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Fluctuation Analysis of Order Positions Under General Cancellations

by

Chen Shen

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Steven Neil Evans, Chair
Professor Lawrence Craig Evans
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Abstract

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In a modern financial market, limit order books are usually managed under the price-time priority rule (i.e. orders are ranked by price and then time/position). Empirical studies of limit order books show that the major component of the order flow occurs in the best bid/ask queue [1], and the order's positional value can be of the same order of magnitude as the bid-ask spread [9]. As a consequence, analyzing the order positions in the best bid/ask queue plays an important role in the field of algorithmic trading.

In this dissertation, we analyze the fluctuation of scaled order positions around their limits in the best bid queue under the general assumption for cancellations, which allows us to generalize the existing result [4, Theorem 15] to include more realistic situations. We first derive the stochastic differential equation that the fluctuation satisfies. Then we prove that the fluctuation is a Gaussian process with mean zero. Furthermore, we show that it has the mean-reverting property, where the mean-reverting level is the same as the fluctuation of scaled best bid queue length multiplied by the order positions relative to the best bid queue length and the mean-reverting speed is proportional to the rate of change of cancellations in the best bid queue.

To my parents.

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Chapter 1

Introduction

In this chapter, we first give an introduction of càdlàg processes that are the mathematical tools to model the dynamics of limit order books in [4]. Then we give an introduction of limit order books, and the dynamics of order positions in the best bid queue. From there, we provide the model framework in [4]. Finally we give an overview of the remainder of this dissertation.

1.1 Càdlàg processes

In this section, we define càdlàg processes in the metric space $(D[0,T], J_1)$. But first, we will need a few more definitions.

Definition 1.1.1. *Let S be a set. A function $d : S \times S \to R$ is called a metric on S if it satisfies the following properties*

(i) (non-negativity) $d(x,y) \geq 0$, and $d(x,y) = 0$ if and only if $x = y$

(ii) (symmetry) $d(x,y) = d(y,x)$

(iii) (triangle inequality) $d(x,y) + d(y,z) \geq d(x,z)$

for any $x, y, z \in S$.

Then (S,d), the set S equipped with the metric d (also known as the distance function) is called a metric space.

Definition 1.1.2. *A function f is called a càdlàg function on $[0,T]$ if it satisfies the following properties*

(i) the left limit $\lim_{s\to t^-} f(s)$ exists

(ii) the right limit $\lim_{s\to t^+} f(s)$ exists and $\lim_{s\to t^+} f(s) = f(t)$

for every $t \in [0, T]$.

Then the collection of all càdlàg functions on $[0, T]$ *is defined as the space* $D[0, T]$.

Definition 1.1.3. *Let* Λ *denote the set of strictly increasing functions* $\lambda : [0, T] \to [0, T]$ *such that both* λ *and its inverse* λ^{-1} *are continuous. Let* e *be the identity function, that is,* $e(t) = t$ *for any* $t \in [0, T]$.

Under these notations, the J_1 *metric between càdlàg functions* x_1 *and* x_2 *on* $[0, T]$ *is defined as* [1]

$$d_{J_1}(x_1, x_2) = \inf_{\lambda \in \Lambda} \{\max\{\|x_1 \circ \lambda - x_2\|, \|\lambda - e\|\}\},$$

where the uniform norm $\|\cdot\|$ *is given as*

$$\|x\| = \sup_{t \in [0,T]} |x(t)|.$$

Remark 1.1.4. *The* J_1 *metric restricted to continuous functions on* $[0, T]$ *coincides with the metric induced by the uniform norm* $\|\cdot\|$. [2]

Definition 1.1.5. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space.* $\boldsymbol{X} = \{X(t, \omega) : t \in [0, T], \omega \in \Omega\}$ *is called a càdlàg process in the metric space* $(D[0, T], J_1)$ [3]*if the following holds.*

(i) $X(t, \cdot)$ *is a random variable on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *for each* $t \in [0, T]$

(ii) $X(\cdot, \omega)$ *is a function in the metric space* $(D[0, T], J_1)$ *for each* $\omega \in \Omega$

1.2 Convergence of càdlàg processes

In this section, we provide the different notions of convergence of càdlàg processes in the space $(D[0, T], J_1)$, and several important theorems that will be used in the proofs in this dissertation.

Let $\boldsymbol{X} = \{X(t, \omega) : t \in [0, T], \omega \in \Omega\}$, $\boldsymbol{X}_n = \{X_n(t, \omega) : t \in [0, T], \omega \in \Omega\}$ be càdlàg processes in the space $(D[0, T], J_1)$ with the common underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In the spirit of [10], the space $(D[0, T], J_1)$ can be used to represent the sample paths of such càdlàg process. Moreover, we can think of the distance $d_{J_1}(\boldsymbol{X}_n, \boldsymbol{X})$ as a random variable with value $d_{J_1}(X_n(\cdot, \omega), X(\cdot, \omega))$ at $\omega \in \Omega$. This motivates defining the different notions of convergence of càdlàg processes in the space $(D[0, T], J_1)$, which are summarized in the following table.

[1] It is shown [2, page 124] that the J_1 metric satisfies the non-negativity and symmetry property as well as the triangle inequality on the space $D[0, T]$, and is therefore well defined on the space $D[0, T]$. As a result, $(D[0, T], J_1)$ is a metric space.

[2] Remark 1.1.4 is also proven in [2, page 124].

[3] In this dissertation, we restrict our attention to càdlàg processes in the metric space $(D[0, T], J_1)$.

Notion	Definition	Notation
Almost sure convergence	$\mathbb{P}(\lim_{n\to\infty} d_{J_1}(\boldsymbol{X}_n, \boldsymbol{X}) = 0) = 1$	$\boldsymbol{X}_n \xrightarrow{a.s.} \boldsymbol{X}$
Convergence in probability	$\lim_{n\to\infty} \mathbb{P}(d_{J_1}(\boldsymbol{X}_n, \boldsymbol{X}) \geq \varepsilon) = 0, \forall \varepsilon \geq 0$	$\boldsymbol{X}_n \xrightarrow{P} \boldsymbol{X}$
Convergence in distribution	$\lim_{n\to\infty} \mathbb{E}(f(\boldsymbol{X}_n)) = \mathbb{E}(f(\boldsymbol{X})), \forall f \in C_b(S)$ [4]	$\boldsymbol{X}_n \Rightarrow \boldsymbol{X}$

Table 1.1: Table summarizing the notions of convergence of càdlàg processes in the space $(D[0,T], J_1)$.

For a sequence of càdlàg processes in the space $(D[0,T], J_1)$, we also have the property that almost sure convergence implies convergence in probability [6, Lemma 3.2], and convergence in probability implies convergence in distribution and the two conditions are equivalent when the limiting process is a constant [6, Lemma 3.7].

Like what we have in real analysis, continuous functions are limit-preserving on a metric space. Specifically, we have the following continuous mapping theorem that can be used on càdlàg processes in the space $(D[0,T], J_1)$.

Theorem 1.2.1. *[10, Theorem 3.4.1] (Continuous mapping theorem) If $\boldsymbol{X}_n \Rightarrow \boldsymbol{X}$ in (S, m) and $g : (S, m) \to (S', m')$ is continuous, then*

$$g(\boldsymbol{X}_n) \Rightarrow g(\boldsymbol{X})$$

in (S', m').

In order to prove Theorem 1.2.3 that will be used in Chapter 3, we state the following theorem from [10].

Theorem 1.2.2. *[10, Theorem 11.5.1] (Continuity of integrals) Suppose that $g : R^k \to R$ is a continuous function and let $f : (D^k[0,T], J_1) \to (C[0,T], U)$ [5] be defined by*

$$f(X)(t) = \int_0^t g(X)(s)ds, \ t \in [0,T],$$

then f is continuous.

Theorem 1.2.3. *Let $\boldsymbol{X}_n$ and $\boldsymbol{X}$ be càdlàg processes in the space $(D[0,T], J_1)$. Suppose that we have the convergence $\boldsymbol{X}_n \Rightarrow \boldsymbol{X}$ in $(D[0,T], J_1)$, then we obtain $\boldsymbol{Y}_n \Rightarrow \boldsymbol{Y}$ in $(D[0,T], J_1)$, where*

$$Y_n(t) = \int_0^t X_n(s)ds \text{ and } Y(t) = \int_0^t X(s)ds$$

for each $t \in [0,T]$.

[4] $C_b(S)$ is the space of bounded, continuous functions on the set S.

[5] $D^k[0,T]$ is the space of càdlàg processes from $[0,T]$ to R^k, and $(C[0,T], U)$ is the space of continuous functions on $[0,T]$ endowed with the uniform norm.

Proof. According to Theorem 1.2.2, we have that the function

$$f : f(X)(t) = \int_0^t X(s)ds,\ t \in [0,T]$$

from $(D[0,T], J_1)$ to $(C[0,T], U)$ is continuous. It follows from the continuous mapping theorem that $\boldsymbol{Y}_n \Rightarrow \boldsymbol{Y}$ in $(C[0,T], U)$, and thus $\boldsymbol{Y}_n \Rightarrow \boldsymbol{Y}$ in $(D[0,T], J_1)$ as $n \to \infty$. □

Theorem 1.2.4. *[10, Theorem 11.4.5] (Joint convergence when one limit is deterministic) Suppose that $\boldsymbol{X}_n \Rightarrow \boldsymbol{X}$ in a separable space (S', m') and $\boldsymbol{Y}_n \Rightarrow y$ in a separable space (S'', m''), where y is deterministic. Then*

$$(\boldsymbol{X}_n, \boldsymbol{Y}_n) \Rightarrow (\boldsymbol{X}, y)$$

in $S' \times S''$.

Remark 1.2.5. *Since the space $(D[0,T], J_1)$ is separable, we are able to use Theorem 1.2.4 on the space $(D[0,T], J_1)$.*

Theorem 1.2.6. *Suppose that we have a sequence of càdlàg processes $\boldsymbol{W}_n$ and we know the convergence $\sqrt{n}\boldsymbol{W}_n \Rightarrow \boldsymbol{W}$ in $(D[0,T], J_1)$ for some càdlàg process $\boldsymbol{W}$. Then we can obtain the convergence*

$$\sqrt{n}\boldsymbol{W}_n^2 \Rightarrow 0$$

in $(D[0,T], J_1)$.

Proof. Since $(\frac{1}{\sqrt{n}}, \sqrt{n}\boldsymbol{W}_n) \Rightarrow (0, \boldsymbol{W})$, we have that

$$\sqrt{n}\boldsymbol{W}_n^2 = \frac{1}{\sqrt{n}}(\sqrt{n}\boldsymbol{W}_n)^2 \Rightarrow 0 \times \boldsymbol{W}^2 = 0$$

by the continuous mapping theorem, because the map

$$(x, f) \mapsto xf^2$$

from $R \times D[0,T]$ to $D[0,T]$ is continuous. □

1.3 Jumps of càdlàg processes

Càdlàg processes include continuous stochastic processes like Brownian motion and discontinuous stochastic processes like Poisson process. When a càdlàg process is discontinuous, there is a good property for its jumps.

