cartan sub-algebra of $\mathfrak{g}$, and Φ is the root system associated to $\mathfrak{h}$. In this case, a Borel sub-algebra can be given by $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi_+} \mathfrak{g}_\alpha$, where Φ_+ is the set of positive roots, see [34] for more details. Furthermore, the positive roots in Φ_+ can be written as a positive integral linear combination of the simple roots $\alpha_1, \ldots, \alpha_r$, and the root lattice $Q = \mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_r$. Inspired by these constructions from Lie algebras, we introduce the following notions:

Definition 5.1.3. Let L be a positive-definite even lattice of rank r, and let V_L be the lattice VOA associated to L.

(1) An abelian sub-monoid $B \leq L$ is called a **Borel-type sub-monoid** if there exists a basis $\{\alpha_1, \ldots, \alpha_r\}$ of L such that $B = \mathbb{Z}_{\geq 0}\alpha_1 \oplus \ldots \oplus \mathbb{Z}_{\geq 0}\alpha_r$. An abelian sub-monoid $P \leq L$ is called a **parabolic-type sub-monoid** if there exists a Borel-type sub-monoid $B \leq L$ such that $B \subseteq P$ (Any parabolic-type sub-monoid is automatically of the Borel-type).

(2) A **Borel-type sub-algebra (or sub-VOA)** of the lattice VOA V_L is a sub-algebra associated to a Borel-type sub-monoid $B \leq L$: $V_B = \bigoplus_{\alpha\in B} M_{\widehat{\mathfrak{h}}}(1,\alpha)$. A **parabolic-type sub-algebra (or sub-VOA)** of V_L is a sub-algebra associated to a parabolic-type sub-monoid $P \leq L$: $V_P = \bigoplus_{\alpha\in P} M_{\widehat{\mathfrak{h}}}(1,\alpha)$.

Observe that both the Borel-type and parabolic-type sub-algebras are of the CFT-type, non-simple (any sub-semigroup $S \leq P$ gives rise to an ideal V_S of V_P by (5.1.8)), and share the same vacuum element $\mathbf{1}$ and Virasoro element ω with the lattice VOA V_L.

For a Borel-type sub-VOA $V_B = \bigoplus_{\alpha\in B} M_{\widehat{\mathfrak{h}}}(1,\alpha)$, where $B = \mathbb{Z}_{\geq 0}\alpha_1 \oplus \ldots \oplus \mathbb{Z}_{\geq 0}\alpha_r$, we may view $M_{\widehat{\mathfrak{h}}}(1,0) \leq V_B$ as an analog of the "Cartan sub-algebra" and $\bigoplus_{\exists n_j>1} M_{\widehat{\mathfrak{h}}}(1, n_1\alpha_1 + \cdots + n_r\alpha_r) \leq V_B$ as the an analog of the "positive-roots part" of a simple Lie algebra $\mathfrak{g}$.

Example 5.1.4. Certain parabolic-type sub-algebras can give rise to the decomposition of the lattice VOA V_L into a direct sum of two vertex Leibniz sub-algebras, which can further give rise to Rota-Baxter operators in the next Chapter. We give some examples of them as follows:

(1) Let L be the rank one positive definite even lattice $L = \mathbb{Z}\alpha$, with $(\alpha|\alpha) = 2N$ for some fixed $N \in \mathbb{Z}_{>0}$. Clearly, $B = \mathbb{Z}\alpha_{\geq 0}$ is a Borel-type sub-monoid, $\mathbb{Z}\alpha_{<0}$ is a sub-semigroup of L, and $L = B \bigsqcup \mathbb{Z}\alpha_{<0}$. Then it follows that $V_{\mathbb{Z}\alpha_{\geq 0}} = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} M_{\widehat{\mathfrak{h}}}(1, m\alpha)$ is a Borel-type sub-algebra, $V_{\mathbb{Z}_{<0}\alpha} := \bigoplus_{m\in\mathbb{Z}_{<0}} M_{\widehat{\mathfrak{h}}}(1, m\alpha)$ is a vertex Leibniz sub-algebras of $V_{\mathbb{Z}\alpha}$, and $V_{\mathbb{Z}\alpha} = V_{\mathbb{Z}_{\geq 0}\alpha} \oplus V_{\mathbb{Z}_{<0}\alpha}$.

(2) Let L be a positive-definite even lattice of rank r, with a basis $\{\alpha_1, \ldots, \alpha_r\}$. Let

$$P := \mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_{r-1} \oplus \mathbb{Z}_{\geq 0}\alpha_r, \quad \text{and} \quad P^- := \mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_{r-1} \oplus \mathbb{Z}_{<0}\alpha_r, \tag{5.1.9}$$

then P is a parabolic-type sub-monoid of L since it contains a Borel-type sub-monoid $\mathbb{Z}_{\geq 0}\alpha_1 \oplus \ldots \oplus \mathbb{Z}_{\geq 0}\alpha_r$, and P^- is a sub-semigroup of L. Moreover, we also have $L = P \cup P^-$ and $P \cap P^- = \emptyset$. Therefore, $V_P = \bigoplus_{\alpha\in P} M_{\widehat{\mathfrak{h}}}(1,\alpha)$ is a parabolic-type sub-algebra of V_L, and $V_{P^-} = \bigoplus_{\beta\in P^-} M_{\widehat{\mathfrak{h}}}(1,\beta)$ is a vertex Leibniz sub-algebra of V_L. Moreover, we have a decomposition $V_L = V_P \oplus V_{P^-}$.

Unlike the lattice VOA itself, the Borel-type and parabolic-type sub-algebras that are not equal to V_L are irrational. Their representations are more like the Heisenberg sub-VOA $M_{\widehat{\mathfrak{h}}}(1,0)$ instead of V_L. For the rest of this subsection, we fix a positive-definite even lattice L of rank r.

Theorem 5.1.5. *The proper Borel-type and parabolic-type sub-algebras of a lattice VOA V_L are all irrational.*

Proof. By Definition 5.1.3, it suffices to show a proper parabolic sub-VOA $V_P \lneq V_L$ is irrational. Assume P contains a Borel-type sub-monoid $\mathbb{Z}_{\geq 0}\alpha_1 \oplus \ldots \oplus \mathbb{Z}_{\geq 0}\alpha_r$. First, we note that there must exist some index $1 \leq j \leq r$ such that for any $n_j < 0$, the element $n_1\alpha_1 + \cdots + n_j\alpha_j + \cdots + n_r\alpha_r$, with $n_k \in \mathbb{Z}$ for any $k \neq j$, is not in P since otherwise, P must be the whole lattice L. Without loss of generality, we assume $j = 1$, then elements in P are of the form $m\alpha_1 + n_2\alpha_2 + \cdots + n_r\alpha_r$, for some $m \geq 0$, and $n_2, \ldots, n_r \in \mathbb{Z}$. In particular, $\mathbb{Z}_{\geq 0}\alpha_1 \subset P$. We let

$$P^1 := \{m\alpha_1 + n_2\alpha_2 + \cdots + n_r\alpha_r \in P : m \geq 1, n_i \in \mathbb{Z}\} \cup \{0 + n_2\alpha_2 + \cdots + n_r\alpha_r \in P : n_i \in \mathbb{Z}\}.$$

It is clear that P^1 is a sub-monoid of P. i.e., $P+P^1 \subseteq P^1$, then by (5.1.8), $V_{P^1} := \bigoplus_{\alpha \in P^1} M_{\widehat{\mathfrak{h}}}(1, \alpha)$ is a submodule of the adjoint module V_P, and $V_P/V_{P^1} \cong M_{\widehat{\mathfrak{h}}}(1, 0)$, which is an irreducible V_P-module. Similarly, if we let

$$P^2 := \{m\alpha_1 + n_2\alpha_2 + \cdots + n_r\alpha_r \in P : m \geq 2, n_i \in \mathbb{Z}\} \cup \{0 + n_2\alpha_2 + \cdots + n_r\alpha_r \in P : n_i \in \mathbb{Z}\},$$

then P^2 is a sub-monoid of both P^1 and P, and $V_{P^2} \subset V_{P^1}$ is V_P-submodule such that $V_{P^1}/V_{P^2} \cong M_{\widehat{\mathfrak{h}}}(1, \alpha_1)$, which is an irreducible V_P-module. Proceed like this, and we can construct a composition series of V_P-modules:

$$V_P \supset V_{P^1} \supset V_{P^2} \supset \ldots V_{P^m} \supset V_{P^{m+1}} \supset \ldots,$$

such that the consecutive quotient are $V_{P^m}/V_{P^{m+1}} \cong M_{\widehat{\mathfrak{h}}}(1, m\alpha_1)$, which is an irreducible V_P-module, for all $m \geq 0$. Note that $M_{\widehat{\mathfrak{h}}}(1, m\alpha_1)$ is not isomorphic to $M_{\widehat{\mathfrak{h}}}(1, m'\alpha_1)$ if $m \neq m'$, since they are not isomorphic as $M_{\widehat{\mathfrak{h}}}(1, 0)$-modules. Thus, V_P has infinitely many non-isomorphic irreducible modules, and so V_P is irrational, see [18, 73]. $\square$

Although the Borel-type and parabolic-type sub-algebras are irrational, we can show that some Borel-type sub-algebras are strongly finitely generated, or equivalently, C_1-cofinite, see Proposition 2.2.3.

Proposition 5.1.6. *Let $B = \mathbb{Z}_{\geq 0}\alpha_1 \oplus \ldots \oplus \mathbb{Z}_{\geq 0}\alpha_r$ be a Borel-type sub-monoid of L such that $(\alpha_i|\alpha_j) \geq 0$, for all $1 \leq i \neq j \leq r$. Then V_B is strongly generated by $U := \{\mathbf{1}, \alpha_i(-1)\mathbf{1}, e^{\alpha_i} : 1 \leq i \leq r\}$. In particular, V_B is C_1-cofinite.*

Proof. Let W be the subspace of $V_{\mathbb{Z}_{\geq 0}\alpha}$ spanned by the following elements:

$$u^1_{-n_1}u^2_{-n_2}\dots u^k_{-n_k}u, \tag{5.1.10}$$

where $u^1,\dots,u^k,u\in U$, and $n_1\geq n_2\geq\cdots\geq n_k\geq 1$. We need to show that each $M_{\widehat{\mathfrak{h}}}(1,n_1\alpha_1+\cdots+n_r\alpha_r)$ is contained in W, for all non-negative integers $n_1,\dots,n_r\geq 0$. Clearly, the Heisenberg sub-VOA $M_{\widehat{\mathfrak{h}}}(1,0)$ is contained in W. Since $M_{\widehat{\mathfrak{h}}}(1,\alpha)=M_{\widehat{\mathfrak{h}}}(1,0)\otimes_{\mathbb{C}}\mathbb{C}e^{\alpha}$, and $M_{\widehat{\mathfrak{h}}}(1,0)$ is strongly generated by $\{\alpha_1(-1)\mathbf{1},\dots,\alpha_r(-1)\mathbf{1}\}$, we only need to show that $e^{n_1\alpha_1+\cdots+n_r\alpha_r}\in W$.

Indeed, first we observe that if $e^{\alpha}\in W$ and $e^{\beta}\in W$, with $(\alpha|\beta)\geq 0$, then

$$\begin{aligned} e^{\alpha}_{-(\alpha|\beta)-1}e^{\beta} &= \mathrm{Res}_z z^{-(\alpha|\beta)-1}E^-(-\alpha,z)E^+(-\alpha,z)e_{\alpha}z^{\alpha}e^{\beta} \\ &= \mathrm{Res}_z z^{-(\alpha|\beta)-1}E^-(-\alpha,z)z^{(\alpha|\beta)}\epsilon(\alpha,\beta)e^{\alpha+\beta} \\ &= \epsilon(\alpha,\beta)e^{\alpha+\beta}\equiv 0 \pmod W. \end{aligned} \tag{5.1.11}$$

Furthermore, since $(\alpha_i|\alpha_j)\geq 0$ for all $1\leq i\neq j\leq r$, and $(\alpha_i|\alpha_i)=2N_i>0$ for all i, we have:

$$(m_1\alpha_1+\cdots+m_r\alpha_r|n_1\alpha_1+\cdots+n_r\alpha_r)=\sum_{i,j=1}^{r}m_in_j(\alpha_i|\alpha_j)\geq 0,$$

for any non-negative integers $m_1,\dots,m_r,n_1,\dots,n_r\geq 0$. In particular, if $e^{m_1\alpha_1+\cdots+m_r\alpha_r}\in W$ and $e^{n_1\alpha_1+\cdots+n_r\alpha_r}\in W$, we have $e^{(n_1+m_1)\alpha_1+\cdots+(n_r+m_r)\alpha_r}\in W$ by (5.1.11). Then it follows from an easy induction that $e^{n_1\alpha_1+\cdots+n_r\alpha_r}\in W$ for any non-negative integers $n_1,\dots,n_r\geq 0$. □

Remark 5.1.7. We believe that certain Borel-type (as well as parabolic-type) sub-algebras that do not satisfy the condition in Proposition 5.1.6 can also be proved to be C_1-cofinite. We will study this problem in the future. However, the Borel-type sub-algebras are **not** C_2-cofinite. We will see this in the next Section.

5.2 The Borel-type sub-algebra $V_{\mathbb{Z}_{\geq 0}\alpha}$

In this Section, we will be focusing on the Borel-type sub-algebra $V_{\mathbb{Z}_{\geq 0}\alpha}$ of the rank one lattice VOA V_L, where $L=\mathbb{Z}\alpha$, with $(\alpha|\alpha)=2N$ for some $N\geq 1$. In this case, the 2-cocycle $\epsilon:\mathbb{Z}\alpha\times\mathbb{Z}\alpha\to\langle\pm 1\rangle$ satisfies $\epsilon(m\alpha,n\alpha)\epsilon(n\alpha,m\alpha)=(-1)^{(m\alpha|n\alpha)}=(-1)^{2Nmn}=1$ for any $m,n\in\mathbb{Z}$. We may assume that $\epsilon(m\alpha,n\alpha)=1$ for all $m,n\in\mathbb{Z}$.

We will show that Zhu's algebra $A(V_{\mathbb{Z}_{\geq 0}\alpha})$ is isomorphic to the associative algebra:

$$\mathbb{C}\langle x,y\rangle/\langle y^2, yx+Ny, xy-Ny\rangle,$$

where $\mathbb{C}\langle x, y\rangle$ is the tensor algebra of $\mathbb{C}x \oplus \mathbb{C}y$. In particular, $A(V_{\mathbb{Z}_{\geq 0}\alpha})$ is a one-dimensional non-abelian nilpotent extension of the polynomial algebra $\mathbb{C}[x]$. We will also show that $\mathrm{gr}A(V_{\mathbb{Z}_{\geq 0}\alpha})$ is isomorphic to $R(V_{\mathbb{Z}_{\geq 0}\alpha})$ as commutative Poisson algebras.

Finally, we will use the $A(V_{\mathbb{Z}_{\geq 0}\alpha})$ and give a complete list of irreducible modules over $V_{\mathbb{Z}_{\geq 0}\alpha}$, and show that the irreducible modules are the same as the irreducible modules over the Heisenberg VOA $M_{\hat{\mathfrak{h}}}(1, 0)$. Then we will be focusing on the special case when $(\alpha|\alpha) = 2$. This case gives us more evidence to justify our choice of the name "Borel-type sub-algebra", and it also leads to an alternative way to construct induced modules over VOAs (cf. [22]).

5.2.1 The Zhu's algebra of $V_{\mathbb{Z}_{\geq 0}\alpha}$

For the clearness and conciseness of our cross-references, we rewrite the formulas about Zhu's algebra $A(V)$ in Section 2.2 at the beginning of this subsection. Recall that

$$a \circ b = \mathrm{Res}_z Y(a, z)b\frac{(1+z)^{\mathrm{wt}a}}{z^2} = \sum_{j\geq 0}\binom{\mathrm{wt}a}{j}a_{j-2}b, \tag{5.2.1}$$

$$a * b = \mathrm{Res}_z Y(a, z)b\frac{(1+z)^{\mathrm{wt}a}}{z} = \sum_{j\geq 0}\binom{\mathrm{wt}a}{j}a_{j-1}b, \tag{5.2.2}$$

$O(V) = \mathrm{span}\{a \circ b : a, b \in V\}$, and $A(V) = V/O(V)$. We have:

$$a * O(V) \subset O(V), \quad \text{and} \quad O(V) * a \subset O(V), \tag{5.2.3}$$

for any $a \in V$, and $A(V)$ is an associative algebra with respect to $*$, with the unit element $[\mathbf{1}]$. Furthermore, we have the following formulas:

$$a * b \equiv \mathrm{Res}_z Y(b, z)a\frac{(1+z)^{\mathrm{wt}b-1}}{z} \pmod{O(V)}, \tag{5.2.4}$$

$$a * b - b * a \equiv \mathrm{Res}_z Y(a, z)b(1+z)^{\mathrm{wt}a-1} \pmod{O(V)}, \tag{5.2.5}$$

for any homogeneous $a, b \in V$. Finally, if $m \geq n \geq 0$, we have:

$$\mathrm{Res}_z Y(a, z)b\frac{(1+z)^{\mathrm{wt}a+n}}{z^{2+m}} \equiv 0 \pmod{O(V)}. \tag{5.2.6}$$

Proposition 5.2.1. *There exists an epimorphism of associative algebras:*

$$F : \mathbb{C}\langle x, y\rangle/\langle y^2, yx + Ny, xy - Ny\rangle \to A(V), \tag{5.2.7}$$

such that $F(x) = [\alpha(-1)\mathbf{1}]$ *and* $F(y) = [e^{\alpha}]$.

Proof. By the definition of $Y(e^\alpha, z)$, it is easy to see that for any $n \geq 0$, we have:

$$e^\alpha_n e^\alpha = 0, \quad e^\alpha_{-1} e^\alpha = \cdots = e^\alpha_{-2N} e^\alpha = 0, \quad \text{and} \quad e^\alpha_{-2N-1} e^\alpha = e^{2\alpha}. \tag{5.2.8}$$

Since $\mathrm{wt} e^\alpha = N$, by (5.2.2) and (5.2.8) we have: $e^\alpha * e^\alpha = \sum_{j\geq 0} \binom{N}{j} e^\alpha_{j-1} e^\alpha = 0$. Hence $[e^\alpha] * [e^\alpha] = 0$ in $A(V)$. By (5.2.6), we have

$$\alpha(-n-2)u + \alpha(-n-1)u \equiv 0 \pmod{O(V)},$$

and $[\alpha(-1)u] = [u] * [\alpha(-1)\mathbf{1}]$, for all $n \geq 0$ and $u \in V$. Thus we have:

$$[\alpha(-n_1-1)\alpha(-n_2-1)\ldots\alpha(-n_k-1)u] = (-1)^{n_1+\cdots+n_k}[u] * [\alpha(-1)\mathbf{1}] * \cdots * [\alpha(-1)\mathbf{1}]$$

in $A(V)$, for any $n_1, \ldots, n_k \geq 0$ and $u \in V$. Thus, $A(V)$ is generated by $[\alpha(-1)\mathbf{1}]$ and $[e^{m\alpha}]$, for all $m \geq 1$. We claim that $[e^{k\alpha}] = 0$ for any $k \geq 2$.

Indeed, for $m \geq 1$, since $e^\alpha_{-2Nm-1} e^{m\alpha} = e^{(m+1)\alpha}$, $e^\alpha_{-n} e^{m\alpha} = 0$ for any $n \leq 2Nm$, and $2Nm + 1 \geq 2$ then by (5.2.6), we have:

$$\begin{aligned} e^{(m+1)\alpha} &= e^\alpha_{-2Nm-1} e^{m\alpha} + \binom{N}{1} e^\alpha_{-2Nm} e^{m\alpha} + \cdots + \binom{N}{N} e^\alpha_{-2Nm-1+N} e^{m\alpha} \\ &= \mathrm{Res}_z Y(e^\alpha, z) e^{m\alpha} \frac{(1+z)^N}{z^{2Nm+1}} \in O(V), \end{aligned}$$

for any $m \geq 1$. Hence $[e^{k\alpha}] = 0$ in $A(V)$ for all $k \geq 2$, and $A(V)$ is generated by $[\alpha(-1)\mathbf{1}]$ and $[e^\alpha]$. Then we have an epimorphism $F : \mathbb{C}\langle x, y\rangle \to A(V)$, such that $F(x) = [\alpha(-1)\mathbf{1}]$ and $F(y) = [e^\alpha]$. Moreover, by (5.2.1) and the definition of $Y(e^\alpha, z)$, we have:

$$\begin{aligned} e^\alpha \circ \mathbf{1} &= e^\alpha_{-2}\mathbf{1} + \binom{N}{1} e^\alpha_{-1}\mathbf{1} + \sum_{j\geq 2} \binom{N}{j} e^\alpha_{j-2}\mathbf{1} \\ &= \mathrm{Res}_z z^{-2} \exp(-\sum_{n<0} \frac{\alpha(n)}{n} z^{-n}) e^\alpha + N e^\alpha + 0 \\ &= \alpha(-1)e^\alpha + N e^\alpha \equiv 0 \pmod{O(V)}, \end{aligned}$$

hence $[e^\alpha] * [\alpha(-1)\mathbf{1}] + N[e^\alpha] = 0$ in $A(V)$. By (5.2.5), we also have:

$$[\alpha(-1)\mathbf{1}] * [e^\alpha] - [e^\alpha] * [\alpha(-1)\mathbf{1}] = [\mathrm{Res}_z Y(\alpha(-1)\mathbf{1}, z)e^\alpha] = [\alpha(0)e^\alpha] = 2N[e^\alpha],$$

and so $[\alpha(-1)\mathbf{1}] * [e^\alpha] - N[e^\alpha] = 0$ in $A(V)$. Therefore, the epimorphism $F : \mathbb{C}\langle x, y\rangle \to A(V)$ factors through the algebra $\mathbb{C}\langle x, y\rangle / \langle y^2, yx + Ny, xy - Ny\rangle$. $\square$

Our next goal is to show that epimorphism (5.2.7) is an isomorphism. We can achieve this goal by determining $O(V)$.

Let O' be the subspace of V spanned by the following elements:

$$\begin{cases} \alpha(-n-2)u + \alpha(-n-1)u, & u \in V, \text{ and } n \geq 0, \\ \alpha(-1)v + Nv, & v \in \bigoplus_{m\geq 1} M_{\widehat{\mathfrak{h}}}(1, m\alpha), \\ M_{\widehat{\mathfrak{h}}}(1, k\alpha), & k \geq 2. \end{cases} \tag{5.2.9}$$

We want to show that $O(V) = O'$. First, we prove the easier part: $O' \subseteq O(V)$. By (5.2.6), clearly we have $\alpha(-n-2)u + \alpha(-n-1)u \in O(V)$, for all $u \in V$ and $n \geq 0$.

Lemma 5.2.2. *For any $k \geq 2$, we have $M_{\widehat{\mathfrak{h}}}(1, k\alpha) \subset O(V)$.*

Proof. By the proof of Proposition 5.2.1, we have $e^{k\alpha} \in O(V)$, for any $k \geq 2$. By (5.2.4), we have $u * \alpha(-1)\mathbf{1} \equiv \alpha(-1)u \pmod{O(V)}$. Now by (5.2.6) and (5.2.3), we have:

$$\begin{aligned} \alpha(-n_1-1)\dots\alpha(-n_r-1)e^{k\alpha} &\equiv (-1)^{n_1+\dots+n_r}\alpha(-1)^r e^{k\alpha} \pmod{O(V)} \\ &\equiv (-1)^{n_1+\dots+n_r} e^{k\alpha} * (\alpha(-1)\mathbf{1}) * \dots * (\alpha(-1)\mathbf{1}) \pmod{O(V)} \\ &\equiv 0 \pmod{O(V)}, \end{aligned}$$

for any $k \geq 2$, $r \geq 1$, and $n_1, \dots n_r \geq 0$, where the last congruence follows from (5.2.3). Thus we have $M_{\widehat{\mathfrak{h}}}(1, k\alpha) \subset O(V)$, for all $k \geq 2$. □

Lemma 5.2.3. *For any $v \in \bigoplus_{m\geq 1} M_{\widehat{\mathfrak{h}}}(1, m\alpha)$, we have $\alpha(-1)v + Nv \in O(V)$.*

Proof. If $m \geq 2$ and $v \in M_{\widehat{\mathfrak{h}}}(1, m\alpha)$, then by Lemma 5.2.2, we have $v \in O(V)$, and

$$\alpha(-1)v + Nv \equiv v * (\alpha(-1)\mathbf{1}) + Nv \equiv 0 \pmod{O(V)},$$

by (5.2.3). Now let $m = 1$, by the proof of Proposition 5.2.1, we have $\alpha(-1)e^{\alpha} + Ne^{\alpha} = e^{\alpha} \circ \mathbf{1} \equiv 0 \pmod{O(V)}$. Let $v = \alpha(-n_1-1)\dots\alpha(-n_r-1)e^{\alpha}$ be a general spanning element of $M_{\widehat{\mathfrak{h}}}(1, \alpha)$, where $r \geq 1$, and $n_1, \dots n_r \geq 0$. Since $[\alpha(-1), \alpha(-p)] = 0$ for all $p \geq 1$, we have:

$$\begin{aligned} \alpha(-1)v + Nv &= \alpha(-n_1-1)\dots\alpha(-n_r-1)(\alpha(-1)e^{\alpha} + Ne^{\alpha}) \\ &\equiv (-1)^{n_1+\dots+n_r}\alpha(-1)^r(\alpha(-1)e^{\alpha} + Ne^{\alpha}) \pmod{O(V)} \\ &\equiv (-1)^{n_1+\dots+n_r}(\alpha(-1)e^{\alpha} + Ne^{\alpha}) * (\alpha(-1)\mathbf{1}) * \dots * (\alpha(-1)\mathbf{1}) \pmod{O(V)} \\ &\equiv 0 \pmod{O(V)}, \end{aligned}$$

where the last congruence follows from (5.2.3) and the fact that $\alpha(-1)e^{\alpha} + Ne^{\alpha} \in O(V)$. □

By Lemma 5.2.2 and Lemma 5.2.3, we have $O' \subseteq O(V)$. Conversely, we need to show that $a \circ u = \mathrm{Res}_z Y(a,z)u((1+z)^{\mathrm{wt} a}/z^2) \in O'$, for any homogeneous $a, u \in V$. First, note that if $a \in M_{\widehat{\mathfrak{h}}}(1, m\alpha)$ and $u \in M_{\widehat{\mathfrak{h}}}(1, n\alpha)$ for some $m, n \geq 1$, then by (5.2.2) and (5.2.4), we have:

$$\mathrm{Res}_z Y(a,z)u\frac{(1+z)^{\mathrm{wt} a}}{z^2} \in M_{\widehat{\mathfrak{h}}}(1,(m+n)\alpha)((z)) \subset O'((z)),$$

since $m+n \geq 2$, and $M_{\widehat{\mathfrak{h}}}(1, k\alpha) \subset O'$ for any $k \geq 2$ by (5.2.9). Thus, we only need to show:

$$a \circ u \in O', \quad \text{for} \quad \begin{cases} a \in M_{\widehat{\mathfrak{h}}}(1,\alpha) \quad \text{and} \quad u \in M_{\widehat{\mathfrak{h}}}(1,0), \\ \text{or} \\ a \in M_{\widehat{\mathfrak{h}}}(1,0) \quad \text{and} \quad u \in M_{\widehat{\mathfrak{h}}}(1,\alpha). \end{cases} \tag{5.2.10}$$

First, we consider the case when $a \in M_{\widehat{\mathfrak{h}}}(1,\alpha)$ and $u \in M_{\widehat{\mathfrak{h}}}(1,0)$. Our strategy is to show $\mathrm{Res}_z Y(e^\alpha, z)u((1+z)^N/z^{2+n}) \in O'$ first, where $u \in M_{\widehat{\mathfrak{h}}}(1,0)$ and $n \geq 0$, and then prove $\mathrm{Res}_z Y(a,z)u((1+z)^{\mathrm{wt} a}/z^2) \in O'$, for $a = \alpha(-n_1)\dots\alpha(-n_r)e^\alpha \in M_{\widehat{\mathfrak{h}}}(1,\alpha)$ by induction.

Lemma 5.2.4. *For any $m \geq 1$, we have $\alpha(-m)O' \subset O'$. For any $u \in M_{\widehat{\mathfrak{h}}}(1,\alpha)$, we have $L(-1)u + L(0)u \in O'$.*

Proof. Since $[\alpha(-m), \alpha(-n)] = 0$ for any $m, n \geq 1$, and $\alpha(-m)M_{\widehat{\mathfrak{h}}}(1, k\alpha) \subset M_{\widehat{\mathfrak{h}}}(1, k\alpha)$, for any $k \geq 0$, we have $\alpha(-m)O' \subset O'$, in view of (5.2.9).

Let $u = \alpha(-n_1)\dots\alpha(-n_r)e^\alpha \in M_{\widehat{\mathfrak{h}}}(1,\alpha)$, where $r \geq 0$ and $n_1, \dots, n_r \geq 1$. Since $L(-1)e^\alpha = (e^\alpha)_{-2}\mathbf{1} = \alpha(-1)e^\alpha$, and $[L(-1), \alpha(-m)] = ma(-m-1)$, we have:

$$\begin{aligned}
&L(-1)\alpha(-n_1)\dots\alpha(-n_r)e^\alpha + L(0)\alpha(-n_1)\dots\alpha(-n_r)e^\alpha \\
&= \alpha(-n_1)\dots\alpha(-n_r)\alpha(-1)e^\alpha + \sum_{j=1}^{r} n_j \cdot \alpha(-n_1)\dots\alpha(-n_j-1)\dots\alpha(-n_r)e^\alpha \\
&\quad + (n_1 + \cdots + n_k + N)\alpha(-n_1)\dots\alpha(-n_r)e^\alpha \\
&= \alpha(-n_1)\dots\alpha(-n_r)(\alpha(-1)e^\alpha + Ne^\alpha) \\
&\quad + \sum_{j=1}^{r}(\alpha(-n_j-1) + \alpha(-n_j))\alpha(-n_1)\dots\widehat{\alpha(-n_j)}\dots\alpha(-n_r)e^\alpha \equiv 0 \pmod{O'},
\end{aligned}$$

since $\alpha(-1)e^\alpha + Ne^\alpha \in O'$, $\alpha(-m)O' \subset O'$ for any $m \geq 1$, and $\alpha(-n-1)v + \alpha(-n)v \in O'$ for all $v \in M_{\widehat{\mathfrak{h}}}(1,\alpha)$ and $n \geq 1$ by (5.2.9). □

Proposition 5.2.5. *Let $u \in M_{\widehat{\mathfrak{h}}}(1,0)$, and $n \geq 0$. We have:*

$$\mathrm{Res}_z Y(e^{\alpha}, z)u\frac{(1+z)^N}{z^{2+n}} \in O'. \tag{5.2.11}$$

Proof. We use induction on the length r of a spanning element $u = \alpha(-n_1)\dots\alpha(-n_r)\mathbf{1}$ of $M_{\widehat{\mathfrak{h}}}(1,0)$, where $n_1,\dots,n_r \geq 1$. The base case is $u = \mathbf{1}$. Note that $e^{\alpha}_{-j-1}\mathbf{1} = \frac{1}{j!}(L(-1)^j e^{\alpha})_{-1}\mathbf{1} = \frac{1}{j!}L(-1)^j e^{\alpha}$, for any $j \geq 0$, and since $e^{\alpha}_{-j-1}\mathbf{1} \in M_{\widehat{\mathfrak{h}}}(1,\alpha)$ for any $j \geq 0$, by Lemma 5.2.4, we have:

$$\begin{aligned}
L(-1)^j e^{\alpha} &\equiv -L(0)L(-1)^{j-1}e^{\alpha} \pmod{O'}\\
&= (-1)(N+j-1)L(-1)^{j-1}e^{\alpha}\\
&\ \vdots\\
&\equiv (-1)^j(N+j-1)(N+j-2)\dots(N+1)Ne^{\alpha} \pmod{O'}.
\end{aligned}$$

Then it follows from the definition of binomial coefficients that

$$\begin{aligned}
Y(e^{\alpha},z)\mathbf{1} &= \sum_{j\geq 0} e^{\alpha}_{-j-1}\mathbf{1}z^j = \sum_{j\geq 0}\frac{1}{j!}L(-1)^j z^j e^{\alpha}\\
&\equiv \sum_{j\geq 0}(-1)^j\frac{(N+j-1)(N+j-2)\dots(N+1)N}{j!}z^j e^{\alpha} \pmod{O'}\\
&= \sum_{j\geq 0}\frac{(-N-j+1)(-N-j+2)\dots(-N-1)(-N)}{j!}z^j e^{\alpha}\\
&= \sum_{j\geq 0}\binom{-N}{j}z^j e^{\alpha} = (1+z)^{-N}e^{\alpha}.
\end{aligned} \tag{5.2.12}$$

Now by (5.2.11) and (5.2.12), and the assumption that $n \geq 0$, we have:

$$\mathrm{Res}_z Y(e^{\alpha},z)\mathbf{1}\frac{(1+z)^N}{z^{2+n}} \equiv \mathrm{Res}_z(1+z)^{-N}\frac{(1+z)^N}{z^{2+n}}e^{\alpha} = \mathrm{Res}_z\frac{1}{z^{2+n}}e^{\alpha} = 0 \pmod{O'}.$$

This finishes the proof of the base case. Assume the conclusion holds for smaller r. Note that for any $m \geq 1$, we have:

$$[\alpha(-m), Y(e^{\alpha},z)] = \sum_{i\geq 0}\binom{-m}{i}Y(\alpha(i)e^{\alpha},z)z^{-m-i} = 2NY(e^{\alpha},z)z^{-m}$$

Then by the fact that $\alpha(-m)O' \subset O'$ in Lemma 5.2.4, the base case and the induction hypothesis, we have:

$$\mathrm{Res}_z Y(e^{\alpha},z)\alpha(-n_1)\dots\alpha(-n_r)\mathbf{1}\frac{(1+z)^N}{z^{2+n}}$$

$$
\begin{aligned}
&= \mathrm{Res}_z \alpha(-n_1)\dots\alpha(-n_r)Y(e^\alpha, z)\mathbf{1}\frac{(1+z)^N}{z^{2+n}}\\
&\quad - \sum_{j=1}^{r} \mathrm{Res}_z \alpha(-n_1)\dots\alpha(-n_{j-1})[\alpha(-n_j), Y(e^\alpha, z)]\alpha(n_{j+1})\dots\alpha(-n_r)\mathbf{1}\frac{(1+z)^N}{z^{2+n}}\\
&\equiv -\sum_{j=1}^{r} 2N\mathrm{Res}_z \alpha(-n_1)\dots\alpha(-n_{j-1})Y(e^\alpha, z)\alpha(-n_{j+1})\dots\alpha(-n_r)\mathbf{1}\frac{(1+z)^N}{z^{2+n+n_j}} \pmod{O'}\\
&\equiv 0 \pmod{O'},
\end{aligned}
$$

where the last congruence follows from the induction hypothesis. This shows that

$$\mathrm{Res}_z Y(e^\alpha, z)\alpha(-n_{j+1})\dots\alpha(-n_r)\mathbf{1}((1+z)^N)/z^{2+n+n_j} \in O'.$$

Hence (5.2.11) holds for any $u \in M_{\widehat{\mathfrak{h}}}(1,0)$ and $n \geq 0$. □

Consider an arbitrary spanning element a of $M_{\widehat{\mathfrak{h}}}(1,\alpha)$, we can write

$$a = \alpha(-n_1)\dots\alpha(-n_r)e^\alpha, \tag{5.2.13}$$

for some $r \geq 0$ and $n_1, \dots, n_r \geq 1$. We want to show that $a \circ u \in O'$, for any $u \in M_{\widehat{\mathfrak{h}}}(1,0)$. If $r = 0$, we have $a = e^\alpha$, and $a \circ u \in O'$ by Proposition 5.2.5.

