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Randomly Perturbed Berezin—Toeplitz Operators

Author
Oltman, Izak

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by

Izak Oltman

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Professor Maciej Zworski, Chair
Professor Michael Christ
Professor Sung-Jin Oh

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Abstract

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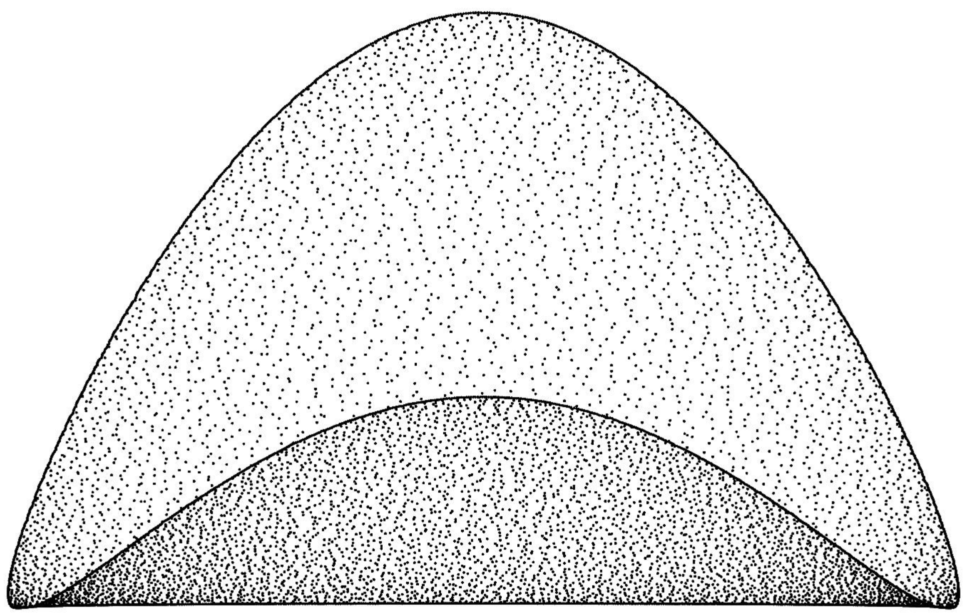
Berezin–Toeplitz operators are quantizations of functions on Kähler manifolds equipped with a positive line bundle L. When the Kähler manifold M is compact, the quantization procedure associates every smooth function $f \in C^\infty(M)$ to a family of matrices $T_N f$ indexed by $N \in \mathbb{N}$, whose size goes to infinity as $N \to \infty$ (corresponding to the semiclassical limit $h \to 0$). Each matrix $T_N f$ acts on the finite-dimensional space of holomorphic sections of the Nth tensor power of L.

In this thesis we study the spectrum of $T_N f + \delta \mathscr{G}_\omega(N)$ where $\delta = \delta(N) > 0$ and $\mathscr{G}_\omega(N)$ is a family of random matrices. Under certain conditions, the spectrum satisfies a probabilistic Weyl law involving f and the volume form on the Kähler manifold. Specifically, as $N \to \infty$, the (normalized) empirical spectral distribution of $T_N f + \delta \mathscr{G}_\omega(N)$ converges weakly almost surely to the (normalized) push-forward by f of the Liouville volume form on M. This generalizes a result of Martin Vogel [Vog20], which considered the case of torii.

Proving this result requires extending the usual calculus of Berezin–Toeplitz operators to an exotic class of functions. The exotic nature of these functions (classical observables) refers to the property that their derivatives are allowed to grow in ways controlled by local geometry and the power of the line bundle. The properties of this quantization are obtained via careful analysis of the kernels of the operators using Melin and Sjöstrand's method of complex stationary phase. For this more exotic class of functions, we obtain a functional calculus result, a trace formula, and a parametrix construction.

To my family and friends.



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Chapter 1

Introduction and statement of results

This thesis generalizes a theorem of Martin Vogel in [Vog20] which proved a probabilistic Weyl law for quantizations of functions on tori. Here we do the same but with the tori replaced by arbitrary compact Kähler manifolds equipped with positive line bundles.

In [Vog20], Vogel considered Toeplitz quantizations of smooth functions on a real $2d$-dimensional torus, which associates every smooth function f on the torus to a family of $N^d \times N^d$ matrices, $T_N f$, for all $N \in \mathbb{N}$ (here N^{-1} is the semi-classical parameter). Vogel proved that if a random matrix with sufficiently small norm is added to $T_N f$, then the spectrum obeys an almost-sure Weyl law as N goes to infinity. This was conjectured by Christiansen and Zworski in [CZ10] and is a major extension of their work.

This result is most striking when the unperturbed matrix is non-self-adjoint. For example, if $f(x) = \cos(2\pi x) + i\cos(2\pi\xi)$, then the quantization is

$$T_N f = \begin{pmatrix} \cos(2\pi/N) & i/2 & 0 & 0 & \cdots & i/2 \\ i/2 & \cos(4\pi/N) & i/2 & 0 & \cdots & 0 \\ 0 & i/2 & \cos(6\pi/N) & i/2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & i/2 & \cos(2(N-1)\pi/N) & i/2 \\ i/2 & 0 & \cdots & 0 & i/2 & \cos(2\pi) \end{pmatrix},$$

which numerically has spectrum contained on two crossing lines in the complex plane. This operator is aptly named the Scottish flag operator and is further described by Embree and Trefethen in [ET05]. Interestingly, despite numerical evidence, there is no proof that the spectrum of $T_N f$ is contained on these crossing lines for general N. However, this is being addressed in ongoing work by the author with Frédéric Klopp and Shengtong Zhang [KOZon]. If randomly perturbed, the spectrum spreads out with density given by the push-forward of the Lebesgue measure on the torus by f. Figure 1.1 plots the spectrum of $T_N f$ with no perturbation and with a small perturbation. We can observe this same effect numerically occurring when computing the spectrum of $T_N f$ for large values of N, suggesting that rounding errors in mathematical software behave like small random perturbations. For $N < 100$, numerically computed eigenvalues appear to lie on two crossed lines. For $N > 100$, the numerically computed spectrum spreads out. For even $N < 12$, the eigenvalues can analytically be shown to lie on the two crossed lines using Mathematica.

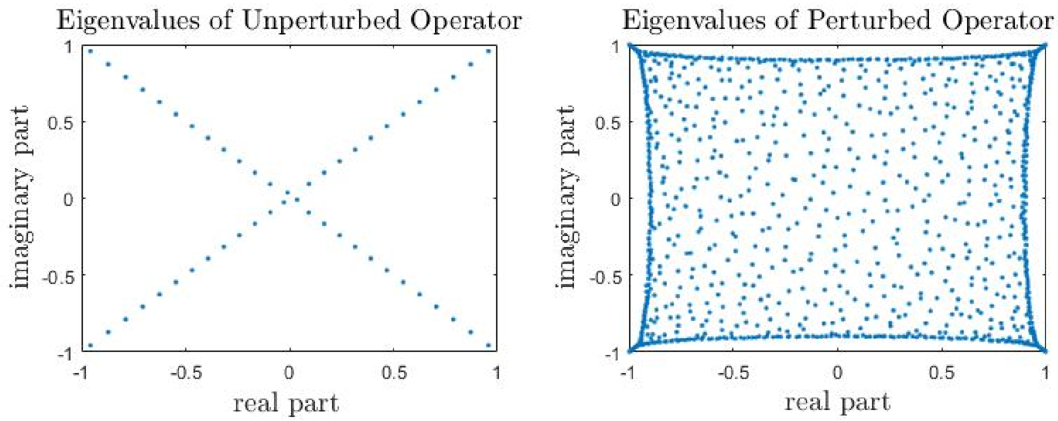

Figure 1.1: Left: Numerically computed eigenvalues of the Scottish flag operator with $N = 50$. Right: Numerically computed eigenvalues of the Scottish flag operator with a small random perturbation with $N = 1000$.

The spectral properties of randomly perturbed non-self-adjoint operators were pioneered by Mildred Hager in [Hag06], in which the operator $hD_x + g(x)\colon H^1(\mathbb{T}^1) \to L^2(\mathbb{T}^1)$ was studied. This result, and numerous subsequent results, were presented by Sjöstrand in [Sjö19]. There are related results describing spectral properties of randomly perturbed Toeplitz matrices, which can be defined as quantizations of symbols on $\mathbb{T}^2$ with symbol independent of x. See Davies and Hager [DH09], Guionnet, Wood and Zeitouni [GWZ14], Sjöstrand and Vogel [SV21a; SV21b], and references given there.

The work in this thesis is a natural generalization of Vogel's result in [Vog20]. We prove a similar result for quantizations of functions on Kähler manifolds (with sufficient structure, as discussed in Chapter 2). These quantizations, called Berezin–Toeplitz operators (or just Toeplitz operators) were first described by Berezin in [Ber75] as a particular type of quantization of symplectic manifolds. Following [Ber75], for every smooth function f on a quantizable Kähler manifold X, we get a family of finite rank operators, $T_N f$, indexed by $N \in \mathbb{N}$ (see [Rou17] for a connection between these quantizations, and quantizations on the torus) which have physical interpretations. Deleporte in [Del19, Appendix A] related this quantization to spin systems in the large spin limit, and Douglas and Klevtsov in [DK10] used path integrals for particles in a magnetic field to describe the Bergman kernel (a key ingredient in constructing $T_N f$).

Next, if we add a small Gaussian-type random perturbation $\mathscr{G}_\omega(N)$ to these operators (see Definition 4.1.3), their empirical spectral measures weakly converge almost surely (see Theorem 4.1.4 in §4.1 for a precise statement). Theorem 4.1.5 states a result about more general random perturbations $\mathscr{W}_\omega(N)$ (see Definition 4.1.3) but with a more restrictive coupling constant. A consequence of Theorem 4.1.4 is the following probabilistic Weyl law.

Theorem 1.0.1 (A probabilistic Weyl law)**.** *Suppose we have a quantizable Kähler manifold X, a function $f \in C^\infty(X;\mathbb{C})$ such that there exists $\kappa \in (0,1]$, $\delta > 0$ so that*

$$\mu_d(\{x \in X : |f(x) - z|^2 \leq t\}) = \mathscr{O}(t^\kappa) \tag{1.0.1}$$

for $t < \delta$ uniformly for $z \in \mathbb{C}$ (where μ_d is the Liouville volume form on X), $\mathscr{G}_\omega(N)$ a family of $N \times N$ Gaussian-type random matrix (see Definition 4.1.3), and an open set $\Lambda \subset \mathbb{C}$. Then for any $p > d/2$, almost surely

$$\left(\frac{2\pi}{N}\right)^d \#\left\{\mathrm{Spec}(T_N f + N^{-p}\mathscr{G}_\omega(N)) \cap \Lambda\right\} \xrightarrow{N\to\infty} \mu_d(x \in X : f(x) \in \Lambda). \tag{1.0.2}$$

Here $\#A$ denotes the number of elements in a set A. It was observed in [CZ10] that if f is real analytic, then (1.0.1) holds. See [CZ10], and references presented there, for further discussion of (1.0.1).

Finer results are expected for describing the spectrum of randomly perturbed Toeplitz operators. In [Vog20], precise statements about the number of eigenvalues were obtained using counting functions of holomorphic functions. Here we only show weak convergence of the empirical measures, but achieve this in a relatively simple way using logarithmic potentials as presented in [SV21c].

Here we present numerical examples to motivate the main result of this thesis. Consider the Kähler manifold $\mathbb{CP}^1$ (complex projective space of dimension 1) which can be identified with the real 2-sphere with coordinates (x_1, x_2, x_3) (see Example 2.2.1 and Appendix B for details of the quantization procedure in this specific case). In Figure 1.2, we compute the spectrum of the quantization of the function $f(x_1, x_2, x_3) \coloneqq x_1 + 2x_2^2 + ix_2$. Before perturbation, the spectrum lies on several lines in the complex plane, somewhat analogous to the Scottish flag operator. However, as a perturbation is added, the spectrum spreads out. This thesis describes the structure of the spectrum of this perturbed operator in the semiclassical limit, as $N \to \infty$. A rotated version of the spectrum of a perturbation of $T_N(x_1 + 2x_2^2 + ix_2)$ with $N = 10{,}000$ is displayed on the dedication page of this thesis.

Numerical verification of this thesis' result can be seen if $f = ix_1 + x_2$ (still on $\mathbb{CP}^1$). Figure 1.3 displays the spectrum of $T_N f$ with a random perturbation added, and plots the number of eigenvalues in circles of increasing radii versus the predicted number of such eigenvalues by (1.0.2). More numerics are presented in Appendix B.3 and animations can be found on the author's website[1].

To put these results in context, we recall that Berezin introduced the concept of Toeplitz operators in [Ber75] to quantize smooth functions (classical observables) on smooth compact symplectic manifolds (classical phase spaces). This generalizes a more straightforward

[1] `https://math.berkeley.edu/~izak/research/toeplitz/movies.html` or YouTube (`https://www.youtube.com/watch?v=tBvpozWA3bY&list=PLlWY1dHxyEOc3ihYylV7Mcx0GevGGduis&pp=gAQBiAQB`)

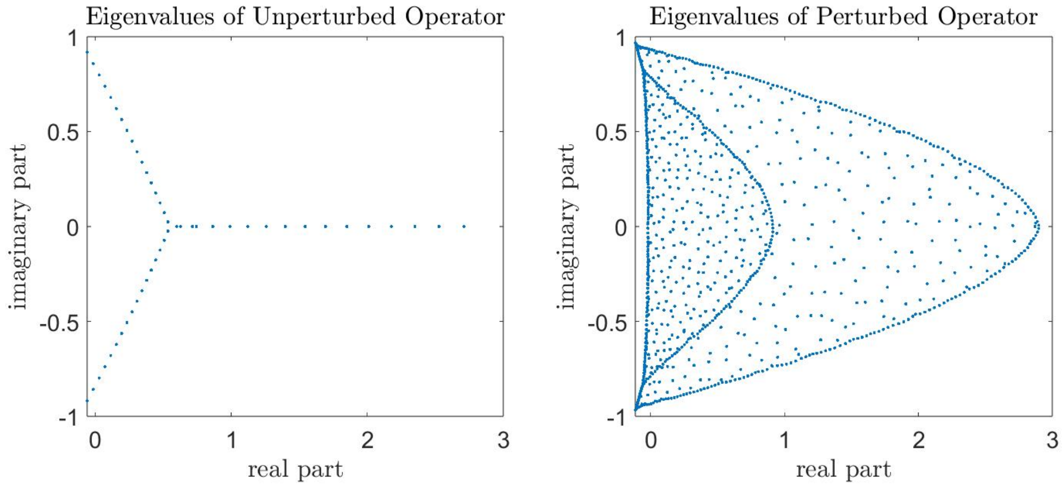

Figure 1.2: Left: Numerically computed eigenvalues of the Toeplitz operator on $\mathbb{CP}^1$ identified with the real 2-sphere with symbol $x_1 + 2x_1^2 + ix_2$ and $N = 50$. Right: Numerically computed eigenvalues of the same operator plus a small random perturbation and $N = 1000$.

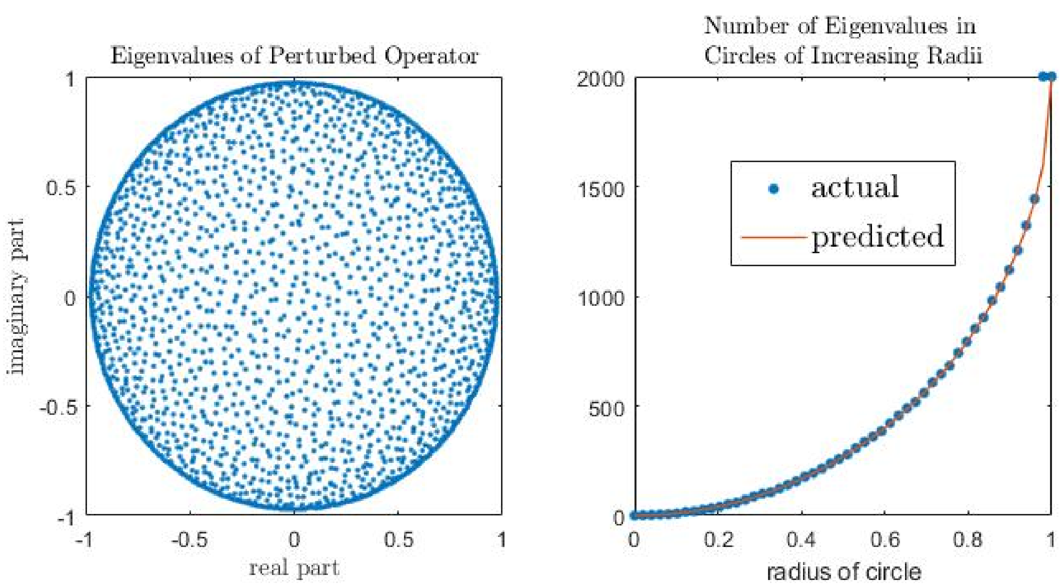

Figure 1.3: Left: Numerically computed eigenvalues of the randomly perturbed Toeplitz operator on $\mathbb{CP}^1$ identified with the real 2-sphere with symbol $ix_1 + x_2$ and $N = 2000$. Right: The number of eigenvalues within circles in the complex plane centered at zero with radii ranging from 0 to 1, plotted against the predicted distribution of eigenvalues from (1.0.2).

quantization of functions on $\mathbb{T}^{2d} := (\mathbb{R}/2\pi\mathbb{Z})^{2d} \times (\mathbb{R}/2\pi\mathbb{Z})^{2d}$. When $d = 1$, we can think of the coordinate on the first circle, $x \in \mathbb{T}$, as the position variable, and the coordinate on the second circle, $\xi \in \mathbb{T}$, as the momentum variable. Functions $F = F(x)$ and $G = G(\xi)$ are quantized as

$$\mathrm{Op}_N(F) \coloneqq \mathrm{diag}\left(F(2\pi j/N)_{j=0}^{N-1}\right), \quad \mathrm{Op}_N(G) \coloneqq \mathscr{F}_N^* \mathrm{diag}\left(G(2\pi j/N)_{j=0}^{N-1}\right) \mathscr{F}_N,$$

where $\mathscr{F}_N$ is the unitary discrete Fourier transform on $\ell^2(\mathbb{Z}_N)$. This can be generalized to arbitrary functions $f \in C^\infty(\mathbb{T}^2)$. If we consider $\mathbb{T}^2$ as a complex curve and take as L the theta bundle over it, one can show that $\mathrm{Op}_N(f) = T_N f + \mathscr{O}(N^{-\infty})$ (see for instance [Rou17]). We should also mention that discretizations used in some numerical schemes correspond to Toeplitz quantization on tori (see for instance [BF22]).

Berezin–Toeplitz quantization of functions on tori have been used as a discrete model of quantum mechanics in both mathematics and physics literature, see for instance [BO23] for a recent application and for pointers in the literature. The physical (rather than purely mathematical) motivation for considering general Kähler manifolds with positive line bundles is less clear. We mention however that Anderson in [And12] and Marché and Paul in [MP15] studied Toeplitz operators in the context of topological quantum field theory. Deleporte [Del19] also used Toeplitz operators to model spin systems in the large spin limit. In [DK10], Douglas and Klevstov derived the Bergman projector parametrix for large N, which is central to the properties of Toeplitz quantization, using path integrals for particles in a magnetic field.

Extending Vogel's result to Berezin–Toeplitz operators required developing an exotic calculus of functions on Kähler manifolds. Specifically we consider operators of the form $\chi(N^{2\delta} T_N f)$ where χ is a smooth cut-off function, $f \in C^\infty(X; \mathbb{C})$, and $\delta \in [0, 1/2)$. This requires a composition formula for Toeplitz operators for functions of the form $N^{2\delta} f$. To develop this calculus, the kernel of $N^{2\delta} T_N f$ is asymptotically expanded using the Bergman kernel approximation from [BBS08]. The resulting integral is then approximated by Melin and Sjöstrand's method of complex stationary phase from [MS75].

This exotic class of functions is described as follows. We let f depend on N, and allow its derivatives to grow in N similarly to the classes $S_\delta(1)$ in the case of quantization on $\mathbb{R}^d \times \mathbb{R}^d$ (see [Zwo12, §4.4]):

$$f \in S_\delta(1) \iff \partial^\alpha f = \mathscr{O}_\alpha(N^{\delta|\alpha|}).$$

In the Kähler setting we consider differentiation on a fixed finite set of coordinate patches (see Definition 3.2.3). As in the quantization on $\mathbb{R}^d \times \mathbb{R}^d$ we need an additional flexibility of allowing order functions in our symbol classes. This is crucial for our main applications in Chapter 4. The order functions, m, are defined on δ scales by demanding that for all $x, y \in X$,

$$m(x) \leq C m(y)(1 + N^\delta \operatorname{dist}(x, y))^{M_0}$$

for some constants $C, M_0 > 0$, and $\delta \in [0, 1/2)$. Then we define $f \in S_\delta(m)$ if and only if $\partial^\alpha f = \mathscr{O}(N^{\delta|\alpha|} m)$ on each coordinate patch.

Chapter 3 develops a calculus of Toeplitz operators quantizing functions belonging to these more exotic symbol classes. A rough formulation is given as follows.

Theorem (An exotic calculus of Berezin–Toeplitz operators)**.** *Suppose that $\delta \in [0, 1/2)$, m_1, m_2 are δ-order functions on a quantizable Kähler manifold X, $f \in S_\delta(m_1)$, and $g \in S_\delta(m_2)$ (see §3.2 for definitions). Then*

1. *The Schwartz kernels of $T_N f$ and $T_N g$ admit asymptotic expansions.*

2. *There exists $h \in S_\delta(m_1 m_2)$ such that $T_N f \circ T_N g = T_N h + \mathscr{O}(N^{-\infty})$.*

The analog of this result in the setting of tori can be obtained by using methods already available for the standard quantization in $\mathbb{R}^d$ (see [CZ10]).

The precise statement for 1 and 2 are given in Theorem 3.3.1 and Theorem 3.3.11 respectively. Applications to functional calculus are given in Theorem 3.4.2 and to trace formulas in Theorem 3.4.5. Coefficients in the expansion of h are given Appendix A. More details are also provided at the end of this section.

An essential ingredient needed to prove this exotic calculus is the asymptotic expansion of the kernel of the Bergman projector Π_N. It was provided by Catlin [Cat99] and Zelditch [Zel98] using the Bergman-Szegö kernel parametrix for strictly pseudoconvex domains obtained by Boutet de Monvel and Sjöstrand [BS75] and extended earlier work by Fefferman [Fef74]. A direct approach to produce a Bergman kernel expansion for powers of positive line bundles was given by Berman, Berndtsson, and Sjöstrand [BBS08], and another direct approach without relying on the Kuranishi trick was provided by Hitrik and Stone [HS22].

Similar exotic calculi have proven themselves useful in PDE problems from mathematical physics, for instance, long time Egorov theorem, resonance counting, or resolvent estimates.

Composition results for Toeplitz operators of uniformly (in N) smooth functions are now standard. They are discussed by Le Floch in [LeF18], Charles in [Cha03], Deleporte in [Del19], Ma and Marinescu in [MM12], and references given there. In Appendix A, we present a direct computation of the second term in the composition formula, proving the classical-quantum correspondence for Toeplitz operators.

1.1 Statement of results

In the following results, we assume (X, ω) is a quantizable Kähler manifold of dimension d with volume form $\mu_d := \omega^{\wedge d}/d!$. The symbol class $S(1)$ as mentioned in the introduction is defined in Definition 4.1.2 and is equivalent to $S_0(1)$ as defined in Definition 3.2.3.

The following two theorems are proven in Chapter 4.

Theorem (Weyl law for Gaussian perturbations)**.** *Suppose $f \in S(1)$ is such that there exists $\kappa \in (0, 1]$ such that*

$$\mu_d(\{x \in X : |f_0(x) - z|^2 \leq t\}) = \mathscr{O}(t^\kappa)$$

as $t \to 0$ uniformly for all $z \in \mathbb{C}$ and $\{\mathscr{G}_\omega(N) : N \in \mathbb{Z}_{\geq 1}\}$ is a family of random operators on $H^0(X, L^N)$ whose matrix elements with respect to a fixed basis are i.i.d. complex Gaussian *random variables with mean* 0 *and variance* 1*. Then for each* $\varepsilon > 0$ *there exists* $\beta = \beta(\varepsilon) \in (0,1)$ *and* $C > 0$ *such that if* $\delta = \delta(N)$ *satisfies*

$$Ce^{-N^{\beta}} < \delta < C^{-1}N^{-d/2-\varepsilon} \tag{1.1.1}$$

then we have almost sure weak convergence of the empirical measures of $T_N f + \delta \mathscr{G}_\omega(N)$ *to* $\mathrm{vol}(X)^{-1}(f_0)_*\mu_d$.

More precisely, if $\lambda_i = \lambda_i(N,\omega)$ *are the (random) eigenvalues of* $T_N f + \delta\mathscr{G}_\omega(N)$*, then for all* $\varphi \in C_0^\infty(\mathbb{C})$

$$\frac{1}{\mathscr{N}}\sum_{i=1}^{\mathscr{N}} \varphi(\lambda_i) \xrightarrow{N\to\infty} \frac{1}{\mathrm{vol}(X)}\int_{\mathbb{C}} \varphi(z)((f_0)_*\mu_d)(\mathrm{d}z) \tag{1.1.2}$$

almost surely, where $(f_0)_*\mu_d$ *is the push-forward of the volume form* μ_d *on* X *by* f_0.

Moreover, for each $\varepsilon > 0$*, the constant* $\beta(\varepsilon)$ *in* (1.1.1) *can be chosen at most strictly less than*

$$\begin{cases} 2\epsilon\kappa & \text{if } \epsilon < \frac{1}{2(\kappa+1)} \\ \frac{\kappa}{\kappa+1} & \text{if } \epsilon \geq \frac{1}{2(\kappa+1)}. \end{cases}$$

Almost sure convergence in the context of random matrices requires explanation. There is a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ such that for each $\omega \in \Omega$ and $N \in \mathbb{N}$, $\mathscr{G}_\omega(N)$ is an $N \times N$ matrix. For each N, the entries of $\mathscr{G}_\omega(N)$ are i.i.d. Gaussian random variables. The expression in (1.1.2) means that there is a full measure set of $\omega \in \Omega$ such that the limit holds.

This theorem can be extended to a more general class of random perturbations, as stated below.

Theorem (Weyl law for more general perturbations)**.** *Suppose* $f \in S(1)$ *is such that there exists* $\kappa \in (0,1]$ *so that*

$$\mu_d(\{x \in X : |f(x) - z|^2 \leq t\}) = \mathscr{O}(t^\kappa)$$

as $t \to 0$ *uniformly for* $z \in \mathbb{C}$ *and* $\{\mathscr{W}_\omega(N) : N \in \mathbb{Z}_{\geq 1}\}$ *is a family of operators on* $H^0(X, L^N)$ *whose entries with respect to a fixed basis are i.i.d. copies of a complex random variable with mean zero and bounded second moment. Then if* $\Lambda \subset \mathbb{C}$ *is an open set, almost surely*

$$\left(\frac{2\pi}{N}\right)^d \#\left\{\mathrm{Spec}(T_N f + N^{-d}\mathscr{W}_\omega(N)) \cap \Lambda\right\} \xrightarrow{N\to\infty} \mu_d(x \in X : f(x) \in \Lambda).$$

In the next series of results (proven in Chapter 3), we assume $\delta \in [0, 1/2)$ is fixed, m_1 and m_2 are two δ-order functions on X, $f \in S_\delta(m_1)$, and $g \in S_\delta(m_2)$ (see Definition 3.2.3).

Theorem (Composition formula)**.** *There exists $h \in S_\delta(m_1 m_2)$ such that*

$$\|T_{N,h} - T_{N,f} \circ T_{N,g}\|_{L^2(X,L^N)\to L^2(X,L^N)} = \mathscr{O}(N^{-\infty}). \tag{1.1.3}$$

Moreover if h is asymptotically written $h \sim \sum_{j=0}^{\infty} N^{-(1-2\delta)j} h_j$ (see Definition 3.2.4) then locally

$$h_0(x) = f(x)g(x) + \mathscr{O}(N^{-(1-2\delta)} m_1(x) m_2(x)) \tag{1.1.4}$$

$$h_1(x) = -\sum_{j,k=1}^{d} (\partial\overline{\partial}\varphi(x))^{j,k} \partial_k f(x) \overline{\partial}_j g(x) + \mathscr{O}(N^{-2(1-2\delta)} m_1(x) m_2(x)) \tag{1.1.5}$$

where $(\partial\overline{\partial}\varphi(x))^{j,k}$ is such that $\sum_k (\partial\overline{\partial}\varphi(x))^{j,k}(\partial_k \overline{\partial}_\ell \varphi(x)) = \delta_{j,\ell}$ for $j, \ell = 1, \ldots, d$.

Theorem (Trace formula)**.** *If $f \sim \sum N^{-(1-2\delta)j} f_j$, then*

$$\operatorname{Tr}(T_{N,f}) = \left(\frac{N}{2\pi}\right)^d \int_X f_0(x)\, \mathrm{d}\mu(x) + \left(\int_X m(x)\, \mathrm{d}\mu(x)\right) \mathscr{O}(N^{d-(1-2\delta)}). \tag{1.1.6}$$

Theorem (Existence of a parametrix)**.** *Suppose $m_1 \geq 1$ and there exists $C > 0$ such that $|f(x)| > Cm_1(x)$ for all $x \in X$. Then there exists $p \in S_\delta(m_1^{-1})$ such that*

$$T_{N,f} \circ T_{N,p} + \mathscr{O}(N^{-\infty}) = T_{N,p} \circ T_{N,f} + \mathscr{O}(N^{-\infty}) = 1. \tag{1.1.7}$$

Theorem (Functional calculus)**.** *Suppose for $x \in X$, $f(x) \in \mathbb{R}_{\geq 0}$ and there exists $C > 0$ such that $|f(x)| \geq C^{-1} m_1(x) - C$. Then for each $\chi \in C_0^\infty(\mathbb{R};\mathbb{C})$, there exists $q \in S_\delta(m^{-1})$ such that*

$$\chi(T_{N,f}) = T_{N,q} + \mathscr{O}(N^{-\infty}) \tag{1.1.8}$$

and the principal symbol of q is $\chi(f_0) + \mathscr{O}(N^{-(1-2\delta)})$ where f_0 is the principal symbol of f.

Equation (1.1.3) is proven in Theorem 3.3.11, equation (1.1.4) is proven in Theorem 3.3.1, equation (1.1.5) is proven in Theorem A.1.1, equation (1.1.6) is proven in Theorem 3.4.5, equation (1.1.7) is proven in Theorem 3.4.1, and equation (1.1.8) is proven in Theorem 3.4.2.

There are two main difficulties in applying Melin and Sjöstrand's method of complex stationary phase to our case. First, the amplitude in our integrals is unbounded in N, and so any almost analytic extension will also be unbounded in N. This growth is carefully controlled by the Gaussian decay of the phase. Second, the critical point of the almost analytically extended phase, which is already an almost analytic extension, must be estimated for proper control of terms in the stationary phase expansion.

Outline of thesis.

1. Chapter 2 reviews preliminaries for this thesis. There is a brief discussion on what quantizations are (§2.1) followed by the procedure of quantizing Kähler manifolds to build Berezin–Toeplitz operators (§2.2).

2. Chapter 3 builds an exotic calculus of Berezin–Toeplitz operators, establishing technical framework to prove a probabilistic Weyl law. This is done by first defining a new symbol class $S_\delta(m)$ (§3.2). In §3.3 we apply the method of complex stationary phase to construct an asymptotic expansion of the kernel of Toeplitz operators whose symbol is in $S_\delta(m)$, which leads to a composition formula (this is also done in the simpler case of $\mathbb{C}$ in §A.2.1). In §3.4, this composition formula is used to prove a parametric construction, a functional calculus, and a trace formula.

3. Chapter 4 proves the probabilistic Weyl law for randomly perturbed Berezin–Toeplitz operators. §4.1 reviews background material and states the main result of this thesis (Theorem 4.1.4). §4.2 reviews logarithmic potentials and reduces Theorem 4.1.4 to proving a probabilistic bound involving logarithmic derivatives of Toeplitz operators. §4.3 sets up a Grushin problem to further reduce the problem to prove probabilistic bounds on spectral properties of self-adjoint operators. §4.4 proves a deterministic bound involving the logarithmic derivative of Toeplitz operators. The technique involves scaling the symbol by a power of N, and therefore relies on the exotic calculus presented in Chapter 3. Finally, §4.5 chooses constants to establish the required probabilistic bound for the almost sure convergence in Theorem 4.1.4. In §4.6, we describe how to extend this result to the more general random perturbations as stated in Theorem 4.1.5.

4. In Appendix A, the second term in the star product of Toeplitz operators is computed.

5. In Appendix B, several Toeplitz operators on $\mathbb{CP}^1$ are explicitly computed and numerics are presented.

Notation. We will use the following notation in this thesis for functions f and g depending on N. We write $f = \mathscr{O}(g)$ if there exists $C > 0$ independent of N such that $|f| \leq Cg$. We write $f = \mathscr{O}(N^{-\infty})$ if for every $M \in \mathbb{N}$, $f = \mathscr{O}(N^{-M})$. Any subscript in the big-O will denote dependence of C of what is in the subscript. We will write $f \lesssim g$ if there exists a $C > 0$ independent of N such that $f \leq Cg$. We similarly write $f \lesssim_\alpha g$ if the constant C depends on a parameter α. We write $f \ll g$ to mean that $Cf \leq g$ for some sufficiently large $C > 0$ independent of N. For a u, v, w elements of a Hilbert space, denote $u \otimes v$ the map that sends w to $u \langle w, v \rangle$. We use the standard multi-index notation with the following twist: if $\alpha \in \mathbb{N}^{2d}$ and $f \in C^\infty(\mathbb{C}^d; \mathbb{C})$, then $\partial^\alpha_{x,\bar{x}} f(x) := \left(\prod_{j,k=1}^d \partial^{\alpha_j}_{x_j} \partial^{\alpha_{k+d}}_{\bar{x}_k} f \right)(x)$, where $\partial_x := \frac{1}{2}(\partial_{\mathrm{Re}(x)} - i\partial_{\mathrm{Im}(x)})$ and $\partial_{\bar{x}} := \frac{1}{2}(\partial_{\mathrm{Re}(x)} + i\partial_{\mathrm{Im}(x)})$ are the holomorphic and anti-holomorphic derivative operators respectively.

Publications. This thesis covers two papers written by the author during his PhD. Chapter 3 follows [Olt22] and chapter 4 follows [Olt23].

Not included in this thesis, but written during the author's PhD are: [BO23], [BOV23], and [BOV24].

Chapter 2

Preliminaries

2.1 Quantization

Before discussing quantizing Kähler manifolds, we discuss briefly what quantizations are. Quantizations are a way of connecting classical and quantum theories. In classical mechanics, a state is represented by a point in a symplectic manifold (or just $\mathbb{R}^{2d}$ for simplicity). A classical observable, like energy, is a smooth function on this symplectic manifold. In quantum mechanics, a state is represented by an element of a Hilbert space (or just an element of $L^2(\mathbb{R}^d)$ for simplicity). A quantum observable is an operator on this Hilbert space. Quantizations are maps from the classical observables to the quantum observables.

Two examples to keep in mind are the quantum and classical harmonic oscillators.

Example 2.1.1 (Harmonic Oscillators)**.** *A **classical** harmonic oscillator is a system modeling the position of a particle $x(t)\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ which has a restoring force proportional to its displacement. That is $x(t)$ must satisfy the ordinary differential equation:*

$$(\partial_t^2 + k^2)x(t) = 0 \tag{2.1.1}$$

for some fixed $k > 0$, which has a basis of solutions $\{\sin(kt), \cos(kt)\}$. The total energy of this system is the sum of the kinetic and potential energy: $\frac{1}{2}(\partial_t x(t))^2 + \frac{1}{2}k^2 x(t)^2$. We then have a classical Hamiltonian $H(p,q)\colon \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ given by $H(p,q) := \frac{1}{2}p^2 + \frac{1}{2}k^2q^2$. Solutions to (2.1.1) can be alternatively found by finding trajectories $(p(t), q(t))\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^2$ solving:

$$\begin{aligned} \partial_t q(t) &= \partial_p H, \\ \partial_t p(t) &= -\partial_q H. \end{aligned}$$

The first equation tells us that $\partial_t q(t) = p$, so that $\partial_t^2 q(t) = -k^2 q(t)$ which coincides with (2.1.1). Another way to view this is: for any fixed energy $E \geq 0$, a solution to the harmonic oscillator is a curve on the ellipse $H = E$ in phase space.

*A **quantum** harmonic oscillator is constructed by first defining the quantized Hamiltonian $\hat{H}$ by replacing p with $\frac{h}{i}\partial_x$ (where $h \in \mathbb{R}_{>0}$ is a small parameter) and q by the operator*

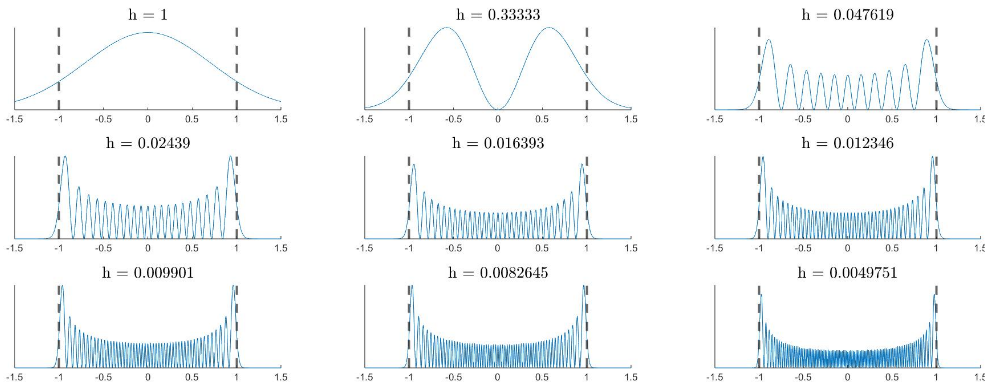

Figure 2.1: Here we solve $(-h^2\partial_x^2 + x^2)\psi(x) = E\psi(x)$ for fixed energy $E = 1$ and decreasing values of h. The blue curve is $|\psi|^2$ (representing the probability of finding a particle at different x values) and the vertical dotted lines are at the classical turning points ($x = \pm 1$). As $h \searrow 0$, the probability of finding the quantum particle is largest near the turning points, and smallest near $x = 0$, similar to the classical harmonic oscillator.

which multiplies by x. That is:

$$\hat{H} := -\frac{1}{2}h^2\partial_x^2 + \frac{1}{2}k^2x^2.$$

Time-independent solutions to the quantum harmonic oscillator are $\psi \in L^2(\mathbb{R})$ solving

$$\hat{H}\psi = E\psi. \tag{2.1.2}$$

Integrating $|\psi|^2$ over a set $A \subset \mathbb{R}$ tells us the probability of finding a quantum state in A. Solutions to (2.1.2) are so-called Hermite functions, and can only be constructed if $E = hk(n + 1/2)$ for some $n \in \mathbb{Z}_{\geq 0}$ [Hal13, Theorem 11.3]. In this way, the quantum system differs from the classical system in that there are only certain quantized allowed energies.