Theorem 1.3.1. *Define the size of jumps*

$$\Delta X(t) = X(t) - X(t-) = X(t) - \lim_{s \to t^-} X(s)$$

for a càdlàg process $\boldsymbol{X}$ *in the space* $(D[0,T], J_1)$*. Then the set* $\{t : |\Delta X(t)| > 0\}$ *is countable. Considering that countable sets have measure zero, for each* $t \in [0,T]$*, we have*

$$\int_0^t \Delta X(s)ds = 0,$$

and thus

$$\int_0^t X(s)ds = \int_0^t X(s-)ds.$$

1.4 Limit order books

What is a limit order book? Nowadays, each exchange maintains a limit order book to display the supply and demand of the price levels for each security traded on the exchange. In the limit order book, there are two types of orders: market orders and limit orders. A market buy/sell order is to buy/sell a security at the best available price. In this sense, when a market buy order comes in, it is executed immediately at the best (lowest) ask price. Similarly, the execution price of a market sell order is the best (highest) bid price. In contrast to market orders, a limit buy/sell order is to buy/sell a security at a price specified by the market participant. If the market does not reach this price, the limit order will not be executed. Another feature of limit orders is that they can be cancelled by market participants anytime before they are filled.

	Market orders	Limit orders
Guaranteed execution	Yes	No
Allowable cancellation	No	Yes

Table 1.2: Table summarizing the differences between market orders and limit orders.

1.5 Dynamics of order positions in the best bid queue

Mathematically, limit order books are considered as a discrete queuing system, and in this dissertation, we focus on the dynamics of order positions in the best bid queue that corresponds to the best bid price.

Given an order in the best bid queue, its dynamic is decreased by the presence of both market orders at the best bid and cancellations at the best bid ahead of it. Specifically, we have the identity

$$\begin{aligned}\Delta(\text{order position at the best bid}) = &-\Delta(\text{market orders at the best bid})\\ &-\Delta(\text{cancellations at the best bid ahead of the order})\end{aligned} \tag{1.1}$$

on the infinitesimal interval $[t, t + \Delta t]$.

Limit orders at the best bid have no influence on this dynamics, but they will increase the length of the best bid queue in the way

$$\begin{aligned} \Delta(\text{best bid queue length}) &= \text{limit orders at the best bid} \\ &\quad - \text{market orders at the best bid} \\ &\quad - \text{cancellations at the best bid} \end{aligned} \tag{1.2}$$

on the interval $[0, t]$.

1.6 Model framework

In this section, we describe the scaling of actual physical queues and then provide a high level description of the model framework in [4]. To this end, we need to introduce a number of definitions and assumptions.

The sequence of inter-arrival times of orders $\{D_i\}_{i\geq 1}$ is assumed [4, Assumption 1] to be a stationary array of positive random variables with

$$\frac{D_1 + D_2 + \cdots + D_i}{i} \to \frac{1}{\lambda}, \text{ in probability}$$

as $i \to \infty$, where λ is a positive constant.

Then we define the order arrival process $\mathbf{N} = (N(t), t \geq 0)$ with

$$N(t) = \max\{m : \sum_{i=1}^{m} D_i \leq t\}.$$

Next, we define the order intensity process $\vec{\boldsymbol{V}}$, where for the ith order, $\vec{V_i} = (V_i^1, V_i^2, \cdots, V_i^6)$ is a six-dimensional vector representing the sizes of limit orders at the best bid, market orders at the best bid, cancellations at the best bid, limit orders at the best ask, market orders at the best ask and cancellations at the best ask respectively.

It's assumed [4, Assumption 2] that $\{\vec{V_i}\}_{i\geq 1}$ is a stationary array of square-integrable random vectors with

$$\frac{\vec{V_1} + \vec{V_2} + \cdots + \vec{V_i}}{i} \to \vec{\bar{V}}, \text{ in probability}$$

as $i \to \infty$, where $\vec{\bar{V}} = (\bar{V}_1, \bar{V}_2, \cdots, \bar{V}_6)$ is a six-dimensional constant vector.

With the help of the assumption [4, Assumption 3] that $\{D_i\}_{i\geq 1}$ is independent of $\{\vec{V_i}\}_{i\geq 1}$, define the scaled order flow process

$$\vec{C}_n(t) = \frac{1}{n} \sum_{i=1}^{N(nt)} \vec{V_i},$$

where $C_n^1(t)$, $C_n^2(t)$ and $C_n^3(t)$ represent the scaled limit orders at the best bid, scaled market orders at the best bid and scaled cancellations at the best bid respectively.

With these scaled order flow processes, we can translate (1.2) into

$$Q_n^b(t) - Q_n^b(0) = C_n^1(t) - C_n^2(t) - C_n^3(t), \tag{1.3}$$

where $Q_n^b(t)$ is defined as the scaled best bid queue length at time t.

Notice that the cancellations ahead of the order in the best bid queue are only a fraction of the cancellations placed in the best bid queue, and this fraction is closely related to the order position relative to the best bid queue length. We characterize this relationship by the function Υ in a way that

$$\text{cancellations at the best bid ahead of the order} = \Upsilon\left(\frac{Z_n(t-)}{Q_n^b(t-)}\right) C_n^3(t), \tag{1.4}$$

where $Z_n(t)$ is defined as the scaled order position in the best bid queue at time t, and the fraction $\frac{Z_n(t-)}{Q_n^b(t-)}$ indicates the order position relative to the best bid queue length immediately before time t. Note that Υ is a random function in practice. However, for the convenience of modelling, we consider Υ the expectation and assume that Υ is deterministic rather than random.

Thus we can translate (1.1) into

$$dZ_n(t) = -dC_n^2(t) - \Upsilon\left(\frac{Z_n(t-)}{Q_n^b(t-)}\right) dC_n^3(t). \tag{1.5}$$

In a rigorous manner, (1.3) and (1.5) only describe the dynamics of $Q_n^b(t)$ and $Z_n(t)$ before any of them hits zero. To extend them for any $t \geq 0$, define the truncated processes

$$\hat{Q}_n^b(t) = Q_n^b(t \wedge \tau_n),\ \hat{Z}_n(t) = Z_n(t \wedge \tau_n),^{6}$$

where

$$\tau_n = \min\Big\{\inf\{t \geq 0 : Q_n^b(t) \leq 0\}, \inf\{t \geq 0 : Z_n(t) \leq 0\}\Big\}.$$

For the convenience of notations, we will use $\boldsymbol{Q}_n^b$ and $\boldsymbol{Z}_n$ rather than $\hat{\boldsymbol{Q}}_n^b$ and $\hat{\boldsymbol{Z}}_n$ from now on.

We refer to (1.4) as the general assumption of cancellations in the best bid queue because it allows for various choices of Υ. For example, when Υ is the identity function, (1.4) becomes the uniform distribution of cancellations. Considering that the closer the order to the best bid queue head, the less likely it is cancelled, we have the following assumption.

Assumption 1.6.1. *Υ is a Lipschitz continuous increasing function from [0,1] to [0,1] with $\Upsilon(0) = 0$ and $\Upsilon(1) = 1$.*

[6] It is shown in [4, Lemma 6] that the truncated processes are well defined, and can be extended for any $t \geq 0$.

For the convenience of the analysis in Chapter 3, we impose an additional assumption for Υ.

Assumption 1.6.2. *Υ is twice continuously differentiable on [0,1] with Lipschitz constant L satisfying*

$$L \leq \inf_{t\in[0,T]} \left| \frac{Q^b(t-)}{2\sqrt{2}C^3(t)} \right|.$$

1.7 Technical results

In this section, we list some results of convergence in [4] that this dissertation builds on.

In order to describe the dynamics before the best bid queue vanishes, we define

$$\tau = \min\Big\{ \lim_{n\to\infty} \inf\{t \geq 0 : Q_n^b(t) \leq 0\}, \lim_{n\to\infty} \inf\{t \geq 0 : Z_n(t) \leq 0\} \Big\}.$$

It can be shown in [4, (2.10) and (2.11)] that τ is a constant.

According to [4, Theorem 31], we have the convergence of scaled order positions

$$\boldsymbol{Z}_n \Rightarrow \boldsymbol{Z}, \text{ in } (D[0,\tau], J_1) \tag{1.6}$$

as $n \to \infty$, where the limiting process $\boldsymbol{Z}$ satisfies the ordinary differential equation

$$dZ(t) = -dC^2(t) - \Upsilon\left(\frac{Z(t-)}{Q^b(t-)}\right) dC^3(t) \tag{1.7}$$

when $t \leq \tau \leq T$. $\boldsymbol{Z}$ can be interpreted as the theoretic order position in the model framework.

Also, we have the convergence for the scaled best bid queue length

$$\boldsymbol{Q}_n^b \Rightarrow \boldsymbol{Q}^b, \text{ in } (D[0,\tau], J_1) \tag{1.8}$$

and its fluctuation around the limit

$$\sqrt{n}(\boldsymbol{Q}_n^b - \boldsymbol{Q}^b) \Rightarrow \boldsymbol{\Psi}^1 - \boldsymbol{\Psi}^2 - \boldsymbol{\Psi}^3, \text{ in } (D[0,\tau], J_1) \tag{1.9}$$

as $n \to \infty$, where $\vec{\boldsymbol{\Psi}} = (\boldsymbol{\Psi}^1, \boldsymbol{\Psi}^2, \cdots, \boldsymbol{\Psi}^6)$ is a six-dimensional correlated Brownian motion with zero drift.

Similarly, we have the convergence

$$\vec{\boldsymbol{C}}_n \Rightarrow \vec{\boldsymbol{C}}, \text{ in } (D^6[0,\tau], J_1), \tag{1.10}$$

and its fluctuation

$$\sqrt{n}(\vec{\boldsymbol{C}}_n - \vec{\boldsymbol{C}}) \Rightarrow \vec{\boldsymbol{\Psi}}, \text{ in } (D^6[0,\tau], J_1), \tag{1.11}$$

as $n \to \infty$, where the second convergence is translated from [4, Theorem 4]. $\vec{C}(t) = (C^1(t), C^2(t), \cdots, C^6(t))$ is a six-dimensional deterministic vector where $C^i(t) = \lambda \bar{V}_i t$.