Assume $r \geq 1$, and we will use induction on the length r of a to show that $a \circ u \in O'$. The base case $a = \alpha(-k)e^\alpha$, with $\mathrm{wt} a = N + k$, is given by the following Lemma:

Lemma 5.2.6. *For any $k \geq 1$, $n \geq 0$, and $u \in M_{\widehat{\mathfrak{h}}}(1,0)$, we have:*

$$\mathrm{Res}_z Y(\alpha(-k)e^\alpha, z)u\frac{(1+z)^{N+k}}{z^{2+n}} \in O'. \tag{5.2.14}$$

Proof. Note that by the Jacobi identity of VOA, it is easy to derive the following formula:

$$Y(\alpha(-1)v, z) = \sum_{j\geq 0} \alpha(-j-1)Y(v,z)z^j + \sum_{j\geq 0} Y(v,z)\alpha(j)z^{-j-1}, \tag{5.2.15}$$

for any $v \in V_{\mathbb{Z}\alpha}$. Now we prove (5.2.14) by induction on k. When $k = 1$, by (5.2.15) we have:

$$
\begin{aligned}
&\mathrm{Res}_z Y(\alpha(-1)e^\alpha, z)u\frac{(1+z)^{N+1}}{z^{2+n}}\\
&= \mathrm{Res}_z \sum_{j\geq 0} \alpha(-j-1)Y(e^\alpha, z)uz^j\frac{(1+z)^{N+1}}{z^{2+n}} + \mathrm{Res}_z \sum_{j\geq 0} Y(e^\alpha, z)\alpha(j)u\frac{(1+z)^{N+1}}{z^{2+n+j+1}}
\end{aligned}
$$

$$= \mathrm{Res}_z\left(\sum_{j\geq 0}\alpha(-j-1)Y(e^{\alpha},z)uz^j\frac{(1+z)^N}{z^{2+n}} + \sum_{j\geq 0}\alpha(-j-1)Y(e^{\alpha},z)uz^{j+1}\frac{(1+z)^N}{z^{2+n}}\right)$$

$$+ \mathrm{Res}_z\sum_{j\geq 0}Y(e^{\alpha},z)\alpha(j)u\frac{(1+z)^N}{z^{2+n+j+1}} + \mathrm{Res}_z\sum_{j\geq 0}Y(e^{\alpha},z)\alpha(j)u\frac{(1+z)^N}{z^{2+n+j}}$$

$$\equiv \mathrm{Res}_z\alpha(-1)Y(e^{\alpha},z)u\frac{(1+z)^N}{z^{2+n}}$$

$$+ \sum_{j\geq 0}(\alpha(-j-2)+\alpha(-j-1))\mathrm{Res}_zY(e^{\alpha},z)uz^{j+1}\frac{(1+z)^N}{z^{2+n}} + 0 \pmod{O'}$$

$$\equiv \mathrm{Res}_z\alpha(-1)Y(e^{\alpha},z)u\frac{(1+z)^N}{z^{2+n}} \pmod{O'},$$

where the first congruence follows from Proposition 5.2.5, as $n + j \geq 0$, and the second congruence follows from (5.2.9). Furthermore, by Proposition 5.2.5 again, we have:

$$\mathrm{Res}_z\alpha(-1)Y(e^{\alpha},z)u\frac{(1+z)^N}{z^{2+n}}$$

$$= \mathrm{Res}_z\left(Y(e^{\alpha},z)\alpha(-1)u\frac{(1+z)^N}{z^{2+n}} + \sum_{j\geq 0}\binom{-1}{j}z^{-1-j}Y(\alpha(j)e^{\alpha},z)u\frac{(1+z)^N}{z^{2+n}}\right)$$

$$\equiv 0 + \mathrm{Res}_z 2NY(e^{\alpha},z)u\frac{(1+z)^N}{z^{2+n+1}} \pmod{O'}$$

$$\equiv 0 \pmod{O'}.$$

This proves (5.2.14) for $k = 1$. Assume (5.2.14) holds for k, then by (5.2.15), and the facts that $[L(-1),\alpha(-k)] = \frac{1}{k}\alpha(-k-1)$ and $\alpha(-k)L(-1)e^{\alpha} = \alpha(-1)\alpha(-k)e^{\alpha}$, we have:

$$\mathrm{Res}_zY(\alpha(-k-1)e^{\alpha},z)u\frac{(1+z)^{N+k+1}}{z^{2+n}}$$

$$= \frac{1}{k}\mathrm{Res}_z\left(Y(L(-1)\alpha(-k)e^{\alpha},z)u\frac{(1+z)^{N+k+1}}{z^{2+n}} - Y(\alpha(-k)L(-1)e^{\alpha},z)u\frac{(1+z)^{N+k+1}}{z^{2+n}}\right)$$

$$= -\frac{1}{k}\mathrm{Res}_z\left(Y(\alpha(-k)e^{\alpha},z)u\frac{d}{dz}\left(\frac{(1+z)^{N+k+1}}{z^{2+n}}\right) + Y(\alpha(-1)\alpha(-k)e^{\alpha},z)u\frac{(1+z)^{N+k+1}}{z^{2+n}}\right)$$

$$= -\frac{N+k+1}{k}\mathrm{Res}_zY(\alpha(-k)e^{\alpha},z)u\frac{(1+z)^{N+k}}{z^{2+n}}$$

$$+ \frac{2+n}{k}\mathrm{Res}_zY(\alpha(-k)e^{\alpha},z)u\frac{(1+z)^{N+k}}{z^{3+n}} + \frac{2+n}{k}\mathrm{Res}_zY(\alpha(-k)e^{\alpha},z)u\frac{(1+z)^{N+k}}{z^{2+n}}$$

$$- \frac{1}{k}\mathrm{Res}_z\sum_{j\geq 0}\alpha(-j-1)Y(\alpha(-k)e^{\alpha},z)uz^j\frac{(1+z)^{N+k+1}}{z^{2+n}}$$

$$
\begin{aligned}
&- \frac{1}{k}\mathrm{Res}_z \sum_{j\geq 0} Y(\alpha(-k)e^{\alpha}, z)\alpha(j)u\frac{(1+z)^{N+k+1}}{z^{2+n+j+1}}\\
&\equiv 0 - \frac{1}{k}\mathrm{Res}_z \sum_{j\geq 0} \alpha(-j-1)\left(Y(\alpha(-k)e^{\alpha}, z)uz^{j} + Y(\alpha(-k)e^{\alpha}, z)uz^{j+1}\right)\frac{(1+z)^{N+k}}{z^{2+n}}\\
&- \frac{1}{k}\mathrm{Res}_z \sum_{j\geq 0} Y(\alpha(-k)e^{\alpha}, z)\alpha(j)u\frac{(1+z)^{N+k}(1+z)}{z^{2+n+1+j}} \pmod{O'}\\
&= -\frac{1}{k}\mathrm{Res}_z\alpha(-1)Y(\alpha(-k)e^{\alpha}, z)u\frac{(1+z)^{N+k}}{z^{2+n}}\\
&- \frac{1}{k}\mathrm{Res}_z \sum_{j\geq 0} \left(\alpha(-j-2) + \alpha(-j-1)\right) Y(\alpha(-k)e^{\alpha}, z)uz^{j+1}\frac{(1+z)^{N+k}}{z^{2+n}}\\
&- \frac{1}{k}\mathrm{Res}_z \sum_{j\geq 0} Y(\alpha(-k)e^{\alpha}, z)\alpha(j)u\frac{(1+z)^{N+k}(1+z)}{z^{2+n+1+j}}\\
&\equiv -\frac{1}{k}\mathrm{Res}_z\alpha(-1)Y(\alpha(-k)e^{\alpha}, z)u\frac{(1+z)^{N+k}}{z^{2+n}} \pmod{O'},
\end{aligned}
$$

where the first congruence follows from the induction hypothesis (5.2.14), and the second congruence follows from (5.2.9) and the induction hypothesis. By the Jacobi identity and the Heisenberg relation $[\alpha(j), \alpha(-k)] = \delta_{j,k}kK$, for any $j \geq 0$, we have:

$$
\begin{aligned}
&- \frac{1}{k}\mathrm{Res}_z\alpha(-1)Y(\alpha(-k)e^{\alpha}, z)u\frac{(1+z)^{N+k}}{z^{2+n}}\\
&= -\frac{1}{k}\mathrm{Res}_z Y(\alpha(-k)e^{\alpha}, z)\alpha(-1)u\frac{(1+z)^{N+k}}{z^{2+n}} - \frac{1}{k}\mathrm{Res}_z \sum_{j\geq 0}\binom{-1}{j}z^{-1-j}Y(\alpha(j)\alpha(-k)e^{\alpha}, z)u\frac{(1+z)^{N+k}}{z^{2+n}}\\
&\equiv 0 - \frac{1}{k}\mathrm{Res}_z(-1)^k kY(e^{\alpha}, z)u(1+z)^k\frac{(1+z)^N}{z^{2+n+1+k}} \pmod{O'}\\
&= -\mathrm{Res}_z(-1)^k\sum_{i\geq 0}\binom{k}{i}Y(e^{\alpha}, z)u\frac{(1+z)^N}{z^{2+n+1+i}} \equiv 0 \pmod{O'},
\end{aligned}
$$

where the first congruence follows from the induction hypothesis, and the second congruence follows from Proposition 5.2.5. Therefore we have:

$$
\mathrm{Res}_z Y(\alpha(-k-1)e^{\alpha}, z)u\frac{(1+z)^{N+k+1}}{z^{2+n}} \in O'.
$$

So (5.2.14) holds for $k+1$, the inductive step is complete. □

Proposition 5.2.7. *For any $u \in M_{\widehat{\mathfrak{h}}}(1,0)$ and $a \in M_{\widehat{\mathfrak{h}}}(1,\alpha)$, we have:*

$$
\mathrm{Res}_z Y(a, z)u\frac{(1+z)^{\mathrm{wt}\, a}}{z^{2+n}} \in O', \tag{5.2.16}
$$

for any $n \geq 0$. In particular, we have $a \circ u \in O'$.

Proof. Write $a = \alpha(-n_1)\dots\alpha(-n_r)e^{\alpha}$ as (5.2.13), where $r \geq 0$ and $n_1, \dots, n_r \geq 1$. We prove (5.2.16) by induction on r. By Proposition 5.2.5 and Lemma 5.2.6, (5.2.16) holds when $a = e^{\alpha}$ or $a = \alpha(-k)e^{\alpha}$. Now let $r \geq 2$. The induction hypothesis is the assumption that

$$\mathrm{Res}_z Y(\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha}, z)u\frac{(1+z)^{N+n_2+\cdots+n_r}}{z^{2+n}} \in O', \tag{5.2.17}$$

for $n_2, \dots, n_r \geq 1$, $n \geq 0$, and $u \in M_{\widehat{\mathfrak{h}}}(1,0)$. First, we claim that

$$\mathrm{Res}_z Y(\alpha(-1)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha}, z)u\frac{(1+z)^{N+n_2+\cdots+n_r+1}}{z^{2+n}} \in O'. \tag{5.2.18}$$

Denote $N + n_2 + \cdots + n_r$ by m, note that $\mathrm{wt}(\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha}) = m$. Then by (5.2.15), (5.2.9), and the induction hypothesis, we have:

$$\begin{aligned}
&\mathrm{Res}_z Y(a(-1)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha}, z)u\frac{(1+z)^{1+m}}{z^{2+n}}\\
&= \mathrm{Res}_z \sum_{j\geq 0} a(-j-1)Y(\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha}, z)uz^j\frac{(1+z)^{1+m}}{z^{2+n}}\\
&\quad + \mathrm{Res}_z \sum_{j\geq 0} Y(\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha}, z)(a(j)u)\frac{(1+z)^{1+m}}{z^{2+n+j+1}}\\
&\equiv \mathrm{Res}_z \sum_{j\geq 0} a(-j-1)Y(\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha}, z)uz^j\frac{(1+z)^{m}}{z^{2+n}}\\
&\quad + \mathrm{Res}_z \sum_{j\geq 0} a(-j-1)Y(a_2(\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha}, z)uz^{j+1}\frac{(1+z)^{m}}{z^{2+n}} \pmod{O'}\\
&= \mathrm{Res}_z a(-1)Y(\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha}, z)u\frac{(1+z)^{m}}{z^{2+n}}\\
&\quad + \mathrm{Res}_z \sum_{j\geq 0} a(-j-2)Y(\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha}, z)uz^{j+1}\frac{(1+z)^{m}}{z^{2+n}}\\
&\quad + \mathrm{Res}_z \sum_{j\geq 0} a(-j-1)Y(\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha}, z)uz^{j+1}\frac{(1+z)^{m}}{z^{2+n}}\\
&= \mathrm{Res}_z [a(-1), Y(\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha}, z)]u\frac{(1+z)^{m}}{z^{2+n}}\\
&\quad + \mathrm{Res}_z Y(\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha}, z)a(-1)u\frac{(1+z)^{m}}{z^{2+n}}\\
&\quad + \mathrm{Res}_z \sum_{j\geq 0}(a(-j-2) + a(-j-1))Y(\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha}, z)uz^{j+1}\frac{(1+z)^{m}}{z^{2+n}}
\end{aligned}$$

$$\equiv \sum_{i\geq 0}\binom{-1}{i}z^{-1-i}\mathrm{Res}_z Y(a(i)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^m}{z^{2+n}} \pmod{O'}$$

$$= \mathrm{Res}_z 2Y(\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha})u\frac{(1+z)^m}{z^{3+n}}$$

$$+\sum_{i\geq 0}\sum_{s=2}^{r}\binom{-1}{i}\mathrm{Res}_z Y(\alpha(-n_2)\dots[\alpha(i),\alpha(-n_s)]\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^m}{z^{2+n+i+1}}$$

$$\equiv \sum_{s=2}^{r}(-1)^{n_s}n_s\mathrm{Res}_z Y(\alpha(-n_2)\dots\widehat{\alpha(-n_s)}\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^m}{z^{2+n+n_s+1}} \pmod{O'}.$$

Denote $\alpha(-n_2)\dots\widehat{\alpha(-n_s)}\dots\alpha(-n_r)e^{\alpha}$ by a_s. Then $m = \mathrm{wt}a_s + n_s$, and by the induction hypothesis (5.2.17), with r replaced by $r-1$, we have:

$$\sum_{s=2}^{r}(-1)^{n_s}n_s\mathrm{Res}_z Y(\alpha(-n_2)\dots\widehat{\alpha(-n_s)}\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^m}{z^{2+n+n_s+1}}$$

$$= \sum_{s=2}^{r}(-1)^{n_s}n_s\mathrm{Res}_z Y(a_s,z)u(1+z)^{n_s}\frac{(1+z)^{\mathrm{wt}a_s}}{z^{2+n+n_s+1}}$$

$$= \sum_{s=2}^{r}\sum_{j\geq 0}\binom{n_s}{j}(-1)^{n_s}n_s\mathrm{Res}_z Y(a_s,z)u\frac{(1+z)^{\mathrm{wt}a_s}}{z^{2+n_s+j+1}} \equiv 0 \pmod{O'},$$

since $n_s + j + 1 \geq 1$. This proves (5.2.18). Now assume that

$$\mathrm{Res}_z Y(\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^{N+k+n_2+\dots+n_r}}{z^{2+n}} \in O', \tag{5.2.19}$$

for some fixed $k \geq 1$, and any $n_2,\dots,n_r \geq 1$ and $n \geq 0$, we want to show that

$$\mathrm{Res}_z Y(\alpha(-k-1)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}} \in O'. \tag{5.2.20}$$

Indeed, by a similar argument as the proof of Lemma 5.2.6, we have:

$$\mathrm{Res}_z Y(\alpha(-k-1)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}}$$

$$= \mathrm{Res}_z \frac{1}{k}Y(L(-1)\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}}$$

$$+ \mathrm{Res}_z \frac{1}{k}Y(\alpha(-k)[L(-1),\alpha(-n_2)\dots\alpha(-n_r)]e^{\alpha},z)u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}}$$

$$+ \mathrm{Res}_z \frac{1}{k}Y(\alpha(-1)\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha})u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}} \tag{5.2.21}$$

$$= -\mathrm{Res}_z \frac{1}{k}Y(\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha},z)u\frac{d}{dz}\left(\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}}\right)$$

$$+\sum_{s=2}^{r}\mathrm{Res}_z\frac{n_s}{k}Y(\alpha(-k)\alpha(-n_2)\dots\alpha(-n_s-1)\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^{N+k+\dots(1+n_s)\dots+n_r}}{z^{2+n}}$$
$$+\mathrm{Res}_z\frac{1}{k}Y(\alpha(-1)\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha})u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}}$$
$$=-\mathrm{Res}_z\frac{1}{k}(N+k+1+\dots+n_r)Y(\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^{N+k+n_2+\dots+n_r}}{z^{2+n}}$$
$$+\mathrm{Res}_z\frac{2+n}{k}Y(\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^{N+k+n_2+\dots+n_r}(1+z)}{z^{2+n+1}}$$
$$+\sum_{s=2}^{r}\mathrm{Res}_z\frac{n_s}{k}Y(\alpha(-k)\alpha(-n_2)\dots\alpha(-n_s-1)\dots\alpha(-n_r)e^{\alpha},z)u\frac{(1+z)^{N+k+\dots(1+n_s)\dots+n_r}}{z^{2+n}}$$
$$+\mathrm{Res}_z\frac{1}{k}Y(\alpha(-1)\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha})u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}}$$
$$\equiv 0+\mathrm{Res}_z\frac{1}{k}Y(\alpha(-1)\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha})u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}}\pmod{O'},$$

where the congruences follow from the induction (on $k\geq 1$) hypothesis (5.2.19). Moreover, by adopting a similar argument as our previous proof of (5.2.18), with the given assumption (5.2.17), we have:

$$\mathrm{Res}_z\frac{1}{k}Y(\alpha(-1)\alpha(-k)\alpha(-n_2)\dots\alpha(-n_r)e^{\alpha})u\frac{(1+z)^{N+k+1+n_2+\dots+n_r}}{z^{2+n}}\in O',$$

with the given assumption (5.2.19). Thus, (5.2.20) is true, and the induction step on $k\geq 1$ and the induction step on the length $r\geq 1$ of $a\in M_{\widehat{\mathfrak{h}}}(1,\alpha)$ are both complete. □

Now we have finished the proof of $a\circ u\in O'$ for the first case in (5.2.10). The second case when $a\in M_{\widehat{\mathfrak{h}}}(1,0)$ and $u\in M_{\widehat{\mathfrak{h}}}(1,\alpha)$ follows from a similar induction process as Lemma 5.2.6 and Proposition 5.2.7 (see also (5.2.1.5) and (5.2.1.6) in [30]), we omit the details of the proof. In particular, for the Borel-type VOA $V=V_{\mathbb{Z}_{\geq 0}\alpha}$, we have:

$$O(V)=O'=\mathrm{span}\Big\{\alpha(-n-2)u+\alpha(-n-1)u,\ \alpha(-1)v+v,\ M_{\widehat{\mathfrak{h}}}(1,k\alpha):\ n\geq 0,\ u\in V,\ v\in\bigoplus_{m\geq 1}M_{\widehat{\mathfrak{h}}}(1,m\alpha),\ k\geq 2\Big\}.\tag{5.2.22}$$

Theorem 5.2.8. *For $V=V_{\mathbb{Z}_{\geq 0}\alpha}$, with $(\alpha|\alpha)=2N$, the epimorphism F given by* (5.2.7) *is an isomorphism of associative algebras. In particular, we have $A(V)\cong\mathbb{C}[x]\oplus\mathbb{C}y$, with*

$$y^2=0,\quad yx=-Ny,\quad xy=Ny.\tag{5.2.23}$$

Proof. We construct an inverse map of F in (5.2.7). Define a linear map:

$$\begin{aligned} G : V &\to \mathbb{C}\langle x, y\rangle/\langle y^2, yx + Ny, xy - Ny\rangle, \\ \alpha(-n_1 - 1)\dots\alpha(-n_r - 1)\mathbf{1} &\mapsto (-1)^{n_1+\dots n_r} x^r, \\ \alpha(-n_1 - 1)\dots\alpha(-n_r - 1)e^{\alpha} &\mapsto (-1)^{n_1+\dots n_r} yx^r = (-1)^{r+n_1+\dots n_r} y, \\ M_{\widehat{\mathfrak{h}}}(1, k\alpha) &\mapsto 0, \end{aligned} \tag{5.2.24}$$

where $r \geq 0$, $n_1, \dots, n_r \geq 0$, and $k \geq 2$, and we use the same symbols x and y to denote their image in the quotient space. Note that G is well-defined, since $V = \bigoplus_{k\geq 0} M_{\widehat{\mathfrak{h}}}(1, k\alpha)$, and $\alpha(-n_1 - 1)\dots\alpha(-n_r - 1)\mathbf{1}$ and $\alpha(-n_1 - 1)\dots\alpha(-n_r - 1)e^{\alpha}$ are basis elements of $M_{\widehat{\mathfrak{h}}}(1, 0)$ and $M_{\widehat{\mathfrak{h}}}(1, \alpha)$, respectively. We claim that $G(O(V)) = 0$.

Indeed, it suffices to show that G vanishes on the spanning elements of $O(V)$ in (5.2.22). By Definition (5.2.24), we already have $G(M_{\widehat{\mathfrak{h}}}(1, k\alpha)) = 0$ for any $k \geq 2$. In particular, we have $G(\alpha(-n-2)u + \alpha(-n-1)u) = G(\alpha(-1)v + Nv) = 0$ if $u, v \in M_{\widehat{\mathfrak{h}}}(1, k\alpha)$ for some $k \geq 2$. If $u = \alpha(-n_1 - 1)\dots\alpha(-n_r - 1)\mathbf{1} \in M_{\widehat{\mathfrak{h}}}(1, 0)$, then by (5.2.24), we have:

$$\begin{aligned} &G(\alpha(-n - 2)u + \alpha(-n - 1)u) \\ &= G(\alpha(-n - 2)\alpha(-n_1 - 1)\dots\alpha(-n_r - 1)\mathbf{1}) + G(\alpha(-n - 1)\alpha(-n_1 - 1)\dots\mathbf{1}) \\ &= (-1)^{n+1+n_1+\dots+n_r} x^{r+1} + (-1)^{n+n_1+\dots+n_r} x^{r+1} = 0. \end{aligned}$$

If $u = \alpha(-n_1 - 1)\dots\alpha(-n_r - 1)e^{\alpha} \in M_{\widehat{\mathfrak{h}}}(1, \alpha)$, by (5.2.24) we have:

$$\begin{aligned} &G(\alpha(-n - 1)u + \alpha(-n - 1)u) \\ &= G(\alpha(-n - 2)\alpha(-n_1 - 1)\dots\alpha(-n_r - 1)e^{\alpha}) + G(\alpha(-n - 1)\alpha(-n_1 - 1)\dots e^{\alpha}) \\ &= (-1)^{n+1+n_1+\dots+n_r} yx^{r+1} + (-1)^{n+n_1+\dots+n_r} yx^{r+1} = 0. \end{aligned}$$

Thus, $G(\alpha(-n-2)u+\alpha(-n-1)u) = 0$ for any $u \in V$. Finally, if $v = \alpha(-n_1-1)\dots\alpha(-n_r-1)e^{\alpha} \in M_{\widehat{\mathfrak{h}}}(1, \alpha)$, then by (5.2.24), we have:

$$\begin{aligned} G(\alpha(-1)v + Nv) &= G(\alpha(-1)\alpha(-n_1 - 1)\dots\alpha(-n_r - 1)e^{\alpha}) + NG(\alpha(-n_1 - 1)\dots\alpha(-n_r - 1)e^{\alpha}) \\ &= (-1)^{n_1+\dots n_r} yx^{r+1} + (-1)^{n_1+\dots n_r} Nyx^r \\ &= (-1)^{n_1+\dots n_r} (yx + Ny)x^r = 0, \end{aligned}$$

since $yx + Ny = 0$. Thus, G in (5.2.24) induces a linear map

$$G : A(V) = V/O(V) \to \mathbb{C}\langle x, y\rangle/\langle y^2, yx + Ny, xy - Ny\rangle, \quad \text{such that}$$

$$G([\alpha(-n_1-1)\dots\alpha(-n_r-1)\mathbf{1}]) = G((-1)^{n_1+\cdots+n_r}[\alpha(-1)\mathbf{1}]^r) = (-1)^{n_1+\cdots+n_r}x^r, \qquad (5.2.25)$$
$$G([\alpha(-n_1-1)\dots\alpha(-n_r-1)e^{\alpha}]) = G((-1)^{n_1+\cdots+n_r}[e^{\alpha}]*[\alpha(-1)\mathbf{1}]^r) = (-1)^{n_1+\cdots+n_r}yx^r,$$

for any $r \geq 0$, $n_1, \dots, n_r \geq 0$, and $k \geq 2$. Since $A(V)$ is spanned by elements of the form $[\alpha(-n_1-1)\dots\alpha(-n_r-1)\mathbf{1}]$ and $[\alpha(-n_1-1)\dots\alpha(-n_r-1)e^{\alpha}]$ because of (5.2.22), it is clear that $G\circ F = \mathrm{Id}$ and $F \circ G = \mathrm{Id}$, in view of (5.2.7) and (5.2.25). □

Recall the definitions of graded algebra $\mathrm{gr}A(V)$ and the C_2-algebra $R(V)$ in Section 2.2. We examined some examples of C_1-cofinite VOAs and proved that $\mathrm{gr}A(V)$ is isomorphic to $R(V)$ for these examples in Section 2.2. For the Borel-type sub-algebra $V_{\mathbb{Z}_{\geq 0}\alpha}$, by Theorem 5.2.8, we also have the isomorphism $\mathrm{gr}A(V_{\mathbb{Z}_{\geq 0}\alpha}) \cong R(V_{\mathbb{Z}_{\geq 0}\alpha})$.

Corollary 5.2.9. $\mathrm{gr}A(V_{\mathbb{Z}_{\geq 0}\alpha})$ *is isomorphic to* $R(V_{\mathbb{Z}_{\geq 0}\alpha})$ *as commutative Poisson algebras. As associative algebras, they are both generated by two elements* X *and* Y*, subject to the relations:* $XY = YX = Y^2 = 0$*. Furthermore,* $\{X, Y\} = 2NY$*.*

Proof. We consider $\mathrm{gr}A(V_{\mathbb{Z}_{\geq 0}\alpha})$ first. With the notations in Theorem 5.2.8, we denote the equivalent classes of x and y in $\mathrm{gr}A(V_{\mathbb{Z}_{\geq 0}\alpha})$ by $\bar{x}$ and $\bar{y}$, respectively. Note that $x = [\alpha(-1)\mathbf{1}] \in F_1A(V_{\mathbb{Z}_{\geq 0}\alpha})$, $y = [e^{\alpha}] \in F_NA(V_{\mathbb{Z}_{\geq 0}\alpha})$, and by Lemma 2.2.1 and (5.2.23), we have $\bar{x}*\bar{y} = \overline{x*y} = \overline{2NY} = 0$ in $\mathrm{gr}A(V_{\mathbb{Z}_{\geq 0}\alpha})$ since $x*y \in F_{N+1}A(V_{\mathbb{Z}_{\geq 0}\alpha})$ but $2Ny \in F_NA(V_{\mathbb{Z}_{\geq 0}\alpha})$. Moreover, we have $\bar{y}*\bar{x} = \bar{x}*\bar{y} = 0$ and $\{\bar{x}, \bar{y}\} = \overline{[\alpha(0)e^{\alpha}]} = 2N\bar{y}$ by Lemma 2.2.1.

Denote $\bar{x}$ and $\bar{y}$ in $\mathrm{gr}A(V_{\mathbb{Z}_{\geq 0}\alpha})$ by X and Y, respectively. Then by Theorem 5.2.8, we have $\mathrm{gr}A(V_{\mathbb{Z}_{\geq 0}\alpha}) \cong \mathbb{C}[X] \oplus \mathbb{C}Y$, with $XY = YX = Y^2 = 0$, and $\{X, Y\} = 2NY$.

Now we consider the C_2-algebra $R(V_{\mathbb{Z}_{\geq 0}\alpha})$. By Proposition 5.1.6, $V_{\mathbb{Z}_{\geq 0}\alpha}$ is strongly generated by $\{\alpha(-1)\mathbf{1}, e^{\alpha}\}$. Then by Proposition 2.2.3, $R(V_{\mathbb{Z}_{\geq 0}\alpha})$ is generated as a commutative algebra by $X = \alpha(-1)\mathbf{1} + C_2(V)$ and $Y = e^{\alpha} + C_2(V)$. By the proof of Proposition 5.2.1, we also have $XY = \alpha(-1)e^{\alpha} + C_2(V) = e^{\alpha}_{-2}\mathbf{1} + C_2(V) = 0$, and $e^{\alpha}_{-1}e^{\alpha} + C_2(V) = 0 + C_2(V) = 0$ in $R(V)$. Moreover, $\{X, Y\} = \alpha(0)e^{\alpha} + C_2(V) = 2N(e^{\alpha} + C_2(V)) = 2NY$. Thus, $R(V_{\mathbb{Z}_{\geq 0}}) \cong \mathbb{C}[X] \oplus \mathbb{C}Y$, with $XY = YX = Y^2 = 0$, and $\{X, Y\} = 2NY$. Therefore, as commutative Poisson algebras. Therefore, $\mathrm{gr}A(V_{\mathbb{Z}_{\geq 0}\alpha}) \cong R(V_{\mathbb{Z}_{\geq 0}\alpha})$ as commutative Poisson algebras □

5.2.2 Irreducible modules of $V_{\mathbb{Z}_{\geq 0}\alpha}$ and the induction

Lemma 5.2.10. *If* $U \neq 0$ *is an irreducible module over* $A(V_{\mathbb{Z}_{\geq 0}\alpha}) \cong \mathbb{C}[x] \oplus \mathbb{C}y$*, then we must have* $y.U = 0$*, and* $U \cong \mathbb{C}e^{\lambda}$ *for some* $\lambda \in \mathfrak{h} = \mathbb{C}\alpha$*, with* $x.e^{\lambda} = (\alpha|\lambda)e^{\lambda}$*.*

Proof. By (5.2.23), $\mathbb{C}y$ is an ideal of $A(V_{\mathbb{Z}_{\geq 0}\alpha})$. Hence $y.U \leq U$ is a submodule, and $y.U$ is either U or 0. If $y.U = U$, then we have $0 = y^2.U = y.U = U$, a contradiction. Thus, $y.U = 0$ and U is an irreducible module over $\mathbb{C}[x]$. We have $U \cong \mathbb{C}[x]/\mathfrak{m}$, for some maximal ideal $\mathfrak{m}$ of $\mathbb{C}[x]$. By the Hilbert's Nullstellensatz, we have $\mathfrak{m} = \langle x - \mu \rangle$ for some $\mu \in \mathbb{C}$. We may choose $\lambda \in \mathfrak{h}$ so that $(\alpha|\lambda) = \mu$. Then $U \cong \mathbb{C}[x]/\langle x - (\alpha|\lambda)\rangle \cong \mathbb{C}e^\lambda$, with $x.e^\lambda = (\alpha|\lambda)e^\lambda$. □

Lemma 5.2.11. *For any irreducible module $W = M_{\widehat{\mathfrak{h}}}(1,\lambda)$ over the Heisenberg VOA $M_{\widehat{\mathfrak{h}}}(1,0)$, W is also an irreducible module over the Borel-type sub-algebra $V_{\mathbb{Z}_{\geq 0}\alpha}$, where $Y_W : V_{\mathbb{Z}_{\geq 0}\alpha} \to$ End$(W)[[z,z^{-1}]]$ satisfies $Y_W(a,z) = 0$, for any $a \in M_{\widehat{\mathfrak{h}}}(1,n\alpha)$ and $n \geq 1$, and $Y_W|_{M_{\widehat{\mathfrak{h}}}(1,0)}$ is given by the action of the Heisenberg VOA $M_{\widehat{\mathfrak{h}}}(1,0)$.*

Proof. By (5.1.8), $(W = M_{\widehat{\mathfrak{h}}}(1,\lambda), Y_W)$, with Y_W defined by the assumption is a well-defined module over the Borel-type sub-algebra $V_{\mathbb{Z}_{\geq 0}\alpha}$. It is clear that W is irreducible. □

Theorem 5.2.12. $\Sigma = \left\{(W = M_{\widehat{\mathfrak{h}}}(1,\lambda), Y_W) : \lambda \in \mathfrak{h} = \mathbb{C}\alpha\right\}$, *with Y_W defined by Lemma 5.2.11, is the complete list of irreducible modules over the rank-one Borel-type sub-algebra $V_{\mathbb{Z}_{\geq 0}\alpha}$.*

Moreover, the fusion rule of the irreducible $V_{\mathbb{Z}_{\geq 0}\alpha}$-modules $M_{\widehat{\mathfrak{h}}}(1,\lambda), M_{\widehat{\mathfrak{h}}}(1,\mu)$, and $M_{\widehat{\mathfrak{h}}}(1,\gamma)$ is the same as the fusion rule of the Heisenberg VOA. i.e., $N\binom{M_{\widehat{\mathfrak{h}}}(1,\gamma)}{M_{\widehat{\mathfrak{h}}}(1,\lambda)\ M_{\widehat{\mathfrak{h}}}(1,\mu)} \cong \delta_{\lambda+\mu,\gamma}$.

Proof. Given a module $(W = M_{\widehat{\mathfrak{h}}}(1,\lambda), Y_W)$ in Σ, the bottom level is $W(0) = \mathbb{C}e^\lambda$, which is an $A(V_{\mathbb{Z}_{\geq 0}\alpha})$-module, with the actions of $x = [\alpha(-1)\mathbf{1}]$ and $y = [e^\alpha]$ given by

$$x.e^\alpha = o(\alpha(-1)\mathbf{1})e^\lambda = (\alpha|\lambda)e^\lambda, \quad y.e^\lambda = o(e^\alpha)e^\lambda = \mathrm{Res}_z z^{N-1} Y_W(e^\alpha,z)e^\lambda = 0.$$

By Lemma 5.2.10, such $A(V_{\mathbb{Z}_{\geq 0}\alpha})$-module $W(0)$, with W varies in Σ, are all the irreducible modules over $A(V_{\mathbb{Z}_{\geq 0}\alpha})$, up to isomorphism. Then by Theorem 2.2.2 in [73], Σ is the complete list of irreducible modules over $V_{\mathbb{Z}_{\geq 0}\alpha}$.

Finally, note that any intertwining operator of modules over the Heisenberg VOA $I \in I\binom{M_{\widehat{\mathfrak{h}}}(1,\gamma)}{M_{\widehat{\mathfrak{h}}}(1,\lambda)\ M_{\widehat{\mathfrak{h}}}(1,\mu)}$ can be naturally lifted up to an intertwining operator $\tilde{I}$ of $V_{\mathbb{Z}_{\geq 0}\alpha}$ since the Jacobi identity of I is

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_{W^3}(a,z_1)I(v,z_2)u - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) I(v,z_2)Y_{W^2}(a,z_1)u \\ &= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) I(Y_{W^1}(a,z_0)v,z_2)u, \end{aligned} \tag{5.2.26}$$

and $Y_{W^i}(a,z) = 0$ for $i = 1,2,3$, if $a \in M_{\widehat{\mathfrak{h}}}(1,n\alpha)$ with $n \geq 1$. Therefore, we can replace I in (5.2.26) by the intertwining operator $\tilde{I}$ of $V_{\mathbb{Z}_{\geq 0}\alpha}$. Conversely, we can also view any intertwining operator $\mathcal{Y} \in I\binom{M_{\hat{\mathfrak{h}}}(1,\gamma)}{M_{\hat{\mathfrak{h}}}(1,\lambda)\ M_{\hat{\mathfrak{h}}}(1,\mu)}$ of $V_{\mathbb{Z}_{\geq 0}\alpha}$ to an intertwining operator $\mathcal{Y}$ of the same type of modules over the Heisenberg VOA. Therefore, the fusion rules of the Borel-type sub-algebra $V_{\mathbb{Z}_{\geq 0}\alpha}$ is the same as the fusion rules of the Heisenberg VOA $M_{\hat{\mathfrak{h}}}(1,0)$. □

Remark 5.2.13. We can also prove the claim about the fusion rules of $V_{\mathbb{Z}_{\geq 0}\alpha}$ by the general fusion rules theorem in Section 4.2. Theorem 5.2.12 is also similar to the simple Lie algebra case. Note that a Borel sub-algebra $\mathfrak{b} = \mathfrak{n}_+ \oplus \mathfrak{h}$ of a simple Lie algebra $\mathfrak{g}$ has the same irreducible modules as the Cartan part $\mathfrak{h}$, and the irreducible modules over (the abelian Lie algebra) $\mathfrak{h}$ are all one-dimensional.