Despite this, the classical and quantum harmonic oscillators are linked in many ways, satisfying the so-called correspondence principle. One way this correspondence shows up is by fixing an energy E, and finding a solution to the classical harmonic oscillator $x(t)$ with energy E. Then for each $h \in \mathbb{R}_{>0}$ such that $1/(hk) - 1/2 \in \mathbb{Z}_{\geq 0}$, we find a solution $\psi_h(x)$ to (2.1.2). Then it can be shown that the behavior of $\psi_h(x)$ approximates the behavior of $x(t)$ as $h \to 0$. Specifically, the probability of finding the quantum particle near the turning points $E = \frac{1}{2}k^2x^2$, is higher than the probability of finding the quantum particle at $x = 0$ [Hal13, Chapter 15]. This agrees with the classical model, as the probability of finding $x(t)$ at the points $E = \frac{1}{2}k^2x^2$ is highest as this is where the particle's velocity vanishes (see also Figure 2.1)

In general, a quantization procedure should take a function on a symplectic manifold and get a family of operators on an associated Hilbert space indexed by a small positive number

h. Such a quantization should have certain natural properties. For one, the procedure should be linear. Real-valued functions should be quantized to self-adjoint operators. And the symplectic structure should be captured in the quantization procedure. Explicitly, if $\hat{f}$ is the quantization of f, then it is natural to require:

$$\{f, g\} = \frac{1}{ih}[\hat{f}, \hat{g}] \tag{2.1.3}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket and $[\cdot, \cdot]$ is the commutator. Such an equality actually cannot hold (see the Groenewold-van Hove Theorem [Gro46]), so instead we require (2.1.3) to hold with an error going to zero in h.

The prototypical example to keep in mind is the Weyl-quantization originally introduced by Hermann Weyl in 1927 [Wey27] (see [Zwo12, §4.2] for a textbook presentation) which quantizes $f \in C^\infty(T^*\mathbb{R}^d)$ as $\mathrm{Op}_h^w(f) \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. In this setting, $f(x, \xi)$ is a smooth function on $\mathbb{R}^{2d}$ where $x \in \mathbb{R}^d$ should be thought of as the position and $\xi \in \mathbb{R}^d$ should be thought of as the momentum. The Weyl-quantization of f applied to a suitable function $u \in C^\infty(\mathbb{R}^d)$ is given by

$$\mathrm{Op}_h^w(f)u(x) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{\frac{i}{h}(x-y)\cdot\xi} f\left(\frac{x+y}{2}, \xi\right) u(y)\, \mathrm{d}\xi\, \mathrm{d}y.$$

In the case of Kähler manifolds, the quantization procedure is slightly more involved. In fact, not every Kähler manifold has the following quantization procedure. Such Kähler manifolds who admit this quantization are called *quantizable*.

2.2 Geometric preliminaries

Here we review the geometric objects needed in this thesis. Two very useful references are Alix Deleporte's thesis [Del19] and Yohann Le Floch's textbook [LeF18]. We also thank Garrett Brown for the helpful discussions as well as the many comments, suggestions, and corrections he made in reading an earlier draft.

The way we get quantized Kähler manifolds goes as follows: we begin with a complex *compact* manifold, *we assume* there exists a positive line bundle, this provides a Kähler metric making the manifold Kähler and allows us to build a Hilbert space and procedure to quantize smooth functions on the manifold.

Definition 2.2.1 (Complex manifold)**.** *M is a complex manifold of dimension d if the following holds.*

1. *M is Hausdorff and second countable.*
2. *There exist a collection of open sets $U_i \subset M$ and homeomorphisms $\rho_j : U_j \to \mathbb{C}^d$ such that $M = \bigcup U_j$.*

3. *If $U_j \cap U_k \neq \emptyset$, then $\rho_j \circ \rho_k^{-1}$ is holomorphic on $\rho_k(U_j \cap U_k)$.*

Such a set $\{U_j, \rho_j\}$ is called an atlas.

On a d-dimensional complex manifold, consider a fixed chart (U, ρ), and let

$$\rho = (z_1, \dots, z_d) = (x_1 + iy_1, \dots, x_d + iy_d)$$

where $x_i = \mathrm{Re}\,(z_i)$ and $y_i = \mathrm{Im}\,(z_i)$. Then the tangent space at each point is a real $2d$-dimensional vector space spanned by the basis element $\partial_{x_j}, \partial_{y_j}$ for $j = 1, 2, \dots, d$. There exists a smooth endomorphism on the tangent bundle, J, called the *complex structure* defined by sending ∂_{x_j} to ∂_{y_j} and ∂_{y_j} to $-\partial_{x_j}$.

The complex structure is diagonalized by choosing a new basis for the complexified tangent space ($T_pM \otimes \mathbb{C}$) by defining $\partial_{z_j} := \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ and $\overline{\partial}_{z_j} := \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$. In this way J sends ∂_{z_j} to $i\partial_{z_j}$ and $\overline{\partial}_{z_j}$ to $-i\overline{\partial}_{z_j}$.

Then we define the holomorphic tangent bundle of M restricted to U as:

$$T^{1,0}M|_U := \mathrm{Span}(\partial_{z_j} : j = 1, \dots, d)$$

and the anti-holomorphic tangent bundle of M restricted to U as:

$$T^{0,1}M|_U := \mathrm{Span}(\overline{\partial}_{z_j} : j = 1, \dots, d).$$

On overlapping atlases, the transition functions are holomorphic, so that these spaces are preserved. Therefore we can globally define $T^{1,0}M$ and $T^{0,1}M$ thus decomposing $TM \otimes \mathbb{C}$ into a holomorphic and anti-holomorphic bundle.

Dual to these local basis vectors ∂_{z_j} and $\overline{\partial}_{z_j}$ are one-forms $\mathrm{d}z_j := \mathrm{d}x_j + i\,\mathrm{d}y_j$ and $\mathrm{d}\bar{z}_j := \mathrm{d}x_j - i\mathrm{d}y_j$ respectively. This allows us to again decompose the cotangent bundle into a holomorphic and anti-holomorphic bundle.

We ultimately want to build a Hilbert space of some sort of holomorphic functions on a complex manifold. Note that if the manifold is compact and connected, then this will consist only of constant functions. We therefore have to be a little more clever, and instead consider holomorphic sections of a well-chosen holomorphic line bundle over our manifold.

Definition 2.2.2 (Holomorphic line bundle)**.** *A holomorphic line bundle over a complex manifold M is a tuple (L, π) where L is a complex manifold, $\pi : L \to M$ is holomorphic such that:*

1. *For each $m \in M$, $\pi^{-1}(m)$ is a 1-dimensional complex vector space.*

2. *There exist open sets $\{U_i : i \in \mathscr{I}\}$ and biholomorphisms $\{\tau_i : i \in \mathscr{I}\}$ where $\tau_i \colon U_i \times \mathbb{C} \to \pi^{-1}(U_i)$ and for each $m \in U_i$, $z \mapsto \tau_i(m, z)$ is a linear isomorphism.*

Here (U_i, τ_i) are called local trivializations.

A line bundle can be completely described by defining its *transition functions*. That is, if (U_i, τ_i), (U_j, τ_j) are local trivializations such that $U_i \cap U_j = \emptyset$, then there exists a smooth function $f_{i,j} \in C^\infty(U_i \cap U_j; \mathbb{C} \setminus 0)$ such that

$$\tau_j(m, 1) = f_{i,j}(m)\tau_i(m, 1)$$

for $m \in U_i \cap U_j$. Such an $f_{i,j}$ is called a *transition function*, and if (U_k, τ_k) is a third local trivialization such that $U_k \cap U_j \cap U_i = \emptyset$, then:

$$f_{i,k} = f_{i,j} f_{j,k} \tag{2.2.1}$$

on $U_k \cap U_j \cap U_i$. It turns out that defining transition functions that satisfy the cocycle relation (2.2.1) completely describes the line bundle. And if such $f_{i,j}'s$ are holomorphic, the line bundle is as well [LeF18, Proposition 3.1.4, 3.1.5].

Definition 2.2.3 (Section of a line bundle)**.** *A section of a line bundle L over M, is a map $s\colon M \to L$ such that for all $m \in M$, $\pi(s(m)) = m$. Denote the space of smooth sections of L over M as $\Omega^0(L, M)$.*

Any smooth section s can be described locally on each trivialization U_i by smooth functions s_i as:

$$s(m) = s_i(m)\tau_i(m, 1)$$

for $m \in U_i$. If $U_i \cap U_j \neq \emptyset$, then for $m \in U_i \cap U_j$

$$s_i(m)\tau_i(m, 1) = s(m) = s_j(m)\tau_j(m, 1) = s_j(m)f_{i,j}(m)\tau_i(m, 1)$$

so we require that $s_i(m) = s_j(m)f_{i,j}(m)$ on $U_i \cap U_j$.

Definition 2.2.4 (Hermitian holomorphic line bundle)**.** *A Hermitian holomorphic line bundle over a complex manifold M is a holomorphic line bundle (L, π) over M with a smoothly varying Hermitian metric h on fibers. That is:*

1. *For each $m \in M$, $h_m(\cdot, \cdot)$ is a Hermitian metric on $\pi^{-1}(m)$.*
2. *If s is a smooth section on M, then $m \mapsto h_m(s(m), s(m))$ is smooth.*

A Hermitian metric on L can be described locally on each trivialization U_i by $h_i(m) := h_m(\tau_i(m, 1), \tau_i(m, 1))$ for $m \in U_i$. Here the polarization identity is used to reconstruct the Hermitian metric.

If U_i, U_j are trivializations with nonempty intersection, then:

$$\begin{aligned} h_i(m) &= h_m(\tau_i(m, 1), \tau_i(m, 1)) = h_m(f_{j,i}(m)\tau_j(m, 1), f_{j,i}(m)\tau_j(m, 1)) \\ &= |f_{j,i}(m)|^2 h_j(m). \end{aligned}$$

Definition 2.2.5 (Strictly plurisubharmonic)**.** *Give a smooth function $f \in C^\infty(U;\mathbb{C})$ for U and open subset of $\mathbb{C}^d$, we say f is strictly plurisubharmonic on U if there exists a $c > 0$ such that for all $t \in \mathbb{C}^d \setminus \{0\}$*

$$\sum_{j,k=1}^{d} \overline{\partial}_{z_j} \partial_{z_k} f(z) \bar{t}_j t_k > c \sum_{j=1}^{d} |t_j|^2$$

for all $z \in U$.

In other words, f is strictly plurisubharmonic on U if in local coordinates the matrix $\partial\overline{\partial} f$ is uniformly positive definite.

Given a smooth function $f \in C^\infty(M;\mathbb{C})$, the exterior derivative d can be written as $d = \partial + \overline{\partial}$ where locally

$$\partial f = \sum_{j=1}^{d} \partial_{z_j} f \, \mathrm{d}z_j \qquad \text{and} \qquad \overline{\partial} f = \sum_{j=1}^{d} \overline{\partial}_{z_j} f \, \mathrm{d}\bar{z}_j.$$

We can compute $\overline{\partial}\partial f$ as:

$$\begin{aligned} \overline{\partial}\left(\sum_{j=1}^{d} \partial_{z_j} f \, \mathrm{d}z_j\right) &= \sum_{j,k=1}^{d} \overline{\partial}_{z_k} \partial_{z_j} f \, \mathrm{d}\bar{z}_k \wedge \mathrm{d}z_j \\ &= -\sum_{j,k=1}^{d} \overline{\partial}_{z_k} \partial_{z_j} f \, \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_j \\ &= -\partial\overline{\partial} f \end{aligned}$$

which is a two-form. We can therefore similarly define a smooth function f to be strictly plurisubharmonic on an open set U if there exists a $c > 0$ such that for all $m \in M$ and $v = \sum v_j \partial_{z_j} \in T^{1,0}_m M$ nonzero,

$$\partial\overline{\partial} f(m)(v, \bar{v}) > c \sum_{j=1}^{d} |v_j|^2.$$

Definition 2.2.6 (Positive Hermitian line bundle)**.** *Given a holomorphic Hermitian line bundle L over M with Hermitian metric h locally given by $h_j(m) = h_m(\tau(m,1), \tau(m,1))$ for $m \in U_j$ over each trivialization (U_j, τ_j), then we say h is positive if*

$$h_j(x) = e^{-\varphi_j(x)}$$

where each φ_j is strictly plurisubhamonic for each j.

Given a positive Hermitian line bundle with Hermitian metric h,

$$\omega := -i\partial\overline{\partial}\log(h_j) = i\partial\overline{\partial}\varphi_j \tag{2.2.2}$$

defines a globally defined symplectic form over M. Indeed, on overlapping trivializations, have $h_i(m) = |f_{i,j}(m)|^2 h_j(m)$, so we require:

$$\partial\overline{\partial}\log(h_i(m)) - \partial\overline{\partial}\log(h_j(m)) = 0$$

which requires showing that

$$\partial\overline{\partial}\log(|f_{i,j}(m)|^2) = 0$$

but this follows using that $f_{i,j}$ is holomorphic.

Now because ω is closed (using that $\mathrm{d} = \partial + \overline{\partial}$) and non-degenerate (because $\log(-h_j)$ is strictly plurisubharmonic) ω is a symplectic form.

One can check that for all $X, Y \in TM$:

$$\omega(JX, JY) = \omega(X, Y) \qquad \text{and} \qquad \omega(X, JX) > 0 \tag{2.2.3}$$

for $X \neq 0$. Indeed we will check the second inequality. Let $X = \sum_{j=1}^{d} a_j\partial_{x_j} + b_j\partial_{y_j}$ so that $JX = \sum_{j=1}^{d} -b_j\partial_{x_j} + a_j\partial_{y_j}$. Note that $\partial_{x_j} = \partial_{z_j} + \overline{\partial}_{z_j}$ and $\partial_{y_j} = i(\partial_{z_j} - \overline{\partial}_{z_j})$ so that

$$X = \sum_{j=1}^{d}(a_j + ib_j)\partial_{z_j} + (a_j - ib_j)\overline{\partial}_{z_j}$$
$$JX = \sum_{j=1}^{d}(-b_j + ia_j)\partial_{z_j} + (-b_j - ia_j)\overline{\partial}_{z_j}$$

We then compute

$$\begin{aligned}\omega(X, JX) &= i\sum_{j,k=1}^{d} \partial_{z_j}\overline{\partial}_{z_k}\varphi\, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k(X, JX)\\ &= i\sum_{j,k=1}^{d} \partial_{z_j}\overline{\partial}_{z_k}\varphi\left((a_j + ib_j)(-ia_k - b_k) - (a_k - ib_k)(-b_j + ia_j)\right).\end{aligned}$$

Let $u_j = a + ib_j$, so that:

$$\begin{aligned}\omega(X, JX) &= i\sum_{j,k=1}^{d} \partial_{z_j}\overline{\partial}_{z_k}\varphi\left(u_j(-i\bar{u}_k) - \bar{u}_k(iu_j)\right)\\ &= 2\sum_{j,k=1}^{d} \partial_{z_j}\overline{\partial}_{z_k}\varphi u_j\bar{u}_k > 0\end{aligned}$$

by plurisubharmonicity of φ.

A symplectic form, ω, satisfying (2.2.3) and such that $d\omega = 0$ (which follows from defining ω in (2.2.2)) is said to be compatible with the complex structure of M. Such a pair (M, ω) is a *Kähler manifold* and φ_j is called the **Kähler potential**.

Given such a Kähler manifold, (M, ω) one can easily check that:

$$g_m(u, v) \coloneqq \omega_m(u, Jv)$$

defines a Riemannian metric on M for $m \in M$, $u, v \in T_m M$.

Remark 2.2.1. *The definition of positive is slightly unmotivated (and confusing) without more context. Very briefly, given a holomorphic Hermitian line bundle, there exists a unique connection ∇ (ie way of taking derivatives of sections) called the Chern connection. One can locally compute the curvature of this connection as $\operatorname{curv}(\nabla) = \partial\overline{\partial}\varphi_j$ (which is notably **not** a positive form). However, $\omega = i\operatorname{curv}(\nabla)$ is a **positive** form in the sense that $\omega(X, JX) > 0$ for $X \in TM$ nonzero. The motivation is that we want to define $g(X, Y) \coloneqq \omega(X, JY)$ to be a Riemannian metric, which requires positivity of ω.*

Definition 2.2.7 (Tensor power of a Line bundle)**.** *Given a holomorphic Hermitian line bundle L over M with Hermitian metric h, for each $N \in \mathbb{N}$, one can define the Nth tensor power of L, denoted by L^N, as a new holomorphic line bundle L^N over M. Moreover, if f_{ij} are transition functions for trivializations of L, then f_{ij}^N are transition functions for L^N and so h^N defines a Hermitian metric for L^N.*

Definition 2.2.8 ($H^0(M, L^N)$)**.** *Given a positive Hermitian holomorphic line bundle L over a d-dimensional complex manifold M with Hermitian metric h, let L^N be the Nth tensor power of L with Hermitian metric h^N. Let ω be the symplectic form given locally by $-i\partial\overline{\partial}(\log(h_j))$. Let $L^2(M, L^N)$ the completion of smooth sections of L^N with respect the the inner-product*

$$\langle u, v\rangle_{L^2_N} \coloneqq \int_M h_m(u(m), v(m)) \frac{\left|\omega^{\wedge d}\right|(\mathrm{d}m)}{d!}.$$

Then $H^0(M, L^N)$ is the subspace of $L^2(M, L^N)$ of holomorphic sections.

In fact, a simple argument [LeF18, Proposition 4.2.3] can be used to show that $H^0(M, L^N)$ is finite-dimensional. More specifically, $\dim(H^0, L^N) \asymp N^d$ (see for instance [LeF18, Theorem 4.2.4]).

Definition 2.2.9 (Bergman projector)**.** *Given the objects in Definition 2.2.8, let Π_N be the orthogonal projection from $L^2(M, L^N)$ to $H^0(M, L^N)$ (called the Bergman projector).*

Definition 2.2.10 (Berezin–Toeplitz operator)**.** *Given the objects in Definition 2.2.8, $f \in C^\infty(M)$. Then the Berezin–Toeplitz operator associated to f is the family of operators, indexed by $N \in \mathbb{N}$, defined as:*

$$T_N f \colon H^0(M, L^N) \ni u \mapsto \Pi_N(fu) \in H^0(M, L^N).$$

Example 2.2.1 (Complex projective space). *In this example, we work through this quantization procedure in the explicit example of the complex projective space of one dimension: $\mathbb{CP}^1$. This should be viewed as the prototypical example of a quantizable Kähler manifold. Indeed, by the Kodaira embedding theorem, every quantizable Kähler is embedded in $\mathbb{CP}^n$ once a sufficiently large power of the line bundle is taken.*

$\mathbb{CP}^1$ is the space $\mathbb{C}^2 \setminus (0,0)$ with the equivalence relation:

$$(z_1, z_2) \equiv (z_1', z_2') \iff \exists c \in \mathbb{C} \setminus \{0\} : (z_1, z_2) = c(z_1', z_2').$$

Elements of $\mathbb{CP}^1$ are denoted by $[(z_1, z_2)]$. We can define an atlas with two charts in the following way.

$$U_1 := \left\{[(z_1, z_2)] \in \mathbb{CP}^1 : z_1 \neq 0\right\}$$
$$U_2 := \left\{[(z_1, z_2)] \in \mathbb{CP}^1 : z_2 \neq 0\right\}$$

and

$$\rho_1 \colon U_1 \ni [(z_1, z_2)] \mapsto \frac{z_2}{z_1} \in \mathbb{C}$$
$$\rho_2 \colon U_2 \ni [(z_1, z_2)] \mapsto \frac{z_1}{z_2} \in \mathbb{C}.$$

Over $\mathbb{CP}^1$ we define the ***holomorphic line bundle*** *$\mathscr{O}(-1)$ (called the tautological line bundle) by essentially associating to every point $(z_1, z_2) \in \mathbb{CP}^1$, the complex line $\lambda(z_1, z_2)$ for $\lambda \in \mathbb{C}$.*

Concretely, define (for $z = (z_1, z_2) \in \mathbb{C}^2$)

$$\mathscr{O}(-1) := \left\{([z], \xi) \in \mathbb{CP}^1 \times \mathbb{C}^2 : \xi = \mathbb{C}z\right\}$$

with the projection $\pi([z], \xi) := [z]$. The trivializations are defined on U_1 and U_2 by:

$$\tau_1 \colon U_1 \times \mathbb{C} \ni ([z], \xi) \mapsto ([z], (\xi, \xi(z_2/z_1))) \in \pi^{-1}(U_1),$$
$$\tau_2 \colon U_2 \times \mathbb{C} \ni ([z], \xi) \mapsto ([z], (\xi(z_1/z_2)), \xi) \in \pi^{-1}(U_2).$$

The transition functions (which completely describe the line bundle) must satisfy:

$$\tau_j([z], 1) = f_{i,j}([z])\tau_i([z], 1)$$

for $[z] \in U_1 \cap U_2$. Replacing j by 1 and i by 2, we have the relation

$$([z], (1, (z_2/z_1))) = f_{2,1}([z])([z], ((z_1/z_2), 1))$$

so that $f_{2,1}([z_1, z_2]) = z_2/z_1$ (and by symmetry $f_{1,2}([z_1, z_2]) = z_1/z_2$).

The line bundle we use for defining Toeplitz operators on $\mathbb{CP}^1$ is the dual of $\mathscr{O}(-1)$, which is denoted by $\mathscr{O}$**(1)**. *This is the line bundle such that each fiber $\mathscr{O}(1)_{[z]}$ is dual to*

$\mathscr{O}(-1)_{[z]}$. Let $\tilde{\tau}_j$ and $\tilde{f}_{i,j}$ be the trivializations and transition functions respectively of $\mathscr{O}(1)$. Let $[z] \in U_1 \cap U_2$ and $\zeta, \tilde{\zeta} \in \mathbb{C}$. Then we have:

$$\begin{aligned} \tilde{\tau}_1([z], \tilde{\zeta}) &= \tilde{f}_{2,1}([z])\tilde{\tau}_2([z], \tilde{\zeta}), \\ \tau_1([z], \zeta) &= f_{2,1}([z])\tau_2([z], \zeta). \end{aligned}$$

Let $\langle \cdot, \cdot \rangle$ the pairing of an element in $\mathscr{O}(-1)_{[z]}$ and its dual. We therefore require:

$$\left\langle \zeta, \tilde{\zeta} \right\rangle = \tilde{f}_{2,1}([z]) f_{2,1}([z]) \left\langle \tilde{\zeta}, \tilde{\zeta} \right\rangle$$

*Therefore the **transitions functions** on $\mathscr{O}(1)$ are:*

$$\tilde{f}_{2,1}([z]) = \frac{1}{f_{2,1}([z])} = \frac{z_1}{z_2}$$

and by symmetry $\tilde{f}_{1,2}([z]) = z_2/z_1$.

*A **Hermitian metric** on $\mathscr{O}(1)$ is locally given by $h_i([z])$ on U_i and must satisfy, for $[z] \in U_1 \cap U_2$:*

$$h_1([z]) = |f_{2,1}([z])|^2 h_2([z]) = \left| \frac{z_1}{z_2} \right|^2 h_2([z]).$$

One example of such a metric is:

$$h_i([z]) := \frac{|z_i|^2}{|z_1|^2 + |z_2|^2}$$

for $i = 1, 2$.

We next check that this is a positive line bundle, which requires checking that

$$\partial_z \overline{\partial}_z (-\log(h_j([z])))(u, \bar{u}) > c|u|^2$$

for $u \in T^{1,0}_{[z]}(\mathbb{CP}^1)$ nonzero. In coordinates $x \in \mathbb{C}$ on U_1, this is:

$$\begin{aligned} \partial_x \overline{\partial}_x (-\log(h_1([z]))) &= \partial \overline{\partial}_x (\log((1 + |x|^2))) \\ &= \partial_x \left(\frac{x}{(1 + |x|^2)} \, \mathrm{d}\bar{x} \right) \\ &= \frac{1}{(1 + |x|^2)^2} \, \mathrm{d}x \wedge \mathrm{d}\bar{x}. \end{aligned}$$

If $u \in T^{1,0}_{[z]}(\mathbb{CP}^1)$, then $u = u_i \partial_x$ and $\bar{u} = \bar{u}_i \overline{\partial}_x$, so that:

$$\partial_x \overline{\partial}_x (-\log(h_1([z])))(u, \bar{u}) = \frac{1}{(1 + |x|^2)^2} |u|^2 > \frac{1}{2} |u|^2.$$

We therefore get a globally defined ***symplectic form*** *(called the Fubini-Study form)* $\omega = i\partial\overline{\partial}\varphi = i(1+|x|^2)^{-2}\,\mathrm{d}x \wedge \mathrm{d}\bar{x}$*. In this way* $(\mathbb{CP}^1, \omega)$ *is a Kähler manifold and* $\mathscr{O}(1)$ *is a so-called* ***prequantum line bundle****.*

The line bundle $\mathscr{O}(N)$*, for* $N \in \mathbb{N}$*, is defined by taking the* N*th tensor power of* $\mathscr{O}(1)$*. The transition functions and Hermitian metric are simply raised to the* N*th power.*

Smooth sections of $\mathscr{O}(N)$ *have the following* L^2 *inner-product:*

$$\langle u, v\rangle_{L^2_N} := \int_{\mathbb{CP}^1} h_{[z]}(u([z]), v(u([z])))\omega([z]).$$

Locally, within U_1*, this inner-product can be written as*

$$\langle u, v\rangle_{L^2_N} := \int_{\mathbb{C}} u_1(x)\overline{v_1(x)}e^{-N\log(1+|x|^2)}\frac{\mathrm{d}x \wedge \mathrm{d}\bar{x}}{(1+|x|^2)^2}.$$

Because U_1 *contains all of* $\mathbb{CP}^1$*, except a single point (which has measure zero), this definition suffices globally. We define* $L^2(\mathbb{CP}^1, \mathscr{O}(N))$ *as the completion of the space of smooth sections with respect to this* L^2 *inner-product. We then define* $H^0(\mathbb{CP}^1, \mathscr{O}(N))$ *as the subspace of holomorphic sections.*

This example is further expanded in Appendix B.3.

2.2.1 Remarks

There are many equivalent ways of constructing Toeplitz operators on Kähler manifolds. An alternative construction is to begin with a Kähler manifold (M, ω), and try to find a holomorphic Hermitian line bundle which produces a sympletic form coinciding with ω. Not every Kähler manifold admits such a line bundle. There are conditions, which will not be discussed here, of when such a line bundle exists. By the Kodaira embedding theorem [Kod54], quantizable Kähler manifolds are exactly Kähler manifolds which can be holomorphicly embedded into a projective space. In the language of complex geometry, the positive line bundle L is called ample. In the language of geometric quantization, the positive line bundle L is called a prequantum line bundle. For the sake of this thesis, we simply call Kähler manifolds quantizable if such a line bundle can be constructed.

By [LeF18, Proposition 3.5.6], a quantizable Kähler manifold with prequantum line bundle L will have a unique Hermitian metric, up to multiplication by a constant. The choice of such a constant varies in the literature. When discussing Bergman kernel asymptotics, the convention is to take $(i/2)\partial\overline{\partial}\varphi = \omega$ for $\varphi = -\log(h)$. When discussing Toeplitz quantizations, the convention is to take $i\partial\overline{\partial}\varphi = \omega$. The latter is more natural in the semiclassical setting (and followed in this thesis). This is the reason why the Bergman kernel used in this thesis (and all literature discussing Toeplitz quantizations) is 2^{-d} times the Bergman kernel in papers discussing the Bergman kernel asymptotics (for instance in [BBS08]).

Chapter 3

An Exotic Calculus

3.1 Introduction

In this chapter, we assume X is a compact, connected, d-dimensional complex manifold with a positively curved holomorphic Hermitian line bundle L. Locally the Hermitian metric is given by $h_\varphi = e^{-\varphi}$. The Hermitian metric gives a globally defined symplectic form on X by $i\partial\overline{\partial}\varphi \coloneqq \omega$. We then get that (X, ω) is a Kähler manifold. For a smooth function $f \in C^\infty(X)$, we denote the Berezin–Toeplitz operator associated to f by $T_{N,f}$ (the f is put in the subscript to more naturally write the Kernel of these operators).

In this chapter, we allow f and its derivatives to grow in N with bounds depending on an order function m on X. This more exotic symbol class is defined in §3.2, and is denoted $S_\delta(m)$ (for m an order function and $\delta \in [0, 1/2)$ a fixed parameter). The main result of this chapter is to obtain asymptotic expansions of the Schwartz kernels of Toeplitz operators whose symbol is in $S_\delta(m)$.

3.2 A new symbol class

For the remainder of this chapter, we fix a finite atlas of neighborhoods $(U_i, \rho_i)_{i\in\mathscr{I}}$ for X.

Definition 3.2.1 (δ-order function on X)**.** *For $\delta \in [0, 1/2)$, a δ-order function on X is a function $m \in C^\infty(X; \mathbb{R}_{>0})$, depending on N, such that there exist $C, M_0 > 0$ so that for all $x, y \in X$*

$$m(x) \leq Cm(y)\left(1 + N^\delta \operatorname{dist}(x, y)\right)^{M_0} \tag{3.2.1}$$

where $\operatorname{dist}(x, y)$ is the is the distance between x and y with respect to the Riemannian metric on X induced by the symplectic form ω.

This chapter will also use δ-order functions on $\mathbb{R}^d$ and $\mathbb{C}^d$, which are defined below.

Definition 3.2.2 (δ-order function on $\mathbb{R}^d$ and $\mathbb{C}^d$)**.** *For $\delta \in [0, 1/2)$, a δ-order function on $\mathbb{R}^d$ (or $\mathbb{C}^d$) is a function $m \in C^\infty(\mathbb{R}^d; \mathbb{R}_{>0})$ (or $C^\infty(\mathbb{C}^d; \mathbb{R}_{>0})$, depending on N, such that there exist $C, M_0 > 0$ so that for all $x, y \in \mathbb{R}^d$ (or $\mathbb{C}^d$)*

$$m(x) \leq Cm(y) \left(1 + N^\delta |x - y|\right)^{M_0}$$

Example 3.2.1. *If $f \in C^\infty(X; \mathbb{R}_{\geq 0})$ and $\delta \in [0, 1/2)$, then $m = N^{2\delta} f + 1$ is a δ-order function on X.*

Proof. First let $x, y \in U_i$ for some $i \in \mathscr{I}$ and define $m_i = m \circ \rho_i^{-1}$ and $f_i = f \circ \rho_i^{-1}$. Using that $m_i \geq 1$, for $\alpha \in \mathbb{N}^{2d}$ with $|\alpha| = 1$,

$$(\partial^\alpha_{x,\bar{x}} m_i)(x) = N^{2\delta} (\partial^\alpha_{x,\bar{x}} f_i)(x) \lesssim N^{2\delta} \sqrt{f_i(x)} \leq N^\delta \sqrt{m_i(x)}.$$

Here we use the fact that if $g \in C^\infty(\mathbb{C}^d; \mathbb{R}_{\geq 0})$ with bounded derivatives and $|\alpha| = 1$ then $|\partial^\alpha_{x,\bar{x}} g(x)| \lesssim \sqrt{g(x)}$ (see for instance [Zwo12, Lemma 4.31]).

If $\alpha \in \mathbb{N}^{2d}$ with $|\alpha| = 2$, then because f is bounded, $(\partial^\alpha_{x,\bar{x}} m_i)(x) \lesssim N^{2\delta}$. So by Taylor expansion, there exists a $C > 0$ such that

$$m_i(x) \leq m_i(y) + C(\sqrt{m_i(y)} |x - y| N^\delta + |x - y|^2 N^{2\delta}) \tag{3.2.2}$$

$$\lesssim (1 + |x - y|^2 N^{2\delta}) m_i(y). \tag{3.2.3}$$

To see this last inequality, let $a = \sqrt{m_i(y)}$ and $b = |x - y| N^\delta$, then using that $a \geq 1$ and $b \geq 0$, the right-hand side of (3.2.2) is:

$$a^2 + C(ab + b^2) \lesssim (a^2 + b^2) \lesssim (a^2 + a^2 b^2)$$

which is the right-hand side of (3.2.3). As X is compact, there exists a $C > 0$ such that

$$\frac{1}{C} |\rho(x) - \rho(y)| \leq \operatorname{dist}(x, y) \leq C |\rho(x) - \rho(y)|$$

for all $x, y \in U_i$. Therefore m satisfies (3.2.1) on the patch U_i with $M_0 = 2$.

For the global statement, pick $x, y \in X$, and consider the minimum number of charts that cover a geodesic from x to y having length $\operatorname{dist}(x, y)$. Next, label these charts $U_0, \ldots, U_M$ where $x \in U_0$ and $y \in U_M$. For each U_i ($i \neq M$), select some $z_i \in U_i \cap U_{i+1}$. Then by the above, we have that

$$m(x) \lesssim m(z_0) \left(1 + \operatorname{dist}(x, z_0) N^\delta\right)^2 \lesssim \cdots \lesssim m(y) \prod_{i=-1}^{M-1} \left(1 + \operatorname{dist}(z_i, z_{i+1}) N^\delta\right)^2$$

(where $z_{-1} := x$ and $z_M := y$). Then by the selection of the charts, $\operatorname{dist}(z_i, z_{i+1}) \leq \operatorname{dist}(x, y)$, so that:

$$m(x) \lesssim m(y) \left(1 + \operatorname{dist}(x, y) N^\delta\right)^{2M}.$$

Therefore we have (3.2.1) for $M_0 = 2 |\mathscr{I}|$ (where $|\mathscr{I}|$ is the number of charts). ∎

Definition 3.2.3 ($S_\delta(m)$)**.** *Let m be a δ-order function on X (with $\delta \in [0, 1/2)$ fixed). Define $S_\delta(m)$ as all smooth functions f on X, which are allowed to depend on N, such that for all $\alpha \in \mathbb{N}^{2d}$, there exists $C_\alpha > 0$ such that*

$$|\partial^\alpha_{x,\bar{x}}(f \circ \rho_i^{-1}(x))| \le C_\alpha N^{\delta|\alpha|} m \circ \rho_i^{-1}(x)$$

for each $i \in \mathscr{I}$ and $x \in \rho_i(U_i)$.

Similarly, by replacing X by $\mathbb{R}^d$ or $\mathbb{C}^d$, we can define $S_\delta(m)$ for functions on $\mathbb{R}^d$ or $\mathbb{C}^d$.

Definition 3.2.4 (Asymptotic expansion of symbols)**.** *For $\delta \in [0, 1/2)$, m a δ-order function on a quantizable Kähler manifold X, functions $f_j \in S_\delta(m)$ for $j \in \mathbb{Z}_{\ge 0}$ and $f \in S_\delta(m)$, we will write $f \sim \sum_0^\infty N^{-(1-2\delta)j} f_j$ if for all $\alpha \in \mathbb{N}^{2d}$, $M \in \mathbb{N}$, and $i \in \mathscr{I}$*

$$\begin{aligned}\partial^\alpha_{x,\bar{x}}\Big(f \circ \rho_i^{-1}(x) - \sum_{j=0}^{M-1} N^{-(1-2\delta)j}(f_j \circ \rho_i^{-1}(x))\Big)& \\ = \mathscr{O}_{\alpha,M}(N^{-M+|\alpha|\delta} m \circ \rho_i^{-1}(x)).&\end{aligned} \tag{3.2.4}$$

By Borel's Theorem (see [Zwo12, Theorem 4.15] for instance), given any $f_j \in S_\delta(m)$ we can always construct such an asymptotic sum.

Proposition 3.2.5 (Borel's Theorem for $S_\delta(m)$)**.** *Fixing $\delta \in [0, 1/2)$, m a δ-order function on a quantizable Kähler manifold X, and $f_j \in S_\delta(m)$ for $j \in \mathbb{Z}_{\ge 0}$, then there exists $f \in S_\delta(m)$ such that (3.2.4) holds and f is unique modulo $\mathscr{O}(N^{-\infty})$ error.*

Proof. On each coordinate patch, U_i, define

$$a_j(x) := N^{-(1-2\delta)j} f_j(\rho_i^{-1}(xN^\delta)) \qquad \text{and} \qquad \tilde{m}_i(x) := m(\rho_i^{-1}(xN^\delta)).$$

In this case $a_j \in S_\delta(\tilde{m}_i)$ for each j, and so by [Zwo12, Theorem 4.15], there exists $a \in S_\delta(\tilde{m}_i)$ such that $a \sim \sum_0^\infty N^{-j} a_j$ with uniqueness modulo $\mathscr{O}(N^{-\infty})$ error. On U_i, we let $f(x) = a(\rho(x)N^{-\delta})$. We can glue each patch together by uniqueness to get a globally defined f. □

Example 3.2.2. *Given a compact Kähler manifold X, smooth functions $f_j \in S_0(1)$ for $j \in \mathbb{Z}_{\ge 0}$, $f_0 \ge 0$, $f \sim \sum_0^\infty N^{-j} f_j$, $\delta \in [0, 1/2)$ and $m = f_0 N^{2\delta} + 1$, then $fN^{2\delta} \in S_\delta(m)$.*

Proof. For each i, let $g_i = N^{2\delta} f \circ \rho_i^{-1}$ and $m_i = m \circ \rho_i^{-1}$. First, for $x \in U_i$

$$|g_i(x)| \lesssim N^{2\delta} f_0 \circ \rho_i^{-1}(x) \lesssim m_i(x).$$

If $\alpha \in \mathbb{N}^{2d}$ with $|\alpha| = 1$, then:

$$|\partial^\alpha_{x,\bar{x}} g_i(x)| \lesssim N^{2\delta} \partial^\alpha_{x,\bar{x}}\left(f_0 \circ \rho_i^{-1}\right)(x) \lesssim N^{2\delta}\sqrt{f_0 \circ \rho_i^{-1}(x)} \lesssim N^\delta m_i(x),$$

using that $0 \le f_0 < mN^{-2\delta}$ and $m \ge 1$. Then, because each f_i is bounded, for all $\alpha \in \mathbb{N}^{2d}$ with $|\alpha| \ge 2$

$$|\partial^\alpha_{x,\bar{x}} g_i(x)| \lesssim N^{2\delta} \partial^\alpha_{x,\bar{x}}\left(f_0 \circ \rho_i^{-1}\right)(x) \lesssim N^{2\delta} \le m(x) N^{2\delta} \le m(x) N^{\delta|\alpha|}.$$

□

3.2.1 Almost analytic extension

When applying the method of complex stationary phase, almost analytic extensions of smooth functions are constructed. We will briefly review results about almost analytic extensions, as well as prove estimates for almost analytic extensions of functions in $S_\delta(\mathbb{R}^d)$. Similar results about almost analytic extensions of functions with growth in N are proven in [MS75, Proposition 1.16].

We recall the notation $\partial_z = \frac{1}{2}(\partial_{\mathrm{Re}(z)} - i\partial_{\mathrm{Im}(z)})$ and $\overline{\partial}_z = \frac{1}{2}(\partial_{\mathrm{Re}(z)} + i\partial_{\mathrm{Im}(z)})$ to denote holomorphic and anti-holomorphic differentiation. Recall $f \in C^\infty(\mathbb{C})$ is holomorphic in an open set $U \subset \mathbb{C}$ if and only if $\overline{\partial}_z f(z) = 0$ for all $z \in U$. To apply the method of complex stationary phase, we would like to take a smooth compactly supported function f on $\mathbb{R}^d$ and extend it to a holomorphic function $\tilde{f}$ on $\mathbb{C}^d$ and apply the Cauchy integral formula. Requiring holomorphy of $\tilde{f}$ is impossible by Liouville's theorem. However, if we relax the condition of holomorphy to $\overline{\partial}\tilde{f}$ vanishing up to infinite order as we approach the real axis, we can apply a variant of the Cauchy integral formula.

For a smooth function $f \in C^\infty(\mathbb{C}^d)$, we write $f = \mathscr{O}(|\mathrm{Im}\,(z)|^\infty)$ to mean that for any $M \in \mathbb{N}$ and compact set $K \subset \mathbb{C}$, there exists $C = C(M, K) > 0$ such that

$$|f(z)| \leq C\,|\mathrm{Im}\,(z)|^M$$

for all $z \in K$.

