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Alex Kinsella

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May 2021

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Dedicated to the memory of my grandfather, John Joseph Kinsella, a physicist whose masterful storytelling inspired joyful exploration.

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Publications

B. Acharya, A. Kinsella, and D. Morrison. "Heterotic Duals of M-Theory on Joyce Orbifolds." To be submitted to JHEP. [arXiv: 2106.03886 [hep-th]]

B. Acharya, A. Kinsella, and E. Eik Svanes. "T^3-Invariant Heterotic Hull-Strominger Solutions." JHEP 01 (2021) 197. [arXiv: 2010.07438 [hep-th]]

S. B. Giddings and A. Kinsella. "Gauge-invariant observables, gravitational dressings, and holography in AdS." JHEP 11 (2018) 074. [arXiv: 1802.01602 [hep-th]]

Abstract

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M-theory and the heterotic string are two dual ways to obtain low-energy effective theories that may be engineered to reproduce the observed particles and forces of our universe. To achieve four-dimensional effective theories, one must compactify a number of extra spatial dimensions. In this dissertation, we study the particle spectra and dualities of the $E_8 \times E_8$ heterotic string and M-theory compactified on special holonomy orbifolds and local models built via torus fibrations.

After an introduction to string theory and special holonomy in Chapter 1, we investigate in Chapter 2 the way in which the heterotic gauge bundle mirrors effects from the M-theory geometry. By fibering the duality between the $E_8 \times E_8$ heterotic string on T^3 and M-theory on K3, we study heterotic duals of M-theory compactified on G_2 orbifolds of the form $T^7/\mathbb{Z}_2^3$. The heterotic backgrounds exhibit point-like instantons that are localized on pairs of orbifold loci, similar to the "gauge-locking" phenomenon seen in Hořava-Witten compactifications. While the instanton configuration looks strange from the perspective of the $E_8 \times E_8$ heterotic string, it may be understood as T-dual $\mathrm{Spin}(32)/\mathbb{Z}_2$ instantons along with winding shifts originating in a dual Type I compactification.

In Chapter 3, we consider local models on $\mathbb{R}^3$ resulting from a reduction of the heterotic string on Calabi-Yau manifolds admitting a Strominger–Yau–Zaslow fibration. Upon reducing the system in the T^3-directions, the Hermitian Yang-Mills conditions can be reinterpreted as a complex flat connection on $\mathbb{R}^3$ satisfying a certain co-closure condition. We give a number of abelian and non-abelian examples, and also compute the back-reaction on the geometry through the α'-corrected heterotic Bianchi identity, which includes an important correction to the equations for the complex flat connection. These are new local solutions to the Hull-Strominger system on $T^3 \times \mathbb{R}^3$. We also propose a method for computing the spectrum of certain non-abelian models, in close analogy with the Morse-Witten complex of the abelian models.

Contents

Chapter 1

Introduction: String Theory and M-Theory on Manifolds of Special Holonomy

String theory is a grand project to describe all fundamental forces and matter via one quantum theory whose fundamental constituents are extended objects. In the original formulation of the theory, the extended objects were limited to one-dimensional strings, but further development demonstrated that it is mathematically necessary to include higher-dimensional objects as well. Although string theory began as a purely physical theory, its modern incarnations are of comparable interest to mathematicians as they are to physicists. In an expansive series of connections and implications, string theory has linked disparate fields of mathematics and provided recipes for extending mathematical understanding seemingly from thin air. This dissertation explores the ways in which compactifications of M-theory and the heterotic string can illuminate the geometry and gauge theory of special holonomy orbifolds and give rise to a 3D analog of the Hitchin system. This introductory chapter is intended to be a survey of the physics background for these topics, including a brief overview of string/M-theory, its compactification on spaces of special holonomy, and the relevant dualities that provide links throughout the web of string/M backgrounds.

In completing the work described in this dissertation, I have relied on many resources. Some textbooks and review papers that were particularly helpful for my work on compactifications on special holonomy spaces were [1, 2, 3, 4, 5]. I found the information compiled on the nLab [1] to be a helpful guide when navigating the literature. From a more philosophical standpoint, I have found Greg Moore's Strings 2014 talk and essay on the path forward for the field of "physical mathematics" to be a central inspiration.

This introduction will include references to some key papers, but will not attempt to be complete in citing the original literature. For a more comprehensive account of the literature, see, for example, the bibliographic discussion in [1].

1.1 Strings, Branes, and M-Theory

Our familiar physical theories, such as electromagnetism and general relativity, describe physical objects and interactions via *particles*, whose worldlines extend in the time dimension but in no spatial dimensions. The route to string theory is to consider instead what a theory of spatially-extended objects would look like. We call these objects *p-branes*, where p is the number of spatial dimensions that the object extends in. So, for example, a particle is a 0-brane, and a string is a 1-brane. A basic tenet of string theory is that the fundamental constituents of the universe are extended in multiple spatial dimensions, but at low energies we see only particles that arise from quantized degrees of freedom of the extended objects.

Which types of p-branes should be included in string theories? All of them! The

[1] URL: https://ncatlab.org

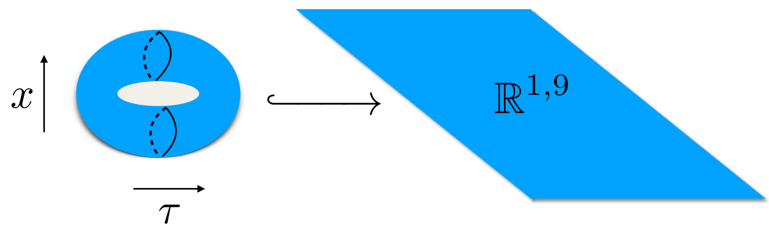

Figure 1.1: In the simplest string backgrounds, the perturbative string may be interpreted via an action principle for embeddings of a 2D string worldsheet into $\mathbb{R}^{1,9}$. The worldsheet has a spatial coordinate x and a Euclidean time coordinate τ. Fluctuations of the worldsheet fields give rise to particles in spacetime.

original perturbative string theories were formulated to include only 1-branes, but non-perturbatively, the theory cannot have only strings. It is forced by various symmetries and dualities to include branes of all other dimensions up to $p = 9$ as well. In general, string theory *tries* to have every type of brane, but only those that are charged under a (generalized) gauge symmetry are stable against decay. Which types of branes are stable depends on which limit of string theory one examines, and this gives rise to different types of low-energy effective theories. We will first briefly describe the perturbative string in generality and then discuss the five consistent supersymmetric string theories as well as M-theory.

1.1.1 String Theory

The original view of string theory is from the *perturbative string*, which is defined by a 2D conformal field theory with a prescribed central charge [6]. The only dynamical brane that appears in perturbation theory is the F1-brane, also known as the *fundamental string*. The prototypical string background (around which we perturb) is where the 2D CFT is a non-linear sigma model that embeds the 2D string worldsheet into a d-dimensional spacetime. Many requirements, including that the theory remain conformal at the quantum level, restrict the spacetime dimension to $d = 26$ for a string with only worldsheet bosons and $d = 10$ for a supersymmetric string. For the purposes of calculation, the worldsheet is usually Wick rotated so that it is a Riemannian manifold. Then calculations may be done using a path integral over maps from the string worldsheet into spacetime (see Figure 1.1). The worldsheet configurations are weighted by the Polyakov action, which measures the area of the string worldsheet. It does this in an indirect way by introducing a worldsheet metric:

$$S_{\text{Polyakov}} = \frac{T}{2} \int d^2\sigma \, \sqrt{-h} h^{ab} g_{\mu\nu}(X) \partial_a X^\mu(\sigma) \partial_b X^\nu(\sigma) \ ,$$

where T is the string tension, h^{ab} is an auxiliary metric on the worldsheet, $g_{\mu\nu}$ is a background metric on spacetime, and $X^\mu(\sigma)$ is the embedding field of the string in spacetime. This action defines a conformal field theory on the worldsheet. We get different CFTs for different choices of background metric $g_{\mu\nu}$. We may also choose other background fields $B_{\mu\nu}$ and ϕ, where $B_{\mu\nu}$ is the *Kalb-Ramond 2-form* and ϕ is the *dilaton*. A perturbative string background may even be defined by a totally different CFT on the worldsheet, as long as it has the correct central charge.

The reason that this formulation is called "perturbative" string theory is that a generic closed string scattering amplitude A takes the schematic form

$$A = \sum_g \lambda^g A_g \ ,$$

where g is the genus of the Riemann surface worldsheet configuration and λ is the string coupling, which is related to the expectation value of the dilaton. This looks like a perturbation series, with λ the small parameter and g the number of loops, so we call this the perturbative string. It is still unclear if this series approximates some "non-perturbative" calculation of string scattering, because we do not yet have a fundamental description of the theory from which to obtain such an exact expression. Much information can be gathered from computing the string partition function for various choices of background fields. The path integral sum over worldsheet topologies will include different contributions depending on if the string is open or closed, as well as oriented or non-oriented.

The worldsheet theory that we have described so far has two parameters: the string coupling λ, which is dimensionless, and the string tension T which is dimensionful. Only dimensionless quantities can really be thought of as parameters, and while the string coupling is dimensionless, the string length is not. To obtain a dimensionless quantity, one must compare the string length to an energy scale or a characteristic radius of a compactification space. There are different effective descriptions of string theory depending on where one lies in the parameter space. These effective descriptions are field theories on the background spacetime.

What we have described of the perturbative string so far is a path integral over maps from the worldvolume into spacetime, weighted by an action that integrates over the string worldsheet. However, there is another way to look at string theory: the fields on the 2D worldsheet give rise to effective 10D fields that at low energies reproduce 10D supergravity. Both the worldsheet perspective and the spacetime perspective describe string theory in certain limits, but neither is able to describe the entire theory. In particular, the worldsheet theory is in principle more fundamental because the string coupling can be large, but only the spacetime theory can include certain semiclassical effects, such as background configurations of higher-dimensional branes.

There are five consistent superstring theories in 10D (see Figure 1.2), which we may distinguish by their numbers of supersymmetries, gauge groups, whether they have open strings, and what types of branes are present [7]. One type of dynamical object in string theories is the D-brane, which has multiple interpretations. Originally it was formulated as a type of topological defect on which open strings may end, but later work also identified it as the fundamental object charged under the RR-fields of the Type II strings [8]. In the Type IIA string, the low-energy excitations include RR-forms of odd degree, which couple to D-branes of even spatial dimension, and vice-versa for the Type IIB string. The five limits of string theory are:

1. Type IIA: $N = 2$ supersymmetry in 10D, no gauge group, contains the closed F1-string, NS5-brane, and D-branes of even spatial dimension (D0,D2,D4,D6,D8).

2. Type IIB: $N = 2$ supersymmetry in 10D, no gauge group, contains the closed F1-string, NS5-brane, and D-branes of odd spatial dimension (D1,D3,D5,D7).

3. $E_8 \times E_8$ Heterotic: $N = 1$ supersymmetry in 10D, gauge group $(E_8 \times E_8) \rtimes \mathbb{Z}_2$, contains the closed F1-string, NS5-brane, and no D-branes.

4. Spin(32)/$\mathbb{Z}_2$ Heterotic: $N = 1$ supersymmetry in 10D, gauge group Spin(32)/$\mathbb{Z}_2$, contains the closed F1-string, NS5-brane, and no D-branes.

5. Type I: $N = 1$ supersymmetry in 10D, gauge group Spin(32)/$\mathbb{Z}_2$, contains the open F1-string and D-branes of odd spatial dimension (D1,D3,D5,D7,D9).

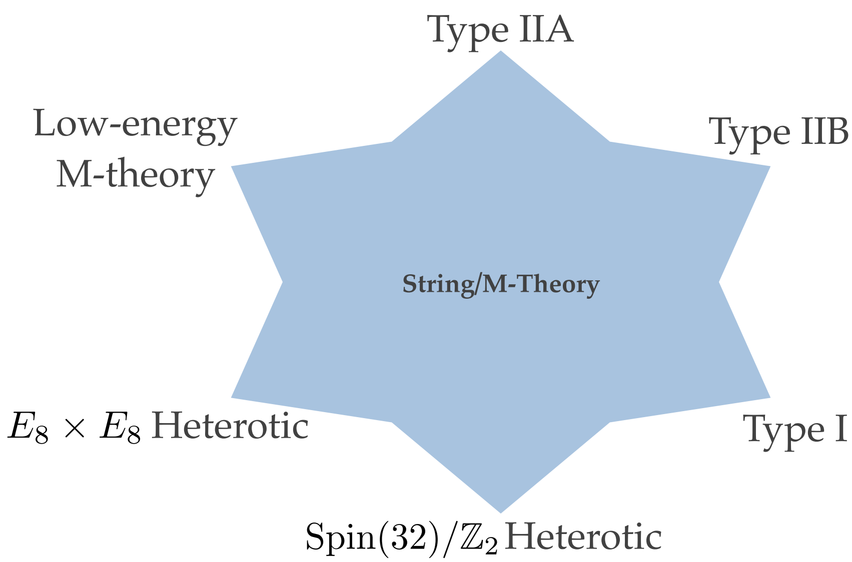

Figure 1.2: A schematic of the space of vacua of string/M-theory. In the interior of the space, the mathematical form of the theory is unknown, but there are six boundary regions in which we have good control: the low-energy limits of the five supersymmetric string theories and M-theory.

These theories are connected by a web of dualities in various dimensions, as further outlined in Section 1.3 below.

1.1.2 M-Theory

The phrase "M-theory" is used in two ways. On the one hand, it can refer to the space of string and brane theories as a whole, including the five stringy low-energy limits. (This space of theories is also sometimes referred to as just "string theory". In this usage, the phrases are interchangeable.) On the other hand, it may be used to denote the strong coupling limit of the Type IIA string, in which the low energy effective theory appears

11-dimensional. In this work, we use the latter meaning of the phrase.

In the strong-coupling limit of the Type IIA string, D0-brane states arrange into the Kaluza-Klein modes required for an eleventh dimension [9]. The bosons of the $N = 1$ supergravity multiplet in 11D are only the metric g and a 3-form field C that is called the C-field. The branes that are charged under the C-field are the M2- and M5-branes, so these are the stable extended objects of the low-energy theory. In the string case, the 10D effective theory is found as the target space into which the 2D string worldsheet is embedded by a nonlinear sigma model. One may expect that M-theory would be interpreted as a sigma model of the 3D M2-brane worldvolume into 11D spacetime, but this interpretation quickly runs into problems, as the quantum theory is not well-defined. One issue is that the worldvolume can grow long, narrow spikes that do not cost energy, and attempting to quantize these fluctuations gives an infinite number of massless particles. For a thorough explanation of this quantization problem, as well as others, see [10, 11, 12, 13]. The issues with the 3D sigma model approach are subtle, as they depend strongly on whether one is considering the bosonic or supersymmetric version of the brane as well as whether the theory is the classical or quantum. Although this type of interpretation has not yet been successful, it may still be a viable path toward a fundamental description of M-theory.

The C-field plays a role in M-theory analogous to the B-field in string theories, but its description is more subtle [14]. The gauge equivalence class (analogous to the holonomies of a gauge field around all closed loops) of the C-field is a *shifted differential character*, a higher-form generalization of the electromagnetism case. Within an equivalence class, the field configuration itself is described in terms of a pair (c, A) of a 3-form field and an E_8 gauge field. The gauge transformations of the field involve both the 3-form and the gauge field. The gauge field is particularly relevant to manifolds with boundary and plays a crucial role in duality with the heterotic string in the half-K3 limit, as will be discussed in Chapter 2. The future of fundamental descriptions of M-theory rests on a better understanding of the nature of the C-field and its charged objects, the M2-brane and M5-brane. The worldvolume theories of these branes are complicated, especially when there are multiple branes stacked atop one another. The theory of multiple M2-branes on an orbifold singularity has been written down in terms of higher algebras [15, 16, 17], but the theory of multiple M5-branes is conjectured to be non-Lagrangian.

1.2 Compactification and Low-Energy Effective Theories

To achieve a description of the observed four-dimensional universe, string theories must be *compactified* on an internal space[2]. This means that we consider a string background of the form

$$X^{9,1} = \mathbb{R}^{3,1} \times Y^6 ,$$

where $X^{9,1}$ is the full 10D spacetime, $\mathbb{R}^{3,1}$ is the uncompactified "visible" 4D spacetime, and Y^6 is a 6-dimensional compact "internal" space. (Similarly, compactification of M-theory involves a 7-dimensional space Y^7.) We then consider perturbative and non-perturbative (to the extent possible) effects about this background configuration. In the cases considered in this dissertation, the metric on X will be a product of the metrics on the individual factors on the right hand side (when it is not a product metric, we say that the compactification is *warped).*

The compactifications for which we have the best tools are those that retain supersymmetry at the compactification scale. For a background to achieve this, it must satisfy equations obtained from variations of the fermion fields. Assuming no background flux,

[2]The simplest compactification spaces are manifolds, but we use the general term 'space' to refer also to orbifolds or even spaces that admit more severe singularities.

the variation of the 10D or 11D gravitino field, $\delta\psi = 0$, implies the condition $\nabla\epsilon = 0$ for a spinor field ϵ. This condition says that the parallel transport of ϵ around any contractible loop by the Levi-Civita connection is trivial. In other words, ϵ is a *parallel spinor field.* The condition that a metric admits a parallel spinor is quite restrictive, and in particular implies that our manifold has *special holonomy*, meaning that the holonomy group of the metric connection is a proper subgroup of $\mathrm{SO}(N)$, where N is the dimension of the manifold.

1.2.1 Spaces of Special Holonomy

For our purposes, the most important spaces of special holonomy will be those in real dimensions 4, 6, and 7, corresponding to K3 surfaces, Calabi-Yau threefolds, and G_2 spaces, respectively. The relevant subgroup inclusions are $\mathrm{SU}(2) < \mathrm{SO}(4)$, $\mathrm{SU}(3) < \mathrm{SO}(6)$, and $G_2 < \mathrm{SO}(7)$. In these cases, the condition of special holonomy may also be phrased in terms of parallel form fields supported on the spaces. For the Calabi-Yau cases, the relevant form fields are the Kähler 2-form ω_K and the holomorphic 3-form Ω, which may be thought of as a complexified version of a volume form. One may always choose local coordinates $z_i = x_i + iy_i$, with $i = 1, 2, 3$ so that these fields take the canonical forms

$$\begin{aligned}\omega_K &= dx^1 \wedge dy^1 + dx^2 \wedge dy^2 + dx^3 \wedge dy^3 \\ \Omega &= dz^1 \wedge dz^2 \wedge dz^3 \ .\end{aligned}$$

Certain form fields allow one to distinguish a class of submanifolds that are said to be *calibrated*: those for which the special holonomy form fields compute the volume. Background configurations of branes extend along on submanifolds of the compact geometry (when the submanifold itself is compact, we say the brane is "wrapped"). The configurations of most interest are those in which the worldvolume theory itself preserves some amount of supersymmetry. We then say that the branes are wrapped on "supersymmetric cycles", and mathematically these cycles correspond to the calibrated submanifolds [18, 19, 20].

In Calabi-Yau threefolds, the cycles calibrated by powers of the Kähler form ω_K are the *complex submanifolds.* Another calibration is given by Re Ω, the real part of the holomorphic volume form, and its calibrated submanifolds are called *special Lagrangians.* As the name suggests, a special Lagrangian submanifold L is Lagrangian, i.e. $\omega_K \big|_L$, and it also has vanishing imaginary part of the holomorphic volume form: Im $\Omega \big|_L = 0$. Special Lagrangian submanifolds are particularly relevant to mirror symmetry and M-theory/heterotic duality, as will be discussed in Section 1.3.

To obtain an effective theory with $N = 1$ SUSY in 4D from M-theory, one must compactify on a 7D space of G_2 holonomy. When this space is smooth, we call it a G_2 *manifold.* One way to characterize these spaces is in terms of a 3-form ω called the G_2 *form*, which, with an appropriate choice of coordinates $x_1, ..., x_7$, may be expressed locally in the canonical form

$$\omega = dx^1 \wedge dx^2 \wedge dx^3 + dx^1 \wedge dx^4 \wedge dx^5 + dx^1 \wedge dx^6 \wedge dx^7 + dx^2 \wedge dx^4 \wedge dx^6 - \\ - dx^2 \wedge dx^5 \wedge dx^7 - dx^3 \wedge dx^4 \wedge dx^7 - dx^3 \wedge dx^5 \wedge dx^6 \ .$$

Physically, this form complexifies the C-field and together they determine the action for membrane instantons.

There are two types of calibrated submanifolds for G_2 manifolds (and similarly for G_2 spaces with singularities). The *associative* submanifolds are those that are calibrated by ω, while *coassociative* submanifolds are those that are calibrated by $\star_g \omega$, where $\star_g$ is the Hodge star with respect to the G_2 holonomy metric g. The G_2 manifolds of primary interest in this dissertation are those with a *coassociative fibration*, meaning that they admit a projection $\pi : X \to B$, where X is the G_2 manifold, B is a 3-manifold base, and the generic fiber is a coassociative submanifold. These are the spaces that participate in a fiberwise M-theory/heterotic duality described further in Section 1.3.2 and Chapter 2.

There are two primary known types of constructions for G_2 manifolds. The first method, due to Joyce [21, 22], constructs flat orbifolds T^7/Γ of G_2 holonomy, inspired by K3 orbifolds such as $T^4/\mathbb{Z}_2$ (see next subsection for more about orbifolds). These spaces are desingularized by a gluing method, creating manifolds with holonomy G_2. The other construction method is by twisted-connected sums, where two *building blocks* are glued together after twisting their asymptotic boundaries so that the correct holonomy is obtained [23, 24].

1.2.2 Orbifolds

While much is known about manifolds of special holonomy, the metrics on these spaces have not been written down in closed form, making calculation of metric-dependent quantities difficult. An exception to this difficulty is at special points in moduli space where the metric is flat almost everywhere. In these cases, the curvature is confined at singular loci. The types of singular spaces that are best-understood are called *orbifolds*, where each patch is homeomorphic to $\mathbb{R}^n/\Gamma$, where Γ is a finite group. At fixed points of the action of Γ, the space has a singularity. The collection of orbifold singularities may be comprised of isolated points, or it may include higher-dimensional loci; in our examples, the orbifold loci will often have the topology of a torus.

The orbifolds that play a role in this work are those of the form T^n/Γ, where T^n is an n-torus and Γ is a product of cyclic groups. We may further specialize to ADE singularities, which are those singularities that come from a finite subgroup of SU(2) acting on an $\mathbb{R}^4$ patch. Through the McKay correspondence, these subgroups are classified by the simply laced Dynkin diagrams, which are those of type A, D, and E. We will primarily be interested in A_1 singularities, which are those locally of the form $\mathbb{R}^4/\mathbb{Z}_2$, where the action of the generator reflects all four coordinates. For example, $T^4/\mathbb{Z}_2$ has 16 isolated A_1 singularities and is an orbifold limit of K3 that is the prototypical special holonomy orbifold. We will also be interested in orbifold limits of Calabi-Yau threefolds of the form $T^6/\mathbb{Z}_2^2$ and G_2 orbifolds of the form $T^7/\mathbb{Z}_2^3$. These spaces have 2- and 3-dimensional loci of A_1 singularities, respectively.

Theories made of only particles are not well-defined on orbifold geometries because the metric singularities create infinities in the predictions of the theory. String theories, however, do make sense on orbifolds because the theory contains extra *twisted sectors* that serve to smooth the singularity in the effective theory. In fact, orbifolds provide some of the simplest and most useful string backgrounds, because they give rise to semi-realistic spectra while having metrics that are flat away from the singularities.

Each of the five supersymmetric string theories has its own characteristics when compactified on an orbifold. In the case of the Type IIA string (as well as M-theory), orbifold singularities are spontaneously wrapped by zero-size D2-branes (M2-branes in the strong-coupling limit) that look like vector particles from the perspective of the low-energy theory. The resulting gauge symmetry is that corresponding to the ADE Dynkin diagram that represents the orbifold singularity [25]. When the Type I string is placed on an orbifold, a tadpole cancellation condition gives rise to D-brane states that provide extra gauge symmetry as well [26, 27]. Meanwhile, in the heterotic string, compactifications on an orbifold may be calculated algorithmically using shifts in the lattices that define the theory. The spectrum of the heterotic string on a K3 orbifold was first described in [28]. The heterotic string can exhibit exotic behavior on an orbifold when the background gauge bundle configuration contains instantons that shrink to zero size and sit on the orbifold loci [29]. See Chapter 2 for much more about point-like instantons.

A more general notion of orbifolds allows one to quotient by actions that are not purely geometric in spacetime. One important example is the *orientifold*, where one quotients by the $\mathbb{Z}_2$ symmetry Ω_P, the parity operator on the 2D string worldsheet. The simplest example of this construction is to quotient a Type IIB string background by Ω_P to obtain a Type I string background; this is one way to define the Type I string. Compactifications of the Type I string on orbifolds mentioned above and discussed in Chapter 2 may be alternatively thought of as Type IIB orientifolds. The action of parity reversal removes the B-field from the spectrum, allowing for strings to break, and thus creating the fundamental open string in the Type I limit. Fixed points of the orientation reversal action create *orientifold planes*, or O-planes. These objects are non-dynamical, but they carry a charge under RR-forms, and tadpole cancellation requires that background D-branes must be included as well to cancel the charge. This allows for rich model-building with the Type I string.

1.2.3 Torus-Invariant Compactifications

One strategy for obtaining tractable low-energy effective theories is to assume a geometric invariance in the background. This is not possible for compact manifolds of special holonomy, which do not admit continuous isometries, but we may avoid this problem by looking at related *non*-compact models. The idea is to "zoom in" on a patch of the compact space and assume that the manifold locally admits a continuous isometry that cannot be extended to the whole space. From this procedure, we obtain the *local model* of a patch that hopefully can be stitched together with local models for other patches into a full theory of the compact space.

The type of local model of interest in this work is one that comes from looking at the heterotic string on a local Calabi-Yau threefold. Specifically, we assume that the portion has the topology $\mathbb{R}^3 \times T^3$ and the background fields are invariant along the T^3-directions, giving an effective low-energy theory on $\mathbb{R}^3$. The form for the local geometry is motivated by the Strominger–Yau–Zaslow (SYZ) conjecture, which says that any Calabi-Yau threefold that participates in mirror symmetry has a fibration where the generic fiber is topologically T^3 and the base is topologically S^3 (see Figure 1.3) [30]. Then, our local model comes from "zooming in" to an $\mathbb{R}^3$ patch of the S^3 base over which all fibers are non-singular.

Following the heterotic equations, called the *Hull-Strominger system*, through this procedure gives rise to a system of three equations on the $\mathbb{R}^3$ base that comprises a flatness condition for a complexified field strength and a stability condition. This 3D system of equations is analogous to the 2D *Hitchin system*, which arises from a similar reduction on the 2D fibers of an elliptic fibration. In chapter 3 of this dissertation, we will discuss the resulting system of 3D PDEs in detail and obtain a new non-abelian solution.

1.3 String/M Dualities

By compactifying the various string theories and M-theory on spaces of special holonomy, one may describe the same low-energy effective theory from many different perspectives. Many of these dualities may be revealed and studied via matching of *moduli spaces*, i.e. the parameter space of vacua of the theory. The phrase 'moduli space' comes from geometry, where a geometric object may have auxiliary structures equipped, such as a metric or a complex structure, and there may be a continuous or discrete space of parameters, or *moduli*, that capture these choices. This moduli space itself is a geometric object, and carries its own natural metric. An example of a geometric moduli space is

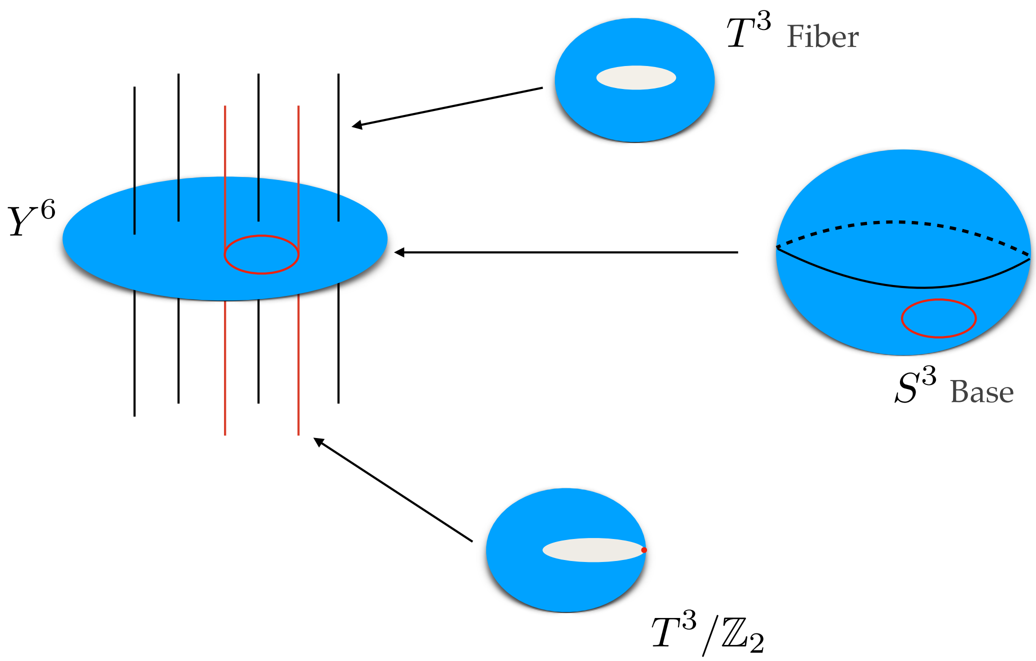

Figure 1.3: A Calabi–Yau threefold Y^6 with a Strominger–Yau–Zaslow (SYZ) fibration. The generic fiber is a special Lagrangian T^3, while the base is topologically S^3. There is a singular locus (red) in the base over which the fibers degenerate to, for example, an orbifold limit of T^3. It is conjectured that all Calabi–Yau threefolds participating in mirror symmetry possess such a fibration.

the moduli space of T^2 thought of as a complex curve.

In string and M-theory, the low-energy incarnations of the theories dictate specific auxiliary structures that must be equipped on geometric objects to enhance their moduli spaces to get their *stringy* incarnations. For example, the low-energy heterotic string requires a choice of background vector bundle with structure group $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$, while the low-energy Type II strings require choices of background higher-form fields of varying degrees. By finding correspondences between these stringy moduli spaces, we may identify dualities between string theories compactified on various spaces.

Dualities between theories compactified on low-dimensional manifolds may be leveraged to more complex situations by *fibering* the duality, where the theories are compactified on spaces that admit a fibration where the generic fiber is the low-dimensional manifold for which the duality is established. This strategy is best justified in the *adiabatic limit*, where the geometry of the base is slowly-varying compared to that of the fiber, so that local models are good approximations to the physics. See [31] for a discussion of criteria for the validity of fibered dualities.