Now we consider the special case of the rank-one Borel-type sub-algebra. Let $L = \mathbb{Z}\alpha$, with $(\alpha|\alpha) = 2$. Then L is the root lattice of type A_1. Then V_L is isomorphic to the affine VOA $L_{\widehat{sl_2}}(1,0)$, where $sl_2 = \mathbb{C}e + \mathbb{C}h + \mathbb{C}f$, and $e^{\alpha} \mapsto e,\ \alpha(-1)\mathbf{1} \mapsto h,\ e^{-\alpha} \mapsto f$, see [28, 30].

Recall that $A(L_{\widehat{sl_2}}(1,0)) \cong U(sl_2)/\langle e^2\rangle$, where $\langle e^2\rangle$ is the two-sided ideal of $A(L_{\widehat{sl_2}}(1,0))$ generated by e^2, and $[a(-1)\mathbf{1}] \mapsto a + \langle e^2\rangle$ for all $a \in sl_2$, see [30]. By applying the Lie bracket $[a,\cdot]$ to e^2 repeatedly, it is easy to show that the following relations hold in $A(L_{\widehat{sl_2}}(1,0))$:

$$eh + e = 0; \quad h^2 - h - 2fe = 0; \quad fh + f = 0; \quad e^2 = f^2 = 0, \tag{5.2.27}$$

where we used the same symbol to denote the equivalent classes. It follows that $A(L_{\widehat{sl_2}}(1,0))$ has a basis $\{1, e, f, h, fe\}$.

Now let A be a sub-algebra of $A(L_{\widehat{sl_2}}(1,0))$ generated by the Borel sub-algebra $\mathfrak{b} = \mathbb{C}e + \mathbb{C}h \leq \mathfrak{g}$. Then by (5.2.27), we have $A = \langle 1, e, h, fe\rangle$. Moreover, by Theorem 5.2.8, we have an epimorphism of associative algebras:

$$A(V_{\mathbb{Z}_{\geq 0}\alpha}) \twoheadrightarrow A = \langle 1, e, h, fe\rangle, \quad x \mapsto h,\ y \mapsto e,\ x^2 - x \mapsto fe. \tag{5.2.28}$$

Since A has more relations than $A(V_{\mathbb{Z}_{\geq 0}\alpha})$, not all irreducible modules over $A(V_{\mathbb{Z}_{\geq 0}\alpha})$ in Theorem 5.2.12 can factor through A.

Let $\mathbb{C}e^{\lambda}$ be an irreducible module over $A(V_{\mathbb{Z}_{\geq 0}\alpha})$, with $x.e^{\lambda} = (\alpha|\lambda)e^{\lambda}$ and $y.e^{\lambda} = 0$. Suppose it can factor through A. Then by (5.2.28), we must have $e.e^{\lambda} = 0$, $h.e^{\lambda} = (\alpha|\lambda)e^{\lambda}$, and $(x^2 - x).e^{\lambda} = ((\alpha|\lambda)^2 - (\alpha|\lambda))e^{\lambda} = fe.e^{\lambda} = 0$. Thus, we have $(\alpha|\lambda)^2 = (\alpha|\lambda)$, and so $\lambda = 0$ or $\alpha/2$. In other words, the only $A(V_{\mathbb{Z}_{\geq 0}\alpha})$-modules that can factor through A are $\mathbb{C}\mathbf{1}$ and $\mathbb{C}e^{\alpha/2}$.

It is an interesting question to investigate the induced modules

$$U_1 := \mathrm{Ind}_A^{A(V_L)}\mathbb{C}\mathbf{1} = A(V_L) \otimes_A \mathbb{C}\mathbf{1}, \quad \text{and} \quad U_2 := \mathrm{Ind}_A^{A(V_L)}\mathbb{C}e^{\alpha/2} = A(V_L) \otimes_A \mathbb{C}e^{\alpha/2}. \tag{5.2.29}$$

More generally, we can introduce a notion of induced modules based on our observations above.

Definition 5.2.14. Let $(V, Y, \mathbf{1}, \omega)$ be a VOA, and $(U, Y, \mathbf{1}, \omega)$ be a sub-VOA of V. Suppose there exists an epimorphism $A(U) \twoheadrightarrow A \leq A(V)$, where A is a sub-algebra of $A(V)$.

Let $W = \bigoplus_{n=0}^{\infty} W(n)$ be an admissible module over the VOA U such that the $A(U)$-module $W(0)$ can factor through A. Define the induced module $\mathrm{Ind}_U^V W$ as follows:

$$\mathrm{Ind}_U^V W := L(A(V) \otimes_A W(0)), \tag{5.2.30}$$

where L is the functor defined in [18].

A natural example of the setups in Definition 5.2.14 is the Borel-type sub-algebra $V_{\mathbb{Z}_{\geq 0}\alpha} \leq V_{\mathbb{Z}\alpha}$, with $(\alpha|\alpha) = 2$ in this subsection. By (5.2.28) and (5.2.29), we have induced modules $L(U_1)$ and $L(U_2)$ of the lattice VOA $V_{\mathbb{Z}\alpha}$. Since U_1 and U_2 are not necessarily irreducible $A(V_{\mathbb{Z}\alpha})$-modules, an interesting question is to find the irreducible module decomposition of $L(U_1)$ and $L(U_2)$.

Remark 5.2.15. On the other hand, an alternative way of constructing induced modules over VOAs was given by Dong and Lin in [22]. They proved that the induction functor Ind constructed in their paper satisfies the usual Frobenius reciprocity. It is natural to compare their inducted modules with our construction (5.2.30). We will take a closer look at this induced module problem in the future.

Chapter 6

Rota-Baxter operators on vertex algebras

This Chapter and the next are the core of this thesis's second part. In this Chapter, we will study in detail the Rota-Baxter operators (RBO) on vertex (operator) algebras as a natural generalization of the Rota-Baxter operators on classical Lie and associative algebras.

In the first Section of this Chapter, after reviewing some basic concepts, we will give the definition and first examples of the (index m) Rota-Baxter operators for vertex (operator) algebras. We call a vertex algebra $(V, Y, \mathbf{1})$, equipped with a Rota-Baxter operator $P : V \to V$, a Rota-Baxter vertex algebra (RBVA), and denote it by $(V, Y, \mathbf{1}, P)$. Since there are infinitely many products on a vertex algebra, the natural generalization of RBO needs to satisfy a quite strong condition. Hence examples of such operators are scarce. Even though the Borel-type sub-algebras from the previous Chapter can provide us with a nontrivial natural example, some other classical examples of vertex algebras, like the rank-one Heisenberg VOA and the Virasoro VOA, only admit trivial examples of such Rota-Baxter operators. Therefore, we use the λ-differential we introduced in Section 2.3 and give a relatively weaker notion of RBO on vertex algebras, which has many examples even on a single vertex algebra. Then we will discuss some basic properties of our definition of RBVAs.

In the second Section of this Chapter, we will study the substructures underlying a Rota-Baxter vertex algebra. In particular, similar to the associative algebra case, we have a so-called dendriform vertex algebra structure $(V, \prec_z, \succ_z, D)$ associated with each RBVA. We will justify our proposed axioms for the dendriform vertex algebra by showing that $(V, \prec_z, \succ_z, D)$ can give rise to a representation of the vertex algebra (without vacuum) (V, Y, D) on itself, and a dendriform vertex algebra can also give rise to the relative Rota-Baxter operators for vertex algebras, which is closely related to the classical Yang-Baxter equation for VOAs in the next Chapter. This Chapter is based on the paper [7].

6.1 The Rota-Baxter vertex algebra

We will study the basics of Rota-Baxter operators on vertex algebras in this Section. Similar to Section 5.2, for the sake of the clearness of our cross-references, we will first write out some of the basic formulas and notions of vertex algebras. Some of them can be found in Section 2.1, while the others were given in different literature. We will give the definitions and examples of ordinary Rota-Baxter operators as well as weak local Rota-Baxter operators for vertex algebras. Then we will study the basic properties of these concepts. We will show that the level-preserving RBOs on a VOA can only be of a special form, and the ordinary RBO on rank-one Heisenberg VOAs and Virasoro VOAs are all trivial. Finally, we will show that an RBVA V can give rise to a new vertex Leibniz algebra (or vertex algebra without vacuum) structure on V.

6.1.1 Definition of Rota-Baxter operators on vertex algebras

Let $(V, Y, \mathbf{1})$ be a vertex algebra. Recall that it satisfies the following properties:

(1) (weak commutativity) For any $a, b \in V$, there exists some integer $k \in \mathbb{N}$ such that

$$(z_1 - z_2)^k Y(a, z_1)Y(b, z_2) = (z_1 - z_2)^k Y(b, z_2)Y(a, z_1). \tag{6.1.1}$$

(2) (weak associativity) For any $a, b, c \in V$, there exists some integer $k \in \mathbb{N}$ (depending on a and c) such that

$$(z_0 + z_2)^k Y(Y(a, z_0)b, z_2)c = (z_0 + z_2)^k Y(a, z_0 + z_2)Y(b, z_2)c. \tag{6.1.2}$$

Moreover, if $Y : V \to (\mathrm{End}V)[[z, z^{-1}]]$ is a linear map that satisfies the truncation property, then the Jacobi identity of Y in the definition of vertex algebra is equivalent to the weak commutativity together with the weak associativity.

Define a translation operator $D : V \to V$ by letting $Da := a_{-2}\mathbf{1}$, for all $a \in V$. Then $(V, Y, D, \mathbf{1})$ satisfies the D-derivative property:

$$Y(Da, z) = \frac{d}{dz}Y(a, z) \qquad (D - \text{derivative property}), \tag{6.1.3}$$

$$[D, Y(a,z)] = \frac{d}{dz}Y(a,z) \qquad (D - \text{bracket derivative property}), \tag{6.1.4}$$

$$Y(a,z)b = e^{zD}Y(b,-z)a \qquad (\text{skew-symmetry}), \tag{6.1.5}$$

where $a, b \in V$. (6.1.3) and (6.1.4) together are called the ***D*-translation invariance property**.

We need the following weaker notions of vertex algebras for our later discussion. The following notion was introduced in [56]:

Definition 6.1.1. A **vertex Leibniz algebra** (V, Y) is a vector space V equipped with a linear map $Y : V \to \text{End}(V)[[z, z^{-1}]]$, satisfying the truncation property and the Jacobi identity.

In particular, a subspace U of a vertex algebra $(V, Y, \mathbf{1})$ is a vertex Leibniz subalgebra with respect to the restricted vertex operator $Y|_U$ if it satisfies $a_n b \in U$, for all $a, b \in U$, and $n \in \mathbb{Z}$. A related notion is the vertex algebra without vacuum (see [37]):

Definition 6.1.2. A **vertex algebra without vacuum** is a vector space V, equipped with a linear map $Y : V \to (\text{End}V)[[z, z^{-1}]]$ and a linear operator $D : V \to V$ satisfying the truncation property, the Jacobi identity, the D-derivative property (6.1.3), and the skew-symmetry (6.1.5). We denote a vertex algebra without vacuum by (V, Y, D).

The following fact is proved by Huang and Lepowsky in [37], see also [56]:

Proposition 6.1.3. *Let V be a vector space, equipped with a linear map $Y : V \to \text{End}(V)[[z, z^{-1}]]$, satisfying the truncation property. If $D : V \to V$ is another linear map that satisfies the D-bracket derivative property* (6.1.4) *and skew-symmetry* (6.1.5)*, then the weak commutativity* (6.1.1) *of Y follows from the weak associativity* (6.1.2).

Since the vertex operator Y can be viewed as the product on a vertex algebra V, and there are infinitely many binary operations corresponding to $Y(-, z)$, we introduce the notion of Rota-Baxter operators on vertex algebras as follows:

Definition 6.1.4. Let $(V, Y, \mathbf{1})$ be a vertex algebra, $\lambda \in \mathbb{C}$ be a fixed complex number, and $m \in \mathbb{Z}$.

(1) An ***m*-ordinary Rota-Baxter operator (RBO) on V of weight λ** is a linear map $P : V \to V$, satisfying the following condition for all $a, b \in V$:

$$(Pa)_m(Pb) = P(a_m(Pb)) + P((Pa)_m b) + \lambda P(a_m b). \tag{6.1.6}$$

We denote the set of m-ordinary RBOs by $\text{RBO}(V)(m)$.

(2) An **ordinary RBO on** V **of weight** λ is a linear map $P : V \to V$ satisfying (6.1.6) for every $m \in \mathbb{Z}$. In other words, P satisfies the following condition for all $a, b \in V$:

$$Y(Pa, z)Pb = P(Y(Pa, z)b) + P(Y(a, z)Pb) + \lambda P(Y(a, z)b). \tag{6.1.7}$$

We denote the set of ordinary RBOs on V by $\mathrm{RBO}(V) = \bigcap_{m\in\mathbb{Z}} \mathrm{RBO}(V)(m)$.

(3) An (m-)ordinary RBO P on V is called **translation invariant** if $PD = DP$, where D is the translation operator: $Da = a_{-2}\mathbf{1}$.

(4) Let V be a VOA, and let P be an (m-)ordinary RBO. P is called **homogeneous of degree** N if $P(V_n) \subseteq V_{n+N}$ for all $n \in \mathbb{N}$. Degree zero RBOs are called **level preserving**.

A **Rota-Baxter vertex algebra (RBVA)** is a vertex algebra $(V, Y, \mathbf{1})$, equipped with an ordinary RBO $P : V \to V$ of weight λ. We denote such an algebra by $(V, Y, \mathbf{1}, P)$. We can similarly define a Rota-Baxter vertex operator algebra $(V, Y, \mathbf{1}, \omega, P)$.

Remark 6.1.5. Although the condition of an m-ordinary RBO on V is very weak and does not have many connections with the substructures of V, it is closely related to the tensor form of Yang-Baxter equations for VOAs, see [6] for more details. In the rest of the paper, we will be focusing on the properties of ordinary RBOs.

It is clear that for any vertex algebra V, $P = -\lambda \mathrm{Id}_V$ satisfies (6.1.7). Hence any vertex algebra can be viewed as an RBVA trivially in this way.

Let $(V, Y, \mathbf{1}, \omega, P)$ be an RBVOA of weight λ, recall that (cf. [12]) the first level $\mathfrak{g} = V_1$ is a Lie algebra, with the Lie bracket $[a, b] = a_0 b$, for all $a, b \in \mathfrak{g}$. Then it follows (6.1.7) that $(\mathfrak{g}, P|_{\mathfrak{g}})$ is a Rota-Baxter Lie algebra. Conversely, if $p : \mathfrak{g} \to \mathfrak{g}$ is an RBO of the Lie algebra $\mathfrak{g}$, and $\mathfrak{g}$ is the first level V_1 of a VOA V, then p can be easily extended to a 0-ordinary RBO $P : V \to V$ by letting $P|_{V_1} = p$ and $P(V_n) = 0$, for all $n \neq 1$, see Example 6.1.10 in [6].

Our definition of the Rota-Baxter operators for vertex algebra is similar to the R-matrix for VOAs in [69]. An R-matrix for a VOA $(V, Y, \mathbf{1}, \omega)$ is defined to be a linear map $R : V \to V$ such that $[R, L(-1)] = 0$, and $Y_R : V \to \mathrm{End}(V)[[z, z^{-1}]]$ defined by $Y_R(a, z) = Y(Ra, z) + Y(a, z)R$ satisfies the Jacobi identity. The following is proved by Xu in [69]:

Proposition 6.1.6. *If a linear map* $R : V \to V$ *satisfies* $[R, L(-1)] = 0$ *and the relation:*

$$Y(Ra, z)R - RY_R(a, z) = \lambda Y(a, z), \tag{6.1.8}$$

the so called "modified Yang-Baxter equation", where $\lambda = 0$ or -1, then R is an R-matrix for V.

Note that R satisfying (6.1.8) with $\lambda = 0$ is the special case of a Rota-Baxter operator of weight 0 in view of (6.1.7). Hence a translation invariant RBO P of weight 0 on a VOA V is an R-matrix of V in the sense of [69].

Let $(A, d, 1_A)$ be a commutative unital differential algebra, recall that A is a vertex algebra (cf. [12]) with the vertex operator Y given by

$$Y(a,z)b = (e^{zd}a) \cdot b, \tag{6.1.9}$$

for all $a, b \in A$, and $\mathbf{1} = 1_A$. The differential operator d of A is the translation operator D in (6.1.4). In particular, let $V = \mathbb{C}[t]$ be the polynomial algebra with variable t, then V is a vertex algebra with the vertex operator:

$$Y(t^m, z)t^n = (e^{z\frac{d}{dt}}t^m) \cdot t^n = \sum_{j\geq 0} \binom{m}{j} t^{m+n-j} z^j, \tag{6.1.10}$$

for all $m, n \in \mathbb{N}$. Then $(\mathbb{C}[t], Y, 1)$ is an ordinary RBVA.

Proposition 6.1.7. *Let $P : \mathbb{C}[t] \to \mathbb{C}[t]$ be the usual (integration) Rota-Baxter operator on $\mathbb{C}[t]$:*

$$P(t^m) = \int_0^t s^m ds = \frac{t^{m+1}}{m+1},$$

for any $m \in \mathbb{N}$. Then $(\mathbb{C}[t], Y, 1, P)$ is an RBVA of weight 0

Proof. For any $m, n \in \mathbb{N}$, by (6.1.10) we have:

$$Y(Pt^m, z)Pt^n = \frac{1}{m+1}\frac{1}{n+1}Y(t^{m+1}, z)t^{n+1} = \frac{1}{m+1}\frac{1}{n+1}\sum_{j\geq 0}\binom{m+1}{j} t^{m+n+2-j} z^j,$$

$$P(Y(Pt^m, z)t^n) = \frac{1}{m+1}P(Y(t^{m+1}, z)t^n) = \frac{1}{m+1}\sum_{j\geq 0}\binom{m+1}{j}\frac{1}{m+n+2-j} t^{m+n+2-j} z^j$$

$$P(Y(t^m, z)Pt^n) = \frac{1}{n+1}P(Y(t^m, z)t^{n+1}) = \frac{1}{n+1}\sum_{j\geq 0}\binom{m}{j}\frac{1}{m+n+2-j} t^{m+n+2-j} z^j.$$

We need to show:

$$\frac{1}{m+1}\frac{1}{n+1}\binom{m+1}{j} = \frac{1}{m+1}\binom{m+1}{j}\frac{1}{m+n+2-j} + \frac{1}{n+1}\binom{m}{j}\frac{1}{m+n+2-j},$$

or equivalently,

$$\frac{m+1-j}{m+n+2-j}\binom{m+1}{j} = \frac{m+1}{m+n+2-j}\binom{m}{j},$$

for all $j \geq 0$. But this follows directly from the definition of the binomial coefficients. Thus, for any $m, n \in \mathbb{N}$ we have: $Y(Pt^m, z)Pt^n = P(Y(Pt^m, z)t^n) + P(Y(t^m, z)Pt^n)$. Hence $(\mathbb{C}[t], Y, 1, P)$ is an RBVA of weight 0. □

Note that both $(\mathbb{C}[t], \frac{d}{dt}, P, 1_{\mathbb{C}[t]})$ and $(A = \bigoplus_{m=0}^{\infty} \mathbb{C}t_m, d, P, 1_A)$ are special cases of the commutative unital differential Rota-Baxter algebras $(A, d, P, 1_A)$. By definition, (A, P) is an RBO of weight 0, and $d \circ P = \mathrm{Id}_A$, see [32] for more details. We have the following fact in general:

Proposition 6.1.8. *Let $(A, d, P, 1_A)$ be an unital commutative differential RBA, and let $Y(a, z)b = (e^{zd}a) \cdot b$. Then we have:*

$$Y(Pa, z)Pb - P(Y(Pa, z)b) - P(Y(a, z)Pb) \in (\ker d)[[z]],$$

for all $a, b \in V$. In particular, $(A, Y, 1_A, P)$ is an RBVA of weight 0 if $\ker d = 0$.

Proof. First we note that $P(a)_{-1}P(b) = P(P(a)_{-1}b) + P(a_{-1}P(b))$ for all $a, b \in A$, since the product of A is given by $x \cdot y = x_{-1}y$ for all $x, y \in A$.

Now assume $n \geq 1$. By (6.1.4) we have $d(a_{-n}b) = (da)_{-n}b + a_{-n}db$ and $(da)_{-n} = nda_{-n-1}$. Moreover, $a - Pd(a) \in \ker d$ for all $a, b \in A$ as $d \circ P = \mathrm{Id}_A$, hence we have:

$$\begin{aligned}
&nP(a)_{-n-1}P(b) - nP(P(a)_{-n-1}b) - nP(a_{-n-1}P(b)) \\
&= (dP(a))_{-n}P(b) - P((dP(a))_{-n}b) - P((da)_{-n}P(b)) \\
&= a_{-n}P(b) - P(a_{-n}b) - P(d(a_{-n}P(b)) - a_{-n}dP(b)) \\
&= a_{-n}P(b) - Pd(a_{-n}P(b)) \equiv 0 \pmod{\ker d}.
\end{aligned}$$

This finishes the proof because $Y(a, z)b = \sum_{n \geq 0}(a_{-n-1}b)z^n$. □

We will give another sufficient condition under which $(A, Y, P, 1_A)$ becomes an RBVA of weight 0 in the next subsection.

6.1.2 The λ-differentials and weak local Rota-Baxter operators

Proposition 6.1.7 indicates that the "right inverse" P of the translation operator D on certain commutative vertex algebras can give rise to ordinary RBOs of weight 0.

However, in the case of non-commutative VOAs, the translation operator $D = L(-1)$ and most of the derivations are *not* invertible globally. They only admit local inverse. On the other hand, by the definition formula (6.1.7), if $P : V \to V$ is an ordinary RBO, then we must have $P(a)_m P(b) \in P(V)$, for all $a, b \in V$ and $m \in \mathbb{Z}$. i.e., $P(V) \subseteq V$ is a vertex Leibniz subalgebra (see Definition 6.1.1). This is also a strong condition imposed on P. If we weaken these conditions, we can construct examples of the Rota-Baxter type operators from the "right inverse" of the λ-differentials on vertex algebra V (see Section 2.3) on a suitable domain.

Definition 6.1.9. Let $(V, Y, \mathbf{1})$ be a vertex algebra, $\lambda \in \mathbb{C}$ be a fixed complex number, and $U \subset V$ be a linear subspace.

(1) A **weak local Rota-Baxter operator (RBO) on** U **of weight** λ is a linear map $P : U \to V$, satisfying the following condition: Whenever $a, b \in U$ and $m \in \mathbb{Z}$ such that $P(a)_m P(b) \in P(U)$, one has $a_m(Pb) + (Pa)_m b + \lambda a_m b \in U$, and

$$(Pa)_m(Pb) = P(a_m(Pb) + (Pa)_m b + \lambda a_m b). \tag{6.1.11}$$

If, furthermore, $U = V$, then P is called a **weak global RBO of weight** λ.

(2) An **ordinary local RBO on** U **of weight** λ is a weak local RBO $P : U \to V$ of weight λ such that $P(U)$ is a vertex Leibniz subalgebra of V. In other words, $P : U \to V$ is a linear map satisfying:

$$Y(Pa, z)Pb = P(Y(Pa, z)b + Y(a, z)Pb + \lambda Y(a, z)b), \tag{6.1.12}$$

for all $a, b \in U$. In particular, if $U = V$, then an ordinary local RBO $P : U = V \to V$ is the same as the ordinary RBO in Definition 6.1.4.

A local RBO (weak or ordinary) $P : U \to V$ is called **translation invariant**, if $DU \subseteq U$ and $PD = DP$ on U. Let V be a VOA, and let $P : U \to V$ be a local RBO. Then P is called **homogeneous of degree** N, if $U \subset V$ is a homogeneous subspace: $U = \bigoplus_{n=0}^{\infty} U_n$, and $P(U_n) \subseteq V_{n+N}$ for all $n \in \mathbb{N}$.

Remark 6.1.10. In equations (6.1.11) and (6.1.12), we do not require $P(a)_m b$ and $a_m P(b)$ to be contained in the domain U of P, and so their right-hand sides cannot separate into the forms of (6.1.6) and (6.1.7).

The following properties of weak and ordinary local RBOs are straightforward:

Proposition 6.1.11. *Let $(V, Y, \mathbf{1})$ be a vertex algebra, and $U \subset V$ be a linear subspace.*

(1) If $P : U \to V$ is a weak (resp. ordinary) local RBO on U of weight $\lambda \neq 0$, then $-P/\lambda$ is a weak (resp. ordinary) local RBO on U of weight -1. If $P : U \to V$ is a weak (resp. ordinary) local RBO of weight 1, then λP is a weak (resp. ordinary) local RBO of weight λ.

(2) Let P be an ordinary local RBO on U of weight λ, then $\tilde{P} = -\lambda \mathrm{Id}_V - P$ is an ordinary local RBO on U of weight λ.

Proof. (1) Let $P : U \to V$ be a weak local RBO of weight $\lambda \neq 0$. Let $a, b \in U$ and $n \in \mathbb{Z}$ satisfy $(-P/\lambda)(a)_n(-P/\lambda)(b) \in (-P/\lambda)(U) = P(U)$, then $P(a)_n P(b) \in P(U)$, and by Definition 6.1.4, we have $a_n P(b) + (Pa)_n b + \lambda a_n b \in U$, and $(Pa)_n(Pb) = P(a_n(Pb) + (Pa)_n b + \lambda a_n b)$. It follows that $a_n(-P/\lambda)(b) + ((-P/\lambda)(a))_n b - a_n b \in U$ and

$$((-P/\lambda)(a))_n((-P/\lambda)(b)) = (-P/\lambda)(a_n(-P/\lambda)(b) + ((-P/\lambda)(a))_n b - a_n b).$$

Thus, $-P/\lambda : U \to V$ is a weak local RBO of weight -1. The proof of the rest is similar, and we omit the details.

(2) Since $P : U \to V$ is an ordinary RBO, by Definition 6.1.4, we have $a_n(Pb) + (Pa)_n b + \lambda a_n b \in U$ for all $a, b \in U$ and $n \in \mathbb{Z}$. It follows that

$$a_n(-\lambda - P)(b) + (-\lambda - P)(a)_n b + \lambda a_n b = -a_n Pb - (Pa)_n b - \lambda a_n b \in U,$$

and it is easy to check that

$$(-\lambda - P)(a)_n(-\lambda - P)(b) = (-\lambda - P)(a_n(-\lambda - P)(b) + (-\lambda - P)(a)_n b + \lambda a_n b),$$

for all $a, b \in U$ and $n \in \mathbb{Z}$. Thus, $(-\lambda - P)(a)_n(-\lambda - P)(b) \in (-\lambda - P)(U)$ for all $a, b \in U$ and $n \in \mathbb{Z}$, and (6.1.7) is satisfied for $\tilde{P} = -\lambda - P$. This shows $\tilde{P} = -\lambda - P$ is an ordinary local RBO on U of weight λ. □

The local inverse of a weak λ-differential (see Definition 2.3.21) of a vertex algebra gives rise to weak local Rota-Baxter operators of weight λ:

Proposition 6.1.12. *Let $(V, Y, \mathbf{1})$ be a vertex algebra, and let $d : V \to V$ be a weak λ-differential. Suppose there exists a linear map $P : U(:= dV) \to V$ such that $d \circ P = \mathrm{Id}_U$. Then $P : U \to V$ is a weak local RBO on U of weight λ.*

Proof. Assume $a, b \in U$ and $n \in \mathbb{Z}$ such that $(Pa)_n(Pb) = P(c) \in P(U)$. Then we have:

$$\begin{aligned} dP(c) &= d((Pa)_n(Pb)) = (dP)(a)_n(Pb) + (Pa)_n(dPb) + \lambda(dPa)_n(dPb) \\ &= a_n(Pb) + (Pa)_n b + \lambda a_n b, \end{aligned}$$

and $dP(c) = c$ since $dP = \mathrm{Id}_U$. Thus, $a_n(Pb) + (Pa)_n b + \lambda a_n b = c \in U$, and

$$(Pa)_n(Pb) = PdP(c) = P(a_n(Pb) + (Pa)_n b + \lambda a_n b).$$

Hence $P : U \to V$ satisfies (6.1.6), and so P is a weak local RBO on U of weight λ. □

Corollary 6.1.13. *Let $(A, d, P, 1_A)$ be an unital commutative differential RBA. Then $(A, Y, 1_A, P)$ with Y given by (6.1.9) is an RBVA of weight 0, if P satisfies $P(a) \cdot P(b) \in P(A)$ and $(d^n a) \cdot P(b) \in P(A)$, for all $a, b \in A$ and $n \in \mathbb{N}$.*

Proof. By (6.1.4) and Definition 2.3.21, $d = D : A \to A$ is an 0-differential of the vertex algebra $(A, Y, 1_A)$. Since $d \circ P = \mathrm{Id}_A$ by the definition of differential RBA, $P : A \to A$ is a weak global RBO of weight 0 on the vertex algebra A by Proposition 6.1.12. If P satisfies the last condition, then $Y(P(a), z)P(b) = P(a) \cdot P(b) + \sum_{j \geq 1} \frac{1}{j!}(d^{j-1}a) \cdot P(b) \in P(A)((z))$, and so $P : A \to A$ is an ordinary RBO of weight 0. □

By (6.1.10), it is easy to check that the conditions in Corollary 6.1.13 are satisfied by $(\mathbb{C}[t], \frac{d}{dt}, P, 1_{\mathbb{C}[t]})$ and $(A = \bigoplus_{m=0}^{\infty} \mathbb{C}t_m, d, P, 1_A)$. This provides us with another proof of Proposition 6.1.7.

There are many examples of weak 0-differentials on vertex operator algebras. We can use them and construct examples of weak local Rota-Baxter operators on general VOAs by Proposition 6.1.12:

Example 6.1.14. Let $(V, Y, \mathbf{1}, \omega)$ be a CFT-type vertex operator algebra. By the main Theorem in [23], the operator $L(-1) : V \to V$ is injective on V_+. Moreover, we have $L(-1)\mathbf{1} = \mathbf{1}_{-2}\mathbf{1} = 0$, and $L(-1)$ is a weak 0-differential by (6.1.3) and (6.1.4).

Let $U = L(-1)V = L(-1)V_+$. Define $P : U \to V$ by letting:

$$P(u) := L(-1)^{-1}u, \tag{6.1.13}$$

for all $u \in L(-1)V_+$. Clearly, P is well-defined and $L(-1)P = \mathrm{Id}_U$. Then by Proposition 6.1.12, $P : U \to V$ given by (6.1.18) is a weak local RBO on $U = L(-1)V_+$ of weight 0, and it is homogeneous of degree -1 and translation invariant.

P is not ordinary since $P(U) = V_+$ is not a vertex Leibniz subalgebra of V.

Example 6.1.15. Let $V = M_{\widehat{\mathfrak{h}}}(k, 0)$ be the level $k \neq 0$ Heisenberg vertex operator algebra of rank r (cf. [29], see also [30]). Recall that $\mathfrak{h}$ is an r-dimensional vertex space, equipped with a nondegenerate symmetric bilinear form $(\cdot|\cdot)$, and $M_{\widehat{\mathfrak{h}}}(k, 0)$ is the Verma module over the infinite-dimensional Heisenberg Lie algebra: $\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$, with

$$[\alpha(m), \beta(n)] = m(\alpha|\beta)\delta_{m+n,0}K, \tag{6.1.14}$$

for all $m, n \in \mathbb{Z}$, where $\alpha(m) = \alpha \otimes t^m$. We have $\widehat{\mathfrak{h}} = \widehat{\mathfrak{h}}_{\geq 0} \oplus \widehat{h}_{<0}$, where $\widehat{h}_{\geq 0} = \mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}K$ and $\widehat{\mathfrak{h}_{<0}} = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]$, and $M_{\widehat{\mathfrak{h}}}(k, 0) = U(\widehat{\mathfrak{h}}) \otimes_{U(\widehat{\mathfrak{h}}_{\geq 0})} \mathbb{C}\mathbf{1}$, with $\alpha(n).\mathbf{1} = 0$ for all $n \geq 0$ and $\alpha \in \mathfrak{h}$, and $K\mathbf{1} = k\mathbf{1}$.

In particular, $\alpha(0) \in \widehat{\mathfrak{h}}$ is a central element by (6.1.14), and $\alpha(0).u = 0$ for all $u \in M_{\widehat{\mathfrak{h}}}(k, 0)$. Fix a nonzero element $\alpha \in \mathfrak{h}$, consider the operator $d = \alpha(1) : V \to V$. For any $u, v \in V = M_{\widehat{\mathfrak{h}}}(k, 0)$ and $n \in \mathbb{Z}$, we have:

$$\begin{aligned}\alpha(1)(u_n v) &= u_n(\alpha(1)v) + [\alpha(1), u_n]v = u_n(\alpha(1)v) + \sum_{j\geq 0}\binom{1}{j}(\alpha(j)u)_{1+n-j}v \\ &= u_n(\alpha(1)v) + (\alpha(1)u)_n v,\end{aligned}$$

since $\alpha(0)u = 0$. Thus, $d = \alpha(1)$ is a weak 0-differential on $M_{\widehat{\mathfrak{h}}}(k, 0)$. By (6.1.14), it is also easy to see that $d = \alpha(1)$ acts as $= k\frac{\partial}{\partial\alpha(-1)}$ on $M_{\widehat{\mathfrak{h}}}(k, 0)$. Hence $\alpha(1)M_{\widehat{\mathfrak{h}}}(k, 0) = M_{\widehat{\mathfrak{h}}}(k, 0)$. Define a linear map $P : \alpha(1)V = V \to V$ as follows:

$$\begin{aligned}&P := \frac{1}{k}\int(\cdot)d\alpha(-1)\mathbf{1} : M_{\widehat{\mathfrak{h}}}(k, 0) \to M_{\widehat{\mathfrak{h}}}(k, 0), \\ &h^1(-n_1)\dots h^k(-n_k)\alpha(-1)^m\mathbf{1} \mapsto \frac{1}{k(m+1)}h^1(-n_1)\dots h^k(-n_k)\alpha(-1)^{m+1}\mathbf{1},\end{aligned} \tag{6.1.15}$$

where $S = \{\alpha = \alpha_1, \alpha_2, \dots \alpha_r\}$ is a basis of $\mathfrak{h}$, and $h^1, \dots, h^k \in S$ are not equal to α. Clearly, we have $dP = \mathrm{Id}_V$, and so $P : V \to V$ is a weak global RBO on $M_{\widehat{\mathfrak{h}}}(k, 0)$ of weight 0 by Proposition

6.1.12. P in (6.1.15) is also homogeneous of degree 1, however, it is not an ordinary RBO since $P(V) = \alpha(-1)M_{\widehat{\mathfrak{h}}}(k,0)$ is not a vertex Leibniz subalgebra.

Example 6.1.16. Let $V = V_{\widehat{\mathfrak{g}}}(k,0)$ be the level k vacuum module vertex operator algebra associated with $\mathfrak{g} = sl(2,\mathbb{C}) = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$ (cf. [30]). $V_{\widehat{\mathfrak{g}}}(k,0) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{g}}_{\geq 0})} \mathbb{C}\mathbf{1}$ is the Weyl vacuum module over the affine Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K$

Since $h = h(-1)\mathbf{1} \in V_1$, $d = o(h) = h(0) : V \to V$ is a 0-differential of V (cf. [14]). Moreover, $V_{\widehat{\mathfrak{g}}}(k,0)$ is a sum of $h(0)$-eigenspaces ([25]):

$$V_{\widehat{\mathfrak{g}}}(k,0) = \bigoplus_{\lambda \in 2\mathbb{Z}} V_{\widehat{\mathfrak{g}}}(k,0)(\lambda),$$

where $V_{\widehat{\mathfrak{g}}}(k,0)(\lambda) = \{v \in V_{\widehat{\mathfrak{g}}}(k,0) : h(0)v = \lambda v\}$ for all $\lambda \in 2\mathbb{Z}$.

Let U be the sum of nonzero eigenspaces of $h(0)$: $U = \bigoplus_{\lambda \in 2\mathbb{Z}\backslash\{0\}} V_{\widehat{\mathfrak{g}}}(k,0)(\lambda)$, and let $P : U \to V$ be given by: $P(u) = \frac{1}{\lambda}u$, for all $u \in V_{\widehat{\mathfrak{g}}}(k,0)(\lambda)$, with $\lambda \neq 0$. Then $dP = \mathrm{Id}_U$, and so $P : U \to V$ is a weak local RBO on U of weight 0. Moreover, P is homogeneous of weight 0, and it is not ordinary since $P(U) = U$ is not a subalgebra.

