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Some Classification Results for Symplectic Fillings of Contact 3-Manifolds

A dissertation submitted in partial satisfaction
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by

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ABSTRACT OF THE DISSERTATION

Some Classification Results for Symplectic
Fillings of Contact 3-Manifolds

by

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We define *splitting surfaces* in contact manifolds, and develop a technique for decomposing the strong or exact symplectic fillings of a contact manifold which admits such a surface, using Eliashberg's strategy of filling by holomorphic discs. We then apply this technique to the symplectic filling classification problem for several families of contact manifolds. In particular, we complete the classification of exact fillings for lens spaces, virtually overtwisted torus bundles, and virtually overtwisted circle bundles over Riemann surfaces. We also produce classification results for contact manifolds obtained by surgery on Legendrian negative cables, and for large families of contact structures on Seifert fibered spaces.

The dissertation of Austin Christian is approved.

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Ko Honda, Committee Chair

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To Kristin,

for everything

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Note. Chapter 5 is a rewrite of [Chr21]. The main theorem of Chapter 3 was obtained jointly with Michael Menke, and first appeared in [CM19]. Chapters 4, 6, 7, and 8 are based on [CL20], which is joint work with Youlin Li. I thank my coauthors for allowing this material to be reproduced here, as well as an anonymous referee for helpful comments.

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CHAPTER 1

Introduction

It is often said that contact geometry is an odd-dimensional cousin of symplectic geometry. One manifestation of this mantra occurs when a symplectic manifold endows its boundary with a contact structure. For instance, say we have a symplectic manifold-with-boundary (W, ω) and a neighborhood U of the boundary ∂W. If there exist a 1-form λ and vector field Z on U such that $\iota_Z \omega = \lambda$ and $\mathcal{L}_Z \omega = \omega$, with Z pointing transversely out of ∂W, then $\ker \lambda$ gives us a contact structure on ∂W. In this case, we say that (W, ω) is a *strong symplectic filling* of the closed contact manifold $(\partial W, \ker \lambda)$.

In Chapter 2.3 we will additionally define *weak symplectic fillings* and *exact symplectic fillings*. Each of these definitions leads to a classification problem: given a contact manifold (M, ξ), can we enumerate the weak, strong, or exact symplectic fillings of (M, ξ)? The purpose of this thesis is to answer questions of this type for a variety of contact manifolds.

Our primary tool for classifying fillings is Theorem 3.2, developed in Chapter 3. This theorem extends [Men18, Theorem 1.1.1] of Menke and allows us to reduce the classification problem for one contact manifold to the same problem for a (hopefully simpler) contact manifold. Namely, Theorem 3.2 tells us that when our contact manifold (M, ξ) admits a *splitting surface*, a symplectic handle may be removed from any of the strong or exact fillings of (M, ξ) to produce a new symplectic manifold which either strongly or exactly fills its boundary. In case the fillings of this new boundary have been classified, we can often lift this classification to an analogous result for (M, ξ). Theorem 3.2 is inspired by a similar result of Eliashberg for contact manifolds obtained by connected sum, and we prove Theorem 3.2 using Eliashberg's *filling by holomorphic discs* technique.

Once Theorem 3.2 is in hand, we set about using it to classify the symplectic fillings of several families of contact manifolds. We start in Chapter 4 by completing the classification of exact fillings of lens spaces, up to diffeomorphism. Lisca classified the exact fillings of *universally tight* lens spaces in [Lis08], and we use Theorem 3.2 to reduce the same problem for *virtually overtwisted* lens spaces to the universally tight case. Our techniques produce a diagrammatic calculus by which one may construct all exact fillings of a given virtually overtwisted lens space from the fillings of a collection of universally tight lens spaces.

In Chapter 5 we reduce the classification of fillings for virtually overtwisted torus bundles to the corresponding classification for lens spaces — the problem which is settled in Chapter 4. We continue in Chapters 6, 7, and 8 to apply Theorem 3.2 to other families of contact manifolds. In Chapter 6 we consider contact manifolds obtained by Legendrian surgery along negative cables of Legendrian knots which have been stabilized both positively and negatively. Chapter 7 sees the classification problem for several families of Seifert fibered spaces reduced to the same problem for lens spaces, and Chapter 8 completes the classification of fillings for virtually overtwisted contact structures on circle bundles over Riemann surfaces.

CHAPTER 2

Background

This chapter introduces the recurring characters of the thesis, and collects several important results which will be cited by later chapters.

2.1 Contact manifolds

The starting point for every discussion in this thesis is a given *contact manifold*, a notion we now define.

Definition. A *contact manifold* is a pair (M, ξ) consisting of a smooth $(2n+1)$-manifold M and a maximally non-integrable hyperplane field $\xi \subset TM$, which we call the *contact structure* on M. If there is a 1-form α on M such that $\xi = \ker \alpha$, then we say that ξ is *cooriented*, and that α is a *contact form* for ξ.

Remark. Unless otherwise stated, we will assume that all contact structures are cooriented. Moreover, we will assume that M is oriented, and that ξ is *positive*, meaning that for any contact form α for ξ, we have $\alpha \wedge (d\alpha)^n > 0$.

Example 2.1. The *standard contact structure* ξ_{std} on $\mathbb{R}^{2n+1}$ is the kernel of the 1-form

$$\alpha_{\text{std}} = dz - \sum_{i=1}^{n} y_i \, dx_i,$$

where $\mathbb{R}^{2n+1}$ has coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n, z$.

Rather famously, the standard contact structure on $\mathbb{R}^{2n+1}$ is in some sense the only contact structure, locally.