Proposition 3.2.6 (Almost analytic extension of $C_0^\infty(\mathbb{R}^d)$ functions)**.** *If $f \in C_0^\infty(\mathbb{R}^d)$, then there exists $\tilde{f} \in C_0^\infty(\mathbb{C}^d)$ such that for $\alpha, \beta \in \mathbb{N}^d$, $|\beta| \geq 1$*

1. $\tilde{f}|_{\mathbb{R}^d} = f$,
2. $\overline{\partial}_z \tilde{f}(z) = \mathscr{O}(|\mathrm{Im}\,(z)|^\infty)$,
3. $\partial_z^\alpha \overline{\partial}_z^\beta \tilde{f}(z) = \mathscr{O}_{\alpha,\beta}(|\mathrm{Im}\,(z)|^\infty)$,
4. $\partial_z^\alpha \tilde{f}(z) = \mathscr{O}_\alpha(1)$.

Moreover, given any neighborhood containing the support of f in $\mathbb{C}^d$, such a $\tilde{f}$ can be constructed to be supported in this neighborhood.

Remark 3.2.1. *While included for clarity, we note that conditions* (2) *and* (3) *are equivalent in Proposition 3.2.6 as proven in [Tre80, Chapter 10, Lemma 2.2].*

A construction in one dimension of such an extension (which is easily generalized to higher dimensions) is

$$\tilde{f}(x+iy) = \frac{\psi(x)}{2\pi} \int_{\mathbb{R}} e^{i(x+iy)\xi} \hat{f}(\xi)\chi(\xi y)\,\mathrm{d}\xi$$

where $\psi, \chi \in C_0^\infty(\mathbb{R})$, with $\chi \equiv 1$ near 0 and $\psi \equiv 1$ on the support of f. See [Tre80, Chapter 10.2] for further discussion.

Almost analytic extensions are not unique. However, if $\tilde{f}$ and $\tilde{g}$ are two almost analytic extensions of f on $\mathbb{R}^d$, then by Taylor expansion, $\partial^\alpha_{z,\bar{z}}(\tilde{f}(z) - \tilde{g}(z)) = \mathscr{O}_\alpha(|\mathrm{Im}\,(z)|^\infty)$ for any $\alpha \in \mathbb{N}^{2d}$.

Furthermore, smooth functions can be extended off any totally real subspace, that is a subspace $V \subset \mathbb{C}^d$ such that $iV \cap V = \{0\}$. In this way if $f \in C_0^\infty(V)$, then there exists $\tilde{f} \in C^\infty(\mathbb{C}^d)$ such that $\overline{\partial}_z \tilde{f}(z) = \mathscr{O}(|\mathrm{dist}(z,V)|^\infty)$. While any holomorphic function is determined by its restriction to a maximally totally real subspace, the same is true for almost analytic extensions modulo $\mathscr{O}(|\mathrm{dist}(z,V)|^\infty)$ error.

Proposition 3.2.7 (Almost analytic extensions of $S_\delta(m)$ functions on $\mathbb{R}^d$)**.** *For a fixed $\delta \in [0, 1/2)$, and given a δ-order function m on $\mathbb{R}^d$, and $f \in S_\delta(m)$, then there exists $C > 0$ and an almost analytic extension $\tilde{f} \in C_0^\infty(\mathbb{C}^d)$ such that for $\alpha, \beta \in \mathbb{N}^d$, $|\beta| \geq 1$,*

1. $\tilde{f}|_{\mathbb{R}^d} = f$,
2. $\overline{\partial}_z \tilde{f}(z) = m(\mathrm{Re}\,(z)) N^\delta \mathscr{O}(\left|\mathrm{Im}\,(z)\, N^\delta\right|^\infty)$,
3. $\partial_z^\alpha \overline{\partial}_z^\beta \tilde{f}(z) = \mathscr{O}_{\alpha,\beta}(|\mathrm{Im}\,(z)\, N^\delta|^\infty) N^{\delta(1+|\alpha|+|\beta|)} m(\mathrm{Re}\,(z))$,
4. $\partial_z^\alpha \tilde{f}(z) = \mathscr{O}_\alpha(N^{\delta|\alpha|}) m(\mathrm{Re}\,(z))$,
5. $\mathrm{supp}\, \tilde{f} \subset \left\{|\mathrm{Im}\,(z)| < CN^{-\delta}\right\}$.

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be such that $\psi_n(x) := \psi(x-n)$ for $n \in \mathbb{Z}^d$ is a smooth partition of unity. Next let $g(x) = f(N^{-\delta}x)$ and $\tilde{m}(x) = m(N^{-\delta}x)$, so that for each $\alpha \in \mathbb{N}^{2d}$, $|\partial^\alpha_{x,\bar{x}} g(x)| \lesssim_\alpha N^{-\delta|\alpha|} N^{\delta|\alpha|} m(N^{-\delta}x) \lesssim_\alpha \tilde{m}(x)$.

For each $n \in \mathbb{Z}^d$, let $\tilde{m}_n = \tilde{m}(n)$ and $g_n(x) = g(x)\psi_n(x)$. By shrinking the support of ψ, we may assume $\mathrm{supp}\, g_n \subset \{|x-n| < 1\}$. Note that $\tilde{m}(x)/\tilde{m}(y) \leq C(1 + |x-y|)^{M_0}$ so that $C^{-1} 2^{-M_0} \tilde{m}_n \leq \tilde{m}(x) \leq C\tilde{m}_n 2^{M_0}$ for all $x \in \mathrm{supp}\, g_n$. Therefore for all $\alpha \in \mathbb{N}^d$, there exists $C_\alpha > 0$ such that $|\partial^\alpha g_n(x)| \leq C_\alpha \tilde{m}_n$. Then, by Proposition 3.2.6, an almost analytic extension of g_n, $\tilde{g}_n$, can be constructed to satisfy the following.

- $\tilde{g}_n|_{\mathbb{R}^d} = g_n$.
- $\overline{\partial}_z \tilde{g}_n(z) = \tilde{m}_n \mathscr{O}(|\mathrm{Im}\,(z)|^\infty)$ (with constants independent of n).
- For each $\alpha, \beta \in \mathbb{N}^d$, $|\beta| \geq 1$: $\partial_z^\alpha \overline{\partial}_z^\beta \tilde{g}_n(z) = \tilde{m}_n \mathscr{O}_{\alpha,\beta}(|\mathrm{Im}\,(z)|^\infty)$.
- For all $\alpha \in \mathbb{N}^d$: $|\partial_z^\alpha \tilde{g}_n(z)| = \tilde{m}_n \mathscr{O}_\alpha(1)$.
- $\mathrm{supp}\, \tilde{g}_n$ is contained within a complex neighborhood of the real ball of radius 1 centered around n.

Since $\sum g_n = g$, it follows that $\sum g_n(xN^\delta) = f(x)$. Therefore the natural choice of an extension of f is $\tilde{f}(z) := \sum \tilde{g}_n(zN^\delta)$.

From this, (1) follows immediately. To see (2), let $\tilde{g}(z) := \tilde{f}(N^{-\delta}z)$, then for all $M \in \mathbb{N}$:

$$|\overline{\partial}\tilde{g}(z)| \leq \sum_{n:z\in \operatorname{supp}\tilde{g}_n} |\overline{\partial}_z \tilde{g}_n(z)| \lesssim_M |\mathrm{Im}\,(z)|^M \sum_{n:z\in \operatorname{supp}\tilde{g}_n} \tilde{m}_n \lesssim_M |\mathrm{Im}\,(z)|^M \tilde{m}(\mathrm{Re}\,(z)).$$

Changing variables, $z \to zN^\delta$, we get:

$$|\overline{\partial} f(z)| \lesssim_M |\mathrm{Im}\,(z)|^M N^{\delta M-1} m(\mathrm{Re}\,(z)).$$

By the same change of variables, (3), (4), and (5) follow similarly. □

3.2.2 Bergman kernels

On the holomorphic Hermitian line bundle L over X, let τ_j be trivializations on the open sets U_j with transition functions g_{jk} defined by: $\tau_j(x,1) = g_{k,j}(x)\tau_k(x,1)$ for $x \in U_j \cap U_k$. Sections on L can be locally written $s(x) = s_j(x)e_j(x)$ where $e_j(x) := \tau_j(x,1)$ and s_j are complex valued functions. A global section s, given by s_j's, must obey the transition rule $s_k(x) = g_{kj}(x)s_j(x)$ for $x \in U_j \cap U_k$. Recall that lengths of elements of L coincide with the Kähler potential φ, that is $\|e_j(x)\| = e^{-\varphi_j(x)}$.

Using the volume form $\mu := \omega^{\wedge d}/d!$, the L^2 inner-product on sections of L is explicitly written

$$\langle u, v\rangle = \sum_j \int_{U_j} \chi_j(x)u_j(x)\bar{v}_j(x)e^{-\varphi_j(x)}\,\mathrm{d}\mu(x),$$

where χ_j is a partition of unity subordinate to U_j, and u, v are smooth sections such that $u = \sum u_j e_j$ and $v = \sum v_j e_j$. In this way, smooth sections can locally be described by smooth functions. Throughout this thesis, as most of the analysis is local, sections are treated as smooth functions and e_j are not written.

Sections of the N^{th} tensor power of L, denoted by L^N, are locally written $s(x) = s_j(x)e_j(x)^{\otimes N}$ where s_j obey the transition rule $s_k(x) = g_{kj}^N(x)s_j(x)$. A Hermitian metric on L^N can be constructed by raising the original metric to the N^{th} power. In this way, there is a natural inner product on sections of L^N coming from the original metric. Given sections u and v on L^N, define

$$\langle u, v\rangle = \sum_j \int_{U_j} \chi_j(x)u_j(x)\bar{v}_j(x)e^{-N\varphi_j(x)}\,\mathrm{d}\mu(x).$$

In the following section, we will write operators on sections as integral kernels. The Bergman projector Π_N is a bounded map from $L^2(X, L^N) \to L^2(X, L^N)$, and by the Schwartz kernel Theorem (see [LeF18, Proposition 6.3.1]), has Schwartz kernel $\Pi_N \in L^2(X \times \bar{X}, L^N \boxtimes \overline{L^N})$.

Here $\bar{X}$ is the manifold X with symplectic form $-\omega$ and complex structure opposite to the complex structure on X. Π_N is a smooth section of $L^N \boxtimes \overline{L^N}$ which is holomorphic in the first argument and anti-holomorphic in the second argument, so we write $\Pi_N \in H^0(X \times \overline{X}, L^N \boxtimes \overline{L^N})$. Similarly, Schwartz kernels of Toeplitz operators also live in $H^0(X \times \overline{X}, L^N \boxtimes \overline{L^N})$. Just as smooth sections can locally be described by smooth functions, smooth sections of $L^N \boxtimes \overline{L^N}$ can be described by smooth functions of two variables. A holomorphic section of $L^N \boxtimes \overline{L^N}$ can be locally defined by a function of two variables which is holomorphic in the first component, and anti-holomorphic in the second component. For this reason, we will locally write Π_N as $\Pi_N(x, \bar{y})$, so that the function $\Pi_N(x, y)$ is holomorphic in both variables. In the remainder of this thesis, this convention will be used for both holomorphic functions, and almost-holomorphic functions.

For a smooth function f on X, the associated Toeplitz operator has kernel

$$T_{N,f}(x, \bar{y}) = \int_X \Pi_N(x, \bar{w}) f(w) \Pi_N(w, \bar{y}) e^{-N\varphi(w)} \, \mathrm{d}\mu(w).$$

This chapter does not include the L^2 weight in the kernel, that is if u is a smooth section on L^N, then

$$T_{N,f}[u](x) := \int_X T_{N,f}(x, \bar{y}) u(y) e^{-N\varphi(y)} \, \mathrm{d}\mu(y).$$

In [BBS08], Berman, Berndtsson, and Sjöstrand provided a direct proof to approximate the Bergman kernel near the diagonal (for each $N_0 \in \mathbb{N}$) by:

$$\Pi_N^{N_0}(x, \bar{y}) = \left(\frac{N}{2\pi}\right)^d e^{N\psi(x,\bar{y})} \left(1 + \sum_1^{N_0} b_i(x, \bar{y})\right) \tag{3.2.5}$$

where $b_m \in C^\infty(X \times X; \mathbb{C})$ are locally almost analytic off $\{(x, \bar{x})\}$, and ψ is an almost analytic extension of φ such that $\psi(x, \bar{x}) = \varphi(x)$. By [BBS08],

$$\Pi_N(x, \bar{y}) = \Pi_N^{N_0}(x, \bar{y}) + \mathscr{O}(N^{d-N_0-1}) e^{\frac{N}{2}(\varphi(x)+\varphi(y))}. \tag{3.2.6}$$

Conversely, away from the diagonal we have the following Lemma proven in [MM14].

Lemma 3.2.8 (Off diagonal decay of Bergman kernel)**.** *For $x, y \in X$, there exists $C, c > 0$ such that:*

$$\|\Pi_N(x, \bar{y})\|_{h_N} \leq C N^d e^{-c\sqrt{N}\,\mathrm{dist}(x,y)}$$

where $\mathrm{dist}(\cdot, \cdot)$ *is the Riemannian distance on X and locally $\|\Pi_N(x, \bar{y})\|_{h_N}$ is*

$$e^{-\frac{N}{2}(\varphi(x)+\varphi(y))} |\Pi_N(x, \bar{y})|. \tag{3.2.7}$$

It should be noted that in [Chr13], Christ proved a stronger decay estimate of Π_N away from the diagonal, but we want to avoid fine analysis at this early stage.

By Taylor expanding $\psi(x, \bar{y})$ near the diagonal, it follows that there exists $C > 0$ such that

$$\operatorname{Re}(\psi(x, \bar{y})) \leq -C|x-y|^2 + \frac{1}{2}(\varphi(x) + \varphi(y)) \tag{3.2.8}$$

(see for instance [HS22, Proposition 2.1]). This, along with Lemma 3.2.8, provides the following global bound.

Lemma 3.2.9 (Global Bergman kernel bound). *There exists $\varepsilon, C, c > 0$ such that:*

$$\|\Pi_N(x, \bar{y})\|_{h_N} \leq \begin{cases} CN^d e^{-CN|x-y|^2} + \mathscr{O}(N^{-\infty}) & \text{if } \operatorname{dist}(x,y) < \varepsilon, \\ CN^d e^{-c\sqrt{N}\varepsilon} & \text{if } \operatorname{dist}(x,y) > \varepsilon. \end{cases}$$

Where, locally $\|\Pi_N(x, y)\|_{h_N}$ is given by (3.2.7) and $|x-y|$ is the distance in coordinates of x and y ($\varepsilon > 0$ is chosen sufficiently small such that x and y are in the same chart).

3.3 Composition of Toeplitz operators with symbols in $S_\delta(m)$

This section estimates the composition of two symbols in $S_\delta(m_1)$ and $S_\delta(m_2)$ where m_1 and m_2 are two δ-order functions on X with $\delta \in [0, 1/2)$ fixed. That is, if $f \in S_\delta(m_1)$ and $g \in S_\delta(m_2)$, then this section constructs $h \in S_\delta(m_1 m_2)$, such that $T_{N,f} \circ T_{N,g} = T_{N,h} + \mathscr{O}(N^{-\infty})$. Here the big-O is in terms of the norm $L^2(X, L^N) \to L^2(X, L^N)$.

This proof will be broken into several steps which rely on the method of complex stationary phase. Before this, we explicitly write out the Schwartz kernel of Toeplitz operators.

3.3.1 An Asymptotic Expansion of the Kernel of a Toeplitz Operator

In this subsection, we apply Melin and Sjöstrand's method of complex stationary phase to obtain an asymptotic expansion of the kernel of Toeplitz operators for functions in $S_\delta(m)$.

Theorem 3.3.1 (Asymptotic expansion of symbols in $S_\delta(m)$). *Suppose $\varepsilon > 0$ is small enough such that if $\operatorname{dist}(x, y) < \varepsilon$, then x and y are contained in the same chart. Let $\Delta_\varepsilon = \{(x, y) \in M \times M : \operatorname{dist}(x, y) < \varepsilon\}$. Fixing $\delta \in [0, 1/2)$, suppose m is a δ-order function on X with constant M_0 in (3.2.1). Then if $f \in S_\delta(m)$, there exist $f_j \in C^\infty(\Delta_\varepsilon; \mathbb{C})$ ($j \in \mathbb{Z}_{\geq 0}$) such that for all $J \in \mathbb{N}$, in local coordinates $T_{N,f}(x, \bar{y})$ is*

$$\left(\frac{N}{2\pi}\right)^d e^{N\psi(x,\bar{y})} \left(\sum_{j=0}^{J-1} N^{-j} f_j(x, \bar{y}) + N^{-J} R_J(x, \bar{y})\right) + e^{\frac{N}{2}(\varphi(x)+\varphi(y))} \mathscr{O}(N^{-\infty}) \tag{3.3.1}$$

where:

$$\begin{aligned} R_J(x,\bar y) &\in N^{2\delta J}(S_\delta(m(x))\cap S_\delta(m(y)), \\ f_j(x,\bar y) &\in N^{2\delta j}\left(S_\delta(m(x))\cap S_\delta(m(y))\right), \\ \operatorname{supp} f_j(x,\bar y) &\subset \left\{\operatorname{dist}(x,y) < CN^{-\delta}\right\}, \\ f_0(x,\bar x) &= f(x), \end{aligned}$$

for some $C>0$. Moreover, in local coordinates, $f_j(x,y)$ are almost analytic off the totally real submanifold $\left\{(x,y)\in\mathbb{C}^d\times\mathbb{C}^d : y=\bar x\right\}$, and if f_j' is another almost analytic extension agreeing with f_j on the diagonal, then the difference of the two kernels is $\exp(\frac{N}{2}(\varphi(x)+\varphi(y))\mathscr{O}(N^{-\infty})$.

Remark 3.3.1. *An alternate asymptotic expansion can be written by bounding R_J in (3.3.1) and absorbing the $\exp((N/2)(\varphi(x)+\varphi(y))\mathscr{O}(N^{-\infty})$ term. That is (with the same quantifiers as in Theorem 3.3.1) $T_{N,f}(x,\bar y)$ can be written*

$$\left(\frac{N}{2\pi}\right)^d e^{N\psi(x,\bar y)}\sum_{j=0}^{J-1} N^{-j} f_j(x,\bar y) \\ + e^{\frac{N}{2}(\varphi(x)+\varphi(y))}\mathscr{O}(N^{d-J(1-2\delta)}\min(m(x),m(y))). \tag{3.3.2}$$

The proof follows the method of complex stationary phase, developed by Melin and Sjöstrand in [MS75], and presented in [Tre80] by Treves. The difficulty is that the amplitude is no longer bounded in N, but lives in $S_\delta(m)$ and so any almost analytic extension is slightly weaker than in [Tre80]. Careful analysis is required to ensure the stationary phase still provides appropriate remainders.

To avoid reproving the method of complex stationary phase, we use the same notation for variables in [Tre80]. Unfortunately, there is no ideal uniform choice of variables. We will have the same variable used for different objects in separate parts of the proof. We begin with (x,y) to denote the argument of the Schwartz kernel. For each (x,y), we get an integral over $w\in\mathbb{C}^d$. We rewrite w in real coordinates, p, and replace (x,y) by t, to use the same notation as in [Tre80, Chapter 10]. After going through complex stationary phase, we replace t by (x,y). This notational choice is summarized in the following table for the reader's convenience.

variable name	space	first reference	step(s) used	comment
x,y	$\mathbb{C}^d$	(3.3.3)	1,5-9	argument of kernel
w	$\mathbb{C}^d$	(3.3.3)	1,5	integrated variable
p	$\mathbb{R}^{2d}$	(3.3.6)	1	realifies w
p	$\mathbb{C}^{2d}$	step 2	2-4	complexifies previous p
t	$\mathbb{R}^{4d}$	(3.3.7)	1-5	realifies $(x,\bar y)$
$\tilde p(t)=\tilde p(x,\bar y)$	$\mathbb{C}^{2d}$	step 2	2,4-8	critical point of $\tilde\Psi$
$p(x,\bar y,z)=p(z_0)$	$\mathbb{C}^{2d}$	step 2	2,3,5,7	point on new contour

Proof. **Step 1: Rewrite $T_{N,f}(x,y)$ locally in real coordinates**

The goal is to write $T_{N,f}(x,y)$ for x near $x_0 \in X$ and $y \in X$. We may assume $x_0 \in U_1$ with $\rho(x_0) = 0$ (recall (U_i, ρ_i) are charts on X). By construction,

$$T_{N,f}(x,\bar{y}) = \int_{U_1} \Pi_N(x,\bar{w}) f(w) \Pi_N(w,\bar{y}) e^{-N\varphi(w)}\, \mathrm{d}\mu(w) + \int_{X\setminus U_1} \Pi_N(x,\bar{w}) f(w) \Pi_N(w,\bar{y}) e^{-N\varphi(w)}\, \mathrm{d}\mu(w). \tag{3.3.3}$$

By Lemma 3.2.9, the second integral is $\exp(\frac{N}{2}(\varphi(x)+\varphi(y)))\mathscr{O}(N^{-\infty})$. If $y \notin U_1$, then by Lemma 3.2.9, (3.3.3) will be $\exp(\frac{N}{2}(\varphi(x)+\varphi(y)))\mathscr{O}(N^{-\infty})$.

We now assume that $x, y \in U_1$ which allows us to work locally. For the remainder of the proof (until the last step) all computations are for x and y in this chart. We therefore replace $\rho(x)$ and $\rho(y)$ by x and y respectively and functions on X are replaced by functions on $\mathbb{C}$ with the same name.

We now rewrite (3.3.3) as

$$\int_{\rho_1 \circ U_1} e^{N\Phi_{x,\bar{y}}(w)} g_{x,\bar{y}}(w)\, \mathrm{d}m(w) + e^{\frac{N}{2}(\varphi(x)+\varphi(y))}\mathscr{O}(N^{-\infty}) \tag{3.3.4}$$

where

$$\begin{aligned} \Phi_{x,\bar{y}}(w) &= \psi(x,\bar{w}) - \varphi(w) + \psi(w,\bar{y}), \\ g_{x,\bar{y}}(w) &= \left(\frac{N}{2\pi}\right)^{2d} f(w) B(x,\bar{w}) B(w,\bar{y}) \mu(w), \\ \frac{\omega^{\wedge d}(w)}{d!} &= \mu(w)\, \mathrm{d}m(w). \end{aligned} \tag{3.3.5}$$

Here $\mathrm{d}m(w)$ is the Lebesgue measure on $\mathbb{C}^d$. Note that locally if

$$\omega = i \sum_{\ell,m=1}^{d} H_{\ell,m}\, \mathrm{d}w_\ell \wedge \mathrm{d}\bar{w}_m$$

then

$$\omega^{\wedge d}/d! = 2^d \det(H)\, \mathrm{dRe}\,(w_1) \wedge \mathrm{dIm}\,(w_1) \wedge \cdots \wedge \mathrm{dRe}\,(w_d) \wedge \mathrm{dIm}\,(w_d)\,.$$

Recall that $H = \partial\bar{\partial}\varphi$ which is locally a positive definite matrix. Therefore, locally, $\mu(w) = 2^d \det(\partial\bar{\partial}\varphi(w))$.

For any $M' \in \mathbb{N}$, the projector Π_N can be estimated with M' terms as in (3.2.6). Indeed, if we let $B(x,\bar{y}) \coloneqq 1 + \sum_1^{M'} b_i(x,\bar{y})$, as in (3.2.5), then this introduces error $\mathscr{O}(N^{2d-2M'-2})\exp(\frac{N}{2}(\varphi(x)+\varphi(y)))$ which is absorbed into the error term in (3.3.4) as M' can be arbitrarily large.

We next change to real coordinates by setting

$$p := (\mathrm{Re}\,(w)\,, \mathrm{Im}\,(w)) \in \mathbb{R}^{2d}, \tag{3.3.6}$$

$$t := (\mathrm{Re}\,(x)\,, \mathrm{Im}\,(x)\,, \mathrm{Re}\,(y)\,, -\mathrm{Im}\,(y)) \in \mathbb{R}^{4d} \tag{3.3.7}$$

and define

$$\Psi(p,t) := \Phi_{x,\bar{y}}(w(p)) - \frac{1}{2}(\varphi(x(t)) + \varphi(y(t)))\colon \mathbb{R}^{2d} \times \mathbb{R}^{4d} \to \mathbb{C}, \tag{3.3.8}$$

$$g(p,t) := g_{x,\bar{y}}(w(p))\chi(w(p))\colon \mathbb{R}^{2d} \times \mathbb{R}^{4d} \to \mathbb{C} \tag{3.3.9}$$

where $\chi \in C_0^\infty(\mathbb{C}^d; [0,1])$ is identically 1 near 0. With this, the first term of (3.3.4) can be written

$$e^{\frac{N}{2}(\varphi(x)+\varphi(y))} \int_{\mathbb{R}^{2d}} e^{N\Psi(p,t)} g(p,t)\,\mathrm{d}p + \int_{\rho_1 \circ U_1} e^{N\Phi_{x,\bar{y}}(w)} g_{x,\bar{y}}(w)(1-\chi(w))\,\mathrm{d}m(w). \tag{3.3.10}$$

By the Taylor expansion of $\mathrm{Re}\,(\Phi_{x,\bar{y}}(w))$ stated in (3.2.8), it immediately follows that the second term in (3.3.10) is $\exp(\frac{N}{2}(\varphi(x)+\varphi(y)))\mathscr{O}(N^{-\infty})$.

Summary of step 1. We have observed that the Schwartz kernel of $T_{N,f}$, written $T_{N,f}(x,\bar{y})$, is concentrated along the diagonal $y = x$. Near any x, we can approximate $T_{N,f}(x,\bar{y})$ as an integral over $\mathbb{R}^{2d}$ of the form $\int \exp(N\psi(p,t))g(p,t)\,\mathrm{d}p$. Here t is a function of x and y, g is a smooth compactly supported function depending on the symbol f, the Bergman kernel, and the density of the volume form on the Kähler manifold X, and Ψ is a sum of phases appearing in the Bergman kernel. We would now like to apply the method of complex stationary phase to approximate this integral.

Step 2: Deform the contour of the main term

Following the method of complex stationary phase presented by Treves in [Tre80, Chapter 10], the first term of (3.3.10) will be estimated by a contour deformation. Let $\tilde{\Psi}$ and $\tilde{g}$ be almost analytic extensions of Ψ and g in the p variable (as described in Propositions 3.2.6 and 3.2.7).

We first observe that there is a unique solution $\tilde{p}(t)$ to $\partial_p\tilde{\Psi}(p,t) = 0$ (where $p \in \mathbb{C}^{2d}$) such that the Hessian $\tilde{\Psi}_{pp}(\tilde{p}(t),t)$ is invertible with real part negative definite. Indeed, by [HS22, Proposition 2.2], $\Psi(p,0)$ has a unique critical point at $p = (0,0)$ with critical value equal to zero such that the real part of the Hessian is a negative definite matrix. By the implicit function theorem (see [Tre80, Chapter 10, Lemma 2.3] for details), there exists a unique smooth function $\tilde{p}(t)$ solving $\partial_p\tilde{\Psi}(\tilde{p}(t),t) = 0$ (here ∂_p is the holomorphic derivative in the p variable). In Lemma 3.3.7, an estimate of $\tilde{p}(t)$ is proven.

The desired contour deformation relies on a particular function q, which is proven to exist in [Tre80, Chapter 10, Lemma 3.2] and is stated here without proof.

Lemma 3.3.2. *There exist $U \subset \mathbb{C}^{2d}$, $V \subset \mathbb{R}^{4d}$ open neighborhoods of 0 and smooth function $q : U \times V \to \mathbb{C}^{2d}$ such that*

1. $q(\tilde{p}(t), t) = 0$,
2. *for each* $t \in V$, $p \mapsto q(p,t)$ *is a diffeomorphism from* U *onto an open neighborhood of zero in* $\mathbb{C}^{2d}$,
3. $\tilde{\Psi}(p,t) - \tilde{\Psi}(\tilde{p}(t),t) + \frac{1}{2}q(p,t)\cdot q(p,t) = \mathscr{O}((|\mathrm{Im}\,(p)| + |\mathrm{Im}\,(\tilde{p}(t))|)^\infty)$.

Let $q(p,t) = (z_1, \ldots, z_{2d}) = (x_1 + iy_1, \ldots, x_{2d} + iy_{2d})$. For each t, let

$$\mathscr{U}(t) \coloneqq q(\operatorname{supp} g \cap \mathbb{R}^{2d}, t) \subset \mathbb{C}^{2d}.$$

For t close to 0, there exists a function ζ such that $\mathscr{U}(t) = \{x + iy : y = \zeta(x,t), x \in \mathscr{U}^{\mathbb{R}}(t)\}$, where $\mathscr{U}^{\mathbb{R}}(t)$ is the projection of $\mathscr{U}(t)$ onto $\mathbb{R}^{2d}$.

For each $s \in [0,1]$, let $\mathscr{U}_s(t) = \{x + is\zeta(x,t) : x \in \mathscr{U}^{\mathbb{R}}(t)\}$. Define the contour:

$$U_s(t) = \{p \in \mathbb{C}^{2d} : q(p,t) \in \mathscr{U}_s(t)\} = q(\cdot,t)^{-1}(\mathscr{U}_s(t)).$$

To ease notation below, let $z_s = z_s(x) \coloneqq x + is\zeta(x,t)$, and $p(z_s) = q(\cdot,t)^{-1}(z_s)$, so that $U_s = \{p(z_s(x)) : x \in \mathscr{U}^{\mathbb{R}}\}$. For a simple example of this contour construction, see Appendix A.2.3. For a schematic drawing of this contour construction, see Figure 3.1.

Observe that $\mathscr{U}_1(t) = \{x + \zeta(x,t) : x \in \mathscr{U}^{\mathbb{R}}(t)\} = \mathscr{U}(t) = q(\operatorname{supp} g \cap \mathbb{R}^{2d}, t)$. So

$$U_1(t) = \{p \in \mathbb{C}^{2d} : q(p,t) \in q(\operatorname{supp} g \cap \mathbb{R}^{2d}, t)\} = \operatorname{supp} g \cap \mathbb{R}^{2d}.$$

Because this contains the support of g, we may rewrite the first integral in (3.3.10) as

$$\int_{\mathbb{R}^{2d}} e^{N\Psi(p,t)} g(p,t)\,\mathrm{d}p = \int_{U_1(t)} e^{N\tilde{\Psi}(p,t)}\tilde{g}(p,t)\,\mathrm{d}p,$$

which, by Stokes' theorem, is

$$\int_{U_0(t)} e^{N\tilde{\Psi}(p,t)}\tilde{g}(p,t)\,\mathrm{d}p^1 \wedge \cdots \wedge \mathrm{d}p^{2d} + \int_W (\overline{\partial}_p(e^{N\tilde{\Psi}(p,t)}\tilde{g}(p,t))) \wedge \mathrm{d}p^1 \wedge \cdots \wedge \mathrm{d}p^{2d} \tag{3.3.11}$$

where $W = \{p \in \mathbb{C}^{2d} : q(p,t) \in \mathscr{U}_s(t), s \in [0,1]\}$.

Summary of step 2. We considered $\int \exp(N\Psi) g\,\mathrm{d}p$ as an integral over $\mathbb{R}^{2d}$ within $\mathbb{C}^{2d}$ by almost analytically extending Ψ and g. We then chose a particular contour deformation and applied Stokes' theorem to rewrite this as two integrals (as in (3.3.11)). We will show that by this choice of contour deformation and almost analytic extensions, the second term in (3.3.11) is negligible.

Step 3: Estimate $\overline{\partial}_p(\exp(N\tilde{\Psi})\tilde{g})$

To control the second term of (3.3.11), $\overline{\partial}$-estimates for $\tilde{\Psi}$ and $\tilde{g}$ are required. Note that $p \mapsto \Psi(p,t) \in C^\infty(\mathbb{R}^{2d})$ (with uniform derivative estimates independent of N as in

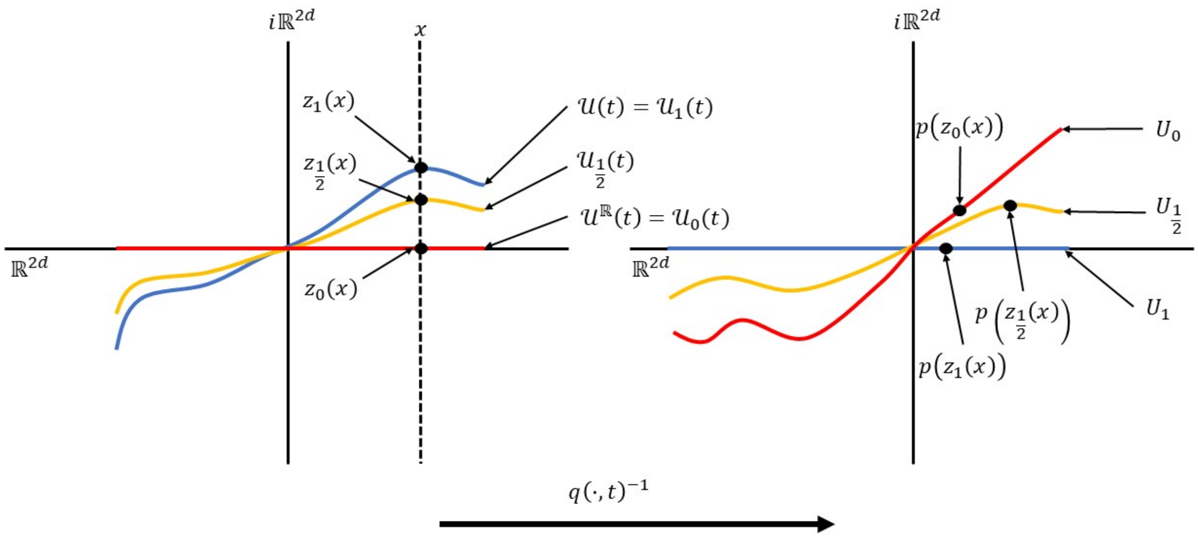

Figure 3.1: Schematic of the contour deformation.

Proposition 3.2.7). Therefore $\tilde{\Psi}$ has the usual $\overline{\partial}_p$ estimates (vanishing to infinite order in the imaginary direction). On the other hand, g has growth in N when differentiated.

Here we define $m_{\mathbb{R}} : \mathbb{R}^{2d} \to \mathbb{R}_{>0}$ by

$$m_{\mathbb{R}}(p_1, p_2) = m \circ \rho_1^{-1}(p_1 + ip_2). \tag{3.3.12}$$

Then, by examining (3.3.5) and (3.3.9), we see that $g(\cdot, t) \in S_\delta(m_{\mathbb{R}})$ (prior to almost analytic extension). Therefore, by Proposition 3.2.7, for all $M \in \mathbb{N}$ and $p \in \mathbb{C}^{2d}$, we have that

$$|\overline{\partial}_p \tilde{g}(p,t)| \lesssim_M N^{\delta - M\delta} \left|\mathrm{Im}\,(p)\right|^M m_{\mathbb{R}}(\mathrm{Re}\,(p)).$$

Therefore:

$$\begin{aligned} |\overline{\partial}_p(e^{N\tilde{\Psi}(p,t)}\tilde{g}(p,t))| &= |(\overline{\partial}_p\tilde{g}(p,t) + N\overline{\partial}_p\tilde{\Psi}(p,t))e^{N\tilde{\Psi}(p,t)}| \\ &\lesssim_M e^{N\mathrm{Re}\left(\tilde{\Psi}(p,t)\right)}\left(N^{\delta-M\delta}m_{\mathbb{R}}(\mathrm{Re}\,(p))\left|\mathrm{Im}\,(p)\right|^M + N\left|\mathrm{Im}\,(p)\right|^M\right) \\ &\lesssim_M e^{N\mathrm{Re}\left(\tilde{\Psi}(p,t)\right)}\left|\mathrm{Im}\,(p)\right|^M (N^{\delta(M+1)}m_{\mathbb{R}}(\mathrm{Re}\,(p)) + N). \end{aligned}$$

As we are integrating over W, we may write $p = p(z_s(x))$ for $x \in \mathscr{U}^{\mathbb{R}}$ and $s \in [0,1]$. Now a bound of $\exp(N\mathrm{Re}\left(\tilde{\Psi}(p(z_s(x)), t)\right))$ is required. For this we apply the following Lemma.

Lemma 3.3.3. *If U and V are small enough, there exists a $C > 0$ such that for all $(x,t) \in U \times V$:*

$$\mathrm{Re}\left(\tilde{\Psi}(p(z_s(x)),t)\right) \le -C|\mathrm{Im}\,(p(z_s(x)))|^2.$$

Proof. This is an upgraded version of [Tre80, Chapter 10, Lemma 3.2], which states that for all $s \in [0,1]$, there exists $C' > 0$ such that

$$\mathrm{Re}\left(\tilde{\Psi}(p(z_s(x)),t)\right) \le -C'(1-s)^{-1}|\mathrm{Im}\,(p(z_s(x)))|^2 - C'\left|\mathrm{Im}\,(\tilde{p}(t))\right|^2. \tag{3.3.13}$$

Observe that for $s = 0$, $p(z_s(x)) \in \mathbb{R}^{2d}$. Then because $\mathrm{Re}\left(\tilde{\Psi}\right)$ has a unique critical point with negative definite Hessian, we see that for s and x near zero, there exists a $C'' > 0$ such that $\mathrm{Re}\left(\tilde{\Psi}(p(z_s(x))),t\right) \le -C''|p(z_s(x)|^2 \le -C''\left|\mathrm{Im}\,(p(z_s(x)))\right|^2$. We can combine this with [Tre80, Chapter 10, Lemma 3.2] to get Lemma 3.3.3. □

Using this lemma, we see that

$$\begin{aligned}|\overline{\partial}_p(e^{N\tilde{\Psi}(p,t)}g(p,t))| &\lesssim_M e^{-CN|\mathrm{Im}(p)|^2}\left|\mathrm{Im}\,(p)\right|^M (N^{\delta(M+1)}m_{\mathbb{R}}(\mathrm{Re}\,(p)) + N)\\ &\lesssim_{M,M'} N^{-M'}\left|\mathrm{Im}\,(p)\right|^{M-2M'}\max(N^{\delta(M+1+M_0)},N)\end{aligned}$$

for any $M' \in \mathbb{N}$. Here we used that $m_{\mathbb{R}}(x) \lesssim N^{\delta M_0}$ and p is bounded on the region we are integrating. Let $M = 2M'$ so that

$$|\overline{\partial}_p(e^{N\tilde{\Psi}(p,t)}g(p,t))| \lesssim_M \max(N^{\delta(M_0+1)-M(1/2-\delta)}, N^{1-M/2}) = \mathscr{O}(N^{-\infty})$$

because $\delta < 1/2$, and M can be made arbitrarily large.

Summary of step 3. In this step, we proved that the second term of (3.3.11) is $\mathscr{O}(N^{-\infty})$. This involved estimating $\overline{\partial}_p$ applied to the integrand. This was controlled by Propositions 3.2.6 and 3.2.7. Because $\tilde{g}$ is a function of our symbol (so its derivatives are unbounded in N), these $\overline{\partial}_p$-estimates are weaker than the $\overline{\partial}_p$-estimates on $\tilde{\Psi}$. Fortunately, by the choice of contour deformation, on the domain of integration the phase behaves like a Gaussian, and destroys all temperate growth in the $\overline{\partial}_p$-estimates.