Let $d_1 := e^{h(0)} - 1 : V \to V$, then d_1 is a 1-differential by Proposition 2.3.23. Let $P_1 : U \to V$ be given by $P_1(u) = \frac{1}{e^{\lambda}-1}u$, for all $u \in V_{\widehat{\mathfrak{g}}}(k,0)(\lambda)$, with $\lambda \neq 0$. Then $d_1 P_1 = \mathrm{Id}_U$, and by Proposition 6.1.12, $P_1 : U \to V$ is a weak local RBO of weight 1.

6.1.3 Properties and further examples of Rota-Baxter vertex algebras

The next theorem generalizes Theorem 1.1.13 in [31] and gives us a systematic way to build examples of RBVA.

Theorem 6.1.17. *Let $(V, Y, \mathbf{1})$ be a vertex algebra, and $P : V \to V$ be a linear map. Then P is an idempotent RBO of weight -1, if and only if V admits a decomposition: $V = V^1 \oplus V^2$ into a direct sum of vertex Leibniz subalgebras V^1 and V^2, and $P : V \to V^1$ is the projection map onto V^1:*

$$P(a^1 + a^2) = a^1,$$

for all $a^1 \in V^1$ and $a^2 \in V^2$, in particular, $V^1 = P(V)$ and $V^2 = \ker P$.

Proof. Let $P : V \to V$ be an idempotent RBO of weight -1. Then $V^1 = P(V) \subseteq V$ is closed under the vertex operator Y, since we have:

$$(Pa)_n(Pb) = P(a_n P(b) + P(a)_n b - a_n b) \in P(V),$$

for all $a, b \in V$ by (6.1.7). Hence $(V^1, Y|_{V^1})$ is a vertex Leibniz subalgebra of V. By Proposition 6.1.11, $V^2 = (1 - P)(V)$ is also a vertex Leibniz subalgebra. Since $P^2 = P$ by assumption, we have $V = V^1 \oplus V^2$. Moreover, for any $a \in V$, we have

$$a = P(a) + (1 - P)(a) = a^1 + a^2,$$

where $a^1 = P(a)$ and $a^2 = (1 - P)(a)$. Note that a^1 and a^2 are unique as the sum is direct. Then $P(a^1 + a^2) = P(a) = a^1$ is the projection onto V^1.

Conversely, suppose V has a decomposition $V = V^1 \oplus V^2$ into vertex Leibniz subalgebras, and $P : V \to V^1$ is the projection. Then for any $a = a^1 + a^2$ and $b = b^1 + b^2$ in V, with $a^i, b^i \in V^i$ for $i = 1, 2$, we have:

$$\begin{aligned}
P(a)_n P(b) &= a^1_n b^1, \\
P((Pa)_n b) &= P(a^1_n b^1 + a^1_n b^2) = a^1_n b^1 + P(a^1_n b^2), \\
P(a_n P(b)) &= P(a^1_n b^1 + a^2_n b^1) = a^1_n b^1 + P(a^2_n b^1), \\
P(a_n b) &= P(a^1_n b^1 + a^1_n b^2 + a^2_n b^1 + a^2_n b^2) = a^1_n b^1 + P(a^1_n b^2) + P(a^2_n b^1).
\end{aligned}$$

It follows that $P(a)_n P(b) = P((Pa)_n b) + P(a_n P(b)) - P(a_n b)$, for all $a, b \in V$. Moreover, clearly we have $P(Pa) = P(a^1) = a^1 = P(a)$. i.e., $P : V \to V$ is an idempotent RBO of weight -1. □

Example 6.1.18. Let $V = V_L$ be the lattice vertex operator algebra associated with the rank one positive definite even lattice $L = \mathbb{Z}\alpha$, with $(\alpha|\alpha) = 2N$ for some $N \in \mathbb{Z}_{>0}$. By Examples 5.1.4 (1), $V_{\mathbb{Z}\alpha} = V_{\mathbb{Z}_{\geq 0}\alpha} \oplus V_{\mathbb{Z}_{<0}\alpha}$ is a decomposition of $V_{\mathbb{Z}\alpha}$ into vertex Leibniz sub-algebras. Then by Theorem 6.1.17, the projection $P : V_{\mathbb{Z}\alpha} \to V_{\mathbb{Z}\alpha_{\geq 0}}$ along $V_{\mathbb{Z}\alpha_{<0}}$ is an ordinary RBO of weight -1. Moreover, P is obviously level-preserving and translation invariant ($PL(-1) = L(-1)P$) since $L(0)$ and $L(-1)$ preserve each $M_{\hat{\mathfrak{h}}}(1, m\alpha)$.

For the higher rank case, let $L = \mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_r$, then by Example 5.1.4 (2), we can choose a parabolic-type sub-monoid $P := \mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_{r-1} \oplus \mathbb{Z}_{\geq 0}\alpha_r$. Then $L = P \sqcup P^1$, where $P^- = \mathbb{Z}\alpha_1 \oplus \ldots \oplus \mathbb{Z}\alpha_{r-1} \oplus \mathbb{Z}_{<0}\alpha_r$, and V_L has a decomposition into vertex Leibniz sub-algebras: $V_L = V_P \oplus V_{P^-}$. By Theorem 6.1.17 again, the projection $P : V_L \to V_P$ along V_{P^-} is an ordinary RBO of weight -1, and P is level-preserving and translation invariant.

Example 6.1.19. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra, and let (W, Y_W) be a weak V-module. It is observed in [50] (see also [27]) that $V \oplus W$ carries a structure of vertex algebra, with vertex operator given by

$$Y_{V\oplus W}(a+v,z)(b+w) = (Y(a,z)b) + (Y_W(a,z)w + Y_{WV}^W(v,z)b), \tag{6.1.16}$$

for all $a, b \in V$ and $v, w \in W$, where Y_{WV}^W is defined by the skew-symmetry formula:

$$Y_{WV}^W(v,z)b = e^{zL(-1)}Y_W(b,-z)v. \tag{6.1.17}$$

We can think of $V \oplus W$ as the semi-direct product $V \rtimes W$ of the vertex operator algebra V with the weak-module W. Since $Y_{V\oplus W}(v,z)w = 0$ for $v, w \in W$, it follows that (V, Y) and $(W, Y_{V\oplus W}|_W)$ are vertex Leibniz subalgebras of $V \rtimes W$. Then by Theorem 6.1.17,

$$P : V \rtimes W \to V, a + v \mapsto a, \tag{6.1.18}$$

is an RBO of weight -1 on the vertex algebra $V \rtimes W$.

If W only has integral weights, then $(V \rtimes W, Y_{V\oplus W}, \mathbf{1}, \omega)$ is a vertex operator algebra (cf. [50]). Then P in (6.1.18) is a level-preserving RBO of weight -1. This example will be used in the discussion of the next section.

Although it is not easy to classify all the Rota-Baxter operators on an arbitrary VOA, we will show that the homogeneous Rota-Baxter operators of non-positive degree on certain CFT-type vertex operator algebras only have very limited choices.

Lemma 6.1.20. *Let $(V, Y, \mathbf{1}, \omega)$ be a VOA of CFT-type, and $P : V \to V$ be a homogeneous RBO of degree $N \leq 0$. Then we have $P(\mathbf{1}) = 0$ or $P(\mathbf{1}) = -\lambda\mathbf{1}$, and $P^2 + \lambda P = 0$.*

Proof. Since $V_n = 0$ for $n < 0$, $V_0 = \mathbb{C}\mathbf{1}$, and $PV_0 \subseteq V_N$ for some $N \leq 0$, we have $P(\mathbf{1}) = \mu\mathbf{1}$ for some $\mu \in \mathbb{C}$. Recall that $\mathbf{1}_{-1}\mathbf{1} = \mathbf{1}$, then by (6.1.7) we have:

$$\begin{aligned} &P(\mathbf{1})_{-1}P(\mathbf{1}) = P(P(\mathbf{1})_{-1}\mathbf{1}) + P(\mathbf{1}_{-1}P(\mathbf{1})) + \lambda P(\mathbf{1}_{-1}\mathbf{1}) \\ &\implies \mu^2\mathbf{1} = \mu^2\mathbf{1} + \mu^2\mathbf{1} + \lambda\mu\mathbf{1}. \end{aligned}$$

Hence μ is either 0 or $-\lambda$. i.e., $P(\mathbf{1}) = 0$ or $-\lambda\mathbf{1}$. Furthermore, again by (6.1.7) we have:

$$P(a)_{-1}P(\mathbf{1}) = P(P(a)_{-1}\mathbf{1}) + P(a_{-1}P(\mathbf{1})) + \lambda P(a_{-1}\mathbf{1}) \tag{6.1.19}$$

for all $a \in V$. If $P(\mathbf{1}) = 0$ then (6.1.19) becomes:

$$0 = P(P(a)) + P(a_{-1}0) + \lambda P(a),$$

and so $P^2(a) + \lambda P(a) = 0$ for all $a \in V$. On the other hand, if $P(\mathbf{1}) = -\lambda \mathbf{1}$, then

$$-\lambda P(a) = P(P(a)) - \lambda P(a) + \lambda P(a),$$

which also implies $P^2(a) + \lambda P(a) = 0$, for all $a \in V$. Therefore, $P^2 + \lambda P = 0$. □

Proposition 6.1.21. *Let $(V, Y, \mathbf{1}, \omega)$ be a VOA of CFT-type, and $P : V \to V$ be a homogeneous RBO of degree $N \leq 0$ and weight $\lambda \neq 0$. Then $V = V^1 \oplus V^2$, where V^1 and V^2 are graded vertex Leibniz subalgebras of V, with $V_n = V_n^1 \oplus V_n^2$ for each $n \in \mathbb{N}$, and*

$$P : V \to V^1, a^1 + a^2 \mapsto -\lambda a^1,$$

for all $a^i \in V^i$, $i = 1, 6.1$. Moreover, we have $P(\mathbf{1}) = 0$ if and only if $V_0^1 = 0$, and $P(\mathbf{1}) = -\lambda \mathbf{1}$ if and only if $V_0^2 = 0$.

Proof. Since $P^2 + \lambda P = 0$ by Lemma 6.1.20, and $\lambda \neq 0$ by assumption, the linear map $-P/\lambda$ is an idempotent. Then by Proposition 6.1.11, $-P/\lambda$ is an RBO on V of weight -1. By Theorem 6.1.17, we have $V = V^1 \oplus V^2$, where $V^1 = (-P/\lambda)(V) = P(V)$ and $V^2 = \ker(-P/\lambda) = \ker P$ are vertex Leibniz subalgebras, and $-P/\lambda$ is the projection:

$$-\frac{P}{\lambda} : V \to V^1, a^1 + a^2 \mapsto a^1.$$

Hence $P(a^1 + a^2) = -\lambda a^1$. Moreover, since $P(V_n) \subset V_{n+N}$ for all $n \in \mathbb{N}$, we have $V^1 = PV = \bigoplus_{m=-N}^{\infty} P(V_m) = \bigoplus_{n=0}^{\infty} V_n^1$, and $V^2 = \bigoplus_{m=-N}^{\infty} \ker(P|_{V_m}) = \bigoplus_{n=0}^{\infty} V_n^2$, where V_n^i is an eigenspace of $L(0)$ of eigenvalue n, and $V_n = V_n^1 \oplus V_n^2$ for each $n \in \mathbb{N}$. Now the last statement is also clear as $V_0^1 \oplus V_0^2 = V_0 = \mathbb{C}\mathbf{1}$. □

Corollary 6.1.22. *Let V be the level one Heisenberg VOA $M_{\widehat{\mathfrak{h}}}(1, 0)$ associated with $\mathfrak{h} = \mathbb{C}\alpha$ or the Virasoro VOA $L(c, 0)$(see [30]), and let $P : V \to V$ be a homogeneous RBO of degree $N \leq 0$ and weight $\lambda \neq 0$. Then P is either 0 or $-\lambda \mathrm{Id}_V$.*

Proof. Let $V = M_{\widehat{\mathfrak{h}}}(k, 0)$ or $L(c, 0)$. By Proposition 6.1.21, $V = V^1 \oplus V^2$ for some graded vertex Leibniz subalgebras $V^1, V^2 \subset V$. But V is generated by a single homogeneous element: $V = M_{\widehat{\mathfrak{h}}}(k, 0)$ is generated as a vertex algebra by $\alpha(-1)\mathbf{1} \in V_1$ and $V_1 = \mathbb{C}\alpha(-1)\mathbf{1} = V_1^1 \oplus V_1^2$, and $V = L(c, 0)$ is generated by $\omega \in V_2 = \mathbb{C}\omega = V_2^1 \oplus V_2^2$. Then the single generator u of V is contained in either V^1 or V^2 for both cases. If $u \in V^1$ then $V = V^1$ and $P = -\lambda \mathrm{Id}_V$; if $u \in V^2$ then $V = V^2$ and $P = 0$. □

Definition 6.1.23. Let $(V, Y, \mathbf{1}, P)$ be an RBVA of weight λ. Define a new linear operator $Y^{\star_P}$: $V \to (\mathrm{End}V)[[z, z^{-1}]]$ as follows:

$$Y^{\star_P}(a,z)b = Y(a,z)Pb + Y(Pa,z)b + \lambda Y(a,z)b. \tag{6.1.20}$$

Note that $Y^{\star_P}$ is a generalization of Y_R in (6.1.8).

Lemma 6.1.24. *$Y^{\star_P}$ satisfies the truncation property and the skew-symmetry* (6.1.5). *If, furthermore, P is translation invariant ($DP = PD$), then $Y^{\star_P}$ also satisfies D-derivative property* (6.1.3) *and the D-bracket derivative property* (6.1.4).

Proof. Given $a, b \in V$, since $Y(a,z)Pb$, $Y(Pa,z)b$, and $\lambda Y(a,z)b$ are all truncated from below, we have $Y^{\star_P}(a,z)b \in V((z))$. Moreover, by (6.1.20) and the skew-symmetry of Y,

$$Y^{\star_P}(a,z)b = e^{zD}Y(Pb,-z)a + e^{zD}Y(b,-z)Pa + \lambda e^{zD}Y(b,-z)a = e^{zD}Y^{\star_P}(b,-z)a.$$

Hence $Y^{\star_P}$ also satisfies the skew-symmetry. Now assume that $DP = PD$, by (6.1.20) and (6.1.3) and (6.1.4) of Y, we have:

$$\begin{aligned} Y^{\star_P}(Da,z)b &= Y(Da,z)Pb + Y(PDa,z)b + \lambda Y(Da,z)b \\ &= \frac{d}{dz}Y(a,z)Pb + Y(DPa,z)b + \lambda\frac{d}{dz}Y(a,z)b = \frac{d}{dz}Y^{\star_P}(a,z)b, \\ [D, Y^{\star_P}(a,z)]b &= DY(a,z)Pb - Y(a,z)PDb + [D, Y(Pa,z)]b + \lambda[D, Y(a,z)]b \\ &= [D, Y(a,z)]Pb + [D, Y(Pa,z)]b + \lambda[D, Y(a,z)]b = \frac{d}{dz}Y^{\star_P}(a,z)b, \end{aligned}$$

then $Y^{\star_P}$ satisfies the D-derivative and D-bracket derivative properties. ◻

The next theorem is the vertex algebra version of Theorem 1.1.17 in [31]. It shows that $Y^{\star_P}$ gives a new structure of a vertex Leibniz algebra (see Definition 6.1.1) or a vertex algebra without vacuum (see Definition 6.1.2) on an RBVA $(V, Y, \mathbf{1}, P)$.

Theorem 6.1.25. *Let $(V, Y, \mathbf{1}, P)$ be an RBVA of weight λ, and $Y^{\star_P}$ be given by* (6.1.20). *Then we have:*

(1) $P(Y^{\star_P}(a,z)b) = Y(Pa,z)Pb$, for all $a, b \in V$.

(2) $(V, Y^{\star_P})$ is a vertex Leibniz algebra. If, furthermore, P is translation invariant, then $(V, Y^{\star_P}, D)$ is a vertex algebra without vacuum.

(3) P is an RBO of weight λ on the vertex Leibniz algebra $(V, Y^{\star_P})$.

Proof. (1) By (6.1.6) and (6.1.20), we have:

$$\begin{aligned} Y(Pa,z)Pb &= P(Y(a,z)Pb) + P(Y(Pa,z)b) + \lambda P(Y(a,z)b) \\ &= P(Y(a,z)Pb + Y(Pa,z)b + \lambda Y(a,z)b) = P(Y^{\star_P}(a,z)b). \end{aligned}$$

(2) By Lemma 6.1.24, to show $(V, Y^{\star_P})$ is a vertex Leibniz algebra, we only need to show that $Y^{\star_P}$ satisfies the Jacobi identity, or equivalently, the weak commutativity and weak associativity, in view of Theorem 2.1.2. For the weak associativity, given $a, b, c \in V$, we need to find an integer $N \in \mathbb{N}$ such that

$$(z_0 + z_2)^N Y^{\star_P}(a, z_0 + z_2) Y^{\star_P}(b, z_2)c = (z_0 + z_2)^N Y^{\star_P}(Y^{\star_P}(a, z_0)b, z_2)c. \tag{6.1.21}$$

Indeed, by (6.1.20), we have the following expansions:

$$\begin{aligned} & Y^{\star_P}(a, z_0 + z_2) Y^{\star_P}(b, z_2)c \\ &= Y^{\star_P}(a, z_0 + z_2)(Y(b, z_2)Pc + Y(Pb, z_2)c + \lambda Y(b, z_2)c) \\ &= Y(a, z_0 + z_2)P(Y(b, z_2)Pc) + Y(Pa, z_0 + z_2)Y(b, z_2)Pc + \lambda Y(a, z_0 + z_2)Y(b, z_2)Pc \\ &\quad + Y(a, z_0 + z_2)P(Y(Pb, z_2)c) + Y(Pa, z_0 + z_2)Y(Pb, z_2)c + \lambda Y(a, z_0 + z_2)Y(Pb, z_2)c \\ &\quad + \lambda Y(a, z_0 + z_2)P(Y(b, z_2)c) + \lambda Y(Pa, z_0 + z_2)Y(b, z_2)c + \lambda^2 Y(a, z_0 + z_2)Y(b, z_2)c, \\ & Y^{\star_P}(Y^{\star_P}(a, z_0)b, z_2)c \\ &= Y^{\star_P}(Y(a, z_0)Pb, z_2)c + Y^{\star_P}(Y(Pa, z_0)b, z_2)c + \lambda Y^{\star_P}(Y(a, z_0)b, z_2)c \\ &= Y(Y(a, z_0)Pb, z_2)Pc + Y(P(Y(a, z_0)Pb), z_2)c + \lambda Y(Y(a, z_0)Pb, z_2)c \\ &\quad + Y(Y(Pa, z_0)b, z_2)Pc + Y(P(Y(Pa, z_0)b), z_2)c + \lambda Y(Y(Pa, z_0)b, z_2)c \\ &\quad + \lambda Y(Y(a, z_0)b, z_2)Pc + \lambda Y(P(Y(a, z_0)b), z_2)c + \lambda^2 Y(Y(a, z_0)b, z_2)c. \end{aligned}$$

From equation (6.1.6), we have

$$\begin{aligned} & Y(a, z_0 + z_2)P(Y(b, z_2)Pc + Y(a, z_0 + z_2)P(Y(Pb, z_2)c) + \lambda Y(a, z_0 + z_2)P(Y(b, z_2)c) \\ &= Y(a, z_0 + z_2)Y(Pb, z_2)Pc, \\ & Y(P(Y(a, z_0)Pb), z_2)c + Y(P(Y(Pa, z_0)b), z_2)c + \lambda Y(P(Y(a, z_0)b), z_2) \\ &= Y(Y(Pa, z_0)Pb, z_2)c. \end{aligned}$$

Now, using the original weak associativity with respect to Y, we can find a common number $N \in \mathbb{N}$, such that the following weak associativities hold at the same time:

$$(z_0+z_2)^N Y(a, z_0+z_2)Y(Pb, z_2)Pc = (z_0+z_2)^N Y(Y(a, z_0)Pb, z_2)Pc,$$

$$(z_0+z_2)^N Y(Pa, z_0+z_2)Y(Pb, z_2)c = (z_0+z_2)^N Y(Y(Pa, z_0)Pb, z_2)c,$$

$$(z_0+z_2)^N Y(Pa, z_0+z_2)Y(b, z_2)Pc = (z_0+z_2)^N Y(Y(Pa, z_0)b, z_2)Pc,$$

$$(z_0+z_2)^N Y(a, z_0+z_2)Y(b, z_2)Pc = (z_0+z_2)^N Y(Y(a, z_0)b, z_2)Pc,$$

$$(z_0+z_2)^N Y(Pa, z_0+z_2)Y(b, z_2)c = (z_0+z_2)^N Y(Y(Pa, z_0)b, z_2)c,$$

$$(z_0+z_2)^N Y(a, z_0+z_2)Y(b, z_2)c = (z_0+z_2)^N Y(Y(a, z_0)b, z_2)c.$$

This shows (6.1.21) by comparing the expansions. The weak commutativity of $Y^{\star_P}$ can be proved by a similar argument. We omit the details of the proof. □

Example 6.1.26. Let $V = V_L$ be a lattice VOA, $L_1 = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_{r-1} \oplus \mathbb{Z}_{\geq 0}\alpha_r \leq L$ be the parabolic-type sub-monoid. Then $L = L_1 \sqcup L_2$, where $L_2 = (L_1)^- = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_{r-1} \oplus \mathbb{Z}_{<0}\alpha_r$. Let $P : V_L \to V_{L_1}$ be the projection RBO in Example 6.1.18. For $a = a_1 + a_2$ and $b = b_1 + b_2$ in V_L, where $a^i, b^i \in V^i$ for $i = 1, 2$, we have

$$P(a_1) = a_1, \quad P(b_1) = b_1, \quad \text{and} \quad P(a_2) = P(b_2) = 0.$$

Then by (6.1.20), with $\lambda = -1$, we have

$$\begin{aligned} Y^{\star_P}(a, z)b &= Y(a, z)Pb + Y(Pa, z)b - Y(a, z)b \\ &= Y(a_1 + a_2)b_1 + Y(a_1, z)(b_1 + b_2) - Y(a_1 + a_2, z)(b_1 + b_2) \\ &= Y(a_1, z)b_1 - Y(a_2, z)b_2. \end{aligned}$$

Since P is translation invariant, by Theorem 6.1.25, $(V_L, Y^{\star_P}, L(-1))$ is a vertex algebra without vacuum, with $Y^{\star_P}$ given by

$$Y^{\star_P}(a, z)b = Y(a_1, z)b_1 - Y(a_2, z)b_2. \tag{6.1.22}$$

Note that the vacuum element $\mathbf{1}$ of V_L is contained in $M_{\hat{\mathfrak{h}}}(1, 0) \subset V_1$, and it cannot be the vacuum element of $(V_L, Y^{\star_P}, L(-1))$ since $Y^{\star_P}(\mathbf{1}, z)a_2 = 0$ for all $a_2 \in V_2$ by (6.1.22).

6.2 Dendriform vertex algebras and Rota-Baxter vertex algebras

In this Section, we will study the substructure underlying a Rota-Baxter vertex algebra $(V, Y, \mathbf{1}, P)$. The substructure is what we call "dendriform vertex algebra". The usual notion of a dendriform algebra was introduced by Loday (cf. [59]). It is a vector space V over a field k, equipped with two binary operators $\prec$ and $\succ$, satsifying

$$(x \prec y) \prec z = x \prec (y \prec z + y \succ z), \tag{6.2.1}$$

$$(x \succ y) \prec z = x \succ (y \prec z), \tag{6.2.2}$$

$$(x \prec y + x \succ y) \succ z = x \succ (y \succ z), \tag{6.2.3}$$

for all $x, y, z \in V$. Given a dendriform algebra $(V, \prec, \succ)$, one can define

$$x \cdot y := x \prec y + x \succ y, \quad \forall x, y \in V. \tag{6.2.4}$$

This is an associative product on V (cf. [59]). Furthermore, the following theorem (Theorem 5.1.4 in [31]) shows that Rota-Baxter algebras can give rise to dendriform algebras:

Theorem 6.2.1. *[31] (a) An Rota-Baxter algebra (R, P) of weight 0 defines a dendriform algebra $(R, \prec_P, \succ_P)$, where $x \prec_P y = xP(y)$, and $x \succ_P y = P(x)y$, for all $x, y \in R$.*
(b) An Rota-Baxter algebra (R, P) of weight λ defines a dendriform algebra $(R, \prec'_P, \succ'_P)$, where $x \prec'_P y = xP(y) + \lambda xy$, and $x \succ'_P y = P(x)y$, for all $x, y \in R$.

The dendriform axioms are the axioms underlying the usual associativity. Note that the associative analog of vertex algebras is the notion of field algebra (cf. [9]), or the nonlocal vertex algebra (cf. [53]). A field algebra $(V, Y, \mathbf{1}, D)$ is a vector space V, equipped with a linear map $Y : V \to \mathrm{End}(V)[[z, z^{-1}]]$, a distinguished vector $\mathbf{1}$, and a linear map $D : V \to V$, satisfying the truncation property, the vacuum and creation properties, the D(-bracket) derivative properties (6.1.3) and (6.1.4), and the weak associativity (6.1.2).

Inspired by the definitions (6.2.1)-(6.2.3), we expect to decompose the vertex operator $Y(\cdot, z)$ into a sum of two operators: $Y(\cdot, z) = Y_{\prec}(\cdot, z) + Y_{\succ}(\cdot, z)$, whose properties are consistent with both the Rota-Baxter type axiom and the weak associativity axiom (6.1.2).

On the other hand, we also expect to have the Jacobi identity from this underlying structure. Since the Jacobi identity follows from weak associativity, together with the D-bracket derivative property and skew-symmetry (see [37]), we will use D and add some additional axioms into our underlying structure so that it leads to the Jacobi identity.

6.2.1 Dendriform field and vertex algebras

Since the axioms of a dendriform are extracted from the associativity axiom, the weak associativity axiom (6.1.2) of a field algebra is enough for our purpose. We introduce the following notion, which can be viewed as a weaker version of both field algebra and vertex Leibniz algebra:

Definition 6.2.2. A **field Leibniz algebra** is a vector space V, equipped with a linear map $Y : V \to (\mathrm{End}V)[[z, z^{-1}]]$, satisfying the truncation property and the weak associativity (6.1.2). We denote a field Leibniz algebra by (V, Y).

An ordinary Rota-Baxter operator on a field Leibniz algebra (V, Y) of weight $\lambda \in \mathbb{C}$ is a linear map $P : V \to V$, satisfying the compatibility formula (6.1.7).

Definition 6.2.3. Let V be a vector space, and $D : V \to V$ be a linear map. Let

$$Y_{\prec}(\cdot, z) : V \to \mathrm{Hom}(V, V((z))),\ a \mapsto Y_{\prec}(a, z),$$
$$Y_{\succ}(\cdot, z) : V \to \mathrm{Hom}(V, V((z))),\ a \mapsto Y_{\succ}(a, z)$$

be two linear operators associated with a formal variable z. For simplicity, we denote $Y_{\prec}(\cdot, z)$ by $\cdot \prec_z \cdot$, and $Y_{\succ}(\cdot, z)$ by $\cdot \succ \cdot$, respectively, and write $Y_{\prec}(a, z)b = a \prec_z b$ and $Y_{\succ}(a, z)b = a \succ_z b$, for all $a, b \in V$. Then

(1) $(V, \prec_z, \succ_z)$ is called a **dendriform field algebra** if for any $a, b, c \in V$, there exists some $N \in \mathbb{N}$ depending on a and c, satisfying:

$$(z_0 + z_2)^N (a \prec_{z_0} b) \prec_{z_2} c = (z_0 + z_2)^N a \prec_{z_0+z_2} (b \succ_{z_2} c + b \prec_{z_2} c), \tag{6.2.5}$$

$$(z_0 + z_2)^N (a \succ_{z_0} b) \prec_{z_2} c = (z_0 + z_2)^N a \succ_{z_0+z_2} (b \prec_{z_2} c), \tag{6.2.6}$$

$$(z_0 + z_2)^N (a \succ_{z_0} b + a \prec_{z_0} b) \succ_{z_2} c = (z_0 + z_2)^N a \succ_{z_0+z_2} (b \succ_{z_2} c). \tag{6.2.7}$$

(2) $(V, \prec_z, \succ_z, D)$ is called a **dendriform vertex algebra** if $(V, \prec_z, \succ_z)$ is a dendriform field algebra, and D, $\prec_z$, and $\succ_z$ satisfy the following compatibility properties:

$$e^{zD}(a \prec_{-z} b) = b \succ_z a, \quad \text{and} \quad e^{zD}(a \succ_{-z} b) = b \prec_z a; \tag{6.2.8}$$

$$D(a \prec_z b) - a \prec_z (Db) = \frac{d}{dz}(a \prec_z b), \quad \text{and} \quad D(a \succ_z b) - a \succ_z (Db) = \frac{d}{dz}(a \succ_z b). \tag{6.2.9}$$

We can regard (6.2.8) as the analog of skew-symmetry (6.1.5) satisfied by the partial operators $\prec_z$ and $\succ_z$. (6.2.9) can be viewed as the D-bracket derivative property (6.1.4) for the partial operators. Similar to Lemma 2.7 in [56], we have the following:

Lemma 6.2.4. *Let V be a vector space, equipped with two operators $\prec_z, \succ_z$: $V \times V \to V((z))$ and a linear operator $D : V \to V$, satisfying the skew-symmetry* (6.2.8)*, then the D-bracket derivative property* (6.2.9) *is equivalent to the following D-derivative property:*

$$(Da) \prec_z b = \frac{d}{dz}(a \prec_z b), \quad \text{and} \quad (Da) \succ_z b = \frac{d}{dz}(a \succ_z b), \tag{6.2.10}$$

for any $a, b \in V$. In particular, a dendriform vertex algebra $(V, \prec_z, \succ_z, D)$ can be defined as a dendriform field algebra $(V, \prec_z, \succ_z)$, satisfying (6.2.8) *and* (6.2.10).

Proof. Similar to the proof of Lemma 2.7 in [56], for any $a, b \in V$, we have:

$$\begin{aligned}
(Da) \prec_z b - \frac{d}{dz}(a \prec_z b) &= e^{zD} b \succ_{-z} Da - De^{zD}(b \succ_z a) - e^{zD}\frac{d}{dz}(b \succ_{-z} a) \\
&= e^{zD}\left(b \succ_{-z} Da - D(b \succ_{-z} a) - \frac{d}{dz}(b \succ_{-z} a)\right), \\
(Da) \succ_z b - \frac{d}{dz}(a \succ_z b) &= e^{zD} b \prec_{-z} Da - De^{zD}(b \prec_z a) - e^{zD}\frac{d}{dz}(b \prec_{-z} a) \\
&= e^{zD}\left(b \prec_{-z} Da - D(b \prec_{-z} a) - \frac{d}{dz}(b \prec_{-z} a)\right).
\end{aligned}$$

Thus, (6.2.9) is equivalent to (6.2.10). □

The axioms of the dendriform are closely related to the properties of Rota-Baxter operators. The following Theorem shows that a weight λ RBVA can give rise to dendriform field algebra, and a weight 0 RBVA can give rise to dendriform vertex algebra:

Theorem 6.2.5. *Let $(V, Y, \mathbf{1}, P)$ be an RBVA of weight λ.*

(1) If $\lambda = 0$, then $(V, Y, \mathbf{1}, P)$ defines a dendriform field algebra $(V, \prec_z, \succ_z)$, where

$$a \prec_z b := Y(a, z)P(b), \qquad a \succ_z b := Y(P(a), z)b, \tag{6.2.11}$$

for all $a, b \in V$. If, furthermore, P is translation invariant, then $(V, \prec_z, \succ_z, D)$ is a dendriform vertex algebra.

(2) *If λ is arbitrary, then $(V, Y, \mathbf{1}, P)$ defines a dendriform field algebra $(V, \prec'_z, \succ'_z)$, where*

$$a \prec'_z b = Y(a,z)P(b) + \lambda Y(a,z)b, \qquad a \succ'_z b = Y(P(a),z)b, \tag{6.2.12}$$

for all $a, b \in V$.

Proof. We first prove part (2). For equation (6.2.5), we have:

$$\begin{aligned}(z_0+z_2)^N(a \prec'_{z_0} b) \prec'_{z_2} c &= (z_0+z_2)^N(Y(Y(a,z_0)P(b) + \lambda Y(a,z_0)b, z_2)P(c)) \\ &\quad + (z_0+z_2)^N(\lambda Y(Y(a,z_0)P(b) + \lambda Y(a,z_0)b, z_2)c).\end{aligned}$$

On the other hand,

$$\begin{aligned}&(z_0+z_2)^N a \prec'_{z_0+z_2} (b \succ'_{z_2} c + b \prec'_{z_2} c) \\ &= (z_0+z_2)^N(Y(a,z_0+z_2)P(Y(P(b),z_2)c + Y(b,z_2)P(c) + \lambda Y(b,z_2)c)) \\ &\quad + (z_0+z_2)^N(\lambda Y(a,z_0+z_2)(Y(P(b),z_2)c + Y(b,z_2)P(c) + \lambda Y(b,z_2)c)) \\ &= (z_0+z_2)^N(Y(a,z_0+z_2)Y(P(b),z_2)P(c) \\ &\quad + \lambda Y(a,z_0+z_2)(Y(P(b),z_2)c + Y(b,z_2)P(c) + \lambda Y(b,z_2)c)).\end{aligned}$$

Take a common N such that the weak associativity for $(a, P(b), P(c))$, $(a, b, P(c))$, $(a, P(b), c)$, and (a, b, c) are satisfied simultaneously, then equation (6.2.5) holds. For equation (6.2.6),

$$\begin{aligned}(z_0+z_2)^N(a \succ'_{z_0} b) \prec'_{z_2} c &= (z_0+z_2)^N(Y(Y(P(a),z_0)b, z_2)P(c) + \lambda Y(Y(P(a),z_0)b, z_2)c), \\ (z_0+z_2)^N a \succ'_{z_0+z_2} (b \prec'_{z_2} c) &= (z_0+z_2)^N Y(P(a), z_0+z_2)(Y(b,z_2)P(c) + \lambda Y(b,z_1)c).\end{aligned}$$

Take a common N such that the weak associativity are satisfied for $(P(a), b, P(c))$ and $(P(a), b, c)$, then equation (6.2.6) holds. Finally, for equation (6.2.7), we have:

$$\begin{aligned}(z_0+z_2)^N(a \succ'_{z_0} b + a \prec'_{z_0} b) \succ'_{z_2} c &= (z_0+z_2)^N Y(P(Y(P(a),z_0)b + Y(a,z_0)P(b) + \lambda Y(a,z_0)b), z_2)c \\ &= (z_0+z_2)^N Y(Y(P(a),z_0)P(b), z_2)c, \\ (z_0+z_2)^N a \succ'_{z_0+z_2} (b \succ'_{z_2} c) &= (z_0+z_2)^N Y(P(a), z_0+z_2)(Y(P(b),z_2)c)\end{aligned}$$

We again take a common N such that the weak associativity is satisfied for $(P(a), P(b), c)$, then equation (6.2.7) holds. This proves (2), and by taking $\lambda = 0$, we see that $(V, \prec_z, \succ_z)$ given by (6.2.11) is a dendriform field algebra. If P is translation invariant ($PD = DP$), then by (6.2.11),

$$e^{zD}(a \prec_{-z} b) = e^{zD}Y(a,-z)P(b) = Y(P(b),z)a = b \succ_z a,$$

$$e^{zD}(a \succ_{-z} b) = e^{zD}Y(P(a), -z)b = Y(b, z)P(a) = b \prec_z a.$$

Moreover, by the D-derivative and bracket derivative properties (6.1.3) and (6.1.4),

$$\begin{aligned} D(a \prec_z b) - a \prec_z (Db) &= DY(a, z)P(b) - Y(a, z)P(Db) = [D, Y(a, z)]P(b) \\ &= \frac{d}{dz}Y(a, z)P(b) = \frac{d}{dz}(a \prec_z b), \\ D(a \succ_z b) - a \succ_z (Db) &= DY(P(a), z)b - Y(P(a), z)Db = [D, Y(P(a), z)]b \\ &= \frac{d}{dz}(Y(P(a), z)b) = \frac{d}{dz}(a \succ_z b). \end{aligned}$$

Hence $(V, \prec_z, \succ_z, D)$ is a dendriform vertex algebra, in view of (6.2.8) and (6.2.9). □

Dendriform field and vertex algebras can also give rise to field Leibniz algebras (see Definition 6.2.2) and vertex algebras without vacuum (see Definition 6.1.2).

