Title
The Unit Group of the Burnside Ring as a Biset Functor for Some Solvable Groups

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THE UNIT GROUP OF THE BURNSIDE RING AS A BISET FUNCTOR FOR SOME SOLVABLE GROUPS

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Abstract

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by

Jamison Blair Barsotti

The theory of bisets has been very useful in progress towards settling the longstanding question of determining units for the Burnside ring. In 2006 Bouc used bisets to settle the question for p-groups. In this paper, we provide a standard basis for the unit group of the Burnside ring for groups that contain an abelian subgroups of index two. We then extend this result to groups G, where G has a normal subgroup, N, of odd index, such that N contains an abelian subgroups of index 2. Next, we study the structure of the unit group of the Burnside ring as a biset functor, $B^{\times}$, on this class of groups and determine its lattice of subfunctors. We then use this to determine the composition factors of $B^{\times}$ over this class of groups. Additionally, we give a sufficient condition for when the functor $B^{\times}$, defined on a class of groups closed under subquotients, has uncountably many subfunctors.

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Chapter 1

Introduction

The principles for the Burnside ring are over 100 years old and can be traced back to William Burnside. However, its ring structure was first considered by Louis Solomon in 1967 ([15]). It was then thoroughly discussed by tom Dieck ([6]) and Dress ([7]) in the following years. The question of describing the unit group of the Burnside ring of a finite group, in terms of data about that finite group can be traced to this time.

Since then, the unit group of the Burnside ring has been studied extensively by Yoshida ([19]), Matsuda ([9], [10]), Yalçın ([17]), and Bouc ([3]), although its rank for a general finite group is still unknown. In [3] Bouc succeeded in computing $B^\times(G)$ for the case where G is a p-group using the theory of bisets. This is obtained through realizing the unit group of the Burnside ring as a biset functor, $B^\times$, on the biset category for finite groups $\mathcal{C}$.

Bouc's result on the units of the Burnside ring for p-groups made use of a deep connection between these units and rational representations of the group at hand, through certain subgroups he referred to as *genetic*. This connection hinged on the Ritter-Segal Theorem ([12] and [13]), which states that the linearization map from the Burnside ring to the ring of rational representations is surjective. Unfortunately, for general finite groups, the Ritter-Segal Theorem is not true (e.g. $C_3 \times Q_8$, see Exercise 4 in 13.1 of [14]). However, another aspect of Bouc's result was to characterize which p-groups were necessary to "generate" the units of the Burnside ring of a p-group, using the perspective that $B^\times$ is a biset functor. In this manuscript, we call such groups *residual with respect to* $B^\times$. The upshot of this approach is that, if there is a valid extension of Bouc's idea of genetic subgroups for general finite groups that is useful to determine the unit group of the Burnside ring, then it will have to involve groups that are residual with respect to $B^\times$.

Chapters $2-4$ of this thesis introduce the necessary background to read the main results. Chapter 2 offers a review of Grothendieck groups, Burnside rings, and biset functors. Chapter 3 addresses an important decomposition that will be used for the main result of Chapter 5. Chapter 4 gives a computational definition of the biset functor $B^\times$ and we state Bouc's Theorem computing $B^\times(P)$, when P is a p-group.

The main contributions of this thesis can be found in Chapters $5-9$. Chapter 5 contains **Theorem 5.1.10**, which is the heart of this thesis. It produces a method for computing $B^\times(G)$ in the case where G has an abelian subgroup of index at most 2. We then extend this result, in **Corollary 5.2.2**, to the class of groups that contain such a G as a normal subgroup of odd index. Throughout the rest of the text, we refer to this class as $\mathcal{C}'$.

Chapter 6 introduces the definition of a *residual* group with respect to a given biset functor and we prove **Theorem 6.2.7**, which determines which groups in $\mathcal{C}'$ are residual with respect to $B^\times$. In Chapter 7, we use these groups to determine the lattice of subfunctors of $B^\times$ over the class $\mathcal{C}'$ (**Theorem 7.1.2**). We end the chapter with a pair of results: **Proposition 7.1.8** which generalizes Corollary 5.2.2 to any subfunctor of $B^\times$ over $\mathcal{C}'$, and **Theorem 7.2.1** which determines the composition factors of $B^\times$ over $\mathcal{C}'$.

Chapters 8 and 9 are concerned with applications and further questions. In particular, one of the main open questions in the theory of bisets is, given a simple biset functor S and a finite group X, what is the evaluation $S(X)$? For a certain class of simple biset functors, **Theorem 8.1.2** determines a method for these computations over $X \in \mathcal{C}'$, in terms of data from X. We also discuss the often studied *exponential map* ε, which is a morphism of the biset functors $B \to B^\times$, where B is the Burnside ring biset functor. Theorem 8.1.2 determines for which $X \in \mathcal{C}'$, the map ε_X is surjective. A particularly nice case of this Theorem 8.1.2 is **Corollary 8.2.4**, which determines the surjectivity of the exponential map for the dihedral group D_{2n}, completely in terms of n.

We note that the material that appears in the article [1] is based on the research done for Chapters $5-8$.

Chapter 2

Background

We assume that the reader has familiarity with the basics of group theory and with the theory of sets that have a group action defined on them. In this document, we are primarily concerned with invariants that can be associated to finite groups. So, whenever we are considering an arbitrary group, it comes with the assumption that it is finite. In this section, we discuss the basics of Grothendieck groups, Burnside rings, and biset functors. We end the section with a brief overview of simple biset functors and composition factors of biset functors.

2.1 Grothendieck Groups

Recall that a *monoid* is a set M, together with an associative binary operation "$*$" and an element $e \in M$, such that $e * m = m * e$ for all $m \in M$. Let $F(M)$ denote the free abelian group with basis M. The *Grothendieck group* of M, with respect to the operation $*$, is the group $\hat{M} := F(M)/H$ where H is the subgroup of $F(M)$ generated by elements of the form

$m * n - n - m$, for $m, n \in M$. There is an additive map τ that takes an element $m \in M$ to its image in $\hat{M}$. Together, $\hat{M}$ and τ have the following universal property: if M' is an abelian group and $f : M \to M'$ is an additive map, then there is a unique group homomorphism $\rho : \hat{M} \to M'$ such that $f = \rho \circ \tau$. If one applies this process to the natural numbers $\mathbb{N}_0$, then $\hat{\mathbb{N}}_0 \cong \mathbb{Z}$. However, since the integers are used in the constructing the free abelian group of a set, we cannot take this as a definition of $\mathbb{Z}$.

2.2 Burnside Rings

Let G be a group. Further, suppose X and Y are *G-sets*, i.e. sets that have a group action defined on them. We can then construct two other G-sets: the disjoint union $X \sqcup Y$ with the obvious action, and the cartesian product $X \times Y$, with the diagonal action $g(x,y) = (gx, gy)$, for any $g \in G$, $x \in X$, and $y \in Y$. Recall that two G-sets X and Y are said to be isomorphic if there is a bijection $f : X \to Y$ that is *G-equivariant*. That is $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. If we denote ${}_G\mathrm{set}$ to be the class of finite G-sets, and let $\overline{{}_G\mathrm{set}}$ denote the set of isomorphism classes of ${}_G\mathrm{set}$, then taking disjoint unions defines a monoid structure on $\overline{{}_G\mathrm{set}}$ where the identity is the class consisting of the empty set.

Definition 2.2.1. Let G be a finite group. The *Burnside group* of G, which we denote by $B(G)$, is defined as the Grothendieck group of $\overline{{}_G\mathrm{set}}$ with respect to taking disjoint unions.

If X is a finite G-set, then we denote the image of X in $B(G)$ by $[X]$.

Recall that every finite G-set X can be written as the disjoint union of transitive G-sets, and each transitive G-set is isomorphic to a coset space G/H, for some subgroup H of

G. Additionally, we also know that if H and K are subgroups of G, $G/H \cong G/K$ as G-sets if and only if there is a $g \in G$ such that $gHg^{-1} = {}^gH = K$. It follows that, if we let $\mathcal{S}_G$ denote a conjugacy class transversal of subgroups of G, the set $\{[G/H]\}_{H \in \mathcal{S}_G}$ will generate the Burnside group. It turns out that this set will be a basis. To see this we have to introduce the *mark homomorphisms*.

Let G be a finite group and X a finite G-set. If H is a subgroup of G, then we denote by $|X^H|$ the number of points in X that are fixed by H. It is clear that $|(X \sqcup Y)^H| = |X^H| + |Y^H|$ for any finite G-sets X and Y. Thus by the universal property of Grothendieck groups, we get a group homomorphism $\phi_H : B(G) \to \mathbb{Z}$, such that $\phi_H([X]) = |X^H|$ for any finite G-set X. For $a \in B(G)$, we will frequently use the abusive notation $|a^H|$ to denote the number $\phi_H(a)$.

Lemma 2.2.2. *Let G be a finite group. Suppose H and K are subgroups of G. Suppose X and Y are finite G-sets. The following hold:*

1. *If $X \cong Y$, then $|X^H| = |Y^H|$.*
2. *If ${}^gK = H$ for some $g \in G$, then $|X^H| = |X^K|$.*
3. *If $|(G/K)^H| \neq 0$, then $H \leqslant {}^gK$ for some $g \in G$.*
4. *$|(G/H)^H| = [N(H) : H]$.*

Proof. (1) is clear. For (2), note that K fixes $x \in X$ if and only if H fixes gx. For (3), suppose that H fixes gK for some $g \in G$. Then for each $h \in H$, we have $hgK = gK \implies h^g \in K$. For (4), notice that H fixes xH if and only if $x \in N(H)$. The result follows. □

Definition 2.2.3. Let G be a finite group. Let $\mathcal{S}_G$ denote a set of representatives of the conjugacy classes of subgroups of G. Then there is a homomorphism

$$\phi : B(G) \to \mathbb{Z}^{|\mathcal{S}_G|}$$

induced by the map that takes a finite G-set X to the column vector $(|X^H|)_{H \in \mathcal{S}_G}$. We call ϕ the *ghost map* of $B(G)$.

The following is a classic result that can be attributed to Burnside.

Theorem 2.2.4 ([4], Theorem 2.4.5)**.** *Let G be a finite group and let X and Y be finite G-sets. The following are equivalent:*

1. *$X \cong Y$ as G-sets.*

2. *$\phi(X) = \phi(Y)$.*

If we again denote $\mathcal{S}_G$ to be a set of representatives of the conjugacy classes of subgroups of G and order $\mathcal{S}_G$ by cardinality. Let M_G be the integer matrix with columns equal to $\phi([G/H])$ for $H \in \mathcal{S}_G$. It is an easy consequence of Lemma 2.2.2 that M_G is nonsingular, which implies that $\{[G/H]\}_{H \in \mathcal{S}_G} \subset B(G)$ is linearly independent. Hence it is a basis. Moreover, ϕ is injective and one can identify $B(G)$ with the integral span of M_G via ϕ. The matrix M_G is called the *table of marks*.

Example 2.2.5. Let $G = S_3 = \langle y, x | y^3 = x^2 = 1, xyx = y^{-1} \rangle$. Then, up to conjugacy, G has 4 subgroups, $\{1\}$, $\langle x \rangle, \langle y \rangle$, and G. The table of marks for G is

$$\begin{pmatrix} 6 & 3 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

More generally, if p is an odd prime and $G = D_{2p}$ is the dihedral group with $2p$ elements, then one can verify that the table of marks for G is

$$\begin{pmatrix} 2p & p & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

There is a natural ring structure on $B(G)$. This is simply the operation induced by the cartesian product of G-sets, where $[X][Y] = [X \times Y]$ for any finite G-sets X and Y. With this multiplication in mind, we call $B(G)$ the *Burnside ring of* G. Since $X \times Y \cong Y \times X$, it follows that $B(G)$ is a commutative ring with identity element equal to the image of the one point set G/G in $B(G)$.

Easy verification shows that for any two G-sets X and Y and any subgroup $H \leqslant G$, we have $|(X \times Y)^H| = |X^H||Y^H|$. It follows that the ghost map, ϕ, is an injective morphism of rings. The following proposition is a consequence of this fact.

Proposition 2.2.6. *Let G be a finite group. Then $B(G)^\times$ is an elementary abelian 2-group.*

Equivalently, we can view $B(G)^\times$ as an $\mathbb{F}_2$-space, with dimension bounded by $|\mathcal{S}_G|$. The ultimate goal would be to construct a basis for $B(G)^\times$ in terms of purely group theoretical data of G. However, even knowing the dimension of $B(G)^\times$ has proved to be a difficult question. The following theorem, first observed by tom Dieck in 1978, illustrates this.

Theorem 2.2.7 ([6]). *Feit-Thompson's Odd Order Theorem is equivalent to the statement that if G has odd order, then $B(G)^\times = \{\pm 1\}$.*

2.3 Bisets

If G and H are finite groups, we consider sets X that are equipped with a left G-action and a right H-action, such that for any $g \in G$, $h \in H$, and $x \in X$ $(gx)h = g(xh)$. Then X is called a (G,H)-*biset*. Equivalently, we may as well have thought of X as a $(G \times H)$-set defined with the action $(g,h) \cdot x = gxh^{-1}$. Conversely, a $(G \times H)$ action on a set X defines a unique (G,H)-biset structure on X. So there is a one-to-one correspondence between (left) $(G \times H)$-sets and (G,H)-bisets. With this identification in mind, we define $B(G,H) := B(G \times H)$.

Suppose G, H, and K are finite groups. If X is a (G,H)-biset and Y is an (H,K)-biset then we construct a (G,K)-biset in the following way: Let $X \times_H Y$ denote the H-orbits of the $(G \times H \times K)$-set $X \times Y$, with action

$$(g,h,k) \cdot (x,y) = (gxh^{-1}, hyk^{-1})$$

for all $g \in G, h \in H, k \in K, x \in X$, and $y \in Y$. Then $X \times_H Y$ has a natural action from $G \times K$.

Definition 2.3.1. If G, H, and K are finite groups, X is a (G,H)-biset and Y is an (H,K)-biset. The (G,K) biset $X \times_H Y$ is called the *tensor product of the bisets X and Y*.

For finite groups G, H, and K, the tensor product of bisets induces a bilinear form

$$- \times_H - : B(G,H) \times B(H,K) \to B(G,H).$$

We use this bilinear form to define the following category.

Definition 2.3.2 ([4], 3.1.1)**.** The *biset category $\mathcal{C}$* of finite groups is the category defined as follows:

- The objects of $\mathcal{C}$ are finite groups.

- If G and H are finite groups, then $\mathrm{Hom}_{\mathcal{C}}(G,H) = B(H,G)$.

- If G, H, and K are finite groups, then the composition $v \circ u$ of morphisms $u \in \mathrm{Hom}_{\mathcal{C}}(G,H)$ and $v \in \mathrm{Hom}_{\mathcal{C}}(H,K)$ is equal to $v \times_H u$.

- For any finite group G, the identity morphism of G in $\mathcal{C}$ is equal to $[G]$, where G is the (G,G)-biset equipped with left and right action induced by the usual multiplication of G.

It is clear that $\mathcal{C}$ is a preadditive category.

2.4 Elementary Bisets

Remark 2.4.1 ([4], 2.3.9)**.** Let G be a finite group. The following bisets are called *elementary bisets*. We will make use of them frequently.

- If H is a subgroup of G, then the set G is an (H,G)-biset given by the usual left and right multiplication in G and is denoted by Res^G_H. We call this *restriction* from G to H.

- If H is a subgroup of G, then the set G is a (G,H)-biset given by the usual left and right multiplication in G. It is denoted by Ind^G_H. We call this *induction* from H to G.

- If N is a normal subgroup of G, then the set G/N is a $(G,G/N)$-biset given by the usual multiplication on the right and projection to G/N then multiplication on the left. It is denoted $\mathrm{Inf}^G_{G/N}$. We call this *inflation* from G/N to G.

- If N is a normal subgroup of G, then the set G/N is a $(G/N,G)$-biset given by the usual multiplication on the left and projection to G/N then multiplication on the right. It is denoted $\mathrm{Def}^G_{G/N}$. We call this *deflation* from G to G/N.

- If $f : G \to H$ is a group isomorphism, then the set H is an (H,G)-biset, where the left action is usual multiplication, and the right action is given by multiplication of the image through f, denoted by $\mathrm{Iso}(f)$ or Iso^G_H if the isomorphism is clear from the context.

We abusively use the same notation for the images of elementary bisets in their respective Burnside groups.

The elementary bisets satisfy various important relations. Notably, they come encoded with the well-known Mackey formula.

Remark 2.4.2. ([4], 1.1.3)

- *Transitivity:*

  1. If H and K are subgroups of G with $K \leqslant H \leqslant G$, then

  $$\mathrm{Res}^H_K \circ \mathrm{Res}^G_H = \mathrm{Res}^G_K, \qquad \mathrm{Ind}^G_H \circ \mathrm{Ind}^H_K = \mathrm{Ind}^G_K.$$

  2. If $\varphi : G \to H$ and $\psi : H \to K$ are group isomorphism, then

  $$\mathrm{Iso}(\psi) \circ \mathrm{Iso}(\varphi) = \mathrm{Iso}(\psi\varphi).$$

  3. If N and M are normal subgroups of G with $N \leqslant M$, then

  $$\mathrm{Inf}^G_{G/N} \circ \mathrm{Inf}^{G/N}_{G/M} = \mathrm{Inf}^G_{G/M}, \qquad \mathrm{Def}^{G/N}_{G/M} \circ \mathrm{Def}^G_{G/N} = \mathrm{Def}^G_{G/M}.$$

Here, $\mathrm{Inf}_{G/M}^{G/N} := \mathrm{Inf}_{(G/N)/(M/N)}^{G/N} \circ \mathrm{Iso}(\alpha^{-1})$ and $\mathrm{Def}_{G/M}^{G/N} := \mathrm{Iso}(\alpha) \circ \mathrm{Def}_{(G/N)/(M/N)}^{G/N}$ where $\alpha : (G/N)/(M/N) \to G/M$ is the canonical isomorphism.

- *Commutation:*

1. If $\varphi : G \to H$ is a group isomorphism, and K is a subgroup of G, then

$$\mathrm{Iso}(\varphi') \circ \mathrm{Res}_K^G = \mathrm{Res}_{\varphi(K)}^H \circ \mathrm{Iso}(\varphi)$$

$$\mathrm{Iso}(\varphi) \circ \mathrm{Ind}_K^G = \mathrm{Ind}_{\varphi(K)}^H \circ \mathrm{Iso}(\varphi'),$$

where $\varphi' : K \to \varphi(K)$ is the restriction of φ.

2. If $\varphi : G \to H$ is a group isomorphism, and N is a normal subgroup of G, then

$$\mathrm{Iso}(\varphi') \circ \mathrm{Def}_{G/N}^G = \mathrm{Def}_{H/\varphi(N)}^H \circ \mathrm{Iso}(\varphi)$$

$$\mathrm{Iso}(\varphi) \circ \mathrm{Inf}_{G/N}^G = \mathrm{Inf}_{H/\varphi(N)}^H \circ \mathrm{Iso}(\varphi'),$$

where $\varphi' : G/N \to H/\varphi(N)$ is the group isomorphism induced by φ.