With the help of Theorem 1.2.4, we are able to get the joint convergence for each subset of the càdlàg processes in Table 1.3 as long as at most one càdlàg process in the subset has random limiting process.

càdlàg process	Limiting process	Property
$\boldsymbol{Z}_n$	$\boldsymbol{Z}$	$\boldsymbol{Z}$ is deterministic
$\boldsymbol{Q}_n^b$	$\boldsymbol{Q}^b$	$\boldsymbol{Q}^b$ is deterministic
$\sqrt{n}(\boldsymbol{Q}_n^b - \boldsymbol{Q}^b)$	$\boldsymbol{\Psi}^1 - \boldsymbol{\Psi}^2 - \boldsymbol{\Psi}^3$	$\boldsymbol{\Psi}$ is a Brownian motion
$\overrightarrow{\boldsymbol{C}}_n$	$\overrightarrow{\boldsymbol{C}}$	$\overrightarrow{\boldsymbol{C}}$ is deterministic
$\sqrt{n}(\overrightarrow{\boldsymbol{C}}_n - \overrightarrow{\boldsymbol{C}})$	$\boldsymbol{\Psi}$	$\boldsymbol{\Psi}$ is a Brownian motion

Table 1.3: Table summarizing the convergence in [4].

1.8 Overview

The remainder of this dissertation is organized as follows.

Chapter 2 surveys some related models from literature. In Chapter 3, we present our contributions: we analyze the fluctuation of scaled order positions in the best bid queue under the general assumption for cancellations.

Chapter 2

Literature

There are a bunch of papers considering a limit order book as a discrete queuing system and analyzing the dynamics of the limit order book in the system. These work can be traced back to Kruk [7], where they establish the dynamics of the number of outstanding orders in a transparent auction. [1]When there are only two price levels in the auction, it has almost the same structure as a limit order book, except that cancellations are not considered. Similar to the dynamics proposed in [7], the joint dynamics of the best bid and ask queue length in a limit order book is derived by Cont and De Larrard in [3]. Instead of analyzing the queue length, Horst and Paulsen establish the joint dynamics of the best bid and ask price and their volume densities in [5].

2.1 Kruk's model

In [7], the author considers a transparent auction with a single asset with a finite number N of possible asset prices $P_1 < P_2 < \cdots < P_N$. Denote the number of outstanding buying and selling orders with price P_i at time t by $B_i(t)$ and $S_i(t)$ respectively. Since buying and selling orders at the price level P_i are executed against each other, it is always true that $B_i(t)S_i(t) = 0$.

In the case of two prices ($N = 2$), considering $B_2(t) = 0$ and $S_1(t) = 0$, the remaining two variables $B_1(t)$ and $S_2(t)$ are similar to the best bid and ask queue length.

Variables in an auction	Analogous variable in a limit order book
Number of outstanding buying orders at P_1	Best bid queue length
Number of outstanding selling orders at P_2	Best ask queue length

Table 2.1: Table summarizing the analogy between an auction and a limit order book in section 2.1.

[1]A transparent auction means all the submitted orders in the auction are visible to all customers.

Under the scaling

$$\hat{B}_1^{(n)}(t) = \frac{1}{\sqrt{n}} B_1(nt) \text{ and } \hat{S}_2^{(n)}(t) = \frac{1}{\sqrt{n}} S_2(nt),$$

the joint dynamics of scaled number of orders is derived [7, Corollary 4.6]

$$\left(\hat{B}_1^{(n)}, \hat{S}_2^{(n)}\right) \Rightarrow Z,$$

where Z is a two-dimensional Brownian motion with drift, reflected instantaneously at the boundary of the non-negative orthant in the direction $(1, 1)$.

2.2 Cont and De Larrard's model

Let $q^b(t)$ and $q^a(t)$ denote the number of outstanding orders on the best bid queue and best ask queue at time t respectively, before any of them hits zero. It is established in [3, Section 4.2 Theorem 2] that the scaled queue size process

$$\left(\frac{q^b(nt)}{\sqrt{n}}, \frac{q^a(nt)}{\sqrt{n}}\right)$$

converges in distribution to a Markov process Q_t in the first quadrant, which behaves like a planar Brownian motion with drift vector

$$(\lambda^b \bar{V}^b, \lambda^a \bar{V}^a)$$

and covariance matrix

$$\begin{pmatrix} \lambda^b v_b^2 & \rho\sqrt{\lambda^a \lambda^b} v_a v_b \\ \rho\sqrt{\lambda^a \lambda^b} v_a v_b & \lambda^a v_a^2 \end{pmatrix}.$$

Notation	Interpretation
$\frac{1}{\lambda^b}$	average duration between orders at the best bid
$\frac{1}{\lambda^a}$	average duration between orders at the best ask
$\bar{V}^b$	expectation of order sizes at the best bid
$\bar{V}^a$	expectation of order sizes at the best ask
v_b	standard deviation of order sizes at the best bid
v_a	standard deviation of order sizes at the best ask
ρ	correlation between order sizes at the best bid and order sizes at the best ask

Table 2.2: Table summarizing the interpretations of notations in section 2.2.

This dynamics allows to study analytically various quantities, such as the conditional distribution of the duration between price moves [3, Section 5.2 Proposition 2] and the probability of a price increase [3, Section 5.3 Theorem 3].

2.3 Horst and Paulsen's model

In [5], they denote the state of a limit order book by

$$S = (B, A, v_b(\cdot), v_a(\cdot)),$$

where

- B is the best bid price,
- A is the best ask price,
- $v_b(x)$ is the volume density at price distance x below the best bid price,
- $v_a(x)$ is the volume density at price distance x above the best ask price.

They then introduce a sequence of limit order book models when the price tick of the model tends to zero and the rate of order arrivals tends to infinity. This sequence of models is indexed by $n \in \mathbb{N}$, where $S_k^{(n)}$ represents the nth system's state after k events occurred in that system.

It is proven in [5, Theorem 1.13] that the process $\{S^{(n)}\}$ converges in probability to a deterministic process, where the volume densities in this deterministic process are given as the unique solution to a partial differential equation [5, (16)] specified by model parameters.

Chapter 3

Fluctuation analysis of order positions under general cancellations

3.1 Motivation

From (1.6), we know that the scaled order positions $\boldsymbol{Z}_n$ converge to the theoretic order position $\boldsymbol{Z}$ under the general assumption for cancellations. This indicates that $\boldsymbol{Z}_n$ can be used to approximate $\boldsymbol{Z}$ in practice. From there, we would like to investigate how good this approximation is. The fluctuation of $\boldsymbol{Z}_n$ around its limit $\boldsymbol{Z}$ under proper scaling, that is, $\sqrt{n}(\boldsymbol{Z}_n - \boldsymbol{Z})$ serves as a measure of the approximation error. As a consequence, it is important to analyze this fluctuation $\sqrt{n}(\boldsymbol{Z}_n - \boldsymbol{Z})$ as $n \to \infty$. This is the focus of the dissertation.

There are two questions.

(i) What is the limiting behavior of the fluctuation? Mathematically, what is the stochastic differential equation that the stochastic process $\sqrt{n}(\boldsymbol{Z}_n - \boldsymbol{Z})$ satisfies in the space $(D[0,\tau], J_1)$ as $n \to \infty$?

(ii) What is the financial interpretation of this stochastic differential equation?

We solve the first question in section 3.2 by establishing the stochastic differential equation (Theorem 3.2.1), and prove it in section 3.4. To get a better sense of this stochastic differential equation, we interpret it in section 3.3. One related property (Proposition 3.2.3) is proven in section 3.5.

3.2 Main results

Theorem 3.2.1. *Given Assumption 1.6.1 and 1.6.2, we have the convergence*

$$\sqrt{n}(\boldsymbol{Z_n} - \boldsymbol{Z}) \Rightarrow \boldsymbol{Y}, \ in\,(D[0,\tau], J_1)$$

as $n \to \infty$, where $Y(t)$ satisfies the stochastic differential equation

$$\begin{aligned} dY(t) = \Upsilon'\left(\frac{Z(t)}{Q^b(t)}\right)\left(\frac{Z(t)(\Psi^1(t)-\Psi^2(t)-\Psi^3(t))}{Q^b(t)} - Y(t)\right)\frac{dC^3(t)}{Q^b(t)} \\ - d\Psi^2(t) - \Upsilon\left(\frac{Z(t)}{Q^b(t)}\right)d\Psi^3(t), \end{aligned} \tag{3.1}$$

with $Y(0) = 0$.

Remark 3.2.2. *We will prove it in section 3.4.*

Proposition 3.2.3. $\boldsymbol{Y}$ *is a Gaussian process with mean zero.*

Remark 3.2.4. *The proof is given in section 3.5.*

3.3 Interpretations

To interpret our result (3.1), we first notice that the stochastic process $\boldsymbol{Y}$ is mean-reverting on $[0, \tau]$.

The mean-reverting level of the stochastic process $\boldsymbol{Y}$ is

$$\frac{\boldsymbol{Z}(\boldsymbol{\Psi}^1 - \boldsymbol{\Psi}^2 - \boldsymbol{\Psi}^3)}{\boldsymbol{Q}^b}.$$

According to Table 1.3, we have the convergence

$$\sqrt{n}(\boldsymbol{Q}_n^b - \boldsymbol{Q}^b) \Rightarrow \boldsymbol{\Psi}^1 - \boldsymbol{\Psi}^2 - \boldsymbol{\Psi}^3,$$

and thus we interpret $\boldsymbol{\Psi}^1 - \boldsymbol{\Psi}^2 - \boldsymbol{\Psi}^3$ as the fluctuation of scaled best bid queue length. In light of this, the mean-reverting level is the same as the fluctuation of scaled best bid queue length multiplied by the order positions relative to the best bid queue length.

The mean-reverting speed of the stochastic process $\boldsymbol{Y}$ is proportional to $\Upsilon'\left(\frac{\boldsymbol{Z}}{\boldsymbol{Q}^b}\right)$. According to (1.4), $\Upsilon\left(\frac{\boldsymbol{Z}}{\boldsymbol{Q}^b}\right)$ is proportional to the rate of cancellations ahead of the order in the best bid queue. In this sense, the mean-reverting speed is proportional to the rate of change of cancellations ahead of the order in the best bid queue.

3.4 Proof of Theorem 3.2.1

As we will see later in this section, the difficulty in the proof of this Theorem 3.2.1 arises when we try to evaluate the difference between two values of the function Υ. We obtain a solution using Taylor's Theorem. We use the linear Taylor polynomial with remainder, and prove that the remainder (proportional to quadratic terms) converges to zero (Proposition

3.4.6 and Proposition 3.4.11). In this sense, quadratic terms vanish as $n \to \infty$, which gives us a substantial simplification. Specifically, we proceed the proof of Theorem 3.2.1 in the following three steps.

(i) Prove Proposition 3.4.6 using Lemma 3.4.1-3.4.5 and Lemma 3.4.7-3.4.10.

(ii) Proof Proposition 3.4.11.

(iii) Prove Theorem 3.2.1 using Proposition 3.4.6 and Proposition 3.4.11.