Theorem 6.2.6. *Let $(V, \prec_z, \succ_z)$ be a dendriform field algebra. Define a linear map $Y : V \to \mathrm{End}(V)[[z, z^{-1}]]$ by*

$$Y(a, z)b := a \prec_z b + a \succ_z b, \tag{6.2.13}$$

for all $a, b \in V$. Then (V, Y) is a field Leibniz algebra. If, furthermore, $(V, \prec_z, \succ_z, D)$ is a dendriform vertex algebra, then (V, Y, D) is a vertex algebra without vacuum.

Proof. Clearly, Y defined by (6.2.13) satisfies the truncation property. We claim that Y satisfies the weak associativity (6.1.2). Indeed, for any $a, b, c \in V$, we have

$$\begin{aligned} &(z_0 + z_2)^N Y(Y(a, z_0)b, z_2)c \\ &= (z_0 + z_2)^k Y(a \prec_{z_0} b + a \succ_{z_0} b, z_2)c \\ &= (z_0 + z_2)^N (a \prec_{z_0} b + a \succ_{z_0} b) \prec_{z_2} c + (a \prec_{z_0} b + a \succ_{z_0} b) \succ_{z_2} c \\ &= (z_0 + z_2)^N ((a \prec_{z_0} b) \prec_{z_2} c + (a \succ_{z_0} b) \prec_{z_2} c + (a \prec_{z_0} b + a \succ_{z_0} b) \succ_{z_2} c), \\ &(z_0 + z_2)^N Y(a, z_0 + z_2)Y(b, z_2)c \\ &= (z_0 + z_2)^k Y(a, z_0 + z_2)(b \prec_{z_2} c + b \succ_{z_2} c) \\ &= (z_0 + z_2)^N (a \prec_{z_0+z_2} (b \prec_{z_2} c + b \succ_{z_2} c) + a \succ_{z_0+z_2} (b \prec_{z_2} c + b \succ_{z_2} c)) \\ &= (z_0 + z_2)^N (a \prec_{z_0+z_2} (b \prec_{z_2} c + b \succ_{z_2} c) + a \succ_{z_0+z_2} (b \prec_{z_2} c) + a \succ_{z_0+z_2} (b \succ_{z_2} c)). \end{aligned}$$

We take a common $N > 0$, such that equations (6.2.5), (6.2.6), and (6.2.7) are satisfied at the same time, then equation (6.1.2) holds. Hence (V, Y) is a field Leibniz algebra.

If $(V, \prec_z, \succ_z, D)$ is a dendriform vertex algebra, then by (6.2.8) and (6.2.9), we have:

$$e^{zD}Y(a,-z)b = e^{zD}(a \prec_{-z} b) + e^{zD}(a \succ_{-z} b) = b \succ_z a + b \prec_z a = Y(b,z)a,$$

for all $a, b \in V$. Hence (V, Y, D) satisfies the skew-symmetry (6.1.5). Moreover,

$$\begin{aligned} D(Y(a,z)b) - Y(a,z)Db &= D(a \prec_z b) + D(a \succ_z b) - a \prec_z (Db) - a \succ_z (Db) \\ &= \frac{d}{dz}(a \prec_z Db) + \frac{d}{dz}(a \succ_z b) = \frac{d}{dz}Y(a,z)b, \end{aligned}$$

for all $a, b \in V$. Hence (V, Y, D) satisfies the D-bracket derivative property (6.1.4). Then by Proposition 6.1.3, Y also satisfies the weak commutativity, and by Theorem 2.1.2, (V, Y, D) satisfies the Jacobi identity. This shows that (V, Y, D) is a vertex algebra without vacuum. □

6.2.2 Equivalent characterization of the dendriform vertex algebra

Theorem 6.2.6 shows that the vertex operator $Y(a,z)b = a \prec_z b + a \succ_z b$ given by a dendriform vertex algebra $(V, \prec_z, \succ_z, D)$ satisfies the Jacobi identity. But $\prec_z$ and $\succ_z$ are operators underlying the vertex operator $Y(\cdot, z)$, it is natural to expect some additional properties to be satisfied by these operators on a dendriform vertex algebra $(V, \prec_z, \succ_z, D)$.

First, we note that $\prec_z$ and $\succ_z$ satisfy an analog of the weak commutativity axiom (6.1.2) of the vertex operators:

Proposition 6.2.7. *Let $(V, \prec_z, \succ_z, D)$ be a dendriform vertex algebra. Then for any $a, b, c \in V$, there exists some $N \in \mathbb{N}$ depending on a and b, such that*

$$(z_1 - z_2)^N a \succ_{z_1} (b \prec_{z_2} c) = (z_1 - z_2)^N b \prec_{z_2} (a \succ_{z_1} c + a \prec_{z_1} c), \tag{6.2.14}$$

$$(z_1 - z_2)^N a \succ_{z_1} (b \succ_{z_2} c) = (z_1 - z_2)^N b \succ_{z_2} (a \succ_{z_1} c). \tag{6.2.15}$$

Conversely, if V is a vector space, equipped with a linear map $D : V \to V$ and two linear operators $\prec_z, \succ_z$: $V \to \mathrm{Hom}(V, V((z)))$, satisfying (6.2.14)*,* (6.2.15)*,* (6.2.8)*, and* (6.2.9)*, then $(V, \prec_z, \succ_z, D)$ is a dendriform vertex algebra.*

Proof. Let $(V, \prec_z, \succ_z, D)$ be a dendriform vertex algebra, then it follows from (6.2.9) that

$$e^{z_0 D} a \prec_z e^{-z_0 D} b = a \prec_{z+z_0} b, \quad \text{and} \quad e^{z_0 D} a \succ_z e^{-z_0 D} b = a \succ_{z+z_0} b, \tag{6.2.16}$$

for all $a, b \in V$. The proof of (6.2.16) is similar to proving the conjugation formula the vertex operator $e^{z_0 D} Y(a,z) e^{-z_0 D} = Y(a, z+z_0)$ from the D-bracket derivative property (6.1.4), see [27, 55], we omit the details.

By (6.2.16) and (6.2.8), we can express each side of (6.2.5) as:

$$\begin{aligned}(z_0+z_2)^N a \prec_{z_0+z_2} (b \succ_{z_2} c + b \prec_{z_2} c) &= (z_0+z_2)^N e^{z_2 D} a \prec_{z_0} e^{-z_2 D}(b \succ_{z_2} c + b \prec_{z_2} c)\\ &= (z_0+z_2)^N e^{z_2 D} a \prec_{z_0} (c \prec_{-z_2} b + c \succ_{-z_2} b), \qquad (6.2.17)\\ (z_0+z_2)^N (a \prec_{z_0} b) \prec_{z_2} c &= (z_0+z_2)^N e^{z_2 D} c \succ_{-z_2} (a \prec_{z_0} b),\end{aligned}$$

where $N \in \mathbb{N}$ depends on a and c. Hence $(z_0+z_2)^N a \prec_{z_0} (c \prec_{-z_2} b + c \succ_{-z_2} b) = (z_0+z_2)^N c \succ_{-z_2} (a \prec_{z_0} b)$, and by replacing (z_0, z_2) with $(z_2, -z_1)$, and replacing (c, a, b) with the ordered triple (a, b, c) in this equation, we have:

$$(z_2 - z_1)^N b \prec_{z_2} (a \prec_{z_1} c + a \succ_{z_1} c) = (z_2 - z_1)^N a \succ_{z_1} (b \prec_{z_2} c),$$

where N depends on a and b. This equation is equivalent to (6.2.14) since $N \geq 0$. Similarly, we can express each side of (6.2.6) as:

$$\begin{aligned}(z_0+z_2)^N a \succ_{z_0+z_2} (b \prec_{z_2} c) &= (z_0+z_2)^N e^{z_2 D} a \succ_{z_0} e^{-z_2 D}(b \prec_{z_2} c)\\ &= (z_0+z_2)^N e^{z_2 D} a \succ_{z_0} (c \succ_{-z_2} b), \qquad (6.2.18)\\ (z_0+z_2)^N (a \succ_{z_0} b) \prec_{z_2} c &= (z_0+z_2)^N e^{z_2 D} c \succ_{-z_2} (a \succ_{z_0} b),\end{aligned}$$

where N depends on a and c. Then $(z_0+z_2)^N a \succ_{z_0} (c \succ_{-z_2} b) = (z_0+z_2)^N c \succ_{-z_2} (a \succ_{z_0} b)$, and by replacing (z_0, z_2) with $(z_1, -z_2)$, and (a, c, b) with (a, b, c), we have N depends on a and b, and $(z_1 - z_2)^N a \succ_{z_1} (b \succ_{z_2} c) = (z_1 - z_2)^N b \succ_{z_2} (a \succ_{z_1} c)$, which is (6.2.15). Finally, we can express each side of (6.2.7) as:

$$\begin{aligned}(z_0+z_2)^N a \succ_{z_0+z_2} (b \succ_{z_2} c) &= (z_0+z_2)^N e^{z_2 D} a \succ_{z_0} e^{-z_2 D}(b \succ_{z_2} c)\\ &= (z_0+z_2)^N e^{z_2 D} a \succ_{z_0} (c \prec_{-z_2} b), \qquad (6.2.19)\\ (z_0+z_2)^N (a \succ_{z_0} b + a \prec_{z_0} b) \succ_{z_2} c &= (z_0+z_2)^N e^{z_2 D} c \prec_{-z_2} (a \succ_{z_0} b + a \prec_{z_0} b),\end{aligned}$$

where N depends on a and c. Then $(z_0+z_2)^N a \succ_{z_0} (c \prec_{-z_2} b) = (z_0+z_2)^N c \prec_{-z_2} (a \succ_{z_0} b + a \prec_{z_0} b)$, and by replacing (z_0, z_2) with $(z_1, -z_2)$, and (a, c, b) with (a, b, c), we have N depends on a and b, and $(z_1 - z_2)^N a \succ_{z_1} (b \prec_{z_2} c) = (z_1 - z_2)^N b \prec_{z_2} (a \succ_{z_1} c + a \prec_{z_1} c)$, which is (6.2.14). The second statement follows by reversing the processes of (6.2.17) and (6.2.18). □

Corollary 6.2.8. *A dendriform vertex algebra is a vector space V, equipped with a linear map $D : V \to V$ and two linear operators $\prec_z, \succ_z : V \to \mathrm{Hom}(V, V((z)))$, satisfying* (6.2.14)*,* (6.2.15)*,* (6.2.8)*, and* (6.2.9)*.*

Remark 6.2.9. Conditions (6.2.14) and (6.2.15) together can give rise to the weak commutativity (6.1.2) of the vertex operator Y defined by (6.2.13): $Y(a,z)b := a \prec_z b + a \succ_z b$. Indeed, by Proposition 6.2.7, there exists $N \in \mathbb{N}$ depending on a and b, such that

$$\begin{aligned}
&(z_1 - z_2)^N Y(a, z_1)Y(b, z_2)c \\
&= (z_1 - z_2)^N (a \prec_{z_1} (b \prec_{z_2} c + b \succ_{z_2} c) + a \succ_{z_1} (b \prec_{z_2} c) + a \succ_{z_1} (b \succ_{z_2} c)) \\
&= (z_1 - z_2)^N (b \succ_{z_2} (a \prec_{z_1} c)) + (z_1 - z_2)^N (b \prec_{z_2} (a \succ_{z_1} c + a \prec_{z_1} c)) + (z_1 - z_2)^N b \succ_{z_2} (a \succ_{z_1} c) \\
&= (z_1 - z_2)^N Y(b, z_2)Y(a, z_1)c.
\end{aligned}$$

We can also obtain an analog of the Jacobi identity for the operators $\prec_z$ and $\succ_z$. We recall the following Lemma 6.1.2 in [54]:

Lemma 6.2.10. *Let U be a vector space, and let $A(z_1, z_2) \in U((z_1))((z_2))$, $B(z_1, z_2) \in U((z_2))((z_1))$, and $C(z_0, z_2) \in U((z_2))((z_0))$. Then*

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)A(z_1, z_2) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)B(z_1, z_2) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)C(z_0, z_2) \tag{6.2.20}$$

holds if and only if there exists $k, l \in \mathbb{N}$ such that

$$(z_1 - z_2)^k A(z_1, z_2) = (z_1 - z_2)^k B(z_1, z_2), \tag{6.2.21}$$

$$(z_0 + z_2)^l A(z_0 + z_2, z_2) = (z_0 + z_2)^l C(z_0, z_2). \tag{6.2.22}$$

Theorem 6.2.11. *Let $(V, \prec_z, \succ_z, D)$ be a dendriform vertex algebra. Then we have three Jacobi identities involving the operators $\prec_z$ and $\succ_z$:*

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)a \succ_{z_1} (b \prec_{z_2} c) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)b \prec_{z_2} (a \succ_{z_1} c + a \prec_{z_1} c) \\
&= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)(a \succ_{z_0} b) \prec_{z_2} c,
\end{aligned} \tag{6.2.23}$$

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)a \succ_{z_1} (b \succ_{z_2} c) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)b \succ_{z_2} (a \succ_{z_1} c) \\
&= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)(a \succ_{z_0} b + a \prec_{z_0} b) \succ_{z_2} c,
\end{aligned} \tag{6.2.24}$$

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)a \prec_{z_1} (b \prec_{z_2} c + b \succ_{z_2} c) - z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)b \succ_{z_2} (a \prec_{z_1} c)$$
$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)(a \prec_{z_0} b) \prec_{z_2} c, \tag{6.2.25}$$

where $a, b, c \in V$, and z_0, z_1, z_2 are formal variables.

Furthermore, (6.2.23)*,* (6.2.24)*, and* (6.2.25) *are mutually equivalent. We call* (6.2.24) *the Jacobi identity for the dendriform vertex algebra* $(V, \prec_z, \succ_z, D)$*.*

Proof. By Proposition 6.2.7 and the formulas (6.2.5)-(6.2.7), we have:

$$(z_0+z_2)^k a \succ_{z_0+z_2} (b \prec_{z_2} c) = (z_0+z_2)^k (a \succ_{z_0} b) \prec_{z_2} c,$$
$$(z_1-z_2)^l a \succ_{z_1} (b \prec_{z_2} c) = (z_1-z_2)^l b \prec_{z_2} (a \succ_{z_1} c + a \prec_{z_1} c),$$

for some $k, l \in \mathbb{N}$. Then $A(z_1, z_2) = a \succ_{z_1} (b \prec_{z_2} c)$, $B(z_1, z_2) = b \prec_{z_2} (a \succ_{z_1} c + a \prec_{z_1} c)$, and $C(z_0, z_2) = (a \succ_{z_0} b) \prec_{z_2} c$ satisfy the conditions (6.2.21) and (6.2.22) in Lemma 6.2.10, then the Jacobi identity (6.2.23) follows from (6.2.20).

Similarly, the Jacobi identity (6.2.24) follows from Lemma 6.2.10 and

$$(z_0+z_2)^k a \succ_{z_0+z_2} (b \succ_{z_2} c) = (z_0+z_2)^k (a \succ_{z_0} b + a \prec_{z_0} b) \succ_{z_2} c,$$
$$(z_1-z_2)^l a \succ_{z_1} (b \succ_{z_2} c) = (z_1-z_2)^l b \succ_{z_2} (a \succ_{z_1} c),$$

for some $k, l \in \mathbb{N}$. The Jacobi identity (6.2.25) follows from Lemma 6.2.10 and

$$(z_0+z_2)^k a \prec_{z_0+z_2} (b \succ_{z_2} c + b \prec_{z_2} c) = (z_0+z_2)^k (a \prec_{z_0} b) \prec_{z_2} c,$$
$$(z_1-z_2)^l a \prec_{z_1} (b \prec_{z_2} c + b \succ_{z_2} c) = (z_1-z_2)^l b \succ_{z_2} (a \prec_{z_1} c),$$

for some $k, l \in \mathbb{N}$. The equivalency of these Jacobi identities essentially corresponds to the S_3-symmetry of the Jacobi identity, see Section 2.7 in [27], and the proof is also similar.

Assume (6.2.23) is true. By the skew-symmetry (6.2.8), we have:

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)a \succ_{z_1} e^{z_2D}(c \succ_{-z_2} b) - z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)e^{z_2D}(a \succ_{z_1} c + a \prec_{z_1} c) \succ_{-z_2} b$$
$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)(a \succ_{z_0} b) \prec_{z_2} c = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)e^{z_2D}c \succ_{-z_2} (a \succ_{z_0} b).$$

Then by (6.2.16) and properties of the formal δ-functions (see Section 6.1.2 in [27]),

$$z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)c \succ_{-z_2} (a \succ_{z_0} b) = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)c \succ_{-z_2} (a \succ_{z_0} b)$$

$$= z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) a \succ_{z_1 - z_2} (c \succ_{-z_2} b) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)(a \succ_{z_1} c + a \prec_{z_1} c) \succ_{-z_2} b$$
$$= z_1^{-1}\delta\left(\frac{z_0 + z_2}{z_1}\right) a \succ_{z_0} (c \succ_{-z_2} b) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)(a \succ_{z_1} c + a \prec_{z_1} c) \succ_{-z_2} b.$$

Change the formal variables $(z_0, z_1, z_2) \mapsto (w_1, w_0, -w_2)$ in the equations above, we have:

$$w_0^{-1}\delta\left(\frac{-w_2 + w_1}{w_0}\right) c \succ_{w_2} (a \succ_{w_1} b)$$
$$= w_0^{-1}\delta\left(\frac{w_1 - w_2}{w_0}\right) a \succ_{w_1} (c \succ_{w_2} b) - w_1^{-1}\delta\left(\frac{w_2 + w_0}{w_1}\right)(a \succ_{w_0} c + a \prec_{w_0} c) \succ_{w_2} b.$$

This equation is the same as (6.2.24) when we change (c, b) into (b, c). This shows the equations (6.2.23) $\iff$ (6.2.24). Similarly, assume (6.2.25) is true. Then

$$z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)(a \prec_{z_0} b) \prec_{z_2} c = z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) e^{z_2 D} e^{z_0 D} e^{-z_0 D} c \succ_{-z_2} (e^{z_0 D} b \succ_{-z_0} a)$$
$$= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) e^{z_1 D} c \succ_{-z_1} (b \succ_{-z_0} a)$$
$$= z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) e^{z_1 D}(b \prec_{z_2} c + b \succ_{z_2} c) \succ_{-z_1} a - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) b \succ_{z_2} e^{z_1 D}(c \succ_{-z_1} a).$$

Hence by the properties of δ-functions, we have:

$$z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) c \succ_{-z_1} (b \succ_{-z_0} a) = z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) c \succ_{-z_1} (b \succ_{-z_0} a)$$
$$= z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)(b \prec_{z_2} c + b \succ_{z_2} c) \succ_{-z_1} a - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) b \succ_{z_2 - z_1} (c \succ_{-z_1} a)$$
$$= z_1^{-1}\delta\left(\frac{z_0 + z_2}{z_1}\right)(b \prec_{z_2} c + b \succ_{z_2} c) \succ_{-z_1} a + z_2^{-1}\delta\left(\frac{-z_0 + z_1}{z_2}\right) b \succ_{-z_0} (c \succ_{-z_1} a).$$

Change the variables $(z_0, z_1, z_2) \mapsto (-w_1, -w_2, w_0)$, the equation above becomes:

$$w_0^{-1}\delta\left(\frac{-w_2 + w_1}{w_0}\right) c \succ_{w_2} (b \succ_{w_1} a)$$
$$= -w_2^{-1}\delta\left(\frac{w_1 - w_0}{w_2}\right)(b \prec_{w_2} c + b \succ_{w_2} c) \succ_{w_2} a + w_0^{-1}\delta\left(\frac{w_1 - w_2}{w_0}\right) b \succ_{w_1} (c \succ_{w_2} a).$$

It is obvious that this equation is the same as (6.2.24). Hence the equations (6.2.25) $\iff$ (6.2.24). Thus (6.2.23), (6.2.24), and (6.2.25) are equivalent. □

Remark 6.2.12. By adding up the three Jacobi identities (6.2.23)-(6.2.25), we can derive the Jacobi identity for the vertex operator $Y(a, z) = a \prec_z b + a \succ_z b$. This provides us with an alternative proof of Theorem 6.2.6

Corollary 6.2.13. *A dendriform vertex algebra is a vector space V, equipped with a linear map $D : V \to V$ and two linear operators $\prec_z, \succ_z$: $V \to \mathrm{Hom}(V, V((z)))$, satisfying* (6.2.8), (6.2.9), *and the Jacobi identity* (6.2.24).

Theorem 6.2.11 also indicates that a dendriform vertex algebra $(V, \prec_z, \succ_z, D)$ defines a module structure on V over its associated vertex algebra without vacuum (V, Y, D) in Theorem 6.2.6. First, we recall the following definition, see Definition 2.9 in [56]:

Definition 6.2.14. Let (V, Y, D) be a vertex algebra without vacuum. A V-module (W, Y_W) is a vector space W, equipped with a linear map $Y_W : V \to \mathrm{End}(W)[[z, z^{-1}]]$, satisfying the truncation property, the Jacobi identity for Y_W in Definition 2.1.5, and

$$Y_W(Da, z) = \frac{d}{dz} Y_W(a, z) \quad \text{for all } a \in V. \tag{6.2.26}$$

Proposition 6.2.15. *Let $(V, \prec_z, \succ_z, D)$ be a dendriform vertex algebra, and let (V, Y, D) be the associated vertex algebra without vacuum, where Y is given by* (6.2.13)*: $Y(a, z)b = a \prec_z b + a \succ_z b$. Let $W = V$, and define:*

$$Y_W : V \to \mathrm{End}(W)[[z, z^{-1}]], \quad Y_W(a, z)b := a \succ_z b, \tag{6.2.27}$$

for all $a \in V$ and $b \in W$. Then (W, Y_W) is a module over (V, Y, D).

Proof. By Definition 6.2.3, clearly Y_W satisfies the truncation property. By the Jacobi identity (6.2.24) of $(V, \prec_z, \succ_z, D)$, (6.2.27), and (6.2.13), we have:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_W(a, z_1)Y_W(b, z_2)c - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) Y_W(b, z_2)Y_W(a, z_1)c$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_W(Y(a, z_0)b, z_2)c,$$

for all $a, b \in V$, and $c \in W = V$. Finally, by Lemma 6.2.4, we have $Y_W(Da, z)b = (Da) \succ_z b = \frac{d}{dz} a \succ_z b$, for all $a \in V$ and $b \in W$. Thus, (W, Y_W) is a module over the vertex algebra without vacuum (V, Y, D), in view of Definition 6.2.14. □

Remark 6.2.16. Since we define Y_W by one of the partial operators $\succ_z$ in (6.2.27), it is natural to consider the vertex operator $\mathcal{Y}$ defined by the other partial operator $\mathcal{Y}(a, z)b = a \prec_z b$.

By the skew-symmetry (6.2.8), we have:

$$\mathcal{Y}(a, z)b = a \prec_z b = e^{zD} b \succ_{-z} a = e^{zD} Y_W(b, -z)a = Y_{WV}^W(a, z)b, \tag{6.2.28}$$

in view of (6.1.17). i.e., $\mathcal{Y} = Y_{WV}^{W}$. It is easy to see that the Jacobi identities (6.2.23) and (6.2.25) correspond to the following equation:

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_W(a, z_1)Y_{WV}^{W}(b, z_2)c - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) Y_{WV}^{W}(b, z_2)Y(a, z_1)c \\
&= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_{WV}^{W}(Y_W(a, z_0)b, z_2)c,
\end{aligned}$$

for all $a, b, c \in V = W$. Moreover, $Y_{WV}^{W}(Da, z)b = (Da) \prec_z b = \frac{d}{dz}a \prec_z b$ by Lemma 6.2.4. Thus, if the vertex algebra without vacuum (V, Y, D) is an underlying structure of some VOA $(V, Y, \mathbf{1}, \omega)$, with $D = L(-1)$, then $Y_{WV}^{W}(a, z)b = a \prec_z b$ is an intertwining operator of type $\binom{W}{W\,V}$.

The notion of relative Rota-Baxter operator is introduced in [6], as the operator form of the classical Yang-Baxter equation for VOAs. Let $(V, Y, \mathbf{1}, \omega)$ be a VOA, and (W, Y_W) be a weak V-module. A relative RBO is a linear map $T : W \to V$ such that

$$Y(Tu, z)Tv = T(Y_W(Tu, z)v) + T(Y_{WV}^{W}(u, z)Tv), \quad \text{for all } u, v \in W. \tag{6.2.29}$$

Corollary 6.2.17. *Let $(V, Y, \mathbf{1}, \omega)$ be a VOA. Assume the underlying vertex algebra without vacuum structure $(V, Y, D = L(-1))$ is induced from a dendriform vertex algebra structure $(V, \prec_z, \succ_z, D)$ by Theorem 6.2.6. Let (W, Y_W) be the weak V-module given by Proposition 6.2.15, then the identity map $T = \mathrm{Id} : W \to V$ is a relative RBO.*

Proof. By (6.2.13), (6.2.27), (6.2.28), and the assumption that $T = \mathrm{Id}$, we have:

$$\begin{aligned}
Y(Tu, z)Tv &= u \prec_z v + u \succ_z v = Y_W(u, z)v + Y_{WV}^{W}(u, z)v \\
&= T(Y_W(Tu, z)v) + T(Y_{WV}^{W}(u, z)Tv),
\end{aligned}$$

for all $u, v \in W$. So $T = \mathrm{Id}$ is a relative RBO, in view of (6.2.29) □

Chapter 7

Vertex operator analog of the classical Yang-Baxter equation

In this last Chapter of the thesis, we will study the analog of the Yang-Baxter equation and its relations with the Rota-Baxter operators on vertex operator algebra as a natural generalization of the classical Yang-Baxter equation for Lie algebras.

7.1 Vertex operator Yang-Baxter equation

In this section, we first recall the background on contragredient modules over VOAs and completed tensor products. We then give the notion of the vertex operator Yang-Baxter equation (VOYBE), followed by the notion of relative Rota-Baxter operators for VOAs as the operator form of the VOYBE.

We will prove that the skew-symmetric solutions to the VOYBE in a VOA U are in one-to-one correspondence with skew-symmetric relative Rota-Baxter operators associated with the coadjoint module U'.

7.1.1 The vertex operator Yang-Baxter equation

Definition 7.1.1. Let $M = \bigoplus_{n=0}^{\infty} M(n)$, $W = \bigoplus_{n=0}^{\infty} W(n)$, and $U = \bigoplus_{n=0}^{\infty}$ be $\mathbb{N}$-graded vector spaces, with $\dim M(n) < \infty$, $\dim W(n) < \infty$, and $\dim U(n) < \infty$, for all $n \in \mathbb{N}$.

(1) Define the **complete tensor products** $M\widehat{\otimes}W$ and $M\widehat{\otimes}W\widehat{\otimes}U$ by

$$M\widehat{\otimes}W := \prod_{p,q=0}^{\infty} M(p)\otimes W(q), \qquad M\widehat{\otimes}W\widehat{\otimes}U := \prod_{p,q,r=0}^{\infty} M(p)\otimes W(q)\otimes U(r). \tag{7.1.1}$$

(2) Let $D(U\widehat{\otimes}U) = \prod_{t=0}^{\infty} U(t)\otimes U(t) \subset U\widehat{\otimes}U$. An element α in $U\widehat{\otimes}U$ is called **diagonal**, if $\alpha = \sum_{t=0}^{\infty}\sum_{i=1}^{p_t} \alpha_i^t \otimes \beta_i^t \in D(U\widehat{\otimes}U)$, where $\alpha_i^t, \beta_i^t \in U(t)$ for all $t \geq 0$ and $i \geq 1$.

(3) A diagonal element α is called **skew-symmetric**, if $\sigma(\alpha) = -\alpha$, where $\sigma : U\widehat{\otimes}U \to U\widehat{\otimes}U$ is defined by $\sigma(\sum_t\sum_i \alpha_i^t\otimes\beta_i^t) = \sum_t\sum_i \beta_i^t\otimes\alpha_i^t$; a skew-symmetric diagonal element α can be written as

$$\alpha = \sum_{t=0}^{\infty}\sum_{i=1}^{p_t}(\alpha_i^t\otimes\beta_i^t - \beta_i^t\otimes\alpha_i^t),$$

where $\alpha_i^t, \beta_i^t \in U(t)$ for all $t \geq 0$ and $i \geq 1$.

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra, and (W, Y_W) be an ordinary V-module, with conformal weight $\lambda \in \mathbb{Q}$. We can construct a semi-direct product vertex algebra $V \rtimes W$ (cf. [50], see also the last section in [27].) As a vector space, $V \rtimes W = V \oplus W$, the vertex operator $Y_{V\rtimes W}$ is given as follows:

$$Y_{V\rtimes W}(a+u, z)(b+v) = (Y(a,z)b) + (Y_W(a,z)v + Y_{WV}^W(u,z)b), \tag{7.1.2}$$

for all $a, b \in V$ and $u, v \in W$, where Y_{WV}^W is defined by the skew-symmetry formula:

$$Y_{WV}^W(v,z)b = e^{zL(-1)}Y_W(b,-z)u. \tag{7.1.3}$$

If W only has integral weights, then $(V \rtimes W, Y_{V\rtimes W}, \mathbf{1}, \omega)$ is a vertex operator algebra. In general, $V \rtimes W$ is only a vertex algebra, and it satisfies all the axioms of a VOA except that $L(0)$ only has integral eigenvalues (see Proposition 2.10 in [50]).

Recall the contragredient modules of a VOA in Section 2.1. Let W be an admissible V-module, and let W' be the graded dual of W: $W' = \bigoplus_{n=0}^{\infty} W(n)^*$. Then $(W', Y_{W'})$ is an admissible V-module, where

$$\langle Y_{W'}(a,z)f, u\rangle = \langle f, Y_W(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})u\rangle, \tag{7.1.4}$$

for all $a \in V$, $f \in W'$ and $u \in W$, see (5.2.4) in [27]. Moreover, the action of $sl(2,\mathbb{C}) = \mathbb{C}L(-1) + \mathbb{C}L(0) + \mathbb{C}L(1)$ satisfies the following properties:

$$\langle L(-1)f, u\rangle = \langle f, L(1)u\rangle, \quad \text{and} \quad \langle L(0)f, u\rangle = \langle f, L(0)u\rangle. \tag{7.1.5}$$

In particular, if (W, Y_W) is an ordinary V-module of conformal weight λ, then $(W', Y_{W'})$ is also an ordinary V-module of the same conformal weight λ, and we can construct the semi-direct product vertex algebra $V \rtimes W'$.

In order to properly define the matrix form of VOYBE, we introduce some new notations of vertex operators based on the definition of contragredient module (7.1.4). Let $(U, Y_U, \mathbf{1}, \omega)$ be a VOA, we define two vertex operators Y'_U and Y'^{op}_U as follows:

$$Y'_U(a,z)b := Y_U(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})b, \tag{7.1.6}$$

$$Y'^{op}_U(a,z)b := Y_U(e^{-zL(1)}(-z^{-2})^{L(0)}a, -z^{-1})e^{zL(1)}b, \tag{7.1.7}$$

for any $a, b, c \in U$.

Definition 7.1.2. Let $(U, Y_U, \mathbf{1}, \omega)$ be a VOA, and $m \in \mathbb{Z}$. We first introduce three m-dot products as follows:

$$\alpha \cdot_m \beta := \mathrm{Res}_z z^m Y_U(\beta, z)\alpha = \beta_m \alpha,$$
$$\alpha \cdot'_m \beta := \mathrm{Res}_z z^m Y'_U(\alpha, z)\beta = \sum_{j\geq 0} \frac{(-1)^{\mathrm{wt}\alpha}}{j!}(L(1)^j\alpha)_{2\mathrm{wt}\alpha - m - j - 2}\beta,$$
$$\alpha \cdot'^{op}_m \beta := \mathrm{Res}_z z^m Y'^{op}_U(\beta, z)\alpha = \sum_{i\geq 0}\sum_{j\geq 0} \frac{(-1)^{\mathrm{wt}\beta + m + i + 1}}{j!i!}(L(1)^j\beta)_{2\mathrm{wt}\beta - m - j - i - 2}L(1)^i\alpha,$$

where $\alpha, \beta \in U$. These products are bilinear.

We denote the grading of the VOA U by $U = \bigoplus_{n=0}^{\infty} U(n)$ and extend the bottom level $U(0)$ by a one-dimensional vector space with a basis element denoted by I, and let

$$\tilde{U} := U \oplus \mathbb{C}I,$$

where $\tilde{U}(0) = U(0) \oplus \mathbb{C}I$, and $\tilde{U}(n) = U(n)$ for all $n \geq 1$. We let I be the identity element with respect to the three m-products:

$$\alpha \cdot_m I = I \cdot_m \alpha = \alpha \cdot'_m I = I \cdot'_m \alpha = \alpha \cdot'^{op}_m I = I \cdot'^{op}_m \alpha = \alpha.$$

Remark 7.1.3. For homogeneous elements $\alpha \in U(s)$ and $\beta \in U(t)$. We observe that $\alpha \cdot_m \beta$, $\alpha \cdot'_m \beta$, and $\alpha \cdot'^{op}_m \beta$ are all homogeneous elements in U, and $\alpha \cdot_m \beta \in U(s+t-m-1)$, $\alpha \cdot'_m \beta \in U(t+m+1-s)$, and $\alpha \cdot'^{op}_m \beta \in U(s+m+1-t)$.

Let $(U, Y_U, \mathbf{1}, \omega)$ be a VOA, and r be a diagonal skew-symmetric two-tensor:

$$r = \sum_{t=0}^{\infty} r^t = \sum_{t=0}^{\infty} \sum_{i=1}^{p_t} \alpha_i^t \otimes \beta_i^t - \beta_i^t \otimes \alpha_i^t,$$

where $r^t = \sum_{i=1}^{p_t} \alpha_i^t \otimes \beta_i^t - \beta_i^t \otimes \alpha_i^t \in U(t) \otimes U(t)$ for every $t \geq 0$. For any $t, s, r \in \mathbb{N}$, we define the elements r_{12}^t, r_{13}^s, and r_{23}^r in $\tilde{U}^{\widehat{\otimes} 3}$ as follows:

$$r_{12}^t := \sum_{i=1}^{p_t} (\alpha_i^t \otimes \beta_i^t \otimes I - \beta_i^t \otimes \alpha_i^t \otimes I), \tag{7.1.8}$$

$$r_{13}^s := \sum_{k=1}^{p_s} (\alpha_k^s \otimes I \otimes \beta_k^s - \beta_k^s \otimes I \otimes \alpha_k^s), \tag{7.1.9}$$

$$r_{23}^r := \sum_{l=1}^{p_r} (I \otimes \alpha_l^r \otimes \beta_l^r - I \otimes \beta_l^r \otimes \alpha_l^r). \tag{7.1.10}$$

Then we define $r_{12} := \sum_{t=0}^{\infty} r_{12}^t \in \tilde{U}^{\widehat{\otimes} 3}$, $r_{13} := \sum_{s=0}^{\infty} r_{13}^s \in \tilde{U}^{\widehat{\otimes} 3}$, and $r_{23} := \sum_{r=0}^{\infty} r_{23}^r \in \tilde{U}^{\widehat{\otimes} 3}$.