Step 4: Reduction to quadratic phase

We now compute the first term in (3.3.11). Define

$$J(N) := \int_{U_0} e^{N\tilde{\Psi}(p,t)}\tilde{g}(p,t)\,\mathrm{d}p^1 \wedge \cdots \wedge \mathrm{d}p^{2d}. \tag{3.3.14}$$

First change variables to integrate over $x \in \mathbb{R}^{2d}$:

$$J(N) = \int_{\mathscr{U}^{\mathbb{R}}} e^{N\tilde{\Psi}(p(x),t)}\tilde{g}(p(x),t)\left(\frac{\partial p}{\partial x}\right)\mathrm{d}x.$$

Next, Taylor expand the phase about the critical point and interpolate the remainder. Define

$$\begin{aligned} iR(x,t) &\coloneqq \tilde{\Psi}(p(x),t) - \tilde{\Psi}(\tilde{p}(t),t) + |x|^2/2, \\ \Psi_s(x,t) &\coloneqq \tilde{\Psi}(\tilde{p}(t),t) - |x|^2/2 + isR(x,t), \\ h(x) &\coloneqq \tilde{g}(p(x),t)(\partial p/\partial x). \end{aligned} \tag{3.3.15}$$

Note that $\Psi_1(x,t) = \tilde{\Psi}(p(x),t)$ and $\Psi_0(x,t) = \tilde{\Psi}(\tilde{p}(t),t) - |x|^2/2$. We would like to prove that $J(N)$ can be estimated using the Ψ_0 phase with $\mathscr{O}(N^{-\infty})$ error.

Using that

$$\int_0^1 NR(x,t)e^{N\Psi_s(x,t)}ds = e^{N\Psi_1(x,t)} - e^{N\Psi_0(x,t)},$$

we get that

$$\begin{aligned} \left|\int_{\mathscr{U}^{\mathbb{R}}} \left(e^{N\Psi_1(x,t)} - e^{N\Psi_0(p(x),t)}\right) h(x)\,\mathrm{d}x\right| &= \left|\int_0^1 \int_{\mathscr{U}^{\mathbb{R}}} h(x)NR(x,t)e^{N\Psi_s(x,t)}\,\mathrm{d}x\,\mathrm{d}s\right| \\ &\leq N\int_0^1 \int_{\mathscr{U}^{\mathbb{R}}} |h(x)R(x,t)|e^{N\mathrm{Re}(\Psi_s(x,t))}\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.3.16}$$

We can control $R(x,t)$ by the following Lemma presented in [Tre80, Chapter 10, Lemma 3.2].

Lemma 3.3.4. *For all $x \in \mathscr{U}^{\mathbb{R}}(t)$, t near 0, $M > 0$, there exist $C_M > 0$ such that $|R(x,t)| \leq C_M(|\mathrm{Im}\,(p(x))| + |\mathrm{Im}\,(\tilde{p}(t))|)^M$.*

Using this, and $\mathrm{Re}\left(\tilde{\Psi}(p(x),t)\right) \leq -C(|\mathrm{Im}\,(\tilde{p}(t))|^2 + |\mathrm{Im}\,(p(x))|^2)$ (by (3.3.13)), we see that for all $M \in \mathbb{N}$, there exists $C_M > 0$, such that

$$|R(x,t)| \leq -C_M\left(\mathrm{Re}\left(\tilde{\Psi}(p(x),t)\right)\right)^M.$$

By expanding $\Psi_s(x,t)$,

$$\begin{aligned} \mathrm{Re}\,(\Psi_s(x,t)) &= (1-s)\mathrm{Re}\left(\tilde{\Psi}(\tilde{p}(t),t)\right) + (1-s)\left(\frac{-|x|^2}{2}\right) + s\mathrm{Re}\left(\tilde{\Psi}(p(x),t)\right) \\ &\leq s\mathrm{Re}\left(\tilde{\Psi}(p(x),t)\right). \end{aligned}$$

Then, since $|h(x)| \leq CN^{\delta M_0}$ (for some $C > 0$), (3.3.16) is bounded by

$$\begin{aligned} CN^{\delta M_0+1}\Bigg(\int_{\mathscr{U}^{\mathbb{R}}}\Bigg(\int_0^{1/2}\Big(-\mathrm{Re}\left(\tilde{\Psi}(p(x),t)\right)\Big)e^{\frac{-N}{4}|x|^2}\,\mathrm{d}s \\ + \int_{\frac{1}{2}}^1 e^{sN\mathrm{Re}\left(\tilde{\Psi}(p(x),t)\right)}\left|\mathrm{Re}\left(\tilde{\Psi}(p(x),t)\right)\right|^M\mathrm{d}s\Bigg)\,\mathrm{d}x\Bigg). \end{aligned}$$

Because $\mathscr{U}^{\mathbb{R}}$ is bounded, and Ψ is bounded, both terms are $\mathscr{O}(N^{-\infty})$.

Therefore:

$$J(N) = e^{N\tilde{\Psi}(\tilde{p}(t),t)} \int_{\mathscr{U}^{\mathbb{R}}} e^{-N|x|^2/2} g(p(x),t) \left(\frac{\partial p}{\partial x}\right) \mathrm{d}x + \mathscr{O}(N^{-\infty}) \tag{3.3.17}$$

Summary of step 4. This step proceeded identically to [Tre80, Chapter 10]. We began with our integral on the constructed contour U_0 in $\mathbb{C}^{2d}$ (3.3.14). We changed variables to integrate over the real variable $x \in \mathbb{R}^{2d}$. We proved that this integral can be approximated by replacing the phase with the critical value of the phase minus a quadratic term. This is to set us up to apply the saddle-point method (also called real stationary phase).

Step 5: Apply the saddle-point method

By the saddle-point method (see for instance [GS94, Exercise 2.4]) for each $J \in \mathbb{N}$ we can now rewrite (3.3.17) as

$$\begin{aligned} &e^{N\tilde{\Psi}(\tilde{p}(t),t)} \int_{\mathbb{R}^{2d}} e^{-N|x|^2/2} h(x) \,\mathrm{d}x + \mathscr{O}(N^{-\infty}) \\ &\qquad = e^{N\tilde{\Psi}(\tilde{p}(t),t)} \left(\frac{2\pi}{N}\right)^d \left(\sum_{j=0}^{J-1} \frac{1}{N^j j! 2^j} \Delta^j h(0) + N^{-J} R_J(t) \right) \end{aligned} \tag{3.3.18}$$

with error bound

$$|R_J(t)| \lesssim_J \sum_{|\alpha|=2(J+1)} \sup_{x\in\mathbb{R}^{2d}} |\partial^\alpha h(x)|,$$

where h is defined in (3.3.15).

We now have to unravel all the definitions of the functions in (3.3.18). First, replace t by $(x, \bar{y})$. From the first four steps, we have shown that for x, y near zero, in local coordinates, $\exp(-(N/2)(\varphi(x) + \varphi(y))T_{N,f}(x, \bar{y})$ is

$$e^{N\tilde{\Psi}(\tilde{p}(x,\bar{y}),x,\bar{y})} \left(\frac{N}{2\pi}\right)^d \left(\sum_{j=0}^{J-1} N^{-j} f_j(x,\bar{y}) + N^{-J} R_J(x,\bar{y}) \right) + \mathscr{O}(N^{-\infty}), \tag{3.3.19}$$

for each $J \in \mathbb{N}$, where:

$$f_j(x,\bar{y}) = (j!2^j)^{-1} \Delta_z^j h(x,\bar{y},0) \tag{3.3.20}$$

with

$$h(x,\bar{y},z) = \tilde{f}(p(x,\bar{y},z))\tilde{g}_2(x,\bar{y},p(x,\bar{y},z)) \det\left(\frac{\partial p(x,\bar{y},z)}{\partial z}\right). \tag{3.3.21}$$

Here we are defining $p(x, \bar{y}, \cdot) \coloneqq q^{-1}(\cdot, x, \bar{y})$ (with q the change of variables defined in Lemma 3.3.2). As usual, the derivatives of the terms in the stationary phase expansion are evaluated at the critical point of the (almost analytically extended) phase. Indeed,

$$p(x, \bar{y}, 0) = q^{-1}(\cdot, x, \bar{y})(0) = \tilde{p}(x, \bar{y})$$

by the first property of q in Lemma 3.3.2.

Recall that $x, y \in \mathbb{C}^d$, $z \in \mathbb{R}^{2d}$, $p \in \mathbb{C}^{2d}$, and $\tilde{g}_2$ is an almost analytic extension of g defined in the following way. We let

$$g_2(w, x, \bar{y}) \coloneqq B(x, \bar{w})B(w, \bar{y}) \det(\partial\overline{\partial}\varphi(w))\chi(w) \colon \mathbb{C}^d_w \times \mathbb{C}^d_x \times \mathbb{C}^d_y \to \mathbb{C}$$

where $B(\cdot, \cdot)$ comes from the Bergman kernel expansion, φ is the Kähler potential and χ is a smooth cut-off function. Then we let $p = (\mathrm{Re}\,(w), \mathrm{Im}\,(w)) \in \mathbb{R}^{2d}$, and define

$$(g_2)_{\mathbb{R}}(p, x, \bar{y}) \coloneqq g_2(w(p), x, \bar{y}) \colon \mathbb{R}^{2d}_p \times \mathbb{C}^d_x \times \mathbb{C}^d_y \to \mathbb{C}$$

and finally let $\tilde{g}_2$ be the almost analytic extension of $(g_2)_{\mathbb{R}}$ in the p variable.

By the support property of almost analytic extensions, we can choose an $\varepsilon > 0$ such that $\tilde{g}_2(x, \bar{y}, p(x, \bar{y}, z)) = 0$ if $|p(x, \bar{y}, z)| > \varepsilon$.

Also observe that when taking derivatives of h with respect to z, everything is uniformly bounded in N, except when derivatives fall on $\tilde{f}(p)$. Therefore by Proposition 3.2.7, for any $\alpha \in \mathbb{N}^{4d}$ and $j \in \mathbb{Z}_{\geq 0}$:

$$|(\partial^{\alpha}_{x,\bar{x},y,\bar{y}} f_j)(x, \bar{y})| \lesssim_{\alpha,j} N^{2\delta j + |\alpha|} m_{\mathbb{R}}(\mathrm{Re}\,(\tilde{p}(x, \bar{y}))), \tag{3.3.22}$$

with $m_{\mathbb{R}}$ defined in (3.3.12).

Summary of step 5. In this step we applied the saddle-point method to obtain an asymptotic expansion of the Schwartz kernel of $T_N f$. To show that this asymptotic expansion makes sense (f_j's belong to appropriate symbol classes and the remainder is controlled) we have to compute derivatives of the terms in the expansion. These terms are almost analytic extensions of functions whose derivatives are unbounded in N. However, we can see that they are bounded by powers of N times the order function evaluated at the critical point of the phase, $\tilde{p}(x, \bar{y})$. We must now estimate $\tilde{p}(x, \bar{y})$ (this will also be used in estimating $\exp(N\tilde{\Psi}(\tilde{p}(x, \bar{y}), x, \bar{y}))$.

Step 6: Estimate critical value of phase

Recall that for each $x, y \in \mathbb{C}^d$ near 0, $\tilde{p}(x, \bar{y})$ is the unique $p \in \mathbb{C}^{2d}$ such that

$$\partial_p \tilde{\Psi}(p, x, \bar{y}) = 0$$

where ∂_p is the holomorphic derivative in the p variable, and $\tilde{\Psi}(p, x, \bar{y})$ is an almost analytic extension of $\Psi(p, x, \bar{y})$ (defined in (3.3.8)) in the p variable.

The goal is to show that $\tilde{p}(x, y)$ is

$$\frac{1}{2}((x + y), -i(x - y)) + \mathscr{O}(|x - \bar{y}|^{\infty}). \tag{3.3.23}$$

This would follow immediately, with no error, if we considered real-analytic Kähler potentials φ. We will show that $\tilde{p}(x, y)$ is almost analytic off of $y = \bar{x}$, and coincides with $\frac{1}{2}((x + y), -i(x - y))$ on $y = \bar{x}$. By uniqueness of almost analytic extensions (modulo appropriate error), this will imply that $\tilde{p}(x, y)$ is (3.3.23).

Lemma 3.3.5. *If $|x - y|$ is sufficiently small, there exists a constant $C > 0$ such that:*

$$|\tilde{p}_1(x, \bar{y}) + i\tilde{p}_2(x, \bar{y}) - x| \leq C|x - y|,$$

where $\tilde{p} = (\tilde{p}_1, \tilde{p}_2) \in \mathbb{C}^d \times \mathbb{C}^d$.

Proof. By Taylor expansion, it suffices to show that $\tilde{p}(x, \bar{x}) = (\mathrm{Re}\,(x), \mathrm{Im}\,(x))$. Before extension, $\partial_p \Psi(p_1, p_2, x, \bar{x}) = 0$ has the unique solution $p_1 = \mathrm{Re}\,(x), p_2 = \mathrm{Im}\,(x)$. Observe that for any $k \in C_0^\infty(\mathbb{R}; \mathbb{C})$ with an almost analytic extension $\tilde{k}$, and any $z_0 \in \mathbb{C}$ with $\mathrm{Im}\,(z_0) = 0$,

$$(\partial_z \tilde{k})(z_0) = (\partial_{\mathrm{Re}(z)} \tilde{k})(z_0) = (\partial k)(\mathrm{Re}\,(z_0))$$

by Theorem 3.2.6.

Applying this observation to $\tilde{\Psi}$ and letting $\tilde{p}_1$ be the extension of p_1 from $\mathbb{R}^d$ to $\mathbb{C}^d$, we see that

$$\begin{aligned} \frac{1}{2}\partial_{\tilde{p}_1} \tilde{\Psi}(\mathrm{Re}\,(x), \mathrm{Im}\,(x), x, \bar{x}) &= \partial_{\mathrm{Re}(\tilde{p}_1)} \tilde{\Psi}(\mathrm{Re}\,(x), \mathrm{Im}\,(x), x, \bar{x}) \\ &= \partial_{p_1} \tilde{\Psi}(\mathrm{Re}\,(x), \mathrm{Im}\,(x), x, \bar{x}) = 0 \end{aligned}$$

and similarly for $\partial_{\tilde{p}_2}$. Because $\tilde{p}$ is unique, the claim is proven. □

Lemma 3.3.6. *If $|x - y|$ is sufficiently small, then*

$$\overline{\partial}_x \tilde{p}(x, y) = \mathscr{O}(|x - y|^\infty) \qquad \text{and} \qquad \overline{\partial}_y \tilde{p}(x, y) = \mathscr{O}(|x - y|^\infty).$$

Before proving Lemma 3.3.6, we give some brief remarks. Proving this lemma is relatively confusing partially due to non-optimal notation (however we try to present a proof as clearly as possible). The difficulty is that we require $\overline{\partial}$ estimate of an almost analytic extension of a function that has been almost analytically extended.

The core of the proof is to show that various $\overline{\partial}$ estimates of $\tilde{\Psi}$ rapidly decay as $|x - \bar{y}|$ goes to zero. Recall the construction of $\tilde{\Psi}$. We began with φ (the Kähler potential), then we almost analytically extended it to ψ such that $\psi(x, \bar{x}) = \varphi(x)$, then we defined $\Psi(p, x, \bar{y}) = \psi(x, \bar{p}) - \varphi(p) + \psi(p, \bar{y}) - \frac{1}{2}(\varphi(x) + \varphi(y))$, then we almost analytically extended this in the p variable. See Figure 3.2 for a schematic diagram of these extensions.

One example to keep in mind is the Kähler manifold $\mathbb{C}$ with symplectic form $\omega = i\, \mathrm{d}x \wedge \mathrm{d}\bar{x}$ with Kähler potential $\varphi(x) = |x|^2$. In this case $\psi(x, \bar{y}) = x\bar{y}$ (which is unique), so that $\Psi(p, x, \bar{y}) = x\bar{p} - |p|^2 + p\bar{y} - \frac{1}{2}(|x|^2 + |y|^2)$. The (unique) almost analytic extension of this in the p variable is $\tilde{\Psi}(p, x, \bar{y}) = x(p_1 - ip_2) - p_1^2 - p_2^2 + (p_1 + ip_2)\bar{y} - \frac{1}{2}(|x|^2 + |y|^2)$ (where $p = (p_1, p_2) \in \mathbb{C}^d \times \mathbb{C}^d$).

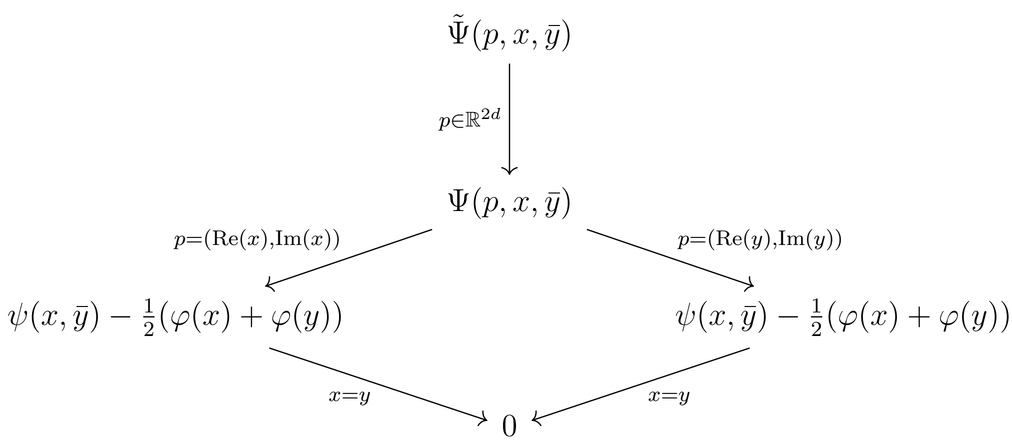

Figure 3.2: Diagram of the restrictions of $\tilde{\Psi}$ to totally real vector spaces. First, we have $\tilde{\Psi}(p,x,\bar{y})\in C^\infty(\mathbb{C}^{2d}_p\times\mathbb{C}^d_x\times\mathbb{C}^d_y)$. This is almost analytic off of $p\in\mathbb{R}^{2d}$, whose restriction to $p\in\mathbb{R}^{2d}$ is $\Psi(p,x,\bar{y})\in C^\infty(\mathbb{R}^{2d}_p\times\mathbb{C}^d_x\times\mathbb{C}^d_y)$. We can restrict Ψ to either $p=(\mathrm{Re}\,(x)\,,\mathrm{Im}\,(x))$ or $p=(\mathrm{Re}\,(y)\,,\mathrm{Im}\,(y))$ to get the same function as shown. When either of these functions are restricted to $x=y$ we get the zero function. Understanding various $\overline{\partial}$ estimates on $\tilde{\Psi}$ is the core part of this step of the proof.

The key properties of $\tilde{\Psi}$ to use is that $\tilde{\Psi}(p,x,\bar{y})=\Psi(p,x,\bar{y})$ when $p\in\mathbb{R}^{2d}$ and $\psi(x,\bar{x})=\varphi(x)$ (ie the extensions agree on certain totally real vector spaces). By Taylor expanding from these totally real vector spaces, we prove Lemma 3.3.6.

Proof. Recall the chain rule for holomorphic differentiation:

$$\overline{\partial}_z(f(g(z))=(\partial_z f)(g(z))\cdot(\overline{\partial}_z g)(z)+(\overline{\partial}_z f)(g(z))\cdot(\overline{\partial}_z\bar{g})(z)$$

for arbitrary $f,g\in C^\infty(\mathbb{C})$.

We can use this when computing $\overline{\partial}_{x_j}\partial_p\tilde{\Psi}$ in conjunction with the implicit function theorem, to see that

$$\overline{\partial}_{x_j}\tilde{p}(x,y)=-(\partial_p\partial_p\tilde{\Psi})^{-1}\left((\overline{\partial}_{x_j}\partial_p\tilde{\Psi})+(\overline{\partial}_p\partial_p\tilde{\Psi})\overline{\partial}_{x_j}\overline{\tilde{p}}(x,y)\right)\tag{3.3.24}$$

where all derivatives of $\tilde{\Psi}$ are evaluated at $(\tilde{p}(x,y),x,y)$. The inverted term is uniformly bounded for x,y close to zero. We now claim the following $\overline{\partial}\tilde{\Psi}$ estimates at $(\tilde{w}_1,\tilde{w}_2,x,y)$ for $\tilde{w}_1,\tilde{w}_2\in\mathbb{C}^d$:

$$\begin{aligned}\overline{\partial}_x\tilde{\Psi}&=\mathscr{O}((|\mathrm{Im}\,(\tilde{w}_1)|+|\mathrm{Im}\,(\tilde{w}_2)|+|x-\mathrm{Re}\,(\tilde{w}_1)-i\mathrm{Re}\,(\tilde{w}_2)|)^\infty),\qquad(3.3.25)\\ \overline{\partial}_y\tilde{\Psi}&=\mathscr{O}((|\mathrm{Im}\,(\tilde{w}_1)|+|\mathrm{Im}\,(\tilde{w}_2)|+|y-\mathrm{Re}\,(\tilde{w}_1)+i\mathrm{Re}\,(\tilde{w}_2)|)^\infty),\\ \overline{\partial}_{\tilde{p}}\tilde{\Psi}&=\mathscr{O}((|\mathrm{Im}\,(\tilde{w}_1)|+|\mathrm{Im}\,(\tilde{w}_2)|)^\infty).\end{aligned}$$

Here we prove (3.3.25). Before extension, the only term in Ψ depending on x is $\psi(x, \bar{w})$ (recall ψ is an almost analytic extension of φ – the Kähler potential satisfying $i\partial\overline{\partial}\varphi = \omega$). In real coordinates, this is $\psi(x_1, x_2, w_1, -w_2)$. An almost analytic extension in the w variable can be written $\tilde{\psi}(x_1, x_2, \tilde{w}_1, -\tilde{w}_2) : \mathbb{C}^{4d} \to \mathbb{C}$. To compute $\overline{\partial}_x\tilde{\psi}$, we Taylor expand about $(x_1, x_2, \mathrm{Re}\,(\tilde{w}_1), -\mathrm{Re}\,(\tilde{w}_2))$ with $K \in \mathbb{N}$ terms, so $\overline{\partial}_x\tilde{\psi}(x_1, x_2, \tilde{w}_1, \tilde{w}_2)$ is

$$\frac{1}{2}(\partial_{x_1} + i\partial_{x_2})\left(\sum_{|\alpha|\leq K}\frac{\partial^\alpha_{w_1,w_2}\tilde{\psi}(x_1, x_2, \mathrm{Re}\,(\tilde{w}_1), -\mathrm{Re}\,(\tilde{w}_2))}{\alpha!}(i\mathrm{Im}\,(\tilde{w}_1), i\mathrm{Im}\,(\tilde{w}_2))^\alpha \right.$$
$$\left. + R_K(x_1, x_2, \tilde{w}_1, \tilde{w}_2)\right).$$

The $\overline{\partial}_x$ operator can be commuted with the $\partial^\alpha_{w_1,w_2}$ operator by Proposition 3.2.6. At $(x_1, x_2, \mathrm{Re}\,(\tilde{w}_1), \mathrm{Re}\,(\tilde{w}_2))$, $\tilde{\psi} = \psi$, and the $\overline{\partial}$ estimates can be made uniform with respect to differentiation of w_i. Recalling how ψ is an almost analytic extension of φ provides the estimate: $(\partial_{x_1} + i\partial_{x_2})\psi(x_1, x_2, y_1, y_2) = \mathscr{O}(|(x_1 + ix_2) - (y_1 - iy_2)|^\infty)$, so that for each α:

$$\frac{1}{2\alpha!}\overline{\partial}_x\partial^\alpha_{w_1,w_2}\tilde{\psi}(x_1, x_2, \mathrm{Re}\,(\tilde{w}_1), -\mathrm{Re}\,(\tilde{w}_2))(i\mathrm{Im}\,(\tilde{w}_1), i\mathrm{Im}\,(\tilde{w}_2))^\alpha$$
$$= |\mathrm{Im}\,(\tilde{w})|^{|\alpha|}\,\mathscr{O}(|x - \mathrm{Re}\,(\tilde{w}_1) - i\mathrm{Re}\,(\tilde{w}_2)|^\infty)$$

while $\overline{\partial}_x R_k = \mathscr{O}(|\mathrm{Im}\,(\tilde{w}))|^{K+1})$. This proves (3.3.25) and the others follow similarly.

Because $\mathrm{Im}\,(\tilde{p}(x, \bar{x})) = 0$, by Taylor expansion: $|\mathrm{Im}\,(\tilde{p}(x, y))|^2 \leq C|x - \bar{y}|^2$ for some $C > 0$. By Lemma 3.3.5, $|x - \mathrm{Re}\,(\tilde{p}_1(x, y)) - i\mathrm{Re}\,(\tilde{p}_2(x, y))\,| \leq C|x - \bar{y}|$. Using this, and the estimate of $\overline{\partial}_x\tilde{\Psi}$, we see that $\overline{\partial}_x\tilde{\Psi}(\tilde{p}(x, y), x, y) = \mathscr{O}(|x - \bar{y}|^\infty)$. This is true for all terms on the right side of (3.3.24), so that $\overline{\partial}_x\tilde{p}(x, y) = \mathscr{O}(|x - \bar{y}|^\infty)$. By an identical argument, $\overline{\partial}_y\tilde{p}(x, y) = \mathscr{O}(|x - \bar{y}|^\infty)$. □

Lemma 3.3.7. *For $|x - \bar{y}|$ sufficiently small*

$$\tilde{p}(x, y) = \left(\frac{1}{2}(x + y), \frac{1}{2i}(x - y)\right) + \mathscr{O}(|x - \bar{y}|^\infty).$$

Proof. From Lemma 3.3.6, we have that $\tilde{p}$ is almost analytic off the diagonal $y = \bar{x}$. The function $(x, y) \mapsto (2^{-1}(x + y), (2i)^{-1}(x - y))$ is holomorphic and agrees with $\tilde{p}$ on $y = \bar{x}$. Therefore by uniqueness (modulo $\mathscr{O}(|x - \bar{y}|^\infty)$ error) of almost analytic extensions, the lemma follows. □

Summary of step 6. In this step we provided an estimate of $\tilde{p}(x, \bar{y})$, the critical point of the phase $\tilde{\Psi}(p, x, \bar{y})$. We will now use this to provide derivative estimates of the terms coming from the stationary phase expansion in (3.3.22).

Step 7: Prove symbol estimates of stationary phase terms

A simple computation shows that from Lemma 3.3.7,

$$\mathrm{Re}\,(\tilde{p}(x,y))_1 + i\mathrm{Re}\,(\tilde{p}(x,y))_2 = \frac{1}{2}(x+\bar{y}) + \mathscr{O}(|x-\bar{y}|^\infty), \tag{3.3.26}$$

where for $a, b \in \mathbb{C}^d$, $c = (a,b) \in \mathbb{C}^{2d}$, we write $c_1 = a$ and $c_2 = b$.

Then, recalling the definition of $m_\mathbb{R}$ from (3.3.12) and using (3.3.26), we get that

$$\begin{aligned} m_\mathbb{R}(\mathrm{Re}\,(\tilde{p}(x,\bar{y}))) &= m\left(\frac{1}{2}(x+y) + \mathscr{O}(|x-y|^\infty)\right) \\ &\lesssim m(x)(1+N^\delta|x-y|)^{M_0}. \end{aligned} \tag{3.3.27}$$

Applying (3.3.27) to the derivative estimate of f_j in (3.3.22), we get that

$$f_j(x,\bar{y}) \in N^{2\delta j} S_\delta(m_\mathbb{R}(\tilde{p}(x,\bar{y}))) \subset N^{2\delta j} S_\delta(m(x)(1+N^\delta|x-y|)^{M_0}). \tag{3.3.28}$$

We similarly have that $|\tilde{p}_1(x,\bar{y}) + i\tilde{p}_2(x,\bar{y}) - y| \le C|x-y|$ (for some$C > 0$), so that

$$f_j(x,\bar{y}) \in N^{2\delta j} S_\delta(m(y)(1+N^\delta|x-y|)^{M_0}).$$

The support of these f_j's are contained in a strip along the diagonal, shrinking with respect to N. Indeed, because $p(x,\bar{y},0) = \tilde{p}(x,\bar{y}) = (2^{-1}(x+\bar{y}), (2i)^{-1}(x-\bar{y})) + \mathscr{O}(|x-y|^\infty)$ we get that $|\mathrm{Im}\,(p(x,\bar{y},0))| \lesssim |x-y|$. Then observe that in (3.3.21), the term $\tilde{f}(p)$ is included, which by Proposition 3.2.7, is supported where $|\mathrm{Im}\,(p)| \lesssim N^{-\delta}$. Therefore, there exists $C > 0$ such that

$$\mathrm{supp}\, f_j(x,\bar{y}) \subset \left\{|x-y| \le CN^{-\delta}\right\}. \tag{3.3.29}$$

But now we can apply (3.3.29) to (3.3.28) to see that:

$$f_j(x,\bar{y}) \in N^{2\delta j}\left(S_\delta(m(x)) \cap S_\delta(m(y))\right).$$

The remainder can be bounded similarly. For each $\alpha \in \mathbb{N}^{4d}$

$$\begin{aligned} |(\partial^\alpha_{x,\bar{x},y,\bar{y}} R_J)(x,\bar{y})| &\lesssim_{\alpha,J} \sum_{|\beta|=2J} \sup_z |(\partial^\alpha_{x,\bar{x},y,\bar{y}}\partial^\beta_z h)(x,\bar{y},z)| \\ &\lesssim_{\alpha,J} N^{2J\delta} N^{|\alpha|\delta} \sup_{z\in\mathrm{supp}\, h(x,\bar{y},\cdot)} m_\mathbb{R}(\mathrm{Re}\,(p(x,\bar{y},z))). \end{aligned}$$

Now $\tilde{g}_2$ (defined in (3.3.21)) is only supported where $|p| < \varepsilon$ so there exists $C > 0$ such that

$$\sup_{z\in\mathrm{supp}\, h(x,\bar{y},\cdot)} m_\mathbb{R}(\mathrm{Re}\,(p(x,\bar{y},z))) \le C\min(m(x), m(y))(1+N^\delta\varepsilon)^{M_0},$$

therefore

$$R_J(x,\bar{y}) \in N^{2\delta J + \delta M_0}(S_\delta(m(x)) \cap S_\delta(m(y))).$$

We can bootstrap this to prove the better remainder bound stated in the Theorem. For any $J \in \mathbb{Z}_{>0}$, we rewrite the sum and remainder in (3.3.19) as

$$\sum_{j=0}^{J-1} N^{-j} f_j(x,\bar{y}) + N^{-J}\left(\sum_{j=J}^{\tilde{J}-1} N^{J-j} f_j(x,\bar{y}) + N^{J-\tilde{J}} R_{\tilde{J}}(x,\bar{y})\right) \tag{3.3.30}$$

where $\mathbb{Z} \ni \tilde{J} > J + (\delta M_0)(1-2\delta)^{-1}$. This choice of $\tilde{J}$ ensures that $N^{J-\tilde{J}} R_{\tilde{J}}(x,\bar{y})$ belongs to $N^{2\delta J}(S_\delta(m(x)) \cap S_\delta(m(y)))$. It is also clear that for each $j = J, \dots, \tilde{J}-1$, $N^{-j+J} f_j(x,\bar{y}) \in N^{2\delta J}(S_\delta(m(x)) \cap S_\delta(m(y)))$. We can therefore define the terms multiplied by N^{-J} in (3.3.30) as the remainder term $R_J(x,\bar{y})$ stated in the Theorem.

Summary of step 7. We showed in step 5 that derivatives of terms in the stationary phase expansion in (3.3.19) are bounded in terms of the order function of f evaluated at the critical point of the almost analytically extended phase, $\tilde{p}(x,\bar{y})$ (see (3.3.22)). From step 6, we estimated $\tilde{p}(x,\bar{y})$ (Lemma 3.3.7) to provide more explicit symbol estimates for f_j's, and the remainder terms, in the stationary phase expansion. We now have a local expansion of the Schwartz kernel of the Toeplitz operator $T_N f$. To prove symbolic calculus results, we need to show this expansion is unique (modulo appropriate error) and that the principal term is the principal part of f.

Step 8: Prove stationary phase terms are almost analytic off the diagonal

Lemma 3.3.8. *We may choose an almost analytic extension of* Ψ *such that* $\Psi(\tilde{p}(t),t)) = \psi(x,\bar{y}) - 2^{-1}(\varphi(x)+\varphi(y))$.

Proof. Because the Toeplitz quantization of the identity is the Bergman kernel, the phase can be recovered up to an appropriate error. Recall that [BBS08] showed:

$$e^{-\frac{N}{2}(\varphi(x)+\varphi(y))} T_{N,1}(x,\bar{y}) \sim e^{N(\psi(x,\bar{y})-\frac{1}{2}(\varphi(x)+\varphi(y))}\left(\frac{N}{2\pi}\right)^d \sum_0^\infty N^{-j} b_j(x,\bar{y}).$$

This must agree, up to $\mathscr{O}(N^{-\infty})$ error, with (3.3.19). It is therefore possible to choose $\tilde{\Psi}$ such that $\tilde{\Psi}(\tilde{p}(t),t) = \psi(x,\bar{y}) - 2^{-1}(\varphi(x)+\varphi(y))$. □

Lemma 3.3.9. *All the* $f_j(x,y)$*'s are almost analytic off of* $y = \bar{x}$.

Proof. When computing $(\overline{\partial}_{x,y}\Delta_z^j)h(x,\bar{y},0)$, observe the following properties of the functions making up h in (3.3.21):

1. When $\overline{\partial}_x$ falls on $\tilde{f}$, we get $(\partial_p \tilde{f})(\overline{\partial}_x \tilde{p}) + (\overline{\partial}_p \tilde{f})(\overline{\partial}_x \overline{\tilde{p}})$. The first term is controlled by almost analyticity of $\tilde{p}$ off of $y = \bar{x}$ while the second term is controlled by almost analyticity of $\tilde{f}$ off of $\tilde{p} \in \mathbb{R}^{2d}$.

2. When $\overline{\partial}_x$ falls on $\tilde{g}_2$, we get $(\overline{\partial}_1\tilde{g}_2) + (\partial_3\tilde{g}_2)(\overline{\partial}_x\tilde{p}) + (\overline{\partial}_3\tilde{g}_2)(\overline{\partial}\overline{\tilde{p}})$ where ∂_i and $\overline{\partial}_i$ are the holomorphic and anti-holomorphic derivatives of the i^{th} argument of g respectively. The first term is controlled by almost analyticity of $B(x, \bar{w})$ off of $w = x$, the second term is controlled by almost analyticity of $\tilde{p}$ off of $y = \bar{x}$, and the third term is controlled by almost analyticity off both totally real manifolds.

3. When $\overline{\partial}_x$ falls on the determinant, we get control by the almost analyticity of $\tilde{p}$ off of $y = \bar{x}$.

Therefore

$$\overline{\partial}_x f_j(x, y) = N^{\delta(1+2j+M_0)}\mathscr{O}((|x - y|N^{\delta})^{\infty}). \tag{3.3.31}$$

□

Note that in this expansion, only knowledge of the kernel along the diagonal is required. Indeed, if f_j and g_j agree along $y = x$, and obey (3.3.31), then by the Gaussian behavior of the phase:

$$\left(\frac{N}{2\pi}\right)^d e^{N\psi(x,\bar{y})} \sum N^{-j}(f_j(x, \bar{y}) - g_j(x, \bar{y})) = e^{\frac{N}{2}(\varphi(x)+\varphi(y))}\mathscr{O}(N^{-\infty}).$$

Summary of step 8. Here we show the terms coming from the stationary phase expansion are almost analytic off the totally real vector space $x = y$ which provides a unique expansion (modulo appropriate error). The final step of the proof is to compute the first term along the diagonal, and prove the global statement.

Step 9: Zeroth order term and global statement

Examining (3.3.20) and the subsequent equations, along the diagonal

$$f_0(x, \bar{x}) = f(x)B(x, \bar{x})B(x, \bar{x}) \det(\partial\overline{\partial}\varphi(x)) \det\left(\frac{\partial q^{-1}(x, \bar{x}, z)}{\partial z}\right)\Big|_{z=0}.$$

This can all be explicitly computed. But note that nothing, except f, on the right-hand side depends on f. And if $f = 1$, then $T_{N,f} = \Pi_N$. By [BBS08], the leading order term of Π_N is 1, therefore, we know that everything on the right-hand side of order N^0, except f, must be 1. Therefore $f_0(x, \bar{x}) = f(x) + \mathscr{O}(N^{-(1-2\delta)}m(x))$. In the appendix, the second term is computed.

We now have proven existence of f_j locally, in a ball of radius ε, around any point $x \in X$. Because each f_j is unique along the diagonal, we can patch together $f_j's$ to construct a global f_j defined near the diagonal. □

3.3.2 Composition of Toeplitz operators

Suppose that $f \in S_\delta(m_1)$ and $g \in S_\delta(m_2)$ for two δ-order functions m_1 and m_2 and $\delta \in [0, 1/2)$. Roughly, this section constructs a function $h \in S_\delta(m_1 m_2)$ such that $T_{N,f} \circ T_{N,g} \approx T_{N,h}$. This h will be written as a star product: $h = f \star g$ following the now standard notation first introduced in [Bay+78]. We first formally construct $f \star g$.

By Theorem 3.3.1, there exist functions f_j and g_j for $j \in \mathbb{Z}_{\geq 0}$ such that

$$T_{N,f}(x, \bar{y}) \sim \left(\frac{N}{2\pi}\right)^d e^{N\psi(x,\bar{y})} \sum_{j=0}^{\infty} N^{-j} f_j(x, \bar{y}),$$
$$T_{N,g}(x, \bar{y}) \sim \left(\frac{N}{2\pi}\right)^d e^{N\psi(x,\bar{y})} \sum_{j=0}^{\infty} N^{-j} g_j(x, \bar{y})$$

in local coordinates.

Define the operators $C_j : C^\infty(X; \mathbb{C}) \to C^\infty(X \times X; \mathbb{C})$ by $C_j(f) \coloneqq B^{-1}(x, \bar{y}) f_j(x, \bar{y})$. Recall that $B = 1 + N^{-1} b_1 + \cdots$ is the amplitude of the Bergman kernel. Note that B is bounded below for N sufficiently large. Explicitly, for x, y contained in a sufficiently small neighborhood U, C_j are such that

$$\left(\frac{N}{2\pi}\right)^{2d} \int_U e^{N\Phi_{x,\bar{y}}(w)} B(x, \bar{w}) B(w, \bar{y}) f(w) \, \mathrm{d}\mu(w)$$

is asymptotically

$$\left(\frac{N}{2\pi}\right)^d e^{N\psi(x,\bar{y})} B(x, \bar{y}) \sum_{j=0}^{\infty} N^{-j} C_j[f(\cdot)](x, \bar{y}).$$

where

$$\Phi_{x,\bar{y}}(w) \coloneqq \psi(x, \bar{w}) - \varphi(w) + \psi(w, \bar{y}).$$

By (3.3.20), C_j are differential operators of order at most $2j$. Using these C_j, we can formally construct $f \star g \in S_\delta(m_1 m_2)$ such that $T_{N,f\star g} = T_{N,f} \circ T_{N,g} + \mathscr{O}(N^{-\infty})$.