1.3.1 Type II T-Duality and Mirror Symmetry

One of the basic string dualities is Type II T-duality, where torus compactifications of the Type IIA and IIB string give identical low-energy physics when the geometry and background fields are related by a transformation that includes inverting the radii of some directions of the compactification space. In this work, we will restrict ourselves to cases where the geometry is a torus orbifold or admits a torus fibration, has a flat background metric (away from orbifold singularities), and has vanishing B-field background. In these cases, the geometric transformation on the affected coordinate is $R \to \alpha'/R$, and the momenta and winding numbers of states in the transformed directions are interchanged.

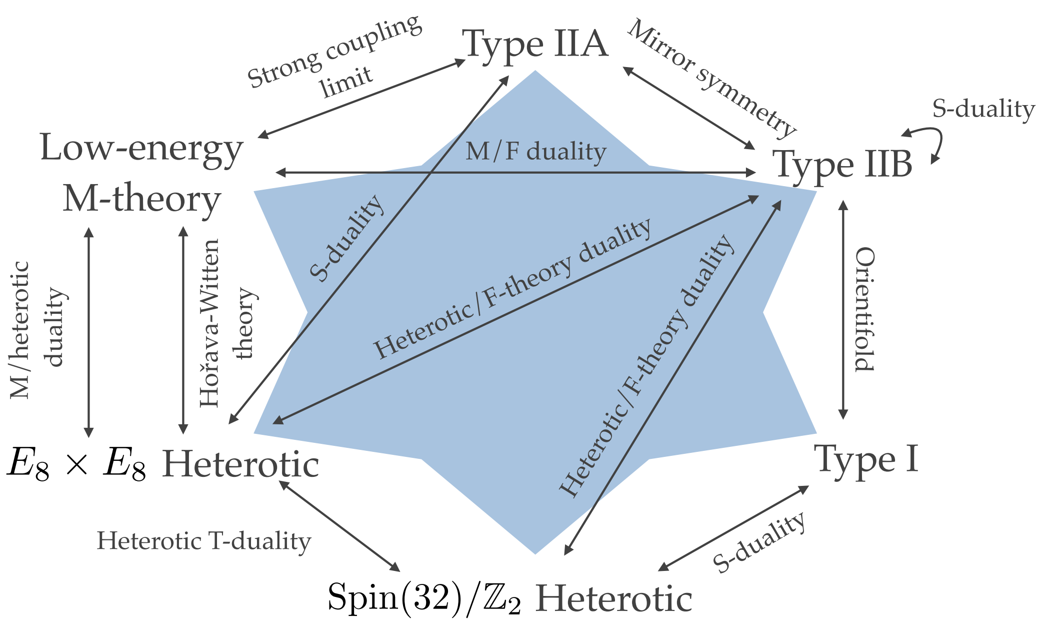

Figure 1.4: A collection of some string/M/F-theory dualities. These relations are useful because they allow one to directly transfer questions between different facets of string theory without needing to pass through the unknown interior of the space of vacua.

When T-duality is applied to an odd number of dimensions, it relates the Type IIA string to the Type IIB string, but when it is applied to an even number of dimensions, it relates either string to itself. The T-duality transformation also affects background D-branes by interchanging Neumann and Dirichlet boundary conditions.

Another famous string duality, called *mirror symmetry,* has been conjectured to be equivalent to T-duality [30]. Mirror symmetry relates the Type IIA string on Calabi-Yau threefold X to the Type IIB string on its mirror Calabi-Yau threefold $\tilde{X}$, where the Hodge diamond of $\tilde{X}$ is that of X reflected along a diagonal (as if in a mirror). Strominger, Yau, and Zaslow conjectured that any Calabi-Yau threefold that admits a mirror manifold also admits a fibration by special Lagrangian tori, and that mirror symmetry is accomplished by applying T-duality along these T^3 fibers. In Chapters 2 and 3, we will be interested in Calabi-Yau threefolds that admit an SYZ fibration, but we will consider the heterotic string on such spaces instead of the Type II strings.

1.3.2 M/Type IIA/Heterotic Duality

A fundamental duality for our work in Chapter 2 is that between the Type IIA string (or, in the correct limit, M-theory) and the $E_8 \times E_8$ heterotic string. Because the Type II strings have twice as much supersymmetry in 10D as do the heterotic strings, a duality requires the heterotic theory to be compactified on a space with twice as many parallel spinors as the IIA space. In terms of 4D supersymmetry, there are three amounts of symmetry often discussed:

- 4D $N = 4$ supersymmetry requires that the IIA string is compactified on K3 $\times T^2$ while the heterotic string is compactified on T^6.
- 4D $N = 2$ supersymmetry is achieved between Type IIA on a Calabi-Yau threefold and heterotic on K3 $\times T^2$.

- 4D $N = 1$ supersymmetry from Type IIA compactified on a Calabi-Yau orientifold and heterotic compactified on a Calabi-Yau threefold.

For excellent reviews on 6D IIA/heterotic duality on special holonomy spaces, see [4, 5].

The above dualities also extend to the M-theory limit, and in the $N = 1$ case, one has a duality between M-theory on a G_2 manifold and heterotic on a Calabi-Yau threefold. This is the central duality we will explore in Chapter 2.

1.3.3 Heterotic-M Theory

Another duality between the heterotic string and M-theory goes by the name of Horava-Witten theory or heterotic-M theory[3] [34, 35]. (Note that this is distinct from the duality between M-theory on K3 and heterotic on T^3 from the previous subsection.) In heterotic-M theory, the strong coupling limit of the $E_8 \times E_8$ heterotic string on a space X is dual to M-theory on the space $X \times S^1/\mathbb{Z}_2$, with the heterotic string coupling dual to the length of the interval $S^1/\mathbb{Z}_2$. Anomaly cancellation shows that each 10D end of the M-theory interval must carry an E_8 gauge theory, so that the heterotic string may be thought of as living on these hyperplanes, and the heterotic weak-coupling limit is achieved when the interval shrinks to zero size and the planes overlap.

1.3.4 Type I/Heterotic Duality

An additional duality that we will make use of in chapter 2 is that between the Type I string and the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string [36]. This is a type of S-duality, meaning that weak string coupling on one side is related to strong string couping on the other side. The two theories share the gauge group $\mathrm{Spin}(32)/\mathbb{Z}_2$, and their low-energy supergravities are related by a simple transformation. To test the duality beyond supergravity, one may

[3] See [32, 33] for reviews of model building using compactifications of this type.

compare the branes in the two theories. The D1-brane of the Type I theory becomes the fundamental heterotic string under the duality, as the two share the same expression for tension. Similarly, the Type I D5-brane becomes the heterotic NS5-brane. There is no heterotic counterpart of the Type I fundamental string, however, because Type I is missing the B-field that would stabilize this type of brane. Thus, they are not stable at strong Type I coupling. We will be interested in this duality in particular when the two sides are compactified on special holonomy orbifolds.

1.3.5 Heterotic T-Duality

A final duality that we make use of in chapter 2 is heterotic T-duality. This duality relates the two heterotic theories when compactified upon T^n for $n \geq 1$. The moduli space in this case is based upon a lattice of signature $(16+n, n)$, and takes the form

$$\mathcal{M}_n^{\text{het}} = \left(\mathrm{O}(16+n,\mathbb{R}) \times \mathrm{O}(n,\mathbb{R})\right) \backslash \mathrm{O}(16+n,n;\mathbb{R})/\mathrm{O}(16+n,n;\mathbb{Z})$$

where the notation indicates quotients by both the subgroups on the left and on the right [37, 38]. The existence of heterotic T-duality leads to the conclusion that the two heterotic theories are distinct only in 10D; when compactified to 9D or below, any heterotic vacuum could be described by a compactification that starts with either gauge group in 10D. In our cases this result is particularly significant because it means that the same heterotic point-like instantons may be describable from either heterotic perspective when the two theories are T-dual.

1.4 Permissions and Attributions

1. The content of Chapter 2 is the result of a collaboration with Bobby Acharya and David R. Morrison [39].

2. The content of Chapter 3 and Appendix A is the result of a collaboration with Bobby Acharya and Eirik Eik Svanes and has previously appeared in the Journal of High Energy Physics: JHEP **01** (2021) 197. It is reproduced here with the permission of the International School of Advanced Studies (SISSA): `https://jhep.sissa.it/jhep/help/JHEP/CR_OA.pdf`.

Chapter 2

Non-Perturbative Heterotic Duals of M-Theory on G_2 Orbifolds

2.1 Introduction and Summary

One of the well-known dualities in string theory relates M-theory compactified on a K3 surface to the $E_8 \times E_8$ heterotic string compactified on a three-torus [40, 9]. It was proposed long ago that this 7D M/heterotic duality could be applied fiberwise over an S^3 base to obtain a 4D duality as well [41, 42, 43]. In this case, M-theory is compactified on a G_2 manifold equipped with a coassociative K3 fibration, while the $E_8 \times E_8$ heterotic string is compactified on a Calabi–Yau threefold equipped with a supersymmetric three-torus fibration (also known as an SYZ fibration [30]).

One way to exhibit the 7D M/heterotic duality is to take the large heterotic volume limit, which corresponds to the "half-K3" limit on the M-theory side [44]. There is a limiting family of K3 metrics in which a long throat of the form $T^3 \times I$ develops, where I is an interval, and the complicated geometry is confined to the two ends. Each complicated end is known as a half-K3 surface and carries a metric known as an ALH instanton [45].

These half-K3 surfaces each determine an E_8 bundle on T^3, together giving a heterotic string gauge background [46].

One can then attempt to find a similar fiberwise picture for a G_2 space X with a coassociative K3 fibration. Under favorable conditions, there will be a family of metrics in which a long throat of the form $Y \times I$ develops, where Y is the SYZ-fibered Calabi–Yau threefold appearing as the heterotic dual. We call this the "half-G_2" limit, and in this chapter we will discuss aspects of M/heterotic duality in this limit that go beyond the perturbative picture of the half-K3 limit. Our goal is to work towards a dictionary between G_2 spaces and the heterotic gauge bundle. We approach this task by trying to answer this question in the simple case of a Joyce orbifold: how is the geometry of the ambient G_2 space reflected by the heterotic bundle, which lives only on a suborbifold? For the simple examples studied in this chapter the topological data on the G_2 side is captured by the configuration of the orbifold singular loci and their intersections with codimension-1 suborbifolds. This data is spread throughout the throat interval in the half-G_2 limit, as opposed to the situation of the half-K3 limit, where the singularities are confined to the ends of the interval. On the heterotic side, this data is represented by point-like instantons on orbifold singularities. We find point-like instanton configurations that looks somewhat exotic from the $E_8 \times E_8$ perspective, but can be understood as T-dual $\mathrm{Spin}(32)/\mathbb{Z}_2$ point-like instantons on an orbifold with a winding shift.

In general, M/heterotic duality shares many properties with heterotic/F-theory duality, and in some cases the two are directly related via a duality chain. This duality was used in [47] to study M-theory on twisted-connected sum G_2 spaces that support fibrations by K3 surfaces that are themselves elliptically fibered. Beyond the twisted-connected sum examples, a generic compactification of M-theory on a K3-fibered G_2 space is not expected to have an F-theory dual, and must be studied in terms of differential geometry instead of complex geometry. In this work we explore M/heterotic duality without the tools of elliptic fibrations on the M-theory side. One useful perspective in this case is duality with the Type I string, where tadpole cancellation conditions give additional computational tools.

It has long been recognized that M-theory needs to be compactified on spaces with singularities in order to produce interesting gauge groups and matter content in the effective theory [48, 42]. Joyce's work [21, 22] is celebrated for demonstrating the existence of nonsingular compact manifolds with holonomy G_2, but ironically, the singular T^7/Γ orbifolds from which Joyce started are more relevant to the physics than their nonsingular cousins. Those orbifolds have flat metrics and a natural G_2 structure encoded in an invariant three-form, which is the limit of the smooth G_2 structures when the resolved singularities are blown back down. In this chapter we will study those orbifolds themselves. The resulting effective theories preserve $N = 1$ supersymmetry and have ADE gauge groups, but the lack of codimension 7 singularities implies that there is no chiral matter, so that these particular Joyce orbifolds cannot produce phenomenologically realistic effective theories in this limit. However, these orbifolds produce a simple laboratory within which to deduce properties of duality that are expected to persist for more realistic examples.

In many of Joyce's orbifolds, there is a fibration by flat Kummer surfaces of the form $T^4/\mathbb{Z}_2$. It is precisely in such an orbifold limit that Ricci-flat metrics on K3 surfaces are easy to construct, because in that limit those metrics are flat. The corresponding fibration is by coassociative cycles of T^7/Γ, with Γ a finite group, and again the coassociative condition is trivial to check because we are working with flat metrics[1]. The geometry of Kummer fibrations of G_2 orbifolds was analyzed in detail by Liu [49], whose work forms part of the foundation upon which we develop heterotic duals.

[1] It is an open question whether on Joyce's resolution of singularities, there are smooth K3 surfaces which resolve the singularities of the Kummer surfaces in such a way as to form a coassociative fibration.

To find the half-G_2 limit, we identify a particular $S^1 \subset T^7$ on which Γ acts as a reflection, so that there is a fibration $T^6/H \to T^7/\Gamma \to S^1/\mathbb{Z}_2$ with H a subgroup of Γ and the ends of the interval $S^1/\mathbb{Z}_2$ the location of the complicated geometry. In all of the examples we consider, the Calabi–Yau threefold Y is also an orbifold T^6/H, and in our $N = 1$ supersymmetric cases, it is an orbifold of a special type known as a Borcea–Voisin orbifold[2] [51, 52]. In fact, our $N = 1$ examples all live on the same Borcea–Voisin orbifold, which is the blow-down limit of the Schoen manifold, in agreement with the results of [47].

Identification of the heterotic dual requires specifying a background gauge bundle with connection on the heterotic Calabi–Yau Y, which is T^6/H or its resolution. Ideally, we would have an algorithmic procedure to determine this bundle from the M-theory data, in analogy to the case of heterotic/F-theory duality [53], but this is made difficult by the fact that the T^3 fibers of Y are not complex submanifolds, so we have instead identified the dual bundle by indirect means. One useful tool is the matching of massless spectra on the two sides. In particular, we may split the heterotic spectrum into a perturbative part and a non-perturbative part, where the former may be seen from a CFT analysis, while the latter comprises the effects that are non-perturbative in the (heterotic) string coupling. These two parts of the dual heterotic spectrum are distinguished on the M-theory side by whether individual components of the singular locus of the G_2 orbifold are transverse to the generic fiber of the K3 fibration or not, in the spirit of [54]. The split refines our analysis of the dual pair, as we must ensure that the heterotic particles have the correct perturbative/non-perturbative origin.

The perturbative spectrum may be obtained by breaking of primordial gauge symmetry by the monodromy of instanton connections sitting on the orbifold singularities.

[2] One of the advantages of this observation is that Gross and Wilson analyzed SYZ fibrations on Borcea–Voisin orbifolds and on their resolutions [50].

We expect the non-perturbative part of the heterotic spectrum to come from these instantons in the singular point-like limit. Such gauge configurations are consistent with heterotic anomaly cancellation conditions and are the best-understood sources of non-perturbative gauge symmetry in heterotic $E_8 \times E_8$ compactifications. The massless particle contributions of point-like instantons are partially understood in simple examples, but distinguishing between different cases can be subtle [55], and there is no complete classification. Some of the point-like instantons that we identify in dual heterotic backgrounds are supported on pairs of orbifold loci and do not look familiar from previous studies of point-like instantons on orbifold singularities. This may be an analog of the gauge locking phenomenon seen in Hořava–Witten compactifications [56, 57, 58] or a freezing of heterotic moduli by a gauge bundle configuration [59]. In the non-singular limit, candidate local descriptions for this type of instanton may be given by $\mathbb{Z}_2$-quotients of instantons on $\mathbb{R}^4$ or a caloron on $\mathbb{R}^3 \times S^1$ [60, 61]. The behavior of the point-like instantons is more clear from a T-dual $\mathrm{Spin}(32)/\mathbb{Z}_2$ perspective [62], where the background is acted upon by a winding shift.

This chapter is organized as follows. Section 2.2 gives an overview of the fundamental M/heterotic duality in 7D and its fibration over a 3D base. Section 2.3 discusses M-theory on G_2 orbifolds and analyzes three examples of K3-fibered G_2 orbifolds that will form the heart of the chapter. In Section 2.4, we examine the dual heterotic geometry, a Borcea–Voisin orbifold, that is dictated by the duality in the half-G_2 limit. In Section 2.5, we survey non-perturbative aspects of the heterotic gauge bundle, and in particular point-like instantons on orbifold singularities. This prepares us to analyze the gauge bundles of our dual heterotic examples in Section 2.6. In Section 2.7, we investigate the nature of the heterotic gauge bundle via an alternative duality chain relating our M-theory setup to Type I compactifications on orbifolds with winding shifts. Finally, in Section 2.8, we interpret our results in terms of Hořava–Witten duals, gauge locking, and frozen moduli and discuss future directions.

2.2 Heterotic/M-Theory Duality

2.2.1 Duality in 7D

To obtain dual low energy effective theories in 4D, we will make use of the duality between the 7D theories arising from the $E_8 \times E_8$ heterotic string on T^3 and M-theory on the compact 4-manifold known as a K3 surface [9]. Evidence for this duality comes in part from the fact that these two compactifications share the same moduli space:[3]

$$\mathcal{M}_{7D} = [\mathrm{SO}(3,19;\mathbb{Z})\backslash\mathrm{SO}(3,19;\mathbb{R})/\mathrm{SO}(3;\mathbb{R}) \times \mathrm{SO}(19;\mathbb{R})] \times \mathbb{R}^+ \ .$$

On the M-theory side, the first factor is interpreted as the moduli space of volume-1 Einstein metrics on K3, while the $\mathbb{R}^+$ factor is the volume. On the heterotic side, the first factor is instead interpreted as the Narain moduli space of heterotic compactifications on T^3, while the $\mathbb{R}^+$ is the string coupling. By comparing the effective actions on each side of the duality, one finds the relation between the $\mathbb{R}^+$ factors

$$e^{3\gamma} = \lambda \ ,$$

where $e^{3\gamma}$ is the volume of the K3 surface and λ is the heterotic string coupling.

There are special points in the moduli space where non-abelian gauge symmetry appears in the 7D theory. From the heterotic side, these points are those at which the holonomy of the flat $E_8 \times E_8$ connection over the T^3 is non-generic. The unbroken gauge symmetry in the effective theory is given by the centralizer of the reduced structure

[3] There are some subtleties concerning the discrete group action which we suppress here.

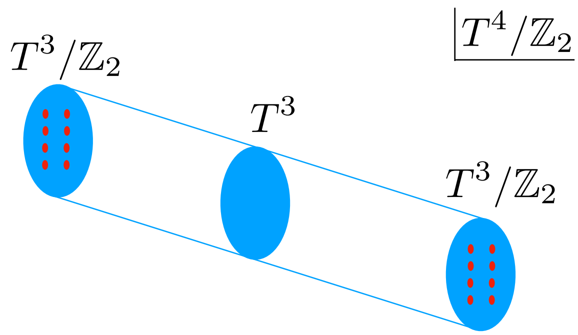

Figure 2.1: The half-K3 limit of $T^4/\mathbb{Z}_2$. The space degenerates into a long throat with cross section T^3, while the 16 orbifold points, which correspond to the complicated geometry of the resolved space, recede to either end of the throat. If we put M-theory on this space, then the dual heterotic theory lives on the central T^3 and has gauge bundle determined by the distant singularities.

group of the gauge bundle with connection. In the case of a flat connection, this is the centralizer of the holonomy group, which is generated by three commuting elements of $E_8 \times E_8$[4]. For a generic choice of these three elements, the gauge symmetry is reduced to the maximal torus $\mathrm{U}(1)^{16}$, but non-generic holonomies give instead ADE gauge groups.

From the view of M-theory, the special points in the moduli space are orbifold limits of K3 that contain ADE singularities [9]. That these singularities give rise to effective non-abelian gauge symmetry can be seen by blowing up an A_1 singularity to give an exceptional $\mathbb{P}^1$: this cycle is dual to a harmonic 2-form, which gives an effective U(1) gauge field upon Kaluza-Klein reduction of the C-field. Wrapping two M2-branes of opposite orientation on the cycle give effective vector particles charged under the U(1). As the $\mathbb{P}^1$ shrinks to zero volume, the charged particles become massless and complete the $\mathfrak{su}(2)$ Lie algebra. A similar argument extends to general ADE singularities.

[4]In this work, we only consider the identity-connected component of the space of flat connections. See [63] for discussion of the other components.

2.2.2 The Half-K3 and Weak Coupling Limits

The heterotic string on T^3 has two primary dimensionless parameters: the dimensionless compactification volume $\frac{\mathrm{vol}T^3}{\alpha'^{3/2}}$ and the string coupling λ. Where possible, we will work in the corner of the 7D parameter space where the compactification volume is large and the string coupling is small. The large volume limit is essential to current investigations into M/heterotic duality because it is where we can differentiate the moduli corresponding to the heterotic geometry and the gauge bundle, so that we may apply a geometric version of the duality [44, 47]. The weak coupling limit allows us to understand the heterotic physics via perturbation theory combined with instanton effects.

Both of these limits have a geometric realization on the M-theory side. Large heterotic volume corresponds to what is called the "half-K3 limit" (see Figure 2.1): the K3 grows a long throat where the geometry is slowly varying and approximately $T^3 \times (-r, r)$ for some $r \in \mathbb{R}$, so that all of the complicated geometry recedes to $\pm r$ [44]. In this limit, the 7D duality is realized by splitting the K3 surface in half, cutting transverse to the throat. This gives us two 4-manifolds with T^3 boundary - these are "half-K3 surfaces". Such a surface may be realized as a rational elliptic surface with a generic divisor (an elliptic curve) removed. The dual heterotic theory is compactified on the T^3 boundary shared by the half-K3 surfaces. The geometry of these surfaces contains the data for the $E_8 \times E_8$ heterotic gauge bundle on T^3. Specifically, the moduli of a half-K3 together with an embedded T^3 is the same as the moduli of an E_8 bundle on T^3. This half-K3 limit is analogous to the stable degeneration limit of 8D F-theory/heterotic duality, where large volume of the heterotic T^2 is dual to a limit in which the F-theory K3 geometry degenerates into the union of two rational elliptic surfaces meeting along the heterotic T^2 [64].

The other parameter is the heterotic string coupling, which corresponds to K3 volume on the M-theory side, with weak heterotic coupling corresponding to zero volume for the K3 surface. Going to this limit takes us out of the regime where 11D supergravity is a reliable approximation to M-theory, but because we are considering highly supersymmetric compactifications, the duality results are expected to persist when we add M2-brane effects. Again, there is an analogous limit in 8D F/het duality: in that case, the heterotic coupling is dual to the area of a section of the elliptic fibration, which may be interpreted as the area of the base of the fibration [64].

2.2.3 Duality in 4D

By fibering the 7D M/heterotic duality adiabatically over an S^3 base, we should be able to obtain dual pairs that give the same 4D effective theory. From the M-theory side, for this theory to have $N = 1$ SUSY, the total space of the K3 fibration must have holonomy G_2. Additionally, we want to look at effective theories with non-abelian gauge symmetry, so that our space will be a G_2 orbifold. In the large heterotic volume limit, the heterotic geometry is determined to be a suborbifold of the G_2 orbifold, and SUSY then requires that it is an SYZ fibration of a Calabi–Yau orbifold (i.e. a special Lagrangian T^3 fibration of such a space over an S^3 base) [41]. The topology of G_2 and Calabi–Yau orbifolds requires that our fibrations have singular fibers (by which we mean fibers with multiple components in their resolution) where the adiabatic assumption will break down[5]. Such fibrations of G_2 manifolds were considered in an adiabatic limit in [65].

The large-volume limits on the heterotic side of the duality requires all geometric radii to be large compared to the relevant dimensionful parameter, which sets up a hierarchy of scales: we require that the T^3 fibers are large compared to $(\alpha')^{3/2}$, but small compared

[5]Because of this violation of the adiabatic assumption, it is not guaranteed that the duality results will persist in 4D. In the notation of [31], our case is of type 2(b), where duality often persists despite the presence of singular fibers.

to the volume of the base[6]. On the M-theory side, the K3 fibers on the G_2 side are also required to be small compared to the volume of the base.

For our 4D duality, we will apply the half-K3 and weak coupling limits fiberwise. This means that we will work in a corner of the G_2 moduli space where each K3 fiber, including the singular fibers, grows a long throat and simultaneously shrinks to small volume. This fiberwise half-K3 limit translates to a "half-G_2" limit, where our G_2 space grows a long throat with a Calabi–Yau threefold fiber that degenerates at the ends. The duality in this limit identifies the generic Calabi–Yau fiber as the heterotic geometry. By introducing a fibration, we also introduce additional possibilities for configurations of singularities in our half-G_2 compared to our half-K3. We will restrict ourselves to orbifold (i.e. codimension four) singularities, which live along a three-dimensional locus. These loci may be confined to the endpoints of the throat interval, in which case we will have a similar picture to the half-K3 limit, but they also may stretch across the throat interval and intersect the generic Calabi–Yau fiber. In the latter case, the singularities are higher codimension in the two boundary fibers and give rise to non-perturbative effects from the perspective of the heterotic compactification.

2.2.4 F-Theory Duals

A useful tool in analyzing the heterotic string and M-theory has been duality with F-theory, so this could be a candidate to use in a search for an algorithmic construction of heterotic duals to given M-theory backgrounds, as was done in [47]. However, in our case, where we are looking at isolated points of enhanced gauge symmetry in moduli space, the fiberwise nature of the data and the complex structures required by the dualities prevent a straightforward implementation of this method.

[6]In our torus-orbifold setup, volumes are to be interpreted as products of radii in the torus covering space.

To see the limitation, consider an M-theory background on a K3-fibered G_2 manifold. If we apply the 7D M/heterotic duality, we obtain bundle and flat connection data on the T^3 fibers of the heterotic geometry Y, i.e. the duality gives the restrictions $E\,\big|_A$ of the heterotic gauge bundle E to each T^3 fiber $A \subset Y$. This by itself is not enough information to reconstruct E—we have the vertical data but not the horizontal data. In the case of an elliptic fibration, where the vertical data is given by a spectral cover, the horizontal data is provided by a line bundle over that spectral cover [53].

In the case of M/heterotic duality, the T^3-fibration of Y is a special Lagrangian fibration, which requires a choice of complex structure where the holomorphic coordinates are made by pairing real coordinates on the base and on the fiber. This means that there is no elliptic curve contained in the T^3 fibers, and therefore we do not have bundle data on any elliptic fibration of Y. Thus an F-theory dual of the heterotic model cannot be used to infer the missing bundle data—the F-theory dual can be constructed only after we are able to determine the bundle by other means.

The complex structure change that would be required for an application of an F-theory dual may be thought of in $N = 2$ language as a movement in the hypermultiplet moduli space. In the case of a generic heterotic gauge bundle, where one would be moving from one generic point of the moduli space to another, an F-theory dual may give the correct answer (although even this generic situation may be complicated by the presence of domain walls in the moduli space). However, our situation deals with non-generic bundles with point-like instantons on orbifold singularities, and a shift in the hypermultiplet moduli space is likely to change the matter spectrum, especially because the bundle moduli of fractional-holonomy point-like instantons are coupled to the geometric moduli of the singular spaces on which they reside [29].

2.3 M-Theory on Joyce Orbifolds

Now we will describe the M-theory backgrounds for which we would like to find candidate heterotic duals. For the purposes of this work, we will think of low-energy M-theory as 11D supergravity supplemented by 7D spectra from M2 branes, as in [66]. Then, an M-theory compactification is specified by a choice of background metric, C-field, and 7D gauge field. Here we will consider G_2 orbifolds X of the form T^7/Γ, where Γ is a finite group, and we will assume vanishing C-field and gauge field backgrounds[7].

The non-abelian factors in the gauge group of the low-energy effective theory may be read off from the locus S of orbifold singularities in X, which comes from the fixed points of elements of Γ. Each connected component of the orbifold locus of codimension four gives rise to gauge symmetry in the effective theory according to the ADE classification of the singularity [9]. In the examples we consider, each component of the singular locus is topologically T^3 or $T^3/\mathbb{Z}_2$. Counting these components on the M-theory side gives the non-abelian gauge symmetry of the low energy theory. The gauge group will have an additional abelian factor $\mathrm{U}(1)^{b^2_\Gamma(X)}$ from the Kaluza-Klein reduction of the M-theory C-field, where $b^2_\Gamma(X)$ counts the number of Γ-invariant harmonic 2-forms on T^7. Isometries of the metric give an additional low-energy abelian gauge symmetry of dimension $b^1_\Gamma(X)$. In our $N=1$ supersymmetric cases, we have $b^1_\Gamma(X)=0$ and $b^2_\Gamma(X)=0$, so that the 4D low-energy gauge group has no abelian factor.

In addition to gauge bosons, the massless spectrum of M-theory on X includes chiral multiplets that may or may not be charged under the gauge symmetry. The number of uncharged chiral multiplets is determined by $b^3_\Gamma(X)$, the number of Γ-invariant harmonic 3-forms on X. The charged matter, meanwhile, is determined by the geometry of the

[7] While background C-field flux on a smooth G_2 manifold necessarily breaks supersymmetry [67], some G_2 orbifolds can support background C-field fluxes and gauge fields at the singular loci that together preserve supersymmetry [68]. It would be interesting to investigate heterotic duals of these cases.

orbifold loci: each codimension four locus component L contributes $b^1(L)$ chiral multiplets valued in the adjoint of the gauge group factor corresponding to L [48]. Intersections of the orbifold loci give rise to more complicated matter representations, but the examples considered in this chapter have non-intersecting loci, so will be limited to adjoint matter. All of the matter in our examples lies in real representations of the gauge group, so the spectra are non-chiral.

Because gauge symmetry and charged matter in the low-energy theory is specified by the orbifold singularities of X, it is independent of a choice of K3 fibration. However, to compare this spectrum to that of a dual heterotic string, we must choose a particular K3 fibration $\pi : X \to Q$ and relate the gauge theory of the 4D effective theory to that of the 7D effective theories on the fibers. For example, the $\mathrm{SU}(2)^{16}$ gauge symmetry on a generic $T^4/\mathbb{Z}_2$ fiber will be reduced to a subgroup in the 4D theory because the relevant components of the orbifold locus intersect the generic fiber at multiple points, so that these singularities appear to be distinct from the perspective of the theory on the fiber, but not from the perspective of X. In other words, the monodromy action of Γ on the singularities of the fiber reduces the gauge group to a subgroup in 4D.

2.3.1 Examples

Now we will discuss details of three M-theory backgrounds that will serve as our examples for which we will identify candidate heterotic duals in the half-G_2 limit. Our G_2 orbifolds are of the form $T^7/\mathbb{Z}_2^3$, where $\mathbb{Z}_2^3$ is generated by elements α, β, and γ. All three examples have the same actions for α and β on T^7, but differ in the action of γ.