3. *(Mackey formula)* If H and K are subgroups of G, then

$$\mathrm{Res}_H^G \circ \mathrm{Ind}_K^G = \sum_{x \in [H \backslash G/K]} \mathrm{Ind}_{H \cap {}^xK}^H \circ \mathrm{Iso}(\gamma_x) \circ \mathrm{Res}_{H^x \cap K}^K,$$

where $[H\backslash G/K]$ is a set of representatives of (H,K)-double cosets in G, and $\gamma_x : H^x \cap K \to H \cap {}^xK$ is the group isomorphism induced by conjugation by x.

4. If N and M are normal subgroups of G, then

$$\mathrm{Def}_{G/N}^G \circ \mathrm{Inf}_{G/M}^G = \mathrm{Inf}_{G/NM}^{G/N} \circ \mathrm{Def}_{G/NM}^{G/M}.$$

Here, $\mathrm{Inf}_{G/NM}^{G/N} := \mathrm{Inf}_{(G/N)/(NM/N)}^{G/N} \circ \mathrm{Iso}(\alpha^{-1})$ and $\mathrm{Def}_{G/NM}^{G/M} := \mathrm{Iso}(\beta) \circ \mathrm{Def}_{(G/M)/(NM/M)}^{G/M}$ where $\alpha : (G/N)/(NM/N) \to G/NM$ and $\beta : (G/M)/(NM/M) \to G/NM$ are the canonical isomorphisms.

5. If H is a subgroup of G and N is a normal subgroup of G, then

$$\mathrm{Def}_{G/N}^{G} \circ \mathrm{Ind}_{H}^{G} = \mathrm{Ind}_{HN/N}^{G/N} \circ \mathrm{Iso}(\varphi) \circ \mathrm{Def}_{H/H\cap N}^{H},$$

$$\mathrm{Res}_{H}^{G} \circ \mathrm{Inf}_{G/N}^{G} = \mathrm{Inf}_{H/H\cap N}^{H} \mathrm{Iso}(\varphi^{-1}) \circ \mathrm{Res}_{HN/N}^{G/N},$$

where $\varphi : H/H\cap N \to HN/N$ is the canonical group isomorphism.

6. If H is a subgroup of G and N is a normal subgroup of G such that $N \leqslant H$, then

$$\mathrm{Res}_{H/N}^{G/N} \circ \mathrm{Def}_{G/N}^{G} = \mathrm{Def}_{H/N}^{H} \circ \mathrm{Res}_{H}^{G}$$

$$\mathrm{Ind}_{H}^{G} \circ \mathrm{Inf}_{H/N}^{H} = \mathrm{Inf}_{G/N}^{G} \circ \mathrm{Ind}_{H/N}^{G/N}.$$

- *Triviality:*

1. If G is a group, then

$$\mathrm{Res}_{G}^{G} = \mathrm{Ind}_{G}^{G} = \mathrm{Def}_{G/\{1\}}^{G} = \mathrm{Inf}_{G/\{1\}}^{G} = \mathrm{Id}.$$

2. If $g \in G$ and $c_g : G \to G$ is the automorphism given by conjugation by g. Then $\mathrm{Iso}(c_g) = \mathrm{Id}$.

Remark 2.4.3. The elementary bisets induce a presentation of $\mathcal{C}$, ([4], Remark 3.1.2). This is a consequence of the Goursat Lemma characterizing subgroups of the direct products of two groups. In particular, given any two objects $G, H \in \mathcal{C}$, for any $\alpha \in B(G,H)$ that is the image of a transitive (G,H)-biset, there is a subquotient A/B of G and a subquotient C/D of H, and an isomorphism $f : C/D \cong A/B$ such that

$$\alpha = \mathrm{Ind}_A^G \circ \mathrm{Inf}_{A/B}^A \circ \mathrm{Iso}(f) \circ \mathrm{Def}_{C/D}^D \circ \mathrm{Res}_C^H.$$

2.5 Biset Functors

If R is a commutative ring with identity and $\mathcal{D}$ is a preadditive subcategory of the biset category $\mathcal{C}$, then the category $R\mathcal{D}$ is the category whose objects are the same as $\mathcal{D}$ and if G and H are two objects in $R\mathcal{D}$, then $\mathrm{Hom}_{R\mathcal{D}}(G,H) = R \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(G,H)$.

Definition 2.5.1 ([4], 3.2.2)**.** A *biset functor* on $\mathcal{D}$ with values in ${}_R\mathrm{Mod}$ is an R-linear functor from $R\mathcal{D}$ to ${}_R\mathrm{Mod}$. The category of biset functors on $\mathcal{D}$ with values in ${}_R\mathrm{Mod}$ whose morphisms are natural transformations is denoted by $\mathcal{F}_{\mathcal{D},R}$.

If we do not specify a ring R, then it is assumed we are over $\mathbb{Z}$.

Proposition 2.5.2 ([4], *Proposition*3.2.8)**.** *Let R be a commutative ring with identity and $\mathcal{D}$ a preadditive subcategory of $\mathcal{C}$.*

1. *The category $\mathcal{F}_{\mathcal{D},R}$ is an R-linear abelian category: if $f : F \to F'$ is a morphism of biset functors, then for any object $G \in \mathcal{D}$*

$$(\mathrm{Ker}(f))(G) = \mathrm{Ker}(f_G) \qquad (\mathrm{Coker}(f))(G) = \mathrm{Coker}(f_G),$$

   where $f_G : F(G) \to F'(G)$ is the evaluation of f at G.

2. *A sequence* $0 \longrightarrow F \xrightarrow{f} F' \xrightarrow{f'} F'' \longrightarrow 0$ *is exact in* $\mathcal{F}_{\mathcal{D},R}$ *if and only if*

$$0 \longrightarrow F(G) \xrightarrow{f_G} F'(G) \xrightarrow{f'_G} F''(G) \longrightarrow 0$$

is an exact sequence of R-modules, for any $G \in \mathcal{D}$.

3. *If I is a set, and* $(F_i)_{i\in I}$ *is a family of objects of* $\mathcal{F}_{\mathcal{D},R}$*, then the direct sum* $\bigoplus_{i\in I} F_i$ *and the direct product* $\prod_{i\in I} F_i$ *exist: for any* $G \in \mathcal{D}$

$$(\bigoplus_{i\in I} F_i)(G) = \bigoplus_{i\in I} F_i(G) \qquad (\prod_{i\in I} F_i)(G) = \prod_{i\in I} F_i(G).$$

For the rest of the paper, we will mostly consider subcategories $\mathcal{D} \subset \mathcal{C}$ that satisfy the following definition.

Definition 2.5.3 ([4], 4.1.7)**.** A class $\mathcal{D}$ of finite groups is said to be closed under taking subquotients if any group isomorphic to a subquotient of an object of $\mathcal{D}$ is in $\mathcal{D}$.

A subcategory $\mathcal{D}$ of $\mathcal{C}$ is said to be *replete* if it is a full subcategory whose class of objects is closed under taking subquotients.

Example 2.5.4. If $\mathcal{D}$ is a replete subcategory of $\mathcal{C}$, the Burnside ring can be viewed as a biset functor on $\mathcal{D}$. Formally, this can be defined as the Yoneda functor $\mathrm{Hom}_{\mathcal{D}}(\{1\}, -)$. It is straightforward to see that for any object $G \in \mathcal{D}$, $\mathrm{Hom}_{\mathcal{D}}(\{1\}, G) = B(G, \{1\}) \cong B(G)$.

Example 2.5.5. If R is a commutative ring with identity and $\mathcal{D}$ is a replete subcategory of $\mathcal{C}$. Suppose $F \in \mathcal{F}_{\mathcal{D},R}$. Let I be a set of objects in $\mathcal{D}$ and for each $H \in I$ let $I_H \subset F(H)$. Then there is a subfunctor of F, denoted $F_{(I_H)_{H\in I}}$, such that for any $X \in \mathcal{D}$

$$F_{(I_H)_{H\in I}}(X) = \sum_{H\in I} \mathrm{Hom}_{R\mathcal{D}}(H, X) I_H.$$

This is *the subfunctor of F generated by the data $(I_H)_{H\in I}$*.

Suppose R is commutative ring with identity and $\mathcal{D}$ is a preadditive subcategory. If F is an object in $\mathcal{F}_{\mathcal{D},R}$, (i.e. a biset functor on $\mathcal{D}$ with values in $_R$Mod) we say that F is a *simple biset functor* if it is a simple object in $\mathcal{F}_{\mathcal{D},R}$.

Proposition 2.5.6 ([4], 4.2.2)**.** *Let R be a commutative ring with identity and let $\mathcal{D}$ be a replete subcategory of $\mathcal{C}$, let $\mathcal{D}'$ be a full subcategory of $\mathcal{D}$.*
If F is a simple object of $\mathcal{F}_{\mathcal{D},R}$, then the restriction of $\mathcal{F}$ to $R\mathcal{D}'$ is either the zero object or a simple object of $\mathcal{F}_{\mathcal{D}',R}$.

If $\mathcal{D}$ is a replete subcategory of $\mathcal{C}$, R is a commutative ring with unity, and $G,H \in \mathcal{D}$, it follows from Remark 2.4.3 that $\mathrm{Hom}_{R\mathcal{D}}(G,H) = \mathrm{Hom}_{R\mathcal{C}}(G,H) = R \otimes_{\mathbb{Z}} B(H,G)$. In this case, we will use the notation $RB(H,G)$ to denote the morphism set $\mathrm{Hom}_{R\mathcal{D}}(G,H)$

Definition 2.5.7. Let G be a finite group. The endomorphism algebra $B(G,G)$ is called *the double Burnside ring*.

It should be noted that, as opposed to the commutative structure of the Burnside ring, the double Burnside ring is only commutative when considering the trivial group. Recall that as abelian groups, $B(G\times G)$ and $B(G,G)$ are isomorphic. To avoid any confusion, we will always use the notation $B(G,G)$ whenever we discuss the double Burnside ring.

Proposition 2.5.8 ([4], 4.3.2)**.** *Let R be a commutative ring with identity, and $\mathcal{D}$ a replete subcategory of $\mathcal{C}$. If G is an object of $\mathcal{D}$, denote I_G to be the R-submodule of $RB(G,G)$ generated by all endomorphisms of G which can be factored through some object H of $\mathcal{D}$ with $|H| < |G|$.*

Then I_G is a two sided ideal of $RB(G,G)$, and there is a decomposition

$$RB(G,G) = A_G \oplus I_G$$

where A_G is an R-subalgebra, isomorphic to the group algebra $ROut(G)$ of the group of outer automorphisms of G.

Remark 2.5.9. We briefly recall the parametrization of simple biset functor on a replete subcategory $\mathcal{D}$ of $\mathcal{C}$. Full details can be found in chapter 4 of [4]:

If S is a simple object of $\mathcal{F}_{\mathcal{D},R}$, then there is a minimal object G of $\mathcal{D}$, with respect to order, such that $S(G) \neq \{0\}$ is a simple $RB(G,G)/I_G \cong R\mathrm{Out}(G)$-module, and for any object H in $\mathcal{D}$ such that $S(H) \neq \{0\}$, G is isomorphic to a subquotient of H. Conversely, if G is an object of $\mathcal{D}$ and V is an $R\mathrm{Out}(G)$-module, then there is a simple object of $\mathcal{F}_{\mathcal{D},R}$, which we denote by $S_{G,V}$, such that $S_{G,V}(G) \cong V$ and for any object H in $\mathcal{D}$, $S_{G,V}(H) \neq 0$ implies that G is isomorphic to a subquotient of H. Two simple objects in $\mathcal{F}_{\mathcal{D},R}$, $S_{G,V}$ and $S_{G',V'}$ are isomorphic if and only if there is a group isomorphism $\varphi : G \to G'$ and an R-module isomorphism $\psi : V \to V'$ such that for all $v \in V$, and all $a \in \mathrm{Out}(G)$, $\psi(a \cdot v) = (\varphi a \varphi^{-1}) \cdot \psi(v)$.

Definition 2.5.10. Let $\mathcal{D}$ be a preadditive subcategory of $\mathcal{C}$. Suppose F is an object of $\mathcal{F}_{\mathcal{D},R}$. A simple functor S is a *composition factor* of F if there are subfunctors $F'' \subset F' \subset F$ on $R\mathcal{D}$, such that $F'/F'' \cong S$.

If G is a finite group we let $\mathcal{C}\downarrow_G$ denote the full subcategory of $\mathcal{C}$ whose objects are subquotients of G.

Definition 2.5.11 ([16])**.** Let G be an object of $\mathcal{D}$ and F an object of $\mathcal{F}_{\mathcal{D},R}$. The functor F has a *composition series over G* if there is a series of subfunctors

$$0 = T_0 \subseteq B_1 \subset T_1 \subseteq \cdots \subseteq B_m \subset T_m \subseteq B_{m+1} = F$$

such that

- T_i/B_i is a simple functor, whose restriction to $\mathcal{C}\downarrow_G$ is nonzero for all $i = 1\ldots,m$.
- $\mathrm{Res}^{\mathcal{D}}_{\mathcal{C}\downarrow_G}(B_{i+1}/T_i) = 0$ for all $i = 0,\ldots,m$.

If such a composition series exists, we will call the set of simple functors T_i/B_i together with their multiplicities the *composition factors of F over G*.

Proposition 2.5.12 ([16], Proposition 3.1)**.** *Let G be a fixed object of $R\mathcal{D}$ and F a biset functor on $R\mathcal{D}$. If F has a composition series over G, then any other composition series over G of F has the same length and the composition factors over G (taken with multiplicities) are the same.*

Theorem 2.5.13 ([2], Theorem 2.42)**.** *If R is a field then every biset functor on $R\mathcal{D}$ with values in ${}_R$mod has a composition series over G, for all objects G in $R\mathcal{D}$.*

Suppose R is a field and F is a biset functor on $\mathcal{D}$ with values in ${}_R$mod. If S is a composition factor of F, then there is an object G of $R\mathcal{D}$ and simple $R\mathrm{Out}(G)$- module V, such that $S \cong S_{G,V}$. It follows from Theorem 2.5.13 that S will be a composition factor of F over G. We can then define the *multiplicity of S* as a composition factor of F to be the number of times it shows up as a composition factor of F over G. Note that choosing a different parametrization for $S \cong S_{G',V'}$ would result in the same multiplicity, since $G \cong G'$ and thus $\mathcal{C}\downarrow_G$ and $\mathcal{C}\downarrow_{G'}$ are the same subcategory of $\mathcal{C}$.

Chapter 3

The Double Burnside Ring

Recall the definition of the double Burnside ring in 2.5.7. It is an important invariant of a finite group and worthy of studying on its own. For our purposes, we will only need special elements that arise from the double Burnside ring. More information about it can by found in Chapter 6 of [4].

3.1 Idempotents of RB(G,G)

For this section, we will assume $\mathcal{D}$ is a replete subcategory of $\mathcal{C}$ and R will denote a commutative ring with unity. Given a biset functor $F \in \mathcal{F}_{\mathcal{D},R}$, and an object $G \in \mathcal{D}$, notice that $F(G)$ has a natural $RB(G,G)$-module structure. In particular, there is a very useful decomposition of the unit element of $RB(G,G)$ into mutually orthogonal idempotents.

Definition 3.1.1. If G is a finite group, and N is a normal subgroup of G, then we define

$$j_N^G := \mathrm{Inf}_{G/N}^G \times_{G/N} \mathrm{Def}_{G/N}^G \in RB(G,G).$$

It is easy to see that the j_N^G is an idempotents of $RB(G,G)$, since $\mathrm{Def}^G_{G/N} \circ \mathrm{Inf}^G_{G/N} = [G/N] = \mathrm{Id} \in RB(G/N, G/N)$, by the elementary biset operations.

Definition 3.1.2 ([4], Definition 6.2.4)**.** Let G be a finite group. If N is a normal subgroup of G, let f_N^G denote the element of $RB(G,G)$ defined by

$$f_N^G = \sum_{N \leq M \trianglelefteq G} \mu_{\trianglelefteq G}(N,M)\, j_N^G$$

where $\mu_{\trianglelefteq G}$ is the Möbius function of the poset of normal subgroups of G.

Proposition 3.1.3 ([4], Proposition 6.2.7)**.** *Let G be a finite group. Then the elements f_N^G, for $N \trianglelefteq G$, are mutually orthogonal idempotents of $RB(G,G)$ and*

$$\sum_{N \trianglelefteq G} f_N^G = \mathrm{Id}_G.$$

Notation 3.1.4 ([4], Definition and Notation 6.3.1)**.** Let F be a biset functor, and let G be an object of $\mathcal{D}$. The set of *faithful elements* of $F(G)$ is the R-submodule

$$\partial F(G) = f_1^G F(G)$$

of $F(G)$.

If F is a biset functor, then for any object G of $\mathcal{D}$, we can use these idempotents to decompose $F(G)$ as an R-module. In particular, we have the following results.

Lemma 3.1.5 ([4], Proposition 6.3.2)**.** *Let $F \in \mathcal{F}_{\mathcal{D},R}$ and G an object of $R\mathcal{D}$.*

1. *If $1 < N \trianglelefteq G$, and $u \in F(G/N)$, then $f_1^G \mathrm{Inf}^G_{G/N} u = 0$.*

2. *$F(G) = \partial F(G) \oplus \sum_{1 < N \trianglelefteq G} \mathrm{Im}\mathrm{Inf}^G_{G/N}$.*

3. $\partial F(G) = \bigcap_{1<N\trianglelefteq G} \mathrm{Ker}\mathrm{Def}^G_{G/N}$.

Proposition 3.1.6 ([4], Proposition 6.3.3)**.** *Let $F \in \mathcal{F}_{\mathcal{D},R}$ and G be an object of $\mathcal{D}$. Then the map $\delta : F(G) \to \oplus_{N\trianglelefteq G} \partial F(G/N)$ defined by*

$$\delta(u) = \bigoplus_{N\trianglelefteq G} f_1^{G/N} \mathrm{Def}^G_{G/N} u$$

is an isomorphism of R-modules. The inverse isomorphism is the map ι defined by

$$\iota\left(\bigoplus_{N\trianglelefteq G} v_N\right) = \sum_{N\trianglelefteq G} \mathrm{Inf}^G_{G/N} v_N.$$

Chapter 4

The Unit Group of the Burnside Ring

A somewhat surprising fact about the unit group of the Burnside ring, is that it too has a biset functor structure. This is surprising in that one cannot simply consider the usual maps induced by the elementary bisets on the Burnside ring. This is because inducing and deflating between Burnside rings is, in general, not multiplicative. It is sufficient for our needs to see how this functor transfers units as viewed as images of the ghost map. The curious reader can find a construction of the unit groups biset functor structure in chapter 11 of [4]. Further information about multiplicative maps for biset functors can be read about in [5].

4.1 The Biset Functor $B^{\times}$

Definition 4.1.1 ([4], 11.2.17)**.** Let G and H be finite groups and U be a finite (H,G)-biset. For $u \in U$ and $L \leqslant H$, let

$$L^u := \{g \in G | \, \exists \, l \in L, \, l \cdot u = u \cdot g\}.$$

Then L^u is a subgroup of G. If $a \in B(G)$, then define

$$T_U(a) = \left(\prod_{u \in [L \backslash U/G]} |a^{L^u}| \right)_{L \in \mathcal{S}_H}$$

where $[L\backslash U/G]$ is a set of representatives of $L\backslash U/G$.

