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Authors
Ogievetsky, O.
Zumino, B.

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Reality in the Differential Calculus on q-euclidean Spaces

O. Ogievetsky[†][1], and B. Zumino[‡][2]

[†] Max-Planck-Institut für Physik und Astrophysik
- Werner-Heisenberg-Institut für Physik
P.O. Box 40 12 12 , D - 8000 Munich 40, Germany

[‡] Department of Physics
University of California, Berkeley
and
Theoretical Physics Group
Physics Division
Lawrence Berkeley Laboratory
1 Cyclotron Road
Berkeley, California 94720

Abstract

The nonlinear reality structure of the derivatives and the differentials for the euclidean q-spaces are found. A real Laplacian is constructed and reality properties of the exterior derivative are given.

[1]On leave of absence from P.N.Lebedev Physical Institute, Theoretical Department, 117924 Moscow, Leninsky prospect 53, CIS

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DISCLAIMER

1 Introduction

In this paper we discuss a new effect appearing in the differential calculus on euclidean q-spaces. Namely, although the conjugation rules for the coordinates look like those in the classical case, the conjugation of the derivatives and the differentials turns out to be non-linear. For the Minkowski q-space these conjugation rules were discussed in [1]. Here we generalize the results of [1] to the higher dimensional euclidean q-spaces. The nonlinearity in the relation between the derivatives and their conjugates turn out to be quite simple. Classically, the derivatives are proportional to the commutator of the Laplacian with the coordinates. On the quantum level these are however two different objects. Our main result is that the commutator of the Laplacian with the coordinates is now proportional to the conjugates of the derivatives. The coefficient of the proportionality is no longer a number. It is a scaling operator introduced in [2]. It q-commutes with all the coordinates and derivatives.

Our treatment relies on the papers [3], [4]. The prescription for the differential calculus on q-spaces, given in [4] works when the commutation relations for coordinates of a q-space are given by a single projector entering the $\hat{R}$-matrix. This is the case for q-orthogonal spaces. Although the $\hat{R}$-matrices have different structure for even and odd dimensional q-orthogonal spaces, the results have the same form in both cases and we treat them simultaneously.

The paper is organized as follows. Section 2 contains basic facts about the orthogonal q-spaces and differential calculus on them. In section 3 we give the reality structure for the derivatives and discuss reality properties of the Laplacian. Section 4 is devoted to the differentials and reality properties of the exterior derivative. Also, there we comment briefly on the relation between two versions of the differential calculus in the SL_q case. In Appendix A we collected relevant relations used in the text.

The q-metric g_{ij} is not symmetric and throughout the text we use the following rules of lowering and raising an index of any one-index quantity m_i:

$$m^i = g^{ij} m_j, \ m_i = g_{ij} m^j. \tag{1.1}$$

Conjugation reverses the order of factors. Finally, we use the notation $\lambda = q - q^{-1}$.

2 Preliminaries

Here we list necessary facts about the $\hat{R}$-matrix for the orthogonal q-group $SO_q(N)$, euclidean q-spaces and differential calculus on q-spaces. For motivations and details we refer to [3],[4]. See also [5] for the discussion of the differential calculus on orthogonal q-spaces.

1. The projector decomposition of the $\hat{R}$-matrix for the orthogonal q-group $SO_q(N)$ is:

$$\hat{R} = qP^+ - q^{-1}P^- + q^{1-N}P^o. \tag{2.1}$$

Here P^+ is the traceless part of the q-analogue of the symmetriser, P^- is the q-analogue of the antisymmetriser and P^o is the trace projector. The projector P^o is built out of the q-metric g_{ij},

$$P^{o\,ij}_{\ kl} = \nu g^{ij} g_{kl}\ , \qquad \nu = \frac{\lambda}{(q^N - 1)(q^{1-N} + q^{-1})}\ . \tag{2.2}$$