The first two generators act as

$$\begin{aligned}\alpha : (x_1, ..., x_7) &\mapsto (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7) \\ \beta : (x_1, ..., x_7) &\mapsto (-x_1, \tfrac{1}{2} - x_2, x_3, x_4, -x_5, -x_6, x_7) \ ,\end{aligned}$$

where each $x_i \sim x_i + 1$ is a coordinate on a circle. Each of these elements fixes 16 T^3's in T^7, while exchanging the fixed tori of the other element in pairs. The element $\alpha\beta$ acts freely on T^7. Quotienting T^7 by the action of $\Gamma_1 = \langle \alpha, \beta \rangle$ gives the G_2 orbifold

$$X_1 = T^7/\Gamma_1 \cong \left(T^6_{123456}/\left\langle \alpha, \beta \right\rangle\right) \times S^1_7 \ ,$$

where subscripts on tori denote their coordinates. At this stage, the orbifold does not have full holonomy G_2, and will preserve $N = 2$ SUSY in 4D, as discussed in the first example below.

The 6-orbifold factor in X_1 is an orbifold limit of a Borcea–Voisin Calabi–Yau threefold with Hodge numbers $(19, 19)$ known as the Schoen manifold[8]. We will discuss this orbifold further in Section 2.4, where it serves as the heterotic geometry in our $N = 1$ examples.

For our M-theory backgrounds, we will quotient the space X_1 further by an action of γ. In our first example, the action of γ is trivial and $N = 2$ SUSY is preserved in 4D, while the remaining examples have nontrivial γ and preserve $N = 1$ SUSY in 4D.

Example 3.1: $N = 2$ **SUSY**

First, we will consider the case where the action of γ is trivial, so that we are compactifying M-theory on the orbifold $X_1 = T^7/\Gamma_1$ above. Ultimately, we are interested in

[8]This orbifold may also be referred to as DW(0-2) [69, 70]

Example Number	γ Action	Low-Energy Gauge Symmetry	Massless Charged Matter ($N = 1$ Language)
3.1	Trivial	$\mathrm{SU}(2)^{16} \times \mathrm{U}(1)^4$	3 adjoint chirals per SU(2)
3.2	Includes shift on x_3	$\mathrm{SU}(2)^{12}$	3 adjoint chirals per SU(2)
3.3	No shift on x_3	$\mathrm{SU}(2)^8 \times \mathrm{SU}(2)^8$	3 adjoint chirals for 8 SU(2) factors and 1 adjoint chiral for other 8 SU(2) factors

Table 2.1: Summary of spectra of M-Theory backgrounds

$N = 1$ SUSY in 4D, where the orbifolds have full holonomy G_2, but non-perturbative features of the half-G_2 limit appear in this simpler situation as well, so it will serve as our first example.

The space X_1 has 16 T^3's of A_1 singularities, with 8 coming from α and 8 coming from β. Its orbifold Betti numbers, by which we mean the counts of independent Γ_1-invariant harmonic forms, are $b^1_{\Gamma_1} = 1$, $b^2_{\Gamma_1} = 3$, and $b^3_{\Gamma_1} = 11$. Thus, the gauge symmetry of the 4D theory is expected to be $\mathrm{SU}(2)^{16} \times \mathrm{U}(1)^4$. The massless matter spectrum is 3 adjoint chirals of each SU(2) plus 11 neutral chiral multiplets, where the count of adjoint chirals comes from $b^1(T^3) = 3$.

There are two immediate coassociative fibrations by Kummer orbifolds:

- The α-fibration $\pi_{567} : T^7/\Gamma_1 \to T^3_{567}/\langle\beta\rangle$ with generic fiber $T^4_{1234}/\langle\alpha\rangle$
- The β-fibration $\pi_{347} : T^7/\Gamma_1 \to T^3_{347}/\langle\alpha\rangle$ with generic fiber $T^4_{1256}/\langle\beta\rangle$

Given a choice of the F-fibration, where F is one of α or β, let $Q_{1,F}$ be the 3-orbifold base of the fibration. In this case, both $Q_{1,\alpha}$ and $Q_{1,\beta}$ are orbifold-equivalent to $S^1 \times P$, where P is the pillow 2-orbifold obtained as the quotient of T^2 by a reflection in both coordinates. Topologically, this base is $S^2 \times S^1$, and it has four non-linking circles of singularities.

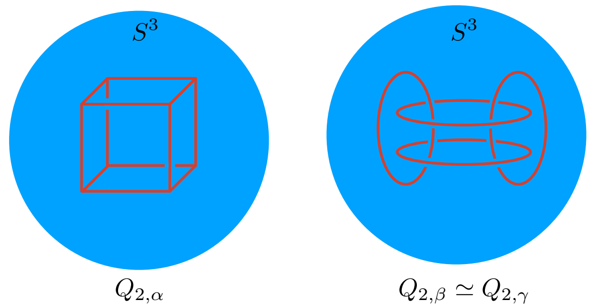

Figure 2.2: The base 3-orbifolds for the α, β, and γ fibrations of the G_2 orbifold X_2. In all cases, the base orbifold is of the form $T^3/\mathbb{Z}_2^2$, and is homeomorphic to a 3-sphere. There is a 1-dimensional locus of singularities that in the case of the α-fibration is the 1-skeleton of a cube, and in the β- and γ-fibrations is a doubled Hopf link. These orbifolds serve as the bases for the fibrations of X_3 as well, with $Q_{3,\alpha} \simeq Q_{3,\beta} \simeq Q_{2\alpha}$ and $Q_{3,\gamma} \simeq Q_{2,\beta} \simeq Q_{2,\gamma}$. The dual heterotic geometries are T^3 fibrations over the same bases.

Each of these fibrations will determine a dual heterotic model. In either case, we want to take the base orbifold to be large compared to both the fiber and the scale set by the gravitational coupling κ, meaning in particular that the S^1_7 factor is large. We are thus in the limit of a strongly-coupled IIA model on T^6_{123456}/Γ_1. By moving in the geometric moduli space to small S^1_7, and thus small IIA coupling, one may apply additional tools of IIA/het duality, but it is possible that the adiabatic assumption is violated in this limit. See Section 2.7 for more discussion of Type IIA duals.

Example 3.2: The Simplest Joyce Orbifold

Next, let us move on to examples that preserve $N = 1$ SUSY in 4D. First, we will consider the Joyce orbifold defined by the third generator

$$\gamma_2 : (x_1, ..., x_7) \mapsto (\tfrac{1}{2} - x_1, x_2, \tfrac{1}{2} - x_3, x_4, -x_5, x_6, -x_7) \ .$$

This orbifold was first considered in [21] and studied further in [49]. Let $\Gamma_2 \cong \langle \alpha, \beta, \gamma_2 \rangle$ and $X_2 = T^7/\Gamma_2$. In this case, the actions of α, β, and γ_2 are symmetric: γ_2 fixes 16 T^3's in T^7, just as α and β do, and it acts freely on the fixed loci of the other elements, as they do on the 16 T^3's fixed by γ_2. Altogether, we find 12 T^3 of A_1 singularities (4 from each of α, β, and γ_2). The orbifold Betti numbers in this case are $b^1_{\Gamma_2} = 0$, $b^2_{\Gamma_2} = 0$, and $b^3_{\Gamma_2} = 7$. Thus in the low energy theory we expect $\mathrm{SU}(2)^{12}$ gauge symmetry with 3 adjoint chirals for each SU(2) and 7 neutral chiral multiplets.

In addition to the two coassociative Kummer fibrations inherited from X_1, the orbifold X_2 has an additional fibration coming from the action of γ_2. These three fibrations are:

- The α-fibration $\pi_{567} : T^7/\Gamma_2 \to T^3_{567}/\langle \beta, \gamma_2 \rangle$ with generic fiber $T^4_{1234}/\langle \alpha \rangle$
- The β-fibration $\pi_{347} : T^7/\Gamma_2 \to T^3_{347}/\langle \alpha, \gamma_2 \rangle$ with generic fiber $T^4_{1256}/\langle \beta \rangle$
- The γ_2-fibration $\pi_{246} : T^7/\Gamma_2 \to T^3_{246}/\langle \alpha, \beta \rangle$ with generic fiber $T^4_{1357}/\langle \gamma_2 \rangle$

Given a choice of the F-fibration, where F is one of α, β, or γ_2, we let $H_{2,F} \cong \mathbb{Z}_2^2$ be the group generated by the two generators of Γ_2 other than F, and we let $Q_{2,F}$ be the 3-orbifold base of the fibration, which is topologically S^3 in all cases. In each case, $H_{2,F}$ will act trivially on one of the 7 coordinates - this is the coordinate that should be chosen as the throat direction in the half-G_2 limit.

Now, let us examine the α-fibration of X_2, following example 3.1 of [49]. We will discuss this first example of a $N = 1$ fibration in detail and be more brief in subsequent examples. The action of $H_{2,\alpha}$ on T^3_{567} has the fixed point loci

$$\begin{aligned}
\mathrm{Fix}(\pi_{567} \circ \beta) &= \left\{x_5 \in \left\{0, \tfrac{1}{2}\right\}, x_6 \in \left\{0, \tfrac{1}{2}\right\}\right\} \\
\mathrm{Fix}(\pi_{567} \circ \gamma_2) &= \left\{x_5 \in \left\{0, \tfrac{1}{2}\right\}, x_7 \in \left\{0, \tfrac{1}{2}\right\}\right\} \\
\mathrm{Fix}(\pi_{567} \circ \beta\gamma_2) &= \left\{x_6 \in \left\{0, \tfrac{1}{2}\right\}, x_7 \in \left\{0, \tfrac{1}{2}\right\}\right\} ,
\end{aligned}$$

which are each 4 disjoint circles. We have

$$\#\left[\mathrm{Fix}(\pi_{567} \circ \beta) \cap \mathrm{Fix}(\pi_{567} \circ \gamma_2) \cap \mathrm{Fix}(\pi_{567} \circ \beta\gamma_2)\right] = 8 \ ,$$

and these 8 points of intersection are the only elements in the intersection of any two of these loci. Because any intersection of the loci involves three circles, and these circles become line intervals $S^1/\mathbb{Z}_2$ under the $H_{2,\alpha}$ quotient, the elements in the intersection correspond to trivalent vertices in the graph of fixed points on the base; the graph is the 1-skeleton of a cube (see Figure 2.2). Denote the base orbifold $T_{567}/H_{2,\alpha}$ by $Q_{2,\alpha}$ and its orbifold locus by $\Sigma_{Q_{2,\alpha}}$.

Let us examine how the singular locus of X lies with respect to the α-fibration. The four components that come from fixed T^3 of α become 4 disjoint multi-sections of π_{567}, so that they provide the 16 A_1 singularities in each Kummer fiber. The remainder of the singular locus lies over $\Sigma_{Q_{2,\alpha}}$. The components coming from fixed T^3 of β project under π_{567} to the edges of $\Sigma_{Q_{2,\alpha}}$ parallel to the x_7 axis, while the components from γ_2 project onto edges parallel to the x_6 axis.

The singular fibers (by which we mean fibers that have multiple components in their resolution) of the α-fibration are those that lie above above $\Sigma_{Q_{2,\alpha}}$. The fibers that project to an edge of Σ_{2,Q_α} are acted upon by one element of $H_{2,\alpha}$ and have multiplicity 2. The fibers lying above a corner of $\Sigma_{Q_{2,\alpha}}$ are acted upon by all of $H_{2,\alpha}$ and have multiplicity 4. Note that $H_{2,\alpha}$ acts trivially on x_4, so that this should be our choice of K3 throat coordinate in this case.

If we consider instead the β-fibration, we find similar results but with a different base orbifold Q_β. In this case, the relevant fixed point loci are

$$\begin{aligned}
\mathrm{Fix}(\pi_{347} \circ \alpha) &= \left\{x_3 \in \left\{0, \tfrac{1}{2}\right\}, x_4 \in \left\{0, \tfrac{1}{2}\right\}\right\} \\
\mathrm{Fix}(\pi_{347} \circ \gamma) &= \left\{x_5 \in \left\{\tfrac{1}{4}, \tfrac{3}{4}\right\}, x_7 \in \left\{0, \tfrac{1}{2}\right\}\right\} \\
\mathrm{Fix}(\pi_{347} \circ \alpha\gamma) &= \varnothing \ .
\end{aligned}$$

This gives us the orbifold locus $\Sigma_{Q_{2,\beta}}$ that is four disjoint circles forming a doubled Hopf link. See example 3.2 of [49] for details. In contrast to the cube locus of the α-fibration, the locus Σ_β has no vertices, so that the singular fibers are of multiplicity 2 only. This makes the monodromy analysis somewhat simpler in the heterotic dual theory. Finally, the γ_2-fibration gives results identical to the β-fibration up to change of coordinates.

Example 3.3: Orbifold Singular Loci

Our second $N = 1$ background is similar to the previous example, except for a shift in the action of γ. This time we define the third group generator

$$\gamma_3 : (x_1, ..., x_7) \mapsto \left(\tfrac{1}{2} - x_1, x_2, -x_3, x_4, -x_5, x_6, -x_7\right) \ ,$$

which is identical to γ_2 except for the lack of shift on x_3. The orbifold defined by this choice of third generator was studied in [22] and used for M-theory compactification in [41]. The element γ_3 still fixes 16 T^3's in T^7, but now $\langle \alpha, \beta \rangle$ does not act freely on these 16 T^3's, and instead orbifolds them to 8 $T^3/\mathbb{Z}_2$'s. The action of $\alpha\beta$ kills two of the harmonic 1-forms on T^3, so that $b^1_{\langle\alpha\beta\rangle}\left(T^3_{246}/\left\langle \alpha\beta \right\rangle\right) = 1$. This modifies the spectrum of massless charged matter.

As before, define $\Gamma_3 = \langle \alpha, \beta, \gamma_3 \rangle$ and $X_3 = T^7/\Gamma_3$. The Betti numbers of X_3 are identical to those of X_2, since the shifts on the coordinates do not affect the harmonic forms. The singular loci of X_3 are 8 T^3 and 8 $T^3/\mathbb{Z}_2$ of A_1 singularities. Thus, we expect low-energy gauge symmetry $\mathrm{SU}(2)^{16}$, with 3 adjoint chiral multiplets each for 8 of these SU(2) factors and 1 adjoint chiral multiplet each for the remaining SU(2) factors. Additionally, there will be 7 neutral chiral multiplets, as in example 3.2.

The coassociative Kummer fibrations are defined in the same way for this example as for example 3.2. The difference is that the base of the β-fibration has changed. The singular loci $\Sigma_{Q_{3,\alpha}}$ and $\Sigma_{Q_{3,\beta}}$ are the 1-skeleton of a cube, as was $\Sigma_{Q_{2,\alpha}}$, while the singular locus $\Sigma_{Q_{3,\gamma_3}}$ is the doubled Hopf link, as was $\Sigma_{Q_{2,\gamma_2}}$.

2.4 The Dual Heterotic Geometry

Given a G_2 orbifold $X = T^7/\Gamma$ with a choice of K3 fibration, we want to identify the dual Calabi–Yau orbifold Y on which to compactify the heterotic string. To obtain Y, we replace the K3 fibers of X by dual T^3 fibers with metric determined by the K3 data. Because we want large heterotic volume, we work in the half-G_2 limit on the M-theory side, where the heterotic geometry is given by the generic fiber transverse to the throat direction. The complex structure on the heterotic orbifold may be determined by demanding that the orbifold group act holomorphically on T^6, and this gives a complex structure compatible with the SYZ condition, which requires that the T^3 fibers are special Lagrangian. Different choices of K3 fibration on the M-theory side give rise to different heterotic geometries, but they are biholomorphic; all of our $N = 1$ examples give orbifold limits of the Schoen manifold [49], similar to the results of [47] for twisted-connected sums. However, the T^3 fibrations of these biholomorphic spaces are inequivalent, and in particular have bases with topologically distinct singular loci, as we saw for the K3 fibrations of the G_2 orbifolds in Section 2.3.

As the heterotic geometry is a fiber of the G_2 orbifold, it intersects the singular loci of the ambient space. In particular, in our examples, each T^3 singular locus of the G_2 orbifold intersects the heterotic geometry either trivially or in two disconnected T^2. (A helpful lower-dimensional picture is to imagine T^2 as a S^1-fibration over an interval that is branched at the two endpoints.) Thus when we have only T^3 singular loci in the G_2 orbifold, the number of components of the heterotic singular locus is twice the number of components of the M-theory singular locus that lie parallel to the throat coordinate. The $T^3/\mathbb{Z}_2$ loci, on the other hand, intersect the heterotic geometry either trivially or in only one T^2, so there is no doubling of loci. The singular loci in the heterotic geometry are expected to give rise to non-perturbative gauge symmetry when they carry point-like instantons, as we will discuss in detail in the following sections.

In the remainder of this section, we will describe the heterotic geometries dual to the examples 3.1, 3.2, and 3.3 that we introduced in the previous section.

Example 4.1: $N = 2$ **SUSY**

In the $N = 2$ case of example 3.1, the α- and β-fibrations are equivalent up to a change of coordinates, so we may study the dual geometry from either perspective. For definiteness, we will choose the α-fibration. Both x_3 and x_4 fit our criteria for the throat coordinate and give biholomorphic results, so we choose x_4 as the throat coordinate, as this is the option that will survive the further γ-action of the $N = 1$ examples. This means that we stretch the x_4 direction and look at our G_2 space as a fibration $\pi_4 : X_1 \to S^1/\langle\alpha\rangle$ over the resulting long interval $S^1/\langle\alpha\rangle \cong \left[0, \frac{1}{2}\right]$. The fiber above a point away from the ends of the interval is our dual geometry $Y_{1,\alpha} = T^6_{123567}/\langle\beta\rangle$. (Note that the action of α only descends to the fibers at $x_4 = 0, \frac{1}{2}$. Away from these points, it serves only to switch the 6-orbifold fiber with an identical “far away” fiber.)

The space $Y_{1,\alpha}$ is constructed as a fibration $\pi_{567} : Y_{1,\alpha} \to Q_{1,\alpha}$ over the same base 3-orbifold $Q_{1,\alpha}$ as on the M-theory side, but with the generic Kummer fiber $T^4_{1234}/\langle\alpha\rangle$

replaced by a flat 3-torus T^3_{123} and with holonomies around the singular fibers determined by those on the M-theory side. The Betti numbers of our space are found to be

$$b^1_{\langle\beta\rangle}(Y_{1,\alpha}) = 2, \quad b^2_{\langle\beta\rangle}(Y_{1,\alpha}) = 7, \quad b^3_{\langle\beta\rangle}(Y_{1,\alpha}) = 12 \ .$$

The complex structure of $Y_{1,\alpha}$ is constrained by the SYZ condition and the holomorphy of the action of β, but, unlike in the $N = 1$ cases below, this information is not enough to fully determine the complex structure—there is an S^2 of complex structures compatible with these conditions.

For another perspective on this space, we may rewrite it as $T^6_{123567}/\langle\beta\rangle \cong (T^4_{1256}/\langle\beta\rangle)\times T^2_{37}$, so we have a trivial fibration of Kummer orbifolds $T^4/\mathbb{Z}_2$ over T^2. From this perspective, we see that the space has 16 T^2's of A_1 singularities, corresponding to $T^2 \sqcup T^2$ cross sections of the 8 T^3 singular loci of the M-theory geometry that come from β. When projected to the base, the singular T^2's project to the singular S^1's of $Q_{1,\alpha}$ in groups of four.

Example 4.2: Duals to Fibrations of X_2

Next, we will examine the dual geometries to fibrations of X_2, studied in example 3.2 above. We will begin with the α-fibration, which is similar to our previous example, but with an additional $\mathbb{Z}_2$ action by γ_2 (see Figure 2.3). Because γ_2 acts nontrivially on x_3, the only coordinate of T^7 that can act as the throat coordinate of the half-G_2 limit is x_4, so the relevant fibration for this limit is $\pi_4 : X_2 \to S^1_4/\langle\alpha\rangle$, where S^1_4 is taken to be large. The fiber above a point away from the ends of the interval is our dual geometry $Y_{2,\alpha} = T^6_{123567}/H_{2,\alpha}$, where, as in example 3.2, $H_{2,\alpha} = \langle\beta, \gamma_2\rangle$. The T^3 fibration dual to the α-fibration of X is $\pi_{567} : Y_{2,\alpha} \to Q_{2,\alpha}$, with generic fiber T^3_{123}. Then π_{567} is an SYZ

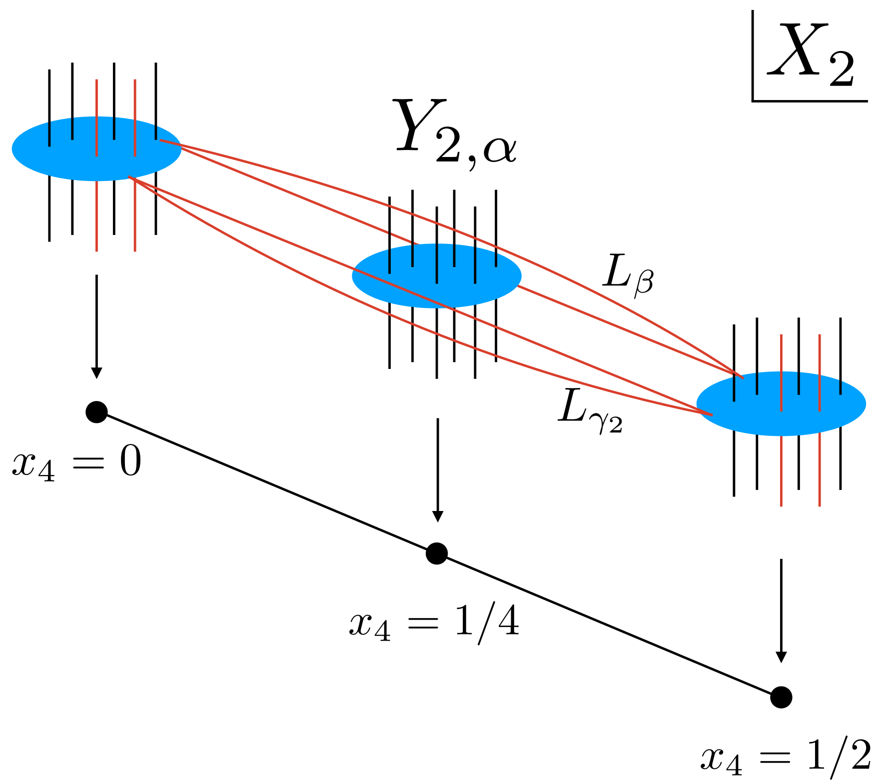

Figure 2.3: A schematic view of the half-G_2 limit of the G_2 orbifold X_2 from example 3.2 with the α-fibration. We have stretched X_2 along the direction of x_4, the throat coordinate. The heterotic dual geometry $Y_{2,\alpha}$ is the inverse image $\pi_4^{-1}\left(\frac{1}{4}\right)$, and is shown with its SYZ fibration of T^3 fibers (black lines) over the 3-orbifold base $Q_{2,\alpha}$ (blue disk). Some of the black lines are singular fibers that do not create singularities in the total space; the singularities in the total space are displayed by red lines. The α-fixed loci (vertical red lines) are confined to the ends of the x_4 interval, while the β-fixed loci L_β and γ_2-fixed loci L_{γ_2} stretch across the interval. These T^3 loci that stretch across the interval intersect $Y_{2,\alpha}$ in a 2-component locus $T^2 \sqcup T^2$. The monodromy action of α on the singular T^2 of $Y_{2,\alpha}$ fixed by β is to travel around a loop in x_4 that begins at $x_4 = \frac{1}{4}$, passes through $x_4 = 0$ or $x_4 = \frac{1}{2}$, and returns to $x_4 = \frac{1}{4}$ along the other leg of L_β, so that the singular T^2's are swapped in pairs.

fibration of the Borcea–Voisin Calabi–Yau orbifold $Y_{2,\alpha}$.

The Betti numbers of this example are

$$b^1_{H_{2,\alpha}}(Y_{2,\alpha}) = 0,\ b^2_{H_{2,\alpha}}(Y_{2,\alpha}) = 3,\ b^3_{H_{2,\alpha}}(Y_{2,\alpha}) = 8\ ,$$

and these will be the same for our remaining $N = 1$ heterotic geometries, which are all homeomorphic.

To see that Y_α is a Borcea–Voisin orbifold, we note that β acts nontrivially only on the 1256 coordinates, and $T^4_{1256}/\langle\beta\rangle$ is a Kummer surface. Furthermore, γ_2 acts as (-1) on the holomorphic 2-form $dz_1 \wedge dz_2$ of the Kummer surface, and if we shift the coordinate on the remaining torus T_{37} to be $w_3 = z_3 - \frac{i}{4}$, then γ_2 acts as $w_3 \mapsto -w_3$, as required.

Because we want an SYZ fibration by the T^3_{123} fibers, the complex structure must pair fiber and base coordinates. Additionally, we demand that $H_{2,\alpha}$ acts holomorphically, and this leaves a unique choice of complex structure:

$$\begin{aligned} z_1 &= ix_1 + x_5 \\ z_2 &= ix_2 + x_6 \\ z_3 &= ix_3 + x_7 \end{aligned}$$

so that our projection map $\pi_{567} : T^6_{123567} \to T^3_{567}$ is

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \mathrm{Re} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

and our group $H_{2,\alpha}$ acts as

$$\begin{aligned}
\beta : (z_1, z_2, z_3) &\mapsto \left(-z_1, \frac{i}{2} - z_2, z_3\right) \\
\gamma : (z_1, z_2, z_3) &\mapsto \left(\frac{i}{2} - z_1, z_2, \frac{i}{2} - z_3\right) \\
\beta\gamma : (z_1, z_2, z_3) &\mapsto \left(z_1 - \frac{i}{2}, \frac{i}{2} - z_2, \frac{i}{2} - z_3\right) .
\end{aligned}$$

Furthermore, if we restrict α to the heterotic geometry, we find the involution

$$\alpha \left|_{Y_{2,\alpha}} : (z_1, z_2, z_3) \mapsto (\overline{z}_1, \overline{z}_2, \overline{z}_3) \right. ,$$

so in the 7D space, α acts as a complex conjugation map between $Y_{2,\alpha}$ and a distant fiber.

The singularities in our threefold are the fixed point loci

$$\begin{aligned}
\text{Fix}(\beta) &= \left\{x_1 \in \left\{0, \tfrac{1}{2}\right\}, x_2 \in \left\{\tfrac{1}{4}, \tfrac{3}{4}\right\}, x_5 \in \left\{0, \tfrac{1}{2}\right\}, x_6 \in \left\{0, \tfrac{1}{2}\right\}\right\} \\
\text{Fix}(\gamma_2) &= \left\{x_1 \in \left\{\tfrac{1}{4}, \tfrac{3}{4}\right\}, x_3 \in \left\{\tfrac{1}{4}, \tfrac{3}{4}\right\}, x_5 \in \left\{0, \tfrac{1}{2}\right\}, x_7 \in \left\{0, \tfrac{1}{2}\right\}\right\} \\
\text{Fix}(\beta\gamma_2) &= \varnothing .
\end{aligned}$$

The first two loci are each 16 disjoint complex curves with $\text{Fix}(\beta) \cap \text{Fix}(\gamma_2) = \varnothing$. The action of β on $\text{Fix}(\gamma_2)$ identifies the curves in pairs, as does the action of γ_2 on $\text{Fix}(\beta)$, so we will have 16 curves of A_1 singularities in $Y_{2,\alpha}$.

Different choices of K3 fibration on the M-theory side give rise to heterotic orbifolds that are biholomorphic, but may have different metrics (determined by the radii of the covering T^6) and different SYZ fibrations. To illustrate this, we will look at the heterotic geometry dual to the β-fibration of X_2. The throat coordinate must now be chosen as x_6, because this is the coordinate that is inverted by β while being fixed by $H_{2,\beta} = \langle\alpha, \gamma\rangle$.

Thus we take S^1_6 to be large and the heterotic geometry $Y_{2,\beta}$ will be realized as the generic fiber of $\pi_6 : X_2 \to S^1_6/\langle\beta\rangle$. This space is again an SYZ fibration with generic fiber T^3, but this time over the base $Q_{2,\beta}$, which we saw in example 3.2 is inequivalent to $Q_{2,\alpha}$, since the singular locus of the former is a doubled Hopf link, while the singular locus of the latter is the 1-skeleton of a cube. Despite the change in base, the total space $Y_{2,\beta} = T^6_{123457}/H_{2,\beta}$ with the complex structure determined by SYZ and $H_{2,\beta}$ is biholomorphic to $Y_{2,\alpha}$. Additionally, the heterotic geometry $Y_{2,\gamma_2} = T^6_{123456}/H_{2,\gamma_2}$ that results from the choice of the γ_2-fibration is biholomorphic to the first two examples and has an SYZ fibration equivalent to that of $Y_{2,\beta}$.

Thus, the choice of fibration of X_2 only affects the metric on the dual heterotic geometry. Because our M/heterotic duality requires a particular geometric limit where the throat direction is stretched and the base of the SYZ fibration is much larger than its fibers, a change in K3 fibration on the M-theory side requires a change of metric on the heterotic side to ensure the correct cycles are large or small. In our torus orbifold cases, this only requires a rescaling of the radii of the covering torus. We will see in the next example that the choice of fibration has other important effects on the heterotic gauge bundle.

Example 4.3: Dual Geometries for Orbifold Singular Loci

Finally, let us look at heterotic dual geometries for X_3, which has singular loci homeomorphic to the nonsingular orbifold $T^3/\mathbb{Z}_2$. Despite this change, we find that the heterotic geometry is again biholomorphic to the one found in example 4.3 for all choices of fibrations.

We begin with the α-fibration, which is similar to the α-fibration of example 4.3 except for the configuration of the singular loci. Our geometry in this case is $Y_{3,\alpha} = T^6_{123567}/H_{3,\alpha}$, where $H_{3,\alpha} = \langle \beta, \gamma_3 \rangle$. The fixed loci of T^6 in this case are

$$\begin{aligned}
\mathrm{Fix}(\beta) &= \left\{x_1 \in \left\{0, \tfrac{1}{2}\right\}, x_2 \in \left\{\tfrac{1}{4}, \tfrac{3}{4}\right\}, x_5 \in \left\{0, \tfrac{1}{2}\right\}, x_6 \in \left\{0, \tfrac{1}{2}\right\}\right\} \\
\mathrm{Fix}(\gamma_3) &= \left\{x_1 \in \left\{\tfrac{1}{4}, \tfrac{3}{4}\right\}, x_3 \in \left\{0, \tfrac{1}{2}\right\}, x_5 \in \left\{0, \tfrac{1}{2}\right\}, x_7 \in \left\{0, \tfrac{1}{2}\right\}\right\} \\
\mathrm{Fix}(\beta\gamma_3) &= \varnothing \ ,
\end{aligned}$$

where the only change relative to the previous example is the x_3 coordinate of the γ_3-loci. As before, each of β and γ_3 acts on the fixed loci of the other to reduce the number of components by a factor of 2. Thus, we again find a Calabi–Yau orbifold of the form $T^6/\mathbb{Z}_2^2$ with 16 A_1 singularities. The 8 T^2 in the γ_3-fixed loci of $Y_{3,\alpha}$ are T^2 cross-sections of the $T^3/\mathbb{Z}_2$ loci in the ambient G_2 orbifold. Note that the $\mathbb{Z}_2$ action does not descend to the T^2 in $Y_{3,\alpha}$ because it is accomplished by the element $\alpha\beta \in \Gamma_3$, which inverts the x_4 coordinate and thus exchanges $Y_{3,\alpha}$ with a different fiber of the half-G_2 limit.