Step 1

The procedure in the first step is to rearrange terms of $\boldsymbol{Z}_n - \boldsymbol{Z}$ to establish the upper bound of $\sqrt{n}(\boldsymbol{Z}_n - \boldsymbol{Z})^2$ (3.2), and then construct the convergence/inequalities for each component in the upper bound. When it is an inequality, we use Lemma 3.4.1 to pass the convergence. Finally, we put all convergence together to prove

$$\int_0^t \sqrt{n}\left(Z_n(s) - Z(s)\right)^2 ds \Rightarrow 0, \text{ in } (D[0,\tau], J_1)$$

as $n \to \infty$.

Let us go into details. First define the stochastic processes on $[0, \tau]$

$$\begin{cases} \boldsymbol{D}_{1,n} = \boldsymbol{C}^2 - \boldsymbol{C}_n^2 \\ \boldsymbol{D}_{2,n} = \Upsilon\left(\dfrac{\boldsymbol{Z}(-)}{\boldsymbol{Q}^b(-)}\right)\boldsymbol{C}^3 - \Upsilon\left(\dfrac{\boldsymbol{Z}(-)}{\boldsymbol{Q}_n^b(-)}\right)\boldsymbol{C}^3 \\ \boldsymbol{D}_{3,n} = \Upsilon\left(\dfrac{\boldsymbol{Z}(-)}{\boldsymbol{Q}_n^b(-)}\right)\boldsymbol{C}^3 - \Upsilon\left(\dfrac{\boldsymbol{Z}(-)}{\boldsymbol{Q}_n^b(-)}\right)\boldsymbol{C}_n^3 \\ \boldsymbol{D}_{4,n} = \Upsilon\left(\dfrac{\boldsymbol{Z}(-)}{\boldsymbol{Q}_n^b(-)}\right)\boldsymbol{C}_n^3 - \Upsilon\left(\dfrac{\boldsymbol{Z}_n(-)}{\boldsymbol{Q}_n^b(-)}\right)\boldsymbol{C}_n^3 \end{cases}$$

and observe that

$$\begin{aligned} \boldsymbol{D}_{1,n} + \boldsymbol{D}_{2,n} + \boldsymbol{D}_{3,n} + \boldsymbol{D}_{4,n} &= \boldsymbol{C}^2 - \boldsymbol{C}_n^2 + \left(\Upsilon\left(\frac{\boldsymbol{Z}(-)}{\boldsymbol{Q}^b(-)}\right)\boldsymbol{C}^3 - \Upsilon\left(\frac{\boldsymbol{Z}_n(-)}{\boldsymbol{Q}_n^b(-)}\right)\boldsymbol{C}_n^3\right) \\ &= \left(-\boldsymbol{C}_n^2 - \Upsilon\left(\frac{\boldsymbol{Z}_n(-)}{\boldsymbol{Q}_n^b(-)}\right)\boldsymbol{C}_n^3\right) - \left(-\boldsymbol{C}^2 - \Upsilon\left(\frac{\boldsymbol{Z}(-)}{\boldsymbol{Q}^b(-)}\right)\boldsymbol{C}^3\right) \\ &= \boldsymbol{Z}_n - \boldsymbol{Z}, \end{aligned}$$

where the first and second component in the second line are the dynamics of $\boldsymbol{Z}_n$ (1.5) and $\boldsymbol{Z}$ (1.7) respectively.

Then we apply the Cauchy-Schwarz inequality to get

$$\begin{aligned} \sqrt{n}(\boldsymbol{Z}_n - \boldsymbol{Z})^2 &= \sqrt{n}(\boldsymbol{D}_{1,n} + \boldsymbol{D}_{2,n} + \boldsymbol{D}_{3,n} + \boldsymbol{D}_{4,n})^2 \\ &\leq 4\sqrt{n}(\boldsymbol{D}_{1,n}^2 + \boldsymbol{D}_{2,n}^2 + \boldsymbol{D}_{3,n}^2 + \boldsymbol{D}_{4,n}^2). \end{aligned} \tag{3.2}$$

Lemma 3.4.1. *Let $\boldsymbol{X}_n$ and $\boldsymbol{Y}_n$ be càdlàg processes in the space $(D[0,T], J_1)$. If they satisfy the inequality $0 \leq \boldsymbol{X}_n \leq \boldsymbol{Y}_n$, and $\boldsymbol{Y}_n \xrightarrow{P} 0$ in $(D[0,T], J_1)$ as $n \to \infty$, then we have $\boldsymbol{X}_n \xrightarrow{P} 0$ in $(D[0,T], J_1)$ as $n \to \infty$* [1].

Proof. Intuitively, it is the squeeze theorem of càdlàg processes in the space $(D[0,T], J_1)$, and the key is to prove that the J_1 metric behaves like the Euclidean distance around 0, that is,

$$d_{J_1}(\boldsymbol{X}_n, 0) \leq d_{J_1}(\boldsymbol{Y}_n, 0).$$

Since $0 \leq \boldsymbol{X}_n \leq \boldsymbol{Y}_n$, we have that

$$\|\boldsymbol{X}_n \circ \lambda - 0\| \leq \|\boldsymbol{Y}_n \circ \lambda - 0\|,$$

and thus

$$\inf_{\lambda \in \Lambda}\{\max\{\|\boldsymbol{X}_n \circ \lambda - 0\|, \|\lambda - e\|\}\} \leq \inf_{\lambda \in \Lambda}\{\max\{\|\boldsymbol{Y}_n \circ \lambda - 0\|, \|\lambda - e\|\}\},$$

which is equivalent to

$$d_{J_1}(\boldsymbol{X}_n, 0) \leq d_{J_1}(\boldsymbol{Y}_n, 0).$$

This implies that

$$\mathbb{P}(d_{J_1}(\boldsymbol{X}_n, 0) \geq \varepsilon) \leq \mathbb{P}(d_{J_1}(\boldsymbol{Y}_n, 0) \geq \varepsilon) \tag{3.3}$$

for any $\varepsilon \geq 0$. Since $\boldsymbol{Y}_n \xrightarrow{P} 0$, we have

$$\lim_{n \to \infty} \mathbb{P}(d_{J_1}(\boldsymbol{Y}_n, 0) \geq \varepsilon) = 0. \tag{3.4}$$

Combining (3.3) and (3.4) yields

$$\lim_{n \to \infty} \mathbb{P}(d_{J_1}(\boldsymbol{X}_n, 0) \geq \varepsilon) = 0,$$

which completes the proof.

□

Lemma 3.4.2. *We have the convergence*

$$\sqrt{n}\boldsymbol{D}_{1,n}^2 \Rightarrow 0, \ \ in\, (D[0,\tau], J_1)$$

as $n \to \infty$.

[1] Notice that $\boldsymbol{Y}_n \xrightarrow{P} 0$ is equivalent to $\boldsymbol{Y}_n \Rightarrow 0$, so we can also apply Lemma 3.4.1 to the convergence in distribution to zero.

Proof. By Table 1.3, we have the convergence

$$\sqrt{n}(\boldsymbol{C}^2 - \boldsymbol{C}_n^2) \Rightarrow \boldsymbol{\Psi}^2 \tag{3.5}$$

in $(D[0,\tau], J_1)$ as $n \to \infty$. Then we apply Theorem 1.2.6 to get that

$$\sqrt{n}\boldsymbol{D}_{1,n}^2 = \sqrt{n}(\boldsymbol{C}^2 - \boldsymbol{C}_n^2)^2 \Rightarrow 0, \text{ in } (D[0,\tau], J_1) \tag{3.6}$$

as $n \to \infty$, which completes the proof. □

Lemma 3.4.3. *We have the inequality*

$$\sqrt{n}\boldsymbol{D}_{2,n}^2 \leq \boldsymbol{A}_n$$

where

$$\boldsymbol{A}_n = L^2 \frac{\boldsymbol{Z}(-)^2(\boldsymbol{C}^3)^2}{(\boldsymbol{Q}^b(-))^2} \frac{\sqrt{n}(\boldsymbol{Q}_n^b(-) - \boldsymbol{Q}^b(-))^2}{(\boldsymbol{Q}_n^b(-))^2}.$$

Proof. We use the Lipschitz property to get

$$\begin{aligned} |\boldsymbol{D}_{2,n}| &= \left| \Upsilon\left(\frac{\boldsymbol{Z}(-)}{\boldsymbol{Q}^b(-)}\right)\boldsymbol{C}^3 - \Upsilon\left(\frac{\boldsymbol{Z}(-)}{\boldsymbol{Q}_n^b(-)}\right)\boldsymbol{C}^3 \right| \\ &\leq L \left| \frac{\boldsymbol{Z}(-)\boldsymbol{C}^3}{\boldsymbol{Q}^b(-)} \right| \left| \frac{\boldsymbol{Q}_n^b(-) - \boldsymbol{Q}^b(-)}{\boldsymbol{Q}_n^b(-)} \right| \end{aligned} \tag{3.7}$$

Squaring and multiplying by $\sqrt{n}$ completes the proof. □

Lemma 3.4.4. *We have the convergence*

$$\sqrt{n}\boldsymbol{D}_{3,n}^2 \Rightarrow 0, \ \textit{in}\, (D[0,\tau], J_1)$$

as $n \to \infty$.

Proof. We first observe that

$$\sqrt{n}\boldsymbol{D}_{3,n}^2 = \Upsilon\left(\frac{\boldsymbol{Z}(-)}{\boldsymbol{Q}_n^b(-)}\right)^2 \cdot \sqrt{n}(\boldsymbol{C}^3 - \boldsymbol{C}_n^3)^2 \leq \sqrt{n}(\boldsymbol{C}^3 - \boldsymbol{C}_n^3)^2 \tag{3.8}$$

By Table 1.3, we have the convergence

$$\sqrt{n}(\boldsymbol{C}^3 - \boldsymbol{C}_n^3) \Rightarrow \boldsymbol{\Psi}^3 \tag{3.9}$$

in $(D[0,\tau], J_1)$, as $n \to \infty$.

According to Theorem 1.2.6, we get that

$$\sqrt{n}(\boldsymbol{C}^3 - \boldsymbol{C}_n^3)^2 \Rightarrow 0, \text{ in } (D[0,\tau], J_1) \tag{3.10}$$

as $n \to \infty$.

It follows immediately from Lemma 3.4.1 that

$$\sqrt{n}\boldsymbol{D}_{3,n}^2 \Rightarrow 0, \text{ in } (D[0,\tau], J_1)$$

as $n \to \infty$.