The $\hat{R}$-matrix has the following symmetry properties:

$$\hat{R}^{-1\ ij}_{\ \ \ kl} = g^{im}\hat{R}^{jn}_{mk}g_{nl} = g_{km}\hat{R}^{mi}_{ln}g^{nj} \tag{2.3}$$

and

$$\hat{R}^{ij}_{kl} = \hat{R}^{kl}_{ij}. \tag{2.4}$$

2. The orthogonal q-space is the algebra with generators x^i, $i = 1, ..., N$ satisfying quadratic relations

$$P^{-\ ij}_{\ \ kl}x^k x^l = 0 \tag{2.5}$$

or

$$\hat{R}^{ij}_{kl}x^k x^l = qx^i x^j - \frac{\lambda}{1 + q^{N-2}}g^{ij}g_{kl}x^k x^l\ . \tag{2.6}$$

The length

$$L = \frac{1}{1 + q^{N-2}}g_{ij}x^i x^j \tag{2.7}$$

is the central element in the algebra of the coordinates, $Lx^i = x^i L$.

The projectors P^+, P^o define the quadratic relations for the differentials ξ^i:

$$P^{+\ ij}_{\ \ kl}\xi^k\xi^l = 0, \tag{2.8}$$

$$P^{o\ ij}_{\ \ kl}\xi^k\xi^l = 0. \tag{2.9}$$

The derivatives ∂_i are defined by

$$\partial_i x^j = \delta^j_i + q\hat{R}^{jk}_{il}x^l\partial_k. \tag{2.10}$$

The commutation relations between x^i and ξ^j are

$$x^i\xi^j = q\hat{R}^{ij}_{kl}\xi^k x^l. \tag{2.11}$$

We need also the commutation relations between ∂_i and ξ^j:

$$\partial_i\xi^j = q^{-1}\hat{R}^{-1\ jk}_{\ \ \ il}\xi^l\partial_k. \tag{2.12}$$

The algebra of the derivatives is

$$P^{-}\,{}^{ij}_{kl}\partial_j\partial_i = 0\ . \tag{2.13}$$

The element

$$\Delta = \frac{1}{1+q^{N-2}} g^{ij}\partial_j\partial_i \tag{2.14}$$

is central in the algebra of the derivatives, $\Delta\partial_i = \partial_i\Delta$.

3. The compact form of $SO_q(N)$ is defined for a real q. In this case we have $\overline{\hat{R}} = \hat{R}$. The conjugation of the coordinates has a form

$$\overline{x^i} = g_{ji}x^j. \tag{2.15}$$

It defines the euclidean q-space. The length L is real under this conjugation, $\overline{L} = L$.

3 Conjugate Derivatives

In this section we find the action of the conjugate derivatives and express the conjugate derivatives in terms of the derivatives themselves. Also, we construct a real Laplacian.

According to [4] the covariant and consistent derivatives are defined by the expression (2.10) involving $\hat{R}$-matrix. One can define another set of consistent and covariant derivatives using $\hat{R}^{-1}$ instead. First of all we show that in the q-orthogonal case this gives the conjugate derivatives.

Lemma.

$$\hat{\partial}_k x^v = \delta^v_k + q^{-1}\hat{R}^{-1}\,{}^{vi}_{kj}x^j\hat{\partial}_i, \tag{3.1}$$

where

$$\hat{\partial}_i = -q^N g_{ik}g^{tk}\overline{\partial^t}. \tag{3.2}$$

Proof. To write relations conjugated to (2.10) in the form (3.1) one finds first a tensor Φ^{ks}_{nv}, inverse to $\hat{R}^{va}_{sb}$ in indices (v, s). Put

$$\Phi^{ks}_{nv} = g_{nl}\hat{R}^{kl}_{uv}g^{us}. \tag{3.3}$$

Using relations (2.3) one finds

$$\Phi^{ks}_{nv}\hat{R}^{va}_{sb} = \hat{R}^{ks}_{nv}\Phi^{va}_{sb} = \delta^k_b\delta^a_n. \tag{3.4}$$

Now one proves (3.1) by a straightforward calculation using (2.15) and (3.2).