The β-fibration gives identical results to the α-fibration (unlike in example 4.2), and the γ_3-fibration gives identical results to that of the γ_2-fibration of example 4.3. Thus, all of our $N = 1$ fibration examples have biholomorphic heterotic geometries. This is not surprising in light of the results of [47], where it was found that all smooth TCS G_2 backgrounds have heterotic duals based on the same Schoen Calabi–Yau. The complexity of heterotic compactifications come from the choices of gauge bundles, and indeed we will see in Section 2.6 that the heterotic duals of the α- and γ-fibrations of example 3.3 have different instanton configurations.

2.5 The Heterotic Gauge Bundle

Now we move on to the more subtle part of the heterotic background: the gauge bundle[9]. The information necessary to construct this bundle is contained in the data of the M-theory metric, C-field background, and 7D gauge field background. Given a K3 fibration of a G_2 manifold, we may apply 7D M/heterotic duality to each fiber to find the restriction of the heterotic gauge bundle to each dual T^3 fiber.

Ideally, the restriction of the bundle to each T^3 fiber, along with the monodromies around the singular fibers, would allow us to reconstruct the gauge bundle over the entire Calabi–Yau space. In the case of an elliptic fibration of a Calabi–Yau manifold, the work of [53] allows one to do exactly that. However, their methods rely on the fact that the elliptic curve is a complex manifold, so their results are not so easily generalized to T^3 fibers. As described in Section 2.4, part of the data required for the gauge bundle reconstruction of [53] is a choice of line bundle over a spectral cover which corresponds in F-theory to an instanton bundle on the background D7-branes. The analogous data in an M-theory compactification would seem to be a background instanton configuration for the gauge theories living on the singular loci, but such backgrounds have not been thoroughly studied.

Reconstructing the bundle in general cases may be possible with better understanding of the special Lagrangian structure of the fibers within the Calabi–Yau, but we do not yet have the tools to work with this data. For now, we will study the gauge bundle from the perspective of the point-like instantons required to cancel anomalies. These instantons give rise to non-perturbative gauge symmetry and matter, and we may attempt to match their spectra with the M-theory side. Insight into instanton behavior may also be found from dual Type I models, where D5-branes play the role of the dual object [62, 71, 72].

[9]Because we are working with orbifolds, we are really constructing gauge *sheaves* or *orbibundles*, but we will continue to informally use the word "bundle" for these objects.

There are at least three levels of checks one may perform to give evidence for a conjectured dual pair:

1. The most coarse check is to ensure that the two sides give the same effective 4D gauge symmetry. In the case of point-like instantons, we may refine this criterion by splitting the gauge symmetry into a perturbative and non-perturbative part from the heterotic perspective, and checking that each part of the gauge symmetry matches with what is given on the M-theory side.

2. Next, one can check that the massless charged matter agrees on the two sides. For point-like instantons on orbifold singularities, the massless spectrum is well-understood only in simple examples.

3. A third level to check is that the low energy effective action agrees on the two sides of the duality. Unfortunately, the action associated to excitations about point-like instantons on orbifold singularities has not been investigated, so there are not currently quantitative checks to be made. However, one can reason qualitatively about the action by considering which modes should be massive or massless at specific points in moduli space.

In this work, we will focus primarily on the coarsest check: the gauge symmetry of the low-energy effective theory. We will start by describing the split between heterotic perturbative and non-perturbative spectra and reviewing some results about spectra of point-like instantons on orbifold singularities.

2.5.1 Perturbative vs. Non-Perturbative Spectra

Although we work in the weak heterotic string coupling limit $\lambda \to 0$ where possible, anomaly cancellation guarantees that near the singular loci of our heterotic geometry, the background will exhibit phenomena that are non-perturbative in the string coupling, such as point-like instantons. Thus the massless spectrum from the heterotic string is best understood as a sum of a perturbative part (the spectrum seen by a 2D CFT description) and a non-perturbative part, which cannot be seen from the CFT perspective. This approach was refined in heterotic orbifold compactifications in [71], where it was argued that because the string worldsheet perspective cannot describe the non-perturbative part of the massless spectrum, the perturbative spectrum is no longer constrained by modular invariance. Instead, the requirement is that the combined perturbative and non-perturbative spectra have no anomalies in the low-energy effective theory.

Relevant examples of perturbative spectra may be constructed from non-singular instantons on orbifold loci. A basic configuration is the SU(2)-instanton on $\mathbb{R}^4/\mathbb{Z}_2$ described in [62], which is obtained as a $\mathbb{Z}_2$-quotient of the standard SU(2)-instanton configuration with $c_2 = 1$ centered at the origin of $\mathbb{R}^4$. If we write $\mathrm{SO}(4) = (\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R)/\mathbb{Z}_2$ and embed the gauge group SU(2) as either $\mathrm{SU}(2)_L$ or $\mathrm{SU}(2)_R$, the resulting SO(4)-connection has a monodromy M on the lens space $S^3/\mathbb{Z}_2$ at infinity given by $M = -I_4$, where I_4 is the rank-4 identity matrix. Denote this connection on $\mathbb{R}^4/\mathbb{Z}_2$ by $\mathcal{A}_0$. We will use this type of instanton in Section 6 to build non-singular bundle configurations on our heterotic orbifolds that reproduce the perturbative spectra seen in our dual M-theory models. When these instantons shrink to zero size, they produce additional effects, as we will discuss in the next subsection. Similar non-singular instantons may be built by starting with calorons, instantons on $\mathbb{R}^3 \times S^1$ periodic up to a gauge transformation [60, 61]. These configurations are made of constituent BPS monopoles and are naturally centered at pairs of points, making them more relevant to the examples at hand.

For M/heterotic duality in 7 non-compact dimensions, the entire spectrum is visible perturbatively in the half-K3 limit, since the moduli space of M-theory on K3 coincides with that of the perturbative heterotic string on T^3. When this duality is fibered over a

3D base, we expect the singular fibers to introduce phenomena that are non-perturbative from the heterotic side. We can identify the effects that come from singular fibers by the same geometric criterion that is used in heterotic/F-theory duality [54]: the gauge symmetry and matter that come from components of the singular locus that meet the generic K3 fiber transversely should be visible perturbatively on the heterotic side, while that coming from components that project to nonzero codimension on the base should come from mechanisms that are invisible to perturbation theory[10]. An alternative characterization used in IIA/heterotic duality is that degenerate K3 fibers on the IIA side that require multiple components in their resolution correspond to non-perturbative effects on the heterotic side [73].

The perturbative dictionary tells us that the data for an E_8 bundle on T^3 is stored in the choice of a half-K3 surface whose boundary is the given T^3. This is analogous to Looijenga's theorem that the data for an E_8 bundle on an elliptic curve is contained in an embedding of the curve into a $k = 8$ del Pezzo surface [74, 75]. Meanwhile, the non-perturbative part of the gauge symmetry will come from point-like instantons sitting on orbifold singularities. Singular gauge bundles coming from point-like instantons on orbifold singularities are not fully understood or classified, but we will review some of what is known.

2.5.2 Point-Like Instantons on Orbifold Singularities

In our flat orbifold examples, the inclusion of point-like instantons is required by the heterotic anomaly cancellation condition:

$$dH = \alpha' \left(\mathrm{tr} F \wedge F - \mathrm{tr} R \wedge R\right) \ ,$$

[10]Note that this rule applies only to matter from singular loci that are codimension-four in the total space, as in our examples. Codimension-seven loci, for instance, give perturbative matter while projecting to nonzero codimension on the base

which for $dH = 0$ forces a gauge bundle for which the second Chern character (i.e. the Poincare dual of the homology class of the instanton distribution) agrees with that of the tangent sheaf of the orbifold (at least in a formal sense). In other words, we are forced to place instantons along the orbifold loci. In the dimensions transverse to the loci, these look like point-like instantons. The right-hand side of the anomaly cancellation condition may be modified non-perturbatively by the presence of background NS5-branes. We work in a limit where any wrapped NS5-branes are represented by point-like instantons [76], so that both perturbative and non-perturbative contributions are contained in the $\mathrm{tr}F \wedge F$ term.

This type of configuration is further motivated by the supersymmetry conditions: because we are working in the half-K3 limit, α' corrections are suppressed, and the supersymmetry condition requires that we have a Hermitian-Yang-Mills connection on our bundle. This condition, in combination with anomaly cancellation, requires the connection to be flat away from the singular loci, while on these loci it has instanton number matching the background metric. To see this, we write the anomaly cancellation condition as $\mathrm{tr}F \wedge F = 0$ and the SUSY D-term equation as $\star F = -\omega \wedge F$, where ω is the Kahler form. Wedging F with both sides and then taking a trace gives us

$$\mathrm{tr}\,(F \wedge \star F) = -\omega \wedge \mathrm{tr}\,(F \wedge F) = 0\ .$$

The left hand side is the norm-squared of the gauge field strength, so it vanishes away from orbifold loci. Together, these conditions tell us that we must place point-like instantons on our orbifold loci, and that there is no freedom to vary the connection away from these loci other than choosing holonomies. It is possible that the gauge fields could have nontrivial profiles along the singular loci, but because we chose a trivial background configuration for the 7D gauge fields on the M-theory side, we expect the profiles to be trivial on the heterotic side as well.

In our $N = 1$ examples, we have additional constraints on the gauge bundles that arise from the properties of the massless spectrum calculated from M-theory:

1. There is no abelian gauge symmetry in the 4D effective theory, meaning no tensor multiplets in a local 6D description near a singular locus.

2. All charged matter in 4D is in the adjoint representation. Because point-like instantons typically come with fundamental multiplets, this suggests that there may be Higgsing of the non-perturbative spectrum.

With these points in mind, we can look at the effects of point-like instantons on the massless spectrum. A point-like instanton comes with extra massless particles that are non-perturbative in the string coupling. There are several ways to understand this phenomenon: one can think of it as a stringy "smoothing" of an apparent geometric singularity via extra massless particles, or as the massless sector of the worldvolume theory of a wrapped NS5-brane or a wrapped M5-brane in a dual theory, or as a theory of tensionless strings. Point-like instantons behave differently in the $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic theories. Because our primary duality gives an $E_8 \times E_8$ model, one may expect that only $E_8 \times E_8$ point-like instantons are relevant. However, the instantons in our backgrounds behave like T-dual $\mathrm{Spin}(32)/\mathbb{Z}_2$ instantons, similar to cases examined in [62, 71].

First let us briefly review what happens when you shrink E_8 point-like instantons to zero size on a smooth 6D geometry [77, 78]. Because this case isn't directly relevant to us, we will just summarize the spectrum: on a smooth point, an E_8 point-like instanton gives rise to an extra massless tensor and no extra gauge symmetry. From the point of view of heterotic-M theory, with M-theory compactified on $Y \times S^1/\mathbb{Z}_2$, where Y is a Calabi–Yau threefold, a point-like instanton may be thought of as an M5-brane wrapped on Y that moves from the interior of the interval to the boundary [79]. In this picture, the VEV of the scalar in the tensor multiplet controls the position of the M5-brane along the interval.

Note that in this case and in the later cases, the extra massless particles can be blocked by the presence of a nontrivial B-field holonomy on the orbifold point [55]. Indeed, to fully specify a heterotic dual, we must choose a background of B-field holonomies on the 2-cycles of our space. The holonomies on the T^3 fibers are determined by the shape of the K3 fibers of the G_2 orbifold, as shown in [80] by matching moduli. There can be no holonomies on the base, as it is homeomorphic to S^3, but there may be B-field holonomies with one leg along a fiber and one leg along the base. This case includes the singular loci as well as any extra 2-cycles of the space.

In our examples, the point-like instantons reside on orbifold points of the geometry. Because this is a worse bundle singularity than the point-like instantons on a smooth point, extra nonperturbative multiplets can arise [29, 81, 55, 82]. For point-like instantons on an orbifold point, the holonomy of the gauge bundle may be nontrivial, since the lens space surrounding the orbifold point has nontrivial fundamental group. The case with trivial holonomy was investigated in [29]. In [81], simple cases of nontrivial holonomy were worked out. It was established in [55] that an $E_8 \times E_8$ point-like instanton with nontrivial holonomy on an orbifold point does not give rise to a tensor multiplet, but retains its non-perturbative gauge symmetry and charged matter. This can be understood from the heterotic-M theory perspective, where a wrapped M5-brane cannot move from the orbifold point into the bulk because it must preserve its holonomy. Thus a point-like instanton with nontrivial holonomy may be thought of as a frozen singularity in the bundle. In some cases, this may be interpreted in terms of fractional M5-branes [83].

In the cases considered in this chapter, the orbifold singularities of the heterotic geometry look locally like an A_1 singularity $\mathbb{C}^2/\mathbb{Z}_2$, so we will review options for fractional $E_8 \times E_8$ instantons on such a space, following section 4.3 of [5]. The only nontrivial option for the holonomy is $\mathbb{Z}_2$, and there are two ways that this may be embedded in E_8, up to conjugacy:

1. It may be embedded so as to have centralizer $(E_7 \times \mathrm{SU}(2))/\mathbb{Z}_2$. This gives instanton number $c_2 = 1/2$ and no tensor multiplet nor gauge symmetry.

2. It may be embedded so as to have centralizer $\mathrm{Spin}(16)/\mathbb{Z}_2$. This gives $c_2 = 1$ and a non-perturbative SU(2), but no tensor.

We may combine these types of instantons to get new examples. For instance, we may place both a trivial holonomy instanton and the $c_2 = 1/2$ instanton on an A_1 singularity to get an instanton with $c_2 = 3/2$ that gives no tensor multiplet, but a non-perturbative SU(2) so that the gauge symmetry in the visible sector becomes $E_7 \times \mathrm{SU}(2)$. This is the situation that corresponds to the tangent sheaf of $\mathbb{C}^2/\mathbb{Z}_2$.

What kinds of instantons are allowed when there are multiple singularities? The case of the tangent sheaf of $T^4/\mathbb{Z}_2$, which has 16 A_1 singularities, is discussed in [55, 84] and has the behavior of 16 independent instantons, each with $c_2 = 3/2$. The behavior of the heterotic backgrounds in our examples suggests that there exist also configurations where the instantons residing on different loci are not independent. In other words, we seem to have instantons that are only semi-localized, so that they spread their instanton number evenly over two loci. In the case of an instanton semi-localized on an $A_1 \oplus A_1$ singularity, the resulting non-perturbative gauge symmetry is only SU(2). The gauge fields localized on the two singularities must take values in the diagonal $\mathfrak{su}(2)$ subalgebra of the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ that would arise from separate instantons on the two loci. A compactification on $T^4/\mathbb{Z}_2$ with 8 such semi-localized instantons suggests that each one has instanton number $c_2 = 3$, the sum of the instanton numbers for each locus. One candidate for these instantons is the singular limit of a $\mathbb{Z}_2$-quotient of an SU(2) caloron.

While our main duality relates M-theory to the $E_8 \times E_8$ heterotic string, we will also be interested in an alternate duality to the Spin(32)/$\mathbb{Z}_2$ string. This dual model involves point-like instantons as well, so we will review some properties of this case. The Spin(32)/$\mathbb{Z}_2$ point-like instantons behave oppositely to the $E_8 \times E_8$ ones with respect to their spectrum: they produce non-perturbative vector multiplets when placed on a smooth point, and augment these with tensor multiplets when placed on orbifold singularities [29, 85]. There are multiple types of Spin(32)/$\mathbb{Z}_2$ instantons, but we are interested in particular in those that live on $\mathbb{Z}_2$ orbifold singularities and participate in the duality with Type I on $T^4/\mathbb{Z}_2$ [27, 26, 62]. In the case that on the Type I side distributes one half-D5-brane at each fixed point, the heterotic background carries a combination of two point-like instantons at each fixed point. Each points has a "hidden" $c_2 = 1$ instanton with no low-energy gauge symmetry or tensor multiplets. On top of this background, there is a configuration of fractional D5-branes, which may also be interpreted as point-like instantons. When the D5-branes are distributed evenly across the fixed points, and in the absence of Wilson lines, the gauge group is SU(16) × U(1), where a rank 16 factor has been removed by a Green-Schwarz-type mechanism [62].

2.5.3 Point-Like Instanton Spectra

Ideally, we would be able to verify that the spectra of our heterotic backgrounds agree with those of their purported M-theory duals. This goal is hampered by the fact that calculating spectra of point-like instantons on orbifold singularities is challenging and still not fully understood in the literature. Existing results are generally based on 6D anomaly cancellation (e.g. [82, 71]) or F-theory duals (e.g. [55, 84]). A pattern seems to emerge that $E_8 \times E_8$ point-like instantons on orbifold singularities do not give rise to adjoint matter; their charged matter appears to be fundamental matter in all existing examples. This provides a challenge for matching such spectra to those of M-theory on our G_2

orbifolds, because the latter have only adjoint matter. The semi-localized instantons suggested in the previous section, perhaps combined with a Wilson line background, likely give rise to matter valued in the adjoint of the diagonal subgroup.

The spectrum of a heterotic orbifold with point-like instantons is not limited to the non-perturbative spectrum of the instanton, but also comprises a perturbative spectrum, split as usual into untwisted and twisted sectors. A recipe for calculating the perturbative spectrum is given in [71], where it is shown that an additional energy term must be included in the left-moving twisted sector mass formula to account for the magnetic flux of the instantons sitting at the fixed point, thought of as wrapped M5-branes. In this chapter we are interested in the non-perturbative gauge sector, so we leave an investigation of the perturbative spectrum using this recipe for future work.

One particularly relevant example appears in section 5 of [71], where anomaly cancellation in an $E_8 \times E_8$ background on $T^4/\mathbb{Z}_3$ is achieved by adding a non-perturbative $\mathrm{SU}(2)^9$ factor to the gauge group along with charged hypermultiplets. This is interpreted as a spectrum arising from frozen fivebranes in the T-dual $\mathrm{Spin}(32)/\mathbb{Z}_2$ theory. We will argue for a similar interpretation of our non-perturbative gauge symmetry in section 7.

2.6 Example Dual Pairs

Equipped with preliminary analysis of the heterotic geometry and gauge bundle, we now explore aspects of our candidate dual pairs. Because we are primarily interested in the non-perturbative aspects of the half-G_2 limit, we will give only a brief description of the perturbative part of the analysis, but we include a construction method for non-singular instantons that replicate the perturbative spectra. We will begin with a description of the 7D duality shared by all three examples, and then discuss the details of each example individually.

Example Number	Fibration	Perturbative Gauge Symmetry	Non-Perturbative Gauge Symmetry
6.1	α, β	$\mathrm{SU}(2)^8 \times \mathrm{U}(1)^4$	$\mathrm{SU}(2)^8$
6.2	α, β, γ	$\mathrm{SU}(2)^4$	$\mathrm{SU}(2)^8$
6.3	α, β	$\mathrm{SU}(2)^4$	$\mathrm{SU}(2)^{12}$
6.3	γ	$\mathrm{SU}(2)^8$	$\mathrm{SU}(2)^8$

Table 2.2: Summary of gauge symmetry in heterotic duals

In all of our examples of M-theory on K3 fibrations, the generic fibers are at the same $\mathbb{Z}_2$ orbifold point in K3 moduli space, so they share the same effective 7D theory. In this case, the heterotic dual background is a flat T^3 with three Wilson lines that branch $E_8 \times E_8$ to $\mathrm{SU}(2)^{16}$ [86, 87]. The only non-gravitational supermultiplet in 7D is the vector multiplet, so there is no charged matter from a 7D perspective. When further compactified on T^3 to 4D, this perturbative spectrum becomes $\mathrm{SU}(2)^{16}$ gauge symmetry with 3 adjoint chiral supermultiplets for each SU(2) (which is just the 4D $N = 4$ vector multiplet in 4D $N = 1$ language). Additionally, there are abelian factors in the gauge group as well as neutral chiral multiplets, but we will ignore these parts of the spectrum, as they are not our primary interest. In the following examples, we will use this 4D perturbative spectrum as a starting point and add in the additional orbifold actions as well as non-perturbative effects.

2.6.1 $N = 2$ Example

First, we will discuss the heterotic dual of the M-theory background of example 3.1, which has a trivial action of γ. There are 16 disjoint T^3's of A_1 singularities in the G_2 orbifold X_1, with 8 coming from α and 8 from β. We saw that there are two choices of coassociative Kummer fibration in this example, but they give equivalent heterotic dual geometries. In either case, half of the singular loci of X_1 have a transverse intersection with the generic fiber, meaning that we expect $\mathrm{SU}(2)^8$ perturbative gauge symmetry and

$\mathrm{SU}(2)^8$ non-perturbative gauge symmetry on the heterotic side.

For definiteness, consider the α-fibration, where we view the M-theory geometry as a $T^4_{1234}/\langle\alpha\rangle$-fibration over $T^3_{567}/\langle\beta\rangle$. In example 4.1, we saw that the dual geometry in this case is a T^3_{123}-fibration over the same base. We may write our heterotic geometry as the trivial Kummer fibration $Y_1 = T^4_{1256}/\langle\beta\rangle \times T^2_{37}$. This space has 16 disjoint T^2's of A_1 singularities, all from β. Note that the SYZ T^3 fibers are not fully contained within the K3 fibers, so that the perturbative Wilson lines along the T^3 fibers prevent the heterotic gauge bundle from factorizing into a K3 component and a T^2 component, which complicates potential applications of IIA/heterotic duality.

From a perturbative orbifold perspective, we have the Wilson lines described above on each T^3_{123} fiber, and we also must determine a $\mathbb{Z}_2$-action of β on the perturbative heterotic gauge bundle. We will assume that β acts by the outer automorphism that swaps the perturbative E_8 factors, as this is the gauge bundle action that corresponds to the geometric origin of the gauge symmetry on the M-theory side: in the G_2 orbifold, the action of β on the fixed loci of α is to swap them in pairs, reducing the resulting non-perturbative gauge symmetry from $\mathrm{SU}(2)^{16}$ to $\mathrm{SU}(2)^8$. This agrees with the choice of the action of β on the heterotic gauge bundle, which will break to the diagonal E_8, and branch this to $\mathrm{SU}(2)^8$ when combined with the Wilson lines. The adjoint chiral multiplets are identified in pairs as well, leaving us with 3 adjoint chirals for each SU(2).

The non-perturbative part of the non-abelian spectrum is the same as the perturbative part: an additional $\mathrm{SU}(2)^8$ with 3 adjoint chiral multiplets each. This part of the spectrum should come from point-like instantons on the β-loci, meaning that we should get $\mathrm{SU}(2)^8$ gauge symmetry from 16 T^2's of A_1 singularities. This appears to be a puzzle, because there is nothing to distinguish 8 of the loci as those that produce gauge symmetry, while the others do not. However, the loci are paired by the monodromy action of α within the ambient space. We illustrate this with an example (see Figure 2.4).

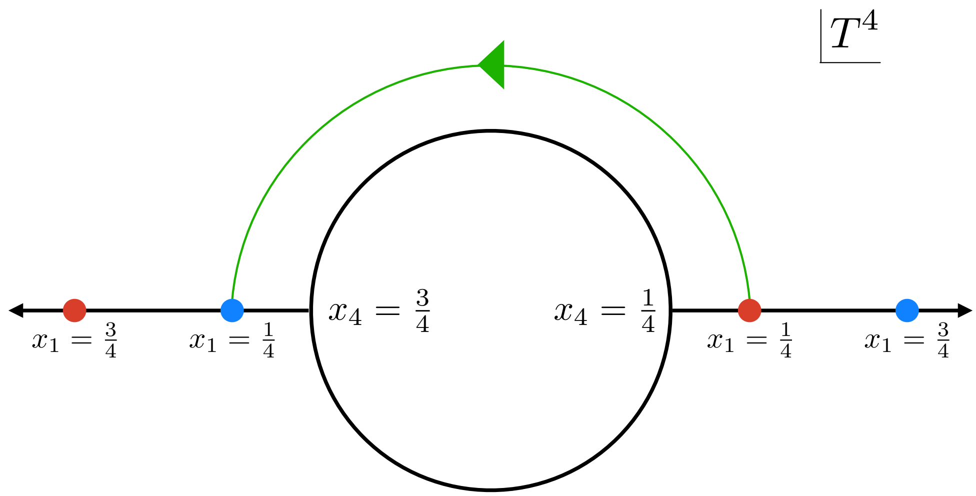

Figure 2.4: The action of α-monodromy on a T^2 singular locus in the $N = 2$ example. Pictured is the T^4 within the covering T^7 that is defined by $x_2 = x_5 = x_6 = 0$. The x_3- and x_7-dimensions are suppressed, so that each colored circle represents a T^2. The fibers $\pi_4^{-1}(1/4)$ and $\pi_4^{-1}(3/4)$ are pictured, represented by the x_1-direction only. The two T^2's represented by red circles are interchanged by the action of α, as are those represented by blue circles. By following the green contour from the $x_4 = 1/4$ fiber to the $x_4 = 3/4$ fiber and applying α, one ends up with a monodromy action by α on the singular loci of the $x_4 = 1/4$ fiber.

Within the heterotic geometry $Y_{1,\alpha} = \pi_4^{-1}\left(\frac{1}{4}\right)$, consider the singular T^2 that is the image of $\left(\frac{1}{4}, 0, x_3, \frac{1}{4}, 0, 0, x_7\right) \subset T^7$, where x_3 and x_7 are the T^2 coordinates. Suppose we translate along the throat direction x_4 to a different Calabi–Yau fiber located at $x_4 = \frac{3}{4}$. Because our T^7 is identified under the action of α, which inverts the first four coordinates, we have ended up back at $x_4 = \frac{1}{4}$, and thus back within $Y_{1,\alpha}$ at the point

$$\left(\frac{3}{4}, 0, -x_3, \frac{1}{4}, 0, 0, x_7\right) .$$

If we perform this translation for every (x_3, x_7), we obtain a monodromy action by α that exchanges these two singular T^2 within $Y_{1,\alpha}$. In general, this monodromy action pairs up the 16 singular T^2 of $Y_{1,\alpha}$. Our task is to reproduce the effect of this geometric action within the heterotic theory itself. The natural guess, given our constraints, is a semi-localized instanton that is evenly distributed over the two T^2, as described in section 5.2. This instanton ought to give rise to an SU(2) gauge symmetry with three adjoint chiral multiplets (or, in $N = 2$ language, an SU(2) gauge symmetry with one adjoint hypermultiplet). Thus we conjecture that the heterotic dual gauge bundle is comprised of 8 instantons of this type distributed across pairs of the singular T^2 loci. This semi-localization may be understood from a T-dual perspective as coming from a winding shift, as we will discuss in the next section.

Although the instanton is distributed over a disconnected locus, the separation is small because of the geometric limits required for our duality with M-theory to be valid. The loci that are paired by the instantons are separated only within the T^3 fiber, which is assumed to be small compared to the base for our duality to hold, as described in section 2. In our example above, the two singular T^2 both lie over $(0, 0, x_7)$ in the base, and their separation in the x_1-direction is infinitesimal compared to the radius of x_7. On the other hand, the separation in the x_1-direction is very large compared to

$\sqrt{\alpha'}$, so the disconnectedness demonstrated by this instanton is small compared to the compactification volume, but large compared to the string scale. The $\mathrm{Spin}(32)/\mathbb{Z}_2$ T-dual model of this configuration is an asymmetric orbifold, as will be discussed below, and thus a (weakly) non-geometric compactification. This non-geometric aspect is not reflected in the geometry of the $E_8 \times E_8$ model, but it leaves a remnant in the gauge bundle.

We may construct candidate configurations that reproduce the perturbative spectrum by deforming away from the point-like instanton limit and building a smooth instanton configuration on the orbifold Y_1 using copies of the connection $\mathcal{A}_0$ described in Section 5.1. We may use the monodromy $M = -I_4$, where I_4 denotes the rank-4 identity matrix, to match the Wilson line monodromies dictated by the half-K3 limit. We will work with the $\mathrm{Spin}(32)/\mathbb{Z}_2$ string for convenience, but the procedure is similar for the $E_8 \times E_8$ string. Consider the triple of $\mathrm{Spin}(32)/\mathbb{Z}_2$-monodromies

$$
\begin{aligned}
W_1 &= (-I_4, -I_4, -I_4, -I_4, I_4, I_4, I_4, I_4) \\
W_2 &= (-I_4, I_4, -I_4, I_4, -I_4, I_4, -I_4, I_4) \\
W_3 &= (I_4, -I_4, I_4, -I_4, I_4, -I_4, I_4, -I_4) \ ,
\end{aligned}
$$

where the notation indicates a block-diagonal matrix in $\mathrm{Spin}(32)/\mathbb{Z}_2$. This triple breaks $\mathrm{Spin}(32)/\mathbb{Z}_2 \to \mathrm{SO}(4)^8$. (In the case of the $E_8 \times E_8$ string, we must instead replace W_1 by the Wilson line that breaks $E_8 \to \mathrm{SO}(16)$.) Let A_W be the flat connection on $(T^3_{123} \times T^3_{567}) \,/\, \langle\beta\rangle$ that has monodromy W_i along the x_i-direction for $i = 1, 2, 3$. We will embed the SO(4)-instanton $\mathcal{A}_0$ into $\mathrm{SO}(4)^8$ and place it at various fixed points of $T^6/\langle\beta\rangle$. Far from the fixed points, the instantons decay and match to the flat connection A_W.

First, embed the connection $\mathcal{A}_0$ in the first four $\mathrm{SU}(2)_L$ factors, and choose vanishing connections for all other SU(2) factors of SO(4)^8. Denote this connection on $\mathbb{R}^4/\mathbb{Z}_2$ by

$$\mathcal{A}_1 = [(g_L, 1)\,(g_L, 1)\,(g_L, 1)\,(g_L, 1)\,(1, 1)\,(1, 1)\,(1, 1)\,(1, 1)\,; W_1] \ ,$$

where the notation indicates which components carry the instantons connections, and that the connection has monodromy W_1 around the x_1 direction. This connection commutes locally with

$$(1, g_R)\,(1, g_R)\,(1, g_R)\,(1, g_R)\,(g_L, g_R)\,(g_L, g_R)\,(g_L, g_R)\,(g_L, g_R) \ ,$$

which generates SU(2)^{12}. We place the connection $\mathcal{A}_1$ on a collection of the sixteen T^2 loci of $\mathbb{R}^4/\mathbb{Z}_2$ singularities to be discussed below.

A similar connection $\mathcal{A}_2$ with monodromy W_2, to be supported on a distinct set of four singular loci, is given by

$$\mathcal{A}_2 = [(1, g_R)\,(1, 1)\,(1, g_R)\,(1, 1)\,(g_L, 1)\,(1, 1)\,(g_L, 1)\,(1, 1)\,; W_2] \ .$$

This connection commutes with a different SU(2)^{12} such that the sum of $\mathcal{A}_1$ and $\mathcal{A}_2$ gives a SO(4)^8-connection whose centralizer is SU(2)^8, generated by

$$(1, 1)\,(1, g_R)\,(1, 1)\,(1, g_R)\,(1, g_R)\,(g_L, g_R)\,(1, g_R)\,(g_L, g_R) \ .$$

Thus this instanton configuration reproduces the desired perturbative gauge symmetry for the $N = 2$ supersymmetric example. The matter spectrum of the candidate instanton configuration is three adjoint chiral multiplets per SU(2) factor, as desired. These arise as the remaining freedom to choose flat connections for the unbroken SU(2) factors: the six directions of the covering T^6 give six adjoints, which form three chiral multiplets.