To achieve this, we again restrict ourselves to (x, y) near (x_0, x_0). In a neighborhood U

of x_0, $(T_{N,f} \circ T_{N,g})(x, \bar{y})$ is formally

$$\begin{aligned}
&\int_U T_{N,f}(x,\bar{w})T_{N,g}(w,\bar{y})e^{-N\varphi(w)}\,\mathrm{d}\mu(w) \\
&\sim \left(\frac{N}{2\pi}\right)^{2d} \int_U B(x,\bar{w})B(w,\bar{y})e^{N\Phi_{x,\bar{y}}(w)} \left(\sum_{k=0}^{\infty} N^{-k} C_k[f](x,\bar{w})\right) \\
&\qquad\qquad\qquad\qquad \cdot \left(\sum_{j=0}^{\infty} N^{-j} C_j[g](w,\bar{y})\right) \mathrm{d}\mu(w) \\
&\sim \left(\frac{N}{2\pi}\right)^{2d} \sum_{j=0}^{\infty} N^{-j} \int_U e^{\Phi_{x,\bar{y}}(w)} B(x,\bar{w})B(w,\bar{y}) \\
&\qquad\qquad\qquad\qquad \cdot \left(\sum_{a+b=j} C_a[f](x,\bar{w})C_b[g](w,\bar{y})\right) \mathrm{d}\mu(w) \\
&\sim \left(\frac{N}{2\pi}\right)^{d} e^{N\psi(x,\bar{y})} B(x,\bar{y}) \sum_{j=0}^{\infty} N^{-j} k_j(x,\bar{y})
\end{aligned}$$

where for each $j \in \mathbb{Z}_{\geq 0}$:

$$k_j(x,\bar{y}) = \sum_{c+d=j} C_d \left[\sum_{a+b=c} C_a[f](x,\cdot)C_b[g](\cdot,\bar{y})\right](x,\bar{y}). \tag{3.3.32}$$

Now suppose that $h \sim \sum N^{-j} h_j$ is such that the terms in $T_{N,h}$'s asymptotic expansion match (3.3.32). Expanding $T_{N,h}$ asymptotically, we see that

$$T_{N,h}(x,\bar{y}) \sim \left(\frac{N}{2\pi}\right)^{d} e^{N\psi(x,\bar{y})} B(x,\bar{y}) \sum_{j=0}^{\infty} N^{-j} \sum_{c+d=j} C_d[h_c](x,\bar{y}). \tag{3.3.33}$$

We simply match the coefficients of N^{-j} of (3.3.33) with (3.3.32) to get the relation

$$\sum_{c+d=j} C_d[h_c](x,\bar{y}) = \sum_{c+d=j} C_d \left[\sum_{a+b=c} C_a[f](x,\cdot)C_b[g](\cdot,\bar{y})\right](x,\bar{y}). \tag{3.3.34}$$

Recall that $C_0[f(\cdot)](x,\bar{x}) = f(x)$, therefore letting $y = x$ and rearranging (3.3.34) gives us:

$$h_j(x) = \sum_{c+d=j} C_d \left[\sum_{a+b=c} C_a[f](x,\cdot)C_b[g](\cdot,\bar{x})\right](x,\bar{x}) - \sum_{d=1}^{j} C_d[h_{j-d}](x,\bar{x}), \tag{3.3.35}$$

which inductively gives the functions h_j, which is sufficient as $h_0(x) = f(x)g(x)$. This provides us with the following Lemma.

Lemma 3.3.10 (Derivative estimates of $(f \star g)_j$)**.** *There exist linear bi-differential operators $\mathscr{C}_j$ of order at most $2j$ such that $h_j(x) = \mathscr{C}_j[f,g](x)$ agrees with h_j written in (3.3.35).*

Proof. This follows by induction on j. When $j = 0$, $h_0 = fg$ and so $\mathscr{C}_0$ is a zero order operator. Next assume this is true for $j-1$ and apply (3.3.35). The first summation in (3.3.35) involves derivatives of order $2j$, and by the induction hypothesis, the second summation involves derivatives of order $2j$. □

To construct h via Borel's Theorem (Proposition 3.2.5) we need to show that for each j, $h_j \in N^{2\delta j} S_\delta(m_1 m_2)$. This follows immediately by Lemma 3.3.10, as

$$h_j(x) = \mathscr{C}_j[f,g](x) \in N^{2j\delta} S_\delta(m_1 m_2).$$

Theorem 3.3.11 (Composition estimate)**.** *For $\delta \in [0, 1/2)$, suppose m_1 and m_2 are two δ-order functions on X (a quantizable Kähler manifold), $f \in S_\delta(m_1)$ and $g \in S_\delta(m_2)$. Then there exists $(f \star g) \in S_\delta(m_1 m_2)$ (constructed via (3.3.35)) such that*

$$\left\| T_{N,(f\star g)} - T_{N,f} \circ T_{N,g} \right\|_{L^2(X,L^N) \to L^2(X,L^N)} = \mathscr{O}(N^{-\infty}).$$

Proof. Let $h = f \star g$ (constructed asymptotically via (3.3.35) and Proposition 3.2.5) and let $K(x,y)$ be the Schwartz Kernel of $T_{N,h} - T_{N,f} \circ T_{N,g}$. By the Schur test:

$$\begin{aligned} &\|T_{N,h} - T_{N,f} \circ T_{N,g}\|^2_{L^2(X,L^N) \to L^2(X,L^N)} \\ &\quad \le \left(\sup_{x \in X} \int_X \|K(x,y)\|_{L^2(X,L^N)} \, \mathrm{d}\mu(y) \right) \left(\sup_{y \in X} \int_X \|K(x,y)\|_{L^2(X,L^N)} \, \mathrm{d}\mu(x) \right). \end{aligned}$$

By Theorem 3.3.1 and (3.3.2), we can approximate the Schwartz kernels of $T_{N,f}$ and $T_{N,g}$ with $J+1$ terms. That is, we may write

$$T_{N,f}(x,\bar{y}) = \underbrace{\left(\frac{N}{2\pi}\right)^d}_{:=c_d} e^{N\psi(x,\bar{y})} \left(\underbrace{\sum_{j=0}^{J} N^{-j} f_j(x,\bar{y})}_{:=F_J(x,\bar{y})} \right) + \underbrace{e^{\frac{N}{2}(\varphi(x)+\varphi(y))} \mathscr{O}(N^{-\tilde{J}} m_1(x))}_{:=R_{f,J}(x,y)}$$

where $\tilde{J} = (J+1)(1-2\delta) - d$. Define G_J and $R_{g,J}$ similarly as an approximation of the kernel of $T_{N,g}$. Then locally:

$$T_{N,f} \circ T_{N,g}(x,\bar{y}) = I_1 + I_2 + I_3,$$

where

$$I_1 := c_d^2 \int_U e^{N(\psi(x,\bar{w})-\varphi(w)+\psi(w,\bar{y}))} F_J(x,\bar{w}) G_J(w,\bar{y})\,\mathrm{d}\mu(w),$$
$$I_2 := c_d \int_U \big(e^{N\psi(x,\bar{w})} F_J(x,\bar{w}) R_{g,J}(w,y) + e^{N\psi(w,\bar{y})} G_J(w,y) R_{f,J}(x,w)\big)\, e^{-N\varphi(w)}\,\mathrm{d}\mu(w),$$
$$I_3 := \int_X e^{\frac{N}{2}(\varphi(x)+\varphi(y))} \mathscr{O}\left(N^{-2\tilde{J}} m_1(x) m_2(x)\right) \mathrm{d}\mu(w).$$

Here $U \subset X$ is a coordinate patch containing x and y which we assume exists, otherwise (by the same reasoning as in the proof of Theorem 3.3.1), $T_{N,f}(x,\bar{y})$ will be $\exp(\frac{N}{2}(\varphi(x)+\varphi(y))\mathscr{O}(N^{-\infty})$ which is bounded by a constant times the I_3 term. Moreover, by the same reasoning, we can just integrate over U, as the integral over $X\setminus U$ will similarly be negligible.

I_3 is $e^{\frac{N}{2}(\varphi(x)+\varphi(y))}\mathscr{O}\left(N^{-2\tilde{J}} m_1(x) m_2(x)\right)$. Using (3.2.8), I_2 is bounded in absolute value by:

$$cN^{d-\tilde{J}+\delta M_0} e^{\frac{N}{2}(\varphi(x)+\varphi(y))} \int_U \left(e^{-Nc|x-w|^2}|F_J(x,\bar{w})| + e^{-Nc|w-y|^2}|G_J(w,\bar{y})|\right) \mathrm{d}\mu(w) \lesssim N^{d-\tilde{J}} e^{\frac{N}{2}(\varphi(x)+\varphi(y))} N^{2M_0\delta}$$

for some positive constant $c > 0$. Here we used that m_1 and m_2 are bounded by a constant times $N^{\delta M_0}$.

We now estimate I_1 using the formal computation of h_j from (3.3.35). We see that

$$I_1 = c_d e^{N\psi(x,\bar{y})} \sum_{j=0}^{2J} N^{-j} (f \star g)_j(x,\bar{y}) + e^{\frac{N}{2}(\varphi(x)+\varphi(y))} \mathscr{O}(N^{-\widetilde{2J}} m_1(x) m_2(x)).$$

Because $h - \sum_{j=0}^{2J} N^{-j} h_j \in N^{-(2J+1)(1-2\delta)} S_\delta(m_1 m_2)$, we see that

$$\begin{aligned} T_{N,h}(x,\bar{y}) &= c_d e^{N\psi(x,\bar{y})} \sum_{j=0}^{2J} N^{-j} T_{N,h_j}(x,\bar{y}) \\ &\quad + e^{\frac{N}{2}(\varphi(x)+\varphi(y))} \mathscr{O}(N^{-(2J+1)(1-2\delta)+d} m_1(x) m_2(x)) \\ &= c_d e^{N\psi(x,\bar{y})} \sum_{j=0}^{2J} N^{-j} (f \star g)_j(x,\bar{y}) \\ &\quad + e^{\frac{N}{2}(\varphi(x)+\varphi(y))} \mathscr{O}(N^{-(2J+1)(1-2\delta)+d} m_1(x) m_2(x)) \\ &= I_1 + e^{\frac{N}{2}(\varphi(x)+\varphi(y))} \mathscr{O}(N^{-\widetilde{2J}} m_1(x) m_2(x)) \end{aligned}$$

for sufficiently large J. Putting these estimates together, we get

$$\begin{aligned} K(x,y) &= T_{N,(f\star g)} - (I_1 + I_2 + I_3) \\ &= e^{\frac{N}{2}(\varphi(x)+\varphi(y))}\Big(\mathscr{O}\Big(N^{-\widetilde{2J}}m_1(x)m_2(x)\Big) \\ &\qquad + \mathscr{O}\Big(N^{d+2M_0\delta-\tilde{J}}\Big) + \mathscr{O}\left(N^{-\widetilde{2J}}m_1(x)m_2(x)\right)\Big) \\ &= e^{\frac{N}{2}(\varphi(x)+\varphi(y))}\mathscr{O}(N^{2M_0\delta+d-\widetilde{J}}). \end{aligned}$$

We therefore get that

$$\begin{aligned} \sup_{x\in X}\int_X |K(x,y)|e^{\frac{-N}{2}(\varphi(x)+\varphi(y))}\,\mathrm{d}\mu(y) = \mathscr{O}(N^{-\infty}), \\ \sup_{y\in X}\int_X |K(x,y)|e^{\frac{-N}{2}(\varphi(x)+\varphi(y))}\,\mathrm{d}\mu(x) = \mathscr{O}(N^{-\infty}) \end{aligned}$$

as J can be made arbitrarily large. □

3.4 Applications of the exotic calculus

This symbol calculus allows us to get a parametrix construction, a functional calculus, and a trace formula.

3.4.1 Parametrix construction

We begin by proving a parametrix construction of Toeplitz operators associated to symbols in $S_\delta(m)$ which are elliptic with respect to m. This follows the usual parametrix construction for psuedo-differential operators (see for instance [GS94, Theorem 4.1]).

Theorem 3.4.1 (Parametrix construction)**.** *Suppose $\delta \in [0, 1/2)$, $m \geq 1$ is a δ-order function on X (a quantizable Kähler manifold), and $f \in S_\delta(m)$ is such that there exists $C > 0$ and $z \in \mathbb{C}$ such that*

$$|f(x) - z| > Cm(x)$$

for all $x \in X$. Then there exists $g \in S_\delta(m^{-1})$ such that

$$T_{N,f-z} \circ T_{N,g} = 1 + \mathscr{O}(N^{-\infty}), \qquad T_{N,g} \circ T_{N,f-z} = 1 + \mathscr{O}(N^{-\infty}),$$

and the principal symbol of g is $(f_0 - z)^{-1} + \mathscr{O}(N^{-(1-2\delta)}m)$ where f_0 is the principal symbol of f.

Proof. Define $s_1(x) \coloneqq (f_0(x) - z)^{-1}$ so that $|s_1(x)| \leq Cm(x)$. For each $\alpha \in \mathbb{N}^{2d}$ locally $\partial^\alpha_{x,\bar{x}} s_1(x)$ can be estimated by the Faà di Bruno formula:

$$\partial^\alpha_{x,\bar{x}} s_1(x) = \sum_{\pi \in \Pi_\alpha} \frac{c_\pi}{(z - f_0)^{|\pi|+1}} \prod_{\beta \in \pi} \partial^\beta_{x,\bar{x}}(z - f_0(x)),$$

where Π_α is the set of partitions on the set $(1, 2, \dots, |\alpha|)$, $\beta \in \pi$ runs through the blocks in the partition π, and c_π is the constant from repeatedly differentiating x^{-1}. For each π, note that $|\beta| = |\pi| \leq |\alpha|$ for $\beta \in \pi$ so that:

$$\begin{aligned}\left| \frac{c_\pi}{(z - f_0(x))^{|\pi|+1}} \prod_{\beta \in \pi} \partial^\beta_{x,\bar{x}}(z - f_0(x)) \right| &\lesssim (m(x))^{-|\pi|-1} N^{|\pi|\delta} (m(x))^{|\pi|} \\ &\lesssim (m(x))^{-1} N^{|\alpha|\delta}.\end{aligned}$$

We therefore have that $s_1 \in S_\delta(m^{-1})$. Next, using Theorem 3.3.11, let $s_2 \in S_\delta(1)$ be such that $T_{N,f-z} \circ T_{N,s_1} = 1 - N^{-(1-2\delta)} T_{N,s_2} + \mathscr{O}(N^{-\infty})$. Then define $s_3 \sim \sum_{j=0}^\infty N^{-j(1-2\delta)} s_2^{\star j}$, where:

$$s_2^{\star j} \coloneqq \underbrace{s_2 \star \cdots \star s_2}_{j \text{ terms}}.$$

By repeatedly applying Theorem 3.3.11, $s_2^{\star j} \in S_\delta(1)$ for all $j \in \mathbb{Z}_{\geq 0}$ so that $s_3 \in S_\delta(1)$. Lastly, define $g \coloneqq s_1 \star s_3 \in S_\delta(m^{-1})$. We can check that

$$\begin{aligned} T_{N,f-z} \circ T_{N,g} &= T_{N,f-z} \circ T_{N,s_1 \star s_3} = T_{N,f-z} \circ T_{N,s_1} \circ T_{N,s_3} + \mathscr{O}(N^{-\infty}) \\ &= (1 - N^{-(1-2\delta)} T_{N,s_2}) \circ T_{N,s_3} + \mathscr{O}(N^{-\infty}). \end{aligned}$$

So that for each $J \in \mathbb{N}$, we have

$$\begin{aligned} T_{N,f-z} \circ T_{N,g} &= (1 - N^{-(1-2\delta)} T_{N,s_2}) \circ \left(\sum_{j=0}^J N^{-(1-2\delta)j} T_{N,s_2} \right) + \mathscr{O}(N^{-(1-2\delta)(J+1)}) \\ &= 1 + \mathscr{O}(N^{-(1-2\delta)(J+1)}). \end{aligned}$$

Therefore $T_{N,f-z} \circ T_{N,g} = 1 + \mathscr{O}(N^{-\infty})$ so that g is a right-parametrix for $f - z$. We can similarly construct g_ℓ as a left-parametrix for $f - z$. But note that

$$T_{N,g_\ell} = T_{N,g_\ell} \circ (T_{N,f-z} \circ T_{N,g} + \mathscr{O}(N^{-\infty})) = T_{N,g} + \mathscr{O}(N^{-\infty}).$$

Therefore g is also a left-parametrix for $f - z$. Lastly, the principal symbol of g (modulo $\mathscr{O}(N^{-(1-2\delta)})$ error, is just the product of the principal terms of s_1 and s_3, which is just the principal term of s_1, which is $(f_0 - z)^{-1}$. □

3.4.2 Functional calculus

Here we present functional calculus of Toeplitz operators using the Helffer–Sjöstrand formula. For symbols bounded uniformly in N, this result is proven in [Cha03, Proposition 12]. Our proof is adapted from results on functional calculus of pseudo-differential operators with symbols in similarly defined symbol classes presented by Dimassi and Sjöstrand in [DS99, Chapter 8].

Theorem 3.4.2 (Functional calculus)**.** *Suppose that $\delta \in [0, 1/2)$, m is a δ-order function on X (a quantizable Kähler manifold) such that $m \geq 1$, and $f \in S_\delta(m)$ is such that*

1. *$f(x) \in \mathbb{R}_{\geq 0}$ for all $x \in X$,*
2. *there exists $C_0 > 0$ such that $|f(x)| \geq C_0^{-1} m(x) - C_0$.*

Then for any $\chi \in C_0^\infty(\mathbb{R}; \mathbb{C})$, there exists $g \in S_\delta(m^{-1})$ such that

$$\chi(T_{N,f}) = T_{N,g} + \mathscr{O}(N^{-\infty}),$$

and the principal symbol of g is $\chi(f_0) + \mathscr{O}(N^{-(1-2\delta)})$ where f_0 is the principal symbol of f.

Proof. Let $\tilde{\chi}$ be an almost analytic extension of χ such that $\overline{\partial}_z \chi(z) = \mathscr{O}(|\mathrm{Im}\,(z)|^\infty)$. Because f is real-valued, [LeF18, Lemma 5.1.3] can be immediately adapted to $S_\delta(m)$ to see that $T_{N,f}$ is self-adjoint. By the Helffer–Sjöstrand formula

$$\chi(T_{N,f}) = \frac{-1}{\pi} \int_{\mathbb{C}} \overline{\partial}_z \tilde{\chi}(z)(z - T_{N,f})^{-1}\, \mathrm{d}m(z)$$

(see for instance [DS99, Theorem 8.1]). For $\mathrm{Im}\,(z) \neq 0$ we aim to construct an approximate inverse of $z - T_{N,f}$ using the parametrix construction stated in Theorem 3.4.1. But first a technical bound must be proven.

Lemma 3.4.3. *If $s_1(z, x) = (z - f_0(x))^{-1}$, with $z \in \mathrm{supp}(\tilde{\chi}(z))$ and $\mathrm{Im}\,(z) \neq 0$, then for all $\alpha \in \mathbb{N}^{2d}$*

$$|\partial^\alpha_{x,\bar{x}} s_1(z, x)| \lesssim |\mathrm{Im}\,(z)|^{-1-|\alpha|}\, N^{|\alpha|\delta}(m(x))^{-1}. \tag{3.4.1}$$

Proof. First we prove a lower bound of $|f(x) + z|$. Write $z = z_1 + iz_2$. Let $C_1 > 0$ be sufficiently large so that for $|z_1 - 1| > C_1$ then $\tilde{\chi}(z) = 0$. Let $C_2 = C_0 - 1 + C_1$ (possibly increasing C_1 so that $C_2 \geq 1$). We may assume that $|z_2| < C_3$ on the support of $\tilde{\chi}$ for some $C_3 > 1$. Then rearranging $f(x) > m(x) C_0^{-1} - C_0$, we see that

$$\begin{aligned}\frac{m(x)}{C_0} &< f(x) + 1 - C_1 + C_2 < |f(x) + 1| - |z_1 - 1| + C_2 \\ &< |f(x) + z_1| + C_2 < \frac{C_2 C_3}{|z_2|}(|f(x) + z_1| + |z_2|) < \frac{2C_2C_3}{|z_2|}(|f(x) + z_1 + iz_2|).\end{aligned}$$

Therefore:

$$|f(x)+z| > \frac{1}{C_0C_2C_3}\,|\mathrm{Im}\,(z)|\,m(x).$$

We can then apply the Faà di Bruno formula in the same way as in the proof of Theorem 3.4.1 to get that for all $\alpha \in \mathbb{N}^{2d}$

$$\partial^\alpha_{x,\bar{x}} s_1(z,x) \lesssim |\mathrm{Im}\,(z)|^{-1-|\alpha|}\,(m(x))^{-1}N^{|\alpha|\delta}$$

which proves (3.4.1). □

We therefore have that $s_1 \in S_\delta(m^{-1})$, but with bounds depending on $|\mathrm{Im}\,(z)|$. We can now apply Theorem 3.4.1 to construct $s_2(z,x) \in S_\delta(m^{-1})$ such that

$$T_{N,z-f}\circ T_{N,s_2} + \mathscr{O}(N^{-\infty}) = T_{N,s_2}\circ T_{N,z-f} + \mathscr{O}(N^{-\infty}) = 1.$$

It can also be shown that for all $\alpha \in \mathbb{N}^{2d}$,

$$|\partial^\alpha_{x,\bar{x}} s_2(z,x)| \lesssim |\mathrm{Im}\,(z)|^{-1-|\alpha|}\,N^{|\alpha|\delta}(m(x))^{-1}.$$

For any $J \in \mathbb{N}$, we can approximate s_2 by a finite sum of elements of $S_\delta(m^{-1})$ (denoted by s_3) such that

$$T_{N,z-f}\circ T_{N,s_3} = 1 + \mathscr{O}(N^{-J}).$$

For such a symbol s_3, for any $\alpha \in \mathbb{N}^{2d}$, there exists $C = C(J,\alpha) > 0$ such that:

$$|\partial^\alpha_{x,\bar{x}} s_3(z,x)| \lesssim |\mathrm{Im}\,(z)|^{-C}\,(m(x))^{-1}. \tag{3.4.2}$$

Therefore:

$$s_4(x) := \frac{-1}{\pi}\int_{\mathbb{C}} \overline{\partial}_z\tilde{\chi}(z)s_3(z,x)\,\mathrm{d}m(z)$$

exists for all x because $\overline{\partial}_z\tilde{\chi} = \mathscr{O}(|\mathrm{Im}\,(z)|^\infty)$. By differentiating s_4, applying (3.4.2), and using $\overline{\partial}_z\tilde{\chi} = \mathscr{O}(|\mathrm{Im}\,(z)|^\infty)$, we also see that $s_4 \in S_\delta(m^{-1})$. We finally check that T_{N,s_4} is an approximation of $\chi(T_{N,f})$. Suppose $u \in H^0(X,L^N)$, then for $x \in X$

$$\begin{aligned}
T_{N,s_4}[u](x) &= \Pi_N\left(\frac{-1}{\pi}\int_{\mathbb{C}} \overline{\partial}_z\tilde{\chi}(z)s_3(z,x)u(x)\,\mathrm{d}m(z)\right)\\
&= \frac{-1}{\pi}\int_{\mathbb{C}} \overline{\partial}_z\tilde{\chi}(z)\Pi_N[s_3(z,x)u(x)]\,\mathrm{d}m(z)\\
&= \left(\frac{-1}{\pi}\int_{\mathbb{C}} \overline{\partial}_z\tilde{\chi}(z)T_{N,s_3}\,\mathrm{d}m(z)\right)[u]\\
&= \left(\frac{-1}{\pi}\int_{\mathbb{C}} \overline{\partial}_z\tilde{\chi}(z)((z-T_{N,f})^{-1}+\mathscr{O}(N^{-J}))\,\mathrm{d}m(z)\right)[u].
\end{aligned}$$

Therefore $T_{N,s_4} = \chi(T_{N,f}) + \mathscr{O}(N^{-J})$. Since J was arbitrary, by Borel's theorem, there exists $g \in S_\delta(m^{-1})$ such that

$$\chi(T_{N,f}) = T_{N,g} + \mathscr{O}(N^{-\infty}).$$

The principal symbol can be easily computed. Unraveling the above, the principal symbol of s_3 is $(z - f_0(x))^{-1}$, so that the principal symbol of g is:

$$\frac{-1}{\pi} \int_{\mathbb{C}} \overline{\partial}_z \tilde{\chi}(z)(z - f_0(x))^{-1} \, \mathrm{d}m(z) = \chi(f_0)$$

by the Cauchy integral formula. □

This can be generalized for Toeplitz operators with a negligible term.

Theorem 3.4.4. *Suppose δ, m, f satisfy the hypotheses of Theorem 3.4.2, and $\{R_N\}_{N\in\mathbb{N}}$ is a family of linear operators mapping $H^0(X, L^N) \to H^0(X, L^N)$ such that $\|R_N\| = \mathscr{O}(N^{-\infty})$ and $T_N f + R_N$ are self-adjoint for all N. Then for any $\chi \in C_0^\infty(\mathbb{R}; \mathbb{C})$, there exists $g \in S_\delta(m^{-1})$ such that:*

$$\chi(T_{N,f} + R_N) = T_{N,g} + \mathscr{O}(N^{-\infty})$$

and the principal symbol of g is $\chi(f_0) + \mathscr{O}(N^{-(1-2\delta)})$ where f_0 is the principal symbol of f.

Proof. Let $\tilde{\chi}$ be an almost analytic extension of χ, so that

$$\chi(T_{N,f} + R_N) = \frac{-1}{\pi} \int_{\mathbb{C}} \overline{\partial}_z \tilde{\chi}(z)(z - T_{N,f} - R_N)^{-1} \, \mathrm{d}m(z).$$

But note that

$$(z - T_{N,f})^{-1} - (z - T_{N,f} - R_N)^{-1} = (z - T_{N,f})^{-1} R_N (z - T_{N,f} - R_N)^{-1}. \tag{3.4.3}$$

Both $(z - T_{N,f})^{-1}$ and $(z - T_{N,f} - R_N)^{-1}$ have operator norm controlled by N to some finite power, so that the right-hand side of (3.4.3) is $\mathscr{O}(N^{-\infty})$. Therefore:

$$\chi(T_{N,f} + R_N) = \frac{-1}{\pi} \int_{\mathbb{C}} \overline{\partial}_z \tilde{\chi}(z)(z - T_{N,f})^{-1} \, \mathrm{d}m(z) + \mathscr{O}(N^{-\infty})$$

and we just follow the rest of the proof of Theorem 3.4.2. □

3.4.3 Trace formula

A critical result required in proving a probabilistic Weyl law for Toeplitz operators in [Olt23] is a trace formula. Fortunately, this is straightforward to compute by the explicit kernel expansion described in Theorem 3.3.1.

Theorem 3.4.5 (Trace formula)**.** *For $\delta \in [0, 1/2)$, suppose m is a δ-order function on X (a quantizable Kähler manifold). Then if $f \in S_\delta(m)$,*

$$\operatorname{Tr}(T_{N,f}) = \left(\frac{N}{2\pi}\right)^d \int_X f_0(x)\,\mathrm{d}\mu(x) + \left(\int_X m(x)\,\mathrm{d}\mu(x)\right) \mathscr{O}(N^{d-(1-2\delta)}). \tag{3.4.4}$$

Proof. By [LeF18, Proposition 6.3.4] and Theorem 3.3.1 (specifically (3.3.2) with $J = 1$),

$$\begin{aligned}\operatorname{Tr}(T_{N,f}) &= \int_X T_{N,f}(x, \bar{x}) e^{-N\varphi(x)}\,\mathrm{d}\mu(x)\\ &= \left(\frac{N}{2\pi}\right)^d \int_X \left(f_0(x, \bar{x}) + \mathscr{O}(N^{-(1-2\delta)} m(x))\right) \mathrm{d}\mu(x).\end{aligned}$$

By Theorem 3.3.1, $f_0(x, \bar{x})$ is $f_0(x) + \mathscr{O}(N^{-(1-2\delta)} m(x))$ and so (3.4.4) follows. □

Chapter 4

Proof of Probabilistic Weyl Law

4.1 Setup of main result

In this chapter, we use a slightly different notation. We let (X, σ) be a compact, connected, $d-$dimensional Kähler manifold with a holomorphic line bundle L with positively curved Hermitian metric locally given by $h = e^{-\varphi}$. That is over each fiber $x \in X$ and $v \in L_x$, $\|v\|_h := e^{-\varphi(x)}|v|$. Given this, the globally defined symplectic form, σ, is related to the Hermitian metric by $i\partial\overline{\partial}\varphi = \sigma$. Fixing local trivializations, φ can be described as a strictly plurisubharmonic smooth real-valued function (called the Kähler potential).

Let L^N be the Nth tensor power of L, which has Hermitian metric $h_N := e^{-N\varphi}$. Let $\mu_d = \sigma^{\wedge d}/d!$ be the Liouville volume form on X. This provides an L^2 structure on sections of L^N. Indeed, if u and v are smooth sections on L^N, then define

$$\langle u, v \rangle_{L^N} := \int_X h_N(u, v)\, \mathrm{d}\mu_d.$$

Define $L^2(X, L^N)$ as the completion of the smooth sections of L^N with respect to this metric. In this L^2 space, let $H^0(X, L^N)$ be the space of holomorphic sections. By the Hirzebruch–Riemann–Roch Theorem (see for instance [Laz17, Theorem 1.1.24]) we have the following.

Proposition 4.1.1. *The dimension of $H^0(X, L^N)$ is finite, and is asymptotically*

$$\left(\frac{N}{2\pi}\right)^d \mathrm{vol}(X) + \mathscr{O}(N^{d-1}).$$

For the remainder of this chapter, denote $\dim(H^0(X, L^N))$ by $\mathscr{N} = \mathscr{N}(N)$. The orthogonal projection from $L^2(X, L^N)$ to $H^0(X, L^N)$ is called the Bergman projector and is denoted by Π_N. Finally, given $f \in C^\infty(X; \mathbb{C})$, the Toeplitz operators associated to f, written $T_N f$, are defined for each $N \in \mathbb{N}$ as $T_N f(u) = \Pi_N(fu)$, where $u \in H^0(X, L^N)$. In this way, $T_N f$ are finite rank operators mapping $H^0(X, L^N)$ to itself. For the remainder of this chapter, we will fix a basis for $H^0(X, L^N)$ so that $T_N f$ (and similar operators) can be considered as matrices.

The class of functions to quantize will often depend on N. To define this symbol class requires local control of functions. Fix a finite atlas of neighborhoods $(U_i, \zeta_i)_{i\in\mathscr{I}}$ for the Kähler manifold X.

Definition 4.1.2 ($S(1)$). *$S(1)$ is the set of all smooth functions f on X taking complex values which can be written asymptotically $f \sim \sum N^{-j} f_j$, where $f_j \in C^\infty(X;\mathbb{C})$ do not depend on N. This tilde means that for all $\alpha \in \mathbb{N}$*

$$\partial_x^\alpha \left(f \circ \zeta_i(x) - \sum_{j=0}^{M} N^{-j} f_j \circ \zeta_i(x) \right) = \mathscr{O}_\alpha(N^{-j-1})$$

for all $i \in \mathscr{I}$, and all $\alpha \in \mathbb{N}^d$. By Borel's theorem, given any $f_j \in S(1)$ not depending on N, there exists $f \in S(1)$ such that $f \sim \sum N^{-j} f_j$.

If $f \sim \sum N^{-j} f_j$, we call f_0 the principal symbol of f, which is unique modulo $\mathscr{O}(N^{-1})$.

We next add a random perturbation to these Toeplitz operators. For this we fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition 4.1.3 ($\mathscr{G}_\omega(N)$ and $\mathscr{W}_\omega(N)$). *For each N, let $\{e_i : i = 1, \dots, \mathscr{N}\}$ be an orthonormal basis of $H^0(X, L^N)$. Define:*

$$\mathscr{G}_\omega(N) \coloneqq \sum_{i,j=1}^{\mathscr{N}} \alpha_{j,k} e_i \otimes e_j \colon H^0(X, L^N) \to H^0(X, L^N)$$

where $\alpha_{j,k}$ are independent identically distributed complex Gaussian random variables with mean zero and variance 1.

Similarly define $\mathscr{W}_\omega(N) = \sum_{i,j=1}^{\mathscr{N}} \tilde{\alpha}_{j,k} e_i \otimes e_j$, with $\tilde{\alpha}_{j,k}$ independent identically distributed copies of a complex random variable with mean zero and bounded second moment.

The ω in the subscript of these objects is to emphasize that these objects are random. That is for each $\omega \in \Omega$ and $N \in \mathbb{N}$, $\mathscr{G}_\omega(N)$ is a finite rank operator. The majority of this chapter describes perturbations by $\mathscr{G}_\omega(N)$ (the Gaussian case), while a brief note at the end concerns the more general perturbations by $\mathscr{W}_\omega(N)$.

This chapter will prove almost sure weak convergence of the empirical distribution of eigenvalues of randomly perturbed Toeplitz operators. The principal symbol of f must also satisfy the property that there exists $\kappa \in (0, 1]$ such that

$$\mu_d(\{x \in X : |f_0(x) - z|^2 \leq t\}) = \mathscr{O}(t^\kappa) \tag{4.1.1}$$

as $t \to 0$ uniformly for all $z \in \mathbb{C}$.

Theorem 4.1.4 (Main result). *Given $f \in S(1)$ which satisfies (4.1.1) and $\mathscr{G}_\omega$, a family of random operators on $H^0(X, L^N)$, as defined in Definition 4.1.3, then for each $\varepsilon > 0$ there exists $\beta = \beta(\varepsilon) \in (0, 1)$ and $C > 0$ such that if $\delta = \delta(N)$ satisfies*

$$Ce^{-N^\beta} < \delta < C^{-1} N^{-d/2-\varepsilon} \tag{4.1.2}$$

then we have almost sure weak convergence of the empirical measures of $T_N f + \delta \mathscr{G}_\omega(N)$ *to* $\mathrm{vol}(X)^{-1}(f_0)_*\mu_d$.

More precisely, if $\lambda_i = \lambda_i(N, \omega)$ *are the (random) eigenvalues of* $T_N f + \delta \mathscr{G}_\omega(N)$*, then for all* $\varphi \in C_0^\infty(\mathbb{C})$

$$\frac{1}{\mathscr{N}} \sum_{i=1}^{\mathscr{N}} \varphi(\lambda_i) \xrightarrow{N \to \infty} \frac{1}{\mathrm{vol}(X)} \int_{\mathbb{C}} \varphi(z)((f_0)_*\mu_d)(\mathrm{d}z)$$

almost surely, where $(f_0)_*\mu_d$ *is the push-forward of the volume form* μ_d *on* X *by* f_0.

Moreover, for each $\varepsilon > 0$*, the constant* $\beta(\varepsilon)$ *in* (4.1.2) *can be chosen at most strictly less than*

$$\begin{cases} 2\epsilon\kappa & \text{if } \epsilon < \frac{1}{2(\kappa+1)} \\ \frac{\kappa}{\kappa+1} & \text{if } \epsilon \geq \frac{1}{2(\kappa+1)} \end{cases}$$

where κ *is defined in* (4.1.1).

We expect Theorem 4.1.4 to hold for a much larger class of random perturbations than described in Definition 4.1.3. Indeed, the only properties of $\mathscr{G}_\omega$ we use is a norm bound (Lemma 4.2.6) and an anti-concentration bound (Proposition 4.3.7). See [VZ21] where Vogel and Zeitouni establish similar logarithmic determinant estimates with these classes of random perturbations, and [BPZ20, Remark 1.3] where Basak, Paquette, and Zeitouni describe random perturbations satisfying these properties.

Here we present a version of Theorem 4.1.4 for the more general random perturbations $\mathscr{W}_\omega(N)$ as described in Definition 4.1.3.

Theorem 4.1.5 (General perturbations)**.** *For* $\mathscr{W}_\omega(N)$ *defined in Definition 4.1.3,* $f \in S(1)$ *satisfying* (4.1.1)*,* $\delta = N^{-d}$*, then the empirical measures of* $T_N f + \delta \mathscr{W}_\omega(N)$ *converge almost surely to* $(\mathrm{vol}(X))^{-1}(f_0)_*\mu_d$.

A proof of this result is presented in §4.6.

Remark 4.1.1. *We expect a wider range of* δ*'s and more general random perturbations in Theorem 4.1.5 should lead to the same conclusion.*

4.2 Probabilistic preliminaries

This chapter uses the probabilistic machinery of logarithmic potentials. A brief overview is presented in this section.

Definition 4.2.1 ($\mathscr{P}(\mathbb{C})$)**.** *Let* $\mathscr{P}(\mathbb{C})$ *be the collection of probability measures* μ *on* $\mathbb{C}$ *such that* $\int \log(1 + |z|)\, \mathrm{d}\mu(z) < \infty$.

Definition 4.2.2 (Logarithmic potential)**.** *For $\nu \in \mathscr{P}(\mathbb{C})$, define the logarithmic potential as:* $U_\nu(z) := \int_{\mathbb{C}} \log |z-w| \, \mathrm{d}\nu(w)$.

Using the fact that $\log|z|$ is the fundamental solution of the Laplacian, it can be shown that, in the sense of distributions, $\Delta U_\nu = 2\pi\nu$, which is the key ingredient in proving the following theorem.

Proposition 4.2.3 (Convergence of random measures by logarithmic Potentials)**.** *([SV21c, Theorem 7.1])*

Given $\{\nu_N\} \subset \mathscr{P}(\mathbb{C})$ random measures such that almost surely $\operatorname{supp} \nu_N \subset \Lambda$ *for $N \gg 1$ (with $\Lambda \Subset \bar{\Lambda} \Subset \Lambda' \Subset \mathbb{C}$) and for almost all $z \in \Lambda'$: $U_{\nu_N}(z) \to U_\nu(z)$ almost surely for some $\nu \in \mathscr{P}(C)$ with* $\operatorname{supp} \nu \subset \Lambda$. *Then almost surely $\nu_N \to \nu$ weakly.*

We wish to use Proposition 4.2.3 to prove almost sure weak convergence of the empirical measures of $T_N f + \delta \mathscr{G}_\omega(N)$.

Definition 4.2.4 (ν_N)**.** *Let σ_N be the spectrum of $T_N f + \delta\mathscr{G}_\omega(N)$. Let*

$$\nu_N = \mathscr{N}^{-1} \sum_{\lambda \in \sigma_N} \hat{\delta}_\lambda$$

where $\delta > 0$ depends on N, and $\hat{\delta}_\lambda$ is the Dirac distribution centered at λ. The logarithmic potentials for these random measures are

$$U_{\nu_N}(z) = \frac{1}{\mathscr{N}} \sum_{\lambda \in \sigma_N} \log|z-\lambda| = \frac{1}{\mathscr{N}} \log|\det(T_N f + \delta\mathscr{G}_\omega(N) - z)|.$$

Definition 4.2.5 (ν)**.** *Let* $\nu = \operatorname{vol}(X)^{-1} (f_0)_* \mu_d$ *(recall μ_d is the volume measure on X) which has logarithmic potential*

$$U_\nu(z) = ⨍_X \log|z - f_0(x)| \, \mathrm{d}\mu_d(x).$$

Where $⨍_X f \, \mathrm{d}\mu_d$ is defined as $\operatorname{vol}(X)^{-1} \int f \, \mathrm{d}\mu_d$.

Claim 4.2.1. *For all N, $\nu_N, \nu \in \mathscr{P}(\mathbb{C})$.*

Proof. For each $N \in \mathbb{N}$

$$\begin{aligned} \int_{\mathbb{C}} \log(1+|z|) \, \mathrm{d}\nu_N(z) &= \frac{1}{\mathscr{N}} \sum_{\lambda \in \sigma_N} \log(1+|\lambda|) \\ &\le \max_{\lambda \in \sigma_N} \log(1+|\lambda|) \\ &\le \log(1 + \|T_N f + \delta\mathscr{G}_\omega(N)\|) < \infty. \end{aligned}$$

And similarly,

$$\int_{\mathbb{C}} \log(1+|z|)\,\mathrm{d}\nu(z) = \frac{1}{\mathrm{vol}(X)} \int_{\mathbb{C}} \log(1+|z|)[(f_0)_* \mu_d](\mathrm{d}z)$$
$$\leq \max_{x\in X} \log(1+|f(x)|) < \infty.$$

□

Let Λ be a neighborhood of $f(X)$. Clearly $\operatorname{supp}\nu \subset \Lambda$, the same is true with probability 1 for ν_N, for sufficiently large N. A standard random matrix lemma is required to show this.

Lemma 4.2.6 (Norm of Gaussian matrix)**.** *([Tao12, Exercise 2.3.3]) There exists $C > 0$ such that*

$$\mathbb{P}(\|\mathscr{G}_\omega(N)\| \leq C\mathscr{N}^{1/2}) \geq 1 - \exp(-\mathscr{N}).$$

If an event has this lower bound of probability, it is said to occur with overwhelming probability.