Comparing (3.1) with (2.10) one sees that the derivatives ∂_i and $\hat{\partial}_i$ act in the same way on the linear functions of x^j but their actions on higher order polynomials do not coincide. Therefore the conjugate derivatives cannot be expressed linearly in terms of the derivatives themselves. It turns out that $\hat{\partial}_i$ can be expressed nonlinearly in ∂_j. To write this expression we need the scalar operators L, Δ and $E = x^i\partial_i$. The commutation relations of these operators with the coordinates and the derivatives are

$$\begin{aligned} Lx^k &= x^k L, \\ \partial_k L &= q^2 L\partial_k + q^{2-N} x_k, \\ \Delta x^k &= q^2 x^k \Delta + q^{2-N}\partial^k, \\ \partial_k \Delta &= \Delta\partial_k, \\ Ex^k &= q^2 x^k E + x^k - q\lambda L\partial^k, \\ \partial_k E &= q^2 E\partial_k + \partial_k - q\lambda x_k \Delta. \end{aligned} \tag{3.5}$$

Finally we will use the operator Λ, introduced in [2]:

$$\Lambda = 1 + q\lambda E + q^N \lambda^2 L\Delta \ . \tag{3.6}$$

It obeys homogeneous relations with both the coordinates and the derivatives,

$$\Lambda x^k = q^2 x^k \Lambda \ , \ \Lambda\partial_k = q^{-2}\partial_k \Lambda \ . \tag{3.7}$$

Now we are ready to formulate the main result of this section.

Theorem.

$$\hat{\partial}_k = q^{N-2}\Lambda^{-1}[\Delta, x_k]. \tag{3.8}$$

Proof. Denote $T_k = q^{N-2}[\Delta, x_k]$. Using (3.5) we can write

$$T_k = \partial_k + q^{N-1}\lambda x_k \Delta \ . \tag{3.9}$$

Compute

$$\begin{aligned} &T_i x^j - q\hat{R}^{-1}{}^{jk}_{il} x^l T_k = \\ &\delta^j_i + q\hat{R}^{jk}_{il} x^l \partial_k + q^{N+1}\lambda x_i(x^j\Delta + q^{-N}\partial^j) - q\hat{R}^{-1}{}^{jk}_{il} x^l\partial_k - q^N\lambda\hat{R}^{-1}{}^{jk}_{il} x^l x_k \Delta \ . \end{aligned} \tag{3.10}$$

For the terms with $xx\Delta$ we have

$$\begin{aligned} &q x_i x^j - \hat{R}^{-1}{}^{jk}_{il} x^l x_k \Delta = \\ &g_{ia}(q x^a x^j - g^{au}\hat{R}^{-1}{}^{jk}_{ul} g_{ks} x^l x^s) = \\ &g_{ia}(q\mathbb{1} - \hat{R})^{aj}_{ls} x^l x^s = \\ &g_{ia}(q - q^{1-N}) P^o{}^{aj}_{ls} x^l x^s = \delta^j_i \lambda L \ . \end{aligned} \tag{3.11}$$

In the second equality we used (2.3). In the third equality we used (2.1), (2.5), and the completness of the set of the projectors P^+, P^-, P^o,

$$\mathbb{1} = P^+ + P^- + P^o \ . \tag{3.12}$$

In the fourth - equations (2.2) and (2.7) were used. For the terms with $x\partial$ we have

$$(q\hat{R}^{jk}_{il} - q\hat{R}^{-1}{}^{jk}_{il} + q\lambda g_{il}g^{jk})x^l\partial_k =$$
$$((q^2-1)P^+ + (q^2-1)P^- + (q^{2-N} - q^N + q\lambda\nu^{-1})P^o)^{jk}_{il}x^l\partial_k = \qquad (3.13)$$
$$q\lambda\delta^j_i E \ ,$$

where ν is given by (2.2). Here in the first equality we used (2.1) and (2.2), and in the second - (3.12). Collecting all terms together we obtain

$$T_i x^j = \delta^j_i\Lambda + q\hat{R}^{-1}{}^{jk}_{il}x^l T_k \ . \qquad (3.14)$$

Now multiplying by Λ^{-1} from the left, using (3.7) and comparing with (3.1) we conclude that the lhs and rhs of (3.8) have the same commutation relations with x^i, which completes the proof.