Theorem 4.1.2 ([4], Corollary 11.2.21)**.** *There exists a unique biset functor $B^\times$ on $\mathcal{C}$ such that $B^\times(G) = B(G)^\times$, for any finite group G, and*

$$B^\times(U) = T_U : B^\times(G) \to B^\times(H)$$

for any finite groups G and H, and any finite (H,G)-biset U.

We then, of course, restrict $B^\times$ to the replete subcategory $\mathcal{D}$. It is worth computing the maps T_U when U is an elementary biset.

Remark 4.1.3. Suppose G is a finite group, H is a subgroup of G, and N is a normal subgroup of G. If U is either Res^G_H or $\mathrm{Inf}^G_{G/N}$, then T_U is equal to the maps $B(U)$ restricted to their respective unit groups. For $U = \mathrm{Ind}^G_H$, we denote $\mathrm{Ten}^G_H := T_U$. If L is a subgroup of G and $x \in G$ then, viewing x as a point in the (G,H)-biset G, we have $L^x = x^{-1}Lx \cap H$ and

$$\mathrm{Ten}^G_H(a) = \left(\prod_{x \in [L\backslash G/H]} |a^{(x^{-1}Lx \cap H)}| \right)_{L \in \mathcal{S}_G} \tag{4.1}$$

for any $a \in B^\times(H)$. If $V = \mathrm{Def}^G_{G/N}$, then for any $b \in B^\times(G)$ and any subgroup K/N of G/N, we have

$$|T_V(b)^{K/N}| = |b^K| = |(b^N)^{K/N}|,$$

where b^N denotes the element in $B^\times(G/N)$ induced by taking N-fixed points on G-sets. Thus $T_V : B^\times(G) \to B^\times(G/N)$ is the map induced by taking N-fixed points on G-sets. In particular, if

$u \in \partial B^\times(G)$, then by Lemma 3.1.5, u is in kernel of all nontrivial deflation maps. This happens if and only if $|u^K| = 1$ for all subgroups K in G that contain a nontrivial normal subgroup of G. This is an important characterization of faithful elements in the unit group of the Burnside ring and will be used many times throughout this work.

4.2 The Unit Group of the Burnside Ring for p-groups

We will end this chapter by stating Bouc's result computing $B(G)^\times$ in the case where G is a p-group for some prime p. First we will need a few definitions. However, the reader is warned that these definitions are presented here without motivations, so they may seem strange. A detailed account of these topics can be read about in Chapters 9 and 11 of [4].

Definition 4.2.1. Let p be a prime and P a finite p-group. A subgroup $S \leqslant P$ is said to be *of normal p-rank* 1 if it does not have a normal subgroup isomorphic to $(C_p)^2$.

Definition 4.2.2. Let p be a prime and P a finite p-group. For any subgroup S, let $Z_p(S)$ denote the priemage of the center of $N_P(S)/S$. S is called *genetic* if $N_P(S)/S$ is of normal p-rank 1 and for any $x \in P$, ${}^xS \cap Z_P(S) \leqslant S$ implies that $x \in N_P(S)$.

Definition 4.2.3. Let p be a prime and P a finite p-group. If S and T are two genetic subgroups of P, we write $S \sim T$ if there exists an $x \in P$, such that ${}^xT \cap Z_P(S) \leqslant S$ and $S^x \cap Z_P(T) \leqslant T$.

Theorem 4.2.4 ([4], 9.6.1)**.** *Let p be a prime and P a finite p-group. The relation $\sim$ from Definition 4.2.3 is an equivalence relation on the set of genetic subgroups of P.*

Definition 4.2.5 ([4], 9.6.11)**.** Let p be a prime and P a finite p-group. A set of representatives of genetic subgroups of P with respect to the equivalence relation $\sim$ from Definition 4.2.3 is called a *genetic basis*.

For simplicity, we state a slightly weaker version of the following theorem but we note that full version establishes an $\mathbb{F}_2$-basis of $B^\times(P)$, whenever P is a p-group, not just the $\mathbb{F}_2$-dimension.

Theorem 4.2.6 ([3], 8.5)**.** *Let p be a prime and P be a finite p-group.*

1. *If $p \neq 2$, then $B^\times(P) = \{\pm 1\}$.*

2. *If $p = 2$, let $\mathcal{H}$ be a genetic basis of P and $\mathcal{N}$ be the subset of $\mathcal{H}$ consisting of subgroups $S \leqslant P$ such that $N_P(S)/S$ is trivial, cyclic of order 2, or dihedral of order at least 16. Then* $\dim_{\mathbb{F}_2}(B^\times(P)) = |\mathcal{N}|$.

Chapter 5

The Unit Group of the Burnside Ring for Some Solvable Groups

The main feature of this chapter is Theorem 5.1.10. In fact, it is the heart of this thesis. If G is a finite group with an abelian subgroup of index 1 or 2, Theorem 5.1.10 provides a standard basis for $B^\times(G)$. The rest of the chapter is dedicated to generalizing this theorem to a larger class of groups and proving a pair of lemmas at the end that will be useful in describing the subfunctors of $B^\times$ on this class of groups, which will be the focus of later chapters.

5.1 Groups With Abelian Subgroups of Index 1 or 2

Since $B(G)$ can be embedded into $\mathbb{Z}^{|\mathcal{S}_G|}$ and this embedding has finite cokernel, extending scalars by $\mathbb{Q}$ we have

$$\mathbb{Q}B(G) := \mathbb{Q} \otimes_{\mathbb{Z}} B(G) \cong \mathbb{Q}^{|\mathcal{S}_G|},$$

The following theorem gives a formula for the primitive idempotents of $\mathbb{Q}B(G)$ in terms of basis elements of $B(G)$. It was originally proved by Gluck ([8]) and independently by Yoshida ([18]).

Theorem 5.1.1 ([4], Theorem 2.5)**.** *Let G be a finite group. If H is a subgroup of G, denote by e_H^G the element of $\mathbb{Q}B(G)$ defined by*

$$e_H^G = \frac{1}{|N_G(H)|}\sum_{K\leq H}|K|\mu(K,H)[G/K],$$

where μ is the Möbius function of the poset of subgroups of G. Then, e_H^G=e_K^G if and only if $H =_G K$, and the elements e_H^G for $H \in \mathcal{S}_G$ are the primitive idempotents of the $\mathbb{Q}$-algebra $\mathbb{Q}B(G)$. Moreover, for $H,K \in \mathcal{S}_G$ one has $(e_H^G)^K = 1$ if $K = H$ and $(e_H^G)^K = 0$ if $H \neq K$.

Notice that for any element $u \in \mathbb{Q}B(G)$, one has $ue_H^G = |u^H|e_H^G$, hence $u = \sum_{H\in\mathcal{S}_G}|u^H|e_H^G$. For $u \in B^\times(G)$, we set $[F_u] := \{S \in \mathcal{S}_G \,|\, |u^H| = -1\}$. If $u = [G/G]$, then $[F_u]$ is the empty set. We then can write

$$u = [G/G] - \sum_{K\in[F_u]} 2e_K^G. \tag{5.1}$$

Lemma 5.1.2 ([3], Lemma 6.8)**.** *Let G be a finite group with $|Z(G)| > 2$, then $\partial B^\times(G)$ is trivial.*

We immediately obtain the following corollary.

Corollary 5.1.3. *Let G be an abelian group, then $\partial B^\times(G)$ is trivial if $|G| > 2$.*

We will make use of Lemma 5.1.2 to determine $\partial B^\times(G)$ for groups G with abelian subgroups of index 2. We begin with a reduction lemma.

Lemma 5.1.4. *Suppose C_2 acts on an abelian group G by automorphisms. Let x denote the generator for C_2. Suppose $|G^{C_2}| \leq 2$. Then x acts by inversion on the odd part $G_{2'}$ of G, and one of the following hold:*

(i) G_2 *is cyclic and* x *acts by inversion on* G_2.

(ii) G_2 *is cyclic of order* $2^n \geq 8$ *and if* $g \in G_2$*, then* ${}^x g = g^{2^{n-1}-1}$.

(iii) $G_2 \cong C_{2^n} \times C_2 = \langle a, b | a^{2^n} = b^2 = [a,b] = 1 \rangle$*, with* ${}^x b = a^{2^{n-1}} b$ *and* ${}^x a = a^e b^{\varepsilon}$*, with* $e \in \{2^n - 1, 2^{n-1} - 1\}$ *and* $\varepsilon \in \{0, 1\}$.

Proof. Let $g \in G$ and set $z = {}^x g g$. Note that $z \in G^{C_2}$. If g has odd order, then so does z. Thus x acts by inversion on g, otherwise $|G^{C_2}| > 2$. If G_2 is cyclic of order 2^n, choose g to be a generator and write ${}^x g = g^d$, with $1 \leq d \leq 2^n$. We may assume that $z^2 = 1$, hence 2^n divides $2(d+1)$. This implies that $d = 2^n - 1$ or $d = 2^{n-1} - 1$, which correspond to cases (i) and (ii) respectively.

Now assume that G_2 is not cyclic. If $G_2 \cong V_4$ then the statement follows by the constraint that $|G^{C_2}| \leq 2$. Assume then that $G_2 \ncong V_4$. Let P be the subgroup of G_2 generated by elements of order 2. In particular, P is an $\mathbb{F}_2$ vector space of dimension equal to the 2-rank of G_2. Let T_x denote the linear transformation on P induced by the action of x on G. Then T_x has minimal polynomial dividing $(\lambda + 1)^2$. Thus, if P has dimension greater than 2, the eigenspace for $\lambda = 1$ has at least dimension 2, thus $|G^{C_2}| > 2$. So we may assume that $G_2 = C_{2^n} \times C_{2^m} = \langle a, b \,|\, a^{2^n} = b^{2^m} = [a,b] = 1 \rangle$, with $n > 1$. Write ${}^x a = a^e b^{\varepsilon}$. If $z = {}^x a a$, then we may again assume that $z^2 = 1$. Thus $e \in \{2^n - 1, 2^{n-1} - 1\}$, and $b^{2\varepsilon} = 1$. Thus ${}^x(a^2) \in \langle a^2 \rangle$ and $a^{2^{n-1}} \in G^{C_2}$. Similarly, we get $b^{2^{m-1}} \in G^{C_2}$. Thus if $m > 1$ or ${}^x b = b$, we have $|G^{C_2}| > 2$. The result follows. $\square$

Recall from the end of Remark 4.1.3 that an element $u \in B^{\times}(G)$ is a faithful element if and only if $|u^K| = 1$ for any $K \leqslant G$ that contains a nontrivial normal subgroup of G. We will use this without citation through the end of the section.

Lemma 5.1.5. *Let $G = C_2 \ltimes N$ where N is an abelian group of order greater than 2. Suppose C_2 acts on N by inversion and that N_2 is cyclic. Then $\partial B^\times(G)$ has $\mathbb{F}_2$-dimension 1. More precisely: If N_2 is trivial, then $\partial B^\times(G)$ is generated by the element*

$$\Phi_G = [G/G] + [G/1] - 2[G/I],$$

where I is a representative of the unique conjugacy class of subgroups of order 2 in G. If N_2 is nontrivial, then $\partial B^\times(G)$ is generated by the element

$$\Phi_G = [G/G] + [G/1] - [G/I] - [G/J],$$

where I and J are representatives of the two non-central conjugacy classes of subgroups of order 2.

Proof. Let x be the generator of C_2 and let $u \in \partial B^\times(G)$. Then $|u^S| = 1$ for any subgroup $S \leqslant G$ which contains a nontrivial normal subgroup. Note that any subgroup of N is normal in G. Let $S \leqslant G$ with $|u^S| = -1$. Then $S \cap N$ is trivial. This means that S is a noncentral subgroup of order 2 or trivial.

Suppose N_2 is trivial. By Sylow's theorem, G has a unique conjugacy class of subgroups of order 2. Let $I = \langle x \rangle$. By Theorem 5.1.1 and Equation 5.1 we may write

$$u = [G/G] + \alpha[G/1] + \beta[G/I],$$

with $\alpha, \beta \in \mathbb{Z}$. Since $Z(G)$ is trivial, I is self-normalizing. Taking fixed points of I and the trivial subgroup, we get $\beta + 1 = \pm 1$ and $\frac{|G|}{2}\beta + |G|\alpha + 1 = \pm 1$. The only integer solutions are $\alpha = \beta = 0$ and $\alpha = 1$ and $\beta = -2$.

Suppose that N_2 is nontrivial and cyclic. Let y denote the generator of N_2. Set $I = \langle x \rangle$ and $J = \langle xy \rangle$. Since $I \neq_G J$, we know that G has at least two conjugacy classes of noncentral subgroups of order 2. Further, since $N_G(I) = Z(G)I$, $N_G(J) = Z(G)J$, and $|Z(G)| = 2$, we have $|N_G(J)| = |N_G(I)| = 4$. Thus $[G : N_G(J)] = [G : N_G(I)] = \frac{|N|}{2}$. Since every noncentral subgroup of order 2 is of the form $\langle xn \rangle$ with $n \in N$, we see that G has exactly two conjugacy classes of noncentral subgroups of order 2. Again by Theorem 5.1.1 and Equation 5.1, we may write

$$u = [G/G] + \alpha[G/1] + \beta[G/I] + \gamma[G/J],$$

where $\alpha, \beta, \gamma \in \mathbb{Z}$. Since $|N_G(I)/I| = |N_G(J)/J| = 2$, taking fixed points of I, J and the trivial subgroup, we have $2\beta + 1 = \pm 1$, $2\gamma + 1 = \pm 1$, and $\frac{|G|}{2}\beta + \frac{|G|}{2}\gamma + |G|\alpha + 1 = \pm 1$. The only integer solutions are $\alpha = \beta = \gamma = 0$ and $\alpha = 1, \beta = \gamma = -1$ and the result follows. □

Notation and Definition 5.1.6. We will refer to groups as described in Lemma 5.1.5 as *pseudodihedral*. We will frequently abuse this definition and say a group is pseudodihedral if it is isomorphic to a pseudodihedral group. If G is trivial, isomorphic to a cyclic group of order 2, or isomorphic to an pseudodihedral group, then $\partial B^{\times}(G)$ is generated by a unique nontrivial element. We will denote this element as Φ_G.

Remark 5.1.7. If G is the trivial group, then $\Phi_G = -1$. If $G = C_2$, then Φ_G can be characterized by its image in the ghost ring, which is given by $|\Phi_G^{\{1\}}| = 1$ and $|\Phi_G^{C_2}| = -1$. Similarly, if G is pseudodihedral, then Lemma 5.1.5 tells us that $|\Phi_G^X| = -1$ if and only if X is a noncentral subgroup of G, of order 2.

It is straightforward to verify that any subgroup or any quotient group of a pseudodihedral group will be either pseudodihedral or abelian. Equivalently, this means that any subquotient of a pseudodihedral group is either abelian or pseudodihedral. This is analogous to a property dihedral groups have.

Suppose $G = C_2 \ltimes N$ is pseudodihedral and x is a generator for C_2. With the exception where $G \cong D_8$, the subgroup N is the unique abelian subgroup with index 2. However, for the case of D_8, the subgroup N can be specified as the unique cyclic subgroup of index 2. Notice that every element of $G - N$ is of the form xn for some $n \in N$. Thus, every element of $G - N$ conjugates elements of N by inversion and every element of $G - N$ has order 2.

Lemma 5.1.8. *Suppose G is abelian or pseudodihedral. If $S \trianglelefteq G$ is a normal subgroup of G, then there is some $X \leqslant G$ such that $G/S \cong X$. Moreover, for every pseudodihedral subgroup of $Y \leqslant G$, there is a quotient of G isomorphic to Y.*

Proof. This is trivial if G is abelian, so we consider the case where G is pseudodihedral. If S is equal to G, or trivial, or if G/S has order 2, then the result is clear. Assume that S is nontrivial and that $|G/S| > 2$. Since G is pseudodihedral, we let N denote the unique abelian subgroup (in the case $G \cong D_8$, we let N denote the unique cyclic subgroup) of index 2 in G. We first prove that $S < N$. For the sake of contradiction, suppose that $x \in S$, such that $x \notin N$. Notice that x will have order 2, conjugation by x inverts elements of N, and $S = \langle x, N \cap S \rangle$. Because of our assumption that $[G : S] > 2$, we can deduce that $[N : S \cap N] > 2$. Thus, there is some $n \in N$ such that $n^2 \notin S \cap N$. The subgroup S is normal in G, so we have that $n^{-1}xn = xn^2 \in S$ and so $x(xn^2) = n^2 \in S$, which is a contradiction. So S is contained in N.

Since N is abelian, there exists $X' \leqslant N$ such that $N/S = \overline{N} \cong X'$, where we use the notation $\overline{g}$ to denote the image of $g \in G$ in the canonical projection of $G \to G/S$, and set $X = \langle x, X' \rangle$, where x is any element not in N. Write $G/S = \langle \bar{x}, \overline{N} \rangle$. We see that G/S is pseudodihedral or isomorphic to $C_2 \times C_2$, since conjugation by $\bar{x}$ inverts elements of $\overline{N}$, the subgroup $\overline{N}$ is abelian, the two part $\overline{N}_2$ of $\overline{N}$ is cyclic, and $\langle \bar{x} \rangle \cap \overline{N}$ is trivial. Similarly one verifies X is pseudodihedral or isomorphic to $C_2 \times C_2$. Since $\overline{N} \cong X'$, then $G/S \cong X$.

To prove the last statement, consider that if Y is a pseudodihedral subgroup of G, then $Y = \langle xn, Y \cap N \rangle$, for some $n \in N$. Since N is abelian, there is some subgroup $M \leqslant N$, such that $Y \cap N \cong N/M$. Moreover, x acts by inversion on N, hence $M \trianglelefteq G$. Thus $G/M \cong Y$. □

Proposition 5.1.9. *Suppose G is a finite group. If $N < G$ is an abelian subgroup with $[G:N] = 2$, then exactly one of the following hold: $\partial B^{\times}(G)$ is trivial, G is isomorphic to C_2, or G is isomorphic to a pseudodihedral group.*

Proof. If N is trivial or of order 2, then G is abelian and we are done by Corollary 5.1.3. We will now suppose that $|N| > 2$. Let $x \in G$ be an element whose image generates G/N, then $\langle x, N \rangle = G$. Denote $x^2 = z$ and note that $z \in N$, in fact $z \in Z(G)$. We may assume that $z^2 = 1$, otherwise the result holds by Lemma 5.1.2. We may also assume $x \notin Z(G)$, otherwise the result holds by Corollary 5.1.3. Since N is abelian, conjugation by x induces an action of G/N on N. Any element of N fixed by this action will be in the center of G. Therefore, by Lemma 5.1.2, we may assume that we are in one of the three cases described in Lemma 5.1.4. Note that Lemma 5.1.4 also allows us to assume N_2 is nontrivial, otherwise $z = 1$ and $G = \langle x \rangle \ltimes N$ is pseudodihedral.