For a fixed $\varepsilon > 0$, we will choose $\delta = \delta(N)$ such that

$$0 < \delta = \mathscr{O}(\mathscr{N}^{-1/2-\varepsilon}). \tag{4.2.1}$$

Lemma 4.2.7 (Borel–Cantelli)**.** *If A_n are events such that $\sum_1^\infty \mathbb{P}(A_n) < \infty$, then the probability that A_n occurs infinitely often is 0.*

Lemma 4.2.8 (Bound of $T_N f$)**.** *Given $f \in S(1)$, then $\|T_N f\|_{L^N \to L^N} \leq \sup|f|$.*

Proof. This follows immediately by writing $T_N f = \Pi_N \circ M_f \circ \Pi_N$ and recalling that Π_N is unitary. □

Claim 4.2.2. *Almost surely,* $\operatorname{supp}\nu_N \subset \Lambda$ *for $N \gg 1$.*

Proof. First note that $\|T_N f + \delta\mathscr{G}_\omega(N)\| \leq \|T_N f\| + \delta\,\|\mathscr{G}_\omega(N)\| \leq \sup f + \mathscr{N}^{-\varepsilon}$ with overwhelming probability (by Lemma 4.2.6, (4.2.1), and Lemma 4.2.8). Let σ_N be the spectrum of $T_N f + \delta\mathscr{G}_\omega$. In this event, for sufficiently large N, $\sigma_N \subset \Lambda$. So if A_N^c is the event that $\sigma_N \subset \Lambda$, then $\mathbb{P}(A_N^c) \geq 1 - e^{-\mathscr{N}}$. Therefore $\sum \mathbb{P}(A_N) < \infty$ and so by Lemma 4.2.7, almost surely $P(A_N^c) = 1$ for $N \gg 1$.

□

Now we recall the standard probability result guaranteeing almost sure convergence of random variables.

Lemma 4.2.9 (Almost sure convergence)**.** *If $\{Y_N\}_{N\in\mathbb{N}}$ and Y are random variables on a probability space $(\Omega, \mathbb{P})$ and ε_N is a sequence of numbers converging to 0 such that*

$$\sum_{N=1}^{\infty} \mathbb{P}(|Y_N - Y| > \varepsilon_N) < \infty,$$

then $Y_N \to Y$ almost surely.

Therefore ν_N and ν satisfy the conditions of Proposition 4.2.3. So it suffices to show that $U_{\nu_N}(z) \to U_\nu(z)$ for almost all z in the bounded set containing Λ. To prove this almost sure convergence, it suffices to apply Lemma 4.2.9 with $Y_N = \mathscr{N}^{-1} \log|\det(T_N f + \delta\mathscr{G}_\omega(N) - z)|$ and $Y = \fint \log|z - f_0(x)| \,\mathrm{d}\mu_d(x)$ for suitably chosen ε_N.

4.3 Setting up a Grushin problem

To control $\log|\det(T_N f + \delta\mathscr{G}_\omega(N) - z)|$ we follow the now standard method of setting up a Grushin problem. This approach was used in [Vog20] and [Hag06], and is comprehensively reviewed in [SZ07].

Let $P = T_N f$ and $\mathscr{H}_N = H^0(X, L^N)$. Define the z-dependent self-adjoint operators $Q = (P-z)^*(P-z)$ and $\tilde{Q} = (P-z)(P-z)^*$. These operators share the same eigenvalues $0 \le t_1^2 \le \cdots \le t_{\mathscr{N}}^2$. We can find an orthonormal basis of eigenvectors of Q for these eigenvalues, denoted by e_i, and similarly, and orthonormal basis of eigenvectors of $\tilde{Q}$ denoted by f_i. These eigenvectors can be chosen such that

$$(P-z)^* f_i = t_i e_i, \qquad (P-z)e_i = t_i f_i, \qquad i = 1, \dots, \mathscr{N}.$$

Next we fix $\rho \in (0, \min(1/2, \varepsilon))$, and define:

$$\alpha := N^{-2\rho}, \qquad A := \max\left\{i \in \mathbb{Z} : t_i^2 \le \alpha\right\}.$$

Definition 4.3.1 ($\mathscr{P}^\delta$)**.** *Let δ_j be the standard basis of $\mathbb{C}^A$, and define the operators $R_+(z) = \sum_1^A \delta_i \otimes e_i : \mathscr{H}_N \to \mathbb{C}^A$ and $R_-(z) = \sum_1^A f_i \otimes \delta_i : \mathbb{C}^A \to \mathscr{H}_N$, where we use the notation $(u \otimes v)(w) = \langle w, v\rangle\, u$. For each $z \in \mathbb{C}$ and $\delta \ge 0$, define*

$$\mathscr{P}^\delta(z) := \begin{pmatrix} P + \delta\mathscr{G}_\omega(N) - z & R_-(z) \\ R_+(z) & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{H}_N \\ \mathbb{C}^A \end{pmatrix} \to \begin{pmatrix} \mathscr{H}_N \\ \mathbb{C}^A \end{pmatrix}. \tag{4.3.1}$$

Lemma 4.3.2. *([Vog20, §5.1]) If $\delta = 0$, then $\mathscr{P}^\delta$, as defined in (4.3.1), is bijective with inverse*

$$\mathscr{E}^0(z) = \begin{pmatrix} \sum_{A+1}^{\mathscr{N}} \frac{1}{t_i} e_i \otimes f_i & \sum_1^A e_i \otimes \delta_i \\ \sum_1^A \delta_i \otimes f_i & -\sum_1^A t_i \delta_i \otimes \delta_i \end{pmatrix} := \begin{pmatrix} E^0(z) & E_+^0(z) \\ E_-^0(z) & E_{-+}^0(z) \end{pmatrix}. \tag{4.3.2}$$

To ease notation, the z in the argument for these operators will often be dropped. Unless specified, all estimates are uniform in z.

Claim 4.3.1 (Invertibility of $\mathscr{P}^\delta$)**.** *$\mathscr{P}^\delta$ is invertible if $\delta \|\mathscr{G}_\omega(N)E^0\| \ll 1$.*

Proof. By computation

$$\mathscr{P}^\delta \mathscr{E}^0 = 1 + \begin{pmatrix} \delta\mathscr{G}_\omega(N)E^0 & \delta\mathscr{G}_\omega(N)E^0_+ \\ 0 & 0 \end{pmatrix} := 1 + K.$$

If $\|K\| < 1$ (which is true given the hypothesis), then $(I+K)^{-1}$ exists as a Neumann series, and we get $\mathscr{P}^\delta\mathscr{E}^0(I+K)^{-1} = I$ (a similar argument shows this is a left inverse as well). $\square$

Lemma 4.3.3 (Norm of E^0)**.** *In the notation of (4.3.2), $\|E^0\| \le \alpha^{-1/2}$.*

Proof. By construction, $E^0 = \sum_{M+1}^{\mathscr{N}}(t_i)^{-1}e_i \otimes f_i$, so that $\|E^0\| = \|E^0 f_{M+1}\| = (t_{M+1})^{-1} \le \alpha^{-1/2}$. $\square$

Lemma 4.3.4 (Norm of E^0_+)**.** *In the notation of (4.3.2), $\left\|E^0_+\right\| = 1$.*

Proof. By construction $E^0_+(z) = \sum_1^M e_i \otimes \delta_i$ which has norm 1. $\square$

These lemmas, along with Lemma 4.2.6, guarantee that if $\delta = \mathscr{O}(\alpha^{1/2}\mathscr{N}^{-1/2})$, then $\mathscr{P}^\delta$ is invertible with overwhelming probability. Denote the inverse of $\mathscr{P}^\delta$ by $\mathscr{E}^\delta$ with the same notation for its components as in (4.3.2).

Define $P^\delta = P + \delta\mathscr{G}_\omega(N)$. By Schur's complement formula, if $P^\delta - z$ is invertible,

$$\det\begin{pmatrix} P^\delta - z & R_- \\ R_+ & 0 \end{pmatrix} = \det(P^\delta - z)\det(-R_+(P^\delta - z)^{-1}R_-).$$

Writing $\mathscr{P}^\delta\mathscr{E}^\delta = 1$, we get that $-R_- = (P^\delta - z)E^\delta_+(E^\delta_{-+})^{-1}$ and $R_+E^\delta_+ = 1$. Therefore $-R_+(P^\delta - z)^{-1}R_- = (E^\delta_{-+})^{-1}$, so that

$$\log|\det(P^\delta - z)| = \log|\det \mathscr{P}^\delta(z)| + \log|\det E^\delta_{-+}(z)|. \tag{4.3.3}$$

Note that $P^\delta - z$ is invertible if and only if E^δ_{-+} is invertible. Therefore (4.3.3) holds even when $P^\delta - z$ is not invertible.

Therefore, to prove Theorem 4.1.4, it suffices to show summability of the probability of the events:

$$\mathscr{A}_N := \left\{ \left| \underbrace{(\mathscr{N})^{-1}(\log|\det \mathscr{P}^\delta| + \log|\det E^\delta_{-+}(z)|) - \fint_X \log|z - f_0(x)|\,\mathrm{d}\mu}_{:=B} \right| > \varepsilon_N \right\}.$$

We let $\varepsilon_N = N^{-\gamma}$ for a suitably chosen $\gamma = \gamma(d, \kappa) > 0$. Expand $B = B_1 + B_2 + B_3$ where:

$$B_1 = \mathscr{N}^{-1} \log |\det \mathscr{P}^0| - \fint_X \log |z - f_0(x)| \, \mathrm{d}\mu(x), \tag{4.3.4}$$

$$B_2 = \mathscr{N}^{-1}(\log |\det \mathscr{P}^\delta| - \log |\det \mathscr{P}^0|), \tag{4.3.5}$$

$$B_3 = \mathscr{N}^{-1} \log |\det E^\delta_{-+}|. \tag{4.3.6}$$

Controlling B_1 requires the most work as it requires utilizing the calculus of Toeplitz operators. However, it is completely deterministic, and remains true for unperturbed operators. B_2 will be easily shown to be negligible. Proving a lower bound on B_3 is the key ingredient in proving Theorem 4.1.4, as it will force the events $\mathscr{A}_N$ to sufficiently small probability. Without a perturbation, B_3 will have no lower bound.

Proving bounds on B_2 and B_3 closely follow [Vog20].

Lemma 4.3.5 (Bound on E_{-+})**.** *In the notation of* (4.3.2), $\left\|E^0_{-+}\right\| \leq \sqrt{\alpha}$.

Proof. By construction, $E^0_{-+} = -\sum_1^A t_j \delta_j \otimes \delta_j$, so $\left\|E^0_{-+}\right\| = |E^0_{-+}(\delta_A)| = t_A \leq \sqrt{\alpha}$. □

Lemma 4.3.6 (Bound on E^δ)**.** *In the notation of* (4.3.2), $\left\|E^\delta\right\| \leq 2\alpha^{-1/2}$ *with overwhelming probability.*

Proof. By the Neumann construction, $\left\|E^\delta\right\| = \|E^0(1 + \delta \mathscr{G}_\omega(N) E^0)^{-1}\| \leq 2 \|E^0\|$ which is bounded by $2\alpha^{-1/2}$ by Lemma 4.3.3. □

Claim 4.3.2 (Bound on B_2)**.** *In the notation of* (4.3.5), $B_2 = \mathscr{O}(\delta \alpha^{-1/2} \mathscr{N}^{1/2})$ *with overwhelming probability.*

Proof. Using Jacobi's formula, $(\log \det A)' = \mathrm{Tr}(A^{-1} A')$, we have that

$$\begin{aligned} \mathscr{N} B_2 = \log |\det \mathscr{P}^\delta| - \log |\det \mathscr{P}| &= \int_0^\delta \frac{d}{d\tau} \log |\det \mathscr{P}^\tau| \, \mathrm{d}\tau \\ &= \int_0^\delta \mathrm{Re} \left(\mathrm{Tr}(\mathscr{E}^\tau \frac{d}{d\tau} \mathscr{P}^\tau) \right) \mathrm{d}\tau = \int_0^\delta \mathrm{Re} \left(\mathrm{Tr}(E^\tau \mathscr{G}_\omega(N)) \right) \mathrm{d}\tau. \end{aligned}$$

Taking absolute values and using properties of trace norms

$$\begin{aligned} |\log |\det \mathscr{P}^\delta| - \log |\det \mathscr{P}^0|| &\leq \delta \sup_{\tau \in [0,\delta]} \|E^\tau\| \, \|\mathscr{G}_\omega(N)\|_{tr} \\ &\leq \mathscr{O}(\delta \alpha^{-1/2} \mathscr{N} \, \|\mathscr{G}_\omega(N)\|), \end{aligned} \tag{4.3.7}$$

where we used Lemma 4.3.6, and Hölder's inequality for the Schatten norm. Recalling the bound on $\mathscr{G}_\omega$, (4.3.7) is $\mathscr{O}(\delta \alpha^{-1/2} \mathscr{N}^{3/2})$ with overwhelming probability. □

The following theorem about singular values of randomly perturbed matrices is required for proving a lower bound of B_3. Given a matrix B, let $s_1(B) \geq s_2(B) \geq \cdots \geq s_N(B)$ be its singular values.

Proposition 4.3.7. *If B is an $N \times N$ complex matrix and $\mathscr{G}_\omega(N)$ is a random matrix with independent identically distributed complex Gaussian entries of mean 0 and variance 1, then there exists $C > 0$ such that for all $\delta > 0$, $t > 0$:*

$$\mathbb{P}(s_N(B + \delta \mathscr{G}_\omega(N)) < \delta t) \leq CNt^2.$$

Proof. See [Vog20, Theorem 23], which is a complex version proven by Sankar, Spielmann, and Teng in [SST06, Lemma 3.2]. □

Claim 4.3.3 (Bound on B_3)**.** *In the notation of (4.3.6), B_3 obeys the probabilistic upper bound*

$$\mathbb{P}(\mathscr{N}^{-1} \log |\det E^\delta_{-+}| < 0) > 1 - e^{-\mathscr{N}}, \tag{4.3.8}$$

for $N \gg 1$. And B_3 obeys the probabilistic lower bound: there exists there exists $C > 0$ such that for all $\delta > 0$

$$\mathbb{P}\left(\mathscr{N}^{-1} \log |\det E^\delta_{-+}| \geq A \mathscr{N}^{-1} \log(\delta t)\right) > 1 - C\mathscr{N}t^2 - e^{-\mathscr{N}}.$$

Proof. First, by the Neumann series construction and choice of δ, with overwhelming probability,

$$\begin{aligned}\left\|E^\delta_{-+}\right\| \leq \left\|E^\delta_{-+} - E^0_{-+}\right\| + \left\|E^0_{-+}\right\| &= \left\|E^0_-(1 - \delta\mathscr{G}_\omega(N)E^0)^{-1}\delta\mathscr{G}_\omega(N)E^0_+\right\| + \left\|E^0_{-+}\right\| \\ &\leq 2\left\|\delta\mathscr{G}_\omega(N)\right\| + \alpha^{1/2} \leq C\alpha^{1/2}.\end{aligned}$$

So, in this event, $\left\|E^\delta_{-+}\right\| \leq C\alpha^{1/2} < 1$ for $N \gg 1$, and therefore $\log\left|\det E^\delta_{-+}\right| < 0$ proving (4.3.8).

For the lower bound, first note that

$$\log\left|\det E^\delta_{-+}\right| = \sum_1^A \log s_j(E^\delta_{-+}) \geq A \log s_A(E^\delta_{-+}).$$

For a matrix B, let $t_1(B)$ be the smallest eigenvalue of $\sqrt{B^*B}$, so $s_A(E^\delta_{-+}) = t_1(E^\delta_{-+})$. Assume that $P - z$ is invertible. Using that $(E^0_{-+})^{-1} = -R_+(P-z)^{-1}R_-$ and properties of singular values of sums and products of trace class operators, we get

$$\begin{aligned}(t_1(E^0_{-+}))^{-1} &= s_1((E^0_{-+})^{-1}) \leq s_1(R_-)s_1(R_+)s_1((P-z)^{-1}) \\ &= \|R_+\|\,\|R_-\|\,s_1((P-z)^{-1}) = s_1((P-z)^{-1}) \\ &= (t_1(P-z))^{-1} = s_{\mathscr{N}}((P-z)^{-1}).\end{aligned}$$

For $\delta = \mathscr{O}(\mathscr{N}^{-1/2}\alpha^{1/2})$, this holds for E^δ_{-+} (the event of a singular matrix has probability zero and the singular values depend continuously on δ) so $s_A(E^\delta_{-+}) = t_1(E^\delta_{-+}) \geq s_{\mathscr{N}}(P + \delta\mathscr{G}_\omega(N) - z)$ with overwhelming probability.

Using Proposition 4.3.7, in the event that $\|\mathscr{G}_\omega(N)\| \leq C\mathscr{N}^{1/2}$ (overwhelming probability) and $s_{\mathscr{N}}(P - z + \delta\mathscr{G}_\omega(N)) > \delta t$ (probability at least $1 - C\mathscr{N}t^2$), we have that $s_A(E^\delta_{-+}) > \delta t$ with probability greater than $1 - C\mathscr{N}t^2 - e^{-\mathscr{N}}$. Therefore

$$\log|\det E^\delta_{-+}| \geq A \log s_A(E^\delta_{-+}) \geq A \log(\delta t)$$

with probability $\geq 1 - e^{-\mathscr{N}} - C\mathscr{N}t^2$. □

4.4 Bound on B_1

This section is devoted to estimating B_1 (as in (4.3.4)) which involves computing the trace of a function of a Toeplitz operator belonging to an exotic symbol class. This closely follows [Vog20], however several simplifications arise partially due to requiring weaker bounds, and several modifications are required as we are working with Toeplitz operators.

Claim 4.4.1 (Bound on B_1). *For $\mathscr{P}$ defined in (4.3.1),*

$$\log|\det \mathscr{P}^0| = N^d \fint_X \log|f_0(x) - z|^2 \,\mathrm{d}\mu + \mathscr{O}(N^{d-\min(2\rho\kappa,(1-2\rho))}\log(N)).$$

Proof. Let's first consider some preliminary reductions in computing $\log|\det \mathscr{P}^0|$. By Schur's complement formula, $|\det \mathscr{P}^0|^2 = |\det(P-z)|^2 |\det E^0_{-+}|^{-2}$. The first term is:

$$|\det(P-z)|^2 = \det Q = \prod_{i=1}^{\mathscr{N}} t_i^2.$$

Because $E^0_{-+} = -\sum_1^A t_j \delta_j \otimes \delta_j$ (recall A is the largest integer such that $t_A^2 \leq \alpha$), the second term is

$$|\det E^0_{-+}|^{-2} = \left(\prod_{i=1}^{A} t_i^2\right)^{-2},$$

therefore

$$|\det \mathscr{P}^0|^2 = \prod_{i=A+1}^{\mathscr{N}} t_i^2 = \alpha^{-A} \prod_{i=1}^{\mathscr{N}} 1_\alpha(t_i^2) = \alpha^{-A} \det 1_\alpha(Q)$$

where $1_\alpha = \max(x, \alpha)$. If χ is a cut-off function identically 1 on $[0,1]$, and supported in $[-1/2, 2]$, then $x + (\alpha/4)\chi(4x/\alpha) \leq 1_\alpha(x) \leq x + \alpha\chi(x/\alpha)$ for $x \geq 0$. Therefore

$$\det\left(Q + 4^{-1}\alpha\chi\left(Q/(4^{-1}\alpha)\right)\right) \leq \det(1_\alpha(Q)) \leq \det\left(Q + \alpha\chi(Q/\alpha)\right). \tag{4.4.1}$$

Now fix $1 \gg \alpha_1 > \alpha$, so that $\log\det(Q + \alpha\chi(Q/\alpha))$ can be written

$$-\int_\alpha^{\alpha_1} \frac{d}{dt} \log\det(Q + t\chi(Q/t))\,\mathrm{d}t + \log\det(Q + \alpha_1\chi(Q/\alpha_1)). \tag{4.4.2}$$

First the integrand is estimated. Let $\psi(t) = (t - t\chi'(t))(1+\chi(t))^{-1}$ so that

$$\frac{d}{dt}\log(x + t\chi(x/t)) = t^{-1}\psi(x/t)$$

for $t > 0$ and $\psi \in C_0^\infty(\mathbb{R}_{\geq 0})$. Therefore, by Jacobi's identity,

$$\frac{d}{dt} \log\det(Q + t\chi(Q/t)) = \mathrm{Tr}(t^{-1}\psi(Q/t)).$$

While morally the same, here we diverge from [Vog20]'s proof to handle this trace term, and must rely on chapter 3. The main issues are that Q is the composition of Toeplitz operators, which may no longer be a Toeplitz operator (but is modulo $\mathscr{O}(N^{-\infty})$ error), Q/t belongs to an exotic symbol class so to compute $\psi(Q/t)$ requires an exotic calculus, and the trace formula (Theorem 3.4.5) has weaker remainder than for quantizations of tori.

Let ρ_t be such that $t = N^{-2\rho_t}$. By Theorem 3.3.11, $Q = T_N q + \mathscr{O}(N^{-\infty})$, where the principal symbol of q is $|f_0 - z|^2$. For each t, Q/t is (modulo $\mathscr{O}(N^{-\infty})$) a Toeplitz operator with symbol in $S_{\rho_t}(m_t)$ where $m_t = q_0/t + 1$, by Example 3.2.2. And so, by Theorem 3.4.2, there exists $q_t \in S_{\rho_t}(m_t^{-1})$, such that $\psi(Q/t) = T_N(q_t) + E_N(t)$. Where q_t has principal symbol $\psi(q/t)$ and $E_N(t) = \mathscr{O}(N^{-\infty})$ (with estimates uniform over t). Therefore

$$\begin{aligned}\int_\alpha^{\alpha_1} \frac{d}{dt} \log\det(Q + t\chi(Q/t))\,\mathrm{d}t &= \int_\alpha^{\alpha_1} \mathrm{Tr}(t^{-1}\psi(Q/t))\,\mathrm{d}t \\ &= \int_\alpha^{\alpha_1} t^{-1}\,\mathrm{Tr}(T_N(q_t) + E_N(t))\,\mathrm{d}t.\end{aligned}$$

The error term is

$$\int_\alpha^{\alpha_1} t^{-1}\,\mathrm{Tr}(E_N(t))dt = \mathscr{O}(N^{-\infty})$$

because $E_N(t)$ is uniformly $\mathscr{O}(N^{-\infty})$. While for each t, Theorem 3.4.5 shows that

$$\mathrm{Tr}(T_N(q_t)) = \left(\frac{N}{2\pi}\right)^d \int_X \psi(q_0/t)\,\mathrm{d}\mu_d(x) + t^{-1}\mathscr{O}(N^{d-1})$$

because m^{-1} is bounded. Therefore

$$\begin{aligned}&\int_\alpha^{\alpha_1} \frac{d}{dt} \log\det(Q + t\chi(Q/t))\,\mathrm{d}t \\ &\qquad = \int_\alpha^{\alpha_1} \left(\int_X \left(\frac{N}{2\pi}\right)^d t^{-1}\psi(q_0/t)\,\mathrm{d}\mu_d(x) + t^{-2}\mathscr{O}(N^{d-1}) \right) \mathrm{d}t \\ &\qquad = \left(\frac{N}{2\pi}\right)^d \int_X \log(q_0 + t\chi(q_0/t))\Big|_{t=\alpha}^{t=\alpha_1} \mathrm{d}\mu(x) + \mathscr{O}(N^{d-1}\alpha).\end{aligned}$$

Next the second term of (4.4.2) is computed. Because α_1 is fixed, Q/α_1 has symbol in $S(1)$. Therefore, by Theorem 3.4.2, $Q + \alpha_1\chi(Q/\alpha_1) = T_N r + E_N$ (with $\|E_N\| = \mathscr{O}(N^{-\infty})$)

where $r \in S(1)$ with principal symbol $q_0 + \alpha_1\chi(q_0/\alpha_1)$. Let $r^t = tr + (1-t) \in S(1)$, so that

$$\log\det(Q + \alpha_1\chi(Q/\alpha_1)) = \int_0^1 \frac{d}{dt}\log\det(T_N r^t + tE_N)\,\mathrm{d}t$$
$$= \int_0^1 \mathrm{Tr}\left(\left(T_N r^t + tE_N\right)^{-1}\left(\frac{d}{dt}T_N r^t + E_N\right)\right)\mathrm{d}t.$$

The principal symbol of r^t is $r_0^1 = t(q_0 + \alpha_1\chi(q_0/\alpha_1)) + (1-t)$. Note that when $x \geq 0$, then $x + \alpha_1\chi(x/\alpha_1) \geq \alpha_1 > 0$. Therefore $(r_0^t) \geq \alpha_1$.

Lemma 4.4.1. *There exists $s(t) \in S(1)$ (with bounds uniform in t) such that $(T_N r^t + tE_N)^{-1} = T_N s(t) + \mathscr{O}(N^{-\infty})$, and the principal symbol of $s(t)$ is $(r_0^t)^{-1}$.*

Proof. By Theorem 3.4.1, there exists a symbol $\ell = \ell(t) \in S(1)$ which inverts (modulo $\mathscr{O}(N^{-\infty})$ error) $T_N r^t$, and has principal symbol $(r_0^t)^{-1}$. But then

$$(T_N r^t + tE_N)T_N\ell = 1 + K$$

with $K = \mathscr{O}(N^{-\infty})$, using that $tE_N = \mathscr{O}(N^{-\infty})$ and $T_N\ell$ has norm bounded independent of N. By Neumann series, for $N \gg 1$, $(1+K)$ is invertible, so that:

$$(T_N r^t + tE_N)(T_N\ell)(1+K)^{-1} = 1.$$

$(T_N\ell)(1+K)^{-1}$ will be a Toeplitz operator, modulo a $\mathscr{O}(N^{-\infty})$ term, with symbol ℓ which has principal symbol $(r_0^t)^{-1}$. By repeating this argument, but left-composing by $T_N\ell$, we get the lemma. □

Clearly $\frac{d}{dt}T_N r^t = T_N(r-1)$ so using Lemma 4.4.1, we get that

$$\left(T_N r^t + tE_N\right)^{-1}\left(\frac{d}{dt}T_N r^t + E_N\right)$$

is (modulo $\mathscr{O}(N^{-\infty})$) a Toeplitz operator with principal symbol $(r_0^t)^{-1}(\frac{d}{dt}r_0^t)$. So by Theorem 3.4.5

$$\mathrm{Tr}\left(\left(T_N r^t + tE_N\right)^{-1}\left(\frac{d}{dt}T_N r^t + E_N\right)\right)$$
$$= \left(\frac{N}{2\pi}\right)^d \int_X (r_0^t)^{-1}\left(\frac{d}{dt}r_0^t\right)\mathrm{d}\mu_d(x) + \mathscr{O}(N^{d-1})$$

which when integrated from $t=0$ to $t=1$ becomes:

$$\left(\frac{N}{2\pi}\right)^d \int_X \log(r_0^1)dx + \mathscr{O}(N^{d-1})$$
$$= \left(\frac{N}{2\pi}\right)^d \int_X \log(q_0 + \alpha_1\chi(q_0/\alpha_1))\,\mathrm{d}\mu_d(x) + \mathscr{O}(N^{d-1}).$$

Therefore (4.4.2) becomes:

$$\left(\frac{N}{2\pi}\right)^d \int_X \log(q_0 + \alpha\chi(q_0/\alpha))\,\mathrm{d}\mu_d + \mathscr{O}(N^{d-1}\alpha^{-1}).$$

A calculus lemma is required to estimate $\int_X \log(q_0 + \alpha\chi(q_0/\alpha))\,\mathrm{d}x$.

Lemma 4.4.2. *Given $q \in C^\infty(X; \mathbb{R}_{\geq 0})$ such that $\mu_d(\{x \in X : q(x) \leq t\}) = \mathscr{O}(t^\kappa)$ as $t \to 0$ for $\kappa \in (0, 1]$, and $\chi \in C_0^\infty((-1/2, 2); [0, 1])$ identically 1 on $[0, 1]$. Then*

$$\int_X \log(q + \alpha\chi(q/\alpha))\,\mathrm{d}\mu_d = \int_X \log(q)\,\mathrm{d}\mu_d + \mathscr{O}(\alpha^\kappa).$$

Proof. Let $g(t) = \log(t + \alpha\chi(t/\alpha))$ and $m(t) = \mu_d(\{x \in X : q(x) \leq t\})$. Then, letting $q_1 = \max q + 2\alpha$,

$$\begin{aligned}
\int_X \log(q + \alpha\chi(q/\alpha)) - \log(\alpha)\,\mathrm{d}\mu_d &= \int_X g(q(x)) - g(0)\,\mathrm{d}\mu_d \\
&= \int_X \int_0^{q(x)} g'(t)\,\mathrm{d}t\,\mathrm{d}\mu_d \\
&= \int_0^{q_1} g'(t) \int_{q(x)>t} \mathrm{d}\mu_d\,\mathrm{d}t \\
&= \int_0^{q_1} g'(t)(\mathrm{vol}(X) - m(t))\,\mathrm{d}t \\
&= \mathrm{vol}(X)(g(q_1) - \log(\alpha)) - \int_0^{q_1} g'(t)m(t)\,\mathrm{d}t.
\end{aligned}$$

So that:

$$\int_X \log(q + \alpha\chi(q/\alpha)\,\mathrm{d}\mu_d = \mathrm{vol}(X)g(q_1) - \int_0^{q_1} g'(t)m(t)\,\mathrm{d}t. \tag{4.4.3}$$

Similarly, if $\tilde{g}(t) = \log(t)$, we get an analogous expression as (4.4.3), that is:

$$\int_X \log(q)\,\mathrm{d}\mu_d = \mathrm{vol}(X)\tilde{g}(q_1) - \int_0^{q_1} \tilde{g}'(t)m(t)\,\mathrm{d}t.$$

Note that $g(q_1) = \tilde{g}(q_1)$. Therefore:

$$\begin{aligned}\left|\int_X \log(q+\alpha\chi(q/\alpha)) - \log(q)\,\mathrm{d}\mu_d\right| &= \left|\int_0^{q_1} (\tilde{g}'(t) - g'(t))m(t)\,\mathrm{d}t\right| \\ &= \left|\int_0^{q_1} \left(\frac{1}{t} - \frac{1+\chi'(t/\alpha))}{t+\alpha\chi(t/\alpha)}\right) m(t)\,\mathrm{d}t\right| \\ &= \left|\int_0^{q_1/\alpha} \left(\frac{1}{s} - \frac{1+\chi'(s)}{s+\chi(s)}\right) m(s\alpha)\,\mathrm{d}s\right| \\ &\lesssim \int_0^2 s^{-1} m(s\alpha)\,\mathrm{d}s \\ &\lesssim \alpha^\kappa \int_0^2 s^{\kappa-1}\,\mathrm{d}s \lesssim \alpha^\kappa.\end{aligned}$$

Here we use that $\chi(0) = 1$ to get a lower bound on $|s+\chi(s)|$, and the fact that $\chi(s) - s\chi'(s)$ is supported in $(0, 2)$. □

Applying this lemma, we get:

$$\log\det(Q+\alpha\chi(Q/\alpha)) = \left(\frac{N}{2\pi}\right)^d \int_X \log(q)\,\mathrm{d}\mu_d(x) + \mathscr{O}(\alpha^\kappa) + \mathscr{O}(N^{d-(1-2\rho)}).$$

Recalling that $(N/2\pi)^d \mathscr{N}^{-1} = \mathrm{vol}(X)^{-1} + \mathscr{O}(N^{-1})$, we get that:

$$\log\det(Q+\alpha\chi(Q/\alpha)) = (\mathscr{N} + \mathscr{O}(N^{-1})) ⨍ \log(q)\,\mathrm{d}\mu_d + \mathscr{O}(N^{d-(1-2\rho)}). \tag{4.4.4}$$

$\int_X \log(q)\,\mathrm{d}\mu_d$ can be uniformly bounded in z, so that the $\mathscr{O}(N^{-1})$ term can be absorbed into $\mathscr{O}(N^{d-(1-2\rho)})$. By (4.4.1), we get the following lower bound by replacing α by $\alpha/4$:

$$\log\det(Q+\alpha\chi(Q/\alpha)) \geq \mathscr{N} ⨍ \log(q)\,\mathrm{d}\mu_d + \mathscr{O}(N^{d-(1-2\rho)}). \tag{4.4.5}$$

Lemma 4.4.3 (Bound on A)**.** *The number of eigenvalues of Q that are less than α is*

$$\mathscr{O}(N^d N^{-\min(2\rho\kappa,(1-2\rho))}).$$

Proof. Let $\psi \in C_0^\infty([-1/2, 3/2]; [0, 1])$ be identically 1 on $[0, 1]$. It then suffices to estimate $\mathrm{Tr}(\psi(Q/\alpha))$. By Theorem 3.4.2, $\psi(Q/\alpha) = T_{N,q_2} + \mathscr{O}(N^{-\infty})$, where $q_2 \in S_\rho(1)$ with principal symbol $\psi(q/\alpha)$.

Then by Theorem 3.4.5

$$\begin{aligned}\mathrm{Tr}(\psi(Q/\alpha)) &= \mathrm{Tr}(T_{N,q_2} + \mathscr{O}(N^{-\infty})) \\ &= (N/2\pi)^d \int_X \psi(q/\alpha)\,\mathrm{d}\mu_d(x) + \mathscr{O}(N^{d-(1-2\rho)}) \\ &\lesssim N^d\alpha^\kappa + N^{d-(1-2\rho)} = \mathscr{O}(N^d N^{-\min(2\rho\kappa,1-2\rho)}).\end{aligned}$$

□

Therefore, putting everything together, we get that

$$\begin{aligned}\log|\det \mathscr{P}^0| &= \frac{1}{2}\log(|\det \mathscr{P}^0|^2) = \frac{1}{2}\log(\alpha^{-A}\det 1_\alpha(Q)) \\ &= \frac{A}{2}\log(1/\alpha) + \frac{1}{2}\log\det(1_\alpha Q)).\end{aligned}$$

(4.4.4) and (4.4.5) provide upper and lower bounds of $2^{-1}\log\det(1_\alpha(Q))$. Then using that $2^{-1}\log q_0 = |f_0 - z|$ and Lemma 4.4.3 we get:

$$\begin{aligned}\left|\log|\det \mathscr{P}^0| - \mathscr{N}⨍_X \log|f_0 - z|d\mu_d\right| &\lesssim A\log(1/\alpha) + \alpha^\kappa + N^{d-(1-2\rho)} \\ &\lesssim N^{d-\min(2\rho\kappa,(1-2\rho))}\log(N) \\ &\qquad + N^{-2\rho\kappa} + N^{d-(1-2\rho)} \\ &\lesssim N^{d-\min(2\rho\kappa,(1-2\rho))}\log(N).\end{aligned}$$

Recall $\mathscr{N}B_1 = \log|\det \mathscr{P}^0| - \mathscr{N}⨍\log|z - f_0(x)|\,\mathrm{d}\mu_d$, so that

$$B_1 = \mathscr{O}(N^{-\min(2\rho\kappa,(1-2\rho))}\log(N)).$$

□

4.5 Summability of $\mathscr{A}_N$

Recall that $\mathscr{A}_N = \{|B(N)| > \varepsilon_N\}$, where $B(N) = B_1 + B_2 + B_3$ with:

$$\begin{aligned}B_1 &= \mathscr{N}^{-1}\log|\det \mathscr{P}^0| - ⨍\log|z - f_0(x)|\,\mathrm{d}\mu_d(x), \\ B_2 &= \mathscr{N}^{-1}(\log|\det \mathscr{P}^\delta| - \log|\det \mathscr{P}^0|), \\ B_3 &= \mathscr{N}^{-1}\log|\det E_{-+}^\delta|.\end{aligned}$$

The following table summarizes the bounds on B_1, B_2, and B_3.

Bound	Probability of Bound	Reference
$B_1 = \mathscr{O}(N^{-\min(2\rho\kappa,(1-2\rho))}\log(N))$	1	Claim 4.4.1
$B_2 = \mathscr{O}(\delta\alpha^{-1/2}\mathscr{N}^{1/2})$	$> 1 - \exp(-\mathscr{N})$	Claim 4.3.2
$B_3 \geq \mathscr{N}^{-1}A\log(t\delta)$	$> 1 - C\mathscr{N}t^2 - \exp(-\mathscr{N})$	Claim 4.3.3
$B_3 < 0$	$> 1 - \exp(-\mathscr{N})$	Claim 4.3.3

Recall that $\rho \in (0, \min(1/2, \varepsilon))$ and $\alpha = N^{-2\rho}$. Theorem 4.1.4 will follow if $\sum \mathbb{P}(\mathscr{A}_N) < \infty$ for $\varepsilon_N = N^{-\gamma}$. Recall that $\delta = \mathscr{O}(N^{-d/2-\varepsilon}) = \mathscr{O}(N^{-d/2}\alpha^{1/2})$. Fix $0 < \gamma < \min(\varepsilon - \rho, 2\rho\kappa, 1 - 2\rho)$.

Then $\mathbb{P}(\mathscr{A}_N) = \mathbb{P}(B > N^{-\gamma}) + \mathbb{P}(B < -N^{-\gamma})$. The first term is:

$$\mathbb{P}(B > N^{-\gamma}) = \mathbb{P}(B_3 > N^{-\gamma} - B_2 - B_1).$$

Because $\gamma < \varepsilon - \rho$ and $B_2 = \mathscr{O}(N^{\rho-\varepsilon})$ (with overwhelming probability), we see that $B_2 = \mathscr{O}(N^{-\gamma})$ (with overwhelming probability). Similarly, because of the bound on B_1 and the choice of γ, $B_1 = \mathscr{O}(N^{-\gamma})$. So if N is sufficiently large, $N^{-\gamma} - B_2 - B_1 \geq CN^{-\gamma} > 0$. But then by Claim 4.3.3, $\mathbb{P}(B > N^{-\gamma}) \leq e^{-N^d}$ for $N \gg 1$.

Similarly, for N sufficiently large, there exists $C_0 \in (0, 1/2)$ such that, $|B_1| + |B_2| < C_0 N^{-\gamma}$, so $\mathbb{P}(B < -N^{-\gamma}) \leq \mathbb{P}(B_3 < -(1 - C_0)N^{-\gamma}) = 1 - \mathbb{P}(B_3 \geq -(1 - C_0)N^{-\gamma})$. By the choice of γ, bound on A from Lemma 4.4.3, and selecting $t = \mathscr{N}^{-2/d-1/2}$, we get for large enough N: $-(1 - C_0)N^{-\gamma} \leq \mathscr{N}^{-1} A \log(\delta t)$ as long as:

$$-N^{-\gamma}(1 - C_0) \leq \mathscr{N}^{-1} A \log(\delta).$$

This requires that $\delta \gg e^{-N^\beta}$ for $\beta = \min(2\rho\kappa, 1 - 2\rho) - \gamma \in (0, 1)$. In this case, by Claim 4.3.3,

$$\begin{aligned}
\mathbb{P}(B_3 > -N^{-\gamma}) &\geq \mathbb{P}(B_3 > A\mathscr{N}^{-1} \log(\delta t)) \\
&\geq 1 - C\mathscr{N} t^2 - e^{-\mathscr{N}} \\
&= 1 - C\mathscr{N}^{-2/d} + e^{-\mathscr{N}}.
\end{aligned}$$

Therefore $\mathbb{P}(B < -N^{-\gamma}) \leq CN^{-2} + e^{-N^d}$ for $N \gg 1$.

With this, $\sum_{N=1}^{\infty} \mathbb{P}(\mathscr{A}_N) = C + \sum_{N \gg 1} \mathbb{P}(A_N) \leq C + \sum_{N \gg 1} (N^{-2} + 2e^{-N^d}) < \infty$ which proves Theorem 4.1.4.