□

Lemma 3.4.5. *We have the inequality*

$$\sqrt{n}\boldsymbol{D}_{4,n}^2 \leq \sqrt{n}(\boldsymbol{Z}_n(-) - \boldsymbol{Z}(-))^2 \boldsymbol{B}_n,$$

where

$$\boldsymbol{B}_n = \frac{1}{8}\left(\frac{\boldsymbol{C}_n^3}{\boldsymbol{Q}_n^b(-)}\right)^2 \left(\frac{\boldsymbol{Q}^b(-)}{\boldsymbol{C}^3}\right)^2.$$

Proof. We use the Lipschitz property to get

$$\begin{aligned} |\boldsymbol{D}_{4,n}| &= \left|\Upsilon\left(\frac{\boldsymbol{Z}(-)}{\boldsymbol{Q}_n^b(-)}\right)\boldsymbol{C}_n^3 - \Upsilon\left(\frac{\boldsymbol{Z}_n(-)}{\boldsymbol{Q}_n^b(-)}\right)\boldsymbol{C}_n^3\right| \\ &\leq L\left|\frac{\boldsymbol{Z}_n(-) - \boldsymbol{Z}(-)}{\boldsymbol{Q}_n^b(-)}\right| \left|\boldsymbol{C}_n^3\right| \end{aligned} \tag{3.11}$$

Squaring and multiplying by $\sqrt{n}$ yields

$$\begin{aligned} \sqrt{n}\boldsymbol{D}_{4,n}^2 &\leq \sqrt{n}(\boldsymbol{Z}_n(-) - \boldsymbol{Z}(-))^2 L^2 \left(\frac{\boldsymbol{C}_n^3}{\boldsymbol{Q}_n^b(-)}\right)^2 \\ &= \sqrt{n}(\boldsymbol{Z}_n(-) - \boldsymbol{Z}(-))^2 \cdot 8L^2 \left(\frac{\boldsymbol{C}^3}{\boldsymbol{Q}^b(-)}\right)^2 \cdot \frac{1}{8}\left(\frac{\boldsymbol{C}_n^3}{\boldsymbol{Q}_n^b(-)}\right)^2 \left(\frac{\boldsymbol{Q}^b(-)}{\boldsymbol{C}^3}\right)^2 \end{aligned} \tag{3.12}$$

By Assumption 1.6.2, we have

$$L \leq \inf_{0 \leq t \leq \tau} \left|\frac{Q^b(t-)}{2\sqrt{2}C^3(t)}\right|$$

and thus

$$8L^2 \cdot \left(\frac{\boldsymbol{C}^3}{\boldsymbol{Q}^b(-)}\right)^2 \leq \left(\frac{\inf\left|\frac{\boldsymbol{Q}^b(-)}{2\sqrt{2}\boldsymbol{C}^3}\right|}{\left|\frac{\boldsymbol{Q}^b(-)}{2\sqrt{2}\boldsymbol{C}^3}\right|}\right)^2 \leq 1 \tag{3.13}$$

Combining (3.12) and (3.13) gives

$$\sqrt{n}\boldsymbol{D}_{4,n}^2 \leq \sqrt{n}(\boldsymbol{Z}_n(-) - \boldsymbol{Z}(-))^2 \frac{1}{8}\left(\frac{\boldsymbol{C}_n^3}{\boldsymbol{Q}_n^b(-)}\right)^2 \left(\frac{\boldsymbol{Q}^b(-)}{\boldsymbol{C}^3}\right)^2. \tag{3.14}$$

So we proved Lemma 3.4.5. □

Hence we have established the convergence/inequalities of each components in the right hand side of (3.2). Notice that the upper bound in Lemma 3.4.5 is related to $(\boldsymbol{Z}_n - \boldsymbol{Z})^2$, which should be combined with the left hand side of (3.2) in order to derive the convergence. This is done in (3.16).

Since we need to pass the convergence to the integral whose integrand involves jumps, We establish Lemma 3.4.7 first, and then use it to prove the convergence of the integral (3.19)-(3.21), which is done in Lemma 3.4.8-3.4.10.

Proposition 3.4.6. *Given Assumption 1.6.1 and 1.6.2, we have the convergence*

$$\int_0^t \sqrt{n}\,(Z_n(s) - Z(s))^2\,ds \Rightarrow 0,\ \ in\,(D[0,\tau], J_1)$$

as $n \to \infty$.

Proof. It is equivalent to show that

$$\int_0^t \sqrt{n}\,(Z_n(s) - Z(s))^2\,ds \xrightarrow{P} 0,\ \text{in}\,(D[0,\tau], J_1)$$

as $n \to \infty$.

Plugging the bounds from Lemma 3.4.1 and 3.4.5 into the inequality (3.2) and separating the jumps of $Z_n(t)$ gives

$$\begin{aligned}
\sqrt{n}(Z_n(t) - Z(t))^2 &\le 4\sqrt{n}D_{1,n}^2(t) + 4A_n(t) + 4\sqrt{n}D_{3,n}^2(t) + \sqrt{n}\,(Z_n(t-) - Z(t-))^2 \cdot 4B_n(t) \\
&= 4\sqrt{n}D_{1,n}^2(t) + 4A_n(t) + 4\sqrt{n}D_{3,n}^2(t) \\
&+ \sqrt{n}\,(Z_n(t) - \Delta Z_n(t) - Z(t))^2 \cdot 4B_n(t) \\
&= 4\sqrt{n}D_{1,n}^2(t) + 4A_n(t) + 4\sqrt{n}D_{3,n}^2(t) + \sqrt{n}(Z_n(t) - Z(t))^2 \cdot 4B_n(t) \\
&+ \sqrt{n}\Delta Z_n^2(t) \cdot 4B_n(t) - 2\sqrt{n}(Z_n(t) - Z(t))\Delta Z_n(t) \cdot 4B_n(t).
\end{aligned} \tag{3.15}$$

We move the term $\sqrt{n}(Z_n(t)-Z(t))^2 \cdot 4B_n(t)$ over so we have everything about $\sqrt{n}(Z_n(t)-Z(t))^2$ in the left hand side, and then divide the inequality by $(1 - 4B_n(t))$ [2]

$$\begin{aligned}
\sqrt{n}(Z_n(t) - Z(t))^2 \le \frac{1}{1 - 4B_n(t)}\big(&4\sqrt{n}D_{1,n}^2(t) + 4A_n(t) + 4\sqrt{n}D_{3,n}^2(t) + \sqrt{n}\Delta Z_n^2(t) \cdot 4B_n(t) \\
&- 2\sqrt{n}(Z_n(t) - Z(t))\Delta Z_n(t) \cdot 4B_n(t)\big).
\end{aligned} \tag{3.16}$$

We integrate (3.16) from 0 to t to obtain

[2] It is legitimate to assume $\boldsymbol{B}_n = \frac{1}{8}\left(\frac{\boldsymbol{C}_n^3}{\boldsymbol{Q}_n^b(-)}\right)^2\left(\frac{\boldsymbol{Q}^b(-)}{\boldsymbol{C}^3}\right)^2 < \frac{1}{4}$, because we will see later that $\bar{\boldsymbol{B}}_n = \frac{1}{8}\left(\frac{\boldsymbol{C}_n^3}{\boldsymbol{Q}_n^b}\right)^2\left(\frac{\boldsymbol{Q}^b}{\boldsymbol{C}^3}\right)^2 \Rightarrow \frac{1}{8}$, which indicates $\|\bar{\boldsymbol{B}}_n - \frac{1}{8}\| \to 0$ as $n \to \infty$.

$$\int_0^t \sqrt{n}(Z_n(s)-Z(s))^2 ds \le \int_0^t \frac{4\sqrt{n}D_{1,n}^2(s)}{1-4B_n(s)}ds + \int_0^t \frac{4A_n(s)}{1-4B_n(s)}ds + \int_0^t \frac{4\sqrt{n}D_{3,n}^2(s)}{1-4B_n(s)}ds$$
$$+ \int_0^t \frac{\sqrt{n}\Delta Z_n^2(s)\cdot 4B_n(s)}{1-4B_n(s)}ds - 2\int_0^t \frac{\sqrt{n}(Z_n(s)-Z(s))\Delta Z_n(s)\cdot 4B_n(s)}{1-4B_n(s)}ds. \tag{3.17}$$

According to Theorem 1.3.1, we can discard the last two integrals in (3.17) whose integrand contains the jumps $\Delta Z_n(s)$, and get that

$$\int_0^t \sqrt{n}(Z_n(s)-Z(s))^2 ds \le \int_0^t \frac{4\sqrt{n}D_{1,n}^2(s)}{1-4B_n(s)}ds + \int_0^t \frac{4A_n(s)}{1-4B_n(s)}ds + \int_0^t \frac{4\sqrt{n}D_{3,n}^2(s)}{1-4B_n(s)}ds \tag{3.18}$$

With the help of Lemma 3.4.1, it suffices to prove that each integral in the right hand side of (3.18) converges to zero

$$t \mapsto \int_0^t \frac{4\sqrt{n}D_{1,n}^2(s)}{1-4B_n(s)}ds \Rightarrow 0 \tag{3.19}$$

$$t \mapsto \int_0^t \frac{4A_n(s)}{1-4B_n(s)}ds \Rightarrow 0 \tag{3.20}$$

$$t \mapsto \int_0^t \frac{4\sqrt{n}D_{3,n}^2(s)}{1-4B_n(s)}ds \Rightarrow 0 \tag{3.21}$$

in $(D[0,\tau], J_1)$ as $n \to \infty$.

To help us better evaluate the integral whose integrand involves jumpy process $\boldsymbol{B}_n$, we establish the following lemma.

Lemma 3.4.7. *Define the "continuous" counterpart of $\boldsymbol{B}_n$ by*

$$\bar{\boldsymbol{B}}_n = \frac{1}{8}\left(\frac{\boldsymbol{C}_n^3}{\boldsymbol{Q}_n^b}\right)^2 \left(\frac{\boldsymbol{Q}^b}{\boldsymbol{C}^3}\right)^2.$$

Then we have the convergence

$$\bar{\boldsymbol{B}}_n \Rightarrow \frac{1}{8}, \ \ in\,(D[0,\tau], J_1)$$

as $n \to \infty$.

Proof. By Table 1.3, we have the joint convergence

$$(\boldsymbol{C}_n^3, \boldsymbol{Q}_n^b) \Rightarrow (\boldsymbol{C}^3, \boldsymbol{Q}^b), \ \text{in}\,(D^2[0,\tau], J_1)$$

as $n \to \infty$.

It follows from the continuous mapping theorem that

$$\bar{\boldsymbol{B}}_n = \frac{1}{8}\left(\frac{\boldsymbol{C}_n^3}{\boldsymbol{Q}_n^b}\right)^2\left(\frac{\boldsymbol{Q}^b}{\boldsymbol{C}^3}\right)^2 \Rightarrow \frac{1}{8},\ \text{in}\,(D[0,\tau], J_1)$$

as $n \to \infty$.