Our first goal is to show that if $\partial B^{\times}(G)$ is nontrivial, then G is isomorphic to D_8 or N has a complement in G. Suppose $u \in \partial B^{\times}(G)$ is any nontrivial element. By Equation 5.1 we can write

$$u = [G/G] - \sum_{K\in[F_u]} 2e_K^G,$$

where $[F_u] = \{S \in \mathcal{S}_G \mid |u^S| = -1\}$. If u is nontrivial, then $[F_u]$ is nonempty. Recall that for any $S \in [F_u]$, the subgroup S does not contain a nontrivial normal subgroup of G. Choose S to be a maximal element of $[F_u]$, then, by Theorem 5.1.1, the coefficient in front of $[G/S]$ in $\beta = \sum_{K\in[F_u]} 2e_K^G$ is equal to $\frac{2}{[N_G(S):S]}$. Since $\beta \in B(G)$, $\frac{2}{[N_G(S):S]}$ is an integer. Thus, $[N_G(S) : S] \leq 2$. If S is trivial, then $N_G(S) = G$ and $[N_G(S) : S] > 4$, since N is nontrivial. Thus S is not trivial.

If $S \leqslant N$, then $[N : S] \leq [N_G(S) : S] \leq 2$. The subgroup S cannot be a normal subgroup, since $|u^S| = -1$. Thus $[N : S] = 2$. Since S does not contain a nontrivial normal subgroup of G, we may assume that $N_{2'}$ is trivial, so N must be a 2-group. Consider each of the three cases in Lemma 5.1.4 applied here. For the first two cases, when N is cyclic, the unique subgroup of order two in N is fixed by the action of G/N. Any maximal subgroup of N will contain this subgroup, which contradicts our assumption that S contains no nontrivial normal subgroup of G. Now, consider the third case of Lemma 5.1.4, where $N = \langle a,b | a^{2^n} = b^2 = [a,b] = 1\rangle$, with ${}^x b = a^{2^{n-1}} b$ and ${}^x a = a^e b^\varepsilon$, with $e \in \{2^n - 1, 2^{n-1} - 1\}$ and $\varepsilon \in \{0,1\}$. Assume for now that $n > 1$. Thus $a^{2^{n-1}} \neq 1$ is fixed by the action of G/N. If $S \neq \langle a\rangle$, then $S\langle a\rangle = N$ implies that $S \cap \langle a\rangle$ is nontrivial since $|N| \geq 8$. In either case, $\langle a^{2^{n-1}}\rangle < S$ and S contains a nontrivial normal subgroup of G.

If $N \cong C_2 \times C_2$, then $|G| = 8$. Thus G is either abelian or isomorphic to D_8, since the quaternion group does not contain a subgroup isomorphic to $C_2 \times C_2$. Hence $G \cong D_8$ or $S \not\leqslant N$.

If $G \ncong D_8$, then we can assume that $S \nleqslant N$. Thus, there is some element in S of the form xa where $a \in N$. Recall that any element of $G - N$ is of the form xb or $x^{-1}b$ for some $b \in N$. Note that since $xN = x^{-1}N$, so conjugation by x and x^{-1} induce the same action. Since N is abelian, it follows that, for any $h \in N$, ${}^{xa}h = {}^{xb}h = {}^{x^{-1}b}h$. This implies S contains all G-conjugates of $N \cap S$. In particular, ${}^{x}(N \cap S)(N \cap S) \leqslant S$ is a normal subgroup of G. Since S contains no nontrivial normal subgroup of G, we may assume that $N \cap S = \{1\}$. This implies that $|S| = 2$ and thus, is a complement to N and $G = S \ltimes N$.

We now prove that if $\partial B^{\times}(G)$ is nontrivial, then $G = S \ltimes N$ is pseudodihedral. Notice that in the previous paragraph, we showed that any maximal element of $[F_u]$ must be a noncentral subgroup of order 2. Let y be a generator of S. We only need to consider cases (*ii*) and (*iii*) from Lemma 5.1.4, otherwise G is pseudodihedral. Note that $N_G(S) = C_G(y)S = Z(G)S$. This establishes that S is contained in a conjugacy class of size $\frac{|G|}{4}$. Each noncentral subgroup of order 2 is generated by an element of the form $ym \in G$ for some unique $m \in N$. Since each conjugacy class of noncentral subgroups of order 2 must also have size $\frac{|G|}{4} = \frac{|N|}{2}$, there are at most two such classes. There will only be two conjugacy classes of noncentral subgroups of order 2 if every element of the form $ym \in G$, where $m \in N$ has order 2. It is easy to verify that this is not so for each of the cases (*ii*) and (*iii*) from Lemma 5.1.4. Thus by Theorem 5.1.1 we can write

$$u = [G/G] + \alpha[G/1] + \beta[G/S],$$

for some integers α and β. Considering fixed points of S and and the trivial group produces equations of the form $2\beta + 1 = \pm 1$ and $\frac{|G|}{2}\beta + |G|\alpha + 1 = \pm 1$. Because G is not abelian, $|G| > 4$

and the only integer solutions are $\alpha = \beta = 0$. The result follows. □

Theorem 5.1.10. *Let G be a finite group with an abelian subgroup of index 1 or 2. Let $\mathcal{N}$ denote the set of normal subgroups of G such that G/N is trivial, isomorphic to C_2, or isomorphic to a pseudodihedral group. Then $\{\mathrm{Inf}_{G/N}^{G}(\Phi_{G/N})\}_{N\in\mathcal{N}}$ is a basis for $B^{\times}(G)$.*

Proof. Note that factor groups of G will also have abelian subgroups of index 1 or 2. If N is normal in G, then Corollary 5.1.3, Proposition 5.1.5, and Proposition 5.1.9 altogether imply that $\partial B^{\times}(G/N)$ is either trivial or generated by $\Phi_{G/N}$. We can apply Proposition 3.1.6 to G and the result follows. □

We remark that this generalizes and unifies some of Matsuda's results on the order of the unit group of the Burnside ring for abelian groups and dihedral groups ([9], Examples 4.5 and 4.8).

5.2 Extending the Main Theorem

We can extend this result to a larger class of groups by the following proposition due to Bouc. We remark that the full proof of this result requires the Feit-Thompson Odd Order Theorem.

Proposition 5.2.1 ([3], Proposition 6.5)**.** *Let G be a finite group, and N be a normal subgroup of odd index in G. Then the group G/N acts on $B^{\times}(N)$ and the maps Res_{N}^{G} and Ten_{N}^{G} induce mutual inverse isomorphisms between $B^{\times}(G)$ and $B^{\times}(N)^{G/N}$.*

Corollary 5.2.2. *Let G be a finite group and N a normal subgroup of odd index in G. Suppose N has an abelian subgroup of index at most 2. If $\mathcal{N}$ and $\mathcal{L} = \{\mathrm{Inf}_{N/K}^{N}(\Phi_{N/K})\}_{K\in\mathcal{N}}$ are respectively the indexing set and basis of $B^{\times}(N)$ described in Theorem 5.1.10 applied to N, then G/N acts on $\mathcal{L}$ and, denoting $L_1,\ldots,L_k$ to be the orbit sums of this action, $\{\mathrm{Ten}_N^G L_i\}_{i=1}^{k}$ is a basis of $B^{\times}(G)$.*

Proof. G acts on N by conjugation, hence on $B(N)$ by ring automorphism, which restricts to an action on $B^{\times}(N)$ by group automorphism. The action of N is trivial, so we obtain an action of G/N on $B^{\times}(N)$. By Proposition 5.2.1 it suffices to show that $\mathcal{L}$ is invariant under this action. Choose $g \in G$ and let c_g denote the automorphism of N given by conjugation by g. Let $u \in B^{\times}(N)$, then $gN \cdot u = \mathrm{Iso}(c_g)(u)$. Hence, if $u = \mathrm{Inf}_{N/K}^{N}(\Phi_{N/K})$ for some $K \in \mathcal{N}$, then the action of gN on u is equal to

$$\mathrm{Iso}(c_g)\circ\mathrm{Inf}_{N/K}^{N}(\Phi_{N/K}) = \mathrm{Inf}_{N/{}^{g}K}^{N}\circ\mathrm{Iso}(\overline{c}_g)(\Phi_{N/K}),$$

where $\overline{c}_g$ is the isomorphism $N/K \to N/{}^{g}K$, induced by c_g. Since $\overline{c}_g$ is an isomorphism, ${}^{g}K \in \mathcal{N}$. Furthermore, $\mathrm{Iso}(\overline{c}_g)(\Phi_{N/K})$ will be in the kernels of all nontrivial deflation maps, so it must be a faithful element of $B^{\times}(N/{}^{g}K)$, hence $\mathrm{Iso}(\overline{c}_g)(\Phi_{N/K}) = \Phi_{N/{}^{g}K}$. Thus $\mathcal{L}$ is invariant under the action of G/N. □

Remark 5.2.3. The previous corollary is a generalization of Theorem 5.1.10, though in general its application will involve a choice of a normal subgroup N of odd index. The statement of the corollary implies that any choice of a normal subgroup N satisfying the hypothesis will produce a basis for $B^{\times}(G)$, and different choices might result in different bases. However, we note that the intersection of such normal subgroups also satisfies the hypothesis, thus we are able to designate a smallest such choice as the standard if needed.

Notation and Remark 5.2.4. It can be easily verified that the full subcategory of $\mathcal{C}$ whose objects are groups with normal subgroups N of odd index, where N has an abelian subgroup of index at most 2, is a replete subcategory of $\mathcal{C}$. We will denote this subcategory by $\mathcal{C}'$.

Example 5.2.5. Suppose G is a finite group and $N \cong D_{2n}$ is a normal subgroup of G with odd index, where D_{2n} denotes the dihedral group of order $2n$. Then N has a unique cyclic subgroup Z, of index 2. Let $d(n)$ denote the number of positive divisors of n. Notice that if $K \trianglelefteq N$, then N/K is dihedral, cyclic of order 2, or trivial, if and only if $|N/K| \neq 4$. Further, if n is odd, every proper normal subgroup of G is a subgroup of Z, and this corresponds exactly to the divisors of n. In the case where $n = 2k$ for some integer $k > 1$, we have three normal subgroups of index 2, namely Z and two that are isomorphic to D_{2k}. Every other proper normal subgroup of N is a proper subgroup of Z and exactly $d(n) - 2$ of these have index other than 4. By Theorem 5.1.10, $|B^\times(N)|$ is equal to $2^{d(n)+1}$ if n is odd and $2^{d(n)+2}$ if n is even.

If n is odd, every normal subgroup of N is characteristic. In the case where $n = 2k$, every normal subgroup is characteristic except for the two normal subgroups isomorphic to D_{2k}. However, since G/N has odd order, it stabilizes these two subgroups. Thus, by Corollary 5.2.2 $\mathrm{Ten}_N^G : B^\times(N) \to B^\times(G)$ is an isomorphism.

The following lemmas will be useful in later chapters.

Lemma 5.2.6. *Let G be an abelian group. Then*

$$B^\times(G) = B(G, \{1\})\Phi_{\{1\}} = \langle \mathrm{Ten}_H^G(-1) \rangle_{H \leqslant G}.$$

Proof. This can be seen by straightforward calculation when G is trivial or C_2. The general case follows from Theorem 5.1.10 and the transitivity of induction. □

Lemma 5.2.7. *Let G be a pseudodihedral group and $H \leqslant G$. If H is trivial or a noncentral group of order 2, then $\mathrm{Res}^G_H(\Phi_G) = -\Phi_H$. If H is isomorphic to a pseudodihedral group then $\mathrm{Res}^G_H(\Phi_G) = \Phi_H$. Moreover, for any subquotient, S, of G we have*

$$B(S,G)\Phi_G = B^\times(S).$$

Proof. We may write $G = \langle x, N\rangle$, where N is abelian and x acts by inversion on N. If H is trivial or a noncentral group of order 2, it is straightforward to verify that $\mathrm{Res}^G_H(\Phi_G) = -\Phi_H$. If H is pseudodihedral, then $H = \langle xy, N\cap H\rangle$ for some $y \in N$. Notice that for any $S \leqslant H$, we have $|\mathrm{Res}^G_H(\Phi_G)^S| = -1$ if and only if S is a noncentral subgroup of H, of order 2 for some. Thus $|\mathrm{Res}^G_H(\Phi_G)^S| = |\Phi^S_H|$ for any $S \leqslant H$, by Remark 5.1.7. It follows that $\mathrm{Res}^G_H(\Phi_G) = \Phi_H$.

To prove the last statement, notice that S is either abelian or pseudodihedral. If S is abelian, this follows easily from Lemma 5.2.6 as long as we can prove it for $S = \{1\}$. For this, it is sufficient to show that $-1 = \Phi_{\{1\}} \in B(\{1\},G)\Phi_G = B^\times(\{1\})$. For this, it is straightforward to verify that

$$\mathrm{Def}^{\langle x\rangle}_{\langle x\rangle/\langle x\rangle} \circ \mathrm{Res}^G_{\langle x\rangle}(\Phi_G) = \Phi_{\langle x\rangle/\langle x\rangle}.$$

In the case when S is pseudodihedral, we first show the lemma holds in the case where $S = G$. Lemma 5.1.8 tells us that each factor group of G is isomorphic to a subgroup of G. If N is a normal subgroup of G and $\partial B^\times(G/N)$ is nontrivial, then let $T \leqslant G$ be isomorphic to G/N and $f : T \to G/N$ is an isomorphism. If G/N has order 2, then choose T to be noncentral. Then $\mathrm{Iso}(f) \circ \mathrm{Res}^G_T(\Phi_G) = \varepsilon * \Phi_{G/N} \in B^\times(G/N)$ (here "$*$" is used to denote the group law of

$B^{\times}(G/N)$), where $\varepsilon = -1 \in B^{\times}(G/N)$ if T is trivial or cyclic of order 2 and $\varepsilon = 1 \in B^{\times}(G/N)$ otherwise. Thus

$$\mathrm{Inf}_{G/N}(\Phi_{G/N}) \in B(G,G)\Phi_G$$

and by Theorem 5.1.10, $B(G,G)\Phi_G = B^{\times}(G)$.

If S is any pseudodihedral subquotient of G, then $S = X/M$ for some $X \leqslant G$ and $M \trianglelefteq X$. By the first part of the lemma and Lemma 5.1.8, we have

$$B(X/M, X/M)\Phi_{X/M} \subseteq B(X/M, G)\Phi_G$$

and the result follows by replacing G in the second paragraph with X/M. □

Chapter 6

Residual groups for $B^\times$

In this chapter, we define a *residual group* with respect to a given biset functor. We then determine the residual groups with respect to $B^\times$ over $\mathcal{C}'$. We will use these groups in the next chapter to describe the subfunctor lattice of $B^\times$ on $\mathcal{C}'$.

For this chapter, we suppose R is a commutative ring with identity and $\mathcal{D}$ is a replete subcategory of $\mathcal{C}$.

6.1 Residual Groups

Notation and Definition 6.1.1. Let F be a biset functor on $\mathcal{D}$ with values in ${}_R\mathrm{Mod}$. For any object G of $\mathcal{D}$, we set $F^<(G)$, to be the E_G-submodule

$$F^<(G) := \sum_{|H|<|G|} \mathrm{Hom}_{R\mathcal{D}}(H,G)F(H) \subseteq F(G),$$

where H runs through objects of $\mathcal{D}$ which have smaller cardinality than G. If G is an object of $\mathcal{D}$, such that $F^<(G) \subsetneq F(G)$, then we say that G is *residual with respect to F*.

It follows by Remark 2.4.3 that

$$F^{<}(G) = \sum_{H} \mathrm{Hom}_{R\mathcal{D}}(H,G)F(H)$$

where H runs over subquotients of G. Thus, if $\mathcal{D}$ and $\mathcal{D}'$ are replete subcategories of $\mathcal{C}$ such that $\mathcal{D} \subset \mathcal{D}'$, then the residual objects of $\mathcal{D}$ are just the residual objects of $\mathcal{D}'$ that are in $\mathcal{D}$.

We denote by $\mathcal{R}_{\mathcal{D},F}$ a set of representatives, up to isomorphism, of objects in $\mathcal{D}$ which are residual with respect to F and call $\mathcal{R}_{\mathcal{D},F}$ a *complete set of residuals* for F in $\mathcal{D}$.

In the next proposition, we use the notation "$H \prec G$", which means H is isomorphic to a subquotient of G.

Proposition 6.1.2. *Suppose $\mathcal{D}$ is a replete subcategory of $\mathcal{C}$ and F is a biset functor on $R\mathcal{D}$. Let $\mathcal{R}_{\mathcal{D},F}$ and $\mathcal{R}'_{\mathcal{D},F}$ be complete sets of residuals for F in $\mathcal{D}$. If $G \in \mathcal{R}_{\mathcal{D},F}$ then $\mathcal{R}'_{\mathcal{D},F}$ has an object G' which is isomorphic to G. Furthermore, for any object $G \in \mathcal{D}$,*

$$F(G) = \sum_{\substack{H \in \mathcal{R}_{\mathcal{D},F} \\ H \prec G}} \mathrm{Hom}_{R\mathcal{D}}(H,G)F(H).$$

Proof. If G is a residual object of $\mathcal{D}$ with respect to F, then any group isomorphic to G will also be residual with respect to F. The second statement follows from easy induction on the order of G and the definition of residual with respect to F. □

Suppose $\mathcal{D} \subset \mathcal{D}'$. If $\mathcal{R}_{\mathcal{D},F}$ is a complete set of residuals for F in $\mathcal{D}$, then there is a complete set of residuals $\mathcal{R}_{\mathcal{D}',F}$ for F in $\mathcal{D}'$ such that $\mathcal{R}_{\mathcal{D},F} \subset \mathcal{R}_{\mathcal{D}',F}$. Also, for any object G of $\mathcal{D}$, we always have that

$$\sum_{1 \neq N \trianglelefteq G} \mathrm{Im}F(\mathrm{Inf}_{G/N}^{G}) \subseteq I_G F(G) \subseteq F^{<}(G) \subseteq F(G),$$

as R-modules.

6.2 Residual Objects of $\mathcal{C}'$ with Respect to $B^\times$

We now consider the biset functor $B^\times \in \mathcal{F}_{\mathcal{C}'}$ where $\mathcal{C}'$ is the subcategory of $\mathcal{C}$ from Remark 5.2.4. To be consistent with the usual biset notation, for any finite groups G, we have opted to denote the group law of $B^\times(G)$ additively throughout the rest of the paper.

Remark 6.2.1. Here we recall a particularly useful consequence of Lemma 5.1.5. Suppose G is pseudodihedral. Then

$$B^\times(G)/(\sum_{1\neq N\trianglelefteq G} \mathrm{ImInf}_{G/N}^{G})$$

is generated by the image of Φ_G, as an $\mathbb{F}_2$-space. In particular, $B^\times(G)/(\sum_{1\neq N\trianglelefteq G}\mathrm{ImInf}_{G/N}^{G})$ has $\mathbb{F}_2$-dimension 1.

Proposition 6.2.2. *Suppose $G \in \mathcal{C}'$.*

(a) If G is residual with respect to $B^\times$ then G is trivial or pseudodihedral.