For $t, s, r \in \mathbb{N}$, and $m \in \mathbb{Z}$, we define the products $r_{12}^t \cdot_m r_{13}^s$, $r_{23}^r \cdot_m' r_{12}^t$, and $r_{13}^s \cdot_m'^{op} r_{23}^r$ by the distribution rule, with respect to the three products in Definition 7.1.2, respectively:

$$\begin{aligned} r_{12}^t \cdot_m r_{13}^s = \sum_{i,k} (&(\alpha_i^t) \cdot_m \alpha_k^s \otimes \beta_i^t \otimes \beta_k^s - (\alpha_i^t) \cdot_m \beta_k^s \otimes \alpha_i^t \otimes \beta_k^s - (\beta_i^t) \cdot_m \alpha_k^s \otimes \alpha_i^t \otimes \beta_k^s \\ &+ (\beta_i^t) \cdot_m \beta_k^s \otimes \alpha_i^t \otimes \alpha_k^s), \end{aligned} \tag{7.1.11}$$

$$\begin{aligned} r_{23}^r \cdot_m' r_{12}^t = \sum_{l,i} (&\alpha_i^t \otimes (\alpha_l^r) \cdot_m' \beta_i^t \otimes \beta_l^r - \beta_i^t \otimes (\alpha_l^r) \cdot_m' \alpha_i^t \otimes \beta_l^r - \alpha_i^t \otimes (\beta_l^r) \cdot_m' \beta_i^t \otimes \alpha_l^r \\ &+ \beta_i^t \otimes (\beta_l^r) \cdot_m' \alpha_i^t \otimes \alpha_l^r), \end{aligned} \tag{7.1.12}$$

$$\begin{aligned} r_{13}^s \cdot_m'^{op} r_{23}^r = \sum_{k,l} (&\alpha_k^s \otimes \alpha_l^r \otimes (\beta_k^s) \cdot_m'^{op} \beta_l^r - \alpha_k^s \otimes \beta_l^r \otimes (\beta_k^s) \cdot_m'^{op} \alpha_l^r - \beta_k^s \otimes \alpha_l^r \otimes (\alpha_k^s) \cdot_m'^{op} \beta_l^r \\ &+ \beta_k^s \otimes \beta_l^r \otimes (\alpha_k^s) \cdot_m'^{op} \alpha_l^r). \end{aligned} \tag{7.1.13}$$

Then we define:

$$r_{12} \cdot_m r_{13} := \sum_{s,t=0}^{\infty} r_{12}^t \cdot_m r_{13}^s, \quad r_{23} \cdot_m' r_{12} := \sum_{r,t=0}^{\infty} r_{23}^r \cdot_m' r_{12}^t, \text{ and } r_{13} \cdot_m'^{op} r_{23} := \sum_{s,r=0}^{\infty} r_{13}^s \cdot_m'^{op} r_{23}^r.$$

Lemma 7.1.4. *$r_{12} \cdot_m r_{13}$, $r_{23} \cdot_m' r_{12}$, and $r_{13} \cdot_m'^{op} r_{23}$ are well-defined elements in $U^{\widehat{\otimes} 3}$. Let $\alpha = r_{12} \cdot_m r_{13} - r_{23} \cdot_m' r_{12} + r_{13} \cdot_m'^{op} r_{23} \in U^{\widehat{\otimes} 3}$. Then we have:*

$$\alpha = \sum_{s,t=0,\ s+t \geq m+1}^{\infty} (r_{12}^s \cdot_m r_{13}^t - r_{23}^t \cdot_m' r_{12}^{t+s-m-1} + r_{13}^{t+s-m-1} \cdot_m'^{op} r_{23}^s), \tag{7.1.14}$$

where $\alpha_{s,t} = (r_{12}^s \cdot_m r_{13}^t - r_{23}^t \cdot_m' r_{12}^{t+s-m-1} + r_{13}^{t+s-m-1} \cdot_m'^{op} r_{23}^s) \in U(t+s-m-1) \otimes U(s) \otimes U(t)$ *for each pair of* $(s,t) \in \mathbb{N} \times \mathbb{N}$ *such that* $s + t \geq m + 1$.

Proof. By Remark 7.1.3, we have:

$$r_{12}^t \cdot_m r_{13}^s \in U(s+t-m-1) \otimes U(t) \otimes U(s), \tag{7.1.15}$$

$$r_{23}^r \cdot_m' r_{12}^t \in U(t) \otimes U(t+m+1-r) \otimes U(r), \tag{7.1.16}$$

$$r_{13}^s \cdot_m'^{op} r_{23}^r \in U(s) \otimes U(r) \otimes U(s+m+1-r). \tag{7.1.17}$$

By Definition 7.1.1, we have $r_{12} \cdot_m r_{13} = \sum_{t,s=0}^{\infty} r_{12}^t \cdot_m r_{13}^s \in \prod_{t,s=0}^{\infty} U(s+t-m-1) \otimes U(t) \otimes U(s)$, which is a linear subspace of $U^{\widehat{\otimes}3}$. Thus, $r_{12}^t \cdot_m r_{13}^s$, and similarly $r_{23}^r \cdot_m' r_{12}^t$ and $r_{13}^s \cdot_m'^{op} r_{23}^r$, are well-defined elements in $U^{\widehat{\otimes}3}$.

Note that $U(s+t-m-1) = 0$ if $s+t-m-1 < 0$. Change the variable (t,s) in (7.1.15) to (s,t), then we have $r_{12} \cdot_m r_{13} = \sum_{s,t=0,\ s+t\geq m+1}^{\infty} r_{12}^s \cdot_m r_{13}^t$, and $r_{12}^s \cdot_m r_{13}^t \in U(t+s-m-1) \otimes U(s) \otimes U(t)$ for each pair of (s,t). Moreover, we observe that there is a one-to-one correspondence between the following sets:

$$\{(t,r) \in \mathbb{N} \times \mathbb{N} : t+m+1-r \geq 0\} \to \{(s_0,t_0) \in \mathbb{N} \times \mathbb{N} : s_0 + t_0 - m - 1 \geq 0\},$$
$$(t,r) \mapsto (s_0,t_0) = (t+m+1-r, r),$$

whose inverse is given by $(s_0,t_0) \mapsto (t,r) = (s_0+t_0-m-1, t_0)$. Change the variables in (7.1.16),

$$r_{23} \cdot_m' r_{12} = \sum_{r,t=0,\ t+m+1-r\geq 0} r_{23}^r \cdot_m' r_{12}^t = \sum_{s_0,t_0=0,\ s_0+t_0-m-1\geq 0}^{\infty} r_{23}^{t_0} \cdot_m' r_{12}^{t_0+s_0-m-1}, \tag{7.1.18}$$

and $r_{23}^{t_0} \cdot_m' r_{12}^{t_0+s_0-m-1} \in U(t_0+s_0-m-1) \otimes U(s_0) \otimes U(t_0)$ for each pair of (s_0,t_0). Finally, there is a one-to-one correspondence between the following sets:

$$\{(r,s) \in \mathbb{N} \times \mathbb{N} : s+m+1-r \geq 0\} \to \{(s_1,t_1) : s_1 + t_1 - m - 1 \geq 0\},$$
$$(r,s) \mapsto (s_1,t_1) = (r, s+m+1-r),$$

whose inverse is given by $(s_1,t_1) \mapsto (r,s) = (s_1, t_1+s_1-m-1)$. Change the variables in (7.1.17),

$$r_{13} \cdot_m'^{op} r_{23} = \sum_{r,s=0,\ s+m+1-r\geq 0}^{\infty} r_{13}^s \cdot_m'^{op} r_{23}^r = \sum_{s_1,t_1=0,\ s_1+t_1-m-1\geq 0}^{\infty} r_{13}^{t_1+s_1-m-1} \cdot_m'^{op} r_{23}^{s_1}, \tag{7.1.19}$$

and $r_{13}^{t_1+s_1-m-1} \cdot_m'^{op} r_{23}^{s_1} \in U(t_1+s_1-m-1) \otimes U(s_1) \otimes U(t_1)$. Now (7.1.14) follows after we replace the variables (s_0,t_0) in (7.1.18) and (s_1,t_1) in (7.1.19) with (s,t). □

Now we give the definition of the Yang-Baxter equation for VOAs:

Definition 7.1.5. Let $(U, Y_U, \mathbf{1}, \omega)$ be a VOA, and r be a diagonal skew-symmetric two-tensor in $U\widehat{\otimes}U$.

(1) Let $m \in \mathbb{Z}$ be a fixed integer. r is called a solution to the **m-vertex operator Yang-Baxter equation (m-VOYBE)** if

$$r_{12} \cdot_m r_{13} - r_{23} \cdot'_m r_{12} + r_{13} \cdot'^{op}_m r_{23} = 0. \tag{7.1.20}$$

(2) r is called a solution to the **vertex operator Yang-Baxter equation (VOYBE)** if it is a solution to every m-VOYBE for $m \in \mathbb{N}$. In other words, we have:

$$r_{12} \cdot_z r_{13} - r_{23} \cdot_z r_{12} + r_{13} \cdot_z r_{23} = 0, \tag{7.1.21}$$

where we let $r_{12} \cdot_z r_{13} := \sum_{m\in\mathbb{Z}}(r_{12} \cdot_m r_{13})z^{-m-1}$, $r_{23} \cdot_z r_{12} := \sum_{m\in\mathbb{Z}}(r_{23} \cdot'_m r_{12})z^{-m-1}$, and $r_{13} \cdot_z r_{23} := \sum_{m\in\mathbb{Z}}(r_{13} \cdot'^{op}_m r_{23})z^{-m-1}$.

7.1.2 Relative Rota-Baxter operators

Now we introduce the notions of relative Rota-Baxter operators for VOA. It is a generalization of the ordinary Rota-Baxter operators for VOA in Chapter 6. It serves as the operator form of the Yang-Baxter equation for VOA, as in the case of the CYBE for Lie algebras. We first fix some notations. Let $(V, Y, \mathbf{1}, \omega)$ be a VOA, and (W, Y_W) be an admissible V-module. Let $m \in \mathbb{Z}$ be a fixed integer. For $a \in V$ and $u \in W$, we write:

$$a_m u = \mathrm{Res}_z z^m Y_W(a, z)u, \quad \text{and} \quad u(m)a = \mathrm{Res}_z z^m Y^W_{WV}(u, z)a. \tag{7.1.22}$$

Definition 7.1.6. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra, and (W, Y_W) be an admissible V-module. Let $T : W \to V$ be a linear map.

(1) Let $m \in \mathbb{Z}$ be a fixed integer, T is called an **m-relative Rota-Baxter operator (m-relative RBO) or an m--operator** if

$$T(u)_m T(v) = T(T(u)_m v) + T(u(m)T(v)), \tag{7.1.23}$$

for any $u, v \in W$, where $T(u)_m T(v) = \mathrm{Res}_z z^m Y(T(u), z)T(v)$, and $T(u)_m v$ and $u(m)T(v)$ are defined by (7.1.22).

(2) T is called a **relative Rota-Baxter operator (relative RBO) or an -operator** if it is a m-relative RBO for every $m \in \mathbb{Z}$, or equivalently, the following equation holds:

$$Y(Tu, z)Tv = T(Y_W(Tu, z)v + Y_{WV}^W(u, z)Tv), \tag{7.1.24}$$

for any $u, v \in W$, where Y_{WV}^W is given by the skew-symmetry formula (7.1.3).

(3) An m-relative Rota-Baxter operator $T : W \to V$ is called **homogeneous of degree** $N \in \mathbb{Z}$ if $T(W(n)) \subseteq V_{n+N}$ for each $n \in \mathbb{N}$. A degree 0 relative m-RBO is called **level preserving**.

Example 7.1.7. Let $V = V_0 \oplus V_+$ be a CFT-type VOA, and let $P : V_1 \to V_1$ be an RBO of the Lie algebra V_1 of weight 0. Extend P to $T : V \to V$ as follows:

$$T(\mathbf{1}) := \mu\mathbf{1}, \quad T|_{V_1} := P, \quad \text{and} \quad T|_{V_n} := 0, \ \forall n \geq 2, \tag{7.1.25}$$

where $\mu \in \mathbb{C}$ is a fixed number. We claim that $T : V \to V$ is a level-preserving 0-relative RBO. Clearly, T is level-preserving. For any $a \in V$, we have:

$$T(\mathbf{1})_0 T(a) = 0 = T(T(\mathbf{1})_0 a) + T(\mathbf{1}_0 T(a)),$$
$$T(a)_0 T(\mathbf{1}) = 0 = T(a_0 T(\mathbf{1})) + T(T(a)_0 \mathbf{1}),$$

since $\mathbf{1}_0 a = a_0 \mathbf{1} = 0$. On the other hand, for any homogeneous elements $a, b \in V_+$, if either $\mathrm{wt} a > 1$ or $\mathrm{wt} b > 1$, then by (7.1.25) and the fact that $T(a)_0 b$ and $a_0 T(b)$ are contained in $V_{\mathrm{wt} a + \mathrm{wt} b - 1}$, we have:

$$T(a)_0 T(b) = 0 = T(T(a)_0 b) + T(a_0 T(b)).$$

Finally, if $a, b \in V_1$, then clearly we have $T(a)_0 T(b) = T(T(a)_0 b) + T(a_0 T(b))$ since $T = P$ on V_1. Thus $T : V \to V$ is a level-preserving 0-relative RBO.

Example 7.1.8. Let $V = M_{\hat{\mathfrak{h}}}(1, 0)$ be the rank-one Heisenberg VOA (cf. [29]), where $\mathfrak{h} = \mathbb{C}\alpha$ and $(\alpha|\alpha) = 1$. Recall that $V_1 = \mathbb{C}\alpha(-1)\mathbf{1}$ and $V_2 = \mathbb{C}\alpha(-1)\alpha \oplus \mathbb{C}\alpha(-2)\mathbf{1}$. Define $T : V \to V$ as follows:

$$T(\mathbf{1}) = T(\alpha(-1)\mathbf{1}) = 0, \quad T|_{V_n} = 0, \ \forall n \geq 3,$$
$$T(\alpha(-1)\alpha) = \alpha(-1)\alpha + \alpha(-2)\mathbf{1}, \quad \text{and} \quad T(\alpha(-2)\mathbf{1}) = -\alpha(-1)\alpha - \alpha(-2)\mathbf{1}.$$

For any $a, b \in V_2$, since $T(a)_1 b$ and $a_1 T(b)$ are contained in V_2, then by a similar argument as the previous example, we can easily show that $T : V \to V$ is a level-preserving 1-RBO.

Example 7.1.9. Let $V = M_{\widehat{\mathfrak{h}}}(1,0)$ be the rank-one Heisenberg VOA, and let $W = M_{\widehat{\mathfrak{h}}}(1,\lambda)$. Recall that $W = M_{\widehat{\mathfrak{h}}}(1,0) \otimes \mathbb{C}e^{\lambda}$, with $W(0) = \mathbb{C}e^{\lambda}$ and $W(1) = \mathbb{C}\alpha(-1)e^{\lambda}$. Define $T : W \to V$ as follows:

$$T(e^{\lambda}) = 0, \quad T(\alpha(-1)e^{\lambda}) = \mathbf{1}, \quad \text{and} \quad T|_{W(n)} = 0,\ \forall n \geq 2. \tag{7.1.26}$$

Then T is homogeneous of degree -1. If $u, v \in W$ are homogeneous, then $(Tu)_0 v$ and $u(0)T(v)$ are contained in $W(\deg u - 2 + \deg v)$, and by (7.1.26), we have:

$$(Tu)_0(Tv) = 0 = T((Tu)_0 v + u(0)Tv).$$

Thus, $T : W \to V$ is a degree -1 homogeneous 0-relative RBO.

7.1.3 From solutions of the VOYBE to the relative RBOs

The Yang-Baxter equation for VOA is formulated from the axiom of a relative RBO for VOAs. Indeed, let $(U, Y_U, \mathbf{1}, \omega)$ be a VOA, and let $U' = \bigoplus_{t=0}^{\infty} U(t)^*$ be the dual adjoint module of U (cf. [27]). By the notations in Definition 7.1.1, we have the following identifications of vector spaces:

$$D(U\widehat{\otimes}U) = \prod_{t=0}^{\infty} U(t) \otimes U(t) \cong \prod_{t=0}^{\infty} \mathrm{Hom}(U(t)^*, U(t)) \cong \mathrm{Hom}_{LP}(U', U), \tag{7.1.27}$$

where $\mathrm{Hom}_{LP}(U', U) \subset \mathrm{Hom}(U', U)$ is the subspace of level-preserving linear maps.

Let $r = \sum_{t=0}^{\infty} r^t = \sum_{t=0}^{\infty} \sum_{i=1}^{p_t} \alpha_i^t \otimes \beta_i^t - \beta_i^t \otimes \alpha_i^t \in D(U\widehat{\otimes}U)$ be a diagonal skew-symmetric element, and let T_r be the corresponding element in $\mathrm{Hom}_{LP}(U', U)$ under the isomorphism (7.1.20),by definition, T_r is given by

$$T_r(f) := \sum_{i=1}^{p_t} \alpha_i^t \langle f, \beta_i^t \rangle - \beta_i^t \langle f, \alpha_i^t \rangle, \tag{7.1.28}$$

for any $t \in \mathbb{N}$ and $f \in U(t)^*$.

Theorem 7.1.10. *With the settings as above, for a given $m \in \mathbb{Z}$, r is an skew-symmetric solution to the m-VOYBE if and only if $T_r : U' \to U$ is a level-preserving m-relative RBO, that is, for any $f, g \in U'$, the following equation holds:*

$$T_r(f)_m T_r(g) = T_r(T_r(f)_m g) + T_r(f(m)T_r(g)). \tag{7.1.29}$$

In particular, $T_r : U' \to U$ is a level-preserving relative RBO if and only if r is a skew-symmetric solution to the VOYBE.

Proof. For given $a, b, c \in U$, with b, c homogeneous, define a linear functional:

$$a \otimes b \otimes c : U' \otimes U' \to U, \quad (a \otimes b \otimes c)(g \otimes f) := a\langle g, b\rangle\langle f, c\rangle,$$

for any homogeneous $f, g \in U'$.

Let $f \in U(t)^*$ and $g \in U(s)^*$, by (7.1.21), $T_r(g) := \sum_{j=1}^{p_s} \alpha_j^s\langle g, \beta_j^s\rangle - \beta_j^s\langle g, \alpha_j^s\rangle$. For a given $m \in \mathbb{Z}$, by Definition 7.1.2 and (7.1.15), we have:

$$\begin{aligned}
T_r(f)_m T_r(g) &= \sum_{i,j}(\alpha_i^t\langle f, \beta_i^t\rangle - \beta_i^t\langle f, \alpha_i^t\rangle)_m(\alpha_j^s\langle g, \beta_j^s\rangle - \beta_j^s\langle g, \alpha_j^s\rangle)\\
&= \sum_{i,j}(\alpha_i^t)_m\alpha_j^s\langle f, \beta_i^t\rangle\langle g, \beta_j^s\rangle - \sum_{i,j}(\alpha_i^t)_m\beta_j^s\langle f, \beta_i^t\rangle\langle g, \alpha_j^s\rangle\\
&\quad - \sum_{i,j}(\beta_i^t)_m\alpha_j^s\langle f, \alpha_i^t\rangle\langle g, \beta_j^s\rangle + \sum_{i,j}(\beta_i^t)_m\beta_j^s\langle f, \alpha_i^t\rangle\langle g, \alpha_j^s\rangle\\
&= \sum_{i,j}((\alpha_j^s) \cdot_m \alpha_i^t \otimes \beta_j^s \otimes \beta_i^t)(g \otimes f) - \sum_{i,j}((\beta_j^s) \cdot_m \alpha_i^t \otimes \alpha_j^s \otimes \beta_i^t)(g \otimes f)\\
&\quad - \sum_{i,j}((\alpha_j^s) \cdot_m \beta_i^t \otimes \beta_j^s \otimes \alpha_i^t)(g \otimes f) + \sum_{i,j}((\beta_j^s) \cdot_m \beta_i^t \otimes \alpha_j^s \otimes \alpha_i^t)(g \otimes f)\\
&= (r_{12}^s \cdot_m r_{13}^t)(g \otimes f).
\end{aligned}$$

By Definition 7.1.2 and (7.1.24), we have $\langle a_m f, b\rangle = \langle f, a_m' b\rangle = \langle f, a \cdot_m' b\rangle$, for any $a, b \in U$ and $f \in U'$. Since $T_r(T_r(f)_m g) \in U(t+s-m-1)^*$, then by Remark 7.1.3, (7.1.21), and (7.1.16), we have:

$$\begin{aligned}
T_r(T_r(f)_m g) &= \sum_k \alpha_k^{t+s-m-1}\langle T_r(f)_m g, \beta_k^{t+s-m-1}\rangle - \sum_k \beta_k^{t+s-m-1}\langle T_r(f)_m g, \alpha_k^{t+s-m-1}\rangle\\
&= \sum_{k,i} \alpha_k^{t+s-m-1}\langle g, (\alpha_i^t) \cdot_m' \beta_k^{t+s-m-1}\rangle\langle f, \beta_i^t\rangle - \sum_{k,i} \alpha_k^{t+s-m-1}\langle g, (\beta_i^t) \cdot_m' \beta_k^{t+s-m-1}\rangle\langle f, \alpha_i^t\rangle\\
&\quad - \sum_{k,i} \beta_k^{t+s-m-1}\langle g, (\alpha_i^t) \cdot_m' \alpha_k^{t+s-m-1}\rangle\langle f, \beta_i^t\rangle + \sum_{k,i} \beta_k^{t+s-m-1}\langle g, (\beta_i^t) \cdot_m' \alpha_k^{t+s-m-1}\rangle\langle f, \alpha_i^t\rangle\\
&= \sum_{k,i}((\alpha_k^{t+s-m-1} \otimes (\alpha_i^t) \cdot_m' \beta_k^{t+s-m-1} \otimes \beta_i^t) - (\alpha_k^{t+s-m-1} \otimes (\beta_i^t) \cdot_m' \beta_k^{t+s-m-1} \otimes \alpha_i^t))(g \otimes f)\\
&\quad - \sum_{k,i}((\beta_k^{t+s-m-1} \otimes (\alpha_i^t) \cdot_m' \alpha_k^{t+s-m-1} \otimes \beta_i^t) + (\beta_k^{t+s-m-1} \otimes (\beta_i^t) \cdot_m' \alpha_k^{t+s-m-1} \otimes \alpha_i^t))(g \otimes f)\\
&= (r_{23}^t \cdot_m' r_{12}^{t+s-m-1})(g \otimes f).
\end{aligned}$$

Finally, by Definition 7.1.2, (7.1.3), (7.1.24), and the fact that $\langle L(-1)f, a\rangle = \langle f, L(-1)a\rangle$, for any $f \in U'$ and $a \in U$ (see (5.2.10) in [27]), we have:

$$\begin{aligned}\langle f(m)a, b\rangle &= \mathrm{Res}_z z^m \langle Y^{U'}_{U'U}(f,z)a, b\rangle = \mathrm{Res}_z z^m \langle e^{zL(-1)} Y_{U'}(a,-z) f, b\rangle \\ &= \mathrm{Res}_z z^m \langle f, Y_U(e^{-zL(1)}(-z^{-2})^{L(0)} a, -z^{-1}) e^{zL(1)} b\rangle = \mathrm{Res}_z z^m \langle f, Y'^{op}_U(a,z) b\rangle \\ &= \langle f, b \cdot'^{op}_m a\rangle,\end{aligned}$$

for any $f \in U'$ and $a, b \in U$. Since $Y^{U'}_{U'U}$ is an intertwining operator (see Section 5.4 in [27]) and T_r is level-preserving, we have $T_r(f(m)T_r(g)) \in U(t+s-m-1)^*$, then

$$\begin{aligned}&T_r(f(m)T_r(g)) = \sum_k \alpha_k^{t+s-m-1}\langle f(m)T_r(g), \beta_k^{t+s-m-1}\rangle - \beta_k^{t+s-m-1}\langle f(m)T_r(g), \alpha_k^{t+s-m-1}\rangle \\ &= \sum_{k,j} \alpha_k^{t+s-m-1}\langle f, (\beta_k^{t+s-m-1}) \cdot'^{op}_m \alpha_j^s\rangle\langle g, \beta_j^s\rangle - \sum_{k,j} \alpha_k^{t+s-m-1}\langle (\beta_k^{t+s-m-1}) \cdot'^{op}_m \beta_j^s\rangle\langle g, \alpha_j^s\rangle \\ &- \sum_{k,j} \beta_k^{t+s-m-1}\langle f, (\alpha_k^{t+s-m-1}) \cdot'^{op}_m \alpha_j^s\rangle\langle g, \beta_j^s\rangle + \sum_{k,j} \beta_k^{t+s-m-1}\langle f, (\alpha_k^{t+s-m-1}) \cdot'^{op}_m \beta_j^s\rangle\langle g, \alpha_j^s\rangle \\ &= \sum_{k,j}((\alpha_k^{t+s-m-1} \otimes \beta_j^s \otimes (\beta_k^{t+s-m-1}) \cdot'^{op}_m \alpha_j^s) - (\alpha_k^{t+s-m-1} \otimes \alpha_j^s \otimes (\beta_k^{t+s-m-1}) \cdot'^{op}_m)\beta_j^s)(g \otimes f) \\ &- \sum_{k,j}((\beta_k^{t+s-m-1} \otimes \beta_j^s \otimes (\alpha_k^{t+s-m-1}) \cdot'^{op}_m \alpha_j^s) + (\beta_k^{t+s-m-1} \otimes \alpha_j^s \otimes (\alpha_k^{t+s-m-1}) \cdot'^{op}_m \beta_j^s))(g \otimes f) \\ &= -(r_{13}^{t+s-m-1} \cdot'^{op}_m r_{23}^s)(g \otimes f).\end{aligned}$$

Thus, for $f \in U(t)^*$ and $g \in U(s)^*$, we have:

$$\begin{aligned}&T_r(f)_m T_r(g) - T_r(T_r(f)_m g) - T_r(f(m)T_r(g)) \\ &= (r_{12}^s \cdot_m r_{13}^t - r_{23}^t \cdot'_m r_{12}^{t+s-m-1} + r_{13}^{t+s-m-1} \cdot'^{op}_m r_{23}^s)(g \otimes f),\end{aligned} \tag{7.1.30}$$

which is contained in $U(t+s-m-1)$. Now T_r satisfies (7.1.29) if and only if $T_r(f)_m T_r(g) - T_r(T_r(f)_m g) - T_r(f(m)T_r(g)) = 0$ for any $s, t \geq 0$ and $f \in U(t)^*$, $g \in U(s)^*$, and by (7.1.30), this is true if and only if

$$\begin{aligned}&\langle T_r(f)_m T_r(g) - T_r(T_r(f)_m g) - T_r(f(m)T_r(g)), h\rangle \\ &= \langle (r_{12}^s \cdot_m r_{13}^t - r_{23}^t \cdot'_m r_{12}^{t+s-m-1} + r_{13}^{t+s-m-1} \cdot'^{op}_m r_{23}^s)(g \otimes f), h\rangle \\ &= \langle (r_{12}^s \cdot_m r_{13}^t - r_{23}^t \cdot'_m r_{12}^{t+s-m-1} + r_{13}^{t+s-m-1} \cdot'^{op}_m r_{23}^s), g \otimes f \otimes h\rangle,\end{aligned}$$

for any $s, t \geq 0$, $f \in U(t)^*$, $g \in U(s)^*$, and $h \in U(t+s-m-1)^*$. Since the element $r_{12}^s \cdot_m r_{13}^t - r_{23}^t \cdot'_m r_{12}^{t+s-m-1} + r_{13}^{t+s-m-1} \cdot'^{op}_m r_{23}^s \in U^{\widehat{\otimes} 3}$ is homogeneous and contained in $U(s+t-m-1) \otimes U(s) \otimes U(t)$

by Lemma (7.1.4), and the space $U(s+t-m-1)\otimes U(s)\otimes U(t)$ is finite dimensional, it follows that T_r satisfies (7.1.29) if and only if $r_{12}^s \cdot_m r_{13}^t - r_{23}^t \cdot'_m r_{12}^{t+s-m-1} + r_{13}^{t+s-m-1} \cdot'^{op}_m r_{23}^s = 0$, for every pair of $(s,t)\in\mathbb{N}\times\mathbb{N}$, with $s+t\geq m+1$, and this is true if and only if

$$r_{12} \cdot_m r_{13} - r_{23} \cdot'_m r_{12} + r_{13} \cdot'^{op}_m r_{23} = 0$$

by Lemma 7.1.4. □

The following Corollary follows immediately from the proof of Theorem 7.1.10. It will be used in Section 4.

Corollary 7.1.11. *With the settings as above, for fixed integers $m\in\mathbb{Z}$ and $s,t\in\mathbb{N}$, the restriction of T_r onto $U(s)^*$ and $U(t)^*$ satisfies* (7.1.29)*, with $f\in U(s)^*$ and $g\in U(t)^*$, if and only if the homogeneous components of $r=\sum_{t=0}^{\infty} r^t$ satisfies*

$$r_{12}^s \cdot_m r_{13}^t - r_{23}^t \cdot'_m r_{12}^{t+s-m-1} + r_{13}^{t+s-m-1} \cdot'^{op}_m r_{23}^s = 0 \tag{7.1.31}$$

in the homogeneous subspace $U(s+t-m-1)\otimes U(s)\otimes U(t)$ of $U^{\widehat{\otimes}3}$.

When $U'\cong U$ as a U-module, let $\varphi: U\to U'$ be an isomorphism, then by Proposition 5.3.6 in [27], there exists a non-degenerate symmetric invariant bilinear form $(\cdot|\cdot): U\times U\to\mathbb{C}$ such that

$$\langle\varphi(a), b\rangle = (a|b), \quad \forall a,b\in U. \tag{7.1.32}$$

Moreover, since $\varphi(Y_U(a,z)b) = Y_{U'}(a,z)\varphi(b)$ and $\varphi L(-1) = L(-1)\varphi$, if $T: U'\to U$ is a relative RBO, then for any $a,b\in U$, we have:

$$\begin{aligned} Y_U((T\varphi)(a),z)(T\varphi)(b) &= T(Y_{U'}((T\varphi)(a),z)\varphi(b)) + T(Y_{U'U}^{U'}(\varphi(a),z)(T\varphi)(b)) \\ &= T\varphi(Y_U((T\varphi)(a),z)b) + T\left(e^{zL(-1)}Y_{U'}((T\varphi)(b),-z)\varphi(a)\right) \\ &= (T\varphi)(Y_U((T\varphi)(a),z)b) + (T\varphi)(Y_U(a,z)(T\varphi)(b)). \end{aligned}$$

Thus, $T\varphi: U\to U$ is an ordinary RBO of weight 0. Clearly, the converse is also true, and by taking $\mathrm{Res}_z z^m$, we have $T: U'\to U$ is an m-relative RBO if and only if $T\varphi: U\to U$ is an m-ordinary RBO of weight 0 (see [7]). Moreover, for $T_r: U'\to U$ given by (7.1.21), denote $T_r\varphi$ by $\tilde{T}_r$, then by (7.1.32) we have:

$$\tilde{T}_r(a) = \sum_{i=1}^{p_t} \alpha_i^t(a|\beta_i^t) - \beta_i^t(a|\alpha_i^t), \tag{7.1.33}$$

for any $a\in U(t)$ and $t\in\mathbb{N}$. Hence we have the following:

Corollary 7.1.12. *Let $\varphi : U \to U'$ be an isomorphism of modules over the VOA U, $m \in \mathbb{Z}$ be an integer, and let $r = \sum_{t=0}^{\infty} \sum_{i=1}^{p_t} \alpha_i^t \otimes \beta_i^t - \beta_i^t \otimes \alpha_i^t \in D(U \widehat{\otimes} U)$ be a diagonal skew-symmetric element. Then r is a solution to the m-VOYBE if and only if $\tilde{T}_r : U \to U$ defined by* (7.1.33) *is an m-ordinary RBO on the VOA U of weight* 0.

In fact, Theorem 7.1.10 gives rise to a one-to-one correspondence between the solutions to the VOYBE and certain relative RBOs. We end the section by making this precise.

Definition 7.1.13. Let $(U, Y_U, \mathbf{1}, \omega)$ be a VOA. We call a level-preserving linear map $T : U' \to U$ **skew-symmetric** if T satisfies

$$\langle T(f), g\rangle = -\langle f, T(g)\rangle, \quad t \in \mathbb{N}, f, g \in U(t)^*.$$

Let $\mathrm{Hom}_{\mathrm{LP}}^{\mathrm{sk}}(U', U)$ denote the subspace of skew-symmetric level-preserving linear maps.

It is easy to check that $T_r : U' \to U$ from (7.1.28) is skew-symmetric. We also state the simple fact:

Lemma 7.1.14. *The linear bijection* (7.1.27) *restricts to a linear bijection*

$$\Phi : \mathrm{SD}(U \widehat{\otimes} U) \cong \mathrm{Hom}_{\mathrm{LP}}^{\mathrm{sk}}(U', U), \quad r \mapsto T_r, \tag{7.1.34}$$

where T_r is defined by (7.1.28)*. The inverse of Φ is given by*

$$\Psi : \mathrm{Hom}_{\mathrm{LP}}^{\mathrm{sk}}(U', U) \to \mathrm{SD}(U \widehat{\otimes} U), \quad T \mapsto \frac{1}{2} \sum_{t=0}^{\infty} \sum_{i=1}^{p_t} \Big(T((v_i^t)^*) \otimes v_i^t - v_i^t \otimes T((v_i^t)^*) \Big), \tag{7.1.35}$$

where $\{v_1^t, \dots, v_{p_t}^t\}$ is a basis for $U(t)$, and $\{(v_1^t)^*, \dots, (v_{p_t}^t)^*\}$ is the dual basis of $U(t)^*$ for $t \in \mathbb{N}$.

Then by Theorem 7.1.10, we obtain

Corollary 7.1.15. *For $m \in \mathbb{Z}$, let $\mathrm{SD}_{\mathrm{sol}}(U \widehat{\otimes} U)(m)$ denote the set of skew-symmetric solutions to the m-VOYBE, and let $\mathrm{RBO}_{\mathrm{LP}}^{\mathrm{sk}}(U', U)(m)$ denote the set of level-preserving skew-symmetric m-relative RBOs. Then Φ in* (7.1.34) *restricts to a bijection*

$$\Phi : \mathrm{SD}_{\mathrm{sol}}(U \widehat{\otimes} U)(m) \leftrightarrow \mathrm{RBO}_{\mathrm{LP}}^{\mathrm{sk}}(U', U)(m), \quad r \mapsto T_r. \tag{7.1.36}$$

7.2 Solving the vertex operator Yang-Baxter equation

Theorem 7.1.10 indicates that we can construct a m-relative RBO $T_r : U' \to U$ from a solution of the m-VOYBE in the VOA U. In fact, we can also use some special m-relative RBOs to reconstruct solutions to the m-VOYBE, just like the case of Lie algebras (cf. [4]) and associative algebras (cf. [8]).

7.2.1 Some actions on the contragredient modules

Let $(V, Y, \mathbf{1}, \omega)$ be a VOA, and (W, Y_W) be an ordinary V-module of conformal weight $\lambda \in \mathbb{Q}$. Since our focusing point is to solve the VOYBE from some special relative RBOs $T : W \to V$, we assume that the conformal weight $\lambda = 0$. Then the contragredient module $(W', Y_{W'})$ is an ordinary V-module of the same conformal weight 0.

Then the semi-direct product $(V \rtimes W', Y_{V\rtimes W'}, \mathbf{1})$ we recalled in section 7.1.1 is a vertex operator algebra, where $Y_{V\rtimes W'}$ is given by (7.1.2), and we can write it as:

$$Y_{V\rtimes W'}(a + f, z)(b + g) = (Y(a, z)b) + (Y_{W'}(a, z)g + Y^{W'}_{W'V}(f, z)b), \tag{7.2.1}$$

for any $a, b \in V$ and $f, g \in W'$. Denote the vertex algebra $(V \rtimes W', Y_{V\rtimes W'}, \mathbf{1}, \omega)$ by $(U, Y_U, \mathbf{1}, \omega)$.

We denote the VOA $(V\rtimes W', Y_{V\rtimes W'}, \mathbf{1}, \omega)$ by $(U, Y_U, \mathbf{1}, \omega)$. Since the conformal weight of W' is 0, the admissible gradation $W' = \bigoplus_{n=0}^{\infty} W(n)^*$ is the same as the $L(0)$-eigenspace gradation, and the gradation on U is given by

$$U(n) := V_n \oplus W(n)^*, \quad n \in \mathbb{N}.$$

Then $U = \bigoplus_{n=0}^{\infty} U(n)$. Observe that for a given level $n \in \mathbb{N}$, the homogeneous space $U(n)\otimes U(n)$ can be decomposed as

$$U(n) \otimes U(n) = (V_n \otimes V_n) \oplus (V_n \otimes W(n)^*) \oplus (W(n)^* \otimes V_n) \oplus (W(n)^* \otimes W(n)^*). \tag{7.2.2}$$

Then we have $V_n \otimes W(n)^* \subset U(n) \otimes U(n)$.