We note that although the conjugation rule (3.8) is nonlinear, on conjugating twice all nonlinearities disappear and $\overline{\overline{\partial_i}} = \partial_i$. The map inverse to (3.8) is

$$\partial_k = (\overline{\Lambda})^{-1}[\overline{\Delta}, x_k] \equiv \Lambda(\hat{\partial}_k - q^{N-3}\lambda x_k\Lambda^{-1}\Delta). \qquad (3.15)$$

To complete the treatment we find the reality properties of the operators E, Δ, Λ. By a somewhat lengthy but straightforward calculation one finds:

$$\overline{\Delta} = q^{-N-2}\Lambda^{-1}\Delta \ , \qquad (3.16)$$

$$\overline{E} = -q^{-N}\Lambda^{-1}((q^N + q\lambda)E + \frac{(q^N-1)(q^{1-N}+q^{-1})}{\lambda} + q^{N+1}\lambda(1+q^{N-2})L\Delta) \ . \qquad (3.17)$$

Therefore, using (A.1), we obtain

$$\overline{\Lambda} = q^{-2N}\Lambda^{-1} \ . \qquad (3.18)$$

Equation (3.16) shows that the Laplace operator Δ built out of the derivatives ∂_i only is not real. However, for

$$\Delta_R = \Lambda^{-1/2}\Delta \qquad (3.19)$$

we have

$$\overline{\Delta_R} = \Delta_R \ . \qquad (3.20)$$

Therefore, Δ_R is a good candidate for a real Laplacian.

4 Conjugate Differentials

In this section we express the conjugate differentials in terms of the differentials themselves, and find the reality property of the exterior derivative.

Conjugating the relation (2.12) and defining $\hat{\xi}^i$ by

$$\overline{\xi^j} = g_{ij}\hat{\xi}^i \ , \tag{4.1}$$

we find using (2.1),

$$\hat{\xi}^i x^j = q\hat{R}^{ij}_{kl} x^k \hat{\xi}^l \ . \tag{4.2}$$

To find the relation between $\hat{\partial}_k$ and ∂_l we used scalar operators obtained by contraction of indices of x^i, ∂_j. Now we need two more scalar operators, the exterior derivative d and the operator

$$W = \xi^i x_i \ . \tag{4.3}$$

The commutation relations of L, Δ, E with ξ^i are simple:

$$\begin{aligned} L\xi^i &= q^2\xi^i L \ , \\ \Delta\xi^i &= q^{-2}\xi^i\Delta \ , \\ E\xi^i &= \xi^i E \ . \end{aligned} \tag{4.4}$$

The new operators d and W have the following commutation relations with x^i, ∂_i, and ξ^i:

$$\begin{aligned} dx^i &= \xi^i + x^i d \ , \\ d\partial_i &= q^2\partial_i d - q^{N-1}\lambda\xi_i\Delta \ , \\ d\xi^i &= -\xi^i d \ , \\ Wx^i &= x^i W - q^{N-1}\lambda\xi^i L \ , \\ \partial_j W &= W\partial_j + q^{N-2}\xi_j - q^{-1}\lambda x_j d + q^{N-1}\lambda\xi_j E \ , \\ W\xi^j &= -q^2\xi^j W \ . \end{aligned} \tag{4.5}$$

As for relations (3.5) we leave a check of these relations to the reader. This check can be reduced to manipulations with the symmetry properties and projector decomposition of the $\hat{R}$-matrix.

We introduce also a quantity

$$U = W + q^{N-3}\lambda L d \ , \tag{4.6}$$

which commutes with all coordinates,

$$Ux^i = x^i U \ , \tag{4.7}$$

while with the derivatives and differentials it obeys

$$\begin{aligned} \partial_j U &= U\partial_j + q^{N-2}\xi_j\Lambda \ , \\ \xi^j U &= -q^{-2}U\xi^j \ . \end{aligned} \tag{4.8}$$

The commutation relations of ξ^i with x^j or ∂_k are homogeneous. Hence rescaling ξ^i by a numerical factor does not change any of them. This shows that the commutation relations with x^j imply the expression of $\hat{\xi}^l$ in terms of ξ^i only up to a factor. Demanding the square of conjugation to be unity, one fixes the absolute value of this factor.