(b) If G is pseudodihedral, then G is residual with respect to $B^\times$ if and only if

$$(B^\times)^<(G) = \sum_{1\neq N\trianglelefteq G} \mathrm{ImInf}_{G/N}^{G}.$$

Proof. By Theorem 5.1.9 and Corollary 5.2.2, G must be trivial, isomorphic to C_2, or pseudodihedral group. But by Lemma 5.2.6, C_2 is not residual.

For any finite group G, we always have

$$\sum_{1\neq N\trianglelefteq G} \mathrm{ImInf}_{G/N}^{G} \subseteq (B^\times)^<(G) \subseteq B^\times(G).$$

When G is pseudodihedral, it follows by Remark 6.2.1 that G is residual if and only if $(B^\times)^<(G) = \sum_{1\neq N \trianglelefteq G} \mathrm{ImInf}^G_{G/N}$. $\square$

Remark 6.2.3. Suppose G is a pseudodihedral group and let N denote its unique abelian subgroup (the unique cyclic subgroup when $G \cong D_8$) of index 2. If H is any maximal subgroup of G other than N, then $H = \langle x, N \cap H \rangle$, where $x \in G - N$ and $N \cap H$ is a maximal subgroup of N. Further, conjugation by x inverts elements of $N \cap H$. So H is pseudodihedral as long as $|N \cap H| > 2$. Notice that $N \cap H$ is a maximal subgroup of N. Therefore, all maximal subgroups of G, other than N, are also pseudodihedral, as long as $\frac{|G|}{2} = |N|$ is not a prime or twice a prime.

Proposition 6.2.4. *Let G be a pseudodihedral group. Let $N < G$ denote the unique abelian subgroup (the unique cyclic subgroup when $G \cong D_8$) with index 2. If G is not residual with respect to $B^\times$ and $|N|$ is not prime or twice a prime, then there is some pseudodihedral maximal subgroup $H < G$, for which $\mathrm{Ten}^G_H(\Phi_H)$ and Φ_G have the same image in $B^\times(G)/(\sum_{1\neq K \trianglelefteq G} \mathrm{ImInf}^G_{G/K})$.*

Proof. Since G is not residual, it follows from Remark 6.2.1 that there is some proper subgroup $H < G$ and an element $u \in B^\times(H)$, such that $\mathrm{Ten}^G_H(u)$ and Φ_G have the same image in

$$B^\times(G)/(\sum_{1\neq K \trianglelefteq G} \mathrm{ImInf}^G_{G/K}).$$

By the transitivity of induction, we can assume that H is a maximal subgroup of G. If $H = N$, then $\partial B^\times(N)$ is trivial by Corollary 5.1.3, since $|N| > 2$. Thus by Lemma 3.1.5 we can write

$$u = \sum_{i=1}^{k} \mathrm{Inf}^N_{N/N_i}(u_i),$$

where $N_1, \ldots, N_k$ are nontrivial normal subgroup of N and $u_i \in B^\times(N/N_i)$, for each $i = 1, \ldots, k$.

Each subgroup of N is normal in G, since G is pseudodihedral. So by Remark 2.4.2

$$\mathrm{Ten}_H^G(u) = \mathrm{Ten}_H^G(\sum_{i=1}^{k} \mathrm{Inf}_{N/N_i}^{N}(u_i)) = \sum_{i=1}^{k} \mathrm{Inf}_{G/N_i}^{G} \mathrm{Ten}_{N/N_i}^{G/N_i}(u_i).$$

Hence $\mathrm{Ten}_H^G(u)$ is trivial in $B^\times(G)/(\sum_{1\neq K\trianglelefteq G} \mathrm{ImInf}_{G/K}^G)$, a contradiction. We can now assume $H \neq N$ and so it is pseudodihedral by assumption (see Remark 6.2.3).

We want to show that $\mathrm{Ten}_H^G(u)$ and $\mathrm{Ten}_H^G(\Phi_H)$ have the same image modulo $\sum_{1\neq K\trianglelefteq G} \mathrm{ImInf}_{G/K}^G$. To this end, we first show that $u \notin \sum_{1\neq S\trianglelefteq H} \mathrm{ImInf}_{H/S}^H$. For the sake of contradiction, suppose $u \in \sum_{1\neq S\trianglelefteq H} \mathrm{ImInf}_{H/S}^H$. If S is any nontrivial normal subgroup of H, then $S\cap N$ cannot be trivial. Otherwise, S is a non-central subgroup of order two and the only elements that normalize S in G, other than S, will be central in G. This would imply $|H| \leq 4$ which contradicts H being pseudodihedral. So, by the transitivity of inflation

$$u \in \sum_{1\neq S\trianglelefteq H} \mathrm{ImInf}_{H/(S\cap N)}^H,$$

where each $S\cap N$ is a nontrivial normal subgroup of G. So again by Remark 2.4.2, we have

$$\mathrm{Ten}_H^G(u) \in \sum_{1\neq K\trianglelefteq G} \mathrm{ImInf}_{G/K}^G,$$

which contradicts that $\mathrm{Ten}_H^G(u)$ and Φ_G have the same image modulo $\sum_{1\neq K\trianglelefteq G} \mathrm{ImInf}_{G/K}^G$.

This allows us to assume that $u = \Phi_H + u'$ where $u' \in \sum_{1\neq S\trianglelefteq H} \mathrm{ImInf}_{H/S}^H$. However, as above, we have

$$\mathrm{Ten}_H^G(u') \in \sum_{1\neq K\trianglelefteq G} \mathrm{ImInf}_{G/K}^G.$$

This implies that $\mathrm{Ten}_H^G(u)$ and $\mathrm{Ten}_H^G(\Phi_H)$ have the same image modulo $\sum_{1\neq K\trianglelefteq G} \mathrm{ImInf}_{G/K}^G$ and the result follows. □

Together with Proposition 6.2.2, the next proposition determines which objects in $\mathcal{C}'$ are residual with respect to $B^{\times}$. It will be used in the next section when we characterize the lattice of subfunctors of $B^{\times}$ over $\mathcal{C}'$ and find its composition factors.

Remark 6.2.5. Before we prove the following proposition, we remark that D_8 is not residual. This follows from the unnecessarily strong machinery given by Theorem 8.5 in [3]. It can also be proved easily from straightforward calculations similar to the case in Proposition 6.2.6, where we consider groups isomorphic to D_{2p}, with p an odd prime. We consider this case settled and do not include it in the next proposition to simplify the characterization.

Proposition 6.2.6. *Suppose G is a pseudodihedral group not isomorphic to D_8. Let N denote the unique abelian subgroup of G with index 2.*

(A) When $N \cong C_p$, the group G is not residual with respect to $B^{\times}$ if and only if $p \equiv 3\,(mod\,4)$.

(B) When N is not simple, the group G is not residual with respect to $B^{\times}$ if and only if $|N_2| = 2$ or N has an element of order p^2 for some odd prime p.

Proof. We frequently make use of (4.1) from Remark 4.1.3. We write it again, for easy reference. If $H \leqslant G$ and $a \in B^{\times}(H)$ then

$$\mathrm{Ten}_H^G(a) = \left(\prod_{g \in [L \backslash G/H]} |a^{(g^{-1}Lg \cap H)}| \right)_{L \in \mathcal{S}_G}, \tag{6.1}$$

where $\mathcal{S}_G$ is a set of representatives of conjugacy classes of subgroups of G. Write $G = C_2 \ltimes N$ and denote the generator of C_2 by x. Since G is pseudodihedral recall that N_2 is cyclic and x acts on N by inversion. We split the rest of the proof into the two cases based on whether or not N is simple.

Part (A):

We first consider the case where $N = C_p$ for some prime p. Then $p \neq 2$ since G is pseudodihedral. By Theorem 5.1.10, $|B^\times(G)| = 2^3$. By Lemma 5.2.6 and Lemma 5.1.8, for any abelian group L, we have

$$B^\times(L) = B(L,\{1\})\Phi_{\{1\}} = \langle \mathrm{Ten}_H^L(-1)\rangle_{H\leqslant L}.$$

Since each proper subgroup of G is abelian, and induction is transitive, the group G is not residual if and only if

$$B^\times(G) = B(G,\{1\})\Phi_{\{1\}} = \langle \mathrm{Ten}_H^G(-1)\rangle_{H\leqslant G}.$$

G has 4 conjugacy classes of subgroups. Denote $\mathcal{S}_G = \{\{1\},\langle x\rangle, C_p, G\}$ a set of representatives of these classes. By (3) (ordering $\mathcal{S}_G$ by increasing cardinality) we compute

$$\mathrm{Ten}_{\{-1\}}^G(-1) = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathrm{Ten}_{\langle x\rangle}^G(-1) = \begin{pmatrix} -1 \\ (-1)^r \\ -1 \\ -1 \end{pmatrix}$$

$$\mathrm{Ten}_{C_p}^G(-1) = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathrm{Ten}_{G}^G(-1) = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

where r is equal to the number of double cosets in $\langle x\rangle\backslash G/\langle x\rangle$. Explicitly, by (3)

$$|\mathrm{Ten}_{\langle x\rangle}^G(-1)^{\langle x\rangle}| = \prod_{g\in[\langle x\rangle\backslash G/\langle x\rangle]}(-1) = (-1)^{|\langle x\rangle\backslash G/\langle x\rangle|}$$

Thus $|B(G,\{1\})(-1)| = |B^\times(G)|$ if and only if $r = |\langle x\rangle\backslash G/\langle x\rangle| = 1+\frac{p-1}{2}$ is even. This happens if and only if $p \equiv 3 \pmod 4$. Hence, G is not residual with respect to $B^\times$ if and only if $p \equiv 3 \pmod 4$.

Part (B):

We now consider the case where N is not simple. When $|N_2| = 2$ or N contains an element of order p^2 for some odd prime p, we will establish that there is some proper subgroup $H < G$ and some $w \in B^\times(H)$, such that $\mathrm{Ten}_H^G(w)$ and Φ_G have the same image in $B^\times(G)/\sum_{1\neq K\trianglelefteq G}\mathrm{ImInf}_{G/K}^G$. On the other hand, if $|N_2| \neq 2$ and N has no element of order p^2, for any odd prime p, then Remark 6.2.3 and Proposition 6.2.4 reduces proving G is residual to showing $\mathrm{Ten}_H^G(\Phi_H)$ is contained in $\sum_{1\neq K\trianglelefteq G}\mathrm{ImInf}_{G/K}^G$, for all pseudodihedral maximal subgroups H.

For the rest of the proof, we let z denote a generator of N_2 and set $I = \langle x\rangle$ and $J = \langle zx\rangle$. Recall that if N_2 is nontrivial, then I and J are representatives of the two conjugacy classes of noncentral subgroups of order two in G. In this case, the group N has a unique maximal subgroup of index 2, which we denote as N'. Further, the group G has exactly two non-abelian maximal subgroups of index 2. These are $H_I := \langle x, N'\rangle$ and $H_J := \langle xz, N'\rangle$. Note that since $G \ncong D_8$, both H_I and H_J are pseudodihedral. (Also note that in the case where N_2 is trivial, $I = J$, and I is a representative of the unique conjugacy class of noncentral subgroups of order two in G)

Claim (i): *If N has an element of order p^2 for some odd prime p, then G is not residual with* *respect to $B^\times$.*

Fix $y \in N_p$ to have maximal order, p^n. Let C be a complement of $\langle y \rangle$ in N. Set $H' = \langle y^p \rangle \times C$ and $H = \langle x, H' \rangle$. By construction H' has no complement in N. Since H' is a maximal subgroup of N we have that H is a maximal subgroup of G. Fix a subgroup $X \leqslant G$. We need to establish that $|\mathrm{Ten}_H^G(\Phi_H)^X| = |\Phi_G^X|$.

Note again that H' has no complement in N. Furthermore, any nontrivial subgroup $S \leqslant N$ will intersect with H', hence H, nontrivially. Thus, for any $g \in G$, if ${}^gX \cap H$ is a noncentral subgroup of order 2 in H, then X is already a noncentral subgroup of order 2 of G. So by (3), we have

$$|\mathrm{Ten}_H^G(\Phi_H)^X| = 1$$

unless X is a noncentral subgroup of order 2.

Since H has index p in G and $p \neq 2$, then both I and J are subgroups of H. Suppose $X = I$. Then $g^{-1}Xg \cap H \neq \{1\}$ if and only if $g^{-1}xg = xg^2 \in H$. Because H has odd index in G, the element $xg^2 \in H$ if and only if $g \in H$. Thus $g^{-1}Xg \cap H \neq \{1\}$ if and only if $XgH = XH$ as double cosets in $X \backslash G / H$. Again by (3),

$$|\mathrm{Ten}_H^G(\Phi_H)^X| = |\Phi_H^X| = -1.$$

When $X = J$, the argument is similar and we get

$$|\mathrm{Ten}_H^G(\Phi_H)^X| = |\Phi_H^X| = -1.$$

Thus, $|\mathrm{Ten}_H^G(\Phi_H)^X| = -1$ if and only if $X \leqslant G$ is a noncentral subgroup of order 2. By Remark 5.1.7, $|\mathrm{Ten}_H^G(\Phi_H)^X| = |\Phi_G^X|$ for any $X \leqslant G$. Thus $\mathrm{Ten}_H^G(\Phi_H) = \Phi_G$. Therefore G is not residual.

Claim (ii): *If* $|N_2| = 2$*, then* G *is not residual with respect to* $B^{\times}$*.*

Since N_2 is not trivial H_I and H_J are defined and we set $v = \mathrm{Ten}_{H_I}^{G}(\Phi_{H_I})$. Note also that $N_{2'}$ is nontrivial, since $|N_2| = 2$ and $|N| > 2$ by G being pseudodihedral. Fix a subgroup X of G. We wish to compute $|v^X|$. If X is trivial then it is clear by (6.1) that $|v^X| = 1$. Suppose X is nontrivial and $X \cap N_{2'} \neq \{1\}$. Since $X \cap N_{2'} = {}^{g}(X \cap N_{2'}) \leqslant {}^{g}X \cap H$ for any $g \in G$, it follows that ${}^{g}X \cap H$ is not a noncentral subgroup of order 2 of H, for any $g \in G$. So $|v^X| = 1$, by (6.1).

Now suppose X is nontrivial and $X \cap N_{2'} = \{1\}$. Then X is conjugate to N_2, I, J, or $JN_2 = IN_2$. We apply (6.1) to each of these subgroups. For N_2 it is easy to verify that

$$|v^{N_2}| = |\Phi_{H_I}^{(N_2 \cap H_I)}| = |\Phi_{H_I}^{\{1\}}| = 1.$$

For I, notice that for any $g \in G$, $g^{-1}Ig \cap H_I = g^{-1}Ig$, since H_I is normal in G and $I < H_I$. Since $|I \backslash G / H_I| = |G/H_I| = 2$, we have that $|v^I| = 1$. Considering J, notice that $g^{-1}Jg \cap H_I = \{1\}$ for any $g \in G$, since $J \not< H_I$, thus $|v^J| = 1$. Lastly, for IN_2, notice that $|IN_2 \backslash G / H_I| = 1$. Hence,

$$|v^{IN_2}| = |\Phi_{H_I}^{IN_2 \cap H_I}| = |\Phi_{H_I}^{I}| = -1.$$

Therefore, v is the unique element of $B^{\times}(G)$ characterized by $|v^X| = -1$ if and only if $X \leqslant G$ is in the same conjugacy class as IN_2.

We now consider the element $u = \mathrm{Inf}_{G/N_2}^{G}(\Phi_{G/N_2})$ and show that $\Phi_G = u + v \in B^{\times}(G)$ (noting that G/N_2 is pseudodihedral, thus Φ_{G/N_2} is defined.) Recall that inflation between unit groups of Burnside rings comes from restricting the inflation ring morphism between Burnside rings. Thus, for any $X \leqslant G$, we have $|u^X| = |\Phi_{G/N_2}^{(XN_2/N_2)}| = -1$ if and only if the image of X through the canonical projection $G \to G/N_2$ is a noncentral subgroup of order 2. In other words, $|u^X| = -1$ if and only if X is conjugate to I, J, or IN_2. This implies that $v + u \in B^\times(G)$ is the unique element characterized by $|(u+v)^X| = -1$ if and only if X is a noncentral subgroup of order 2. It follows that $u + v = \Phi_G$, by Remark 5.1.7. Therefore G is not residual with respect to $B^\times$. (We note that $u + v = \Phi_G$ would still be true if we had set $v = \mathrm{Ten}_{H_J}^G(\Phi_{H_J})$.)

Claim (iii): *If p is an odd prime that divides $N_{2'}$ and N has no element of order p^2, then $\mathrm{Ten}_H^G(\Phi_H)$ is contained in $\sum_{1 \neq K \trianglelefteq G} \mathrm{ImInf}_{G/K}^G$, for any subgroup $H < G$ of index p.*

Suppose $H \leqslant G$ has index p. Denote $H' = H \cap N$, then H' is a maximal subgroup of N with index p. Furthermore, there is some $n \in N_p$ such that $H = \langle xn, H' \rangle$. So for $h = n^{\frac{1-p}{2}}$, we have ${}^hH = \langle x, H' \rangle$, since ${}^h(xn) = xn^{1+p-1} = x$. If $c_h : G \to G$ denotes the automorphism induced by conjugation with h, then

$$\mathrm{Iso}(c_h) \circ \mathrm{Ten}_H^G(\Phi_H) = \mathrm{Ten}_{c_h(H)}^G(\mathrm{Iso}(c_h)(\Phi_H)) = \mathrm{Ten}_{c_h(H)}^G(\Phi_{c_h(H)}),$$

since the element $\mathrm{Iso}(c_h)(\Phi_H) \in B^\times(c_h(H))$ is clearly a nontrivial faithful element and thus $\mathrm{Iso}(c_h)(\Phi_H) = \Phi_{c_h(H)}$. It follows that $\mathrm{Ten}_H^G(\Phi_H)$ is in the image of $\sum_{1 \neq K \trianglelefteq G} \mathrm{ImInf}_{G/K}^G$ if and only if $\mathrm{Ten}_{c_h(H)}^G(\Phi_{c_h(H)})$ is in the image of $\sum_{1 \neq K \trianglelefteq G} \mathrm{ImInf}_{G/K}^G$. So it is sufficient for us to assume $H = \langle x, H' \rangle$ and show that $\mathrm{Ten}_H^G(\Phi_H)$ is in the image of $\sum_{1 \neq K \trianglelefteq G} \mathrm{ImInf}_{G/K}^G$.

Write $|N_p| = p^m$. Our assumption implies that N_p is an elementary abelian p-group of rank m. Thus, every maximal subgroup of N_p has $\frac{(p^m-1)-(p^{m-1}-1)}{p-1} = p^{m-1}$ complements in N_p. It follows that every maximal subgroup of N with index p has p^{m-1} complements in N. Let $Q_1, \cdots, Q_{p^{m-1}}$ denote the complements of H' in N.