Recall that a linear map $T : W \to V$ is called level-preserving if it satisfies $T(W(n)) \subseteq V_n$ for $n \in \mathbb{N}$. We let $\mathrm{Hom}_{\mathrm{LP}}(W, V)$ denote the space of level-preserving linear maps from W to V. For $n \in \mathbb{N}$, let $\{v^n_1, v^n_2, \dots, v^n_{p_n}\}$ be a basis of $W(n)$, and let $\{(v^n_1)^*, (v^n_2)^*, \dots, (v^n_{p_n})^*\}$ be the dual basis of $W(n)^*$. Similar to (7.1.20), there is a natural isomorphism of vector spaces

$$\mathrm{Hom}_{\mathrm{LP}}(W, V) = \prod_{t=0}^{\infty} \mathrm{Hom}(W(t), V_t) \cong \prod_{t=0}^{\infty} V_t \otimes W(t)^* \subset \prod_{t=0}^{\infty} U(t)\widehat{\otimes} U(t) = D(U\widehat{\otimes} U),$$

where $T \in \mathrm{Hom}_{\mathrm{LP}}(W, V)$ corresponds to $\sum_{t=0}^{\infty}\sum_{i=1}^{p_t} T(v_i^t) \otimes (v_i^t)^*$.

From now on, we fix a level-preserving linear map $T : W \to V$. By Eq. (7.2.2), we have $T = \sum_{t=0}^{\infty}\sum_{i=1}^{p_t} T(v_i^t) \otimes (v_i^t)^* \in V_t \otimes W(t)^*$. Hence we have $\sigma(T) = \sum_{t=0}^{\infty}\sum_{i=1}^{p_t} (v_i^t)^* \otimes T(v_i^t) \in W(t)^* \otimes V_t$. Therefore we take the skew-symmetrization

$$r := r_T := T - \sigma(T) = \sum_{t=0}^{\infty}\sum_{i=1}^{p_t} T(v_i^t) \otimes (v_i^t)^* - (v_i^t)^* \otimes T(v_i^t) \in \mathrm{SD}((V \rtimes W')\widehat{\otimes}(V \rtimes W')). \quad (7.2.3)$$

In order to relate that m-VOYBE for such r with certain axioms satisfied by T, we need the following Lemmas:

Lemma 7.2.1. *With the setting as above, for homogeneous $a \in V$, we have:*

$$\sum_i T(v_i^t) \otimes a'_m (v_i^t)^* = \sum_j T(a_m v_j^{t+m+1-\mathrm{wt}a}) \otimes (v_j^{t+m+1-\mathrm{wt}a})^*, \quad (7.2.4)$$

$$\sum_i a'^{op}_m (v_i^t)^* \otimes T(v_i^t) = \sum_j (v_j^{t+m+1-\mathrm{wt}a})^* \otimes T((v_j^{t+m+1-\mathrm{wt}a})(m)a), \quad (7.2.5)$$

$$\sum_k a_m (v_k^s)^* \otimes T(v_k^s) = \sum_j (v_j^{\mathrm{wt}a-m-1+s})^* \otimes T(a'_m v_j^{\mathrm{wt}a-m-1+s}), \quad (7.2.6)$$

$$\sum_k (v_k^s)^*(m)a \otimes T(v_k^s) = \sum_j (v_j^{\mathrm{wt}a-m-1+s})^* \otimes T(a'^{op}_m v_j^{\mathrm{wt}a-m-1+s}). \quad (7.2.7)$$

Proof. Note that $a'_m(v_i^t)^* \in W(t+m+1-\mathrm{wt}a)^*$, then by (7.1.6), (7.1.7), and Definition 7.1.2, we have:

$$\begin{aligned}
&\sum_i T(v_i^t) \otimes a'_m (v_i^t)^* \\
&= \sum_i \sum_j T(v_i^t) \otimes \langle a'_m (v_i^t)^*, v_j^{t+m+1-\mathrm{wt}a}\rangle (v_j^{t+m+1-\mathrm{wt}a})^* \\
&= \sum_i \sum_j T(v_i^t) \otimes \mathrm{Res}_z z^m \langle Y_{W'}(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})(v_i^t)^*, v_j^{t+m+1-\mathrm{wt}a}\rangle (v_j^{t+m+1-\mathrm{wt}a})^* \\
&= \sum_i \sum_j T(v_i^t) \otimes \mathrm{Res}_z z^m \langle (v_i^t)^*, Y_W(a,z) v_j^{t+m+1-\mathrm{wt}a}\rangle (v_j^{t+m+1-\mathrm{wt}a})^* \\
&= \sum_i \sum_j T(\langle (v_i^t)^*, a_m v_j^{t+m+1-\mathrm{wt}a}\rangle v_i^t) \otimes (v_j^{t+m+1-\mathrm{wt}a})^* \\
&= \sum_j T(a_m v_j^{t+m+1-\mathrm{wt}a}) \otimes (v_j^{t+m+1-\mathrm{wt}a})^*.
\end{aligned}$$

This proves (7.1.19). Since we also have $a'^{op}_m (v_i^t)^* \in W(t+m+1-\mathrm{wt}a)^*$, then

$$\sum_i a'^{op}_m (v_i^t)^* \otimes T(v_i^t)$$

$$= \sum_i \sum_j \langle a_m'^{op}(v_i^t)^*, v_j^{t+m+1-\mathrm{wt}a} \rangle (v_j^{t+m+1-\mathrm{wt}a})^* \otimes T(v_i^t)$$

$$= \sum_{i,j} \mathrm{Res}_z z^m \langle Y_{W'}(e^{-zL(1)}(-z^{-2})^{L(0)}a, -z^{-1}) e^{zL(1)}(v_i^t)^*, v_j^{t+m+1-\mathrm{wt}a} \rangle (v_j^{t+m+1-\mathrm{wt}a})^* \otimes T(v_i^t)$$

$$= \sum_{i,j} \mathrm{Res}_z z^m \langle (v_i^t)^*, e^{zL(-1)} Y_W(a, -z) v_j^{t+m+1-\mathrm{wt}a} \rangle (v_j^{t+m+1-\mathrm{wt}a})^* \otimes T(v_i^t)$$

$$= \sum_{i,j} \mathrm{Res}_z z^m \langle (v_i^t)^*, Y_{WV}^W(v_j^{t+m+1-\mathrm{wt}a}, z)a \rangle (v_j^{t+m+1-\mathrm{wt}a})^* \otimes T(v_i^t)$$

$$= \sum_{i,j} (v_j^{t+m+1-\mathrm{wt}a})^* \otimes T(\langle (v_i^t)^*, (v_j^{t+m+1-\mathrm{wt}a})(m)a \rangle v_i^t)$$

$$= \sum_j (v_j^{t+m+1-\mathrm{wt}a})^* \otimes T((v_j^{t+m+1-\mathrm{wt}a})(m)a).$$

This proves (7.2.5). The proof of (7.2.6) and (7.2.7) are similar, just observe that $a_m(v_k^s)^*$ and $(v_k^s)^*(m)a$ are both contained in $W(\mathrm{wt}a - m - 1 + s)^*$, we omit the details of the proof. ◻

Lemma 7.2.2. *With the setting as above, for any homogeneous $a \in V$, we have:*

$$\sum_l ((v_l^r)^*)'(m)a \otimes T(v_l^r) = \sum_j \mathrm{Res}_z z^{-m-2-4\lambda}(-1)^{m+1}(v_j^{\mathrm{wt}a+m+1-r})^* \otimes \\ T\left(Y_{WV}^W(e^{z^{-1}L(1)}(-z^2)^{L(0)} v_j^{\mathrm{wt}a+m+1-r}, z)e^{-z^{-1}L(1)}a\right), \tag{7.2.8}$$

where $((v_l^r)^*)'(m)a = \mathrm{Res}_z z^m Y_{W'V}^{W'}(e^{zL(1)}(-z^{-2})^{L(0)}(v_l^r)^*, z^{-1})a$.

$$\sum_l T(v_l^r) \otimes ((v_l^r)^*)'^{op}(m)a = \sum_j \mathrm{Res}_z z^{-m-2-4\lambda}(-1)^{m+1}(v_j^{\mathrm{wt}a+m+1-r})^* \otimes \\ T\left(Y_{WV}^W(e^{-z^{-1}L(1)}(-z^2)^{L(0)} v_j^{\mathrm{wt}a+m+1-r}, -z)a\right), \tag{7.2.9}$$

where $((v_l^r)^*)'^{op}(m)a = \mathrm{Res}_z z^m Y_{W'V}^{W'}(e^{-zL(1)}(-z^{-2})^{L(0)}(v_l^r)^*, -z^{-1})e^{zL(1)}a$.

Proof. We prove (7.2.8) first. Since the conformal weight of W is 0, we have:

$$(-z^2)^{L(0)}(v_j^{\mathrm{wt}a+m+1-r}) = (-1)^{\mathrm{wt}a+m+1-r+\lambda} z^{2\mathrm{wt}a+2m+2-2r+2\lambda} v_j^{\mathrm{wt}a+m+1-r},$$
$$(-z^{-2})^{L(0)}(v_l^r)^* = (-1)^{r+\lambda} z^{-2r-2\lambda}(v_r^l)^*,$$

where we fix a root of unity to define $(-1)^\lambda$. Moreover, $((v_l^r)^*)'(m)a \in W(\mathrm{wt}a + m + 1 - r)^*$ for any $r \geq 0$, then we have:

$$\sum_l ((v_l^r)^*)'(m)a \otimes T(v_l^r) = \sum_{j,l} \langle ((v_l^r)^*)'(m)a, v_j^{m+1-r+\mathrm{wt}a} \rangle (v_j^{m+1-r+\mathrm{wt}a})^* \otimes T(v_l^r)$$

$$= \sum_{j,l} \text{Res}_z z^m (v_j^{\text{wt}a+m+1-r})^* \otimes \langle Y_{W'V}^{W'}(e^{zL(1)}(-z^{-2})^{L(0)}(v_l^r)^*, z^{-1})a, v_j^{\text{wt}a+m+1-r}\rangle T(v_l^r)$$
$$= \sum_{j,l} \text{Res}_z z^m (v_j^{\text{wt}a+m+1-r})^* \otimes \langle e^{z^{-1}L(1)} Y_{W'}(a, -z^{-1}) e^{zL(1)}(-z^{-2})^{L(0)}(v_l^r)^*, v_j^{\text{wt}a+m+1-r}\rangle T(v_l^r)$$
$$= \sum_{j,l} \text{Res}_z z^m (v_j^{\text{wt}a+m+1-r})^* \otimes \langle (-z^{-2})^{L(0)}(v_l^r)^*,$$
$$e^{zL(-1)} Y_W(e^{-z^{-1}L(1)}(-z^2)^{L(0)}a, -z) e^{z^{-1}L(1)} v_j^{\text{wt}a+m+1-r}\rangle T(v_l^r)$$
$$= \sum_{j,l} \text{Res}_z z^m (v_j^{\text{wt}a+m+1-r})^* \otimes \langle (-1)^{r+\lambda} z^{-2r-2\lambda}(v_l^r)^*,$$
$$Y_{WV}^W(e^{z^{-1}L(1)} v_j^{\text{wt}a+m+1-r}, z) e^{-z^{-1}L(1)} (-1)^{\text{wt}a} z^{2\text{wt}a} a\rangle T(v_l^r)$$
$$= \sum_{j,l} \text{Res}_z z^{-m-2-4\lambda} (-1)^{m+1} (v_j^{\text{wt}a+m+1-r})^* \otimes \langle (v_l^r)^*,$$
$$Y_{WV}^W(e^{z^{-1}L(1)}(-z^2)^{L(0)} v_j^{\text{wt}a+m+1-r}, z) e^{-z^{-1}L(1)} a\rangle T(v_l^r)$$
$$= \sum_{j} \text{Res}_z z^{-m-2-4\lambda} (-1)^{m+1} (v_j^{\text{wt}a+m+1-r})^* \otimes T\left(Y_{WV}^W(e^{z^{-1}L(1)}(-z^2)^{L(0)} v_j^{\text{wt}a+m+1-r}, z) e^{-z^{-1}L(1)} a, \right).$$

This proves (7.2.8), the proof of (7.2.9) is similar, we omit the details. □

7.2.2 Strong relative RBO and the solutions in the semi-direct product

The level-preserving linear map $T : W \to V$ has the coadjoint map:

$$T^* : V' \to W', \quad \langle T^*(f), u\rangle := \langle f, T(u)\rangle, \tag{7.2.10}$$

for any homogeneous $f \in V'$ and $u \in V$. Clearly, T^* is also a level-preserving map between the contragredient modules W' and V'.

We define an intertwining operator $Y_{WW'}^{V'} \in I\binom{V'}{W\,W'}$ (see Section 5.4 in [27]) as follows: for $u \in W$, $u^* \in W'$, and $a \in V$, let

$$\langle Y_{WW'}^{V'}(u, z)u^*, a\rangle := z^{-4\lambda}\langle u^*, Y_{WV}^W(e^{zL(1)}(-z^{-2})^{L(0)}u, z^{-1})a\rangle, \tag{7.2.11}$$

where λ is the conformal weight of W. Then we use it to define $Y_{W'W}^{V'} \in I\binom{V'}{W'\,W}$ by the skew-symmetry formula:

$$Y_{W'W}^{V'}(u^*, z)u := e^{zL(-1)} Y_{WW'}^{V'}(u, -z)u^*. \tag{7.2.12}$$

We define the intertwining operators $Y_{VW'}^{W'}$ and $Y_{VV'}^{V'}$ by the adjoint formulas (see (5.5.4) in [27]), then define $Y_{W'V}^{W'}$ and $Y_{V'V}^{V'}$ by the skew-symmetry formula.

Definition 7.2.3. Let $(V, Y, \mathbf{1}, \omega)$ be a VOA, and (W, Y_W) be an ordinary V-module of conformal weight $\lambda \in \mathbb{Q}$. Let $T \in \mathrm{Hom}_{LP}(W, V)$, and $T^* : V' \to W'$ be given by (7.2.10).

(1) Let $m \in \mathbb{Z}$. T is called a m**-strong relative Rota-Baxter operator**, if T is a level-preserving relative RBO, and T and T^* satisfy the following compatibility axioms for any $u \in W$ and $f \in V'$:

$$\mathrm{Res}_z z^m \left(Y_{W'}(T(u), z)T^*(f) - T^*(Y_{VV'}^{V'}(T(u), z)f) - T^*(Y_{WW'}^{V'}(u, z)T^*(f)) \right) = 0, \quad (7.2.13)$$

$$\mathrm{Res}_z z^m \left(Y_{W'V}^{W'}(T^*(f), z)T(u) - T^*(Y_{V'V}^{V'}(f, z)T(u)) - T^*(Y_{W'W}^{V'}(T^*(f), z)u) \right) = 0. \quad (7.2.14)$$

(2) T is called a **strong relative Rota-Baxter operator**, if T is a m-strong relative Rota-Baxter operator.

Remark 7.2.4. (1) In Definition 7.2.3, if $T^* : V' \to W'$ is commutative with the operator $L(-1)$, then (7.2.14) follows from (7.2.13) since $Y_{W'V}^{W'}$, $Y_{V'V}^{V'}$, and $Y_{W'W}^{V'}$ are defined by the skew-symmetry formulas.
(2) If $W = V$ the adjoint V-module and $V \cong V'$, then we have $\lambda = 0$, and all of the intertwining operators in (7.2.13) and (7.2.14) are Y_V. In this case, a m-strong relative RBO $T : V \to V$ is just an ordinary m-RBO on the VOA V.

The following Theorem shows that a strong relative RBO can give rise to solutions to the VOYBE.

Theorem 7.2.5. *Let V be a VOA, W be an (ordinary) V-module, $U = V \rtimes W'$ be the semi-direct product vertex algebra, $T \in \mathrm{Hom}_{LP}(W, V)$ be a level-preserving linear operator, and r be $T - T^{21} \in U^{\widehat{\otimes} 2}$ as* (7.2.3)*. Define r_{12}, r_{13}, and r_{23} as in* (7.1.8),(7.1.9)*, and* (7.1.10)*:*

$$r_{12} := \sum_{t=0}^{\infty} \sum_{i=1}^{p_t} T(v_i^t) \otimes (v_i^t)^* \otimes I - (v_i^t)^* \otimes T(v_i^t) \otimes I,$$

$$r_{13} := \sum_{s=0}^{\infty} \sum_{k=1}^{p_s} T(v_k^s) \otimes I \otimes (v_k^s)^* - (v_k^s)^* \otimes I \otimes T(v_k^s),$$

$$r_{23} := \sum_{r=0}^{\infty} \sum_{l=1}^{p_r} I \otimes T(v_l^r) \otimes (v_l^r)^* - I \otimes (v_l^r)^* \otimes T(v_l^r).$$

Let $m \in \mathbb{Z}$. Then r satisfies the m-VOYBE (7.1.20) *in the vertex algebra $U = V \rtimes W'$ if and only if $T : W \to V$ is a strong m-relative RBO.*

In particular, r is a solution to the VOYBE in the vertex algebra U if and only if $T : W \to V$ is a strong relative RBO.

Proof. By Definition 7.1.2, (7.2.1), formulas (7.2.6) and (7.2.7) in Lemma 7.2.1, and the fact that $(v_i^t)^* \cdot_m (v_k^s)^* = 0$ in U for all t, s, i, k, we have:

$$\begin{aligned}
r_{12} \cdot_m r_{13} &= \sum_{s,t=0}^{\infty} \sum_{i,k} \big(T(v_k^s)_m T(v_i^t) \otimes (v_i^t)^* \otimes (v_k^s)^* - T(v_k^s)_m (v_i^t)^* \otimes T(v_i^t) \otimes (v_k^s)^* \\
&\quad - (v_k^s)^*(m) T(v_i^t) \otimes (v_i^t)^* \otimes T(v_k^s) \big) \\
&= \underbrace{\sum_{s,t=0}^{\infty} \sum_{i,k} T(v_k^s)_m T(v_i^t) \otimes (v_i^t)^* \otimes (v_k^s)^*}_{(11)} - \underbrace{\sum_{s,t=0}^{\infty} \sum_{k,j} (v_j^{t-m-1+s})^* \otimes T(T(v_k^s)'_m v_j^{t-m-1+s}) \otimes (v_k^s)^*}_{(12)} \\
&\quad - \underbrace{\sum_{s,t=0}^{\infty} \sum_{i,j} (v_j^{s-m-1+t})^* \otimes (v_i^t)^* \otimes T(T(v_i^t)'^{op}_m v_j^{s-m-1+t})}_{(13)} \\
&= (11) + (12) + (13).
\end{aligned}$$

By Definition 7.1.2, (7.2.8) in Lemma 7.2.2, (7.2.4) in Lemma 7.2.1, and the fact that $((v_l^r))^* \cdot'_m (v_i^t)^* = 0$ in U for all t, r, i, l, we have:

$$\begin{aligned}
&- r_{23} \cdot'_m r_{12} \\
&= \sum_{t,r=0}^{\infty} \sum_{i,l} \big(- T(v_i^t) \otimes T(v_l^r)'_m (v_i^t)^* \otimes (v_l^r)^* + (v_i^t)^* \otimes T(v_l^r)'_m T(v_i^t) \otimes (v_l^r)^* \\
&\quad - (v_i^t)^* \otimes ((v_l^r)^*)'(m) T(v_i^t) \otimes T(v_l^r) \big) \\
&= \underbrace{- \sum_{t,r=0}^{\infty} \sum_{j,l} T(T(v_l^r)_m v_j^{t+m+1-r}) \otimes (v_j^{t+m+1-r})^* \otimes (v_l^r)^*}_{(21)} \\
&\quad + \underbrace{\sum_{t,r=0}^{\infty} \sum_{i,l} (v_i^t)^* \otimes T(v_l^r)'_m T(v_i^t) \otimes (v_l^r)^*}_{(22)} - \underbrace{\sum_{t,r=0}^{\infty} \sum_{i,j} \Big(}_{} \\
&\quad \underbrace{\mathrm{Res}_z (-1)^{m+1} z^{-m-2-4\lambda} (v_i^t)^* \otimes (v_j^{t+m+1-r})^* \otimes T\left(Y_{WV}^W (e^{z^{-1}L(1)} (-z^2)^{L(0)} v_j^{t+m+1-r}, z) e^{-z^{-1}L(1)} T(v_i^t) \right) \Big)}_{(23)} \\
&= (21) + (22) + (23).
\end{aligned}$$

Finally, by Definition 7.1.2, (7.2.9) in Lemma 7.2.2, (7.2.5) in Lemma (7.2.1), and the fact that $(v_k^s)^* \cdot'^{op}_m (v_l^r)^* = 0$ for all s, t, k, l, we have:

$$r_{13} \cdot'^{op}_m r_{23}$$

$$= \sum_{s,r=0}^{\infty}\sum_{k,l}\Big(-T(v_k^s)\otimes (v_l^r)^*\otimes T(v_l^r)'^{op}_m (v_k^s)^* - (v_k^s)^*\otimes T(v_l^r)\otimes ((v_l^r)^*)'^{op}(m)T(v_k^s)$$
$$+ (v_k^s)^*\otimes (v_l^r)^*\otimes T(v_l^r)'^{op}_m T(v_k^s)\Big)$$
$$= -\underbrace{\sum_{s,r=0}^{\infty}\sum_{j,l} T((v_j^{s+m+1-r})(m)T(v_l^r))\otimes (v_l^r)^*\otimes (v_j^{s+m+1-r})^*}_{(31)} - \underbrace{\sum_{s,r=0}^{\infty}\sum_{k,j}\Big(\mathrm{Res}_z(-1)^{m+1}z^{-m-2-4\lambda}(v_k^s)^*\otimes T\left(Y_{WV}^W(e^{-z^{-1}L(1)}(-z^2)^{L(0)}v_j^{s+m+1-r},-z)T(v_k^s)\right)\otimes (v_j^{s+m+1-r})^*\Big)}_{(32)}$$
$$+ \underbrace{\sum_{s,r=0}^{\infty}\sum_{k,l}(v_k^s)^*\otimes (v_l^r)^*\otimes T(v_l^r)'^{op}_m T(v_k^s)}_{(33)}$$
$$= (31) + (32) + (33).$$

Then we have $r_{12}\cdot_m r_{13} - r_{23}\cdot'_m r_{12} + r_{13}\cdot'^{op}_m r_{23} = (11)+(12)+(13)+(21)+(22)+(23)+(31)+(32)+(33)$, and we have:

$$(11)+(21)+(31)$$
$$= \sum_{s,t=0}^{\infty}\sum_{k,i} T(v_k^s)_m T(v_i^t)\otimes (v_i^t)^*\otimes (v_k^s)^* - \sum_{r,t=0}^{\infty}\sum_{l,j} T(T(v_l^r)_m v_j^{t+m+1-r})\otimes (v_j^{t+m+1-r})^*\otimes (v_l^r)^*$$
$$- \sum_{r,s=0}^{\infty}\sum_{j,l} T((v_j^{s+m+1-r})(m)T(v_l^r))\otimes (v_l^r)^*\otimes (v_j^{s+m+1-r})^*.$$

Change the variables as in proof Lemma 7.1.4, and observe that $T(v_k^s)_m T(v_i^t)\in V_{s-m-1+t}=0$ for all s,k if $s+t<m+1$, then we have:

$$(11)+(21)+(31)$$
$$= \sum_{s,t=0,s+t\geq m+1}^{\infty}\sum_{k,i} T(v_k^s)_m T(v_i^t)\otimes (v_i^t)^*\otimes (v_k^s)^* - \sum_{s,t=0,s+t\geq m+1}^{\infty}\sum_{k,i} T(T(v_k^s)_m v_i^t)\otimes (v_i^t)^*\otimes (v_k^s)^*$$
$$- \sum_{s,t=0,s+t\geq m+1}^{\infty}\sum_{k,i} T((v_k^s)(m)T(v_i^t))\otimes (v_i^t)^*\otimes (v_k^s)^*$$
$$= \sum_{s,t=0,s+t\geq m+1}^{\infty}\sum_{k,i}\Big(T(v_k^s)_m T(v_i^t) - T(T(v_k^s)_m v_i^t) - T((v_k^s)(m)T(v_i^t))\Big)\otimes (v_i^t)^*\otimes (v_k^s)^*,$$

which is contained in the subspace $\prod_{s,t=0,s+t\geq m+1}^{\infty} V_{s+t-m-1}\otimes W(t)^*\otimes W(s)^*\subset U^{\widehat{\otimes}3}$. Since $(v_i^t)^*$ and $(v_k^s)^*$ are basis elements of $W(t)^*$ and $W(s)^*$, respectively, then we have:

$$(11)+(21)+(31)=0 \iff T(v_k^s)_m T(v_i^t) - T(T(v_k^s)_m v_i^t) - T((v_k^s)(m)T(v_i^t)) = 0,$$

for all $s, t \in \mathbb{N}$, and $1 \leq k \leq p_s, 1 \leq i \leq p_t$. Since (v_i^t), $1 \leq i \leq p_t$, are basis elements of $W(t)$ for each $t \geq 0$, it follows that

$$(11) + (21) + (31) = 0 \iff T(u)_m T(v) = T(T(u)_m v) + T(u(m)T(v)), \tag{7.2.15}$$

for all $u, v \in W$. i.e., $T : W \to V$ is an m-relative RBO.

We perform a similar examination for the other terms in $r_{12} \cdot_m r_{13} - r_{23} \cdot'_m r_{12} + r_{13} \cdot'^{op}_m r_{23}$. First, we note that

$$\begin{aligned}
&(12) + (22) + (32)\\
&= -\sum_{s,t=0}^{\infty} \sum_{k,j} (v_j^{t-m-1+s})^* \otimes T(T(v_k^s)'_m v_j^{t-m-1+s}) \otimes (v_k^s)^* + \sum_{t,r=0}^{\infty} \sum_{i,l} (v_i^t)^* \otimes T(v_l^r)'_m T(v_i^t) \otimes (v_l^r)^* - \sum_{s,r=0}^{\infty}\\
&\sum_{k,j} \mathrm{Res}_z \Big((-1)^{m+1} z^{-m-2-4\lambda} (v_k^s)^* \otimes T\left(Y_{WV}^W (e^{-z^{-1}L(1)}(-z^2)^{L(0)} v_j^{s+m+1-r}, -z) T(v_k^s)\right) \otimes (v_j^{s+m+1-r})^* \Big).
\end{aligned}$$

Change the variables $(t - m - 1 + s, s) \mapsto (t, r)$ in (12) and $(s, s + m + 1 - r) \mapsto (t, r)$ in (32), and observe that $T(v_l^r)'_m T(v_i^t) \in V_{t+m+1-r} = 0$ if $r - t > m + 1$, so we have:

$$\begin{aligned}
&(12) + (22) + (32)\\
&= -\sum_{t,r=0, r-t\leq m+1}^{\infty} \sum_{l,i} (v_i^t)^* \otimes T(T(v_l^r)'_m v_i^t) \otimes (v_l^r)^* + \sum_{t,r=0, r-t\leq m+1}^{\infty} \sum_{l,i} (v_i^t)^* \otimes T(v_l^r)'_m T(v_i^t) \otimes (v_l^r)^*\\
&- \sum_{t,r=0, r-t\leq m+1}^{\infty} \sum_{l,i} \Big(\mathrm{Res}_z (-1)^{m+1} z^{-m-2-4\lambda} (v_i^t)^* \otimes T\left(Y_{WV}^W (e^{-z^{-1}L(1)}(-z^2)^{L(0)} v_l^r, -z) T(v_i^t)\right) \otimes (v_l^r)^* \Big),
\end{aligned}$$

which is contained in the subspace $\prod_{t,r=0, r-t\leq m+1}^{\infty} W(t)^* \otimes V_{t+r-m-1} \otimes W(r)^* \subset U^{\widehat{\otimes} 3}$. Then

$$\begin{aligned}
(12) + (22) + (32) = 0 \iff & -T(T(v_l^r)'_m v_i^t) + T(v_l^r)'_m T(v_i^t) - \mathrm{Res}_z (-1)^{m+1} z^{-m-2-4\lambda} (\\
& T(Y_{WV}^W (e^{-z^{-1}L(1)}(-z^2)^{L(0)} v_l^r, -z) T(v_i^t))) = 0 \text{ in } V_{t+r-m-1},
\end{aligned}$$

for all $r, t \in \mathbb{N}$, and $1 \leq l \leq p_r$, $1 \leq i \leq p_t$.

Let $f \in V_{t+m+1-r}^* \subset V'$, we apply $\langle -, f \rangle$ to the terms on the right-hand side. Then by (7.1.6), (7.2.10), and (7.2.11), we have:

$$\begin{aligned}
\langle T(T(v_l^r)'_m v_i^t), f \rangle &= \langle T(v_l^r)'_m v_i^t, T^*(f) \rangle = \mathrm{Res}_z z^m \langle Y_W(e^{zL(1)}(-z^{-2})^{L(0)} T(v_l^r), z^{-1}) v_i^t, T^*(f) \rangle\\
&= \mathrm{Res}_z z^m \langle v_i^t, Y_{W'}(T(v_l^r), z) T^*(f) \rangle,\\
\langle T(v_l^r)'_m T(v_i^t), f \rangle &= \langle Y_V(e^{zL(1)}(-z^{-2})^{L(0)} T(v_l^r), z^{-1}) T(v_i^t), f \rangle = \langle T(v_i^t), Y_{VV'}^{V'}(T(v_l^r)) f \rangle
\end{aligned}$$

$$= \langle v_i^t, T^*(Y_{VV'}^{V'}(T(v_l^r))f)\rangle.$$

Recall that $\mathrm{Res}_w g(w) = \mathrm{Res}_z g(h(z))h'(z)$, see (1.1.3) in [73]. Let $h(z) = -z^{-1}$, then by (7.2.11),

$$\begin{aligned}
&\mathrm{Res}_z(-1)^{m+1}z^{-m-2-4\lambda}\langle T(Y_{WV}^W(e^{-z^{-1}L(1)}(-z^2)^{L(0)}v_l^r, -z)T(v_i^t)), f\rangle\\
&= \mathrm{Res}_z(-1)^{m+1}z^{-m-2}\langle T(v_i^t), Y_{WW'}^{V'}(v_l^r, -z^{-1})T^*(f)\rangle\\
&= \mathrm{Res}_z - h(z)^m h'(z)\langle v_i^t, T^*(Y_{WW'}^{V'}(v_l^r, h(z))T^*(f))\rangle\\
&= -\mathrm{Res}_w w^m\langle v_i^t, T^*(Y_{WW'}^{V'}(v_l^r, w)T^*(f))\rangle.
\end{aligned}$$

Since an element $\alpha \in V_{t+r-m-1}$ is 0 if and only if $\langle \alpha, f\rangle = 0$ for all $f \in V_{t+r-m-1}^*$, we have:

$$\begin{aligned}
&(12) + (22) + (32) = 0 \iff\\
&\mathrm{Res}_z z^m(\langle v_i^t, -Y_{W'}(T(v_l^r), z)T^*(f) + T^*(Y_{VV'}^{V'}(T(v_l^r), z)f) + T^*(Y_{WW'}^{V'}(v_l^r, z)T^*(f))\rangle) = 0,
\end{aligned}$$

for all $r, t \in \mathbb{N}$, $1 \le l \le p_r$, $1 \le i \le p_t$, and $f \in V_{t+r-m-1}^*$. But v_i^t, $1 \le i \le p_t$, are basis elements of $W(t)$, so we have:

$$\begin{aligned}
&(12) + (22) + (32) = 0 \iff\\
&\mathrm{Res}_z z^m\left(Y_{W'}(T(u), z)T^*(f) - T^*(Y_{VV'}^{V'}(T(u), z)f) - T^*(Y_{WW'}^{V'}(u, z)T^*(f))\right) = 0,
\end{aligned} \tag{7.2.16}$$

for all $u \in W$ and $f \in V'$. i.e., T and T^* satisfy (7.2.13).

Finally, we examine the sum (13) + (23) + (33), note that

$$\begin{aligned}
&(13) + (23) + (33)\\
&= -\sum_{s,t=0}^{\infty}\sum_{i,j}(v_j^{s-m-1+t})^* \otimes (v_i^t)^* \otimes T(T(v_i^t)_m'^{op}v_j^{s-m-1+t}) - \sum_{t,r=0}^{\infty}\sum_{i,j}\Big(\\
&\mathrm{Res}_z(-1)^{m+1}z^{-m-2-4\lambda}(v_i^t)^* \otimes (v_j^{t+m+1-r})^* \otimes T\left(Y_{WV}^W(e^{z^{-1}L(1)}(-z^2)^{L(0)}v_j^{t+m+1-r}, z)e^{-z^{-1}L(1)}T(v_i^t)\right)\Big)\\
&+ \sum_{s,r=0}^{\infty}\sum_{k,l}(v_k^s)^* \otimes (v_l^r)^* \otimes T(v_l^r)_m'^{op}T(v_k^s).
\end{aligned}$$

Change the variables $(s - m - 1 + t, t) \mapsto (s, r)$ in (13) and $(t, t + m + 1 - r) \mapsto (s, r)$ in (23), and observe that $T(v_l^r)_m'^{op}T(v_k^s) \in V_{s+m+1-r} = 0$ if $r - s > m + 1$, we have

$$\begin{aligned}
&(13) + (23) + (33)\\
&= -\sum_{s,r=0, r-s\le m+1}^{\infty}\sum_{k,l}(v_k^s)^* \otimes (v_l^r)^* \otimes T(T(v_l^r)_m'^{op}v_k^s) - \sum_{s,r=0, r-s\le m+1}^{\infty}\sum_{k,l}\Big(
\end{aligned}$$

$$\mathrm{Res}_z(-1)^{m+1}z^{-m-2-4\lambda}(v_k^s)^*\otimes(v_l^r)^*\otimes T\left(Y_{WV}^W(e^{z^{-1}L(1)}(-z^2)^{L(0)}v_l^r,z)e^{-z^{-1}L(1)}T(v_k^s)\right)\Big)$$

$$+\sum_{s,r=0,r-s\le m+1}^{\infty}\sum_{k,l}(v_k^s)^*\otimes(v_l^r)^*\otimes T(v_l^r)_m'^{op}T(v_k^s),$$

which is contained in the subspace $\prod_{s,r=0,r-s\le m+1}^{\infty}W(s)^*\otimes W(r)^*\otimes V_{r+s-m-1}\subset U^{\widehat{\otimes}3}$. Then

$$(13)+(23)+(33)=0\iff -T(T(v_l^r)_m'^{op}v_k^s)+T(v_l^r)_m'^{op}T(v_k^s)-\mathrm{Res}_z(-1)^{m+1}z^{-m-2-4\lambda}(\\ T(Y_{WV}^W(e^{z^{-1}L(1)}(-z^2)^{L(0)}v_l^r,z)e^{-z^{-1}L(1)}T(v_k^s))=0\text{ in }V_{r+s-m-1}.$$

Let $f\in V_{r+s-m-1}^*$, and apply $\langle -,f\rangle$ to the terms on the right-hand side of this equivalency, then by (7.1.7), (7.2.10), we have:

$$\begin{aligned}\langle T(T(v_l^r)_m'^{op}v_k^s),f\rangle&=\langle T(v_l^r)_m'^{op}v_k^s,T^*(f)\rangle\\&=\mathrm{Res}_z z^m\langle Y_W(e^{-zL(1)}(-z^{-2})^{L(0)}T(v_l^r),-z^{-1})e^{zL(1)}v_k^s,T^*(f)\rangle\\&=\mathrm{Res}_z z^m\langle v_k^s,e^{zL(-1)}Y_{W'}(T(v_l^r),-z)T^*(f)\rangle\\&=\mathrm{Res}_z z^m\langle v_k^s,Y_{W'V}^{W'}(T^*(f),z)T(v_l^r)\rangle\\ \langle T(v_l^r)_m'^{op}T(v_k^s),f\rangle&=\mathrm{Res}_z z^m\langle Y_V(e^{-zL(1)}(-z^{-2})^{L(0)}T(v_l^r),-z^{-1})e^{zL(1)}T(v_k^s),f\rangle\\&=\mathrm{Res}_z z^m\langle v_k^s,T^*(e^{zL(-1)}Y_{VV'}^{V'}(T(v_l^r),-z)f)\rangle\\&=\mathrm{Res}_z z^m\langle v_k^s,T^*(Y_{V'V}^{V'}(f,z)T(v_l^r))\rangle.\end{aligned}$$

By (7.2.11), (7.2.12), and (1.1.3) in [73], we have:

$$\begin{aligned}&\mathrm{Res}_z(-1)^{m+1}z^{-m-2-4\lambda}\langle T(Y_{WV}^W(e^{z^{-1}L(1)}(-z^2)^{L(0)}v_l^r,z)e^{-z^{-1}L(1)}T(v_k^s),f\rangle\\&=\mathrm{Res}_z(-1)^{m+1}z^{-m-2}\langle e^{-z^{-1}L(1)}T(v_k^s),Y_{WW'}^{V'}(v_l^r,z^{-1})T^*(f)\rangle\\&=\mathrm{Res}_z(-1)^{m+1}z^{-m-2}\langle v_k^s,T^*(Y_{W'W}^{V'}(T^*(f),-z^{-1})v_l^r)\rangle\\&=-\mathrm{Res}_w w^m\langle v_k^s,T^*(Y_{W'W}^{V'}(T^*(f),w)v_l^r)\rangle.\end{aligned}$$

It follows that

$$(13)+(23)+(33)=0\iff\\ \mathrm{Res}_z z^m\langle v_k^s,-Y_{W'V}^{W'}(T^*(f),z)T(v_l^r)+T^*(Y_{V'V}^{V'}(f,z)T(v_l^r))+T^*(Y_{W'W}^{V'}(T^*(f),z)v_l^r)\rangle=0,$$

for all $r,s\in\mathbb{N}$, $1\le k\le p_s$, $1\le l\le p_r$, and $f\in V_{r+s-m-1}^*$. Again because v_k^s, $1\le k\le p_s$, are basis elements of $W(s)^*$, we have:

$$(13)+(23)+(33)=0\iff\\ \mathrm{Res}_z z^m\left(Y_{W'V}^{W'}(T^*(f,z)T(u))-T^*(Y_{V'V}^{V'}(f,z)T(u))-T^*(Y_{W'W}^{V'}(T^*(f),z)u)\right)=0,\tag{7.2.17}$$

for all $f \in V'$ and $u \in W$. i.e., T and T^* satisfy (7.2.14).