This method of building instanton configurations creates the correct perturbative spectrum, but it is not immediately clear how to place the summands $\mathcal{A}_1$ and $\mathcal{A}_2$ on the correct T^2 loci as dictated by the half-G_2 limit. In the point-like limit, we expect a $\mathbb{Z}_2$ symmetry such that every SU(2)-instanton is associated to a pair of T^2 loci. However, placing separate $\mathcal{A}_0$ instantons on these loci does not give the correct counting of c_2. The instanton configuration that behaves appropriately in the point-like limit likely begins with an instanton on $(\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$ that does not arise from local $\mathbb{R}^4/\mathbb{Z}_2$ instantons. Such a solution may be built from a $\mathbb{Z}_2$-quotient of a configuration of calorons, which are instantons on $\mathbb{R}^3 \times S^1$ that are made from pairs of BPS monopoles [60, 61]. With the correct choice of parameters, the caloron is symmetric between pairs of points, and in the point-like limit it may provide a candidate building block for the singular gauge configuration required for this heterotic dual model.

2.6.2 Simplest $N = 1$ Example

We continue to our first $N = 1$ example, which is similar in most regards to the $N = 2$ example. In this case, we have a G_2 orbifold X_2 with 12 T^3 of A_1 singularities and three possible choices of K3 fibration. Although the base 3-orbifold of the fibration differs for the different choices, our analysis of the heterotic gauge bundle is unaffected by this change. For our analysis, we will choose the α-fibration, which gives the heterotic geometry $Y_{2,\alpha} = T^6_{123567}/\langle \beta, \gamma_2 \rangle$ described in example 3.2.

For the perturbative part of the spectrum, in addition to the T^3 Wilson lines described above, we must choose an action of $H_{2,\alpha} = \langle \beta, \gamma_2 \rangle$ on the perturbative gauge bundle. We choose β to act as the outer automorphism of $E_8 \times E_8$ as in example 6.1, while γ_2 must act in a way that swaps two $\mathrm{SU}(2)^4$ factors within the $\mathrm{SU}(2)^8$ subgroup of E_8 that is preserved by the Wilson lines. These group actions accomplish the monodromy seen on the G_2 orbifold side, where β and γ_2 each act on the 16 fixed loci of α so as to identify them in fours. There are two $\mathbb{Z}_2$ elements of E_8 (corresponding to nodes on the Dynkin diagram with Dynkin label 2), familiar from $T^4/\mathbb{Z}_2$ orbifolds, that are candidates for the action of γ_2. The computation of the perturbative spectrum must additionally take into account shifts in left-moving energy from point-like instantons, as described in section 5.3.

Now we investigate the non-perturbative spectrum. The heterotic geometry $Y_{2,\alpha}$ has 16 T^2 of A_1 loci, half from β and half from γ_2. As in the previous example, we must produce $\mathrm{SU}(2)^8$ non-perturbative gauge symmetry from these 16 loci. Again, the monodromy action of α in the ambient space interchanges the β-loci in pairs, and now they interchange the γ_2-loci in pairs as well. Thus we again expect the gauge bundle to be made of 8 semi-localized instantons that reside on pairs of T^2 and come with 3 adjoint chirals each.

The most intuitive description of this gauge bundle configuration (and that of the previous example) is via a "sequential orbifold", where the monodromy action of α on the β- and γ_2-loci is captured by a heterotic orbifold by the full Γ_2 (instead of only the subgroup $H_{2,\alpha}$ that acts nontrivially on the geometry). To make sense of this prescription, the elements of the orbifold group are taken to act in a certain order, where α acts upon the non-perturbative $H_{2,\alpha}$-orbifold: we think of the model as $X_2/\Gamma_2 = (X_2/H_{2,\alpha}) / \langle\alpha\rangle$. Because Γ_2 is abelian, we are free to order the elements in this way, although a fully satisfactory interpretation of this model would consider the non-perturbative effects of all of Γ_2 at once.

Because α acts to swap the heterotic geometry with another fiber of $\pi_4 : X_{2,\alpha} \to S^1_4/\langle\alpha\rangle$, only $H_{2,\alpha}$ descends to the heterotic geometry, which we identify with the orbifold $Y_{2,\alpha} = T^6_{123567}/H_{2,\alpha}$. Nonetheless, we may think of this string background as a Γ_2 background where α acts trivially on the geometry, but has a nontrivial action on the gauge bundle, identifying SU(2) factors in pairs. The action of α on the gauge bundle may be thought of as identifying components of the connection that take values in pairs of $\mathfrak{su}(2)$ summands. These Lie algebra summands correspond to SU(2) factors of the gauge group that arise non-perturbatively from fixed loci of β and γ_2, so for this interpretation to reproduce the intuitive picture from the 7D geometry, we must choose a specific order for the orbifold actions. We construct an orbifold background on $T^6/H_{2,\alpha}$ with a non-perturbative spectrum from standard point-like instantons, such as those found on the tangent sheaf, and then act on the resulting theory with a further orbifold action by α that identifies components of the resulting connection.

Given these results, we can ask how they inform our understanding of the half-G_2 map. In the 7D case of the half-K3 limit, the heterotic gauge symmetry may be read off from the complicated geometry at the ends of the interval, because all singularities were isolated, and therefore able to be moved to the complicated ends. In our half-G_2 limit, this remains true for the perturbative gauge symmetry, since those loci are transverse to the generic fiber, but the singular fibers that give rise to the non-perturbative gauge symmetry necessarily stretch all the way across the interval (see Figure 2.3). In the example at hand, each singular T^3 that stretches across the interval intersects the generic fiber in two components, while it intersects the end fiber in only one component. This means that looking only at the complicated ends of the interval will not determine the heterotic gauge bundle configuration, because this information would not tell you which pairs of T^2 loci in the heterotic geometry join into one in the complicated end. In other words, to reconstruct the α-monodromy, one must look at the entire interval to follow the loci through the 6D fibers. So we conclude that the information of the heterotic gauge bundle may be spread throughout the half-G_2 interval, even when the metric in the bulk of the interval is trivial.

We may again consider non-singular instanton configurations that reproduce the correct perturbative spectrum. In this case, we add a third summand to the instanton configuration:

$$\mathcal{A}_3 = [(1,1)\,(1,g_R)\,(1,1)\,(1,g_R)\,(1,1)\,(g_L,1)\,(1,1)\,(g_L,1)\,;W_3] \ .$$

Then the centralizer of the sum of $\mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_3$ is $\mathrm{SU}(2)^4$, embedded in $\mathrm{SO}(4)^8$ as

$$(1,1)\,(1,1)\,(1,1)\,(1,1)\,(1,g_R)\,(1,g_R)\,(1,g_R)\,(1,g_R) \ .$$

Again, we get three chiral multiplets per unbroken SU(2) from freedom to specify flat connections on the covering T^6.

2.6.3 Orbifold Singular Locus Example

Lastly, we will look at our $N = 1$ example with $T^3/\mathbb{Z}_2$ singular loci, which exhibits different point-like instanton behavior than the previous examples and also varying bundle configurations for different choices of fibration. We will first consider the α-fibration, in which case we have 8 singular T^2 loci from β and an additional 8 from γ_3. The β-loci come from the intersection of 4 T^3 loci with the heterotic geometry, while the γ_3-loci come from the intersection with 8 $T^3/\mathbb{Z}_2$ loci. So we expect $\mathrm{SU}(2)^4$ gauge symmetry with 3 adjoint chirals per SU(2) from the 8 β-loci while we expect $\mathrm{SU}(2)^8$ gauge symmetry with only 1 adjoint chiral per SU(2) from the 8 γ_3-loci. Thus it is clear that the two loci support different types of point-like instantons.

We can understand the difference between the loci based on the monodromy actions in the ambient space. The action of α on the β-loci is identical to the previous example, but it does not interchange the γ_3-loci, as it did for the γ_2-loci in the that case. To see this, we will consider an example locus in the covering space. The throat coordinate is

x_4, and the heterotic geometry is $Y_{3,\alpha} = \pi_4^{-1}\left(\frac{1}{4}\right)$. Consider the γ_3-locus

$$L = \left(\frac{1}{4}, x_2, 0, \frac{1}{4}, 0, x_6, 0\right) \; ,$$

where x_2 and x_6 can vary. We must keep in mind that this T^2 in the covering space represents the same T^2 as if we act upon this with β:

$$\beta L = \left(\frac{3}{4}, \frac{1}{2} - x_2, 0, \frac{1}{4}, 0, -x_6, 0\right) \; .$$

Because x_2 and x_6 are free coordinates, the only change is in the x_1 coordinate. On the other hand, we may consider the effect of α-monodromy on L. We shift along the throat coordinate to $x_4 = \frac{3}{4}$ and apply α, which gives us

$$\alpha L_{x_4+\frac{1}{2}} = \left(\frac{3}{4}, -x_2, 0, \frac{1}{4}, 0, x_6, 0\right) \; .$$

We see that the α-monodromy accomplishes the *same* interchange of the γ_3-loci in the covering space as does β, so the action on the γ_3-loci in $Y_{3,\alpha}$ is trivial. Because of this, each $T^3/\mathbb{Z}_2$ intersects the heterotic geometry only once, and therefore the associated instantons are fully localized on a single T^2.

However, the monodromy of α does eliminate harmonic one-forms on $T^3/\mathbb{Z}_2$ (as can be seen by the action of $\alpha\beta$ on either of the end-fibers of the x_4-interval), so that the instanton should come with only one adjoint chiral multiplet. In $N = 2$ language, the resulting gauge theory should be pure $N = 2$ SU(2) SYM. The existing 6D point-like instanton classification does not appear to include a $c_2 = 3/2$ instanton that gives non-perturbative gauge symmetry with no charged matter, so this gauge bundle configuration may also be previously undescribed. Note that the charged matter could be blocked by a B-field holonomy, as in [55], but this would block the gauge symmetry as well.

Theory	Perturbative $\mathrm{SU}(2)^8$	Non-perturbative $\mathrm{SU}(2)^8$
M	8 T^3 α-loci	8 T^3 β-loci
IIA	D6-branes on orientifold planes	8 T^2 β^*-loci
I	Subgroup of D9-brane $\mathrm{Spin}(32)/\mathbb{Z}_2$	D5-branes on 16 singularities with winding shift
SO(32)	Subgroup of primordial $\mathrm{Spin}(32)/\mathbb{Z}_2$	Point-like instantons on 16 singularities with winding shift
E_8	Subgroup of primordial $E_8 \times E_8$	T-dual point-like instantons on 16 singularities

Table 2.3: Origin of non-abelian gauge symmetry in the $N = 2$ model at each stage of the duality chain. "Perturbative" and "Non-perturbative" labels refer to the string coupling of the heterotic theories.

The β-fibration of X_3 gives identical results, but the γ_3-fibration provides a heterotic dual with a different gauge background. In this case, the geometry is $Y_{3,\gamma_3} = T^6_{123456}/\langle\alpha,\beta\rangle$, which has singular loci as in example 6.2. The non-perturbative part of the spectrum should be described, as in that case, by 8 semi-localized instantons on pairs of loci. The difference this time is in the perturbative part of the compactification: as discussed for the α-fibration, the monodromy actions of α and β on the γ_3 loci in the T^7 covering space are identical. Therefore, in the γ_3-fibration, where the γ_3 loci give rise to perturbative gauge symmetry on the heterotic side, the actions of α and β on the perturbative gauge bundle must be chosen accordingly. In particular, if we choose α to act on the perturbative gauge bundle as the outer automorphism of $E_8 \times E_8$, we must choose β as an element of E_8 that commutes with the resulting $\mathrm{SU}(2)^8$, but reduces the charged matter spectrum from 3 adjoint chirals per SU(2) to 1 adjoint chiral per SU(2).

2.7 An Alternate Duality Chain via Type I

To understand the gauge symmetry and particle spectrum seen in our M-theory orbifold backgrounds, it is informative to look at another chain of dualities that relates

M-theory to the Spin(32)/$\mathbb{Z}_2$ heterotic string. The point-like instanton effects we have seen in heterotic dual models look odd from the $E_8 \times E_8$ perspective, but may be better understood as Spin(32)/$\mathbb{Z}_2$ point-like instantons, which naturally appear with symplectic gauge groups and without tensor multiplets. The appearance of T-dual Spin(32)/$\mathbb{Z}_2$ point-like instantons in $E_8 \times E_8$ heterotic string theories was found in a similar setup in [62], where they resolve confusions that arose from mistakenly attributing their effects to $E_8 \times E_8$ point-like instantons. They were also found to explain the spectrum of an $E_8 \times E_8$ compactification in [71]. Our duality chain begins with M-theory, proceeds to a IIA orientifold, then a T-dual Type I theory, and finally an S-dual Spin(32)/$\mathbb{Z}_2$ heterotic model. The latter theory may be related to the $E_8 \times E_8$ heterotic string theory by an additional T-duality.

2.7.1 $N = 2$ Example

Beginning with our $N = 2$ example of section 3.1, if we take the x_4-direction as the M-theory circle, we may obtain a dual theory from Type IIA on T^6_{123567} orientifolded by the group

$$\Gamma_1^* = \left\langle (-1)^{F_L} \alpha^* \Omega, \beta^* \right\rangle = \left\langle (-1)^{F_L} R_{123}\Omega, R_{1234}\sigma_2 \right\rangle ,$$

where F_L is the left-moving fermion number, Ω is the worldsheet parity operator, $\alpha^* = \alpha \big|_{123567}$, and similarly for β^* [88]. We also write the action in terms of the reflection operator R, which flips the coordinates shown in its subscripts, and the shift operator σ_i that performs an order-two shift on coordinate x_i. In this IIA background, an $\mathrm{SU}(2)^8$ gauge symmetry arises from the D6-branes required to cancel the RR charges created by O6-planes along the 123-directions. An additional $\mathrm{SU}(2)^8$ gauge symmetry comes from D2-branes wrapped on the loci of A_1 singularities created by β^*, which are exchanged in pairs by α^*. In choosing the x_4 direction as the M-theory circle, requiring a weakly-coupled Type IIA dual would violate the limits in which we previous formulated our M/heterotic duality. Before, we chose the x_4 direction as the throat direction of the half-G_2 limit and required it to be large compared to the other dimensions of the K3 fiber. Thus, if we want to compare our IIA model directly to M-theory in the half-G_2 limit, we must work with strong IIA coupling. We could instead choose the x_7 direction as the M-theory circle, but this radius would also be required to be large due to the adiabatic limit.

Next, we apply T-duality along the 123-directions to obtain a Type I dual. This perspective gives a conceptual advantage because the entire spectrum is expected to be visible perturbatively on the Type I side, and the tadpole cancellation conditions give a powerful tool for computations. Early examples of spectrum computations using this method include [26, 89, 90, 91, 72]. In our case, T-duality gives Type IIB on $T^6_{\hat{1}\hat{2}\hat{3}567}$ orientifolded by the dual group

$$\tilde{\Gamma}_1^* = \left\langle \Omega, \tilde{\beta}^* \right\rangle = \langle \Omega, R_{1234}\tilde{\sigma}_2 \rangle \ ,$$

where $\tilde{\beta}^*$ has a winding shift in the x_2 direction instead of the momentum shift in β^* (signified by the tilde on $\tilde{\sigma}_2$). The hat notation on the torus coordinates signifies that the radii of the first three coordinates of the torus are inverted by T-duality. The operation also transforms the D6-branes to D9-branes that generate an $\mathrm{SU}(2)^8$ gauge symmetry as a subgroup of $\mathrm{Spin}(32)/\mathbb{Z}_2$. Meanwhile, the possible presence of D-branes at the A_1 singularities, and the resulting gauge symmetry, is complicated by the presence of the winding shift.

Momentum and winding shifts were originally discussed in the heterotic context in [92], and their effects were studied in the Type I context in [93, 94], where they give rise to supersymmetry breaking via stringy variants of the Scherk-Schwarz mechanism [95].

In these Type I models, the shifts take place in directions along which the reflections do not act. In our case, the shifts are in directions that are acted upon by the reflection, but they cannot be removed by coordinate redefinitions. The role of the Type I winding shift may be understood via its dual action in the Type IIA model. Relative to the IIA model without a shift, the momentum shift on x_2 blocks the appearance of a second sector of D6-branes that would intersect the first sector of D6-branes. Thus, it cuts in half the gauge symmetry and reduces the matter spectrum. This is exactly the behavior that we want to attribute to the semi-localized point-like instantons in the $E_8 \times E_8$ heterotic dual. Aside from the winding shift, our Type I model is similar to the $\mathbb{Z}_2$-orbifold of Type I considered in [27, 26]. A variant of this model with a momentum shift was considered in [91].

The last step of the duality chain is an S-duality to the Spin(32)/$\mathbb{Z}_2$ heterotic string. The Type I D9-brane gauge symmetry becomes the perturbative gauge symmetry $\mathrm{SU}(2)^8$ within the primordial Spin(32)/$\mathbb{Z}_2$ gauge group. The other $\mathrm{SU}(2)^8$ is non-perturbative and is expected to come from Spin(32)/$\mathbb{Z}_2$ point-like instantons effects. The background orbifold is unchanged when passing from Type I to the heterotic string, so the heterotic dual inherits the winding shift, which interacts with the point-like instantons to create the $\mathrm{SU}(2)^8$ gauge symmetry.

The $E_8 \times E_8$ heterotic string may be reached by a final T-duality between the two heterotic string theories. From this perspective, the instanton configuration appears to be spread across two disconnected singular loci. This duality chain provides a sequence that transforms the geometric data from the G_2 space into the bundle data of the $E_8 \times E_8$ heterotic compactification. At the initial M-theory stage, there are 8 singular loci that give rise to a rank-8 gauge group. In the final $E_8 \times E_8$ heterotic stage, the same rank-8 gauge group comes from 16 singular loci. In the intervening Type I and Spin(32)/$\mathbb{Z}_2$ heterotic stages, the compactification is weakly non-geometric due to the winding shift, so there isn't a clear answer to the number of singular loci, but the winding shift accomplishes the same rank-8 gauge group as the initial and final stages.

An alternative duality chain may be obtained in this $N = 2$ case by starting with a different Type IIA limit. Our M-theory background is $T^7/\langle\alpha,\beta\rangle$, where none of the elements in the orbifold group act on the final coordinate, x_7. Thus, we may take this coordinate as the M-theory circle and obtain a IIA dual on $T^6_{123456}/\langle\alpha,\beta\rangle$, which is again the orbifold limit of the Borcea–Voisin manifold of Hodge numbers $(19, 19)$. The geometric limits discussed in section 3 require that the radius of x_7 is large, meaning that this IIA dual is strongly-coupled. For our purposes, the only relevant non-perturbative effects are the massless states that arise from wrapped D2-branes on the orbifold singularities.

Type I and heterotic duals to this model were considered in [96], where it was found that the Type I dual includes momentum or winding shifts along the invariant T^2. This is in contrast to the Type I duals found in our duality chain above, where these shifts were along a direction of a T^4 on which the orbifold group acts nontrivially. The massless states in the heterotic dual of [96] were found to all be of non-perturbative origin, suggesting that this heterotic dual is distinct from the one obtained in the half-G_2 limit, which has a mixture of perturbative and non-perturbative gauge symmetry. This second duality chain is not available in the $N = 1$ cases, because there is no coordinate on which the M-theory orbifold group acts trivially, so we may not obtain a IIA orbifold dual in the same manner.

An additional Type IIB dual may be obtained by applying T-duality along only the x_3-direction instead of the x_{123}-directions. In this case, we find Type IIB compactified on $T^6_{12\hat{3}567}/\langle\Omega R_{12}, R_{1234}\sigma_2\rangle$. Cancellation of the O7-plane charge created at fixed points of ΩR_{12} will create a D7-brane background, so this dual model should be expressible in terms of F-theory, along the lines of [47].

2.7.2 The $N = 1$ Examples

In the $N = 1$ cases, we also must take into account the nontrivial action of γ as we go through the steps of the duality chain. A similar Type I orbifold was studied in [90], and further examples are given in [71, 72]. A similar duality chain was considered for M-theory on Spin(7) orbifolds in [97]. Our model differs from that of [90] by the inclusion of winding shifts in multiple directions that avoid an intersecting brane interpretation and reduce the rank of the gauge symmetry. In the $N = 1$ cases, discrete torsion is a nontrivial choice in the orbifold backgrounds as well. In our cases, it is expected to be present, as in [98].

For the IIA dual of our M-theory model on $T^7/\langle\alpha,\beta,\gamma_2\rangle$ of example 3.2, we take x_4 to be the M-theory direction, so that we obtain the dual theory IIA on T^6_{123567} orientifolded by

$$\Gamma_2^* = \left\langle (-1)^{F_L}\,\alpha^*\Omega, \beta^*, \gamma_2^* \right\rangle = \left\langle (-1)^{F_L}\,R_{123}\Omega, R_{1256}\sigma_2, R_{1357}\sigma_1\sigma_3 \right\rangle \ .$$

This is the dual model labeled as “Orientifold B” in [88]. Applying T-duality in the 123-directions gives us Type IIB on $T^6_{\hat{1}\hat{2}\hat{3}567}$ orientifolded by

$$\tilde{\Gamma}_2^* = \left\langle \Omega, \tilde{\beta}^*, \tilde{\gamma}_2^* \right\rangle = \langle \Omega, R_{1256}\tilde{\sigma}_2, R_{1357}\tilde{\sigma}_1\tilde{\sigma}_3 \rangle \ .$$

The winding shifts persist in the S-dual Spin(32)/$\mathbb{Z}_2$ heterotic model as well. If we apply T-duality to convert this to an $E_8 \times E_8$ heterotic model, we end up with an instanton configuration that looks locally similar to the $N = 2$ case.

The M-theory background of example 3.3, which lives on the space $T^7/\langle\alpha,\beta,\gamma_3\rangle$, is similarly dual to Type IIB on T^6_{123567} orientifolded by

$$\tilde{\Gamma}_3^* = \left\langle \Omega, \tilde{\beta}^*, \tilde{\gamma}_3^* \right\rangle = \langle \Omega, R_{1256}\tilde{\sigma}_2, R_{1357}\tilde{\sigma}_1 \rangle \ ,$$

where the only difference from the previous example is the lack of a winding in the x_3-direction. Thus, while the instantons in models 6.2 and 6.3 look rather different from the $E_8 \times E_8$ heterotic perspective, the models differ on the Spin(32)/$\mathbb{Z}_2$ side only by the inclusion of a winding shift on one coordinate, just as they differed on the M-theory side by only a momentum shift. Explicit calculations of the effect of winding shifts on the $T^6/\mathbb{Z}_2^2$ background of [90] would further explain the instanton effects, but is beyond the scope of this work.

2.8 Discussion

To better understand the types of point-like instantons that appear in our $E_8 \times E_8$ backgrounds, we may compare examples 6.2 and 6.3, our two $N = 1$ cases. These examples live on the same Calabi–Yau orbifold, so the difference in their non-perturbative gauge symmetry cannot come from any mechanism that depends on the geometry alone. For example, one might expect that the superpotential contributions from worldsheet instantons could lift gauge bundle moduli in a way that differentiates the two cases. However, the presence of worldsheet instanton effects at lowest order is controlled only by the existence of rigid rational curves, so it is a property only of the geometry [99]. Thus, if we are to appeal to some part of the heterotic background to explain the differences in non-perturbative behavior, it must be the background gauge field or B-field. A particularly attractive mechanism is Wilson line backgrounds. We have already specified the perturbative Wilson line background via the half-K3 limit, but there may be additional Wilson line effects involving the non-perturbative part of the gauge group, and these may break this part of the gauge symmetry in the low energy effective theory. To further understand the behavior of the non-perturbative spectra in our examples, we will discuss the relation to two other heterotic phenomena: Hořava–Witten duals and coupled heterotic moduli.

2.8.1 Gauge Locking in Hořava–Witten Duals

As observed in [41], Hořava–Witten theory [34, 35] suggests that our heterotic models should have an additional M-theory dual on a background of the form $T^6/H \times S^1/\mathbb{Z}_2$. Then, via the heterotic string, we should have an M-theory/M-theory duality between compactifications on G_2 spaces and Hořava–Witten compactifications. One interesting aspect of this duality is how the heterotic point-like instantons are represented on each side. In the heterotic duality with Hořava–Witten theory, point-like instantons on orbifold singularities are thought of as fractional M5-branes wrapped on the singularity. On the other hand, in the duality with M-theory on G_2, the instantons correspond to M2-branes wrapped on degenerate K3 fibers. This is an example of electromagnetic duality for the C-field that interchanges M2 and M5 branes [100, 101]. Thus, Hořava–Witten theory offers an electromagnetically dual perspective from which to investigate our phenomena.

In the dual pairs of examples 6.1 and 6.2, we found that the M-theory geometry dictates a spectrum that looks subtle from the $E_8 \times E_8$ heterotic side, where gauge symmetries from different singular loci are united. This phenomenon is familiar from studies of heterotic orbifolds via Hořava–Witten theory, where it has been found that 7-planes stretching between the 10-plane ends of the M-theory interval can carry gauge degrees of freedom that "lock" together, reducing to a smaller subgroup [56, 57, 58, 102]. An example considered first in [56] and later in [58] is a heterotic compactification on $T^4/\mathbb{Z}_2$ with perturbative gauge group $\mathrm{SO}(16) \times E_7 \times \mathrm{SU}(2)$ (up to $\mathbb{Z}_2$ quotients). The point-like instantons required to cancel the magnetic charge of the 16 A_1 singularities would naively contribute a non-perturbative gauge symmetry of $\mathrm{SU}(2)^{16}$, but it can be shown by duality with F-theory that all SU(2) factors are broken to a common diagonal

SU(2), denoted SU(2)*, so that the full gauge group is $\mathrm{SO}(16)\times E_7\times \mathrm{SU}(2)^*$. In this sense, all of the non-perturbative SU(2) factors and the perturbative SU(2) factor are "locked" together. The M-theory mechanism invoked to describe this phenomenon is nonzero G-flux required by anomaly cancellation, deforming the Hořava–Witten geometry away from a metric product. The gauge locking explains how the perturbative twisted spectrum can include matter charged under both E_8 factors, even though they are separated at either end of the Hořava–Witten interval—the singular 7-planes carry the gauge quantum numbers between the two ends.

In [102], similar phenomena were found for the Hořava–Witten picture of a heterotic $T^6/\mathbb{Z}_3$ orbifold. In this case, the effective theory is 4D and the states charged under the two E_8 factors are not localized to one side. Instead, the states that carry the bifundamental representation of SU(3) subgroups of the two E_8 factors are spread over the length of the interval in a meson-like configuration.

These Hořava–Witten phenomena—gauge locking and delocalized bundle configurations—are very similar to the semi-localized instantons that we observe in our examples, so it is possible that they are incarnations of the same type of phenomenon seen from dual perspectives. However, our examples do not have a topological defect analogous to an orbifold 7-plane to carry quantum numbers between matter loci. Additionally, the gauge locking is achievable on heterotic backgrounds that lack a momentum shift, so its interpretation in a dual Type I model may be quite different from that of the semi-localized instantons. The relation between these phenomena is an interesting question for future work.

2.8.2 Coupled Heterotic Moduli

An important feature of heterotic compactifications is that the moduli space does not factorize into complex structure and gauge bundle moduli: the two are coupled by the fact that the gauge bundle must remain holomorphic, so that a particular bundle configuration is compatible with only certain deformations of the complex structure [59]. This may allow our semi-localized point-like instantons to lift moduli that are unphysical from the M-theory perspective by coupling bundle moduli to the Kahler and complex structure moduli of the loci on which they are supported. For instance, in example 6.2, because the T^3 loci of the G_2 orbifold intersect the heterotic geometry in $T^2 \sqcup T^2$ loci, the T^2 loci cannot be blown up or deformed independently, but must have their moduli coupled, as they are part of the same T^3 locus in the ambient space. Thus, coupling of these moduli by semi-localized instantons of the gauge bundle looks quite natural. In this sense, we may think of the singularities of the heterotic orbifold as "partially frozen", since the directions of moduli space that correspond to independent resolutions of singular loci have become massive.

2.8.3 Future Directions

This chapter is based on the half-G_2 limit and point-like instantons on orbifold singularities, neither of which has been fully understood in the literature. Consequently, there are many directions in which this work can be taken to deepen our knowledge of non-perturbative aspects of M/heterotic duality.

- As discussed in previous sections, there are several perturbative and non-perturbative spectrum computations that would elucidate the relations between our M-theory, heterotic, and Type I backgrounds, but were beyond the scope of this work. Of particular interest would be a calculation of the Type I spectra with the effects of winding shifts, as described in section 7, as well as a calculation of the heterotic spectra taking into account Wilson lines and the lack of modular invariance, as in [71].

- In this chapter, we restricted ourselves to A_1 singularities, but there exist examples of G_2 orbifolds with other ADE singularities. How does the half-G_2 map operate in those situations? The choice of a throat coordinate was made simple by the fact that the elements of Γ acted as reflections, but the choice may not be so obvious if the group elements act in more complicated ways.

- A next step in the understanding of the half-G_2 map would be to consider more general M-theory backgrounds that include nontrivial profiles for the C-field and 7D gauge fields. Additionally, studying G_2 orbifolds with intersecting codimension 4 singularities and/or codimension 7 singularities will allow for a greater variety of matter representations. The Type I tadpole cancellation conditions in the alternate duality chain of section 7 give another way to look at the presence or absence of singularities in the G_2 moduli space.

- The examples of G_2 orbifolds that we look at in this chapter are non-generic in the sense that they have *multiple* K3 fibrations, giving us extra tools to work with in determining the heterotic gauge bundle. In particular, extra K3 fibrations on the M-theory side will guarantee a K3-fibration on the heterotic side (in the half-G_2 limit), which simplifies our treatment of point-like instantons by increasing the amount of supersymmetry in the local theory. Eventually, the half-G_2 map should be generalized to K3-fibered G_2 orbifolds that have only one fibration and dual heterotic orbifolds that only enjoy an SYZ fibration.

- Reconstruction of heterotic gauge bundles from fiberwise data on a T^3 fibration is not yet well-understood, but progress is being made in that direction via the 3D Hitchin system and related spectral cover descriptions of heterotic gauge bundles [103, 104, 105, 106]. These methods give a promising route toward a rigorous algorithm for constructing non-perturbative heterotic duals of M-theory backgrounds.

Chapter 3

T^3-Invariant Heterotic Hull-Strominger Solutions

3.1 Introduction

Since the early days of string theory, heterotic compactifications have been a fruitful road towards realistic models of particle physics, beginning with the seminal paper of Candelas, Horowitz, Strominger, and Witten [107]. It is common practice to look for supersymmetric solutions by compactifying on a six-dimensional Calabi–Yau manifold with a gauge bundle, at least as a zeroth-order geometry. However, α'-corrections generically induce torsion [108, 109], whereby the geometry is described by a more complicated set of equations known as the Hull-Strominger system. The geometric features of the torsional Hull-Strominger system are much more mysterious than their Calabi–Yau cousins. Part of the purpose of this chapter is to shed light on the local geometry of these solutions by considering a certain dimensional reduction of the geometry and studying local solutions to the system in this limit. In fact, the reduced solutions we present constitute new T^3-invariant local solutions of the Hull-Strominger system.