We again use (6.1) to compute $\mathrm{Ten}_H^G(\Phi_H)$. Let $X \leqslant G$. If $X \in \{\{1\}, Q_1, \ldots, Q_{p^{m-1}}\}$, then it is easy to compute that $|\mathrm{Ten}_H^G(\Phi_H)^X| = 1$. If $X \cap H'$ is nontrivial, then ${}^g(X \cap H') \leqslant {}^gX \cap H$ is not a noncentral subgroup of order 2 of H for any $g \in G$. Thus again, $|\mathrm{Ten}_H^G(\Phi_H)^X| = 1$. The for the remaining cases, we may assume $X \cap H' = \{1\}$ and $X \nleqslant N$. It is straightforward to verify this implies X is conjugate to one of the subgroups in the set $\{I, IQ_1, \ldots, IQ_{p^{m-1}}, J, JQ_1, \ldots, JQ_{p^{m-1}}\}$.

Since $I \leqslant H$, we argue as in the proof of claim (i) that $g^{-1}Ig \cap H \neq \{1\}$ if and only if $IgH = IH$ as double cosets in $I \backslash G/H$. Thus

$$|\mathrm{Ten}_H^G(\Phi_H)^I| = -1,$$

and similarly

$$|\mathrm{Ten}_H^G(\Phi_H)^J| = -1.$$

Suppose that $X \in \{IQ_1, \ldots, IQ_{p^{m-1}}, JQ_1, \ldots, JQ_{p^{m-1}}\}$. Then $|X \backslash G/H| = 1$, and thus

$$|\mathrm{Ten}_H^G(\Phi_H)^X| = |\Phi_H^{X \cap H}| = -1,$$

since $X \cap H$ is either I or J.

In summary, $\mathrm{Ten}_H^G(\Phi_H)$ is the element of $B^\times(G)$ characterized by $|\mathrm{Ten}_H^G(\Phi_H)^X| = -1$ if and only if X is conjugate to one of the subgroups in the set

$\{I, IQ_1, \ldots, IQ_{p^{m-1}}, J, JQ_1, \ldots, JQ_{p^{m-1}}\}$.

Now consider the element

$$\omega = \sum_{i=1}^{p^{m-1}} \mathrm{Inf}_{G/Q_i}^{G}(\Phi_{G/Q_i}) \in B^{\times}(G)$$

Then $|\omega^{IQ_i}| = |\omega^{JQ_i}| = -1$ for all $i = 1,\dots,p^{m-1}$. Also, $|\omega^I| = |\omega^J| = (-1)^{p^{m-1}} = -1$. For any other subgroup $X \leqslant G$ that is not conjugate with I, J, IQ_i, or JQ_i for $i = 1,\dots,p^m$, it is easy to verify that $|\omega^X| = 1$. Thus $|\omega^X| = |\mathrm{Ten}_H^G(\Phi_H)^X|$ for any subgroup $X \leqslant G$, which implies that $\omega = \mathrm{Ten}_H^G(\Phi_H)$. In particular, we have shown that $\mathrm{Ten}_H^G(\Phi_H)$ is an element of $\sum_{1\neq K\trianglelefteq G} \mathrm{ImInf}_{G/K}^G$.

Claim (iv): *If* $|N_2| > 2$ *then* $\mathrm{Ten}_{H_I}^G(\Phi_{H_I})$ *and* $\mathrm{Ten}_{H_J}^G(\Phi_{H_J})$ *are the trivial element in* $B^{\times}(G)$*.*

We prove this for $\mathrm{Ten}_{H_I}^G(\Phi_{H_I})$ and note that the proof is nearly the same for $\mathrm{Ten}_{H_J}^G(\Phi_{H_J})$. Fix $X \leqslant G$. It suffices to show that $|\mathrm{Ten}_{H_I}^G(\Phi_{H_I})^X| = 1$. Notice that since $|N_2| > 2$, this implies that $H_I \cap N$ has no complement in N. So if there is an element $g \in G$ such that ${}^gX \cap H_I$ is noncentral of order 2 in H_I, then $X \cap N$ is trivial, hence X is noncentral of order 2 in G. If $|\mathrm{Ten}_{H_I}^G(\Phi_{H_I})^X| = -1$, then there is such a $g \in G$, hence X is noncentral of order 2. However, by (3) we can verify that $|\mathrm{Ten}_{H_I}^G(\Phi_{H_I})^I| = 1$ and $|\mathrm{Ten}_{H_I}^G(\Phi_{H_I})^J| = 1$ (for the same reasons as in claim (ii)). This means that $|\mathrm{Ten}_{H_I}^G(\Phi_{H_I})^X| = 1$ and we are done.

Claims (i) and (ii) prove that if $|N_2| = 2$ or N has an element of order p^2 for some odd prime p, then G is not residual. On the other hand when $|N_2| \neq 2$ and N has no element of order p^2, claims (iii) and (iv) prove that $\mathrm{Ten}_H^G(\Phi_H)$ is contained in $\sum_{1\neq K\trianglelefteq G}\mathrm{ImInf}_{G/K}^G$, for all pseudodihedral maximal subgroups H. Therefore, applying Proposition 6.2.4 implies G is residual with respect to $B^\times$.

□

Consequentially, we get the following theorem.

Theorem 6.2.7. *Let $\mathcal{R}$ denote the set of finite groups consisting of the trivial group, dihedral groups D_{2p} where p is prime and $p\equiv 1\,(mod\,4)$, and all pseudodihedral groups $C_2\ltimes N$, excluding D_8, where N is not simple, $|N_2|\neq 2$, and $N_{2'}$ has no element of order q^2, for any odd prime q. Then $\mathcal{R}$ is a complete set of residuals for $B^\times$ in $\mathcal{C}'$. Moreover, for any $G\in\mathcal{C}'$,*

$$B^\times(G)=\sum_{\substack{H\in\mathcal{R}\\ H\prec G}}B(G,H)\Phi_H.$$

Proof. That $\mathcal{R}$ is a complete set of residuals follows by Remark 6.2.5, Proposition 6.2.2, and Proposition 6.2.6. The last statement follows from Proposition 6.1.2 and Lemma 5.2.7. □

Chapter 7

Subfunctors of $B^\times$

The goal of this chapter is to parametrize the subfunctors of $B^\times$ over $\mathcal{C}'$. We can do this in terms of sets of the groups that are residual with respect to $B^\times$ found in Chapter 6. For the rest of the section, we let $\mathcal{R}_{\mathcal{C}',B^\times}$ be a complete set of residuals for $B^\times$ in $\mathcal{C}'$.

7.1 The Subfunctor Lattice of $B^\times$ over $\mathcal{C}'$

Notation 7.1.1. For any $I \subset \mathcal{R}_{\mathcal{C}',B^\times}$, set $F_I \subseteq B^\times$ to be the subfunctor generated by $\{\Phi_X | X \in I\}$. In other words, F_I is the subfunctor of $B^\times$ such that, for any $G \in \mathcal{C}'$

$$F_I(G) = \sum_{X \in I} B(G,X)\Phi_X,$$

if I is nonempty and F_I is the trivial biset functor if I is empty. Note that we may construct $F_I \subseteq B^\times \in \mathcal{F}_{\mathcal{D}}$, where $\mathcal{D}$ is any replete subcategory of $\mathcal{C}$.

Recall that when $I = \mathcal{R}_{\mathcal{C}',B^\times}$, Theorem 6.2.7 implies $F_I = B^\times \in \mathcal{F}_{\mathcal{C}'}$. If $I \subset \mathcal{R}_{\mathcal{C}',B^\times}$, we denote

$$\bar{I} := \{X \in \mathcal{R}_{\mathcal{C}',B^\times} | X \prec H \in I\},$$

i.e, $\bar{I}$ consists of the elements of $\mathcal{R}_{\mathcal{C}',B^\times}$ which are isomorphic to subquotients of elements of I. We say $\bar{I}$ is the *residual subquotient closure* of I. If $I = \bar{I}$ then I is said to be *closed under residual subquotients*. Recall that $\mathcal{R}_{\mathcal{C}',B^\times}$ consists of the trivial group and certain pseudodihedral groups. Thus if $X \in I$ then every quotient of X is isomorphic to a subgroup of X. So if we had defined $\bar{I}$ to be the subset of $\mathcal{R}_{\mathcal{C}',B^\times}$ consisting of elements isomorphic to subgroups of elements from I, it would result in the same set.

Theorem 7.1.2. *Suppose F is a subfunctor of $B^\times \in \mathcal{F}_{\mathcal{C}'}$. Set*

$$I_F := \{X \in \mathcal{R}_{\mathcal{C}',B^\times} | F(X) = B^\times(X)\},$$

then $F_{I_F} = F$ and $I_F = \overline{I_F}$. Moreover, if $\mathcal{A} = \{\bar{J} | J \subset \mathcal{R}_{\mathcal{C}',B^\times}\}$ and $\mathcal{B}$ is the set of subfunctors of $B^\times$ over $\mathcal{C}'$, let $a : \mathcal{A} \to \mathcal{B}$ be the map sending $J \mapsto F_J$ and $b : \mathcal{B} \to \mathcal{A}$ the map sending $F \to I_F$, then a and b are isomorphisms of posets, inverse to each other.

Proof. If F is trivial, then I is empty and this case is clear. Suppose F is nontrivial, there exists $X \in \mathcal{C}'$ and $u \in F(X) \subset B^\times(X)$, such that u is nontrivial. Hence, there is a $K \leqslant X$, such that $|u^K| = -1$. This implies that $\mathrm{Def}^K_{K/K} \circ \mathrm{Res}^X_K(u) = \Phi_{K/K} \in B(K/K,X)F(X) \subset F(K/K) \cong F(\{1\})$. Further, $F(\{1\}) \subset B^\times(\{1\})$. However, since $\Phi_{\{1\}} \in B(\{1\})$ generates $B^\times(\{1\})$, we have $F(\{1\}) = B^\times(\{1\})$. Thus I contains the trivial group. Moreover, by Lemma 5.2.6, we have $F_I(X) = F(X) = B^\times(X)$ for abelian X.

It is clear that $F_I \subseteq F$. Fix a nonabelian object $G \in \mathcal{C}'$. We start with the case when G has an abelian subgroup of index 2 and show $F_I(G) = F(G)$. In this case, Proposition 3.1.6

implies it is sufficient to show $\partial F_I(G) = \partial F(G)$, since having an abelian subgroup of index at most 2 is closed under taking quotients. For any object $X \in \mathcal{C}'$, we have

$$\partial F_I(X) \leqslant \partial F(X) \leqslant \partial B^\times(X).$$

We also know that $\partial B^\times(G)$ is either trivial or has $\mathbb{F}_2$-dimension 1, by Lemma 5.1.5 and Proposition 5.1.9. So we only need to show that when $\partial F(G)$ is nontrivial $\partial F_I(G)$ is also nontrivial.

When $\partial F(G)$ is nontrivial, $\partial B^\times(G)$ is nontrivial. Moreover, since $G \neq \{1\}$, if $\partial F(G)$ is nontrivial, then Proposition 5.1.9 implies G is pseudodihedral and $\Phi_G \in F(G)$. Thus, Proposition 5.2.7 tells us $B^\times(S) = B(S,G)\Phi_G \subset F(S)$, for any $S \prec G$. This implies $F(S) = B^\times(S)$, for any $S \prec G$. So I contains an isomorphic copy of any subquotient of G that is residual with respect to $B^\times$. Hence, $\Phi_S \in F_I(S)$ for all $S \prec G$ that are residual with respect to $B^\times$. Thus, applying Theorem 6.2.7

$$F(G) = B^\times(G) = \sum_S B(G,S)\Phi_S \subset F_I(G)$$

where S runs over all the subquotients in G which are residual with respect to $B^\times$. It follows that $F(G) = F_I(G)$ and in particular $\partial F(G) = \partial F_I(G)$.

We have so far shown that $F(G) = F_I(G)$, whenever $G \in \mathcal{C}'$ has an abelian subgroup of index at most 2. If G is a general object of $\mathcal{C}'$, then there is a normal subgroup $N \trianglelefteq G$ with odd index in G, containing an abelian subgroup of index at most 2. By the above argument $F_I(N) = F(N)$. Since $F \subseteq B^\times$, the isomorphism in Proposition 5.2.1 given by $\mathrm{Ten}_N^G : B^\times(N)^{G/N} \to B^\times(G)$ restricts to an isomorphism $F(N)^{G/N} \to F(G)$. Since $F_I(N)^{G/N} = F(N)^{G/N}$, we have $F_I(G) = F(G)$.

That $I = \overline{I}$ follows from Proposition 5.2.7. What is left is to establish that if $I \subset R_{\mathcal{C}',B^\times}$ is closed under residual subquotients, and

$$I' = \{X \in \mathcal{R}_{\mathcal{C}',B^\times} | F_I(X) = B^\times(X)\},$$

then $I = I'$. Again, Proposition 5.2.7 implies that $I \subset I'$. If I is empty, this is clear. Suppose I is not empty and for the sake of contradiction that there is some $G \in I'$, such that $G \notin I$. Hence

$$B^\times(G) = F_I(G) = \sum_{X \in I} B(G,X)\Phi_X.$$

However, since G is not isomorphic to a subquotient of any element of I, it follows from Remark 2.4.3 that for any $\varphi \in B(G,X)$, we have $\varphi(\Phi_X) \in (B^\times)^<(G)$. But this implies that $B^\times(G) = (B^\times)^<(G)$, which is a contradiction. Hence $I = I'$. It is easy to verify that this bijection is an isomorphism of posets. □

Remark 7.1.3. Theorem 7.1.2 has a formal generalization. If $\mathcal{D}$ is any replete subcategory of $\mathcal{C}$ contained in $\mathcal{C}'$, and $\mathcal{R}_{\mathcal{D},B^\times}$ is a complete set of residuals for $B^\times$ in $\mathcal{D}$ (for example, we can choose $\mathcal{R}_{\mathcal{D},B^\times}$ to be the set of $G \in \mathcal{R}$ such that G is an object of $\mathcal{D}$, where $\mathcal{R}$ is the set from Theorem 6.2.7), then the lattice of subfunctors of $B^\times \in \mathcal{F}_\mathcal{D}$ is isomorphic as a poset to $\{\overline{J} | J \subset \mathcal{R}_{\mathcal{D},B^\times}\}$. The proof is the same, with the exception of using the symbol "$\mathcal{D}$" whenever there is a "$\mathcal{C}'$".

In Theorem 9.5 of [3], Bouc characterized the subfunctors of $B^\times$ over the class of 2-groups (and through inflation, nilpotent groups). In our terminology, Bouc showed that the lattice of subfunctors of $B^\times$ is uniserial and the nontrivial proper subfunctors are in bijection with the sets $\overline{\{D_{2^n}\}}$ for $n > 3$. Together with Bouc's result, Theorem 7.1.2 can be used to show the structure of the lattice of subfunctors of $B^\times$ over the full subcategory of $\mathcal{C}$, whose objects are nilpotent groups or groups in $\mathcal{C}'$. Furthermore, it is easy to verify that the lattice of subfunctors of $B^\times$ is not uniserial over $\mathcal{C}'$. Additionally, Theorem 7.1.2 gives us a sufficient condition for when the biset functor $B^\times$, defined on a replete subcategory $\mathcal{D} \subset \mathcal{C}$ has uncountably many subfunctors.

Corollary 7.1.4. *Consider the following sets of groups:*

$$\mathcal{S}_0 := \{D_{2p} |\ p\ prime\ ,\ p \equiv 1\,(mod\,4)\},$$

$$\mathcal{S}_1 := \{C_2 \ltimes (C_p \times C_p) |\ C_2 \ltimes (C_p \times C_p)\ is\ pseudodihedral\ ,\ p\ prime\ ,\ p \equiv 3\,(mod\,4)\},$$

$$\mathcal{S}_2 := \{D_{2pq} |\ p \neq q;\ p,q\ prime\ ;\ p,q \equiv 3\,(mod\,4)\},$$

and

$$\mathcal{S}_r := \{D_{2rp} |\ p\ prime\ ,\ p \equiv 3\,(mod\,4)\},$$

where $r = 4$ or r is a prime congruent to 1 modulo 4. If $\mathcal{D}$ is a replete subcategory of $\mathcal{C}$ containing infinitely many objects from one of the sets S_0, S_1, S_2 or S_r, where $r = 4$ or r is a prime congruent to 1 modulo 4, then $B^\times \in \mathcal{F}_\mathcal{D}$ has uncountably many subfunctors.

Proof. Let $\mathcal{R}$ be the complete set of residuals with respect to $B^\times$ from Theorem 6.2.7. Let $\{G_p\}_{p\in\mathcal{K}}$ be the assumed set of groups from the statement, whose elements are objects of $\mathcal{D}$, indexed by the infinite set $\mathcal{K}$. For any subset $\Pi \subset \mathcal{K}$, we denote

$$I_\Pi = \overline{\{G_p\}_{p\in\Pi}} \subset \mathcal{R}.$$

For any two subsets $\Pi, \Pi' \subset \mathcal{K}$, we have $I_\Pi = I_{\Pi'}$ if and only if $\Pi = \Pi'$. It follows from Remark 7.1.3 that $F_{I_\Pi} = F_{I_{\Pi'}}$ if and only if $I_\Pi = I_{\Pi'}$. □

In fact, the sufficient condition in Corollary 7.1.4 is also a necessary condition for any replete subcategory of $\mathcal{C}$ contained in $\mathcal{C}'$. However, before we give its proof, we need a technical lemma about countability. We include a proof of it for completeness, though the reader may wish to skip it.

Before we state the following lemma, we clarify our terminology. Consider a set I of n-tuples with entries from $\mathbb{N}$ with the property that, if $(a_1,\ldots,a_n) \in I$ and $(b_1,\ldots,b_n)$ is any other n-tuple with entries in $\mathbb{N}$ such that $b_i \leq a_i$ for each $i = 1,\ldots,n$, then $(b_1,\ldots,b_n) \in I$. In this case, we say that I is *closed under the product ordering from below*. In the lemma, by $\mathbb{N}$, we mean the subset of $\mathbb{Z}$ consisting of the positive integers and 0.

Lemma 7.1.5. *Let n be a positive integer. There are only countably many subsets of $\mathbb{N}^n$ that are closed under the product ordering from below.*

Proof. We will prove this by showing that a nonempty subset of $\mathbb{N}^n$, closed under the product ordering from below, can be parametrized by a finite subset of $(\mathbb{N}_\infty)^n$, where $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$ and ∞ is some maximal element, with respect to the regular ordering. Since $(\mathbb{N}_\infty)^n$ is countable, and the finite subsets of a countable set form a countable set, the result will follow.

For each $I \subset \mathbb{N}^n$ that is closed under the product ordering from below, there is a unique $I^* = I \cup I_\infty \subset (\mathbb{N}_\infty)^n$, where

$$I_\infty = \{\mathbf{x} \in (\mathbb{N}_\infty)^n - \mathbb{N}^n |\, \mathbf{y} < \mathbf{x} \implies \mathbf{y} \in I, \forall \mathbf{y} \in \mathbb{N}^n\}.$$

Note that if I is finite, then $I_\infty = \emptyset$ and $I^* = I$. Furthermore, it is straightforward to verify that I^* is closed under the product ordering from below in $(\mathbb{N}_\infty)^n$ and $I^* \cap \mathbb{N}^n = I$. Moreover, for every $(x_1,\ldots,x_n) \in I^*$, there is a maximal element $(a_1,\ldots,a_n) \in I^*$ such that $(x_1,\ldots,x_n) \leq (a_1,\ldots,a_n)$.