Hence we proved Lemma 3.4.7. □

With the help of Lemma 3.4.7, we prove the convergence (3.19)-(3.21) one by one.

Lemma 3.4.8. *(Proof of (3.19)) We have the convergence*

$$t \mapsto \int_0^t \frac{4\sqrt{n}D_{1,n}^2(s)}{1-4B_n(s)}ds \Rightarrow 0,\ \ in\,(D[0,\tau], J_1)$$

as $n \to \infty$.

Proof. Observe that

$$\int_0^t \frac{4\sqrt{n}D_{1,n}^2(s)}{1-4B_n(s)}ds = \int_0^t \frac{4\sqrt{n}D_{1,n}^2(s)}{1-4\bar{B}_n(s)}ds$$

by Theorem 1.3.1.

Since we have the convergence

$$\sqrt{n}\boldsymbol{D}_{1,n}^2 \Rightarrow 0,\ \bar{\boldsymbol{B}}_n \Rightarrow \frac{1}{8},$$

by Theorem 1.2.4 and the continuous mapping theorem, we get

$$\frac{4\sqrt{n}\boldsymbol{D}_{1,n}^2}{1-4\bar{\boldsymbol{B}}_n} \Rightarrow 0.$$

We apply Theorem 1.2.3 to obtain

$$t \mapsto \int_0^t \frac{4\sqrt{n}D_{1,n}^2(s)}{1-4\bar{B}_n(s)}ds \Rightarrow 0,\ \text{in}\,(D[0,\tau], J_1)$$

as $n \to \infty$.

The the proof of (3.19) is therefore complete.

□

Lemma 3.4.9. *(Proof of (3.20)) We have the convergence*

$$t \mapsto \int_0^t \frac{4A_n(s)}{1-4B_n(s)}ds \Rightarrow 0,\ \ in\,(D[0,\tau], J_1)$$

as $n \to \infty$.

Proof. Define the "continuous" counterpart of $\boldsymbol{A}_n$ by

$$\bar{\boldsymbol{A}}_n = L^2 \frac{\boldsymbol{Z}^2(\boldsymbol{C}^3)^2}{(\boldsymbol{Q}^b)^2} \frac{\sqrt{n}(\boldsymbol{Q}_n^b - \boldsymbol{Q}^b)^2}{(\boldsymbol{Q}_n^b)^2}.$$

According to Theorem 1.3.1, we have

$$\int_0^t \frac{4A_n(s)}{1-4B_n(s)} ds = \int_0^t \frac{4\bar{A}_n(s)}{1-4\bar{B}_n(s)} ds.$$

In order to evaluate the integral in the right hand side, we establish the convergence of $\bar{\boldsymbol{A}}_n$.

By Table 1.3, we have the joint convergence

$$(\boldsymbol{Q}_n^b, \boldsymbol{Q}_n^b - \boldsymbol{Q}^b, \sqrt{n}(\boldsymbol{Q}_n^b - \boldsymbol{Q}^b)) \Rightarrow (\boldsymbol{Q}^b, 0, \boldsymbol{\Psi}^1 - \boldsymbol{\Psi}^2 - \boldsymbol{\Psi}^3), \text{ in } (D^3[0,T], J_1) \tag{3.22}$$

as $n \to \infty$.

It follows from the continuous mapping theorem that

$$\bar{\boldsymbol{A}}_n = L^2 \frac{\boldsymbol{Z}^2(\boldsymbol{C}^3)^2}{(\boldsymbol{Q}^b)^2} \frac{\sqrt{n}(\boldsymbol{Q}_n^b - \boldsymbol{Q}^b)^2}{(\boldsymbol{Q}_n^b)^2} = L^2 \frac{\boldsymbol{Z}^2(\boldsymbol{C}^3)^2}{(\boldsymbol{Q}^b)^2} \frac{(\boldsymbol{Q}_n^b - \boldsymbol{Q}^b) \cdot \sqrt{n}(\boldsymbol{Q}_n^b - \boldsymbol{Q}^b)}{(\boldsymbol{Q}_n^b)^2} \Rightarrow 0$$

as $n \to \infty$.

At this point, we have the convergence

$$\bar{\boldsymbol{A}}_n \Rightarrow 0, \ \bar{\boldsymbol{B}}_n \Rightarrow \frac{1}{8},$$

and by Theorem 1.2.4 and the continuous mapping theorem, we get the convergence

$$\frac{4\bar{\boldsymbol{A}}_n}{1-4\bar{\boldsymbol{B}}_n} \Rightarrow 0.$$

We apply Theorem 1.2.3 to obtain

$$t \mapsto \int_0^t \frac{4\bar{A}_n(s)}{1-4\bar{B}_n(s)} ds \Rightarrow 0, \text{ in } (D[0,\tau], J_1)$$

as $n \to \infty$, which completes the proof of (3.20). □

Lemma 3.4.10. *(Proof of (3.21)) We have the convergence*

$$t \mapsto \int_0^t \frac{4\sqrt{n}D_{3,n}^2(s)}{1-4B_n(s)} ds \Rightarrow 0, \ \textit{in } (D[0,\tau], J_1)$$

as $n \to \infty$.

Proof. Notice that

$$\int_0^t \frac{4\sqrt{n}D_{3,n}^2(s)}{1-4B_n(s)}ds = \int_0^t \frac{4\sqrt{n}D_{3,n}^2(s)}{1-4\bar{B}_n(s)}ds$$

by Theorem 1.3.1.

Since we have the convergence

$$\sqrt{n}\boldsymbol{D}_{3,n}^2 \Rightarrow 0,\ \bar{\boldsymbol{B}}_n \Rightarrow \frac{1}{8},$$

by Theorem 1.2.4 and the continuous mapping theorem, we get

$$\frac{4\sqrt{n}\boldsymbol{D}_{3,n}^2}{1-4\bar{\boldsymbol{B}}_n} \Rightarrow 0.$$

We apply Theorem 1.2.3 to obtain

$$t \mapsto \int_0^t \frac{4\sqrt{n}D_{3,n}^2(s)}{1-4\bar{B}_n(s)}ds \Rightarrow 0,\ \text{in}\,(D[0,\tau],J_1)$$

as $n \to \infty$, which completes the proof of (3.21). □

We proved Lemma 3.4.8-3.4.10 and thus we proved Proposition 3.4.6. □

Step 2

Proposition 3.4.11. *We have the convergence*

$$t \mapsto \int_0^t \sqrt{n}\left(\frac{Z_n(s)}{Q^b(s)} - \frac{Z_n(s)}{Q_n^b(s)}\right)^2 ds \Rightarrow 0,\ \ in\,(D[0,\tau],J_1)$$

as $n \to \infty$.

Proof. Notice that

$$\begin{aligned}\sqrt{n}\left(\frac{\boldsymbol{Z}_n}{\boldsymbol{Q}^b} - \frac{\boldsymbol{Z}_n}{\boldsymbol{Q}_n^b}\right)^2 &= \sqrt{n}\frac{\boldsymbol{Z}_n^2}{(\boldsymbol{Q}^b)^2(\boldsymbol{Q}_n^b)^2}\left(\boldsymbol{Q}_n^b - \boldsymbol{Q}^b\right)^2 \\ &= \frac{\boldsymbol{Z}_n^2}{(\boldsymbol{Q}^b)^2(\boldsymbol{Q}_n^b)^2}\left(\boldsymbol{Q}_n^b - \boldsymbol{Q}^b\right)\cdot\sqrt{n}\left(\boldsymbol{Q}_n^b - \boldsymbol{Q}^b\right).\end{aligned} \tag{3.23}$$

By Table 1.3, we have the joint convergence

$$(\boldsymbol{Q}_n^b, \boldsymbol{Z}_n) \Rightarrow (\boldsymbol{Q}^b, \boldsymbol{Z})$$

in $(D^2[0,\tau],J_1)$ as $n \to \infty$. Then we can apply the continuous mapping theorem to get that

$$\frac{\boldsymbol{Z}_n^2}{(\boldsymbol{Q}^b)^2(\boldsymbol{Q}_n^b)^2} \Rightarrow \frac{\boldsymbol{Z}^2}{(\boldsymbol{Q}^b)^2(\boldsymbol{Q}^b)^2} \tag{3.24}$$

in $(D[0,\tau], J_1)$ as $n \to \infty$, where this limit processing is deterministic.

By Table 1.3, we have the convergence

$$\sqrt{n}(\boldsymbol{Q}_n^b - \boldsymbol{Q}^b) \Rightarrow \boldsymbol{\Psi}^1 - \boldsymbol{\Psi}^2 - \boldsymbol{\Psi}^3$$

in $(D[0,T], J_1)$ as $n \to \infty$. It follows from Theorem 1.2.6 that

$$\sqrt{n}\left(\boldsymbol{Q}_n^b - \boldsymbol{Q}^b\right)^2 \Rightarrow 0 \tag{3.25}$$

in $(D[0,\tau], J_1)$ as $n \to \infty$.

Combining (3.23), (3.24) and (3.25) gives the convergence

$$\sqrt{n}\left(\frac{\boldsymbol{Z}_n}{\boldsymbol{Q}^b} - \frac{\boldsymbol{Z}_n}{\boldsymbol{Q}_n^b}\right)^2 \Rightarrow 0, \text{ in } (D[0,T], J_1)$$

as $n \to \infty$.

Integrating it from 0 to t completes the proof.

□

Step 3

Based on Proposition 3.4.6 and Proposition 3.4.11, we are ready to do the last step to finish the proof of Theorem 3.2.1.

The key is to divide $\sqrt{n}(\boldsymbol{Z}_n - \boldsymbol{Z})$ into several terms (3.26) after rearrangements, and establish the convergence of the three building blocks in (3.26). Specifically, we have the following convergence.

(i) The convergence of the second component in the left hand side in (3.26) is provided in (3.27).

(ii) The convergence of the first and second components in the right hand side in (3.26) is taken care of in (3.33).

(iii) The third component in the right hand side in (3.26) is done in (3.32) via (3.28) and (3.31).