Another route to physically realistic effective theories is via M-theory on singular G_2 holonomy spaces [42]. Specifically, G_2 holonomy spaces with codimension-7 singularities sitting in codimension-4 orbifold loci may give a geometry on which M-theory can produce realistic models of particle physics. One tool to investigate M-theory compactifications is the duality between M-theory on a K3 surface and the $E_8 \times E_8$ heterotic string on T^3 [9]. This duality is simplest in the limit of large heterotic volume, which corresponds on the M-theory side to the half-K3 limit, where the K3 surface is stretched along one direction (analogous to the stable degeneration limit of F-theory). This duality may be adiabatically fibered over a 3D base to obtain 4D effective theories: when the G_2 holonomy space of the M-theory geometry carries a coassociative K3 fibration, we expect it to be dual to the $E_8 \times E_8$ heterotic string compactified on a Calabi–Yau threefold with a fibration by special Lagrangian 3-tori, known as an SYZ fibration [30]. The conditions of $N = 1$ supersymmetry additionally require that the heterotic background gauge field must satisfy the Hermitian Yang-Mills equations. The equations we study in this chapter may be understood as an approach to this duality from the heterotic side, where they give the lowest order α'-corrections to the heterotic large volume (i.e. M-theory half-K3) limit. This is a step towards the α'-corrected heterotic dual of Donaldson's local adiabatic limit of co-associative fibered G_2 manifolds [65], applicable in the M-theory setting.

Compact spaces of the required type for physically realistic effective theories are not yet available: on the M-theory side, no compact G_2 holonomy spaces with codimension-7 singularities sitting inside orbifold loci have been constructed, and on the heterotic side, Hermitian Yang-Mills connections over compact SYZ fibrations are not well-understood. Thus we are currently limited to working with local models of such geometries. The Hull-Strominger system, when reduced on the fibers of a local model $T^3 \times \mathbb{R}^3$ of the SYZ geometry, gives an α'-corrected version of the equations for a stable complex flat connection in 3D, which were introduced in [68]. These equations give a fruitful playground for understanding the matter spectrum of G_2 (and heterotic) compactifications, and have been studied in various ways in the literature. A method for computing spectra of solutions via Morse cohomology was given in [103], and the method was extended and applied in [104] to reduced models of twisted connected sum G_2 holonomy spaces [110, 23, 24]. The cohomology method was further extended to local G_2 holonomy spaces in [111]. The first non-abelian solution was given in [105], where an $SU(3)$ solution was constructed from the M-theory perspective via T-branes [112]. The authors of [105] point out that such non-abelian solutions can allow for chiral zero-modes, even if the upstairs G_2 geometry locally has only a codimension six singularity, and they present a local example with an explicit construction of such a chiral mode.

In this chapter, we study non-abelian stable complex flat connections from a heterotic perspective. We present a Morse-Witten type cohomology that can be used to compute the index of such solutions. If we make some additional assumptions for the types of non-abelian solutions considered, we can also use this cohomology to compute the spectrum. We also investigate how α'-corrections coming from the local reduced Hull-Strominger system correct and change the tree-level solutions. We consider how α'-corrections may modify the generic behavior of solutions near sources, and how the metric and D-term stability condition receive non-trivial corrections due to the heterotic Bianchi-Identity close to the sources. In particular, for generic non-abelian solutions, we are forced to introduce a two-form field which alters the equations in an interesting way.

The chapter is organized as follows. In Section 3.2, we introduce the Hull-Strominger system, reduce it on a local model for an SYZ-fibered Calabi–Yau threefold, and compute the α'-corrections to the resulting 3D equations for a stable complex flat connection. In Section 3.3, we explore solutions to this system of equations, beginning with abelian solutions and then examining a non-abelian monopole-type solution. In Section 3.4, we introduce a method for computing the chiral spectrum or chiral index of non-abelian solutions that become asymptotically abelian near sources of the Higgs field, or at least remain well-behaved. In Section 3.5, we present examples of spectrum computations. In Section 3.6, we give our conclusions and further directions.

3.2 The Hull-Strominger System

We start by first recalling the α'-corrected system of equations that must be satisfied by the geometry, gauge bundle, dilaton, and B-field of an $N = 1$ heterotic background. The manifolds of interest admit an $SU(3)$ structure (X, Ω, ω), where Ω is a complex nowhere vanishing three-form and ω is a real two-form of maximal rank. The form Ω is also required to be locally decomposable, which implies that it endows X with an almost complex structure J [113]. Moreover, Ω is a $(3, 0)$ form with respect to J.

The forms Ω and ω now satisfy the usual $SU(3)$ structure relations

$$\tfrac{i}{||\Omega||^2}\Omega \wedge \overline{\Omega} = \tfrac{1}{6}\omega \wedge \omega \wedge \omega \, , \quad \omega \wedge \Omega = 0 \, . \tag{3.1}$$

The first equation identifies the volume forms defined by the two structure forms, while the second relation says that ω is of type $(1, 1)$ with respect to J. For supersymmetric heterotic compactifications to four dimensional Minkowski space,

$$\mathcal{M}_{9,1} = \mathcal{M}_{3,1} \times X \, , \tag{3.2}$$

the internal geometry X is required to satisfy the relations [108, 109, 114]

$$\mathrm{d}(e^{-2\phi}\Omega) = 0 \tag{3.3}$$

$$\mathrm{d}(e^{-2\phi}\omega \wedge \omega) = 0 \tag{3.4}$$

$$i(\overline{\partial} - \partial)\omega = H \ , \tag{3.5}$$

where ϕ denotes the heterotic dilaton and H is given by

$$H = \mathrm{d}B + \tfrac{\alpha'}{4}\left(\omega_{CS}(A) - \omega_{CS}(\nabla)\right) \ . \tag{3.6}$$

Here A is the gauge connection of a vector bundle with structure group contained in either $E_8 \times E_8$ or $SO(32)$, while ∇ is an $\mathrm{End}(TX)$ valued connection which will play less of a role for us[1]. The two-form B is the heterotic Kalb-Ramond field, which has to transform under gauge transformations in order that the flux H remains gauge invariant [121]. The three form then satisfies the heterotic Bianchi identity

$$\mathrm{d}H = 2i\partial\overline{\partial}\omega = \tfrac{\alpha'}{4}\left(\mathrm{tr}\, F \wedge F - \mathrm{tr}\, R \wedge R\right) \ , \tag{3.7}$$

where F and R are the curvatures of A and ∇ respectively. For our local explicit solutions, the last term on the right hand side will be of cubic order in α' and will hence be dropped from now on. However, this term will be important when one considers compact global issues.

The first condition (3.3) implies that the complex structure defined by Ω is *integrable*, so (X, J) is a complex manifold. The second condition (3.4) is known as the conformally balanced condition, while the third condition (3.5) identifies the heterotic NS three-form

[1]The freedom to choose ∇ has been discussed extensively in the literature, see e.g. [115, 116, 117, 118, 119, 120].

flux in terms of the internal geometry. In addition, the gauge bundle is required to satisfy the supersymmetry conditions

$$F \wedge \Omega = 0\,, \quad \omega \wedge \omega \wedge F = 0\,, \tag{3.8}$$

often referred to as the Hermitian Yang-Mills conditions. Indeed the first condition implies that F is of type $(1,1)$, which means that the bundle is *holomorphic*. The second condition is the Yang-Mills constraint, which implies that the bundle is poly-stable by the Donaldson–Uhlenbeck–Yau theorem [122, 123].

The system of equations (3.3)-(3.5) together with the heterotic Bianchi identity (3.7) and the Hermitian Yang-Mills conditions (3.8) are often referred to as the Hull-Strominger system. It is (perturbatively) accurate modulo cubic corrections in α'.

3.2.1 Reducing the Hull-Strominger system on $T^3 \times M_3$

We are interested in solutions of the Hull-Strominger system on local models for Calabi–Yau manifolds, or more generally torsional models solving the Hull-Strominger system. In particular, we assume that the internal geometry has a special Lagrangian T^3 fibration [30]. Locally we can model such a geometry as

$$X = T^3 \times M_3\,. \tag{3.9}$$

We will also assume the fibers T^3 to be sufficiently small, so that we can have our solution depend nontrivially only on the M_3 coordinates. In this chapter we will take M_3 to be either S^3 or $\mathbb{R}^3$ for local models of M_3. Let us proceed to consider such local models where $M_3 = \mathbb{R}^3$.

The tree-level ($\alpha' = 0$) geometry then consists of a Calabi–Yau metric on this space, with a local complex structure given by

$$\mathrm{d}z^i = \mathrm{d}x^i + i\mathrm{d}\phi^i \,, \tag{3.10}$$

where $\{x^1, x^2, x^3\}$ are the coordinates on M_3, while $\{\phi^1, \phi^2, \phi^3\}$ are the coordinates of the three-torus. Other ansatze for the geometry are of course possible. In particular, we discuss an ansatz in appendix A with relevance for local heterotic/F-theory duality, which also turns out to result in a α'-corrected version of the t'Hooft-Polyakov monopole [124, 125].

Locally on $T^3 \times \mathbb{R}^3$ we have the corresponding complex top-form and Hermitian form given by

$$\Omega_0 = \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \,, \quad \omega_0 = \lambda^2 \sum_{i=1}^{3} \mathrm{d}x^i \wedge \mathrm{d}\phi^i = g_{0ij}\mathrm{d}x^i \wedge \mathrm{d}\phi^j \,, \tag{3.11}$$

where λ denotes a constant size parameter. This model $SU(3)$ structure corresponds to a flat tree-level metric. We will use this as our model for explicit $\mathbb{R}^3$ computations throughout this chapter, leaving reductions to more generic curved backgrounds for future work.

Let us then reduce the Hermitian Yang-Mills equations on this system. We first expand

$$A = A^x_i \mathrm{d}x^i + A^\phi_i \mathrm{d}\phi^i \,, \tag{3.12}$$

where A^x and A^ϕ are assumed to depend only on the non-compact coordinates. We have

$$\mathrm{d}_A = \mathrm{d} + A \,, \quad F = \mathrm{d}A + A \wedge A \,. \tag{3.13}$$

Plugging this into the holomorphic constraint gives

$$F^x = A^\phi \wedge A^\phi \tag{3.14}$$

$$\mathrm{d}_{A^x} A^\phi = 0 \,, \tag{3.15}$$

where d_{A^x} is the exterior covariant derivative with respect to the connection A^x, and where $A^\phi = A^\phi_i \, \mathrm{d}x^i$ transforms as a one-form on $\mathbb{R}^3$ due to a topological twist. This twisting occurs because the T^3 fiber is a special Lagrangian, so that its normal bundle is isomorphic to its tangent bundle [48]. The F-term equations (3.14) and (3.15) can be recast as the equation for a complex flat connection

$$\mathcal{A} = A^x + i\, A^\phi \,, \quad F_{\mathcal{A}} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0 \,. \tag{3.16}$$

Reducing the Yang-Mills conditions gives a co-closure constraint on A^ϕ. At zeroth order in α', the equation reads

$$d^\dagger_{A^x} A^\phi = 0 \,. \tag{3.17}$$

where the dagger denotes an adjoint taken with respect to the tree-level metric from (3.11). This equation can be viewed as a stability condition on the flat connection [126]. Together, the F-term equation (3.16) and D-term equation (3.17) comprise the equations for a stable complex flat connection in 3D.

Including α'-corrections modifies the D-term co-closure equation while preserving the F-term flatness condition, as we will see in the next subsection.

3.2.2 The Back-Reacted Geometry

Now we will consider α'-corrections to our equations. Instanton configurations of the type (3.14)-(3.24) will back-react on the geometry through the heterotic Bianchi identity (3.7) and supersymmetry conditions (3.3)-(3.5). Such back-reactions will only become relevant close to sources, and so we restrict ourselves to a local patch containing the source.

A short computation using (3.14)-(3.15) reveals that for the reduced geometry, the Pontryagin class may be written as

$$\mathrm{tr}\, F\wedge F = -\mathrm{d}_x\tilde{\mathrm{d}}_x\mathrm{tr}\left(A^\phi\wedge\tilde{A}^\phi\right)+\tfrac{2}{3}\,\mathrm{d}_x\mathrm{tr}\left(\tilde{A}^\phi\wedge\tilde{A}^\phi\wedge\tilde{A}^\phi\right)-\tfrac{2}{3}\,\tilde{\mathrm{d}}_x\mathrm{tr}\left(A^\phi\wedge A^\phi\wedge A^\phi\right)\,, \tag{3.18}$$

where $\mathrm{d}_x = \mathrm{d}x^i\partial_i$, $\tilde{\mathrm{d}}_x = \mathrm{d}\phi^i\partial_i$ and $\tilde{A}^\phi = \mathrm{d}\phi^i A^\phi_i$. Using the Hodge decomposition of M_3, it follows that

$$\tfrac{2}{3}\,\tilde{\mathrm{d}}_x\mathrm{tr}\left(A^\phi\wedge A^\phi\wedge A^\phi\right) = \tilde{\mathrm{d}}_x\mathrm{d}_x B \tag{3.19}$$

for some real two-form $B\in\Omega^2(M_3)$. We then find

$$\mathrm{tr}\, F\wedge F = \mathrm{d}_x\tilde{\mathrm{d}}_x\Big(B+\tilde{B}-\mathrm{tr}\left(A^\phi\wedge\tilde{A}^\phi\right)\Big)\,, \tag{3.20}$$

where $\tilde{B}=\frac{1}{2}B_{ij}\mathrm{d}\phi^{ij}$. From the Bianchi identity (3.7), it follows that ω is corrected to

$$\omega = \omega_0 - \tfrac{\alpha'}{16}\Big(\mathrm{tr}\left(A^\phi\wedge\tilde{A}^\phi\right)-B-\tilde{B}\Big)\,, \tag{3.21}$$

where ω_0 is the tree-level solution, which satisfies

$$\mathrm{d}_x\tilde{\mathrm{d}}_x\omega_0 = 0\,. \tag{3.22}$$

Locally on $\mathbb{R}^3$ we will take ω_0 to be given by (3.11) above, i.e. corresponding to the flat metric. From ω we get a corrected metric

$$g_{mn} = g_{0mn} - \tfrac{\alpha'}{16}\mathrm{tr}\left(A^\phi_m A^\phi_n\right) , \tag{3.23}$$

on M_3, where g_0 is the metric derived form ω_0.

Before we consider the α'-correction to (3.17), let us first make an observation concerning the two-form B we introduced above. It is tempting to speculate that by giving an appropriate α'-correction to the complex structure ansatz (3.10), we can absorb B, leaving a Hermitian form as in (3.11) but with $g_{0ij} = \lambda^2\delta_{ij}$ replaced by g_{ij}. However, it turns out that if the deformed complex structure is to remain integrable, then B must be a closed two-form if it is to be cancelled by such a deformation. As only the exterior derivative of B appears in the Bianchi identity, we see that such a deformation cannot absorb B.

The α'-corrections to the D-term supersymmetry equation, which is the reduction of the Yang-Mills condition for the gauge sector, reads

$$g^{mn}\nabla^x_m A^\phi_n = \tfrac{\alpha'}{16}B^{nq}A^\phi_n A^\phi_q , \tag{3.24}$$

where indices are raised and lowered using g_{mn}, and ∇^x_m acts as the Levi-Civita connection on space-time indices. Note at zeroth order in α' this is just the co-closure condition, but for general non-abelian connections where $B \neq 0$, the equation gets corrected even at first order in α'. The appearance of the Levi-Civita connection for the α'-corrected metric does not come directly from reduction of the Hermitian Yang-Mills equations, but is implied by the topological twist discussed in the previous subsection. Interestingly, the additional terms in the covariant derivative required by the Levi-Civita connection vanish to first order in α' when contracted with the inverse metric, so the different choices of connection are equivalent to the order to which the Hull-Strominger system is corrected.

We also need to check the conformally balanced condition (3.4). A somewhat lengthy computation gives the dilaton factor as

$$\mathrm{d}_x\phi = \tfrac{\alpha'}{64}\mathrm{d}_x\left(\mathrm{tr}(A^\phi_m A^{\phi m}) + \tfrac{\alpha'}{32}\mathrm{tr}(A^\phi_m A^\phi_n)\mathrm{tr}(A^{\phi m}A^{\phi n}) + \tfrac{\alpha'}{32}\kappa\, f - \tfrac{\alpha'}{32}B_{mn}B^{mn}\right) , \qquad (3.25)$$

where we have used (3.24), and where one must also choose B so that

$$\nabla^m B_{mn} = 0 \, . \qquad (3.26)$$

Here indices are again raised and the adjoint is taken with respect to the metric g. We are free to assume B is co-exact, given in terms of a function f as

$$B_{np} = \nabla^m(f\,\epsilon_{mnp}) \, . \qquad (3.27)$$

The constant κ is the part of $\mathrm{tr}(A^\phi \wedge A^\phi \wedge A^\phi)$ proportional to the volume form

$$\tfrac{2}{3}\,\mathrm{tr}(A^\phi \wedge A^\phi \wedge A^\phi) = \mathrm{d}_x B + \kappa * 1 \, . \qquad (3.28)$$

On $\mathbb{R}^3$ this can be absorbed in $\mathrm{d}_x B$, as any three-form on $\mathbb{R}^3$ is exact by the Poincare lemma.

Finally, we can also check that (3.3) is satisfied, provided that we take

$$\Omega = e^{2\phi}\Omega_0 \, . \qquad (3.29)$$

Additionally, it can be verified that the $SU(3)$ conditions (3.1) are also satisfied.

3.3 Solutions to the Reduced System

The natural place to begin a search for stable complex flat connections is smooth solutions of finite energy. However, in analogy to the 4D Hitchin system on $\mathbb{R}^4$, it seems likely that any smooth, finite energy stable complex flat connection is gauge equivalent to the trivial solution. We will show below that this is true if one assumes strong enough fall-off conditions on the Higgs field. The implications are that finite energy solutions must not be smooth, which we take to mean that the Higgs field has singularities along a configuration of sources.

To see that the claim is true for strong enough fall-off conditions, we look at a corollary of the Bochner-Weitzenboch identity that holds for solutions of the complex Yang-Mills equations (and thus for solutions of (3.16)-(3.17)):

$$\frac{1}{2}\Delta \left|A^{\phi}\right|^2 - \left|\nabla_{A^x} A^{\phi}\right|^2 - \left|A^{\phi} \wedge A^{\phi}\right|^2 = 0$$

where $A^x + iA^{\phi}$ satisfies the complex Yang-Mills equations and we have specialized to $\mathbb{R}^3$ to drop a term involving the Ricci curvature [127]. We then integrate this equation over $\mathbb{R}^3$. The Laplacian term may be integrated via Stoke's theorem, and with strong enough fall-off conditions on A^{ϕ}, this tells us that the integral of this term vanishes. In that case, we have that the sum of the other two terms vanishes. Because these terms are negative semidefinite, we conclude that

$$\nabla_{A^x} A^{\phi} = 0, \quad A^{\phi} \wedge A^{\phi} = 0$$

The second equation tells us that F^x is zero (via the F-term equation) and because we are on $\mathbb{R}^3$, we may then choose a gauge in which $\nabla_{A^x} = \mathrm{d}$, and then the above equations imply that A^{ϕ} is trivial.

It seems likely that the 6D finite energy condition, when reduced to 3D, will provide strong enough fall-off conditions for the vanishing of the integrated Laplacian above. To prove this hypothesis would require additional analysis of 3D gauge theory that we leave for future work.

The above argument applies only to the tree level $\alpha' = 0$ equations. For the α'-corrected system, we do not have access to a Bochner-Weitzenboch identity, so cannot rule out smooth solutions in the same way. However, in the cases examined below, including α'-corrections to a solution that is singular at tree level does not smooth out the singularity, and instead worsens the singularities (at least at $O(a')$, where the solution is reliable).

3.3.1 Abelian Solutions

The simplest solutions to the reduced Hull-Strominger system are found by taking the gauge group to be abelian. It turns out that, as we will see below, these are also relevant to certain non-abelian solutions with sources. Indeed, we will argue that for a certain type of non-abelian solution, the part of the connection which sees the source effectively becomes abelian. It thus makes sense to investigate closer the local geometry of such solutions near sources. These solutions were also extensively studied as local models for M-theory compactifications on G_2 manifolds in [42, 103, 104, 105, 128].

The Higgs field of an abelian solution satisfies $A^\phi \wedge A^\phi = 0$, which by (3.14) implies $F^x = 0$. Assuming we are on a simply connected space M_3, we may then choose $A^x = 0$.[2] Thus we can decouple the gauge and tangent bundle factors of A^ϕ and search for $A^\phi \in \Omega^1(M_3)$ satisfying the F-term and D-term equations

$$\mathrm{d}A^\phi = \mathrm{d}^\dagger A^\phi = 0\,, \tag{3.30}$$

[2] Our space will either be S^3 or the local model $\mathbb{R}^3$.

where the adjoint is taken with respect to the α'-corrected metric. Thus we are looking for harmonic 1-forms on M_3. We will work in a patch, so that the closure condition on A^ϕ implies by the Poincare lemma that we can set

$$A^\phi = \mathrm{d}\psi \,, \tag{3.31}$$

for some real function ψ. The co-closure condition then becomes

$$\mathrm{d}^\dagger \mathrm{d}\psi = 0 \,, \tag{3.32}$$

and so we are looking for harmonic functions on $\mathbb{R}^3$, which may blow up at sources. Note that the metric is given as

$$g_{mn} = g_{0mn} - \tfrac{\alpha'}{8}\,\partial_m\psi\partial_n\psi \,, \tag{3.33}$$

where the factor of 2 relative to (3.23) comes from the trace, and the form of g_0 is given by (3.11). The dilaton is

$$\phi = \tfrac{\alpha'}{64}\, g^{mn}\partial_m\psi\partial_n\psi + \tfrac{\alpha'^2}{32\times 64}\left(g^{mn}\partial_m\psi\partial_n\psi\right)^2 \,, \tag{3.34}$$

where we have set the constant part of the dilaton to zero. Let us go on to consider some common local solutions on $\mathbb{R}^3$ and their α' corrections.

At zeroth order, the radially symmetric solution is the standard monopole harmonic function

$$\psi(r) = \tfrac{C}{r} + \mathcal{O}(\alpha') \,, \tag{3.35}$$

for some constant C corresponding to the charge of the monopole. Given the monopole solution (3.35) for ψ, we find that the dilaton is

$$\phi = \frac{\alpha' \, C^2}{64\lambda^2 \, r^4} + \mathcal{O}(\alpha'^2) \, . \tag{3.36}$$

The only component of the metric that is corrected is g_{rr}:

$$g_{rr} = \lambda^2 + \tfrac{\alpha' C^2}{8r^4} + \mathcal{O}(\alpha'^2) \, . \tag{3.37}$$

Note that both the dilaton and metric blow up as $r \to 0$.

Exact Solution and large charge limit

Local solutions to torsional heterotic compactifications and the Hull-Strominger system with abelian bundles have been studied from different perspectives before [129, 130, 131, 132]. The benefits of studying abelian bundles is that a particular double scaling limit can be employed, where the charge of the gauge field is sent to infinity, while α' is sent to zero in a controlled manner. This results in a finite correction to the geometry at first order in α', while higher corrections vanish. The solution is one-loop exact.

We consider the exact solution to the reduced Hull-Strominger system with a radially-symmetric potential field $\psi(r)$. If we solve equation (3.24) using the α'-corrected metric, we find

$$\psi'(r) = -\tfrac{C}{\sqrt{r^4 - \alpha' \frac{C^2}{8\lambda^2}}} \tag{3.38}$$

This Higgs field no longer blows up at the origin, but at a non-zero radius. From this Higgs field, we can calculate the metric, which again is corrected only in its g_{rr} component:

$$g_{rr} = \tfrac{\lambda^2}{1 - \alpha' \frac{C^2}{8\lambda^2 r^4}} \, . \tag{3.39}$$

This Higgs field, metric, and Riemann curvature blow up at the finite radius $r_0^4 = \frac{\alpha' C^2}{8\lambda^2}$, indicating that there is a spherical source at this radius. The Higgs field becomes imaginary inside $r = r_0$, so our solution is unphysical inside this radius and cannot tell us about the interior. The radius r_0 depends on the size parameter λ such that in the large volume limit $\lambda \to \infty$, the singularity becomes concentrated near the origin. We also see from (3.39) that the appropriate double scaling limit to consider when sending α' to zero is to rescale the charge so that $\alpha' C^2$ remains finite.

There are no further corrections to the Higgs field or metric from the Hull-Strominger system. However, outside of the large charge limit the D-term equation itself is expected to receive further corrections at $O(\alpha'^2)$, so the exact solution to the $O(\alpha')$ equation may not be physically reliable at higher orders in the SUGRA expansion. One may also extend this analysis to the case of multiple monopole sources in a straightforward way.

We can also consider the solution close to a one-dimensional source. This is effectively a two-dimensional problem with cylindrical symmetry. The zeroth order solution now reads

$$\psi(\rho) = C \log(\rho) + \mathcal{O}(\alpha') \, , \tag{3.40}$$

where now C denotes the charge density along the source. The results for this case are very similar to the monopole case, but with the substitution $r^2 \to \rho$. In particular, the fully corrected Higgs field and metric are now

$$\psi'(r) = -\frac{C}{\sqrt{\rho^2 - \alpha' \frac{C^2}{8\lambda^2}}} \tag{3.41}$$

and

$$g_{\rho\rho} = \frac{\lambda^2}{1 - \alpha' \frac{C^2}{8\lambda^2 \rho^2}} \tag{3.42}$$

This solution blows up at a nonzero radius in the plane transverse to $\rho = 0$, indicating a cylindrical source surrounding this line.

One may also examine solutions of the reduced Hull-Strominger system with negative α', which have interesting behaviors, though their physical relevance is less clear. In this case, for the radially-symmetric solution, we find that the Higgs field is everywhere smooth, while the metric has a curvature singularity at the origin, but is smooth at $r = r_0$. We may imagine smoothly adjusting α' from a positive to a negative value and tracking the behavior of the Higgs field singularity along the way: for $\alpha' > 0$, there is a singular horizon at $r = r_0(\alpha')$, which contracts as we decrease α'. When $\alpha' = 0$, the singularity sits at the origin in the Higgs field only, and when we continue to $\alpha' < 0$, the singularity moves instead to the metric only, where it becomes a curvature singularity at the origin.

3.3.2 $SU(N)$ Solutions

Let us now look for solutions to the reduced Hull-Strominger system for non-abelian gauge fields. In particular, we will restrict ourselves to gauge group $SU(N)$, and our main example will have an $SU(2)$ gauge group. We will first consider the configurations at tree-level in α', and then discuss α'-corrections at the end of the section.

Our complex connection on $\mathbb{R}^3$ is given by

$$\mathrm{d}_{A^x} = \mathrm{d} + A^x \,, \tag{3.43}$$

where A^x is a one-form valued in the Lie-algebra of $SU(2)$, i.e. the span of the anti-Hermitian Pauli matrices (in math conventions). A complex connection $\mathcal{A}$ can now be constructed as

$$\mathcal{A} = A^x + i\, A^\phi \,, \tag{3.44}$$

where we require both A^x and A^ϕ to be anti-Hermitian, valued in $\mathfrak{su}(2)$ with legs now on the three-dimensional base. This implies that $\mathcal{A}$ takes values in the Lie algebra of $SL(2,\mathbb{C})$. The flatness condition on $\mathcal{A}$ then implies that

$$\mathcal{A} = G^{-1}\mathrm{d}G \,, \tag{3.45}$$

where $G \in SL(N,\mathbb{C})$, at least locally. Because we are working on a local model, we may take (3.45) as our ansatz. We thus have the real and imaginary parts of $\mathcal{A}$ given by

$$A^x = \tfrac{1}{2}\left(G^{-1}\mathrm{d}G - \mathrm{d}G^\dagger (G^\dagger)^{-1}\right) \tag{3.46}$$

$$A^\phi = -\tfrac{i}{2}\left(G^{-1}\mathrm{d}G + \mathrm{d}G^\dagger (G^\dagger)^{-1}\right) \,. \tag{3.47}$$

We now use the polar decomposition which states that any invertible matrix can be represented as

$$G = HU \,, \tag{3.48}$$

where U is unitary and H is Hermitian matrix of positive eigenvalues. When G has unit determinant, which we will assume, both H and U may be chosen to have unit determinant as well, so that $U \in SU(N)$. A transformation

$$G \to G\tilde{U} \,, \tag{3.49}$$

where $\tilde{U} \in SU(N)$ transforms the gauge field and Higgs field as

$$A^x \to \tilde{U}^\dagger A^x \tilde{U} + \tilde{U}^\dagger \mathrm{d}\tilde{U} \tag{3.50}$$

$$A^\phi \to \tilde{U}^\dagger A^\phi \tilde{U} \,. \tag{3.51}$$

This then corresponds to usual gauge transformations. We can use this to make G Hermitian, since there is always a gauge where

$$G = H \, . \tag{3.52}$$

In this gauge, the gauge field and Higgs field read

$$A^x = \tfrac{1}{2}[H^{-1}, \mathrm{d}H] \tag{3.53}$$

$$A^\phi = -\tfrac{i}{2}\{H^{-1}, \mathrm{d}H\} \, , \tag{3.54}$$

where curly brackets denote the anti-commutator. An anti-Hermitian matrix of unit determinant has $N^2 - 1$ degrees of freedom (the dimension of $SU(N)$). The co-closure condition

$$d^\dagger_{A^x} A^\phi = 0 \tag{3.55}$$

then gives a non-linear second order differential equation to be solved for H.

$SU(2)$ Solutions

Now let us specialize to $N = 2$. We may further parameterize H as

$$H = H_0 + c\, I_2 \, , \tag{3.56}$$

where H_0 is traceless, $c \in \mathbb{R}$, and I_2 is the rank-2 identity matrix. We expand H_0 in Pauli matrices as

$$H_0 = \sum_i a_i \sigma_i \, , \tag{3.57}$$

The condition that H has unit determinant is then

$$c^2 - \sum_i a_i a_i = 1 \, . \tag{3.58}$$

We are hence left with an overall number of four parameters $\{c, a_1, a_2, a_3\}$ describing the complex flat connection, subject to the constraint (3.58).

In terms of H, the one-forms A^x and A^ϕ now read

$$A^x = -\tfrac{1}{2}[H_0, \mathrm{d}H_0] \tag{3.59}$$

$$A^\phi = i\left(H_0 \mathrm{d}c - c\, \mathrm{d}H_0\right) \, , \tag{3.60}$$

where we have used the relation (3.58), which implies that

$$\mathrm{d}(H_0^2) = \mathrm{d}(c^2) I_2 \, . \tag{3.61}$$

We now come to the tree-level stability equation

$$\partial_i A_i^\phi + [A_i^x, A_i^\phi] = 0 \, . \tag{3.62}$$

Plugging in (3.59) and (3.60) into this equation, we find the following nonlinear differential equation

$$-c^2 \Delta a_m + (\Delta c) a_m + 2a^2 \partial_i c \partial_i a_m - \partial_i c \partial_i (a^2) a_m - c \partial_i (a^2) \partial_i a_m + 2c(\partial_i a_j)(\partial_i a_j) a_m = 0 \, , \tag{3.63}$$

where $a^2 = a_j a_j$ and $\Delta = \partial_i \partial_i$. Multiplying by c and using (3.58), this can be simplified a bit to

$$-c^2\Delta a_m + (c\Delta c)a_m - \tfrac{1}{2}\partial_i(a^2)\partial_i(a^2)a_m - \partial_i(a^2)\partial_i a_m + 2c^2(\partial_i a_j)(\partial_i a_j)a_m = 0\,. \quad (3.64)$$

Recall that the function c is determined by the a_i's through (3.58).