Let M be a set of elements, from $(\mathbb{N}_\infty)^n$, that are pairwise noncomparable with respect to the product ordering. Set I_M to be the closure, from below, with respect to the product ordering. If $I \subset \mathbb{N}^n$ is closed under the product ordering from below, and M is the set of maximal elements in I^*, our construction guarantees that $I_M = I^*$. Thus, it suffices to show that I^* has only finitely many maximal elements.

We do this inductively on n. If $n = 1$, this is clear, since every nonempty proper subset of $\mathbb{N}$, closed under the product ordering from below has one maximal element and $\mathbb{N}^* = \mathbb{N}_\infty$ also has one maximal element. For the general case, suppose $n > 1$. Let $I \subset \mathbb{N}^n$ be a proper subset, closed under the product ordering from below, then there is a maximal $a \in \mathbb{N}$, such that $(a,\ldots,a) \in I^*$. For each $i \in \{1,\ldots,n\}$ and each $m \in \{1,\ldots,a\}$, let I_i^m be the subset of I^* consisting of all n-tuples with m in the ith component. Notice that every element of I^* is contained in at least one I_i^m, for some $i \in \{1,\ldots,n\}$ and $m \in \{0,\ldots,a\}$. Otherwise, there is an element $(x_1,\ldots,x_n) \in I^*$ where $(x_1,\ldots,x_n) \notin I_i^m$, for all $i = 1,\ldots,n$ and all $m = 0,\ldots,a$, hence $x_i > a$ for all i. But this means that $(a+1,\ldots,a+1) \in I^*$, which is a contradiction. Furthermore, for any $i = 1,\ldots,n$ and all $m = 0,\ldots,a$, the image of I_i^m in $(\mathbb{N}_\infty)^{n-1}$, created by deleting the ith entry in each element of I_i^m, is closed under the product ordering from below. Thus, by induction, I_i^m has finitely many maximal elements, all with the entry m in the ith component. If we denote by M_i^m the set of maximal elements of the set I_i^m for every $i = 1,\ldots,n$

and all $m = 0, \dots, a$, and denote

$$M_I = \bigcup_{1 \le i \le n, 0 \le m \le a} I_i^m,$$

then M_I is a finite set. Moreover, M_I will contain all the maximal elements of I^*, and the result follows.

□

Theorem 7.1.6. *Suppose $\mathcal{D}$ is a replete subcategory of $\mathcal{C}'$. The biset functor $B^\times$ over $\mathcal{D}$ has uncountably many subfunctors if and only if $\mathcal{D}$ has infinitely many objects from one of the sets S_0, S_1, S_2 or S_r, where $r = 4$ or r is a prime congruent to 1 modulo 4, from Corollary 7.1.4.*

Proof. The "if" direction follows from Corollary 7.1.4. What is left is to prove that if $\mathcal{D}$ contains only finitely many objects from each of the sets described in Corollary 7.1.4, then the lattice of subfunctors of $B^\times$ over $\mathcal{D}$ is at most countable. Suppose $\mathcal{D}$ contains only finitely many objects from each set described in Corollary 7.1.4. Let $\mathcal{U}$ denote the objects contained in $\mathcal{D}$ from the union of all these sets. Since $\mathcal{D}$ contains only finitely many objects from S_0, it follows that $\mathcal{U}$ must be finite. Notice that if $G \in \mathcal{C}'$ is residual with respect to $B^\times$ and p is an odd prime dividing the order of $|G|$, then G has a subquotient (in fact, a subgroup) X isomorphic to a group from one of the sets in Corollary 7.1.4, such that $p||X|$. Thus, by the finiteness of the set $\mathcal{U}$, there are odd primes $p_1, \dots, p_r$ such that, for any $X \in \mathcal{D}$, residual with respect to $B^\times$, we have natural numbers $n_0, n_1 \dots, n_r$ and a pseudodihedral group

$$G_{(n_0,n_1,\dots,n_r)} := C_2 \ltimes (C_{2^{n_0}} \times (C_{p_1})^{n_1} \times \cdots \times (C_{p_r})^{n_r}),$$

with $X \prec G_{(n_0,n_1,\dots,n_r)}$.

By Remark 7.1.3, we may assume that every object in $\mathcal{D}$ is a subquotient of some object which is residual with respect to $B^{\times}$, since this does not change the poset structure. Suppose $\mathcal{D}'$ is a full subcategory of $\mathcal{C}$ consisting of subquotients of the groups $G_{(n_0,n_1\ldots,n_r)}$, for any $(r+1)$-tuple of natural numbers $(n_0,n_1\ldots,n_r)$. Then $\mathcal{D}\subset\mathcal{D}'$ and the restriction functor $\mathcal{F}_{\mathcal{D}'}\to\mathcal{F}_{\mathcal{D}}$ induces a surjective morphism between the poset of subfunctors of $B^{\times}$ over $\mathcal{D}'$ and the poset of subfunctors of $B^{\times}$ over $\mathcal{D}$. Thus, it suffices to show that $B^{\times}$ over $\mathcal{D}'$ has only countably many subfunctors.

Let $\mathcal{R}$ be the complete set of residuals with respect to $B^{\times}$ in $\mathcal{C}'$ form Theorem 6.2.7. To see that $B^{\times}$ over $\mathcal{D}'$ has only countably many subfunctors, we first set $\mathcal{R}_{\mathcal{D}',B^{\times}}$ be the complete set of residual with respect to $B^{\times}$ in $\mathcal{D}'$, consisting of elements of $\mathcal{R}$ that are in $\mathcal{D}'$. Using Theorem 7.1.2, the result follows if the set $\{\bar{J}|J\subset\mathcal{R}_{\mathcal{D}',B^{\times}}\}$ is countable. To see this, let $J\subset\mathcal{R}_{\mathcal{D}',B^{\times}}$ and consider the set $\{(a_0,a_1,\ldots,a_r)\in\mathbb{N}|G_{(a_0,a_1,\ldots,a_r)}\prec G_{(n_0,n_1,\ldots,n_r)}\in\bar{J}\}$. This set is closed under the product ordering on $\mathbb{N}^{r+1}$. Moreover, this induces an injection into the set of subsets of $\mathbb{N}^{r+1}$ that are closed under the product ordering. The result follows by Lemma 7.1.5.

□

It is natural to ask if we can relax the condition in Theorem 7.1.6, for $\mathcal{D}$ to be a subcategory of $\mathcal{C}'$.

Question 7.1.7. Let $\mathcal{D}$ be any replete subcategory of $\mathcal{C}$. Is the condition in Corollary 7.1.4 a necessary condition for the biset functor $B^{\times}$ over $\mathcal{D}$ to have uncountably many subfunctors?

The next result generalizes Corollary 5.2.2.

Proposition 7.1.8. *Suppose $I \subset \mathcal{R}_{\mathcal{C}',B^\times}$ is closed under residual subquotients. Let G be an object of $\mathcal{C}'$, and H a normal subgroup in G with odd index, such that H has an abelian subgroup of index at most 2. Let $\mathcal{N}$ be the set of all normal subgroups $N \trianglelefteq H$, such that H/N is trivial, cyclic of order 2, or pseudodihedral, such that every suquotient of H/N that is residual with respect to $B^\times$ is isomorphic to an element in I. Set $\mathcal{L} = \{\mathrm{Inf}_{H/N}^{H}\Phi_{H/N}\}_{N\in\mathcal{N}}$, then $\mathcal{L}$ is a basis for $F_I(H)$ and G/H acts on $\mathcal{L}$. If $L_1,\ldots,L_k$ denote the orbit sums of this action, then $\{Ten_H^G L_1\}_{i=1}^{k}$ is a basis for $F_I(G)$.*

Proof. The only thing we need to prove is that $\mathcal{L}$ is a basis of $F_I(H)$, the rest follows by replacing $B^\times$ by F_I in the proof of Corollary 5.2.2. We can use Proposition 3.1.6 to decompose

$$F_I(H) \cong \bigoplus_{N\trianglelefteq H} \partial F_I(H/N).$$

We have $F_I(H/N) \subset B^\times(H/N)$, hence $\partial F_I(H/N) \subset \partial B^\times(H/N)$. Thus $\partial F_I(H/N)$ is trivial or $\partial F_I(H/N)$ is generated by $\Phi_{H/N}$. If X is a pseudodihedral groups such that every subquotient residual with respect to $B^\times$ is isomorphic to an element of I, then $B^\times(X) = F_I(X)$. Thus $\langle \mathrm{Inf}_{H/N}^{H}\Phi_{H/N}\rangle_{N\in\mathcal{N}} \subset F_I(H)$.

Now, suppose $M \trianglelefteq H$ and H/M has a subquotient S, such that S is isomorphic to an element $X \in \mathcal{R}_{\mathcal{C}',B^\times} - I$. For the sake of contradiction, suppose $\Phi_{H/M} \in \partial F_I(H/M)$. Then by Proposition 5.2.7,

$$B^\times(X) \cong B^\times(S) = B(S, H/M)\Phi_{H/M} \subseteq F_I(S) \cong F_I(X),$$

and by dimension, we have $B^\times(X) = F_I(X)$. Hence $X \in I$, which is a contradiction. Therefore, $\langle \mathrm{Inf}_{H/N}^{H}\Phi_{H/N}\rangle_{N\in\mathcal{N}}$ must be all of $F_I(H)$ □

In general, computing the dimension of the evaluations of simple biset functors is difficult. In the next section the explicit form of Proposition 7.1.8 will give us an easy way to compute the dimension of $S_{G,\mathbb{F}_2}(H)$ when $G,H \in \mathcal{C}'$ and G is residual with respect to $B^\times$.

7.2 Composition Factors

We end this Chapter by detailing which simple biset functors show up as composition factors of $B^\times \in \mathcal{F}_{\mathcal{C}'}$. Before we state the following theorem, we clarify a technicality. Let $\mathcal{D}$ be any replete subcategory of $\mathcal{C}$. Note that $B^\times(G)$ is always an elementary abelian 2-group, thus we may view $B^\times$ as a functor in $B^\times \in \mathcal{F}_{\mathcal{D},\mathbb{F}_2}$. So there is no ambiguity when we discuss the multiplicity of composition factors of $B^\times$.

Theorem 7.2.1. *The composition factors of $B^\times \in \mathcal{F}_{\mathcal{C}'}$ are parametrized exactly by the simple biset functors $S_{G,\mathbb{F}_2}$, where $G \in \mathcal{R}_{\mathcal{C}',B^\times}$. Furthermore, each composition factor has multiplicity 1.*

Proof. Let $F_{I'} \subseteq F_I \subseteq B^\times$ be two subfunctors of $B^\times \in \mathcal{F}_{\mathcal{C}'}$ such that $F_I/F_{I'}$ is simple, where I and I' are the corresponding subsets of $\mathcal{R}_{\mathcal{C}',B^\times}$ from Theorem 7.1.2. We may then assume that $F_I/F_{I'} \cong S_{G,V}$ where $G \in \mathcal{C}'$ and V is a simple $\mathbb{F}_2\mathrm{Out}(G)$-module, such that $S_{G,V}(G) \cong V$. By Theorem 7.1.2, $I' \subsetneq I$. Moreover, we claim that $I - I'$ is a one element set consisting of a group that is isomorphic to G. Indeed, suppose $H_1, H_2 \in I - I'$ such that $H_1 \neq H_2$. We may assume that H_1 is not a subquotient of H_2. Let $I'' = \overline{I' \cup \{H_2\}}$. Thus $I' \subsetneq I'' \subsetneq I$, which implies that $F_I/F_{I'}$ is not simple, by Theorem 7.1.2. Furthermore, if H is the unique element in $I - I'$, then for any $X \in \mathcal{C}'$ such that $|X| < |H|$, we have $F_{I'}(X) = F_I(X)$, by Proposition 7.1.8. Thus $S_{G,V}(X)$

is trivial. Again by by Proposition 7.1.8, we have $F_{I'}(H) \subsetneq F_I(H)$. So H is a minimal object for $S_{G,V}$ and by Remark 2.5.9, $H \cong G$. We can now assume $H = G$. Through evaluation

$$V \cong S_{G,V}(G) \cong F_I/F_{I'}(G) \cong B^{\times}(G)/(B^{\times})^{<}(G),$$

has dimension 1 as an $\mathbb{F}_2$-space by Proposition 6.2.2 and Proposition 5.1.5, since G is residual. Furthermore, $B^{\times}(G)/(B^{\times})^{<}(G)$ is generated by the image of the element $\Phi_G \in B^{\times}(G)$. By the proof of Proposition 4.3.2 in [4], the action of $\mathrm{Out}(G)$ on V as an $\mathbb{F}_2\mathrm{Out}(G)$-module, is given by $f \cdot v = \mathrm{Iso}(f)(v)$ for any $f \in \mathrm{Out}$ and $v \in V$. Since Φ_G is the unique faithful element in $B^{\times}(G)$, it is fixed by $\mathrm{Iso}(f)$ for any $f \in \mathrm{Out}(G)$. Thus V is isomorphic to the trivial $\mathbb{F}_2\mathrm{Out}(G)$-module.

That $S_{G,\mathbb{F}_2}$ has multiplicity 1 is easy to see by a simple construction. Let $I = \overline{\{G\}} \subset \mathcal{R}_{\mathcal{C}',B^{\times}}$. Then choose $\{\{1\}\} = I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_n \subsetneq I_{n+1} = I$ to be a filtration such that $|I_{i+1} - I_i| = 1$ and I_i is closed under residual subgroups, for each $i = 1, \ldots, n$. Note that this forces $I_{n+1} - I_n = \{G\}$. Thus

$$F_{I_1} \subseteq F_{I_1} \subset F_{I_2} \subseteq F_{I_2} \subset \cdots \subseteq F_{I_n} \subset F_{I_{n+1}} \subseteq B^{\times}$$

is a composition series for $B^{\times}$ over G. It is easy to verify that $F_{I_{n+1}}/F_{I_n}$ is the only composition factor isomorphic to $S_{G,\mathbb{F}_2}$, since $G \notin I_i$ for any $i = 1, \ldots, n$. □

Chapter 8

Applications to Simple Biset Functors

8.1 Computing $S_{G,\mathbb{F}_2}(H)$ for G residual with respect to $B^\times$

For this section, we again set $\mathcal{R}_{\mathcal{C}',B^\times}$ to be a complete set of residuals for $B^\times$ in $\mathcal{C}'$.

If G and H are finite groups and k is a field, the following is a characterization due to Bouc of the k-module $S_{G,k}(H)$.

Proposition 8.1.1 ([4], proposition 4.4.6, page 71)**.** *Let $\mathcal{D}$ be a replete subcategory of $\mathcal{C}$. For any objects G and H of $\mathcal{D}$, let $kB_H(G)$ be the k-vector space with basis the set of conjugacy classes of sections (T,S) of G such that $T/S \cong H$.*

Then the module $S_{H,k}(G)$ is isomorphic to the quotient of $kB_H(G)$ by the radical of *the k-valued symmetric bilinear form on $kB_H(G)$ defined by*

$$\langle (B,A)|(T,S)\rangle_H = |\{h \in B\backslash G/T \,|\, (B,A) - ({}^hT, {}^hS)\}|_k,$$

where $(B,A) - (T,S)$ means that

$$B \cap S = A \cap T \qquad (B \cap T)A = B \qquad (B \cap T)S = T.$$

In particular, the k-dimension of $S_{H,k}(G)$ is equal to the rank of this bilinear form.

It is generally difficult to determine the k-dimension of $S_{G,k}(H)$ in terms of other data about G and H. However, we can apply the results from the the previous section to determine this for $S_{G,\mathbb{F}_2}(H)$, where $G \in \mathcal{R}_{\mathcal{C}',B^\times}$ and $H \in \mathcal{C}'$.

Theorem 8.1.2. *Suppose $G \in \mathcal{R}_{\mathcal{C}',B^\times}$ and $H \in \mathcal{C}'$. Set $r = \dim_{\mathbb{F}_2}(F_I(H))$ and $l = \dim_{\mathbb{F}_2}(F_J(H))$, where $I = \overline{\{G\}}$, $J = I - \{G\}$. Then $\dim_{\mathbb{F}_2}(S_{G,\mathbb{F}_2}(H)) = r - l$.*

Proof. Notice that $\bar{J} = J$. Since $J \subset I$, Theorem 7.1.2 F_I/F_J is simple. Proposition 7.1.8 implies that G is a minimal object for F_I/F_J. Thus by Remark 2.5.9 and Theorem 7.2.1, $F_I/F_J \cong S_{G,\mathbb{F}_2}$ and the result follows. □

Theorem 8.1.2, together with Proposition 7.1.8, gives us a way to compute $S_{G,\mathbb{F}_2}(H)$, where $G \in \mathcal{R}_{\mathcal{C}',B^\times}$ and $H \in \mathcal{C}'$. To illustrate this, we cover the cases where G is trivial or dihedral and H is dihedral.

Example 8.1.3. Consider the dihedral group D_{2k}, where $k \geq 3$. Let $p_1, \ldots, p_r$ denote the prime divisors of k that are congruent to 3 modulo 4. Set m_i to be the power of the p_i-part of k, for

$i = 1, \dots, r$ and m_0 the power of the 2-part of k. We show that

$$\dim_{\mathbb{F}_2}(S_{\{1\},\mathbb{F}_2}(D_{2k})) = \begin{cases} 2 + m_1 + \cdots + m_r & \text{if} \quad m_0 = 0 \\ 4 + 2(m_1 + \cdots + m_r) & \text{if} \quad m_0 = 1 \\ 5 + 2(m_1 + \cdots + m_r) & \text{if} \quad m_0 > 1 \end{cases}$$

By Proposition 7.1.8, $\dim_{\mathbb{F}_2}(S_{\{1\},\mathbb{F}_2}(D_{2k}))$ is equal to the number of normal subgroups $N \trianglelefteq D_{2k}$, such that D_{2k}/N is trivial, cyclic of order 2, or dihedral (nonabelian), and there are no nontrivial subquotients of D_{2k}/N residual with respect to $B^\times$. If D_{2k}/N is dihedral and there are no nontrivial subquotients of D_{2k}/N residual with respect to $B^\times$, Theorem 6.2.7 implies that $D_{2k}/N \cong D_8$ or $D_{2k}/N \cong D_{2l}$, where $l = 2^\varepsilon p^s$, where p is a prime congruent to 3 modulo 4, and $\varepsilon \in \{0, 1\}$. The following cases are straightforward to verify:

- If $m_0 = 0$, then there is one normal subgroup of D_{2k} with trivial quotient, one normal subgroup of D_{2k} of index 2, and $m_1 + \cdots + m_r$ normal subgroups $N \trianglelefteq D_{2k}$, such that D_{2k}/N is dihedral and there are no nontrivial subquotients of D_{2k}/N residual with respect to $B^\times$.

- If $m_0 = 1$, then there is one normal subgroup of D_{2k} with trivial quotient, three normal subgroups of D_{2k} of index 2, and $2(m_1 + \cdots + m_r)$ normal subgroups $N \trianglelefteq D_{2k}$, such that D_{2k}/N is dihedral and there are no nontrivial subquotients of D_{2k}/N residual with respect to $B^\times$.