Theorem.

$$\hat{\xi}^i = \sigma q^N \Lambda(\xi^i + q^{-1}\lambda x^i d - q^{1-N}\lambda U\hat{\partial}^i) \ , \tag{4.9}$$

where σ is a pure phase, $\sigma = e^{i\phi}$.

Proof. Again, once the rhs of (4.9) is written, one can check (using the projector decomposition and the symmetries of the $\hat{R}$-matrix) that it has the same commutation relations with x^i as $\hat{\xi}^j$ do.

To prove that σ is a pure phase, we find the square of the conjugation. To this end we need the expressions for $\overline{d}$ and $\overline{W}$. A straightforward calculation shows that

$$\overline{d} = -\sigma q^N(\Lambda d - q\lambda U\Delta), \tag{4.10}$$

and

$$\overline{W} = \sigma q^N(q^{2-N}U + q^{N-5}\lambda\Lambda L d - q^N\lambda^2 U L\Delta) \ . \tag{4.11}$$

It then follows that

$$\overline{U} = \sigma U \ . \tag{4.12}$$

Conjugating (4.12) we find that σ is a pure phase as stated. One more check shows that the square of the conjugation is unity on ξ^i as well. This finishes the proof.

The mapping inverse to (4.9) is

$$\xi^i = \overline{\sigma} q^N \overline{\Lambda}(\hat{\xi}^i + \lambda q x^i \overline{d} + q^3\lambda\overline{U}\partial^i). \tag{4.13}$$

As in the discussion of the reality properties of the Laplacian, one may build a combination of operators which reduces to d in the classical limit and has a linear conjugation law. One choice is

$$d_0 = \Lambda^{1/2}d + (1-q)\Lambda^{-1/2}U\Delta \ . \tag{4.14}$$

Then

$$\overline{d_0} = -\sigma d_0 \ . \tag{4.15}$$

However this choice destroys the fundamental nilpotency property of the exterior derivative. Another possibility is simply to take

$$d_1 = \frac{1}{2}(d - \overline{\sigma}\overline{d}) = \frac{1}{2}((1 + q^N\Lambda)d - q\lambda U\Delta) \ . \tag{4.16}$$

Then

$$\overline{d_1} = -\sigma d_1 \ , \tag{4.17}$$

as in (4.15). Moreover, one finds

$$d\bar{d} + \bar{d}d = 0 \ . \tag{4.18}$$

Therefore

$$d_1^2 = 0 \ . \tag{4.19}$$

We note that rescaling ξ^i by a phase one can eliminate the factor σ in the above formulas.

To conclude, we stress once more that the mappings (3.8) and (4.9) are covariant under the quantum group $SO_q(N)$.

Remark. In the $\mathrm{SL}_q(n, \mathbb{R})$ case the action of the conjugate derivatives is given by the $\hat{R}$-matrix itself and therefore the conjugation rules for the derivatives are linear [4], [6]. However still there is another set of covariant and consistent derivatives defined with the help of $\hat{R}^{-1}$,

$$\partial'_i x^j = \delta_i^j + q^{-1}\hat{R}^{-1}\,{}^{jk}_{il} x^l \partial'_k \ , \tag{4.20}$$

and one may ask how they are related to the original ones. Using the relation

$$\hat{R} = \hat{R}^{-1} + \lambda \ , \tag{4.21}$$

valid in the SL case, we may rewrite the action of the original derivatives in the form

$$\partial_i x^j = \delta_i^j + q\hat{R}^{-1}\,{}^{jk}_{il} x^l \partial_k + q\lambda\delta_i^j x^l \partial_l = \mu_1 \delta_i^j + q\hat{R}^{-1}\,{}^{jk}_{il} x^l \partial_k \ , \tag{4.22}$$

where $\mu_1 = 1 + q\lambda E$. The operator μ_1 is multiplicative [2]:

$$\mu_1 x^i = q^2 x^i \mu_1 \ , \quad \mu_1 \partial_i = q^{-2}\partial_i \mu_1 \ . \tag{4.23}$$

Multiplying (4.22) by μ_1^{-1} from the left we see that the operators $\mu_1^{-1}\partial_i$ satisfy the same commutation relations with the coordinates as ∂'_i. Therefore we may set

$$\partial'_i = \mu_1^{-1}\partial_i \ . \tag{4.24}$$

This is the needed relation between the original and primed derivatives.