Note that for given $p, q, r \in \mathbb{N}$, the subspaces $V_p \otimes W(q)^* \otimes W(r)^*$, $W(p)^* \otimes V_q \otimes W(r)^*$, and $W(p)^* \otimes W(q)^* \otimes V_r$ are in direct sum within the vector space $U(p) \otimes U(q) \otimes U(r)$. Furthermore, by Definition 7.1.1 we have $U^{\widehat{\otimes} 3} = \prod_{p,q,r=0}^{\infty} U(p) \otimes U(q) \otimes U(r)$, and by our discussion above,

$$\begin{aligned}
(11) + (21) + (31) &\in \prod_{s,t=0, s+t \geq m+1}^{\infty} V_{s+t-m-1} \otimes W(t)^* \otimes W(s)^* \subset U^{\widehat{\otimes} 3},\\
(12) + (22) + (32) &\in \prod_{t,r=0, r-t \leq m+1}^{\infty} W(t)^* \otimes V_{t+r-m-1} \otimes W(r)^* \subset U^{\widehat{\otimes} 3},\\
(13) + (23) + (33) &\in \prod_{s,r=0, r-s \leq m+1}^{\infty} W(s)^* \otimes W(r)^* \otimes V_{r+s-m-1} \subset U^{\widehat{\otimes} 3}.
\end{aligned}$$

Then it is easy to see that

$$\begin{aligned}
& r_{12} \cdot_m r_{13} - r_{23} \cdot'_m r_{12} + r_{13} \cdot'^{op}_m r_{23} = 0\\
\iff & (11) + (21) + (31) = (12) + (22) + (32) = (13) + (23) + (33) = 0,
\end{aligned}$$

and by (7.2.15), (7.2.16), and (7.2.17), we see that $r_{12} \cdot_m r_{13} - r_{23} \cdot'_m r_{12} + r_{13} \cdot'^{op}_m r_{23} = 0$ if and only if $T : W \to V$ is an m-relative RBO, and T and its coadjoint T^* satisfy (7.2.13) and (7.2.14). i.e., $T : W \to V$ is a strong m-relative RBO. Now the proof is complete. □

Remark 7.2.6. If we want to drop the condition that the conformal weight λ of W is 0, then $U = V \rtimes W'$ is a $\mathbb{Q}$-graded vertex algebra. We have to adjust the definitions of $\alpha \cdot'_m \beta$ and $\alpha \cdot'^{op}_m \beta$ in Definition 7.1.2 to accommodate the appearance of the term $z^{-2\lambda}$ in $Y'_U(u^*, z)a = Y^{W'}_{W'V}(e^{zL(1)}(-z^{-2})^{L(0)}u^*, z^{-1})a$, where $u^* \in W'$ and $a \in V$. Although we might still be able to find a way to make things work, this is not our focusing point for solving the VOYBE, so we make the assumption that $\lambda = 0$.

With notations similar to those in Corollary 7.1.15, we obtain an embedding

$$\mathrm{StrRBO}_{\mathrm{LP}}(W, V)(m) \hookrightarrow \mathrm{SD}_{\mathrm{sol}}((V \rtimes W') \rtimes (V \rtimes W'))(m), \quad m \in \mathbb{Z}. \tag{7.2.18}$$

Here $\mathrm{StrRBO}_{\mathrm{LP}}(W, V)(m)$ is the set of level-preserving strong m-RBO from the V-module W to V, and the set on the right-hand side is the set of skew-symmetric solutions to the m-VOYBE in the VOA $V \rtimes W'$.

7.2.3 The coadjoint case

In this subsection, we let $(U, Y_U, \mathbf{1}, \omega)$ be a VOA. Consider the case when (W, Y_W) is the coadjoint module $(U', Y_{U'})$, and T is a level-preserving map $T : U' \to U$. By (7.2.10) we have:

$$T^* : U' \to (U')' = U,\ \langle T^*(f), g\rangle = \langle f, T(g)\rangle, \quad f, g \in U'.$$

In particular, by Definition 7.1.13 and (7.2.10), if T is symmetric (resp. skew-symmetric), then we have $T^* = T$ (resp. $T^* = -T$).

Lemma 7.2.7. *The intertwining operator* $\mathcal{Y}_{U'U}^{U'}$ *given by* (7.2.11) *is the same as the vertex operator* $Y_{U'U}^{U'}$ *that is given by the skew-symmetry formula with respect to* $Y_{U'}$.

Proof. Let $a, b \in U$ and $f \in U'$ be homogeneous elements. Then by (7.2.11) we have:

$$\begin{aligned}
&\langle \mathcal{Y}_{U'U}^{U'}(f, z)a, b\rangle = \langle a, Y_{U'U}^{U'}(e^{zL(1)}(-z^{-2})^{L(0)} f, z^{-1})b\rangle \\
&= \langle e^{z^{-1}L(1)}a, Y_{U'}(b, -z^{-1})e^{zL(1)}(-z^{-2})^{L(0)} f\rangle \\
&= \langle e^{zL(-1)}Y_U(e^{-z^{-1}L(1)}(-z^2)^{L(0)}b, -z)e^{z^{-1}L(1)}a, (-z^{-2})^{L(0)} f\rangle \\
&= (-1)^{\mathrm{wt}(f)+\mathrm{wt}(b)} z^{2\mathrm{wt}(b)-2\mathrm{wt}(f)} \langle Y_U(e^{z^{-1}L(1)}a, z)e^{-z^{-1}L(1)}b, f\rangle \\
&= (-1)^{\mathrm{wt}(f)+\mathrm{wt}(b)} z^{2\mathrm{wt}(b)-2\mathrm{wt}(f)} \sum_{j\geq 0}\sum_{i\geq 0} \frac{z^{\mathrm{wt}(f)-\mathrm{wt}(b)-\mathrm{wt}(a)}}{i!j!}(-1)^j \langle (L(1)^i a)_{\mathrm{wt}(a)+\mathrm{wt}(b)-\mathrm{wt}(f)-i-j-1}(L(1)^j b), f\rangle \\
&= \sum_{j\geq 0}\sum_{i\geq 0} \frac{z^{-\mathrm{wt}(a)+\mathrm{wt}(b)-\mathrm{wt}(f)}}{i!j!}(-1)^{\mathrm{wt}(f)+j-\mathrm{wt}(b)} \langle (L(1)^i a)_{\mathrm{wt}(a)+\mathrm{wt}(b)-\mathrm{wt}(f)-i-j-1}(L(1)^j b), f\rangle \\
&= (-1)^{\mathrm{wt}(a)} z^{-2\mathrm{wt}(a)} \langle f, Y_U(e^{-zL(1)}a, -z^{-1})e^{zL(1)}b\rangle \\
&= \langle f, Y_U(e^{-zL(1)}(-z^{-2})^{L(0)}a, -z^{-1})e^{zL(1)}b\rangle \\
&= \langle Y_{U'}(a, -z)f, e^{zL(1)}b\rangle = \langle Y_{U'U}^{U'}(f, z)a, b\rangle.
\end{aligned}$$

Hence $\mathcal{Y}_{U'U}^{U'}(f, z)a = Y_{U'U}^{U'}(f, z)a$ for $f \in U'$ and $a \in U$, and so $\mathcal{Y}_{U'U}^{U'} = Y_{U'U}^{U'}$. □

It is also easy to check that the rest of the intertwining operators appearing in (7.2.13) and (7.2.14), with $V = U$, satisfy $Y_{W'} = Y_U$, $Y_{UU'}^{U'} = Y_{U'}$, $Y_{W'U}^{W'} = Y_U$, and $\mathcal{Y}_{W'W}^{U'} = Y_{U'}$. In particular, if T is skew-symmetric: $T = -T^*$, then both (7.2.13) and (7.2.14) become

$$\mathrm{Res}_z z^m \left(-Y_U(T(f), z)T(g) + T(Y_{U'}(T(f), z)g) + T(Y_{U'U}^{U'}(f, z)T(g))\right) = 0,$$

which is the condition that T is an m-relative RBO. Hence we have the following conclusion.

Lemma 7.2.8. *For $m \in \mathbb{Z}$, any level-preserving skew-symmetric m-relative RBO $T : U' \to U$ is strong.*

Furthermore, given a symmetric linear map $T : U' \to U$, we note that the skew-symmetrization $r = T - \sigma(T)$ given by (7.2.3) is a nonzero element in $(U \rtimes U)^{\widehat{\otimes}2}$, due to the fact that the tensor form of T and $\sigma(T)$ are in linearly independent subspaces, in view of (7.2.2). Then by (7.2.18), we have an embedding:

$$\mathrm{RBO}^{\mathrm{sk}}_{\mathrm{LP}}(U', U)(m) \hookrightarrow \mathrm{SD}_{\mathrm{sol}}((U \rtimes U)\widehat{\otimes}(U \rtimes U))(m), \tag{7.2.19}$$

where $\mathrm{RBO}^{\mathrm{sk}}_{\mathrm{LP}}(U', U)(m)$ is the set of skew-symmetric level-preserving m-relative RBOs $T : U' \to U$. Combining (7.2.19) with (7.1.36), we have an embedding

$$\mathrm{SD}_{\mathrm{sol}}(U\widehat{\otimes}U)(\cong \mathrm{RBO}^{\mathrm{sk}}_{\mathrm{LP}}(U', U)(m)) \hookrightarrow \mathrm{SD}_{\mathrm{sol}}((U \rtimes U)\widehat{\otimes}(U \rtimes U))(m), \tag{7.2.20}$$

which leads to the following conclusion.

Corollary 7.2.9. *Let $(U, Y_U, \mathbf{1}, \omega)$ be a VOA and $m \in \mathbb{Z}$. Every skew-symmetric solution r of the m-VOYBE in U is a skew-symmetric solution to the m-VOYBE in $U \rtimes U$, via the embedding* (7.2.20).

7.3 Relations with the classical Yang-Baxter equation

Theorems 7.1.10 and 7.2.5 are generalizations of the classical results about the relationship between relative RBOs (also known as -operators) and the CYBE for (finite-dimensional) Lie algebras (cf. [4], see also [45, 64]).

7.3.1 Solutions of CYBE and relative RBOs

We first observe some facts about the first-level Lie algebra of a VOA and its modules. Let V be a VOA, W be an admissible V-module, and $T \in \mathrm{Hom}_{\mathrm{LP}}(W, V)$ be a level-preserving linear operator.Recall the first level V_1 of the VOA V is a Lie algebra with respect to the Lie bracket

$$[a, b] = a_0 b, \quad a, b \in V_1,$$

and $W(1)$ is a module over the Lie algebra V_1, with respect to

$$\rho : V_1 \to \mathfrak{gl}(W(1)), \quad \rho(a)u = a_0 u = \mathrm{Res}_z Y_W(a, z)u, \quad a \in V_1, u \in W(1). \tag{7.3.1}$$

Suppose that V_1 consists of quasi-primary vectors, that is, $L(1)V_1 = 0$ (see [27] as well as [50]). Then by (7.3.1), for $u^* \in W(1)^*$, $v \in W(1)$ and $a \in V_1$, we have

$$\langle a_0 u^*, v\rangle = \left\langle u^*, \sum_{j\geq 0} \frac{(-1)^{\mathrm{wt}(a)}}{j!}(L(1)^j a)_{2\mathrm{wt}(a)-j-2} v\right\rangle = \langle u^*, -a_0 v\rangle = -\langle u^*, \rho(a)v\rangle.$$

Therefore, the first level $W(1)^*$ of the contragredient V-module W' is the dual module of the Lie algebra V_1-module $W(1)$:

$$\rho^* : V_1 \to \mathfrak{gl}(W(1)^*), \quad \rho^*(a)u^* = a_0 u^* = \mathrm{Res}_z Y_W'(a,z)u^*, \quad a \in V_1, u^* \in W(1)^*. \tag{7.3.2}$$

Let $(U, Y_U, \mathbf{1}, \omega)$ be a VOA such that $U(1)$ consists of quasi-primary vectors and let $r = \sum_{t=0}^{\infty} r^t = \sum_{t=0}^{\infty}\sum_{i=1}^{p_t} \alpha_i^t \otimes \beta_i^t - \beta_i^t \otimes \alpha_i^t$ be a diagonal skew-symmetric two-tensor in $D(U \widehat{\otimes} U)$. Let

$$R := r^1 = \sum_{i=1}^{p_1} \alpha_i^1 \otimes \beta_i^1 - \beta_i^1 \otimes \alpha_i^1 \in U(1) \otimes U(1). \tag{7.3.3}$$

Then R is a skew-symmetric two tensor in the Lie algebra $U(1)$.

Lemma 7.3.1. *If r is a skew-symmetric solution to the 0-VOYBE in U, then R is a skew-symmetric solution to the CYBE in the Lie algebra $U(1)$.*

Proof. For any $\alpha, \beta \in U(1)$, since $L(1)\alpha = L(1)\beta = 0$, by Definition (7.1.2), we have

$$\alpha \cdot_0 \beta = [\beta, \alpha], \quad \alpha \cdot_0' \beta = (-1)\alpha_0\beta = [\beta, \alpha], \quad \alpha \cdot_0'^{\mathrm{op}} \beta = \beta_0\alpha = [\beta, \alpha]. \tag{7.3.4}$$

By Lemma 7.1.4, if r is a solution to the 0-VOYBE, then taking the projection of $r_{12} \cdot_0 r_{13} - r_{23} \cdot_0' r_{12} + r_{13} \cdot_0'^{\mathrm{op}} r_{23}$ onto the homogeneous subspace $U(1) \otimes U(1) \otimes U(1)$, we obtain

$$r_{12}^1 \cdot_0 r_{13}^1 - r_{23}^1 \cdot_0' r_{12}^1 + r_{13}^1 \cdot_0'^{\mathrm{op}} r_{23}^1 = 0.$$

By (7.3.4) and the definitions in (7.1.11)–(7.1.13), we have

$$r_{12}^1 \cdot_0 r_{13}^1 = [R_{13}, R_{12}], \quad -r_{23}^1 \cdot_0' r_{12}^1 = [R_{23}, R_{12}], \quad r_{13}^1 \cdot_0'^{\mathrm{op}} r_{23}^1 = [R_{23}, R_{13}],$$

where R_{12}, R_{13}, and R_{23} are elements in the universal enveloping algebra $\mathcal{U}(U(1))$ defined from R in (7.3.3) in the conventional way. Hence we have $[R_{12}, R_{13}] + [R_{12}, R_{23}] + [R_{13}, R_{23}] = 0$, which is the CYBE (see (1.2.5)). $\square$

Let $T_r : U' \to U$ be given by (7.1.28). Consider the restriction map

$$T_R := T_r|_{U(1)^*} : U(1)^* \to U(1). \tag{7.3.5}$$

Lemma 7.3.2. *If $T_r : U' \to U$ is a 0-relative RBO, then T_R is a relative RBO of the Lie algebra $U(1)$ with respect to the module $U(1)^*$.*

Proof. Let $f, g \in U(1)^*$, and $a, b \in U(1)$. Since $L(-1)a_1 f \in U(1)^*$, and $U(1)$ consists of quasi-primary vectors, by (7.1.5) we have $\langle L(-1)(a_1 f), b\rangle = \langle a_1 f, L(1)b\rangle = 0$. Moreover, since $a_j f \in U(1-j)^* = 0$ for $j \geq 2$, by (7.1.22) and (7.3.2) we derive

$$\begin{aligned}\langle f(0)a, b\rangle = \mathrm{Res}_z \langle e^{zL(-1)} Y_{U'}(a, -z) f, b\rangle &= \sum_{j\geq 0} \langle (-1)^j \frac{L(-1)^j}{j!}(a_j f), b\rangle \\ &= \langle a_0 f, b\rangle - \langle L(-1)(a_1 f), b\rangle = \langle \rho^*(a)(f), b\rangle.\end{aligned}$$

Hence $f(0)a = \rho^*(a)(f)$. If $T_r : U' \to U$ is a 0-relative RBO, then by (7.3.1) and the definition formulas (7.1.23) and (7.3.5), we have

$$\begin{aligned}0 &= T_r(f)_0 T_r(g) - T_r(T_r(f)_0 g) - T_r(f(0)T_r(g)) \\ &= [T_R(f), T_R(g)] - T_R(\rho^*(T_R(f))(g)) - T_R(\rho^*(T_R(g))(f)),\end{aligned}$$

for all $f, g \in U(1)$. Thus $T_R : U(1)^* \to U(1)$ is a relative RBO. □

With the notations in this subsection, by Lemma 7.3.1, Lemma 7.3.2 and Theorem 7.1.10, we have the following diagram.

$$\begin{array}{ccc} \text{s.-s. } r \text{ is solution to 0-VOYBE in } U & \xrightarrow{\text{Lem. 7.3.1}} & \text{s.-s. } R \text{ is solution to the CYBE in } U(1) \\ \text{Thm. 7.1.10} \Updownarrow & & \Updownarrow [45] \\ T_r \text{ is a 0-relative RBO of } U & \xrightarrow{\text{Lem. 7.3.2}} & T_R \text{ is a relative RBO of } U(1) \end{array}$$

where "s.-s." in this diagram is the abbreviation of "skew-symmetric". In particular, the classical result about constructing a relative RBO from a solution of the CYBE [45] can be viewed as a corollary of Theorem 7.1.10.

7.3.2 Solving the CYBE from relative RBOs

We can also recover the process of using relative RBOs to produce solutions of the CYBE in the semi-direct product Lie algebras [4] by restricting the corresponding process for the VOYBE to the first levels.

In this subsection, we let $(V, Y, \mathbf{1}, \omega)$ be a VOA, and (W, Y_W) be a V-module of conformal weight 0. Let $U = V \rtimes W'$. Let $T \in \mathrm{Hom}_{\mathrm{LP}}(W, V)$. For $r = T - \sigma(T) = \sum_{t=0}^{\infty} r^t = \sum_{t=0}^{\infty} \sum_{i=1}^{p_t} T(v_i^t) \otimes (v_i^t)^* - (v_i^t)^* \otimes T(v_i^t) \in U^{\widehat{\otimes}2}$ in (7.2.3), we let

$$\mathcal{R} := r^1 = \sum_{i=1}^{p_1} T(v_i^1) \otimes (v_i^1)^* - (v_i^1)^* \otimes T(v_i^1) \in (V_1 \oplus W(1)^*) \otimes (V_1 \oplus W(1)^*) \tag{7.3.6}$$

be the homogeneous part of r in $U(1) \otimes U(1)$, where $\{v_1^1, \dots, v_{p_1}^1\}$ is a basis of $W(1)$ while $\{(v_1^1)^*, \dots, (v_{p_1}^1)^*\}$ the dual basis of $W(1)^*$.

Lemma 7.3.3. *Assume that V_1, $W(1)$ and $W(1)^*$ are spanned by quasi-primary vectors. Then the operations $\cdot_0, \cdot_0'$, and $\cdot_0'^{\mathrm{op}}$ in Definition 7.1.2 satisfy*

$$\begin{aligned}
&a_0 b = [a, b], && a_0' b = -[a, b], && a_0'^{\mathrm{op}} b = [a, b];\\
&a_0 v^* = \rho^*(a) v^*, && a_0' v^* = -\rho^*(a) v^*, && a_0'^{\mathrm{op}} v^* = \rho^*(a) v^*;\\
&v^*(0) a = -\rho^*(a) v^*, && (v^*)'(0) a = \rho^*(a) v^*, && (v^*)'^{\mathrm{op}}(0) a = -\rho^*(a) v^*,
\end{aligned}$$

for $a, b \in V_1$ and $v^* \in W(1)^*$, where ρ^* is given by (7.3.2).

Proof. Since $\mathrm{wt}(a) = \mathrm{wt}(b) = 1$, and $L(1)a = L(1)b = L(1)v^* = 0$, the equations on first two rows follow immediately from (7.1.17), (7.1.18) and (7.3.2). Let $u \in W(1)$. By assumption we have $L(1)u = 0$, and $\mathrm{wt}(u) = \mathrm{wt}(v^*) = 1$. Then

$$\begin{aligned}
\langle v^*(0)a, u\rangle &= \mathrm{Res}_z \langle Y_{W'V}^{W'}(v^*, z)a, u\rangle = \mathrm{Res}_z \langle Y_{W'}(a, -z)v^*, e^{zL(1)}u\rangle = \langle -a_0 v^*, u\rangle = \langle -\rho^*(a)v^*, u\rangle,\\
\langle (v^*)'(0)a, u\rangle &= \mathrm{Res}_z \langle Y_{W'V}^{W'}(e^{zL(1)}(-z^{-2})v^*, z^{-1})a, u\rangle = \mathrm{Res}_z (-1) z^{-2} \langle Y_{W'}(a, -z^{-1})v^*, e^{z^{-1}L(1)}u\rangle\\
&= \mathrm{Res}_z (-1) z^{-2} \langle \sum_{n\in\mathbb{Z}} a_n v^* (-1)^{n+1} z^{n+1}, u\rangle = \langle \rho^*(a)v^*, u\rangle,\\
\langle (v^*)'^{\mathrm{op}}(0)a, u\rangle &= \mathrm{Res}_z \langle Y_{W'V}^{W'}(e^{-zL(1)}(-z^{-2})^{L(0)}v^*, -z^{-1})e^{zL(1)}a, u\rangle\\
&= \mathrm{Res}_z (-1) z^{-2} \langle Y_{W'}(a, z^{-1})v^*, e^{z^{-1}L(1)}u\rangle\\
&= \mathrm{Res}_z (-1) z^{-2} \langle \sum_{n\in\mathbb{Z}} a_n v^* z^{n+1}, u\rangle = \langle -\rho^*(a)v^*, u\rangle.
\end{aligned}$$

Since $v^*(0)a$, $(v^*)'(0)a$ and $(v^*)'^{\mathrm{op}}(0)a$ are contained in $W(1)^*$, we arrive at the conclusion. □

Recall (cf. [4]) that $V_1 \oplus W(1)^*$ carries a semi-direct product Lie algebra structure:

$$[a + u^*, b + v^*] = [a, b] + \rho^*(a)v^* - \rho^*(b)u^*, \quad a, b \in V_1,\ u^*, v^* \in W(1)^*. \tag{7.3.7}$$

Proposition 7.3.4. *Assume that V_1, $W(1)$ and $W(1)^*$ are spanned by quasi-primary vectors. Let $r = T - \sigma(T) \in U^{\widehat{\otimes}2}$ be as in (7.2.3), and let $\mathcal{R} = r^1$ be as in (7.3.6). If r is a skew-symmetric solution to the 0-VOYBE in $V \rtimes W'$ (see Definition 7.1.5), then $\mathcal{R} \in (V_1 \rtimes W(1)^*)^{\otimes 2}$ is a skew-symmetric solution to the classical Yang-Baxter equation in the Lie algebra $V_1 \rtimes W(1)^*$*

$$[\mathcal{R}_{12}, \mathcal{R}_{13}] + [\mathcal{R}_{12}, \mathcal{R}_{23}] + [\mathcal{R}_{13}, \mathcal{R}_{23}] = 0. \tag{7.3.8}$$

Proof. Define a projection map on $U^{\widehat{\otimes}3}$ by

$$p_{1,1,1} : U\widehat{\otimes}U\widehat{\otimes}U \to U(1) \otimes U(1) \otimes U(1), \quad p_{1,1,1}\Big(\sum_{q,s,t=0}^{\infty} \sum_{i,j,k} \alpha_i^q \otimes \beta_j^s \otimes \gamma_k^t\Big) = \sum_{i,j,k} \alpha_i^1 \otimes \beta_j^1 \otimes \gamma_k^1,$$

where the sums over i, j, k are finite, and $\alpha_i^q \in U(r)$, $\beta_j^s \in U(s)$, and $\gamma_k^t \in U(t)$, for $q, s, t \geq 0$, and $i, j, k \geq 1$. By (7.1.15)-(7.1.17), the computations in the proof of Theorem 7.2.5, together with Lemma 7.3.3, we derive

$$\begin{aligned}
&p_{1,1,1}(r_{12} \cdot_0 r_{13}) = r_{12}^1 \cdot_0 r_{13}^1 \\
&= \sum_{i,k} \Big(T(v_k^1)_0 T(v_i^1) \otimes (v_i^1)^* \otimes (v_k^1)^* - T(v_k^1)_0 (v_i^1)^* \otimes T(v_i^1) \otimes (v_k^1)^* - (v_k^1)^*(0) T(v_i^1) \otimes (v_i^1)^* \otimes T(v_k^1)\Big) \\
&= \sum_{i,k} \Big(- [T(v_i^1), T(v_k^1)] \otimes (v_i^1)^* \otimes (v_k^1)^* - \rho^*(T(v_k^1))(v_i^1)^* \otimes T(v_i^1) \otimes (v_k^1)^* \\
&\quad + \rho^*(T(v_i^1))(v_k^1)^* \otimes (v_i^1)^* \otimes T(v_k^1)\Big) = -[\mathcal{R}_{12}, \mathcal{R}_{13}], \\
&p_{1,1,1}(-r_{23} \cdot_0' r_{12}) = -r_{23}^1 \cdot_0' r_{12} \\
&= \sum_{i,l} \Big(- T(v_i^1) \otimes T(v_l^1)_0' (v_i^1)^* \otimes (v_l^1)^* + (v_i^1)^* \otimes T(v_l^1)_0' T(v_i^1) \otimes (v_l^1)^* \\
&\quad - (v_l^1)^* \otimes ((v_l^1)^*)'(0) T(v_i^1) \otimes T(v_l^1)\Big) \\
&= \sum_{i,l} \Big(T(v_i^1) \otimes \rho^*(T(v_l^1))(v_i^1)^* \otimes (v_l^1)^* + (v_i^1)^* \otimes [T(v_i^1), T(v_l^1)] \otimes (v_l^1)^* \\
&\quad - (v_i^1)^* \otimes \rho^*(T(v_i^1))(v_l^1)^* \otimes T(v_l^1)\Big) = -[\mathcal{R}_{12}, \mathcal{R}_{23}], \\
&p_{1,1,1}(r_{13} \cdot_0^{\prime\mathrm{op}} r_{23}) = r_{13}^1 \cdot_0^{\prime\mathrm{op}} r_{23} \\
&= \sum_{k,l} \Big(- T(v_k^1) \otimes (v_l^1)^* \otimes T(v_l^1)_0^{\prime\mathrm{op}} (v_k^1)^* - (v_k^1)^* \otimes T(v_l^1) \otimes ((v_l^1)^*)^{\prime\mathrm{op}}(0) T(v_k^1) \\
&\quad + (v_k^1)^* \otimes (v_l^1)^* \otimes T(v_l^1)_0^{\prime\mathrm{op}} T(v_k^1)\Big) \\
&= \sum_{k,l} \Big(- T(v_k^1) \otimes (v_l^1)^* \otimes \rho^*(T(v_l^1))(v_k^1)^* + (v_k^1)^* \otimes T(v_l^1) \otimes \rho^*(T(v_k^1))(v_l^1)^*
\end{aligned}$$

$$- (v_k^1)^* \otimes (v_l^1)^* \otimes [T(v_k^1), T(v_l^1)]\Big) = -[\mathcal{R}_{13}, \mathcal{R}_{23}].$$

Since the projection $p_{1,1,1}$ is clearly a linear map, we have

$$0 = p_{1,1,1}(r_{12} \cdot_0 r_{13} - r_{23} \cdot_0 r_{12} + r_{13} \cdot_0 r_{23}) = -[\mathcal{R}_{12}, \mathcal{R}_{13}] - [\mathcal{R}_{12}, \mathcal{R}_{23}] - [\mathcal{R}_{13}, \mathcal{R}_{23}].$$

Hence $\mathcal{R} \in (V_1 \rtimes W(1)^*)^{\otimes 2}$ is a solution to the classical Yang-Baxter equation. □

On the other hand, consider the restriction of the level-preserving linear map $T \in \mathrm{Hom}_{\mathrm{LP}}(W, V)$ to the first level. We write

$$\mathcal{T} := T|_{W(1)} : W(1) \to V_1. \tag{7.3.9}$$

Proposition 7.3.5. *Let V_1 and $W(1)$ be spanned by quasi-primary vectors, and let $T \in \mathrm{Hom}_{\mathrm{LP}}(W, V)$. If either one of the conditions (7.1.23), (7.2.13), or (7.2.14) with $m = 0$ holds, then $\mathcal{T} : W(1) \to V_1$ is a relative RBO. In particular, if T is a 0-strong relative RBO, then $\mathcal{T}$ is a relative RBO.*

Proof. Let $u, v \in W(1)$ and $f \in V_1^*$. Assume that T satisfies (7.1.23). Then we have

$$\begin{aligned}
[\mathcal{T}(u), \mathcal{T}(v)] &= \mathcal{T}(u)_0\mathcal{T}(v) = \mathcal{T}(\mathcal{T}(u)_0 v) + \mathcal{T}(u(0)\mathcal{T}(v)) \\
&= \mathcal{T}(\rho(\mathcal{T}(u))v) + \mathcal{T}\Big(\sum_{j\geq 0} \frac{(-1)^{j+1}}{j!} L(-1)^j (\mathcal{T}(v))_j u\Big) \\
&= \mathcal{T}(\rho(\mathcal{T}(u))v) - \mathcal{T}(\rho(\mathcal{T}(v))u) + \mathcal{T}(L(-1)\mathcal{T}(v)_1 u).
\end{aligned}$$

Note that $L(-1)\mathcal{T}(v)_1 u \in W(1)$, and $\langle L(-1)\mathcal{T}(v)_1 u, w\rangle = \langle \mathcal{T}(v)_1 u, L(1)w\rangle = 0$, for all $w \in W(1)$. Hence $L(-1)\mathcal{T}(v)_1 u = 0$. Then we have $\mathcal{T}(u(0)\mathcal{T}(v)) = -\mathcal{T}(\rho(\mathcal{T}(v))u)$ and $[\mathcal{T}(u), \mathcal{T}(v)] = \mathcal{T}(\rho(\mathcal{T}(u))v) - \mathcal{T}(\rho(\mathcal{T}(v))u)$. Thus $\mathcal{T}$ is a relative RBO.

Assume T satisfies (7.2.13). Note that $\mathcal{T}^* = T^*|_{V_1^*} : V_1^* \to W(1)^*$. Then by (7.2.11) and the definition of $Y_{VV'}^{V'}$ and $Y_{W'}$ in Section 5.4 in [27], we obtain

$$\begin{aligned}
0 &= \mathrm{Res}_z \left\langle Y_{W'}(T(u), z)T^*(f) - T^*(Y_{VV'}^{V'}(T(u), z)f) + T^*(\mathcal{Y}_{WW'}^{V'}(u, z)T^*(f)), v\right\rangle \\
&= \mathrm{Res}_z \left\langle \mathcal{T}^*(f), Y_W(e^{zL(1)}(-z^{-2})^{L(0)}\mathcal{T}(u), z^{-1})v\right\rangle - \mathrm{Res}_z \left\langle f, Y_V(e^{zL(1)}(-z^{-2})^{L(0)}\mathcal{T}(u), z^{-1})\mathcal{T}(v)\right\rangle \\
&\quad + \mathrm{Res}_z \left\langle \mathcal{T}^*(f), Y_{WV}^{W}(e^{zL(1)}(-z^{-2})^{L(0)}u, z^{-1})\mathcal{T}(v)\right\rangle \\
&= -\langle f, \mathcal{T}(\mathcal{T}(u)_0 v)\rangle + \langle f, \mathcal{T}(u)_0\mathcal{T}(v)\rangle - \langle f, \mathcal{T}(u(0)\mathcal{T}(v))\rangle \\
&= \langle f, [\mathcal{T}(u), \mathcal{T}(v)] - \mathcal{T}(\rho(\mathcal{T}(u))v) + \mathcal{T}(\rho(\mathcal{T}(v))u)\rangle.
\end{aligned}$$

Hence $\mathcal{T}$ is a relative RBO. Finally, assume that T satisfies (7.2.14). Then

$$
\begin{aligned}
0 &= \mathrm{Res}_z \left\langle Y_{W'V}^{W'}(T^*(f), z)T(u) - T^*(Y_{V'V}^{V'}(f, z)T(u)) + T^*(\mathcal{Y}_{W'W}^{V'}(T^*(f), z)u), v \right\rangle \\
&= \mathrm{Res}_z \langle Y_{W'}(\mathcal{T}(u), -z)\mathcal{T}^*(f), v\rangle - \langle Y_{V'}(\mathcal{T}(u), -z)f, \mathcal{T}(v)\rangle + \langle \mathcal{Y}_{WW'}^{V'}(u, -z)\mathcal{T}^*(f), \mathcal{T}(v)\rangle \\
&= \langle f, \mathcal{T}(\mathcal{T}(u)_0 v)\rangle - \langle f, \mathcal{T}(u)_0\mathcal{T}(v)\rangle + \langle f, \mathcal{T}(u(0)\mathcal{T}(v))\rangle,
\end{aligned}
$$

and so $\mathcal{T}$ is a relative RBO. □

The proof of Proposition 7.3.5 immediately gives an easy way to construct 0-strong relative RBOs like in Example 7.1.7.

Corollary 7.3.6. *Let V_1 and $W(1)$ be spanned by quasi-primary vectors, and let $\phi : W(0) \to V_0 = \mathbb{C}\mathbf{1}$ be an arbitrary linear map. Then a relative RBO $\mathcal{T} : W(1) \to V_1$ of the Lie algebra V_1 can be extended to a 0-strong relative RBO $T : W \to V$ by letting*

$$T|_{W(0)} := \phi, \quad T|_{W(1)} := \mathcal{T}, \text{ and } T|_{W(n)} := 0, \quad n \geq 2.$$

Now apply Propositions 7.3.4 and 7.3.5 and assume that V_1, $W(1)$ and $W(1)^*$ are spanned by quasi-primary vectors. Then we have another diagram that illustrates the relationship between the 0-VOYBE and 0-strong relative RBO of VOAs on the one hand and the CYBE and the relative RBO of Lie algebras on the other.

$$
\begin{array}{ccc}
T \text{ is a 0-strong relative RBO of VOA} & \xrightarrow{\text{Prop. 7.3.5}} & \mathcal{T} \text{ is a relative RBO of Lie algebra} \\
\text{Thm. 7.2.5}\Big\Updownarrow & & \Big\Updownarrow[4] \\
r = T - T^{21} \text{ is a solution to 0-VOYBE} & \xrightarrow{\text{Prop. 7.3.4}} & \mathcal{R} = \mathcal{T} - \mathcal{T}^{21} \text{ is a solution to CYBE}
\end{array}
$$

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