Note that if $\varepsilon > (2(\kappa+1))^{-1}$, then we can select $\rho = (2(\kappa+1))^{-1}$ and choose γ arbitrarily small, so that $\beta = \kappa(\kappa+1)^{-1} - \gamma$. While if $\varepsilon < (2(\kappa+1))^{-1}$, then the maximum β can be is $2\epsilon\kappa$. Therefore we have:

$$\beta < \begin{cases} 2\epsilon\kappa & \text{if } \epsilon < \frac{1}{2(\kappa+1)}, \\ \frac{\kappa}{\kappa+1} & \text{if } \epsilon \geq \frac{1}{2(\kappa+1)}. \end{cases}$$

4.6 General random perturbations

In this section, we provide a discussion about how to modify the proof of Theorem 4.1.4 (Gaussian random perturbations) to prove Theorem 4.1.5 (more general random perturbations). We also deduce Theorem 1.0.1 (stated in §1.1) from Theorem 4.1.5.

Proof. Under the assumptions of $\mathscr{W}_\omega(N)$ (see Definition 4.1.3), we have the following probabilistic norm bound:

$$\mathbb{E}[\|\mathscr{W}_\omega(N)\|^2] = \sum_{i,j=1}^{\mathscr{N}} \mathbb{E}[|(\mathscr{W}_\omega(N))_{i,j}|^2] = \mathscr{O}(\mathscr{N}^2), \tag{4.6.1}$$

as well as the following anti-concentration bound (from [TV09, Theorem 3.2]): for $\gamma_0 \geq 1/2$, $A_0 \geq 0$, there exists a $c > 0$ such that if M is a deterministic matrix with $\|M\| \leq \mathscr{N}^{\gamma_0}$ then

$$\mathbb{P}(s_{\mathscr{N}}(M + \mathscr{W}_\omega(N)) \leq \mathscr{N}^{-(2A_0+1)\gamma_0}) \leq c\left(\mathscr{N}^{-A_0+o(1)} + \mathbb{P}(\|\mathscr{W}_\omega(N)\| \geq \mathscr{N}^{\gamma_0})\right). \quad (4.6.2)$$

Recall, for an $N \times N$ matrix A, we denote $s_1 \geq s_2 \geq \cdots \geq s_N(A)$ the singular values of A.

From (4.6.1), and Markov's inequality, we get

$$\mathbb{P}(\|\mathscr{W}_\omega(N)\| \geq N^{d-1}) = \mathscr{O}(N^{-2})$$

therefore if $\delta = N^{-d}$ then $\delta \|\mathscr{W}_\omega(N)\| = \mathscr{O}(N^{-1})$ with probability at least $1 - CN^{-2}$. From this, Claim 4.2.2 (the supports of the random empirical measures being contained in a bounded set for $N \gg 1$) will follow by an identical argument.

Next, with probability at least $1 - CN^{-2}$, we have $\delta \|\mathscr{W}_\omega(N)\| \alpha^{1/2} \ll 1$. In this event, we can build our perturbed Grushin problem the same way as in Section 4.3.

Next, we have to modify the estimate of B_2 which was estimated in Claim 4.3.2. For this, we simply modify (4.3.7) with a weaker estimate on the probability $\|\mathscr{W}_\omega(N)\|$ is small. Specifically, we see there exists $C > 0$ such that

$$\mathbb{P}(B_2 = \mathscr{O}(\alpha^{-1/2}N^{-1})) > 1 - CN^{-2}.$$

The final modification is in estimating $B_3 = \mathscr{N}^{-1}\log|\det E_{-+}^\delta|$. We see, by the same argument presented in Section 4.3, that

$$\mathbb{P}(B_3 < 0) \geq 1 - CN^{-2}.$$

To prove a lower bound, we go through the same argument, to get that:

$$\log|\det E_{-+}^\delta| \geq A \log|s_{\mathscr{N}}(T_N f - z + \delta\mathscr{W}_\omega(N))|.$$

Next, let

$$K_0 := \sup_{z\in\Lambda} \|T_N f - z\| = \mathscr{O}(1)$$

(recall Λ is a neighborhood of $f(X)$). By (4.6.2) (with $\gamma_0 = 1$ and $A_0 = 2$), we have (for $N \gg 1$)

$$\begin{aligned}
&\mathbb{P}(s_{\mathscr{N}}(T_N f - z + \delta\mathscr{G}_\omega(N)) \leq N^{-7d}) \\
&\quad = \mathbb{P}(s_{\mathscr{N}}(\delta^{-1}K_0^{-1}(T_N f - z) + K_0^{-1}\mathscr{W}_\omega(N)) \leq (N^d)^{-(2A_0+1)\gamma_0}) \\
&\quad \leq c(N^{-2d+o(1)} + \mathbb{P}(\|K_0^{-1}\mathscr{W}_\omega(N)\| \geq N^{-d})) \\
&\quad \leq cN^{-2}.
\end{aligned}$$

Here we use that $\left\|\delta^{-1}K_0^{-1}(T_N f - z)\right\| \le N^d$. With this, we can proceed as in Section 4.5, with weaker probabilistic estimates. We choose $\rho \in (0, 1/2)$, and $0 < \gamma < \min(2\rho\kappa, 1 - 2\rho)$. Writing $\mathbb{P}(\mathscr{A}_N) = \mathbb{P}(B > N^{-\gamma}) + \mathbb{P}(B < -N^{-\gamma})$, we see that

$$\mathbb{P}(B > N^{-\gamma}) \le CN^{-2}$$

for $N \gg 1$. Similarly, in the event $s_{\mathscr{N}}(T_N f - z + \delta \mathscr{W}_\omega(N)) \ge N^{-7d}$, we have (for $N \gg 1$)

$$A \log |s_{\mathscr{N}}(T_N f - z + \delta \mathscr{W}_\omega(N))| \le N^{d-\gamma}$$

so that

$$\mathbb{P}(B_3 > -N^{-\gamma}) \ge \mathbb{P}(B_3 > A\mathscr{N}^{-1} \log |s_{\mathscr{N}}(T_N f - z + \delta \mathscr{W}_\omega(N))|) \ge 1 - CN^{-2}.$$

Therefore $\mathbb{P}(B < -N^{-\gamma}) \le CN^{-2}$ for $N \gg 1$. With this, $\sum_1^\infty \mathbb{P}(\mathscr{A}_N) < \infty$, and we have almost sure weak convergence of the empirical measures of $T_N f + \delta \mathscr{W}_\omega(N)$ to $\mathrm{vol}(X)^{-1}(f_0)_* \mu_d$. □

Proposition 4.6.1. *Theorem 4.1.5 implies the probabilistic Weyl law (Theorem 1.0.1) stated in the introduction.*

Proof. Here we prove Theorem 1.0.1 for the general random perturbation case, but the Gaussian case (with a less restrictive coupling constant) follows similarly.

For $\Lambda \subset \mathbb{C}$ given in the hypothesis, let

$$A_N := \frac{\mathrm{vol}(X)}{\mathscr{N}} \# \left\{ \mathrm{Spec}(T_N f + N^{-d} \mathscr{W}_\omega(N)) \cap \Lambda \right\}.$$

It suffices to show that for each $\varepsilon > 0$

$$\mathbb{P}\left(\limsup_{N \to \infty} |A_N - \mu_d(f \in \Lambda)| > \varepsilon \right) = 0.$$

We may assume Λ is bounded. If not, let $\tilde{\Lambda}$ be an open, bounded neighborhood of $f(X)$. Recall that almost surely $\mathrm{Spec}(T_N f + \delta \mathscr{W}_\omega(N)) \subset \tilde{\Lambda}$ for $N \gg 1$. Therefore if

$$\tilde{A}_N = (\mathrm{vol}(X)/\mathscr{N}) \# \left\{ \mathrm{Spec}(T_N f + N^{-d} \mathscr{W}_\omega(N)) \cap \Lambda \cap \tilde{\Lambda} \right\},$$

then

$$\mathbb{P}\left(\limsup_{N \to \infty} |A_N - \mu_d(f \in \Lambda)| > \varepsilon \right) = \mathbb{P}\left(\limsup_{N \to \infty} |\tilde{A}_N - \mu_d(f \in \Lambda)| > \varepsilon \right).$$

Now relabel $\Lambda \cap \tilde{\Lambda}$ as Λ. For each $\varepsilon > 0$, let $K \subset \Lambda$ be a compact set such that $m(\Lambda \setminus K) < \varepsilon$. Similarly, let $U \supset \Lambda$ be an open set such that $m(U \setminus \Lambda) < \varepsilon$ (here m denotes the Lebesgue measure). Define $\varphi, \psi \in C_0^\infty(\mathbb{C}; [0,1])$ such that $\operatorname{supp}\varphi, \operatorname{supp}\psi \subset U$, $\varphi(x) \equiv 1$ for $x \in K$, and $\psi(x) \equiv 1$ for $x \in \Lambda$. We then have that for $\lambda_j \in \operatorname{Spec}(T_N f + N^{-d}\mathscr{W}_\omega(N))$

$$\frac{\operatorname{vol}(X)}{\mathscr{N}} \sum_{j=1}^{\mathscr{N}} \varphi(\lambda_i) - \frac{\operatorname{vol}(X)}{\mathscr{N}} \#\{\lambda_i \in \Lambda \setminus K\} \leq A_N \leq \frac{\operatorname{vol}(X)}{\mathscr{N}} \sum_{j=1}^{\mathscr{N}} \psi(\lambda_i). \tag{4.6.3}$$

By Theorem 4.1.5, the lower bound of (4.6.3) convergences almost surely to

$$\int_{\mathbb{C}} \varphi(z)(f_* \mu_d)(\mathrm{d}z) = \mu_d(f \in \Lambda) + \mathscr{O}(\varepsilon^\kappa).$$

Here the term involving $\#\{\lambda_i \in \Lambda \setminus K\}$ is easily handled by a similar argument of constructing upper and lower bounds with bump functions and using that $\Lambda \setminus K$ is an open set whose measure is $\mathscr{O}(\varepsilon)$. Similarly, the upper bound of (4.6.3) converges almost surely to $\mu_d(f \in \Lambda) + \mathscr{O}(\varepsilon^\kappa)$ (where the constant in $\mathscr{O}(\varepsilon^\kappa)$ is deterministic). Therefore there exists $C > 0$ such that

$$\mathbb{P}\left(\limsup_{N\to\infty} |A_N - \mu_d(f \in \Lambda)| > C\varepsilon^\kappa\right) = 0.$$

Because $\varepsilon > 0$ is arbitrary, this implies A_N converges almost surely to $\mu_d(f \in \Lambda)$. Then, because $\mathscr{N} = \operatorname{vol}(X)(N/2\pi)^d + \mathscr{O}(N^{d-1})$, $(N/2\pi)^d \operatorname{vol}(X)\mathscr{N}^{-1} A_N$ converges almost surely to $\mu_d(f \in \Lambda)$. $\square$

Appendix A

Computation of the second term in the star product

The goal of this section is to compute the second term in the star product of Toeplitz operators. Indeed, by Theorem 3.3.11 we know that if f and g are symbols in $S_\delta(m)$, then there exists a symbol $h \sim \sum N^{-j} h_j$, such that $T_{N,f} \circ T_{N,g} = T_{N,h} + \mathscr{O}(N^{-\infty})$. It is straightforward to show that $h_0 = fg$ (modulo $\mathscr{O}(N^{-(1-2\delta)})$).

A.1 General quantizatable Kähler manifolds

This section directly computes h_1 (modulo $\mathscr{O}(N^{-2(1-2\delta)}m)$ error).

In this section, for vectors $u, v \in \mathbb{C}^d$, we write $\langle u, v\rangle \coloneqq \sum u_i v_i$. For functions $f \in C^\infty(\mathbb{C}^d)$, we denote by $\nabla_x f$ the vector in $\mathbb{C}^d$ whose j^{th} component is $\partial_{x_j} f$. We similarly denote by $\nabla_{\bar{x}} f$ the vector whose j^{th} component is $\overline{\partial}_{x_j} f$.

Theorem A.1.1. *Given $\delta \in [0, 1/2)$, suppose m_1, m_2 are two δ-order functions on X (a quantizable Kähler manifold with Kähler potential φ), $f \in S_\delta(m_1)$, $g \in S_\delta(m_2)$, and $h = f \star g$. Then locally*

$$h_1(x) = -\sum_{j,k=1}^{d} (\partial\overline{\partial}\varphi(x))^{j,k} \partial_k f(x) \overline{\partial}_j g(x) + \mathscr{O}(N^{-2(1-2\delta)} m(x)), \tag{A.1.1}$$

where $(\partial\overline{\partial}\varphi(x))^{j,k} \in \mathbb{C}^{d\times d}$ is such that $\sum_k (\partial\overline{\partial}\varphi(x))^{j,k} (\partial_k \overline{\partial}_\ell \varphi(x)) = \delta_{j,\ell}$.

Remark A.1.1. *From this, we get the classical-quantum correspondence of Toeplitz operators. Indeed, by (A.1.1) the principal symbol of $[T_{N,f}, T_{N,g}]$ is*

$$N^{-1}(-\left\langle (\partial\overline{\partial}\varphi)^{-1} \nabla_x f, \nabla_{\bar{x}} g \right\rangle + \left\langle (\partial\overline{\partial}\varphi)^{-1} \nabla_x g, \nabla_{\bar{x}} f \right\rangle) + \mathscr{O}(N^{-2(1-2\delta)} m). \tag{A.1.2}$$

Note that the Poisson bracket of f and g is $\{f, g\} = \omega(X_f, X_g)$, where X_f and X_g are the Hamiltonian vector fields of f and g respectively and ω is the symplectic form on X. If we *write* $\omega = \sum W_{i,j}\,\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j$, *then*

$$\overline{\partial}_{x_i} f = \sum_{j=1}^{d} W_{j,i}\,\mathrm{d}z_j(X_f), \qquad \partial_{x_i} f = -\sum_{j=1}^{d} W_{i,j}\,\mathrm{d}\bar{z}_j(X_f),$$

with identical identities relating g *and* X_g*. Therefore*

$$\begin{aligned}\omega(X_f, X_g) &= \sum_{i,j} W_{i,j}\,\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j(X_f, X_g)\\ &= \sum_{i,j} W_{i,j}(\mathrm{d}z_i(X_f)\,\mathrm{d}\bar{z}_j(X_g) - \mathrm{d}z_i(X_g)\,\mathrm{d}\bar{z}_j(X_f))\\ &= \left\langle W^t(W^t)^{-1}\nabla_{\bar{x}} f, -W^{-1}\nabla_x g\right\rangle - \left\langle W^t(W^t)^{-1}\nabla_{\bar{x}} g, -W^{-1}\nabla_x f\right\rangle\\ &= \left\langle W^{-1}\nabla_x f, \nabla_{\bar{x}} g\right\rangle - \left\langle W^{-1}\nabla_x g, \nabla_x f\right\rangle.\end{aligned}$$

Now, because $W_{i,j} = i\partial_i\overline{\partial}_j\varphi$*, we see from* (A.1.2) *that*

$$[T_{N,f}, T_{N,g}] = \frac{1}{iN} T_{N,\{f,g\}} + \mathscr{O}(N^{-2(1-2\delta)}m).$$

The method to prove Theorem A.1.1 is to compute the Schwartz kernel of the asymptotic expansion of $T_{N,f} \circ T_{N,g}$ and find a symbol that agrees with this kernel. By the almost analytic properties of the kernel, it suffices to work exclusively on the diagonal. Along the diagonal, the method of stationary phase has more explicit formulae. This section will use a stationary phase expansion presented in [Hör83].

Proof. Estimates on the error term in (A.1.1) were established in Theorem 3.3.1. For a simpler proof, we assume that $f, g \in S_0(1)$.

Near $x_0 \in X$, we choose a normal coordinate system $(z^1(x), \ldots, z^d(x)) \in \mathbb{C}^d$. In this way, $\partial_{z_j}\partial_{\bar{z}_k}\varphi(z(x_0)) = \delta_{j,k}$ and $\partial^\alpha_{z,\bar{z}}\partial_{z_j}\partial_{\bar{z}_k}\varphi(z(x_0)) = 0$ for all $j, k = 1, \ldots, d$ and $\alpha \in \mathbb{N}^{2d}$ with $|\alpha| = 1$.

Let C_j be the differential operators coming from stationary phase:

$$\left(\frac{N}{2\pi}\right)^{2d} \int_{\mathbb{C}^d} u(w) e^{N\Phi_{x\bar{y}}(w)}\,\mathrm{d}\mu(w) \sim \left(\frac{N}{2\pi}\right)^{d} e^{N\psi(x,\bar{y})} \sum_0^\infty N^{-j} C_j[u](x, \bar{y}),$$

with $u \in C^\infty(\mathbb{C}^d; \mathbb{C})$, $\Phi_{x,\bar{y}}(w) = \psi(x, \bar{w}) - \varphi(w) + \psi(w, \bar{y})$, and $\mu(w) = \omega^{\wedge d}/d!$. When computing $T_{N,f}$, terms of order $\mathscr{O}(N^{-2})$ are not needed to compute the second term in the expansion. In this case the functions coming from the Bergman kernel expansion can be approximated as $B(x, \bar{y}) = 1 + N^{-1}b_1(x, \bar{y}) + \mathscr{O}(N^{-2})$, so that the amplitude in the kernel of $T_{N,f}$ is $f(w)(1 + N^{-1}(b_1(x, \bar{w}) + b_1(w, \bar{y})) + \mathscr{O}(N^{-2})$. In this way:

$$\begin{aligned}f_0(x, \bar{y}) &= C_0[f](x, \bar{y}),\\ f_1(x, \bar{y}) &= C_0[f(\cdot)(b_1(x, \cdot) + b_1(\cdot, \bar{y})](x, \bar{y}) + C_1[f](x, \bar{y}),\end{aligned}$$

and on the diagonal:

$$\begin{aligned} f_0(x,\bar{x}) &= f(x), \\ f_1(x,\bar{x}) &= 2f(x)b_1(x) + C_1[f](x,\bar{x}). \end{aligned} \tag{A.1.3}$$

If we are given $T_{N,f}$ and $T_{N,g}$, then the first term in the expansion of $T_{N,f} \circ T_{N,g}$ along the diagonal will be $f(x)g(x)$. While the second term is

$$C_0[f_1(x,\cdot)g_0(\cdot,\bar{y}) + f_0(x,\cdot)g_1(\cdot,\bar{y})](x,\bar{y}) + C_1[f_0(x,\cdot)g_0(\cdot,\bar{y})](x,\bar{y}).$$

Along the diagonal, this is

$$(2fb_1 + C_1[f])g + f(2gb_1 + C_1[g]) + C_1[f_0(x,\cdot)g_0(\cdot,\bar{x})], \tag{A.1.4}$$

with all C_j operators evaluated at $(x,\bar{x})$ and functions evaluated at x. Suppose $T_{N,f} \circ T_{N,g} = T_{N,h} + \mathscr{O}(N^{-\infty})$ for some $h \sim \sum N^{-j}h_j$. The N^0 order term of $T_{N,h}(x,\bar{x})$ is $C_0(h_0) = h_0(x)$, so that $h_0 = f(x)g(x)$. The N^{-1} order term is

$$C_0[h_0(b_1(x,\cdot) + b_1(\cdot,\bar{y})) + h_1](x,\bar{y}) + C_1[h_0](x,\bar{y}).$$

Along the diagonal this is:

$$2h_0(x)b_1(x) + h_1(x) + C_1[fg](x,\bar{x}). \tag{A.1.5}$$

Setting (A.1.5) equal to (A.1.4), and solving for h_1 gives the relation

$$\begin{aligned} h_1(x) &= 4fgb_1 - 2fgb_1 - C_1[fg] + C_1[f_0(x,\cdot)g_0(\cdot,\bar{x})] + C_1[f]g + fC_1[g] \\ &= 2fgb_1 + C_1[f_0(x,\cdot)g_0(\cdot,\bar{x}) - f(\cdot)g(\cdot)] + gC_1[f] + fC_1[g] \end{aligned} \tag{A.1.6}$$

with all C_j operators evaluated at $(x,\bar{x})$, f,g evaluated at x, and b_1 evaluated at $(x,\bar{x})$.

Recall that $i\partial\overline{\partial}\varphi = \omega$ and in normal coordinates $\omega(x) = iH$ with H a positive definite, real, self-adjoint, invertible matrix, such that $H(x_0) = 1$.

Lemma A.1.2. *On the diagonal:*

$$C_1[u] = L_1[u \det H]$$

with $L_1 = -\langle \nabla_z, \nabla_{\bar{z}} \rangle + A$, *and*

$$A = -2^{-1}\langle \nabla_z, \nabla_{\bar{z}} \rangle \det H(x_0) - b_1(x_0)$$

Proof. By [Hör83, Theorem 7.7.5]

$$\int_{\mathbb{R}^{2d}} u(w)e^{N\Phi_{x,\bar{x}}(w)}\,\mathrm{d}w \sim \frac{e^{N\Phi_{x,\bar{x}}(x)}(2\pi)^d}{\sqrt{\det(-N\Phi''_{x\bar{x}}(x))}} \sum_{j=0}^{\infty} N^{-j}L_j u, \tag{A.1.7}$$

with:

$$L_1 u = \sum_{\nu=1}^{3} i^{-1+\nu} 2^{-\nu} \left\langle (\Phi''_{x,\bar{x}})^{-1}(x) D, D \right\rangle^{\nu} (\mathfrak{g}_x^{\nu-1} u)(w)/((\nu-1)!\nu!),$$

with derivatives evaluated at x, and

$$\mathfrak{g}_x(w) = \Phi_{x,\bar{x}}(w)/i - \Phi_{x,\bar{x}}(x)/i + \langle i\Phi_{x,\bar{x}}(x)''(w-x), w-x \rangle /2.$$

By computation

$$\Phi''_{x_0,\bar{x}_0}(x_0) = -4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so that

$$\det(-N\Phi''_{x_0,\bar{x}_0}(x_0))^{-1/2} = (4N)^{-d}.$$

Therefore, (A.1.7) simplifies to:

$$\left(\frac{\pi}{2N}\right)^d \sum_{j=0}^{\infty} N^{-j} L_j(u) \tag{A.1.8}$$

Observe that $\left\langle \Phi''_{x_0,x_0}(x_0)^{-1} D, D \right\rangle = 4^{-1}\Delta_w$ (using the notation that $D = i^{-1}\nabla$). Let $\mathfrak{g} := \mathfrak{g}_{x_0}$, and note that $\mathfrak{g}$ vanishes to third order at x_0. Then we compute that

$$L_1 u = \frac{1}{8}\Delta_w u + c_2((\nabla_w)^2 \mathfrak{g})u + c_3(\nabla_w(\Delta_w \mathfrak{g})) \cdot \nabla_w u + c_4((\Delta_w)^3(\mathfrak{g}^2)u) \tag{A.1.9}$$

for some constants c_2, c_3, c_4. Observe that $(\nabla_w(\Delta_w \mathfrak{g}))$ evaluated at x_0 will be a linear combination of first derivatives of the entries of $H(x)$, which are all zero because we are using normal coordinates. Therefore (A.1.9), evaluated at x_0, reduces to:

$$L_1 u = \left(\frac{1}{8}\Delta_w + A\right) u = \left(\frac{1}{2}\nabla_z \cdot \nabla_{\bar{z}} + A\right) u \tag{A.1.10}$$

for some constant A, and using the complex variable $z = w_1 + iw_2$.

The operators C_j along the diagonal can be recovered from L_j. Indeed, by matching powers of N and using that $\mathrm{d}\mu = \omega^\wedge d/d! = \det(H) 2^d \, \mathrm{d}m(w)$ (see [LeF18, Lemma 2.6.2]), we see, by (A.1.8),

$$C_j[u] = L_j[u \det H]. \tag{A.1.11}$$

The constant A can be computed by recalling that if $f = 1$, the Toeplitz operator is just the Bergman projector. So letting $f = 1$ in (A.1.3), we get that $C_0[2b_1] + C_1[1] = b_1$, which can be rearranged as $b_1 = -C_1(1)$. Then since $\det H(x_0) = 1$, and using (A.1.11) and (A.1.10), we get

$$b_1 = -A - 2^{-1} \langle \nabla_z, \nabla_{\bar{z}} \rangle \det H(z). \tag{A.1.12}$$

□

Because we are using normal coordinates, $\nabla \det H = 0$, therefore using (A.1.12),

$$\begin{aligned} C_1[u] &= (2^{-1}\langle \nabla_z, \nabla_{\bar{z}}\rangle + A)(u \det H) \\ &= Au + 2^{-1}\langle \nabla_z, \nabla_{\bar{z}}\rangle u + 2^{-1}(\langle \nabla_z, \nabla_{\bar{z}}\rangle \det(H))u \\ &= 2^{-1}\langle \nabla_z, \nabla_{\bar{z}}\rangle u - b_1 u. \end{aligned}$$

Then (A.1.6) becomes, after canceling all fgb_1's,

$$2^{-1}(\langle \nabla_z, \nabla_{\bar{z}}\rangle (f_0(x,\cdot)g_0(\cdot,\bar{x}) - f(\cdot)g(\cdot)) - g\langle \nabla_z, \nabla_{\bar{z}}\rangle f - f\langle \nabla_z, \nabla_{\bar{z}}\rangle g) \tag{A.1.13}$$

Now note that f_0 and g_0 are almost holomorphic in the first argument, and almost anti-holomorphic in the second coordinate. They can be treated as holomorphic and anti-holomorphic as we are on the diagonal. So (A.1.13) becomes, after applying the product rule and canceling terms:

$$-2^{-1}\langle \nabla_z f, \nabla_{\bar{z}} g\rangle . \tag{A.1.14}$$

Finally, if we use arbitrary holomorphic coordinates x, and let $J = Dx/Dz$ be the Jacobian relating the x coordinates to the normal coordinates, then (A.1.14) is

$$-2^{-1}\left\langle J^t\nabla_x f, (\bar{J})^t\nabla_{\bar{x}} g\right\rangle = -2^{-1}\left\langle \bar{J}J^t\nabla_x f, \nabla_{\bar{x}} g\right\rangle . \tag{A.1.15}$$

Because we used normal coordinates, J must satisfy

$$2I = J^t(\partial_x\overline{\partial}_x\varphi)\bar{J} = J^t(H)\bar{J},$$

so that $J^T = 2\bar{J}^{-1}H^{-1}$, so that (A.1.15) becomes:

$$-\left\langle H^{-1}\nabla_x f, \nabla_{\bar{x}} g\right\rangle .$$

Then, because $H = \partial\overline{\partial}\varphi$, we get our theorem. □

A.2 Example: $\mathbb{C}$

In this section we prove an asymptotic expansion of the kernel of Toeplitz operators on $\mathbb{C}$ and compute the second term in the star product of operators.

A.2.1 Asymptotic expansion of Toeplitz operators

A simple (although not compact) Kähler manifold is $\mathbb{C}$ with symplectic form $\omega = i\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$. Considering holomorphic sections of powers of the trivial line bundle, the quantum space for each N can be identified with all holomorphic functions f such that

$$\int_{\mathbb{C}} |f|^2 e^{-N|z|^2}\,\mathrm{d}m(z) < \infty.$$

This space is called the Bargmann space with L^2 structure

$$\langle f, g\rangle := 2\int_{\mathbb{C}} f(z)\bar{g}(z)e^{-N|z|^2}\,\mathrm{d}m(z).$$

In this case, the Bergman kernel is

$$\Pi_N(x,\bar{y}) = \frac{N}{2\pi}\exp(Nx\bar{y}).$$

The Kähler potential is $\varphi(y) = |y|^2$ with analytic extension $\psi(x,\bar{y}) = x\bar{y}$ (see for example [LeF18, Example 7.2.2]). So if $f \in S_\delta(1)$ for a fixed $\delta \in [0, 1/2)$, the kernel of the Toeplitz operator $T_{N,f}$ is

$$T_{N,f}(x,\bar{y}) = \left(\frac{N}{2\pi}\right)^2 \int_{\mathbb{C}} f(w)\exp(N(x\bar{w} - |w|^2 + w\bar{y}))2\,\mathrm{d}m(w).$$

We write this as an integral over $\mathbb{R}^2$ by letting $w = w_1 + iw_2$. Completing the square of the phase, this integral is:

$$\left(\frac{N}{2\pi}\right)^2 e^{Nx\bar{y}} \int_{\mathbb{R}^2} e^{N(-(w_1-a)^2-(w_2-b)^2)} f(w_1 + iw_2)2\,\mathrm{d}w_1\,\mathrm{d}w_2 \tag{A.2.1}$$

with $a = 2^{-1}(x + \bar{y})$ and $b = (2i)^{-1}(x - \bar{y})$, which is approximately true for quantizable Kähler manifolds[1]. Note by the Gaussian decay in the integrand of (A.2.1) it suffices to assume f is compactly supported as anything away from a or b will be exponentially small in N.

We may now integrate (A.2.1) as an iterated integral. Let's first integrate over w_1. For R sufficiently large, let $a = a_1 + ia_2$, and rewrite the inner integral in (A.2.1) as

$$\int_{-R}^{R} e^{-N(w_1-ia_2)^2} f(w_1 + a_1 + iw_2)\,\mathrm{d}w_1. \tag{A.2.2}$$

Let $f_{\mathbb{R}}(x, y) = f(x+iy)$, so the integrand has the term $f_{\mathbb{R}}(w_1+a_1, w_2)$. By Stokes' Theorem, (A.2.2) is

$$\int_{-R}^{R} e^{-Nw_1^2}\tilde{f}_{\mathbb{R}}(w_1 + ia_2 + a_1, w_2)\,\mathrm{d}w_1 + \iint_{\Omega_a} e^{-N(z-ia_2)^2}\overline{\partial}_z\tilde{f}_{\mathbb{R}}(z + a_1, w_2)\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$$

[1] In the general case, this (a, b) is $\tilde{p}(t)$. Much of the trouble with the method of complex stationary phase is that the phase is not holomorphic, and therefore the extension is not unique, and so the critical point is no longer unique. However, when the phase is not holomorphic, the critical point still approximately takes this form.

where $\tilde{f}_{\mathbb{R}}$ is an extension of $f_{\mathbb{R}}$ to $\mathbb{C}^2$, and $\Omega_a = \{x + iy : x \in [-R, R], y \in [0, a_2]\}$. Ignoring the second term for the moment, we now integrate the first term over w_2, by the same reasoning (possibly increasing R), we get

$$\int_{-R}^{R}\int_{-R}^{R} e^{-N(w_1^2+w_2^2)}\tilde{f}_{\mathbb{R}}(w_1+a, w_2+b)\,\mathrm{d}w_1\,\mathrm{d}w_2 \\ + \int_{-R}^{R}\iint_{\Omega_b} e^{-N(w_1^2+(z-ib_2))}\overline{\partial}_z\tilde{f}_{\mathbb{R}}(w_1+a, z+b)(\mathrm{d}z\wedge\mathrm{d}\bar{z})\,\mathrm{d}w_1. \tag{A.2.3}$$

The first term in (A.2.3) is estimated using the method of steepest descent (for example see [GS94, Exercise 2.4]), as

$$\left(\frac{\pi}{N}\right)\sum_{k=0}^{M-1}\frac{N^{-k}}{4^k k!}\Delta^k\tilde{f}_{\mathbb{R}}(a,b) + S_M(f,N),$$

with

$$|S_M(f,N)| \le C_N N^{-M-1}\sum_{|\alpha|=2M}\sup|\partial^\alpha\tilde{f}_{\mathbb{R}}|.$$

Here $\Delta\tilde{f}_{\mathbb{R}}(x,y) \coloneqq (\partial^2_{\mathrm{Re}(x)} + \partial^2_{\mathrm{Re}(y)})(\tilde{f}_{\mathbb{R}}(x,y))$. If we compute the kernel on the diagonal, $x = y$, then all derivatives are tangential to the totally real submanifold which $\tilde{f}_{\mathbb{R}}$ is extended from and we evaluate the derivatives at $(\mathrm{Re}\,(x), \mathrm{Im}\,(x))$. So when $x = y$, the first term in (A.2.3) is

$$\left(\frac{\pi}{2N}\right)\sum_{k=0}^{M}\frac{N^{-k}}{4^k k!}(\partial_u^2+\partial_v^2)^k(f(u+iv))\Big|_{\substack{u=\mathrm{Re}(x)\\ v=\mathrm{Im}(x)}} + S_M(f,N) \\ = \left(\frac{\pi}{2N}\right)\sum_{k=0}^{M}\frac{N^{-k}}{k!}(\partial\overline{\partial})^k f(x) + S_M(f,N). \tag{A.2.4}$$

A.2.2 Controlling error terms

Next we show that the error terms

$$I_1 \coloneqq \left(\frac{N}{2\pi}\right)^2 e^{Nx\bar{y}}\int_{\mathbb{R}} e^{-N(w_2-b)^2}\iint_{\Omega_a} e^{-N(z-ia_2)^2}\overline{\partial}_z\tilde{f}_{\mathbb{R}}(z+a_1, w_2)\,\mathrm{d}z\wedge\mathrm{d}\bar{z}\,\mathrm{d}w_2,$$

$$I_2 \coloneqq \left(\frac{N}{2\pi}\right)^2 e^{Nx\bar{y}}\int_{\mathbb{R}}\iint_{\Omega_b} e^{-N(w_1^2+(z-ib_2))}\overline{\partial}_z\tilde{f}_{\mathbb{R}}(w_1+a, z+b)\,\mathrm{d}z\wedge\mathrm{d}\bar{z}\,\mathrm{d}w_1,$$

are $\exp(-\frac{N}{2}(|x|^2+|y|^2))\mathscr{O}(N^{-\infty})$. First note that $2\mathrm{Re}\,(x\bar{y}) = -|x-y|^2 + |x|^2 + |y|^2$. Let $\varepsilon = x - y$, so that $a = \mathrm{Re}\,(x) - \bar{\varepsilon}/2$, $b = \mathrm{Im}\,(x) - \bar{\varepsilon}/(2i)$, $a_2 = \mathrm{Im}\,(\varepsilon)/2$, and $b_2 = \mathrm{Re}\,(\varepsilon)/2$.

Therefore, $|I_1|$ is bounded by

$$\left(\frac{N}{2\pi}\right)^2 e^{\frac{N}{2}(|x|^2+|y|^2)}\Big|\int_{\mathbb{R}} e^{-N(w_2-b)^2}\int_{-R}^{R}\int_0^{\frac{\mathrm{Im}(\varepsilon)}{2}} e^{-N(z_1+iz_2-i\mathrm{Im}(\varepsilon)/2)^2-N|\varepsilon|^2/2)} \cdot\overline{\partial}_z\tilde{f}_{\mathbb{R}}(z+a_1,w_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}w_2\Big|.$$

We then apply $\overline{\partial}$ estimates for $\tilde{f}_{\mathbb{R}}$. That is for each $M\in\mathbb{N}$, there exists $C_M>0$ so that $\overline{\partial}_z\tilde{f}_{\mathbb{R}}(a,b)\leq C_M N^{\delta M_0}(|\mathrm{Im}\,(a)|+|\mathrm{Im}\,(b)|)^M$. Fixing, M, the inner integral is bounded by

$$C_M\int_0^{\frac{\mathrm{Im}(\varepsilon)}{2}} e^{N(z_2-\mathrm{Im}(\varepsilon)/2)^2-N|\varepsilon|^2/2}N^{\delta M_0}z_2^M\,\mathrm{d}z_2 \tag{A.2.5}$$

Expanding the exponential, we see that

$$\begin{aligned}(\text{A.2.5})&\leq C_M N^{\delta M_0}e^{-\frac{N}{4}\mathrm{Im}(\varepsilon)^2}\int_0^{\frac{\mathrm{Im}(\varepsilon)}{2}}\exp\left(Nz_2^2-z\mathrm{Im}\,(\varepsilon)\,N\right)\mathrm{d}z_2\\
&\leq C_M N^{\delta M_0}e^{-\frac{N}{4}\mathrm{Im}(\varepsilon)^2}\frac{\mathrm{Im}\,(\varepsilon)^{M+1}}{N^{M+1}}\int_0^{\frac{N}{2}}t^M\exp\left(t\mathrm{Im}\,(\varepsilon)^2\left(\frac{t}{N}-1\right)\right)\mathrm{d}t\\
&\leq C_M N^{\delta M_0}e^{-\frac{N}{4}\mathrm{Im}(\varepsilon)^2}\frac{\mathrm{Im}\,(\varepsilon)^{M+1}}{N^{M+1}}\int_0^{\frac{N}{2}}t^M e^{-t\mathrm{Im}(\varepsilon)^2/2}\,\mathrm{d}t\\
&\lesssim_M e^{-\frac{N}{4}\mathrm{Im}(\varepsilon)^2}\frac{N^{\delta M_0}}{N^{M+1}\mathrm{Im}\,(\varepsilon)^{M+1}}\int_0^{\frac{\mathrm{Im}(\varepsilon)^2N}{4}}e^{-t}t^M\,\mathrm{d}t\\
&\lesssim_M N^{\delta M_0-M-1}.\end{aligned}$$

Therefore:

$$\begin{aligned}|I_1|&\lesssim_M N^{\delta M_0-M-1+2}e^{\frac{N}{2}(|x|^2+|y|^2)}\int_{\mathbb{R}}e^{-N(w_2-b)^2}\int_{-R}^{R}e^{-Nz_1^2}\,\mathrm{d}z_1\,\mathrm{d}w_2\\
&\lesssim N^{\delta M_0-M+1}e^{\frac{N}{2}(|x|^2+|y|^2)},\end{aligned}$$

so that $I_1=e^{\frac{N}{2}(|x|^2+|y|^2)}\mathscr{O}(N^{-\infty})$. An identical argument is used to show the same bound for I_2.

A.2.3 Using the notation presented in Treves

It is instructive to see how the change of variables presented by Treves in [Tre80], and used in §3.3, applies to this simple example. Let's consider a symbol f to quantize. Then, as in (3.3.10),

$$T_{N,f}(x,\bar{y})=\left(\frac{N}{2\pi}\right)e^{\frac{N}{2}(|x|^2+|y|^2)}\int_{\mathbb{R}^2}e^{N\Psi(p,t)}g(p,t)\,\mathrm{d}p,$$

with $x = (t_1 + it_2), y = (t_3 + it_4)$, $w = p_1 + ip_2$, and

$$\begin{aligned}\Psi(p,t) &:= x\bar{w} - |w|^2 + w\bar{y} - \frac{1}{2}(|x|^2 + |y|^2)\\ &= x\bar{w} - |w|^2 + w\bar{y} - \frac{1}{2}(|x|^2 + |y|^2)\\ &= (t_1 + it_2)(p_1 - ip_2) - p_1^2 - p_2^2 + (p_1 + ip_2)(t_3 - it_4)\\ &\qquad\qquad - \frac{1}{2}(t_1^2 + t_2^2 + t_3^2 + t_4^2),\\ g(p,t) &:= 2f(p_1 + ip_2).\end{aligned}$$

The critical point for this phase when it is holomorphically extended (note Ψ is real-analytic, so the extension is unique) is

$$\tilde{p}(t) = \left(\frac{1}{2}(t_1 + it_2 + t_3 - it_4), \frac{1}{2i}(t_1 + it_2 - t_3 + it_4)\right),$$

so that

$$\tilde{\Phi}_{pp}(\tilde{p}(t)) = -2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

In this case, we change variables, as in Lemma 3.3.2,

$$\begin{aligned}q(p,t) &= \sqrt{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}(p - \tilde{p}(t))\\ &= \sqrt{2}\begin{pmatrix}p_1 - \frac{1}{2}(t_1 + it_2 + t_3 - it_4)\\ p_2 - \frac{1}{2i}(t_1 + it_2 - t_3 + it_4)\end{pmatrix} : \mathbb{C}^2 \times \mathbb{R}^4 \to \mathbb{C}^2.\end{aligned}$$

Then the new contour is $\{p \in \mathbb{C}^2 : q(p,t) = w \in \mathbb{R}^2\}$. For each $w \in \mathbb{R}^2$, we see that

$$\begin{aligned}w_1 &= \sqrt{2}\mathrm{Re}\,(p_1) - \frac{1}{\sqrt{2}}(t_1 + t_3), & w_2 &= \sqrt{2}\mathrm{Re}\,(p_2) - \frac{1}{\sqrt{2}}(t_2 + t_4),\\ 0 &= \mathrm{Im}\,(p_1) + \frac{1}{\sqrt{2}}(-t_2 + t_4), & 0 &= \mathrm{Im}\,(p_2) + \frac{1}{\sqrt{2}}(t_1 - t_3).\end{aligned}$$

Therefore the new contour is

$$U_0 = \left\{ p(w) := \begin{pmatrix}\frac{1}{2}(t_1 + t_3) + \frac{w_1}{\sqrt{2}} - \frac{i}{2}(-t_2 + t_4)\\ \frac{1}{2}(t_2 + t_4) + \frac{w_2}{\sqrt{2}} - \frac{i}{2}(t_1 - t_3)\end{pmatrix} : (w_1, w_2) \in \mathbb{R}^2 \right\}.$$

The real stationary phase is applied to the amplitude $g(p(w))\det(\frac{\partial p}{\partial w})$, which is, after replacing t with its definition,

$$\frac{1}{2}\tilde{g}\left(\frac{1}{2}(x + \bar{y}) + \frac{w_1}{\sqrt{2}}, \frac{1}{2i}(x - \bar{y}) + \frac{w_2}{\sqrt{2}}\right).$$

So that, ignoring constants

$$f_j(x,y) = (\partial^2_{\mathrm{Re}(w_1)} + \partial^2_{\mathrm{Re}(w_2)})^j \tilde{g}(w_1,w_2)|_{\substack{w_1=\frac{1}{2}(x+\bar{y}) \\ w_2=\frac{1}{2}(x-\bar{y})}}.$$

Along the diagonal, this agrees with the computation in (A.2.4) (with constants which can be shown to be equal). This provides a (non-unique) asymptotic expansion of $T_{N,f}$ using complex stationary phase, and possibly sheds light on how this method works in general.