This is a rather complicated nonlinear differential equation, and it is not practical to find a general solution. One might try to simplify matters by, for example, assuming that the field c can be taken to be constant. This leads to the simpler equation

$$\Delta a_m - 2(\partial_i a_j)(\partial_i a_j)a_m = 0\,, \quad (3.65)$$

where by (3.58) we have used that a^2 will also be constant in this case. Contracting this equation by a_m and using (3.58) again, we find

$$(\partial_i a_j)(\partial_i a_j)(2a^2 + 1) = 0\,, \quad (3.66)$$

which can only be satisfied if $\partial_i a = 0$. We conclude that in order to have nontrivial solutions to (3.64), we need $\partial_i c \neq 0$.

3.3.3 Monopole-Type Solution

We would like to find a Hermitian matrix that satisfies the constraint (3.58) and solves equation (3.62). As we have seen, the general equation is quite complicated, but the hope is that we can find a clever parameterization of H which solves the system. To get a foothold, let us consider again (3.58). A convenient parameterization of c and

$a^2 = a_i a_i$ then reads

$$c = \cosh(u(x^i)) \tag{3.67}$$

$$a^2 = \sinh(u(x^i)) \tag{3.68}$$

for some function $u(x^i)$. We will assume that $u(x^i)$ is radially symmetric, so that it depends only on the radius r. We then write

$$H_0 = \sinh(u(r))\tilde{a}_i\sigma_i \; , \tag{3.69}$$

where we have introduced the normalized $\tilde{a}_i$ so that $\tilde{a}_i\tilde{a}_i = 1$. We want to allow the $\tilde{a}_i$ to depend on coordinates other than r, since otherwise the field A^ϕ will square to zero and the curvature is flat everywhere. We may try a simple ansatz inspired by the t'Hooft-Polyakov monopole [124, 125]:

$$\tilde{a}_i = \hat{x}^i = \tfrac{x_i}{r} \; . \tag{3.70}$$

In this case, the eigenvalues of H are given by $e^{\pm u}$.

Plugging this ansatz into the equations and solving the system using Mathematica, we find that the reduced Hull-Strominger system is indeed solved provided the function u satisfies the equation

$$2r^2\,\Delta u(r) = \sinh(4u(r)) \; . \tag{3.71}$$

where Δ is the Laplacian. Equation (3.71) becomes even simpler if we view it in terms of the inverse variable

$$t = \tfrac{1}{r} \; , \tag{3.72}$$

which gives

$$2t^2\, u''(t) = \sinh(4u(t))\,. \tag{3.73}$$

Note that this equation implies that the second derivative of $u(t)$ always takes the sign of $u(t)$. In particular, a solution that tends to zero at large r must necessarily blow up at some small r. To see this, assume that $u(t) \to 0$ as $t \to 0$. Then the equation to first order in t is

$$4u(t) - 2t^2\, u''(t) = 0\,, \tag{3.74}$$

with solution

$$u(t) = C_1\, t^2 + \tfrac{C_2}{t}\,. \tag{3.75}$$

To avoid a singularity at $t = 0$, we must set $C_2 = 0$. The remaining solution will continue to grow for larger t, i.e. as r tends to zero.

Equation (3.75) is a good approximation for the solution in a region of t when $u(t)$ is small, but the non-linear effects in (3.73) from the hyperbolic sine will sooner or later come into play. Numerical results suggest that these non-linear effects force the solution to blow up at finite t. Thus any solution that tends to zero at $t = 0$ will at best be defined on an interval of the form $(0, t_1)$, while a solution that tends to zero at $t = \infty$ is at best defined on an interval (t_1, ∞), where the solution blows up at $t_1 > 0$. Switching back to the r coordinate, it is also interesting to note that for solutions that tend to zero at the origin, we have u growing linearly in r away from zero. Hence, the eigenvalues of H are not smooth at the origin for such solutions, although it can be checked that the complex flat connection is nonetheless well-defined and smooth. See Figure 3.1 for a plot of an example solution.

We have sources where $u(r) \to \pm\infty$, meaning that the sources are spherical, as in the earlier α'-corrected abelian example. We will consider later what happens to the solution

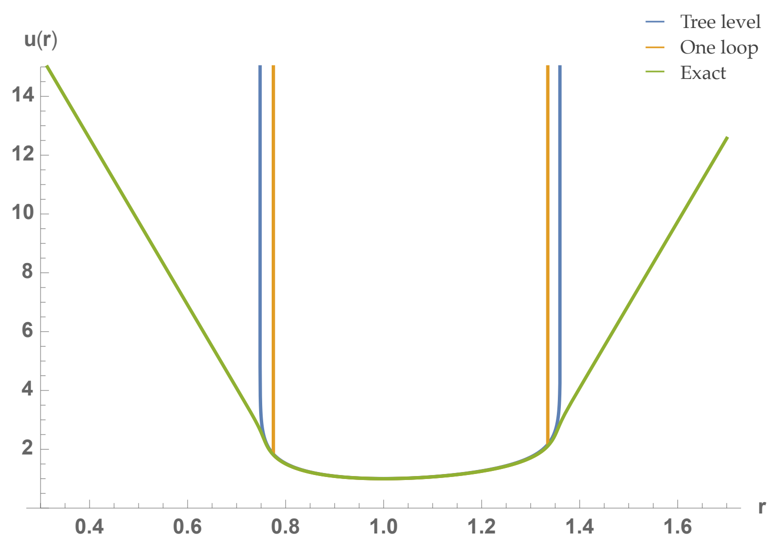

Figure 3.1: A particular solution to the D-term equation for the boundary condition $u(1) = 1$, $u'(1) = 0$, with $\alpha' = 0.01$. At zeroth order and first order in α', the solutions show similar behavior, with two finite-r singularities. The exact solution instead removes the small-r singularity and pushes the other to $r \to \infty$.

as we approach such sources.

α'-Corrections

Now we will add the α'-corrections to the monopole-type solution. To do so, we keep the same ansatz for the complex flat connection, but we modify our D-term equation to (3.24). Again, this matrix equation reduces to an ODE for the function $u(r)$:

$$\Delta_{\tilde{g}} u(r) = \frac{512 r^2 \sinh(4u(r)) + 8\alpha' \sinh^2(2u(r))\big(4u(r) + \sinh(4u(r))\big)}{\big(32 r^2 + \alpha' \sinh^2(2u(r))\big)^2} , \tag{3.76}$$

where $\Delta_{\tilde{g}}$ is the Laplacian with respect to the α'-corrected metric defined in (3.23).

Unlike the abelian case, it is difficult to define a large-charge limit. The α'-corrections are generically not one-loop exact, and we find qualitatively different results when solving the equations to $\mathcal{O}(\alpha')$ or exactly in α'. This is as expected, because the Hull-Strominger system includes only the first order correction in α', and higher order effects are expected to enter from supergravity and gauge theory sectors beginning at $\mathcal{O}(\alpha'^2)$.

Numerical solutions of the equation truncated to first order in α' suggest that the singularity behavior for solutions to the one-loop equation are similar to that of the uncorrected equation, with generic solutions existing on an interval (r_1, r_2) (see Figure 3.1). Meanwhile, numerical solutions to the exact D-term equation have no singularities in $u(r)$ or its first derivative for finite r, although $u(r)$ blows up linearly as $r \to \infty$. The solution approximates the zeroth order solution and α'-corrected solution well in the interval (r_1, r_2). Thus, when compared to the $\alpha' = 0$ solution, the exact solution to the α'-corrected equation seems to smear the non-abelian sources such that the finite-r singularities disappear. The exact α'-corrected metric exhibits interesting behavior as well, as it becomes approximately anti-de Sitter outside of the region (r_1, r_2). However, these behaviors must be interpreted with caution because of the unknown higher-order α'-corrections.

For additional analysis of the D-term equation, we will work in a different gauge, where instead of choosing the complex gauge transformation G to be Hermitian, we choose it to be of the form

$$G = UD \tag{3.77}$$

with $U \in \mathrm{SU}(2)$ and D a diagonal matrix with positive eigenvalues. For the monopole-type solution, these matrices are

$$D = \begin{pmatrix} e^{u(r)} & 0 \\ 0 & e^{-u(r)} \end{pmatrix}, \quad U = \begin{pmatrix} \cos(\theta/2) & e^{i\phi}\sin(\theta/2) \\ -e^{-i\phi}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \tag{3.78}$$

where (θ, ϕ) are the usual angles of $\mathbb{R}^3$. The function $u(r)$ satisfies the same D-term equation (3.76). In this gauge, the complex flat connection is

$$\mathcal{A} = \begin{pmatrix} -iu'(r)\mathrm{d}r - \sin^2(\theta/2)\mathrm{d}\phi & \frac{1}{2}e^{i\phi}e^{-2u(r)}(-i\mathrm{d}\theta + \sin\theta\mathrm{d}\phi) \\ \frac{1}{2}e^{-i\phi}e^{2u(r)}(i\mathrm{d}\theta + \sin\theta\mathrm{d}\phi) & iu'(r)\mathrm{d}r + \sin^2(\theta/2)\mathrm{d}\phi \end{pmatrix}. \tag{3.79}$$

Near sources for the Higgs field, both $u(r)$ and $u'(r)$ blow up, so some components of $\mathcal{A}$ will blow up as well. We can examine the rates at which components of $\mathcal{A}$ blow up near the sources at $r = r_1$ and $r = r_2$ to determine the behavior of $\mathcal{A}$. In particular, the ratio of $u'(r)$ to $e^{2u(r)}$ determines whether the diagonal or off-diagonal components of $\mathcal{A}$ dominate in the limit. The behavior of $u(r)$ is controlled by the D-term equation, so we see that α'-corrections may influence the behavior of $\mathcal{A}$ near the sources. Numerical results suggest that

$$\mid u'(r)/e^{2u(r)} \mid \to c \tag{3.80}$$

for a constant c that is positive when $u(r)$ satisfies the tree-level or one-loop D-term equation. Thus, all terms in $\mathcal{A}$ are of the same order, and the solution is fully non-abelian near the singularities. We may also consider the exact solution to the α'-corrected D-term equation, for which there is no singularity at r_1 and $r_2 \to \infty$. In this case, we find that $c = 0$ for the singularity at ∞, so that the off-diagonal components dominate. Furthermore, the lower left component of $\mathcal{A}$ dominates the top right one, so that for a fixed (θ, ϕ), the connection sits asymptotically in an abelian subalgebra of $\mathfrak{sl}(2;\mathbb{C})$. However, the dependence of the differentials on θ and ϕ ensure that $\mathcal{A}$ sits in a different abelian subalgebra at every point on the celestial sphere.

The behavior of the Higgs field near sources determines what methods may be used to calculate its spectrum, as will be discussed in the next section. In the present case of the monopole-type solution, its non-abelian behavior near sources prevents us from calculating its spectrum directly, but we may reliably calculate its chiral index, as will be described below.

3.4 Localized Chiral Matter

Now we will consider the matter spectrum associated to non-abelian solutions to the reduced Hull-Strominger system. In the abelian case, the spectrum is computed via the relative cohomology of M_3 with respect to the sources for the Higgs field. For non-abelian solutions, the computation will not always be so straightforward, but in some cases we may apply the same techniques as in the abelian case.

For our spectrum computations, we will assume a complex flat connection $\mathcal{A}$ with trivial holonomy on the three-manifold M_3. We write our connection as $\mathcal{A} = G^{-1}\mathrm{d}G$ with a polar-decomposed gauge transformation $G = \tilde{U}H$, where $\tilde{U}$ is unitary and H is Hermitian with non-negative eigenvalues. We may additionally choose a gauge in which $G = UD$ for a different unitary U and a diagonal D.[3] We assume that $\mathcal{A}$ satisfies the D-term equation, so that it provides a solution to the reduced Hull-Strominger system. In this chapter we will assume that G is globally defined, but singular at points. The implications of this are discussed further below. This resembles the common assumption made for abelian Higgs fields $\phi = \mathrm{d}f$ where f is a global but singular function; other generalizations of this abelian setup have been studied in, for example, [103, 111].

The effect of α'-corrections on the spectrum computation is only to modify the metric on M_3, which may modify the spectrum, but the method itself is independent of which order in α' we consider. Thus, we will not choose a particular order in α' for this section. We leave the case of $\mathcal{A}$ with nontrivial monodromies for future work.

[3] Note that we will use the same symbols $\mathcal{A}$ and G for the connection and $SU(N)$-valued function for different choices of gauge. Because we will fix a certain gauge in each instance, this should not cause ambiguity.

3.4.1 Behavior Near Sources

Our analysis of the matter spectrum is dependent on the asymptotic form of the Higgs field near its sources. We may classify the behavior of such solutions near a source into three cases:

- **Type 1:** The connection $\mathcal{A}$ becomes abelian near the source, meaning that there exists an abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that the norm of $\mathcal{A}$ becomes dominated by the components along $\mathfrak{h}$ as one approaches the source. In this case, we may calculate the spectrum using methods analogous to those for an abelian solution.

- **Type 2:** The connection $\mathcal{A} = G^{-1}\mathrm{d}G$ does not become abelian near the source, but U remains nonsingular, where U is the unitary matrix in $G = UD$. In this case, the tools we develop to compute the spectrum using relative cohomology don't apply, but we may at least compute the chiral index reliably. We can do this by smoothly deforming away U to obtain an abelian Higgs field, which may change the spectrum, but not its index.

- **Type 3:** The connection $\mathcal{A}$ does not become abelian near the source and U becomes singular. In this case, there is currently nothing we can say about the spectrum.

In this section, we will first consider the computation of the spectrum for Type 1 solutions, and use deformation theory to address the chiral spectrum of Type 2 solutions. The cohomology methods used in this section are a generalization of those introduced for flat solutions in [103] and [104] to certain non-abelian solutions.

For a Type 1 solution with the Hermitian gauge choice, the flat connection may be written as $\mathcal{A} = H^{-1}\mathrm{d}H$ in terms of a Hermitian matrix H which approaches the factorized form

$$H \rightarrow \begin{pmatrix} \tilde{D} & 0 \\ 0 & \tilde{H} \end{pmatrix}, \tag{3.81}$$

where $\tilde{D}$ is diagonal and gives rise to the abelian part of the connection, which blows up near the source. The other block, $\tilde{H}$, is Hermitian and gives rise to the rest of the connection, which may remain non-abelian near the source, but has a vanishing contribution to the norm of $\mathcal{A}$ in this limit. This is the form we will assume a Type 1 solution approaches close to any source.

Note that for the particular set of equations studied here, i.e. non-abelian solutions to the reduced Hull-Strominger system, we do expect stringy corrections to the equations when we approach the sources. While the flatness condition will be unaffected, as it is derived from an F-term, the D-term stability equation will receive corrections. There is hence a small tubular neighborhood around any source wherein we do not know what equations we are solving, except that the complex connection remains flat. Since finding the true equations to solve is beyond the scope of this chapter, we will instead model the true solution within this neighborhood by a flat connection of the above form, i.e. with an abelian part containing the sources, plus a non-abelian part commuting with the singular part. This can always be done since the boundary of the tubular neighborhood is a Riemann surface Σ, and all maps from Σ into $SU(N)$ are homotopy equivalent.

3.4.2 Matter Field Excitations

As in the case of abelian solutions, fermions can be represented as poly-forms $\psi_{\mathcal{R}} \in \Omega^*(M_3, E_{\mathcal{R}})$, where $E_{\mathcal{R}}$ is the vector bundle associated to the representation $\mathcal{R}$ of $SU(N)$ or its complexification, $SL(N;\mathbb{C})$. Chiral fermions correspond to odd forms while anti-chiral fermions are even. They solve the Dirac equation

$$\mathcal{D}_{\mathcal{A}}\psi_{\mathcal{R}} = \mathrm{d}_{\mathcal{A}}\psi_{\mathcal{R}} + \mathrm{d}^{\dagger}_{\mathcal{A}}\psi_{\mathcal{R}} = 0\,. \tag{3.82}$$

We are interested in solutions that are localized appropriately away from the boundary of the geometry[4] or any singularities of the complex flat connection. We can then further impose that

$$\mathrm{d}_{\mathcal{A}}\psi_{\mathcal{R}} = 0\,, \quad \mathrm{d}^{\dagger}_{\mathcal{A}}\psi_{\mathcal{R}} = 0\,, \tag{3.83}$$

which follows from a simple integration by parts argument. We may also assume without loss of generality that $\psi_{\mathcal{R}}$ has a given form degree p. These equations are equivalent to

$$\mathrm{d}\left(G^{\alpha}\cdot\psi^{\alpha}_{\mathcal{R}}\right) = 0\,, \quad \mathrm{d}\left((G^{\alpha\dagger})^{-1}\cdot *\psi^{\alpha}_{\mathcal{R}}\right) = 0\,, \tag{3.84}$$

in a local patch $\mathcal{U}_{\alpha}$, where $G^{\alpha} \in \Gamma(\mathcal{U}^{\alpha}, SL(N;\mathbb{C}))$ is the local gauge transformation that gives rise to the flat connection $\mathcal{A}$. Here the dot denotes the action of G on the given representation. (For example, if $\psi_{\mathcal{R}}$ is in the fundamental representation, it is just matrix multiplication on a vector, whereas if $\mathcal{R} = \mathrm{Ad}(SU(N))$ it is the adjoint action.) The $*$ denotes the three-dimensional Hodge-star. The polar decomposition in a local patch is given as

$$G^{\alpha} = HU^{\alpha}\,, \quad (G^{\alpha\dagger})^{-1} = H^{-1}U^{\alpha}\,. \tag{3.85}$$

Hence, solving the equations (3.84) is equivalent to solving the equations

$$\mathrm{d}(H\cdot\tilde{\psi}_{\mathcal{R}}) = 0\,, \quad \mathrm{d}\left(H^{-1}\cdot *\tilde{\psi}_{\mathcal{R}}\right) = 0\,, \tag{3.86}$$

[4] We will define more clearly what we mean by appropriately localized away from boundaries below.

for $\tilde{\psi}_{\mathcal{R}} = U^\alpha \cdot \psi^\alpha_{\mathcal{R}}$. Note that we have dropped the superscript α as $\tilde{\psi}_{\mathcal{R}}$ is a *global object*, i.e. $\tilde{\psi}^\alpha_{\mathcal{R}} = \tilde{\psi}^\beta_{\mathcal{R}}$ on overlaps $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$. We will also drop the tilde from here on.

Our matrix function H is globally defined because we have assumed that $\mathcal{A}$ has trivial monodromies. But even if H were not global, we must still have

$$(H^\alpha)^{-1}\mathrm{d}H^\alpha = (H^\beta)^{-1}\mathrm{d}H^\beta \tag{3.87}$$

on overlaps $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$. Using this and the fact that H^α and H^β are positive definite matrices, we see that

$$\mathrm{d}(H^\alpha \cdot \psi_{\mathcal{R}}) = 0 \quad \Leftrightarrow \quad \mathrm{d}(H^\beta \cdot \psi_{\mathcal{R}}) = 0 \tag{3.88}$$

on overlaps. With this in mind we may as well take H^α and try to extend it to a full global solution H. The obstructions for doing so will be the monodromies of $\mathcal{A}$, which can be trivialized by removing a submanifold of positive codimension, analogous to a branch cut. Such an operation will modify the boundary conditions for the Higgs field, which affects the spectrum.

Now we will go to a gauge where $G = UD$, where U is unitary and D is diagonal. Again, because $\mathcal{A}$ has trivial monodromies, both U and D are globally defined. In this gauge we have

$$\mathrm{d}\left((UD) \cdot \psi_{\mathcal{R}}\right) = 0\,, \quad \mathrm{d}\left((UD^{-1}) \cdot *\psi_{\mathcal{R}}\right) = 0\,, \tag{3.89}$$

or equivalently

$$\mathrm{d}_A(D \cdot \psi_{\mathcal{R}}) = 0\,, \quad \mathrm{d}_A(D^{-1} \cdot *\psi_{\mathcal{R}}) = 0\,, \tag{3.90}$$

where $A = U^{-1}\mathrm{d}U$ is a flat connection. If D is regular on M_3, then solutions to the set of equations (3.86) are counted by the cohomology

$$H^*_{\mathrm{d}_A}(M_3;\mathcal{R}) \cong H^*_{\mathrm{d}}(M_3;\mathcal{R}) \ , \tag{3.91}$$

where the isomorphism is due to the fact that A is a globally trivial flat connection. However, as in the abelian case [103], if the eigenvalues of D blows up, we need to restrict to solutions $\psi_{\mathcal{R}}$ with appropriate vanishing properties at those singular loci. This means that we should compute a relative cohomology.

3.4.3 The Relative Cohomology

We are now in a position to define the relative cohomology in question. Let us rewrite the closure equation in analogy with the abelian case as

$$\mathrm{d}(D\tilde{\psi}) = 0 \ , \tag{3.92}$$

where we have defined

$$\tilde{\psi} = (D^{-1}UD)\cdot\psi \ , \tag{3.93}$$

and we have dropped the $\mathcal{R}$-label on the fields. Note that because U factorizes at the singularities, and in particular becomes the identity matrix for the eigenvalues corresponding to sources, the $SL(N;\mathbb{C})$-valued invertible matrix $D^{-1}UD$ is regular over the three-manifold. Indeed, in the region of singularities U becomes block-diagonal and the identity matrix for the given eigenvalues of D which blow up or vanish. As in the abelian case, we will require the component $\tilde{\psi}_i$ (or equivalently ψ_i) to vanish where the corresponding eigenvalue λ_i blows up.

The co-closure equation reads

$$\mathrm{d}(D^{-1} * \tilde{H} \cdot \tilde{\psi}) = 0\,, \tag{3.94}$$

where we have defined the Hermitian metric on the bundle

$$\tilde{H} = DUD^{-2}U^{\dagger}D\,. \tag{3.95}$$

We note that by the above reasoning, the metric $\tilde{H}$ is also regular over M_3. In particular, it also approaches a block-diagonal form near the sources where it becomes the identity matrix for the given eigenvalues of D which blow up or vanish. Again, as in the abelian case we require the component $*\tilde{\psi}_i$ (or equivalently $*\psi_i$) to vanish where the corresponding eigenvalue λ_i goes to zero.

We now come to defining the relative cohomology. Let us first comment on what we take to be the domain of the component $\tilde{\psi}_i$ of $\tilde{\psi}$ corresponding to the ith eigenvalue of D. As in the abelian case, we will take this to be $M_3^i = M_3 \setminus \Delta_i^+ \cup \Delta_i^-$, where Δ_i^+ and Δ_i^- are small tubular neighborhoods of the positive and negative sources corresponding to $\lambda_i \to \infty$ and $\lambda_i \to 0$ respectively. We can then define the inner-product

$$(\psi^1, \psi^2) = \sum_i \int_{M_3^i} \overline{\psi}_i^1 \wedge *(\tilde{H} \cdot \psi^2)_i\,, \tag{3.96}$$

where the bar denotes complex conjugation. We will denote $\partial\Delta_i^{\pm} = \Sigma_i^{\pm}$. Note that as we approach these boundaries, $(\tilde{H} \cdot \psi^2)_i \to \psi_i^2$.

Note that the harmonic types of $\mathcal{R}$-valued k-forms $\tilde{\psi}$ we are considering (where $\tilde{\psi}$ vanishes when restricted to Σ^+ and $*\tilde{\psi}$ vanishes when restricted to Σ^-)[5] form part of a

[5]Note that a form vanishing when restricted to a sub-manifold Σ does not imply that its Hodge-dual will vanish as well, because normal components of the form might still be non-zero.

Hodge-type decomposition of forms

$$\{\tilde{\psi}\} \oplus \{\mathrm{d}_D\beta\} \oplus \{\mathrm{d}_D^{\tilde{\dagger}}\gamma\}\,, \tag{3.97}$$

with respect to the above inner-product. Here β is an $\mathcal{R}$-valued $(k-1)$-form and γ is an $\mathcal{R}$-valued $(k+1)$-form. However, this does not span all the allowed forms, as restrictions are put on $\tilde{\psi}$, β, and γ at the boundaries. Here $\mathrm{d}_D = D^{-1} \circ \mathrm{d} \circ D$, and $\mathrm{d}_D^{\tilde{\dagger}} = \tilde{H}^{-1} D \circ \mathrm{d}^{\dagger} \circ D^{-1}\tilde{H}$ is the adjoint of d_D with respect to the above inner product. In addition to the above restrictions on $\tilde{\psi}$, we also restrict β_i to vanish at the positively charged boundaries Σ_i^+, while $*\gamma_i$ vanishes at the negatively charged boundaries Σ_i^-. We can confirm that the individual components are orthogonal with respect to the inner product. For example,

$$(\tilde{\psi}, \mathrm{d}_D^{\tilde{\dagger}}\gamma) = \sum_i \int_{M_3^i} \overline{\tilde{\psi}}_i \wedge *(D\mathrm{d}^{\dagger} D^{-1}\tilde{H}\cdot\gamma)_i = -\sum_i \int_{\Sigma_i^+\cup\Sigma_i^-} \overline{\tilde{\psi}}_i \wedge *(\tilde{H}\cdot\gamma)_i\,,$$

where we have integrated by parts and used that $\tilde{\psi}$ is harmonic and so d_D-closed. Because the Hermitian metric $\tilde{H}$ becomes the identity on the boundaries, we end up with

$$(\tilde{\psi}, \mathrm{d}_D^{\tilde{\dagger}}\gamma) = -\sum_i \int_{\Sigma_i^+\cup\Sigma_i^-} \overline{\tilde{\psi}}_i \wedge *\gamma_i = 0\,, \tag{3.98}$$

since $\tilde{\psi}_i$ vanishes at Σ_i^+ while $*\gamma_i$ vanishes at Σ_i^-. It can be checked that the other terms in the Hodge decomposition are similarly orthogonal.

We then claim that the $\tilde{\psi}$ that give rise to stable complex flat connections are in one-to-one correspondence with the relative cohomology classes of d_D, which acts on forms that vanish on the positive boundaries exactly as in the abelian case. Indeed, consider a d_D-closed form α which vanishes at the positive boundaries and is orthogonal with respect to the above inner product to the set $\{\mathrm{d}_D\beta\}$ in the above Hodge decomposition. We require

$$0 = (\mathrm{d}_D\beta, \alpha) = (\beta, \mathrm{d}_D^{\tilde{\dagger}}\alpha) + \sum_i \int_{\Sigma_i^-} \overline{\beta}_i \wedge *\alpha_i \,. \tag{3.99}$$

If this is to vanish for all β (which vanish appropriately at positive boundaries), we see that we need to require $\mathrm{d}_D^{\tilde{\dagger}}\alpha = 0$ in addition to $*\alpha_i$ vanishing at the negative boundaries Σ_i^-. Hence, α is harmonic with respect to the above definition. The one-to-one correspondence between harmonic forms and cohomology classes follows.

So we see that to find the spectrum we proceed just as in the abelian case. We find the chiral and anti-chiral modes are counted using the eigenvalues of $\log(D)$ in the appropriate representation as Morse functions when computing the relative cohomology. We may hence compute the number of chiral zero-modes $N_{\chi\mathcal{R}}$ and anti-chiral zero-modes $N_{\overline{\chi}\mathcal{R}}$ in this representation as

$$N_{\chi\mathcal{R}} = \sum_{i=1}^{n_\mathcal{R}} h^1(M_3^i, \Sigma_i^+) \tag{3.100}$$

$$N_{\overline{\chi}\mathcal{R}} = \sum_{i=1}^{n_\mathcal{R}} h^2(M_3^i, \Sigma_i^+) \,, \tag{3.101}$$

where we also note that, as in the abelian case, the zeroth and third order cohomologies vanish. Indeed, for the zeroth order cohomology, for instance, we are again looking for constant functions which vanish at the boundaries of the charged regions, or global harmonic functions if the Morse functions for the given representation are regular. This can only happen if the function vanishes. Further details of how to compute these cohomologies for a given source configuration are given in [103, 104].

The index counting the net chirality is then

$$\mathrm{Index}(\mathcal{D}_\mathcal{R}) = N_{\chi\mathcal{R}} - N_{\overline{\chi}\mathcal{R}} \tag{3.102}$$

for the Dirac operator $\mathcal{D}_{\mathcal{R}}$ in the representation $\mathcal{R}$. We finish this section by remarking that even if the above-mentioned factorization of the complex flat connection into an abelian and a non-abelian part near the charged sources should not happen for a particular solution, we still expect the net chiral index to be counted by equation (3.102). Indeed, as we will see below, given a complex flat connection in terms of an $SL(N;\mathbb{C})$-valued matrix $G = UD$, assuming a regular U-matrix so we are not infinitely far away from a diagonal G in deformation space, and keeping the behavior of D near the sources fixed, we can always smoothly deform G to be diagonal. On general grounds we expect the index to be topological, and hence insensitive to such deformations. In the end, we expect the index to compute the number of massless modes in $\mathcal{R}$, and we expand on this in Section 3.5.

A Comment on Yukawa Couplings

Methods for computing Yukawa couplings have been developed in the case of stable abelian complex flat connections using the gradient flow trajectories of the Morse functions [103, 104]. These Yukawa couplings are given by multi-linear maps from the cohomologies computing the chiral spectrum into $\mathbb{C}$. For example, the third order couplings are of the form

$$\text{Yuk}(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3) : H^1(M_3; \mathcal{R}_1) \times H^1(M_3; \mathcal{R}_2) \times H^1(M_3; \mathcal{R}_3) \to \mathbb{C}\,, \tag{3.103}$$

where there must be at least one non-trivial singlet in the tensor product $\mathcal{R}_1 \otimes \mathcal{R}_2 \otimes \mathcal{R}_3$. The Yukawa couplings in the non-abelian case are given by similar multi-linear maps. Using the isomorphism between the cohomologies, we can hence map a non-abelian Yukawa to an abelian Yukawa, where we can use the gradient flow method to compute the coupling.

3.4.4 Deforming Solutions

Having seen in the previous section that abelian and non-abelian solutions are closely related in the way the chiral spectrum is computed, one may wonder if one can deform a given non-abelian background to an abelian one. Equivalently, are the non-abelian solutions in question simply deformations of abelian ones? We will argue here in the affirmative of this, at least for non-abelian solutions that are well-behaved enough. As before, assume an $SU(N)$-valued function $G = UH$ in polar decomposed form, where U is unitary and H is a positive matrix. We may write $H = \tilde{U}^{-1}D\tilde{U}$ for a unitary matrix $\tilde{U}$ and a diagonal matrix D of positive eigenvalues. Now let $U' = U\tilde{U}^{-1}$, so that $G = U'DU$. Our 'well-behaved' assumption is then that U' is non-singular, which corresponds to the Type 2 classification described at the beginning of this section.