Proof. We use the dynamics of $\boldsymbol{Z}_n$ (1.5) and $\boldsymbol{Z}$ (1.7) and do the following rearrangements

$$\begin{aligned}
d(Z_n(t) - Z(t)) &= -d(C_n^2(t) - C^2(t)) - \Upsilon\left(\frac{Z_n(t-)}{Q_n^b(t-)}\right) dC_n^3(t) + \Upsilon\left(\frac{Z(t-)}{Q^b(t-)}\right) dC^3(t) \\
&= -d(C_n^2(t) - C^2(t)) - \Upsilon\left(\frac{Z_n(t-)}{Q_n^b(t-)}\right) d(C_n^3(t) - C^3(t)) \\
&+ \left[\Upsilon\left(\frac{Z(t-)}{Q^b(t-)} - \Upsilon\left(\frac{Z_n(t-)}{Q_n^b(t-)}\right)\right] dC^3(t) \\
&= -d(C_n^2(t) - C^2(t)) - \Upsilon\left(\frac{Z_n(t-)}{Q_n^b(t-)}\right) d(C_n^3(t) - C^3(t)) \\
&+ \left[\Upsilon\left(\frac{Z(t-)}{Q^b(t-)}\right) - \Upsilon\left(\frac{Z_n(t-)}{Q^b(t-)}\right)\right] dC^3(t) \\
&+ \left[\Upsilon\left(\frac{Z_n(t-)}{Q^b(t-)}\right) - \Upsilon\left(\frac{Z_n(t-)}{Q_n^b(t-)}\right)\right] dC^3(t)
\end{aligned}$$

Then we multiply it by $\sqrt{n}$, integrate it and rewrite it as

$$\begin{aligned}
&\sqrt{n}(Z_n(t) - Z(t)) - \int_0^t \sqrt{n}\left[\Upsilon\left(\frac{Z(s-)}{Q^b(s-)}\right) - \Upsilon\left(\frac{Z_n(s-)}{Q^b(s-)}\right)\right] dC^3(s) \\
&= -\sqrt{n}(C_n^2(t) - C^2(t)) - \int_0^t \Upsilon\left(\frac{Z_n(s-)}{Q_n^b(s-)}\right) d\sqrt{n}(C_n^3(s) - C^3(s)) \\
&+ \int_0^t \sqrt{n}\left[\Upsilon\left(\frac{Z_n(s-)}{Q^b(s-)}\right) - \Upsilon\left(\frac{Z_n(s-)}{Q_n^b(s-)}\right)\right] dC^3(s),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\sqrt{n}(Z_n(t) - Z(t)) - \int_0^t \sqrt{n}\left[\Upsilon\left(\frac{Z(s)}{Q^b(s)}\right) - \Upsilon\left(\frac{Z_n(s)}{Q^b(s)}\right)\right] dC^3(s) \\
&= -\sqrt{n}(C_n^2(t) - C^2(t)) - \int_0^t \Upsilon\left(\frac{Z_n(s-)}{Q_n^b(s-)}\right) d\sqrt{n}(C_n^3(s) - C^3(s)) \\
&+ \int_0^t \sqrt{n}\left[\Upsilon\left(\frac{Z_n(s)}{Q^b(s)}\right) - \Upsilon\left(\frac{Z_n(s)}{Q_n^b(s)}\right)\right] dC^3(s).
\end{aligned} \tag{3.26}$$

By Assumption 1.6.1 and 1.6.2, $\Upsilon \in C^2[0,1]$, we can thus apply Taylor's Theorem to get

$$\Upsilon\left(\frac{Z_n(s)}{Q^b(s)}\right) = \Upsilon\left(\frac{Z(s)}{Q^b(s)}\right) + \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\frac{Z_n(s) - Z(s)}{Q^b(s)} + R_{n,1}(s),$$

where we have

$$|R_{n,1}(s)| \leq K\left[Z_n(s) - Z(s)\right]^2$$

for some positive constant K.

Multiplying by $\sqrt{n}$ and integrating yields

$$\int_0^t \sqrt{n}\left[\Upsilon\left(\frac{Z(s)}{Q^b(s)}\right)-\Upsilon\left(\frac{Z_n(s)}{Q^b(s)}\right)\right]dC^3(s)+\int_0^t \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\cdot\sqrt{n}(Z_n(s)-Z(s))\cdot\frac{dC^3(s)}{Q^b(s)}$$
$$=R_{n,2}(t),$$

where we have

$$|R_{n,2}(t)|\le K\int_0^t \sqrt{n}\left[Z_n(s)-Z(s)\right]^2 ds.$$

By Proposition 3.4.6, the right hand side converges to zero. Hence we have the convergence

$$\int_0^t \sqrt{n}\left[\Upsilon\left(\frac{Z(s)}{Q^b(s)}\right)-\Upsilon\left(\frac{Z_n(s)}{Q^b(s)}\right)\right]dC^3(s)+\int_0^t \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\cdot\sqrt{n}(Z_n(s)-Z(s))\frac{dC^3(s)}{Q^b(s)}\Rightarrow 0 \tag{3.27}$$

in $(D[0,\tau],J_1)$ as $n\to\infty$.

Similarly, we apply Taylor's Theorem to $\Upsilon\left(\frac{Z_n(s)}{Q^b(s)}\right)-\Upsilon\left(\frac{Z_n(s)}{Q^b_n(s)}\right)$ to get

$$\int_0^t \sqrt{n}\left[\Upsilon\left(\frac{Z_n(s)}{Q^b(s)}\right)-\Upsilon\left(\frac{Z_n(s)}{Q^b_n(s)}\right)\right]dC^3(s)+\int_0^t \sqrt{n}\Upsilon'\left(\frac{Z_n(s)}{Q^b(s)}\right)\left(\frac{Z_n(s)}{Q^b_n(s)}-\frac{Z_n(s)}{Q^b(s)}\right)dC^3(s)$$
$$=R_{n,3}(t),$$

where we have

$$|R_{n,3}(t)|\le L\int_0^t \sqrt{n}\left(\frac{Z_n(s)}{Q^b_n(s)}-\frac{Z_n(s)}{Q^b(s)}\right)^2 ds$$

for some positive constant L.

With the help of Proposition 3.4.11, we know that the right hand side converges to zero, and thus we obtain the convergence

$$\begin{aligned}&\int_0^t \sqrt{n}\left[\Upsilon\left(\frac{Z_n(s)}{Q^b(s)}\right)-\Upsilon\left(\frac{Z_n(s)}{Q^b_n(s)}\right)\right]dC^3(s)\\&+\int_0^t \sqrt{n}\Upsilon'\left(\frac{Z_n(s)}{Q^b(s)}\right)\left(\frac{Z_n(s)}{Q^b_n(s)}-\frac{Z_n(s)}{Q^b(s)}\right)dC^3(s)\Rightarrow 0\end{aligned} \tag{3.28}$$

in $(D[0,\tau],J_1)$ as $n\to\infty$.

To analyze the convergence of the second integral in (3.28), observe that

$$\sqrt{n}\Upsilon'\left(\frac{Z_n(s)}{Q^b(s)}\right)\left[\frac{Z_n(s)}{Q^b_n(s)}-\frac{Z_n(s)}{Q^b(s)}\right]=\Upsilon'\left(\frac{Z_n(s)}{Q^b(s)}\right)\frac{Z_n(s)}{Q^b_n(s)Q^b(s)}\sqrt{n}(Q^b(s)-Q^b_n(s)). \tag{3.29}$$

By Table 1.3, we have the joint convergence $(\boldsymbol{Q}_n,\boldsymbol{Z}_n,\sqrt{n}(\boldsymbol{Q}_n-\boldsymbol{Q}))\Rightarrow(\boldsymbol{Q},\boldsymbol{Z},\boldsymbol{\Psi}^1-\boldsymbol{\Psi}^2-\boldsymbol{\Psi}^3)$ in $(D^3[0,\tau],J_1)$ as $n\to\infty$. This implies the convergence

$$\Upsilon'\left(\frac{Z_n(s)}{Q^b(s)}\right)\frac{Z_n(s)}{Q^b_n(s)Q^b(s)}\sqrt{n}(Q^b(s)-Q^b_n(s))\Rightarrow-\Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\frac{Z(s)(\Psi^1(s)-\Psi^2(s)-\Psi^3(s))}{(Q^b(s))^2}. \tag{3.30}$$

We combine (3.29) and (3.30), and then apply Theorem 1.2.3 to get

$$\begin{aligned}&\int_0^t \sqrt{n}\Upsilon'\left(\frac{Z_n(s)}{Q^b(s)}\right)\left[\frac{Z_n(s)}{Q_n^b(s)}-\frac{Z_n(s)}{Q^b(s)}\right]dC^3(s)\\ &\Rightarrow \int_0^t -\Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\frac{Z(s)(\Psi^1(s)-\Psi^2(s)-\Psi^3(s))}{Q^b(s)}\frac{dC^3(s)}{Q^b(s)}\end{aligned} \tag{3.31}$$

in $(D[0,\tau],J_1)$ as $n\to\infty$.

Subtracting (3.31) from (3.28) gives

$$\begin{aligned}&\int_0^t \sqrt{n}\left[\Upsilon\left(\frac{Z_n(s)}{Q^b(s)}\right)-\Upsilon\left(\frac{Z_n(s)}{Q_n^b(s)}\right)\right]dC^3(s)\\ &\Rightarrow \int_0^t \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\frac{Z(s)(\Psi^1(s)-\Psi^2(s)-\Psi^3(s))}{Q^b(s)}\frac{dC^3(s)}{Q^b(s)}\end{aligned} \tag{3.32}$$

in $(D[0,\tau],J_1)$ as $n\to\infty$.

Given the joint convergence $(\boldsymbol{Q}_n,\boldsymbol{Z}_n,\sqrt{n}(\boldsymbol{C}_n-\boldsymbol{C}))\Rightarrow(\boldsymbol{Q},\boldsymbol{Z},\boldsymbol{\Psi})$ in $(D^8[0,\tau],J_1)$ as $n\to\infty$, and the fact that Υ is twice continuously differentiable from $[0,1]$ to $[0,1]$, with the help of [4, Theorem 10] we can get that

$$\begin{aligned}&-\sqrt{n}(C_n^2(t)-C^2(t))-\int_0^t \Upsilon\left(\frac{Z_n(s-)}{Q_n^b(s-)}\right)d\sqrt{n}(C_n^3(s)-C^3(s))\\ &\Rightarrow -\Psi^2(t)-\int_0^t \Upsilon\left(\frac{Z(s-)}{Q^b(s-)}\right)d\Psi^3(s)=-\Psi^2(t)-\int_0^t \Upsilon\left(\frac{Z(s)}{Q^b(s)}\right)d\Psi^3(s)\end{aligned} \tag{3.33}$$

In fact, the first part of (3.33) is because the only change from [4, page 18] to the term here is that we apply the function Υ to $\frac{Z_n(s-)}{Q_n^b(s-)}$ and $\frac{Z(s-)}{Q^b(s-)}$, and this is plausible for [4, Theorem 10], which comes from [8, Theorem 5.4]. For sake of completeness, we state the theorem here. But first, we need some definitions.

Define $h_\delta:[0,\infty)\to[0,\infty)$ by $h_\delta(r)=(1-\frac{\delta}{r})^+$. Define $J_\delta:D^m[0,\tau]\to D^m[0,\tau]$ by $J_\delta(x)(t)=\sum_{s\le t}h_\delta(|x(s)-x(s-)|)(x(s)-x(s-))$.