A.2.4 Computing the second term in the star product

In this section we let $f,g \in C_0^\infty(\mathbb{C};\mathbb{C})$ and determine $(f \star g)_1$. This is a simpler version of Appendix A, however in this case, the symplectic form is constant and in the Bergman kernel expansion, $b_j = 0$ for $j \geq 1$. First, if f is a symbol, then

$$T_{N,f}(x,\bar{y}) \sim \left(\frac{N}{2\pi}\right) e^{Nx\bar{y}} \sum_{j=0}^{\infty} N^{-j} C_j(f)(x,\bar{y})$$

with

$$C_j[f](x,\bar{y}) = \frac{1}{4^k k!}(\partial^2_{\mathrm{Re}(w_1)} + \partial^2_{\mathrm{Re}(w_2)})^j \tilde{f}(w_1,w_2)|_{(w_1,w_2)=\tau(x,\bar{y})}$$

where $\tau(x,\bar{y}) = 2^{-1}(x+\bar{y}, i^{-1}(x-\bar{y}))$. Importantly, when $y = x$ this becomes:

$$\begin{aligned} C_j[f(\cdot)](x,\bar{x}) &= \frac{1}{4^j j!}(\partial^2_{\mathrm{Re}(w_1)} + \partial^2_{\mathrm{Re}(w_2)})^j \tilde{f}(w_1,w_2)|_{(x,y)=\tau(x,\bar{x})} \\ &= \frac{1}{j!}(\partial\bar{\partial})^j f(x). \end{aligned}$$

Now we may write the first few terms of $(f \star g)$:

$$(f \star g)_0(x) = C_0[f_0(x,\cdot)g_0(\cdot,\bar{x})](x,\bar{x}) = f_0(x,\bar{x})g_0(x,\bar{x}) = f(x)g(x),$$

$$\begin{aligned} (f \star g)_1(x) &= C_0[f_1(x,\cdot)g_0(\cdot,\bar{x})](x,\bar{x}) + C_0[f_0(x,\cdot)g_1(\cdot,\bar{x})](x,\bar{x}) \\ &\qquad + C_1[f_0(x,\cdot)g_0(\cdot,\bar{x})](x,\bar{x}) - C_1[h_0(\cdot)](x,\bar{x}) \\ &= \partial\overline{\partial} f(x)g(x) + f(x)\partial\overline{\partial} g(x) + C_1[f_0(x,\cdot)g_0(\cdot,\bar{x})](x,\bar{x}) - \partial\overline{\partial}(f(x)g(x)) \\ &= C_1[f_0(x,\cdot)g_0(\cdot,\bar{x})](x,\bar{x}) - \partial f(x)\overline{\partial} g(x) - \overline{\partial} f(x)\partial g(x). \end{aligned}$$

Note:

$$\begin{aligned} C_1[f_0(x,\cdot)g_0(\cdot,\bar{x})](x,\bar{x}) &= \partial_w\overline{\partial}_w[\tilde{f}(\tau(x,\bar{w})\tilde{g}(w,\bar{x})]|_{w=x} \\ &= \left(\frac{1}{2}\partial_1\tilde{f} + \frac{i}{2}\partial_2\tilde{f}\right)\left(\frac{1}{2}\partial_1\tilde{g} - \frac{i}{2}\partial_2\tilde{g}\right) \\ &= \overline{\partial} f(x)\partial g(x) \end{aligned}$$

where $\partial_i \tilde{f}$ is the holomorphic derivative of $\tilde{f}$ with respect to its ith component. Here we use that f_0 is almost anti-holomorphic in the second argument and g_0 is almost holomorphic in the first argument. The error terms are absorbed in the $\mathscr{O}(N^{-2})$ error. We therefore get:

$$(f \star g)_1(x) = -\partial f(x) \overline{\partial} g(x).$$

Appendix B

Computation of Toeplitz operators on $\mathbb{CP}^1$

In this appendix, we explicitly compute the Toeplitz operators on $\mathbb{CP}^1$.

The Hilbert space we use is the holomorphic sections of tensor powers of the dual of the tautological line bundle. In local coordinates, the Kähler potential is $\varphi(z) \coloneqq \log(1+|z|^2)$, so that the symplectic form is $i\overline{\partial}\partial\varphi(z) = i(1+|z|^2)^{-2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$. Smooth sections of the Nth tensor power are identified with smooth functions on $\mathbb{C}$. The Hilbert space then has the inner-product given by:

$$\langle f, g\rangle_{L^2(M,L^N)} \coloneqq \int_{\mathbb{C}} \frac{f(z)\overline{g(z)}}{(1+|z|^2)^{N+2}} 2\,\mathrm{d}m(z)$$

for $f, g \in C^\infty(\mathbb{C})$. Here we use that $|\mathrm{d}z \wedge \mathrm{d}\bar{z}| = 2\,|\mathrm{dRe}\,(z) \wedge \mathrm{dIm}\,(z)| = 2\,\mathrm{d}m(z)$.

B.1 Computing Bergman kernel

The space of holomorphic sections is identified with polynomials of degree less than N.

An orthonormal basis is given by $\left\{c_k z^k : k = 0, 1, \ldots, N-1\right\}$, where $c_k \coloneqq \left\|z^k\right\|^{-2}_{L^2(M,L^N)}$. Explicitly,

$$\begin{aligned}
\left\|z^k\right\|^2_{L^2(M,L^N)} &= 2\int_{\mathbb{C}} \frac{|z|^{2k}}{(1+|z|^2)^{N+2}}\,\mathrm{d}m(z) \\
&= 4\pi \int_0^\infty \frac{r^{2k+1}}{(1+|r|^2)^{N+2}}\,\mathrm{d}r \\
&= \frac{2\pi k!(N-k)!}{(N+1)!} = \frac{2\pi}{(N+1)\binom{N}{k}}
\end{aligned}$$

so that:

$$c_k := \sqrt{\frac{N+1}{2\pi}\binom{N}{k}}.$$

Then the Bergman kernel is:

$$\begin{aligned}
\Pi_N(z,\bar{w}) &= \sum_{k=0}^{N} e_k(z)e_k(\bar{w}) \\
&= \sum_{k=0}^{N} c_k^2 z^k \bar{w}^k \\
&= \frac{N+1}{2\pi}\sum_{k=0}^{N}\binom{N}{k} z^k \bar{w}^k \\
&= \frac{N+1}{2\pi}(1+z\bar{w})^N \\
&= \frac{N+1}{2\pi}\exp\left(N\log(1+z\bar{w})\right).
\end{aligned}$$

Note that $\psi(z,w) := \log(1+zw)$ is the analytic extension of $\varphi(z) = \log(1+|z|^2)$.

B.2 Explicit computation of matrices

We now choose a chart for $\mathbb{CP}^1$ that contains all but one point of $\mathbb{CP}^1$. We then identify $\mathbb{C}$ with the two-sphere without the north pole via the assignment:

$$\mathbb{C} \ni z \mapsto \frac{1}{1+|z|^2}\left(2\mathrm{Re}\left(z\right), 2\mathrm{Im}\left(z\right), 1-|z|^2\right) := (x_1, x_2, x_3) \in \mathbb{S}^2 \subset \mathbb{R}^3.$$

Then for each $N \in \mathbb{N}$, we compute $T_N(x_i)$ for $i = 1, 2, 3$. For $0 \le \ell \le N$:

$$\begin{aligned}
(T_N(x_1))(z^\ell)(x) &= \frac{N+1}{2\pi} \int_{\mathbb{C}} x_1 w^\ell e^{N\psi(x,\bar{w})} e^{-N\varphi(w)} \mu_1(w) \\
&= \frac{N+1}{\pi} \int_{\mathbb{C}} \frac{w+\bar{w}}{1+|w|^2} w^\ell (1+x\bar{w})^N (1+|w|^2)^{-N} (1+|w|^2)^{-2}\, \mathrm{d}m(w) \\
&= \frac{N+1}{\pi} \int_{\mathbb{C}} \frac{w+\bar{w}}{(1+|w|^2)^{N+3}} w^\ell (1+x\bar{w})^N\, \mathrm{d}m(w) \\
&= \frac{N+1}{\pi} \sum_{j=0}^{N} \binom{N}{j} x^j \int_{\mathbb{C}} \frac{(w+\bar{w}) w^\ell \overline{w}^j}{(1+|w|^2)^{N+3}}\, \mathrm{d}m(w) \\
&= \frac{N+1}{\pi} \sum_{j=0}^{N} \binom{N}{j} x^j \int_0^{2\pi} \int_0^\infty \frac{r^{j+\ell+2}(e^{i\theta(\ell+1-j)} + e^{i\theta(\ell-1-j)})}{(1+r^2)^{N+3}}\, \mathrm{d}r\, \mathrm{d}\theta \\
&= 2(N+1) \left(1_{\ell<N} x^{\ell+1} \binom{N}{\ell+1} \int_0^\infty \frac{r^{2\ell+3}}{(1+r^2)^{N+3}}\, \mathrm{d}r \right. \\
&\qquad \left. + 1_{\ell \ge 0} x^{\ell-1} \binom{N}{\ell-1} \int_0^\infty \frac{r^{2\ell+1}}{(1+r^2)^{N+3}}\, \mathrm{d}r \right) \\
&= (N+1) \left(1_{\ell<N} x^{\ell+1} \binom{N}{\ell+1} \frac{(\ell+1)!(N-\ell)!}{(N+2)!} \right. \\
&\qquad \left. + 1_{\ell \ge 0} x^{\ell-1} \binom{N}{\ell-1} \frac{\ell!(N-\ell+1)!}{(N+2)!} \right) \\
&= 1_{\ell<N} \frac{x^{\ell+1}(N-\ell)}{N+2} + 1_{\ell \ge 0} \frac{x^{\ell-1}\ell}{N+2}.
\end{aligned}$$

Now we use the normalization constants to get the entries of the matrix representation of $T_N(x_1)$.

For each $0 < \ell \le N$, the $(\ell-1, \ell)$ entry of the matrix of $T_N(x_1)$ is:

$$\begin{aligned}
\frac{c_\ell}{c_{\ell-1}} \frac{\ell}{N+2} &= \frac{\ell}{N+2} \frac{\sqrt{\binom{N}{\ell}}}{\sqrt{\binom{N}{\ell-1}}} \\
&= \frac{\ell}{N+2} \frac{\sqrt{N-\ell+1}}{\sqrt{\ell}}.
\end{aligned}$$

Similarly, for each $0 \le \ell < N$, the $(\ell+1, \ell)$ entry of the matrix $T_N(x_1)$ is:

$$\frac{c_\ell}{c_{\ell+1}} \frac{N-\ell}{N+2} = \frac{N-\ell}{N+2} \frac{\sqrt{\ell+1}}{\sqrt{N-\ell}}.$$

All the other entries of the matrix are zero. We therefore get that:

$$T_N(x_1) = \frac{1}{N+2}\begin{pmatrix} 0 & \sqrt{N} & 0 & \cdots & & 0 \\ \sqrt{N} & 0 & \sqrt{2(N-1)} & & & 0 \\ 0 & \sqrt{2(N-1)} & 0 & \sqrt{3(N-2)} & & \vdots \\ \vdots & & \sqrt{3(N-2)} & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 0 & \sqrt{N} \\ 0 & 0 & \cdots & 0 & \sqrt{N} & 0 \end{pmatrix}$$

where the off-diagonals are $\sqrt{\ell(N-\ell+1)}$. $T_N(x_1)$ can be constructed in Matlab with the following code stored as the variable $X1$.

```
e = (N+2)^(-1)*sqrt((1:N).*(N:-1:1)); % off-diagonals
X1 = diag(e',1)+diag(e',-1);
```

We can similarly compute the matrix representation of $T_N(x_2)$ by replacing $w+\bar{w}$ in the integrand by $-i(w-\bar{w})$. We will then get that $T_N(x_2)$ is

$$\frac{1}{N+2}\begin{pmatrix} 0 & i\sqrt{N} & 0 & \cdots & & 0 \\ -i\sqrt{N} & 0 & i\sqrt{2(N-1)} & & & 0 \\ 0 & -i\sqrt{2(N-1)} & 0 & i\sqrt{3(N-2)} & & \vdots \\ \vdots & & -i\sqrt{3(N-2)} & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 0 & i\sqrt{N} \\ 0 & 0 & \cdots & 0 & -i\sqrt{N} & 0 \end{pmatrix}.$$

In Matlab, this can be coded and stored under the variable $X2$ in the following way.

```
e = (N+2)^(-1)*sqrt((1:N).*(N:-1:1)); % off-diagonals
X2 = diag(1i*e',1)+diag(-1i*e',-1);
```

Lastly, for $0 \leq \ell \leq N$:

$$\begin{aligned}
(T_N(x_3))(z^\ell)(x) &= \frac{N+1}{2\pi}\sum_{j=0}^{N}\binom{N}{j}x^j\int_{\mathbb{C}}\frac{(1-|w|^2)w^\ell\bar{w}^j}{(1+|w|^2)^{N+3}}\,\mathrm{d}m(w) \\
&= \frac{N+1}{\pi}\sum_{j=0}^{N}\binom{N}{j}x^j\int_0^{2\pi}\int_0^{\infty}\frac{r^{\ell+j+1}e^{i\theta(\ell-j)}-r^{3+\ell+j}e^{i\theta(\ell-j)}}{(1+r^2)^{N+3}}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 2(N+1)\binom{N}{\ell}x^\ell\left(\int_0^{\infty}\frac{r^{2\ell+1}-r^{2\ell+3}}{(1+r^2)^{N+3}}\,\mathrm{d}r\right) \\
&= 2(N+1)\binom{N}{\ell}x^\ell\frac{(N-2\ell)\ell!(N-\ell)!}{2(N+2)!} \\
&= \frac{N-2\ell}{N+2}.
\end{aligned}$$

Therefore the matrix representation of $T_N(x_3)$ is:

$$T_N(x_3) = \frac{1}{N+2}\begin{pmatrix} N & 0 & \cdots & & \cdots & 0 \\ 0 & N-2 & 0 & & & \vdots \\ \vdots & 0 & N-4 & \ddots & & \\ & & \ddots & & \ddots & \vdots \\ & & & \ddots & -N+2 & 0 \\ 0 & \cdots & & \cdots & 0 & -N \end{pmatrix}.$$

In Matlab, this can be coded and stored under the variable $X3$ in the following way.

```
X3 = diag((N+2)^(-1)*(N:-2:-N));
```

We can then check that:

$$T_N(x_1 + ix_2) = \frac{1}{N+2}\begin{pmatrix} 0 & 0 & 0 & \cdots & & 0 \\ \sqrt{N} & 0 & 0 & & & 0 \\ 0 & \sqrt{2(N-1)} & 0 & 0 & & \vdots \\ \vdots & & \sqrt{3(N-2)} & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \sqrt{N} & 0 \end{pmatrix}.$$

We can compute the Toeplitz quantizations of powers of these x_i's, but the computations become more tedious. Here we compute $T_N(x_1^2)$.

Define for $k, N \in \mathbb{Z}$, $0 \le k < 2 + N$:

$$B_{k,N} := \int_0^\infty \frac{x^{2k+1}}{(1+x^2)^{N+4}}\,\mathrm{d}x = \frac{k!(N-k+2)!}{2(N+3)!}.$$

For $0 \le \ell \le N$, we have:

$$\begin{aligned} T_N(x_1^2)(x) &= \frac{N+1}{\pi}\int_{\mathbb{C}} \frac{(w+\bar{w})^2}{(1+|w|^2)^{N+4}} w^\ell (1+x\bar{w})^N \,\mathrm{d}m(w) \\ &= \frac{N+1}{\pi}\sum_{j=0}^{\ell} x^j \binom{N}{j} \int_{\mathbb{C}} \frac{(w^2+2|w|^2+\bar{w}^2) w^\ell \bar{w}^j}{(1+|w|^2)^{N+4}}\,\mathrm{d}m(w) \\ &= 2(N+1)\Bigg(1_{\ell\le N-2} x^{2+\ell}\binom{N}{\ell+2} B_{\ell+2,N} + 2x^\ell \binom{N}{\ell} B_{\ell+1,N} \\ &\qquad\qquad + 1_{\ell \ge 2} x^{\ell-2}\binom{N}{\ell-2} B_{\ell,N}\Bigg). \end{aligned}$$

Therefore the matrix representation of $T_N(x_1^2)$ will have exactly three nonzero diagonals. For $\ell \geq 2$, the $(\ell-2,\ell)$ entry will be:

$$2(N+1)\binom{N}{\ell-2}B_{\ell,N}\frac{c_\ell}{c_{\ell-2}} = 2(N+1)\binom{N}{\ell-2}\frac{\ell!(N-\ell+2)!}{2(N+3)!}\sqrt{\frac{\binom{N}{\ell}}{\binom{N}{\ell-2}}}$$
$$= \frac{\sqrt{(\ell-1)\ell(N-\ell+1)(N-\ell+2)}}{(N+3)(N+2)}.$$

On the diagonal, the (ℓ,ℓ) entry is:

$$4(N+1)\binom{N}{\ell}B_{\ell+1,N} = 4(N+1)\binom{N}{\ell}\frac{(\ell+1)!(N-\ell+1)!}{2(N+3)!}$$
$$= \frac{2(\ell+1)(N-\ell+1)}{(N+3)(N+2)}.$$

The matrix should be self-adjoint (because $x_1^2 \in \mathbb{R}$), but as a partial verification of this computation, we compute the upper diagonal. For $\ell \leq N-2$, the $(\ell+2,\ell)$ entry of $T_N(x_1^2)$ is

$$2(N+1)\binom{N}{\ell+2}B_{\ell+2,N}\frac{c_\ell}{c_{\ell+2}} = 2(N+1)\binom{N}{\ell+2}\frac{(\ell+2)!(N-\ell)!}{2(N+3)!}\sqrt{\frac{\binom{N}{\ell}}{\binom{N}{\ell+2}}}$$
$$= \frac{\sqrt{(\ell+2)(\ell+1)(N-\ell)(N-\ell-1)}}{(N+3)(N+2)}$$

and so we see the nonzero off-diagonals are the same, verifying that the matrix is self-adjoint.

The matrix representation of $T_N(x_1^2)$ can be coded in Matlab in the following way and stored as the variable $Y1$.

```
l = (2:N);
e2 = sqrt((l-1).*l.*(N-l+1).*(N-l+2));
e2 = e2/((N+3)*(N+2)); % off-diagonal
l = (0:N);
e1 = (l+1).*(N-l+1);
e1 = 2*e1/((N+3)*(N+2)); % diagonal
Y1 = diag(e1)+diag(e2,-2)+diag(e2,2);
```

Because $T_N(x_1^2) = T_N(x_1)T_N(x_1) + \mathscr{O}(N^{-1})$, we can numerically verify the construction of $T_N(x_1^2)$ by computing $N\,\|T_N(x_1^2) - (T_N(x_1))^2\|$, this is done in Figure B.1.

B.3 Numerics

In this section, we compute the spectrum of Toeplitz operators on $\mathbb{CP}^1$ numerically with and without random perturbations.

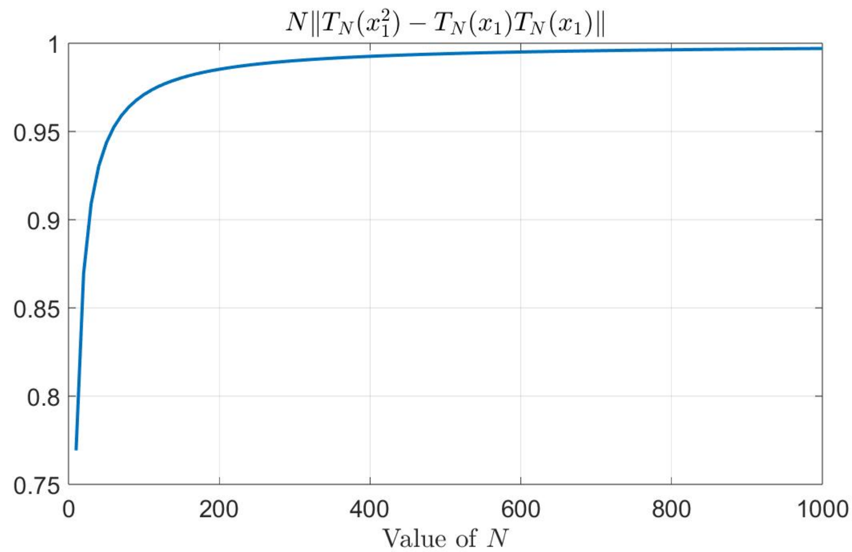

Figure B.1: Numerical computation of $N\,\|T_N(x_1^2) - (T_N(x_1))^2\|$.

As an example of how this is coded, here is code for computing the spectrum of $T_N(x_1 + ix_2)$ with small random perturbation with $N = 1000$.

```
N = 1000;
e = ((N+2))^(-1)*sqrt((1:N).*(N:-1:1));
A = diag(e',1)+diag(e',-1);
B = diag(1i*e',1)+diag(-1i*e',-1);
R = N^(-3)*(randn(N+1)+1i*randn(N+1));
ev = eig(A+1i*B+R);
plot(real(ev),imag(ev),'.');
```

The output is plotted in Figure B.2.

Interestingly, if the random perturbation is replaced by a matrix with i.i.d. uniform-$(0,1)$ random variables, there is an absence of eigenvalues near the real strip $[0,1]$. This is plotted in Figure B.3. This particular random perturbation is not included in the scope of this thesis. An interesting direction is to prove why this occurs.

For Figures B.4, B.5, and B.6, we plot the numerically computed spectrum of $T_N(x_1 + 2x_1^2 + ix_2)$ with and without random perturbation. In Figure B.4, no perturbation is added and the spectrum is computed for increasing values of N. As N increases, the spectrum spreads out with density matching the predicted probabilistic Weyl law. This matches the intuition that rounding errors in Matlab act like small random perturbations.

In Figure B.5, we plot the spectrum of the same matrix with a random perturbation ($N = 2,000$, $\delta = N^{-3}$). The density of random eigenvalues should converge to the push-forward of the surface measure on the sphere by the symbol of the operator. This density is approximated by sampling $100,000$ points uniformly on the sphere, and plotting their

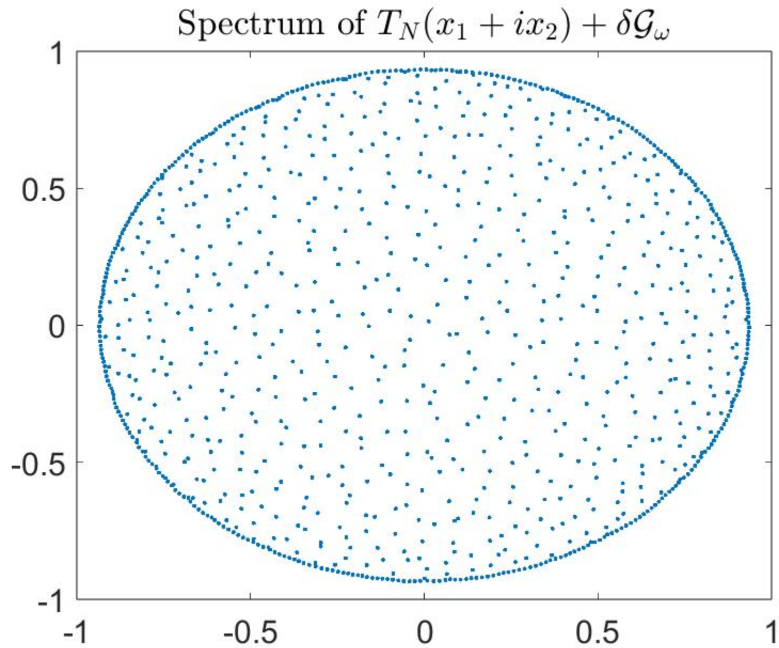

Figure B.2: Spectrum of $T_N(x_1 + ix_2) + \delta\mathscr{G}_\omega(N)$ with $N = 1,000$ and $\delta = N^{-3}$.

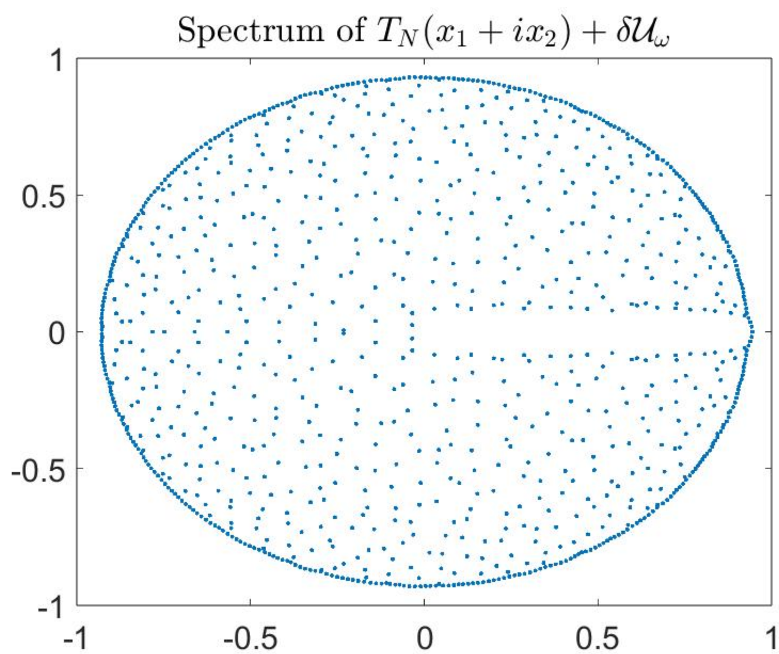

Figure B.3: Spectrum of $T_N(x_1 + ix_2) + \delta\mathscr{U}_\omega(N)$ with $N = 1,000$ where $\mathscr{U}_\omega(N)$ has entries given by i.i.d. uniform $(0, 1)$ random variables and $\delta = N^{-3}$.

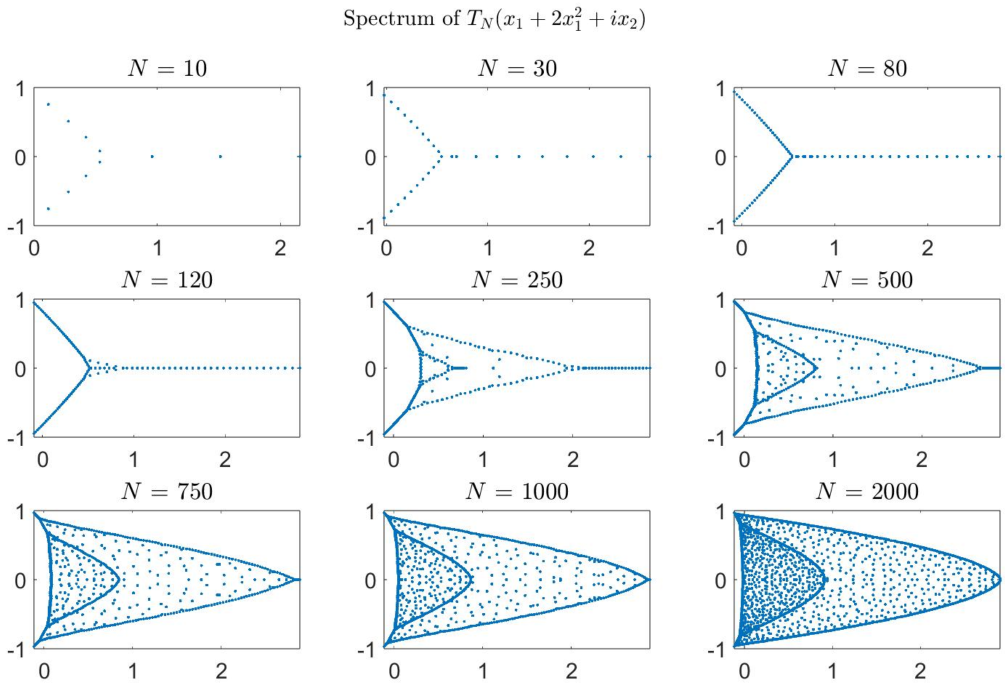

Figure B.4: Spectrum of $T_N(x_1+2x_1^2+ix_2)$ with **no** random perturbation at various values of N. As N increases, this matches the spectrum of this operator with a random perturbation, suggesting that small rounding errors in Matlab act like random perturbations. Conjecturally, the spectrum should lie on lines (as it does up to $N = 80$), and for larger N, this is **not** the actual spectrum.

image under the symbol. This stochastic approximation of the density is plotted on the left. Visually the two densities agree but are quantitatively compared in Figure B.6. In Figure B.6, we compare the densities within rectangles and plot the results.

In Figures B.7 and B.8 the operator $T_N(x_1 + ix_1^2)$ is computed with and without random perturbation for various values of N. In this case, the operator is normal and is therefore spectrally stable under random perturbations (Which is demonstrated in the figures). Furthermore, the Weyl law states that with random perturbation, the spectrum is contained only on the curve $\mathrm{Im}\,(z) = \mathrm{Re}\,(z)^2$ which is consistent.

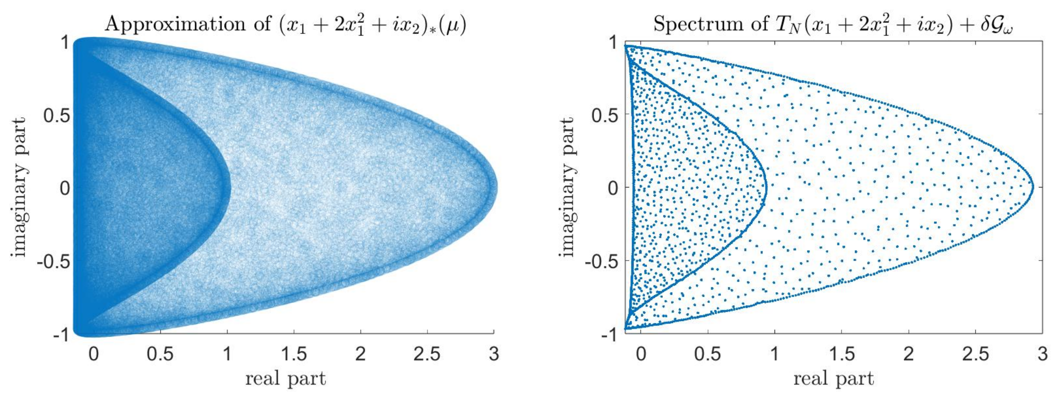

Figure B.5: On the left is a stochastic approximation of the push-forward of the surface measure on the sphere by the function $x_1 + 2x_1^2 + ix_2$ as a density on $\mathbb{C}$. This is computed by sampling 100,000 points uniformly randomly on the sphere and plotting the image under this function. On the right is the spectrum of $T_N(x_1 + 2x_1^2 + ix_2) + \delta\mathscr{G}_\omega(N)$ with $N = 2,000$ and $\delta = N^{-3}$. This thesis' result states that these two distributions should agree as $N \to \infty$.

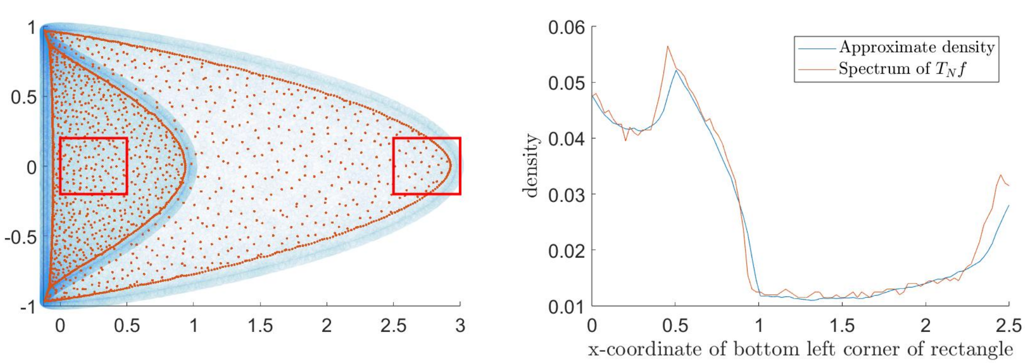

Figure B.6: Here we quantitatively compare the two densities on $\mathbb{C}$ as constructed in Figure B.5. This is done by comparing the density of points in a rectangle of width 0.5, height 0.4, and lower-left corner $(x, -0.2)$ for x ranging between 0 and 2.5. The first and last rectangle in the range is plotted on the left. On the right, these two densities are compared.

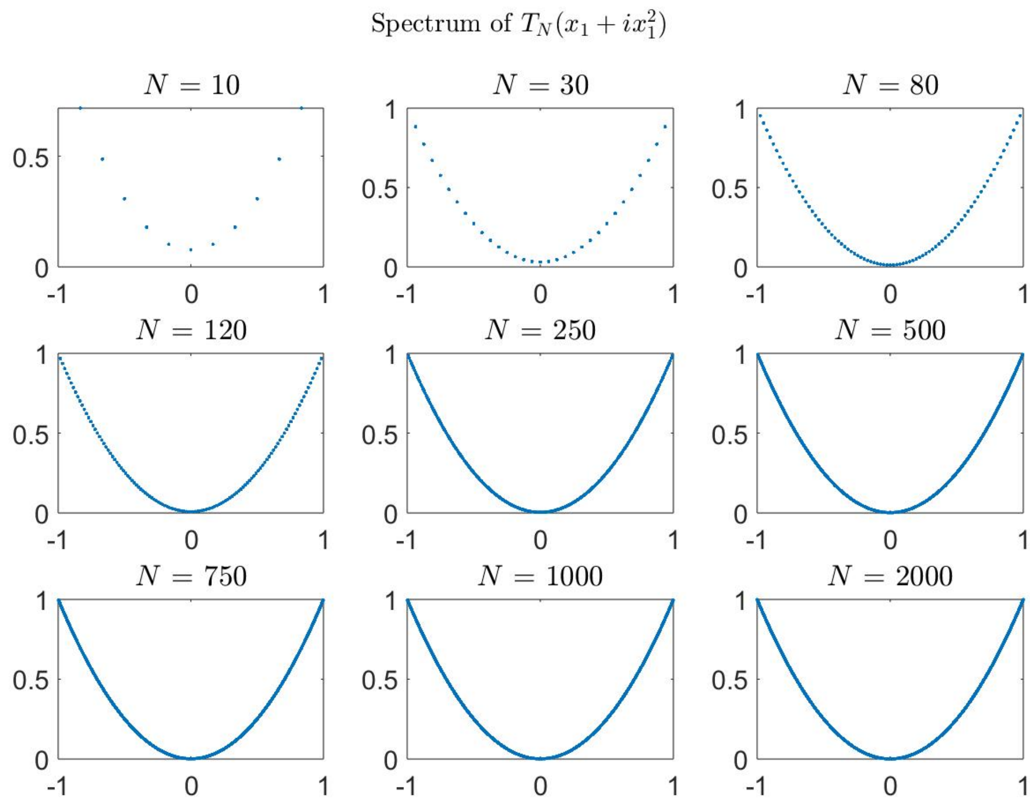

Figure B.7: Here we plot the spectrum of $T_N(x_1+ix_1^2)$ with **no** random perturbation. Unlike Figure B.4, the spectrum does not spread out as N increases. This should be expected. For one, these matrices turn out to be normal operators, and so are stable under small perturbation. For two, the Weyl law for this symbol states that under perturbations, the spectrum should lie on the curve $\text{Im}\,(z) = \text{Re}\,(z)^2$.

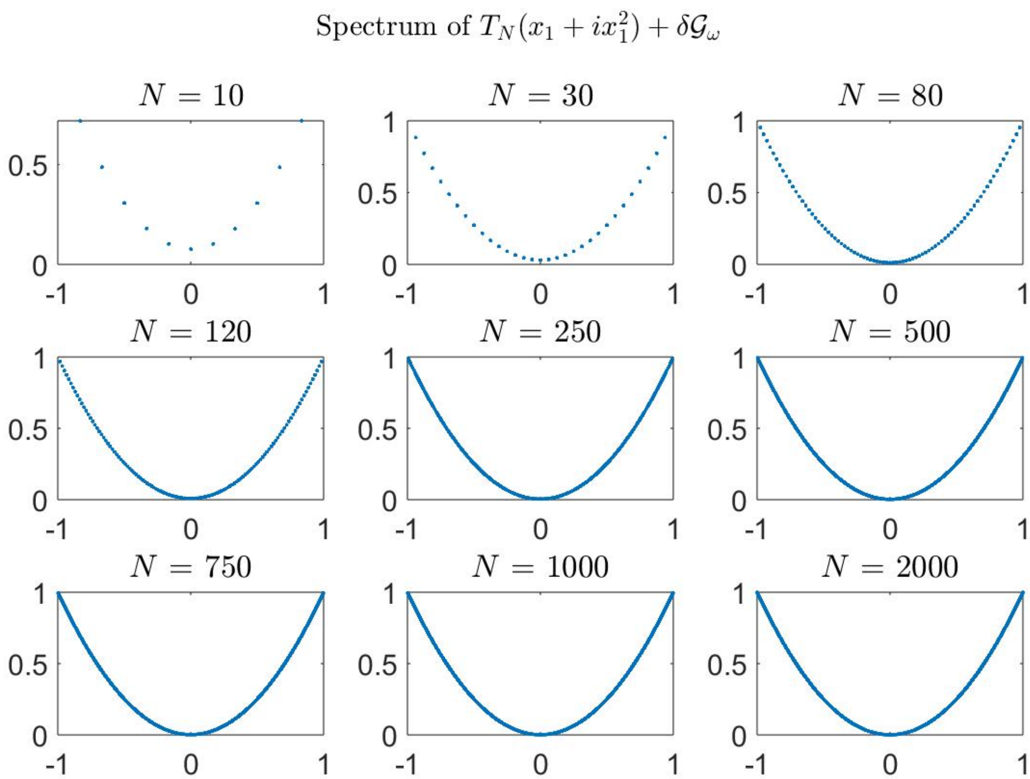

Figure B.8: Here we plot the spectrum of $T_N(x_1 + ix_1^2)$ **with** a random perturbation for $\delta = N^{-3}$. As is expected by the probabilistic Weyl law, there is no change to the spectrum.

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