To turn a non-abelian complex flat connection into an abelian one is then equivalent to turning off U' (as we may eliminate the other unitary matrix, U, by a gauge transformation). We note that a given U' corresponds to a map from M_3 into $SU(N)$ and hence has a representative homotopy class in $[M_3, SU(N)]$. For $M_3 = S_3$ this is $\pi_3(SU(N)) = \mathbb{Z}$. If the representative of U' is the trivial class, we can always turn off U' by deformation. If the class of U' is non-trivial, we can deform U' to be the identity matrix inside of a tubular neighborhood of the singularities of the eigenvalues of D. Indeed, we assume the singularities have co-dimension at least one in M_3, and U' can always be trivialized on a graph or Riemann surface. Simultaneously, we deform D to be the identity matrix outside of the tubular neighborhood. Thus, at every point, either U' or D is the identity matrix so that the deformed matrices commute. We may then write $G = DU'U$, and the unitary factor $U'U$ may be removed by a global gauge transformation, leaving G as diagonal and thus giving rise to an abelian connection.

In the deformations involved in the above steps, the chiral spectrum may be modified, but the chiral index should remain the same because it is invariant under smooth deformation. Thus, in the case of Type 2 non-abelian complex connections, we expect that the chiral index of the spectrum may be reliably computed by an abelian deformation.

The above describes a way of deforming a given non-abelian flat connection to an abelian one, or equivalently a deformation of an abelian solution to a non-abelian one. This is a solution to the Maurer-Cartan equation

$$\mathrm{d}_{\mathcal{A}}\alpha + \alpha \wedge \alpha = 0\,, \tag{3.104}$$

where α is the deformation of the connection. We keep the part of the connection which blows up at a source fixed at this source under such deformations. Recall that this part is assumed to become abelian, commuting with the rest of the connection. For the deformed connection to satisfy (3.17), we must also impose that the deformation of the co-closure condition holds. Before we consider this equation, we note that $\mathcal{A}$ should be thought of as a *holomorphic function* on the complex parameter space of complex connections. Indeed, as shown in [68], reduction of the six-dimensional holomorphic Chern-Simons functional gives the superpotential

$$W(\mathcal{A}) = \int_{M_3} \mathrm{tr}\left(\mathcal{A} \wedge \mathrm{d}\mathcal{A} + \tfrac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right)\,. \tag{3.105}$$

Hence, when considering such a holomorphic deformation Δ, we should set $\Delta\mathcal{A} = \alpha$ and $\Delta\mathcal{A}^\dagger = 0$. Imposing this on the co-closure condition for such deformations, we find

$$(\mathrm{d}_{\mathcal{A}})^\dagger \alpha = 0\,. \tag{3.106}$$

This equation imposes that the $\mathrm{d}_{\mathcal{A}}$-exact part of α in the Hodge decomposition

$$\alpha = \alpha_h + \mathrm{d}_{\mathcal{A}}\beta + \mathrm{d}_{\mathcal{A}}^{\dagger}\gamma \tag{3.107}$$

vanishes. Given a solution to the Maurer-Cartan equation (3.104), we can always do a Laurent series-type expansion

$$\alpha(Z) = Z^A \alpha_A + \tfrac{1}{2} Z^A Z^B \alpha_{AB} + \dots\,, \tag{3.108}$$

where Z^A denote holomorphic coordinates on the moduli space. If α then contains a d_A-exact part, it can be checked that this can be removed by redefining α order by order in this expansion, thus also solving the co-closure equation. Holomorphy and the Laurent theorem then guarantees that the resulting series can be re-summed.

We see that when considering a deformation in a holomorphic direction Δ, the first order deformations α_A are harmonic and hence counted by the relative cohomologies

$$N_{Ad(N)} = \sum_{i=1}^{n_{Ad(N)}} h^1(M_3^i; f^i_{\mathrm{Ad}(N)})\,, \tag{3.109}$$

where $f^i_{\mathrm{Ad}(N)}$ are the Morse functions of the adjoint representation $Ad(N)$ of $SU(N)$. In this chapter we have restricted to flat backgrounds of the form $\mathcal{A} = G^{-1}\mathrm{d}G$ where G is global (but with singularities) as a direct generalization of abelian solutions. We want to preserve this ansatz under deformations, so in general not all of the above deformations will be considered. To find the relevant deformations, consider part of the long exact sequence used to compute $H^i(M_3^i, \Sigma_i^+)$:

$$\dots \to H^q(M_3^i, \Sigma_i^+) \xrightarrow{i_p^*} H^q(M_3^i) \xrightarrow{p_p^*} H^q_{\mathrm{d}}(\Sigma_i^+) \xrightarrow{\alpha_p} H^{q+1}(M_3^i, \Sigma_i^+) \to \dots\,. \tag{3.110}$$

By the long exact sequence we have

$$H^1(M_3^i, \Sigma_i^+) \cong \mathrm{Im}(\alpha_0) \oplus \mathrm{Im}(i_1^*) \,. \tag{3.111}$$

The deformations which preserve the global triviality of the complex flat connection correspond to the modes in the image of the connecting homomorphism α_0 in the above direct sum. Assuming each M_3^i are connected, there are precisely $n_i^+ - 1$ such modes, where n_i^+ denotes the number of connected positively charged regions in M_3^i, i.e. the number of components where the given Morse function $f^i_{Ad(N)}$ blows up. As explained above, these directions are necessarily unobstructed.

Writing the deformation α as an $N \times N$-matrix, there is a Morse function corresponding to each element α_{pq}, which takes the form

$$f_{pq} = f_p - f_q \,, \tag{3.112}$$

where $f_p = \log(\lambda_p)$ is the Morse function of the pth eigenvalue of $\log(D)$ as an $N \times N$ matrix. We are hence counting the number of positive sources for these Morse functions. Note in particular that starting with an abelian solution, the Morse functions for the diagonal directions that keep the solution abelian vanish, and we need to turn on off-diagonal non-abelian directions in order to deform the solution non-trivially.

Had we instead considered real deformations $\Delta + \overline{\Delta}$ of the complex flat connection, the zeroth order co-closure equation becomes

$$\left(\mathrm{d}^\dagger_{\mathcal{A}}\alpha - \mathrm{d}^\dagger_{\mathcal{A}^\dagger}\alpha^\dagger\right) - [\alpha^m, \alpha^\dagger_m] = 0 \,. \tag{3.113}$$

At first glance, this equation may put potential obstructions on the deformations α. However, as we argued above, the zeroth order Morse cohomologies vanish. Using this fact, this equation may also be solved order by order by the usual methods of perturbative deformation theory.

3.5 Matter Spectrum Examples

Before we move to discuss examples, let us first see why in the end it is the index of the Dirac operator $\mathcal{D}_{\mathcal{A}}$ which counts the number of massless states in the final low-energy theory. In a heterotic compactification on a six-dimensional $SU(3)$ structure manifold X, the zero-modes in a representation $\mathcal{R}$ are counted by the cohomology $H^{(0,1)}_{\overline{\partial}_A}(X;\mathcal{R})$. Reality considerations of the decomposition of the gauge group implies that for $(0,1)$-modes in $\mathcal{R}$ we also expect to look for $(0,1)$-modes in $\overline{\mathcal{R}}$, i.e. counted by $H^{(0,1)}_{\overline{\partial}_A}(X;\overline{\mathcal{R}})$. Such modes can couple in the superpotential. For example, a fundamental mode a^c can couple with an anti-fundamental mode b_d via an adjoint mode $\alpha^d{}_c \in H^{(0,1)}_{\overline{\partial}_A}(X;\mathrm{End}(V))$ in a Yukawa coupling of the form

$$\mathrm{Yuk}(a,b,\alpha) = \int_X a^c \wedge b_d \wedge \alpha^d{}_c \wedge \Omega \,, \tag{3.114}$$

where Ω is the holomorphic top-form on X. Generically, such couplings are expected to remove modes in $\mathcal{R}$ and $\overline{\mathcal{R}}$ in pairs, such that the true massless spectrum is counted by the index of $\overline{\partial}_A$ in the given representation.

Lets see how this works in the reduced three-dimensional setting. In the three-dimensional theory, a $(0,1)$-mode in the representation $\mathcal{R}$ reduces to a one-form mode on the three-manifold M_3, i.e. an element of the cohomology $H^1(M_3;\mathcal{R})$. Note then that in six dimensions, taking the Hodge dual of a $\overline{\partial}_A$-harmonic $(0,1)$-modes in $\overline{\mathcal{R}}$ correspond to a ∂_A-harmonic $\overline{\mathcal{R}}$-valued $(2,3)$-form which also corresponds to a ∂_A-harmonic $\overline{\mathcal{R}}$-valued $(2,0)$-form. Complex conjugation then gives a $\overline{\partial}_A$-harmonic $\mathcal{R}$-valued $(0,2)$-form. In the reduction these give rise to $\mathcal{R}$-valued two-form modes on the three-dimensional space, i.e. elements of the cohomology $H^2(M_3;\mathcal{R})$. We therefore expect couplings between such one- and two-form modes, and the true massless spectrum is computed by the index of $\mathrm{d}_{\mathcal{A}}$ in the $\mathcal{R}$-representation.

3.5.1 Example: Breaking patterns of $SU(4)$

To get a feel for how this goes, let us consider breaking $SU(4)$ to $SU(3)$ by turning on an abelian Higgs field corresponding to an $SU(4)$ generator of the form $\log(D) = \mathrm{diag}(f, f, f, -3f)$, and consider the spectrum of $\mathrm{Ad}(SU(4))$. A similar example breaking $SU(6)$ to $SU(5)$ is given in [103]. We have

$$\mathrm{Ad}(SU(4)) = \mathrm{Ad}(SU(3)) + \mathbf{3}_1 + \overline{\mathbf{3}}_{-1} + \mathbf{1}\,. \tag{3.115}$$

The subscripts denote the charges under the (broken) $U(1)$. For such a configuration, one finds that the adjoint action of $SU(4)$ gives f as the Morse function counting modes in the fundamental representation $\mathbf{3}$, while $-f$ counts the modes in the anti-fundamental one $\overline{\mathbf{3}}$.

We can instead consider breaking $SU(4) \to U(1)$ by turning on a non-trivial $SU(3)$ within $SU(4)$, e.g. by choosing a non-abelian configuration with $\log(D) = \mathrm{diag}(f_1, f_2, -f_1 - f_2, 0)$. Chiral and anti-chiral modes of positive charge (transforming in the fundamental of the $SU(3)$) are then computed by the Morse cohomologies of f_1, f_2 and $f_3 = -f_1 - f_2$, while $-f_i$'s compute negatively charged modes. For example, one could imagine a model on S^3 where f_1 has a single positive point source and a single negative point source, while f_2 has a single positive source overlapping the one of f_1 and a single negative point source. Consider positively charged matter. A straight forward computation in relative cohomology shows that we get

$$h^1(M_3; f_3) = 1 , \tag{3.116}$$

while all other $h^{1/2}(M_3; f_i)$ vanish. Hence, such a model would give a single chiral mode of positive charge, and the chiral index in this representation is one.

3.5.2 Example: Monopole-type Solution

Consider the monopole-type solution on $\mathbb{R}^3$ described in Section 3.3.3 above. We focus for now on the zeroth and first order (non-exact) solutions, where the eigenvalues have finite radius singularities. For this case we have

$$D = \begin{pmatrix} e^u & 0 \\ 0 & e^{-u} \end{pmatrix} , \quad U = \begin{pmatrix} \cos(\theta/2) & e^{i\phi}\sin(\theta/2) \\ -e^{-i\phi}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} , \tag{3.117}$$

where (θ, ϕ) are the usual angles of $\mathbb{R}^3$ and u satisfies the equation (3.71) in the tree-level case or the $O(\alpha')$ part of (3.76) in the α'-corrected case. Set $\lambda_1 = e^u$ and $\lambda_2 = e^{-u}$. We would like to compute the spectrum of the fundamental representation in $SU(2)$. Note that we then need to consider the modes of both $f_1 = u$ and $f_2 = -u$. We will consider some examples of configurations of u for this setup. But before we do, we note that the zeroth order solution in α' for this ansatz does not satisfy the assumption that the solution becomes abelian near the sources (though the exact α'-corrected solution does satisfy this criteria). The individual cohomology computations at zeroth order might hence be less trustworthy. Note however that as the matrix U is regular, we still expect the index computation to be reliable as we can smoothly deform U to the identity.

To avoid issues concerning the U being ill-defined at $r = 0$, let us consider an example where u tends to negative infinity at a small finite r and blows up at a larger r. The space is now a three-dimensional annulus, that is, an open ball with a hole at the origin.

It is easy to compute both

$$h^1(M_3; f_{1/2}) = 0\,, \quad h^2(M_3; f_{1/2}) = 0\,, \tag{3.118}$$

and so this geometry has no chiral or anti-chiral modes. Let us also consider a u that blows up at a small r and a large r. The space is again a three-dimensional annulus. There are again no zero modes for degree zero and three, but we now find for f_1

$$h^1(M_3; f_1) = 1\,, \quad h^2(M_3; f_1) = 0\,. \tag{3.119}$$

so we have a chiral zero mode. From a relative cohomology perspective, this mode corresponds to exact forms $\mathrm{d}f$ where f approaches different constant values on the boundaries $\partial^+ M_3$. However, if we consider $f_2 = -u$ we find

$$h^1(M_3; f_2) = 0\,, \quad h^2(M_3; f_2) = 1\,, \tag{3.120}$$

so the net chirality or index of the solution is zero.

Next, we consider a solution of f_1 that vanishes at infinity and blows up at finite radius. In order to avoid complications regarding harmonic modes on non-compact geometries, we assume that the solution can be embedded in a large three-sphere, with a flat metric where the solution is non-trivial. Hence our space is S^3 minus an open ball. Again, we find

$$h^1(M_3; f_{1/2}) = 0\,, \quad h^2(M_3; f_{1/2}) = 0\,, \tag{3.121}$$

and so the chiral index of the spectrum is trivial for this solution as well.

3.5.3 Chiral index of $SU(2)$ representations

The vanishing of the chiral index for $SU(2)$ representations is actually much more general. Indeed, consider some $SU(2)$ solution which we have deformed to an abelian solution by the procedure described above. Then, given a normalizable chiral mode $\psi_0 \in \Omega^1(M_3, \mathbf{2})$ of the form

$$\psi_0 = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} , \tag{3.122}$$

that solves the Dirac equation, it is clear that the anti-chiral mode

$$\tilde{\psi}_0 = \begin{pmatrix} *\psi_b \\ *\psi_a \end{pmatrix} \tag{3.123}$$

will solve the Dirac equation as well. Hence the Chiral index of the fundamental representation of $SU(2)$ solutions vanishes. It can be checked that this is also the case for the adjoint representation $\mathbf{3}$, and is thus also true for the singlet as

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} + \mathbf{3}\,. \tag{3.124}$$

Inductively, it is easy to convince oneself that the chiral index will vanish for all higher irreducible representations as well, by taking higher tensor products with the fundamental representation. Hence it does not appear that $SU(2)$ solutions on $\mathbb{R}^3$ or S^3 support a non-trivial chiral index in any representation.

One can imagine embedding $SU(2)$ into a larger group, and in this way achieve a spectrum of non-vanishing chiral index. For example, let us assume that in the example above, the fundamental representation is also charged with respect to a $U(1)$ whose Morse function is $2u$.[6] The solution where g tends to infinity at a small and a large r would then have two chiral modes and zero anti-chiral modes in the representation $\mathbf{2}_1$.

3.5.4 Example: Non-abelian T-brane solution of Barbosa et al.

Of course, it may happen that the complex flat connection cannot be written as $\mathcal{A} = G^{-1}\mathrm{d}G$ for a global (but singular) G. This is the case when $\mathcal{A}$ has monodromies of various sorts. An example of this kind turns out to be the non-abelian local solution of [105]. Let us briefly discuss this example now.

The non-abelian explicit solution constructed in [105] is an $SU(3)$ example on $\mathbb{R}^3$ with coordinates (x, y, t). The authors consider the decomposition $SU(3) \to SU(2) \times U(1)$ and the complex connection $\mathcal{A} = A^x + iA^\phi$ with

$$A^x = \begin{pmatrix} \frac{1}{2}\left(\partial_z f\,\mathrm{d}z - \partial_{\overline{z}} f\,\mathrm{d}\overline{z}\right) & 0 & 0 \\ 0 & -\frac{1}{2}\left(\partial_z f\,\mathrm{d}z - \partial_{\overline{z}} f\,\mathrm{d}\overline{z}\right) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.125}$$

$$A^\phi = \begin{pmatrix} \frac{i}{3}\mathrm{d}h & -v\overline{z}e^{-f(z,\overline{z})}\mathrm{d}\overline{z} + \varepsilon\, e^{f(z,\overline{z})}\mathrm{d}z & 0 \\ vze^{-f(z,\overline{z})}\mathrm{d}z - \varepsilon\, e^{f(z,\overline{z})}\mathrm{d}\overline{z} & \frac{i}{3}\mathrm{d}h & 0 \\ 0 & 0 & -\frac{2i}{3}\mathrm{d}h \end{pmatrix}, \tag{3.126}$$

where ε and v are real constants, $h(z, \overline{z}, t)$ is the function

$$h = \tfrac{\kappa}{8}(z+\overline{z})^2 - \tfrac{\kappa}{2}t^2 \tag{3.127}$$

for a real constant κ, and the D-term condition demands that $f(z, \overline{z})$ satisfies the differ-

[6]This Morse-function will generically obey a different co-closure equation, but we take it to be $2u$ for illustration purposes.

ential equation

$$\tfrac{1}{4}\left(f_{rr} + \tfrac{1}{r} f_r\right) = \varepsilon^2\, e^{2f} - v^2 r^2 e^{-2f}\,, \tag{3.128}$$

where $r = |z|$. In the case where $\varepsilon \neq 0$, this equation may be transformed into a Painlevé III differential equation, while in the $\varepsilon = 0$ case it becomes a modified Liouville equation. We may attempt to apply our methods to solutions of this form, but we will see that we run into obstacle for computing the spectrum for both choices of ε.

Case 1: $\varepsilon \neq 0$

To analyze this solution, we want to examine limiting forms of the connection near the sources. In the case of $\varepsilon \neq 0$, the only source is at infinity, so we consider the solution at large distances in the (x, y)-plane and at large t, with the intention of studying the asymptotic behavior of the Morse functions to determine a charge distribution at infinity. If we take the limit where $r = |z|$ and t are large, then the asymptotic behavior of the Painlevé III transcendental reveals the approximate form

$$\mathcal{A} \to \begin{pmatrix} -\tfrac{1}{3}\mathrm{d}h + \tfrac{\mathrm{d}z}{8z} - \tfrac{\mathrm{d}\overline{z}}{8\overline{z}} & -ipr^{-1/2}\overline{z}\mathrm{d}\overline{z} + ipr^{1/2}\mathrm{d}z & 0 \\ ipr^{-1/2}z\mathrm{d}z - ipr^{1/2}\mathrm{d}\overline{z} & -\tfrac{1}{3}\mathrm{d}h - \tfrac{\mathrm{d}z}{8z} + \tfrac{\mathrm{d}\overline{z}}{8\overline{z}} & 0 \\ 0 & 0 & \tfrac{2}{3}\mathrm{d}h \end{pmatrix}, \tag{3.129}$$

where $p = \sqrt{\epsilon v}$. With this asymptotic form of $\mathcal{A}$, the equation $\mathrm{d}G = G\mathcal{A}$ can be solved explicitly, and we find the asymptotic gauge transformation

$$G = \tfrac{i+\sqrt{3}}{2} \begin{pmatrix} e^{i\theta/4 - h(z,t)/3}\cosh s & -e^{-i\theta/4 - h(z,t)/3}\sinh s & 0 \\ ie^{i\theta/4 - h(z,t)/3}\sinh s & -ie^{-i\theta/4 - h(z,t)/3}\cosh s & 0 \\ 0 & 0 & e^{2h(z,t)/3} \end{pmatrix}, \tag{3.130}$$

where

$$s = \tfrac{4}{3} p r^{3/2} \sin(3\theta/2)\ . \tag{3.131}$$

As discussed above, in order to compute the spectrum as if the solution were abelian we need to check that the solution becomes asymptotically abelian at the sources. By examining the asymptotic form of $\mathcal{A}$, we see that it will be dominated by the diagonal, and thus asymptotically abelian, as long as we approach infinity in a direction where $\cos\theta \neq 0$. The directions in which the source does not become asymptotically abelian are of measure zero on the celestial sphere, but it is unclear if this fully justifies the use of abelian methods.

However, we also see that despite the fact that $\mathcal{A}$ is a flat connection on $\mathbb{R}^3$, our G is not single-valued due to the fractional dependence on θ, and instead should be defined on a four-sheeted cover. This means that we cannot work with a globally defined G, so that we cannot apply our method for computing spectra or the chiral index. It would be interesting to generalize the methods we have proposed to this setting.

As another route of investigation, we may examine the behavior of the solution near the origin. A solution for G to fourth order in the coordinates on $\mathbb{R}^3$ is given by

$$\begin{pmatrix} G_{11} & G_{12} & 0 \\ \overline{G_{12}} & G_{22} & 0 \\ 0 & 0 & G_{33} \end{pmatrix}\ , \tag{3.132}$$

where the entries are given by

$$G_{11} = 1 + \tfrac{k^2}{2} z\overline{z} - \tfrac{kl}{6}(2z^3 + \overline{z}^3) + \tfrac{\kappa}{24}(4t^2 - (z+\overline{z})^2)$$
$$G_{12} = ikz - \tfrac{il}{2}\overline{z}^2 + \tfrac{ik^3}{2} z^2\overline{z} + \tfrac{i\kappa}{24} kz(4t^2 - (z+\overline{z})^2)$$
$$G_{22} = 1 + \tfrac{k^2}{2} z\overline{z} - \tfrac{kl}{6}(z^3 + 2\overline{z}^3) + \tfrac{\kappa}{24}(4t^2 - (z+\overline{z})^2)$$
$$G_{33} = 1 - \tfrac{\kappa}{12}(4t^2 - (z+\overline{z})^2) \ ,$$

where the constants k and l are defined in terms of the constants ϵ, v, c, and κ of [105] as follows:

$$k = c\,\epsilon^{1/3} v^{1/3} \tag{3.133}$$
$$l = \frac{\epsilon^{2/3} v^{2/3}}{c} \ . \tag{3.134}$$

This G is single-valued, so we could, in theory, use it to compute the saddle point behavior of Morse functions near the origin, which could give a hint toward their behavior at the source. However, the matrix D in the gauge-fixed polar decomposition $G = UD$ is not smooth at the origin in this case, which again hampers the use of our methods.

From this, we may calculate the Morse functions in a neighborhood of the origin as

$$\tilde{f}_1 = \tfrac{1}{8}\left(\kappa\left((z+\overline{z})^2 - 4t^2\right) + \tfrac{8kz\overline{z} - 2l\left(z^3 + \overline{z}^3\right)}{\sqrt{z\overline{z}}}\right) \tag{3.135}$$
$$\tilde{f}_2 = \tfrac{1}{8}\left(\kappa\left((z+\overline{z})^2 - 4t^2\right) - \tfrac{8kz\overline{z} - 2l\left(z^3 + \overline{z}^3\right)}{\sqrt{z\overline{z}}}\right) \ . \tag{3.136}$$

These local Morse functions share the same saddle point behavior as the asymptotic Morse functions, reinforcing our confidence in their behavior. However, there may be additional saddle points at intermediate radii that are not seen by either of these approximations.

Case 2: $\varepsilon = 0$

We may also consider the $\varepsilon = 0$ case, where the D-term equation for $u(r)$ becomes a modified Liouville equation whose solution has a singularity at $r_0 = 1/\sqrt{v}$. This means that the domain of the solution is all of $\mathbb{R}^3$ exterior to a cylinder at $r = r_0$ that extends infinitely in the positive and negative t directions. Our method for computing the chiral matter is hampered in this case by the presence of a nontrivial monodromy around any loop enclosing the cylinder. The Wilson loop around a circular loop L_r at fixed r may be computed as

$$W(\mathcal{A}, L_r) = \mathrm{Tr}\exp\left(-\int_0^{2\pi} \mathrm{d}\theta \mathcal{A}_\theta\right) = 1 + 2\cos\tfrac{4\pi v^2 r^4}{1 - v^2 r^4} \ , \tag{3.137}$$

where the path ordering is trivial because the quantity in the exponent lies in an abelian subalgebra. In particular, this holonomy comes from the gauge field, while the Higgs field has trivial holonomy. As r approaches r_0, the value of the Wilson loop oscillates rapidly, so that the gauge field has a singularity at r_0 as does the Higgs field.

A consequence of nontrivial monodromy is that we cannot find a global solution G to the equation $\mathcal{A} = G^{-1}\mathrm{d}G$, which again means that the direct computation of the chiral spectrum would require more sophisticated techniques than those presented in the previous section.

Besides investigating the spectrum, we may also consider the first order α'-corrections to this example. For any choice of ε, the 2-form B introduced in (3.19) vanishes, so that the corrected D-term equation (3.24) remains in the same form as in the $\alpha' = 0$ case, but with respect to an α'-corrected metric given by (3.23). All components of the metric receive nontrivial corrections, so that the co-closure condition becomes a complicated differential equation.

3.6 Conclusions and Outlook

In this chapter, we have considered heterotic string compactifications on $SU(3)$-structure manifolds and their reduction to the equations for a stable complex flat connection on a three-dimensional submanifold. We saw that upon reduction of the heterotic equations, the complex connection remains flat, but the D-term co-closure condition gets corrected even at first order in α' due to torsional effects of the 6D Hull-Strominger geometry and the non-trivial heterotic Bianchi identity. In this chapter we have studied local solutions on $\mathbb{R}^3$ for the α'-corrected system, including both abelian and non-abelian examples of bundles and their back-reaction on the geometry. These solutions constitute new local T^3-invariant solutions to the Hull-Strominger system. It would be interesting to consider the reduced Hull-Strominger system also for more generic compact three-manifolds, and to investigate both the physical and mathematical/topological implications of these corrections.

We also introduced a way of computing the spectrum (or at least the index) of a particular set of non-abelian solutions with a non-flat gauge field, with the caveat that the complex flat connection $\mathcal{A} = G^{-1}\mathrm{d}G$ is given by a *global but singular matrix* G, similar to the common assumption made for abelian Higgs fields $\phi = \mathrm{d}f$ where f is a global but singular function.[7] Assuming an appropriate behavior of G near singularities (Type 1 or 2 in the classification of Section 4.1), the index computation resembles that of abelian Higgs fields. We find that a monopole-type nonabelian solution has vanishing chiral index, while the non-abelian local example of [105] does not have the correct behavior for our current methods to apply.

Another avenue to explore is to ask what these α'-corrections correspond to on the M-theory side, following the M-theory/heterotic duality. Given that α'-corrections corre-

[7] It would be interesting to see how our methods may generalize to the case when $\mathcal{A}$ has non-trivial monodromies.

spond to higher curvature corrections from a supergravity point of view, it is conceivable that they correspond to higher curvature corrections on the M-theory side as well, possibly with an incorporation of four-form G-flux. Since α'-corrections are vital for understanding properties of heterotic compactifications such as the moduli problem, Yukawa couplings, and moduli metric, both in the three-dimensional reduced system and also the upstairs Hull-Strominger geometry [133, 134, 135, 136, 137, 138, 139, 140], a better understanding of these corrections might therefore lead to better insight into such issues on the M-theory side as well.

Recently, much progress has been made in the study of the heterotic moduli problem [133, 134, 135, 136, 141, 140, 142, 143, 144, 145]. It would be interesting to reduce the corresponding moduli structures to the three-dimensional setting as well, and study the resulting equations generalizing that of the moduli problem of a complex flat connection. Indeed, the geometric structures and moduli problem in particular are expected to retain much of the important physical and mathematical properties of the upstairs geometry, hence the reason for doing such a reduction in the first place. Moreover, the three-dimensional system has the advantage of being much more explicit, which gives hope for a more hands on approach for understanding the geometric structures and moduli, particularly in terms of explicit solutions. It was recently discovered that the moduli problem of the Hull-Strominger system is governed by an interesting quasi-topological theory which has flavors of both Kodaira-Spencer and Donaldson-Thomas theory [142]. A reduction of this theory to three dimensions opens the door to more explicit computations, such as, for example, its partition function on the three-sphere.

Appendix A

An $N = 2$ Solution: The t'Hooft-Polyakov Monopole

Instead of the complex structure of the SYZ fibration, we can alternatively endow $\mathbb{R}^3 \times T^3$ with the following complex structure,

$$\begin{aligned} z^1 &= x^1 + i\, x^2 \\ z^2 &= x^3 + i\, \phi^1 \\ z^3 &= \phi^2 + i\, \phi^3\,. \end{aligned} \tag{A.1}$$

That is, the coordinate z^3 is the complex coordinate of an elliptic curve. This is more reminiscent of the local geometric structure required by heterotic duality with F-theory, and may thus be important for studying the local nature of this duality. The duality is again done fiber-wise, with the elliptic curve now playing the role of T^3 in the Hull-Strominger system reduction. We also point out that the reduced geometry we study in this example now preserves $N = 2$ supersymmetry rather than $N = 1$, and so a continuous deformation to the reduced system of a stable complex flat connection is unlikely to exist.

A reduction of the holomorphic Yang-Mills equations to $\mathbb{R}^3$, assuming a flat connection on the elliptic curve spanned by $\{\phi^2, \phi^3\}$, is then given by

$$F^x = *_3 \mathrm{d}_{A^x}\psi \,, \tag{A.2}$$

where we have defined $A^y = \psi(x)\mathrm{d}\phi^1$. This is precisely the equation satisfied by the t'Hooft-Polyakov monopole [124, 125]. The reduction of the Bianchi identity (3.7) now becomes

$$*_3\Delta_{\mathrm{d}} e^{\Phi} = \tfrac{\alpha'}{2}\mathrm{tr}\left(\mathrm{d}_{A^x}\psi \wedge *_3\mathrm{d}_{A^x}\psi\right) = \tfrac{\alpha'}{2}\mathrm{d} *_3 \left(\mathrm{tr}\left(\psi \mathrm{d}_{A^x}\psi\right)\right) = \tfrac{\alpha'}{4}\mathrm{d} *_3 \mathrm{d}\left(\mathrm{tr}\,\psi^2\right) \;, \tag{A.3}$$

where Φ is the dilaton, and the corresponding three-dimensional metric is conformally flat given by $g_{ij} = e^{\Phi}\delta_{ij}$. In the second equality, we have used (A.2) and the Bianchi identity for F^x. Equation (A.3) is solved by

$$e^{\Phi} = -\tfrac{\alpha'}{4}\,\mathrm{tr}\,\psi^2 + C \,, \tag{A.4}$$

for some constant C which can be thought of as an overall volume modulus.

The geometric system in this case is far simpler than the α'-corrected equations for a stable complex flat connection, and can indeed be solved exactly if we have solutions to (A.2), for example the exact solution of [146]. This is perhaps not surprising due to the enhanced supersymmetry.

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