Let Y_n be a sequence of stochastic processes adapted to $\mathcal{F}_t$. Define $Y_n^\delta=Y_n-J_\delta(Y_n)$. Let $Y_n^\delta=M_n^\delta+A_n^\delta$ be a decomposition of Y_n^δ into an $\mathcal{F}_t$-local martingale and a process with finite variation.

Let $T_1[0,\tau]$ denote the collection of non-decreasing mappings λ from $[0,\tau]$ to $[0,\tau]$ such that $\lambda(h+t)-\lambda(t)\le h$ for all $0\le t\le h+t\le\tau$ and $\lambda(0)=0$.

Theorem 3.4.12. *Suppose that* $(\boldsymbol{U}_n,\boldsymbol{X}_n,\boldsymbol{Y}_n)$ *satisfies*

$$X_n(t)=U_n(t)+\int_0^t F_n(X_n,s-)dY_n(s),$$

$(\boldsymbol{U}_n,\boldsymbol{Y}_n)\Rightarrow(\boldsymbol{U},\boldsymbol{Y})$ *in* $(D[0,T],J_1)$ *and that for some* $0<\delta\le\infty$, $\boldsymbol{Y}_n$ *satisfies the condition*

For each $\alpha > 0$, there exists stopping times τ_n^α such that $P(\tau_n^\alpha \le 1) \le \frac{1}{\alpha}$ and $\sup_n \mathbb{E}[[M_n^\delta]_{t\le\tau_n^\alpha} + T(A_n^\delta)_{t\le\tau_n^\alpha}] < \infty$, where $[M_n^\delta]_{t\le\tau_n^\alpha}$ denotes the total quadratic variation of M_n^δ up to time τ_n^α and $T(A_n^\delta)_{t\le\tau_n^\alpha}$ denotes the total variation of A_n^δ up to time τ_n^α.

Assume that $\{F_n\}$ and F have representations

$$F_n \circ \lambda = G_n(x \circ \lambda, \lambda),\ \ F \circ \lambda = G(x \circ \lambda, \lambda)$$

for $(x, \lambda) \in D^k[0, \tau] \times T_1[0, \tau]$, where $\{G_n\}$ and G satisfy the conditions

(i) For each compact subset $\mathcal{H} \subset D^k[0, \tau] \times T_1[0, \tau]$ and $0 \le t \le \tau$, $\sup_{(x,\lambda)\in\mathcal{H}} \sup_{s\le t} |G_n(x, \lambda, s) - G(x, \lambda, s)| \to 0$

(ii) For $\{(x_n, \lambda_n)\} \in D^k[0, \tau] \times T_1[0, \tau]$, $\sup_{s\le t} |x_n(s) - x(s)| \to 0$ and $\sup_{s\le t} |\lambda_n(s) - \lambda(s)| \to 0$ for each $0 \le t \le \tau$ imply $\sup_{s\le t} |G(x_n, \lambda_n, s) - G(x, \lambda, s)| \to 0$.

If there exists a global solution X of

$$dX(t) = U(t) + \int_0^t F(X, s-)dY(s),$$

and the local uniqueness holds, then

$$(\boldsymbol{U}_n, \boldsymbol{X}_n, \boldsymbol{Y}_n) \Rightarrow (\boldsymbol{U}, \boldsymbol{X}, \boldsymbol{Y}).$$

Considering Υ is twice continuously differentiable from $[0, 1]$ to $[0, 1]$, we know that the functions $\bar{G}_n$ and $\bar{G}$ defined by

$$\Upsilon \circ F_n \circ \lambda = \bar{G}_n(x \circ \lambda, \lambda),\ \ \Upsilon \circ F \circ \lambda = \bar{G}(x \circ \lambda, \lambda)$$

satisfy the conditions in Theorem 3.4.12, and thus we can apply Theorem 3.4.12 to obtain the first part of (3.33).

Combining (3.27), (3.32) and (3.33) yields

$$\begin{aligned}
&\sqrt{n}(Z_n(t) - Z(t)) + \int_0^t \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right) \cdot \sqrt{n}(Z_n(s) - Z(s)) \cdot \frac{dC^3(s)}{Q^b(s)} \\
&\Rightarrow \int_0^t \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right) \frac{Z(s)(\Psi^1(s) - \Psi^2(s) - \Psi^3(s))}{Q^b(s)} \frac{dC^3(s)}{Q^b(s)} - \Psi^2(t) - \int_0^t \Upsilon\left(\frac{Z(s)}{Q^b(s)}\right) d\Psi^3(s).
\end{aligned}$$

in $(D[0, \tau], J_1)$ as $n \to \infty$.

Hence we obtain

$$\sqrt{n}(\boldsymbol{Z}_n - \boldsymbol{Z}) \Rightarrow \boldsymbol{Y}$$

in $(D[0,\tau], J_1)$ as $n \to \infty$, where $Y(t)$ satisfies

$$\begin{aligned} &dY(t) + \Upsilon'\left(\frac{Z(t)}{Q^b(t)}\right) \cdot Y(t) \cdot \frac{dC^3(t)}{Q^b(t)} \\ &= \Upsilon'\left(\frac{Z(t)}{Q^b(t)}\right) \frac{Z(t)(\Psi^1(t) - \Psi^2(t) - \Psi^3(t))}{Q^b(t)} \frac{dC^3(t)}{Q^b(t)} - d\Psi^2(t) - \Upsilon\left(\frac{Z(t)}{Q^b(t)}\right) d\Psi^3(t) \end{aligned}$$

which is equivalent to (3.1),

$$\begin{aligned} dY(t) &= \Upsilon'\left(\frac{Z(t)}{Q^b(t)}\right)\left(\frac{Z(t)(\Psi^1(t) - \Psi^2(t) - \Psi^3(t))}{Q^b(t)} - Y(t)\right)\frac{dC^3(t)}{Q^b(t)} \\ &- d\Psi^2(t) - \Upsilon\left(\frac{Z(t)}{Q^b(t)}\right) d\Psi^3(t). \end{aligned}$$

The proof of Theorem 3.2.1 is therefore complete. □

3.5 Proof of Proposition 3.2.3

Proof. We recall from section 1.7 that $C^3(t) = \lambda \bar{V}_3 t$ for constant λ and $\bar{V}_3$.

We first move the term $-\Upsilon'\left(\frac{Z(t)}{Q^b(t)}\right) Y(t) \frac{dC^3(t)}{Q^b(t)}$ in (3.1) over so that everything about Y is in the left hand side

$$\begin{aligned} dY(t) + \Upsilon'\left(\frac{Z(t)}{Q^b(t)}\right) Y(t) \frac{dC^3(t)}{Q^b(t)} &= \Upsilon'\left(\frac{Z(t)}{Q^b(t)}\right)\left(\frac{Z(t)(\Psi^1(t) - \Psi^2(t) - \Psi^3(t))}{Q^b(t)}\right)\frac{dC^3(t)}{Q^b(t)} \\ &- d\Psi^2(t) - \Upsilon\left(\frac{Z(t)}{Q^b(t)}\right) d\Psi^3(t). \end{aligned} \tag{3.34}$$

We then multiply (3.34) by the integrating factor $e^{\int_0^t \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\frac{dC^3(s)}{Q^b(s)}}$ to get the differential form of $e^{\int_0^t \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\frac{dC^3(s)}{Q^b(s)}} \cdot Y(t)$

$$\begin{aligned} &d\left(e^{\int_0^t \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\frac{dC^3(s)}{Q^b(s)}} \cdot Y(t)\right) \\ &= e^{\int_0^t \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\frac{dC^3(s)}{Q^b(s)}} dY(t) + e^{\int_0^t \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\frac{dC^3(s)}{Q^b(s)}} \Upsilon'\left(\frac{Z(t)}{Q^b(t)}\right) Y(t) \frac{dC^3(t)}{Q^b(t)} \\ &= e^{\int_0^t \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\frac{dC^3(s)}{Q^b(s)}} \left(dY(t) + \Upsilon'\left(\frac{Z(t)}{Q^b(t)}\right) Y(t) \frac{dC^3(t)}{Q^b(t)}\right) \\ &= e^{\int_0^t \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\frac{dC^3(s)}{Q^b(s)}} \Upsilon'\left(\frac{Z(t)}{Q^b(t)}\right)\left(\frac{Z(t)(\Psi^1(t) - \Psi^2(t) - \Psi^3(t))}{Q^b(t)}\right)\frac{dC^3(t)}{Q^b(t)} \\ &- e^{\int_0^t \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\frac{dC^3(s)}{Q^b(s)}} d\Psi^2(t) - e^{\int_0^t \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\frac{dC^3(s)}{Q^b(s)}} \Upsilon\left(\frac{Z(t)}{Q^b(t)}\right) d\Psi^3(t). \end{aligned} \tag{3.35}$$

Integrating (3.35) from 0 to t yields

$$\begin{aligned}
&e^{\int_0^t \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\frac{dC^3(s)}{Q^b(s)}} \cdot Y(t) \\
&= \int_0^t e^{\int_0^s \Upsilon'\left(\frac{Z(u)}{Q^b(u)}\right)\frac{dC^3(u)}{Q^b(u)}} \Upsilon'\left(\frac{Z(s)}{Q^b(s)}\right)\left(\frac{Z(s)(\Psi^1(s)-\Psi^2(s)-\Psi^3(s))}{Q^b(s)}\right)\frac{dC^3(s)}{Q^b(s)} \\
&- \int_0^t e^{\int_0^s \Upsilon'\left(\frac{Z(u)}{Q^b(u)}\right)\frac{dC^3(u)}{Q^b(u)}} d\Psi^2(s) - \int_0^t e^{\int_0^s \Upsilon'\left(\frac{Z(u)}{Q^b(u)}\right)\frac{dC^3(u)}{Q^b(u)}} \Upsilon\left(\frac{Z(s)}{Q^b(s)}\right) d\Psi^3(s).
\end{aligned} \tag{3.36}$$

Notice the only random elements in the left hand of (3.36) are $\boldsymbol{\Psi}^1$, $\boldsymbol{\Psi}^2$ and $\boldsymbol{\Psi}^3$. Recall from section 1.7 that $\boldsymbol{\Psi}^1$, $\boldsymbol{\Psi}^2$ and $\boldsymbol{\Psi}^3$ are Gaussian processes with zero drift. Considering that integral of Gaussian processes with respect to time are also Gaussian processes, we get that $\boldsymbol{Y}$ is a Gaussian process. Also, given that $\boldsymbol{\Psi}^1$, $\boldsymbol{\Psi}^2$ and $\boldsymbol{\Psi}^3$ have zero drift, we have $\mathbb{E}(\boldsymbol{\Psi}^i) = 0$ for $1 \leq i \leq 3$, and thus $\mathbb{E}(\boldsymbol{Y}) = 0$.

Hence we proved that $\boldsymbol{Y}$ is a Gaussian process with mean zero.

□

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