The primed differentials, defined by

$$x^i \xi'^j = q^{-1}\hat{R}^{-1}\,{}^{ij}_{kl} \xi'^k x^l \ , \tag{4.25}$$

can be expressed in terms of the original differentials as well. One checks that the quantities $\mu_1(\xi^i + q^{-1}\lambda x^i d)$ satisfy the same commutation relations with the coordinates as ξ'^i. As in the q-orthogonal case this gives the relation between the primed and original differentials up to an overall numerical factor c,

$$\xi'^i = c\mu_1(\xi^i + q^{-1}\lambda x^i d) \ . \tag{4.26}$$

One now checks that μ_1 commutes with ξ^i. Therefore we also find the relation

$$d' \equiv \xi'^i \partial'_i = c\mu_1(\xi^i + q^{-1}\lambda x^i d)\mu_1^{-1}\partial_i = c(d + q\lambda E d) = c\mu_1 d \tag{4.27}$$

between the primed and original exterior derivatives.

A Useful Formulas

Here we collect various identities and commutation relations needed for checks and proofs of the statements in Sections 2 and 3.

The action of the scalar operators on the coordinates, derivatives and differentials was given in the text. Here are some useful commutators between the scalar operators themselves.

$$\begin{aligned} EL &= q^2 LE + (q^{2-N}+1)L \,, \\ \Delta E &= q^2 E\Delta + (q^{2-N}+1)\Delta \,, \\ \Delta L &= q^4 L\Delta + q^{4-N}E + \frac{q^{2-N}-q^{2-2N}}{q^{-1}\lambda} \,, \\ dL &= q^{-N+2}W + Ld \,, \\ dW &= -Wd \,, \\ dE &= q^2 Ed + d - q\lambda W\Delta \,, \\ \Delta W &= W\Delta + q^{-N}d \,, \\ \Delta U &= U\Delta + \Lambda d \,, \\ \Delta d &= q^{-2}d\Delta \,, \\ dU &= -q^{-2}Ud \,. \end{aligned} \tag{A.1}$$

Also, the following summation rules

$$\begin{aligned} \partial^i x_i &= \frac{(q^N-1)(q^{1-N}+q^{-1})}{\lambda} + q^N E \,, \\ \partial^i \xi_i &= q^{-N}d \,, \\ x^i \xi_i &= q^{2-N}W \,, \\ \hat{\xi}^i \hat{\partial}_i &= -\bar{d} \,, \\ \hat{\partial}^i \hat{\partial}_i &= (1+q^{N-2})q^{N-2}\Lambda^{-1}\Delta \end{aligned} \tag{A.2}$$

were used.

Remark. The operators

$$e = q^{N-1}L \,, \quad h = q^N(E + \frac{1-q^{-N}}{q\lambda}) \text{ and } f = q^{N-1}\Delta \tag{A.3}$$

satisfy the relations of the q-deformed $sl(2)$-algebra,

$$\begin{aligned} q^{-1}he - qeh &= (q+q^{-1})e \ , \\ q^{-1}fh - qhf &= (q+q^{-1})f \ , \\ q^{-2}fe - q^2ef &= h \ . \end{aligned} \tag{A.4}$$

Note that the operator entering Λ,

$$q^N\Lambda = 1 + \lambda\tilde{h} \quad , \quad \tilde{h} = qh + q^2\lambda ef \tag{A.5}$$

has an algebraic meaning as well. We have

$$\begin{aligned} q^{-2}\tilde{h}e - q^2e\tilde{h} &= (q+q^{-1})e \ , \\ q^{-2}f\tilde{h} - q^2\tilde{h}f &= (q+q^{-1})f \ , \\ q^{-1}fe - qef &= \tilde{h} \ , \end{aligned} \tag{A.6}$$

which is another form of the $sl_q(2)$-algebra.

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