- If $m_0 > 1$, then there is one normal subgroup of D_{2k} with trivial quotient, three normal subgroups of D_{2k} of index 2, and $2(m_1+\cdots+m_r)+1$ normal subgroups $N \trianglelefteq D_{2k}$, such that D_{2k}/N is dihedral and there are no nontrivial subquotients of D_{2k}/N residual with respect to $B^\times$.

Set $D_2 \cong C_2$ and $D_4 \cong C_2 \times C_2$. It is straightforward to verify $\dim_{\mathbb{F}_2}(S_{\{1\},\mathbb{F}_2}(D_2)) = 2$ and $\dim_{\mathbb{F}_2}(S_{\{1\},\mathbb{F}_2}(D_4)) = 4$. There is then a function $s : \mathbb{N} \to \mathbb{N}$ defined as $s(n) = \dim_{\mathbb{F}_2}(S_{\{1\},\mathbb{F}_2}(D_{2n}))$. Let ϕ denote Euler's ϕ-function, and set m to be the number of positive divisors $d|n$, such that $\phi(d) \equiv 2 \pmod 4$. Recall that for any $x \in \mathbb{N}$, $\phi(x) \equiv 2 \pmod 4$ if and only if $x = 4$, x is the power of a prime congruent to 3 modulo 4, or x is twice the power of a prime congruent to 3 modulo 4. Thus $s(n) = m+2$ if n is odd and $s(n) = m+4$ if n is even.

Example 8.1.4. Suppose D_{2n} is a dihedral group residual with respect to $B^\times$. If n is not a power of 2, by Theorem 6.2.7 we can decompose

$$n = 2^{n_0} p_1 \cdots p_r$$

where $n_0 \neq 1$, and the p_i pairwise coprime odd primes, for $i = 1, \ldots, r$. Moreover, if $n_0 = 0$ and $r = 1$, then $p_1 \equiv 1 \pmod 4$.

Let $k \geq 3$ and consider the dihedral group D_{2k}. Let m_0 denote the exponent of the 2-part of k. If n is a power of 2 (recall that in this case $n \geq 8$), then we set $m = 1$. Otherwise, set $m = m_1 \cdots m_r$, where m_i denotes the power of the p_i-part of k, for $i = 1, \ldots, r$. We show that

$$\dim_{\mathbb{F}_2}(S_{D_{2n},\mathbb{F}_2}(D_{2k})) = \begin{cases} 0 & \text{if} \quad n \nmid k \\ m & \text{if} \quad n|k;\ n_0 \neq 0 \text{ or } m_0 = 0 \\ 2m & \text{if} \quad n|k;\ n_0 = 0 \text{ and } m_0 \neq 0 \end{cases}$$

If $n \nmid k$, then D_{2n} does not show up as a subquotient of D_{2k}, hence by Remark 2.5.9, $S_{D_{2n},\mathbb{F}_2}(D_{2k}) = 0$.

Suppose $n|k$. Set $I = \overline{\{D_{2n}\}}$ and $J = I - \{D_{2n}\}$. Theorem 8.1.2 and Proposition 7.1.8 tell us that $\dim_{\mathbb{F}_2}(S_{D_{2n},\mathbb{F}_2}(D_{2k}))$ is equal to the number of normal subgroups $N \trianglelefteq D_{2k}$, such that every subquotient, residual with respect to $B^\times$, of D_{2k}/N is in I and at least one is not in J. In other words, every subquotient, residual with respect to $B^\times$, of D_{2k}/N is in I and D_{2n} is a subquotient of D_{2k}/N. When $n_0 \neq 0$, this means that D_{2k}/N must be isomorphic to a dihedral group D_{2l_N} such that

$$l_N = 2^{n_0} p_1^{n_1} \dots p_r^{n_r},$$

with $1 \le n_i \le m_i$, $i = 1, \dots, r$. When $n_0 = 0$, this means that D_{2k}/N must be isomorphic to a dihedral group D_{2l_N}

$$l_N = 2^{\varepsilon} p_1^{n_1} \dots p_r^{n_r},$$

with $1 \le n_i \le m_i$, $i = 1, \dots, r$ and $\varepsilon \in \{0, 1\}$. By properties of dihedral groups, there is a bijective correspondence between normal subgroups of $N \trianglelefteq D_{2k}$ of this type and divisors $d|k$, such that $\frac{k}{d} = l_N$. The formula for $\dim_{\mathbb{F}_2}(S_{D_{2n},\mathbb{F}_2}(D_{2k}))$ follows by counting such divisors of k.

8.2 Surjectivity of the Exponential Map $B \to B^\times$

For any finite group G, we define the exponential map

$$\varepsilon_G : B(G) \to B^\times(G)$$

to be the group homomorphism induced by sending

$$[G/H] \mapsto \mathrm{Ten}_H^G(-1),$$

for any $H \leqslant G$.

Remark 8.2.1. It is straightforward to show that the collection of all these group morphisms ε_G is a morphism $\varepsilon : B \to B^\times$ of biset functors. Yoshida proves this in Lemma 3.2 of [19]. Yalçın also discusses this map in Section 7 of [17]. The way we define it in this paper is in the spirit of Bouc 9.8 of [3].

The next result is proved by Bouc 9.7 in [3] for p-groups. It is likely known to experts but we include a proof for the reader, since we make use of it to determine when the exponential map is surjective for objects in $\mathcal{C}'$.

Proposition 8.2.2. *Let G be a finite group, then* $\mathrm{Im}(\varepsilon_G) \cong S_{1,\mathbb{F}_2}(G)$ *as* $\mathbb{F}_2$*-modules.*

Proof. If $k = \mathbb{F}_2$ and $H = \{1\}$, then in reference to Proposition 8.1.1 we can identify $\mathbb{F}_2 B_1(G)$ with $\mathbb{F}_2 B(G)$ by associating (K,K) to $[G/K] \in \mathbb{F}_2 B(G)$. Since $B^\times(G)$ is an $\mathbb{F}_2$ vector space, $\varepsilon_G : B(G) \to B^\times(G)$ is equal to the map $\overline{\varepsilon} \circ \pi$, where $\pi : B(G) \to \mathbb{F}_2 B(G)$ is the projection map and $\overline{\varepsilon} : \mathbb{F}_2 B(G) \to B^\times(G)$ is again the group homomorphism induced by sending $[G/H] \to \mathrm{Ten}_H^G(-1)$. It suffices to show that the kernel of $\overline{\varepsilon}$ is equal to the radical of the symmetric bilinear form in Proposition 8.1.1, on $\mathbb{F}_2 B_1(G)$. For $\mathbb{F}_2 B_1(G)$, this form is defined by

$$\langle (K,K) | (S,S) \rangle_1 = |\{h \in K \backslash G / S\}|_{\mathbb{F}_2},$$

since the requirement that $(K,K) - ({}^hS, {}^hS)$, is true for any $K, S \leqslant G$ and $h \in G$. Thus, if $K_1, \ldots, K_n$ are subgroups of G and $X = (K_1,K_1) + \cdots + (K_n,K_n) \in \mathbb{F}_2 B_1(G)$, then X is in the radical our bilinear form if and only if

$$\sum_{i=1}^{n} |\{h \in S\backslash G/K_i\}|$$

is even for all subgroups $S \leqslant G$. Similarly by the formula in Remark 4.1.3 the element associated to X in $\mathbb{F}_2 B(G)$ is in the kernel of of ε_G if and only if

$$\sum_{i=1}^{n} |\{h \in S\backslash G/K_i\}|$$

is even for all subgroups $S \leqslant G$. □

We can now determine when the exponential map $\varepsilon_G : B(G) \to B^{\times}(G)$ is surjective for $G \in \mathcal{C}'$.

Theorem 8.2.3. *Suppose $G \in \mathcal{C}'$. Let $N \trianglelefteq G$ be a normal subgroup with odd index having an abelian subgroup of index at most 2. Let $\mathcal{R}_{\mathcal{C}',B^{\times}}$ be a complete set of residuals for $B^{\times}$ in $\mathcal{C}'$. Then $\varepsilon_G : B(G) \to B^{\times}(G)$ is surjective if and only if no nontrivial quotient of N is isomorphic to an element of $\mathcal{R}_{\mathcal{C}',B^{\times}}$.*

Proof. By Proposition 7.1.8, we have $S_{1,\mathbb{F}_2}(G) = B^{\times}(G)$ if and only if no nontrivial subquotient of a pseudodihedral quotient of N is isomorphic to an element of $\mathcal{R}_{\mathcal{C}',B^{\times}}$.

Suppose $M \trianglelefteq N$, such that N/M is pseudodihedral and N/M has a nontrivial subquotient isomorphic to an element of $\mathcal{R}_{\mathcal{C}',B^{\times}}$. Lemma 5.1.8 implies that N/M has a quotient isomorphic to a nontrivial element of $\mathcal{R}_{\mathcal{C}',B^{\times}}$, thus N has a nontrivial quotient isomorphic to an element of $\mathcal{R}_{\mathcal{C}',B^{\times}}$. It follows that N/M will have a subquotient isomorphic to an element of $\mathcal{R}_{\mathcal{C}',B^{\times}}$ if and only if N/M has a quotient isomorphic to an element of $\mathcal{R}_{\mathcal{C}',B^{\times}}$.

Hence $S_{1,\mathbb{F}_2}(G) = B^\times(G)$ if and only if no nontrivial quotient of N is isomorphic to an element of $\mathcal{R}_{\mathcal{C}',B^\times}$. The result follows from Proposition 8.2.2. □

A particularly nice case of the above result is when we consider dihedral groups. In the following result, $\phi : \mathbb{N} \to \mathbb{N}$ denotes Euler's ϕ-function.

Corollary 8.2.4. *For any positive integer $n \geq 3$, $\varepsilon_{D_{2n}} : B(D_{2n}) \to B^\times(D_{2n})$ is surjective if and only if $\phi(n) \equiv 2\,(mod\,4)$.*

Proof. Proposition 8.2.2, implies that $\varepsilon_{D_{2n}}$ will be surjective if and only if

$$\dim_{\mathbb{F}_2}(S_{\{1\},\mathbb{F}_2}(D_{2n})) = \dim_{\mathbb{F}_2}(B^\times(D_{2n})).$$

Recall that by Example 5.2.5, if we use $d : \mathbb{N} \to \mathbb{N}$, to denote the function that counts the number of positive divisors of any $n \in \mathbb{N}$, then $\dim_{\mathbb{F}_2}(B^\times(D_{2n})) = d(n)+1$ if n is odd, and $\dim_{\mathbb{F}_2}(B^\times(D_{2n})) = d(n)+2$ if n is even. By Example 8.1.3, the function $s : \mathbb{N} \to \mathbb{N}$, where $s(n) = S_{\{1\},\mathbb{F}_2}(D_{2n})$, is equal to the number of positive divisors $d|n$, such that $\phi(d) \equiv 2\,(\mathrm{mod}\,4)$, plus 2 or 4, depending on whether n is odd or even, respectively. If $n \geq 3$, one can verify that $s(n) = \dim_{\mathbb{F}_2}(B^\times(D_{2n}))$ if and only if $\phi(n) \equiv 2\,(\mathrm{mod}\,4)$.

□

Chapter 9

More Residual Groups

We would like to have a complete list of finite groups that are residual with respect to $B^{\times}$. Theroem 6.2.7 shows that the smallest group, residual with respect to $B^{\times}$ is D_{10}. One might wonder naively if Theroem 6.2.7 describes all of them. Unfortunately, this is not so. In this chapter we show that S_4 and A_5 are both residual with respect to $B^{\times}$. S_4 is the smallest group that falls outside of the list in Theroem 6.2.7, and A_5 is the smallest simple group. It would be of interest to answer the following questions:

Question 9.0.1. Are all nonabelian simple groups residual with respect to $B^{\times}$?

Question 9.0.2. Are all groups S_n, with $n \geq 4$, residual with respect to $B^{\times}$?

Lemma 9.0.3. *Let G be a finite group and $\mathcal{S}_G := \{H_1, \ldots, H_n\}$ a set of representatives of subgroups of G with respect to conjugation. Let $M = (a_{ij})$ be the matrix over $\mathbb{F}_2$ with entries defined by*

$$a_{ij} = \sum_{[H_i \backslash G / H_j]} 1 \pmod 2.$$

Then

$$\mathrm{rank}_{\mathbb{F}_2}(M) = \dim_{\mathbb{F}_2}(\mathrm{Im}\varepsilon_G) = \dim_{\mathbb{F}_2}(S_{\{1\},\mathbb{F}_2}(G)).$$

Proof. The first equality follows from Remark 4.1.3 (4.1) and the definition of the exponential map $B(G) \to B^\times(G)$. The second equality comes from Proposition 8.2.2. □

Lemma 9.0.4. *Let G be a nontrivial finite group. Suppose for every proper subquotient H of G we have that* $\varepsilon_H : B(H) \to B(H)^\times$ *is surjective. Then G is not residual with respect to* $B^\times$ *if and only if* $\varepsilon_H : B(G) \to B(G)^\times$ *is surjective.*

Proof. If $\varepsilon_H : B(G) \to B(G)^\times$ is surjective, then G is clearly not residual. Conversely, say G is not residual. Then

$$B^\times(G) = (B^\times)^<(G) = \sum_H B(G,H)B^\times(H)$$

as H runs over proper subquotients of G. It suffices to check that $\mathrm{Ten}_H^G(u) \in \mathrm{Im}(\varepsilon_G)$ if H is a subgroup of G and $u \in B^\times(H)$ and that $\mathrm{Inf}_H^G(u) \in \mathrm{Im}(\varepsilon_G)$ if $u \in B^\times(H)$ and there is a normal subgroup N such that $G/N \cong H$. The first case is obvious by our assumption and transitivity of induction. For the second case, because $u \in \mathrm{Im}(\varepsilon_{G/N})$ we can write

$$u = \mathrm{Ten}_{G/N}^{G/N}(-1)^i \mathrm{Ten}_{H_1/N}^{G/N}(-1) \cdots \mathrm{Ten}_{H_r/N}^{G/N}(-1),$$

where $r \in \mathbb{N}_0$, $i \in \{0,1\}$ and H_j are proper subgroups of G for $j = 1,\ldots,r$. Notice that $\mathrm{Ten}_{G/N}^{G/N}(-1)^i = (-1)^i$ and $\mathrm{Inf}_{G/N}^G((-1)^i) = (-1)^i = \mathrm{Ten}_G^G(-1)^i$. Hence

$$\mathrm{Inf}_{G/N}^G(u) = \mathrm{Ten}_G^G(-1)^i \mathrm{Ten}_{H_1}^G(\mathrm{Inf}_{H_1/N}^{H_1}(-1)) \cdots \mathrm{Ten}_{H_r}^G(\mathrm{Inf}_{H_r/N}^{H_r}(-1))$$

$$\mathrm{Ten}_G^G(-1)^i \mathrm{Ten}_{H_1}^G(-1) \cdots \mathrm{Ten}_{H_r}^G(-1) \in \mathrm{Im}(\varepsilon_G),$$

where the first equality comes from the commutation properties of induction and inflation bisets and the second equality from the fact that inflation on the unit group of the Burnside ring comes from restricting the inflation ring morphism on the Burnside ring. The result follows. □

We can relax the hypothesis of Lemma 9.0.4 to just knowing that the exponential map is surjective on maximal subgroups of G and minimal normal subgroups of G. Lemma 9.0.3 gives us a straightforward way to compute the $\mathbb{F}_2$-dimension of the image of the exponential map. Joining this with the previous lemma, we get a tool to find *minimal groups residual with respect to* $B^{\times}$, that is, groups G that are residual with respect to $B^{\times}$, but any proper subquotient H of G is not residual with respect to $B^{\times}$.

Proposition 9.0.5. *S_4 is residual with respect to $B^{\times}$.*

Proof. We note that our calculations for this proof are run through the computer algebra system GAP. In GAP, the rank of the matrix in Lemma 9.0.3 is easily computable for a given group G, and this computation is straightforward to verify. Additionally, we make use of the GAP algorithm for the rank of the unit group of the Burnside ring in [11] and compare our results.

The maximal subgroups of S_4 are isomorphic to D_6, D_8, and A_4 and S_4 has a unique minimal normal subgroup, whose factor group is isomorphic to D_6. GAP computes that the dimensions of the images of the exponential map of D_6, D_8, and A_4 are 3, 5, and 2 respectively, which is the rank of $B^{\times}(D_6)$, $B^{\times}(D_8)$, and $B^{\times}(A_4)$ respectively, by [11]. Note that, again by GAP and [11], $\dim_{\mathbb{F}_2}(\operatorname{Im}\varepsilon_{S_4}) = 5$ but $\dim_{\mathbb{F}_2}(B^{\times}(S_4)) = 6$. Thus, we can apply Lemma 9.0.4 and see that S_4 is residual with respect to $B^{\times}$. □

We take a similar approach to show that A_5 is also residual with respect to $B^{\times}$. Since

A_5 is simple, then

$$(B^\times)^<(A_5) = \sum_H B(G,H)B^\times(H)$$

where H runs over a set of representatives of subgroups of A_5, with respect to conjugation. In fact, we can refine this to

$$(B^\times)^<(A_5) = \sum_{H\in\mathcal{S}_{A_5}^{\max}} \mathrm{Im}(\mathrm{Ten}_H^G),$$

where $\mathcal{S}_{A_5}^{\max}$ a set of representatives of maximal subgroups of A_5, with respect to conjugation. So it is enough to show that

$$\sum_{H\in\mathcal{S}_{A_5}^{\max}} \mathrm{Im}(\mathrm{Ten}_H^G) \neq B^\times(A_5)$$

in order to establish that A_5 is residual with respect to $B^\times$.

Proposition 9.0.6. *A_5 is residual with respect to $B^\times$.*

Proof. A_5 has three maximal subgroups up to conjugacy. There isomorphism classes are A_4, D_6, and D_{10}. The exponential map for A_4 and D_6 is surjective and easy computation verifies the image of the exponential map for D_{10} has codimension 1 with complement generated by $\Phi_{D_{10}}$ (see part A of the proof of Proposition 6.2.6). Thus as $\mathbb{F}_2$-spaces, we have

$$(B^\times)^<(A_5) = \sum_{H\in\mathcal{S}_{A_5}^{\max}} \mathrm{Im}(\mathrm{Ten}_H^G) = \mathrm{Im}(\varepsilon_{A_5}) + \langle \mathrm{Ten}_{D_{10}}^{A_5}(\Phi_{D_{10}})\rangle.$$

Thus, $\dim_{\mathbb{F}_2}((B^\times)^<(A_5)) \leq \dim_{\mathbb{F}_2}(\mathrm{Im}((\varepsilon_{A_5}))) + 1$. However, by computation in GAP we have $\dim_{\mathbb{F}_2}(\mathrm{Im}(\varepsilon_{A_5})) = 3$ and by [11] we have $\dim_{\mathbb{F}_2}(B^\times(A_5)) = 5$. It follows that A_5 is residual with respect to $B^\times$. □

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