Theorem 2.2 (Darboux's theorem). *Let (M, ξ) be a contact manifold of dimension $2n+1$. For any point p in M and any contact form α for ξ, there are coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n, z$ on a neighborhood $U \subset M$ of p such that*

$$\alpha|_U = dz - \sum_{i=1}^{n} y_i \, dx_i,$$

with p having coordinates $(0, \ldots, 0)$.

In contrast with the situation for symplectic structures on even-dimensional spheres, all odd-dimensional spheres admit a standard contact structure.

Example 2.3. The *standard contact structure* ξ_{std} on the sphere S^{2n+1} is obtained by realizing S^{2n+1} as the unit sphere in $\mathbb{R}^{2n+2}$ and restricting the contact form

$$\alpha_{\text{std}} = \sum_{i=1}^{n+1} (x_i \, dy_i - y_i \, dx_i),$$

where $\mathbb{R}^{2n+2}$ has coordinates $x_1, y_1, \ldots, x_{n+1}, y_{n+1}$.

We will generally be interested in two notions of equivalence for contact structures: *contactomorphism* and *contact isotopy*.

Definition. A *contactomorphism* between contact manifolds (M_0, ξ_0) and (M_1, ξ_1) is a diffeomorphism $\varphi \colon M_0 \to M_1$ with the property that $\varphi_*(\xi_0) = \xi_1$. Given a contact manifold (M, ξ), we say that another contact structure ξ' on M is *isotopic* to ξ if there exists a contactomorphism $\varphi \colon (M, \xi) \to (M, \xi')$ which is isotopic to the identity on M.

Example 2.4. For any point p in the standard contact sphere $(S^{2n+1}, \xi_{\text{std}})$, one can show that the contact manifolds $(S^{2n+1} \setminus \{p\}, \xi_{\text{std}})$ and $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$ are contactomorphic.

Remark. Though the above definitions are stated for contact manifolds of any (odd) dimension, we will restrict our attention to contact manifolds of dimension three, unless otherwise specified.

With these notions of equivalence established, we immediately encounter the classification question: how many contact structures exist on a given (closed, oriented) smooth manifold M, up to either contactomorphism or isotopy? Eliashberg discovered in [Eli89] that there is an important class of contact structures for which the classification question is answered (up to isotopy) by purely topological considerations.

Definition. Let (M, ξ) be a contact 3-manifold. We say that the contact structure ξ is *overtwisted* if there is a smoothly embedded disc $D \subset M$ for which the distributions ξ and TD agree along ∂D. If no such embedded disc exists, we say that ξ is *tight*.

Eliashberg showed that the contact isotopy class of an overtwisted contact structure is determined by its homotopy class as a hyperplane distribution.

Theorem 2.5 ([Eli89])**.** *Let M be a smooth, closed, oriented 3-manifold. Then two cooriented overtwisted contact structures on M are isotopic if and only if they are homotopic as hyperplane distributions.*

Remark. The notion of overtwistedness also exists for higher-dimensional contact structures, though this generalization took some time to navigate. See [BEM15] and [CMP19].

In combination with earlier results of Lutz [Lut77] and Martinet [Mar71], Theorem 2.5 settles the classification problem for overtwisted contact structures. The case of tight structures has proven more difficult, but significant process has been made in many cases. For instance, later chapters of this thesis will rely on classification results due to Honda, presented in [Hon00a] and [Hon00b].

We close this section by presenting a dichotomy for tight contact structures which will be needed in stating some of our results.

Definition. Let (M, ξ) be a tight contact 3-manifold, $\tilde{\pi}\colon \tilde{M} \to M$ the universal cover of M. We will say that ξ is *universally tight* if the contact manifold

$$(\tilde{M}, \tilde{\xi} := \tilde{\pi}^* \xi)$$

is tight. We will call ξ *virtually overtwisted* if there is some finite cover $\overline{\pi}\colon \overline{M} \to M$ of M for which the contact manifold

$$(\overline{M}, \overline{\xi} := \overline{\pi}^*\xi)$$

is overtwisted.

Remark. It is not immediately obvious that every tight structure on a 3-manifold must be either universally tight or virtually overtwisted. However, every tight contact structure on a manifold with residually finite fundamental group must fall into one of these categories, and work of Hempel ([Hem87]) along with the geometrization conjecture shows that every 3-manifold has this property.

2.2 Legendrian knots

By definition, a contact structure is *maximally non-integrable*; for a contact 3-manifold (M, ξ), this means that there are no embedded surfaces in M which are everywhere tangent to ξ. However, contact manifolds do admit embedded knots which are everywhere tangent to ξ, and these play an important role in 3-dimensional contact topology.

Definition. A *Legendrian knot* (or *Legendrian link*) in a contact 3-manifold (M, ξ) is an oriented knot (or link) in M which is everywhere tangent to ξ.

We will generally consider Legendrian knots up to Legendrian isotopy.

Definition. Legendrian knots L and L' are said to be *Legendrian isotopic* if there is a 1-parameter family L_t, $t \in [0,1]$, of Legendrian knots with $L_0 = L$ and $L_1 = L'$.

In Section 2.7.2 we will discuss surgery diagrams for contact 3-manifolds, in which Legendrian knots in (S^3, ξ_{std}) play an important role. These may be treated as Legendrian knots in $(\mathbb{R}^3, \xi_{\text{std}})$, for which we have the *front projection*.

Definition. The *front projection* of a Legendrian knot L in $(\mathbb{R}^3, \xi_{\text{std}} = \ker(dz - y\,dx))$ is the image $\pi(L) \subset \mathbb{R}^2$ of L under the map $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\pi(x, y, z) := (x, z)$.

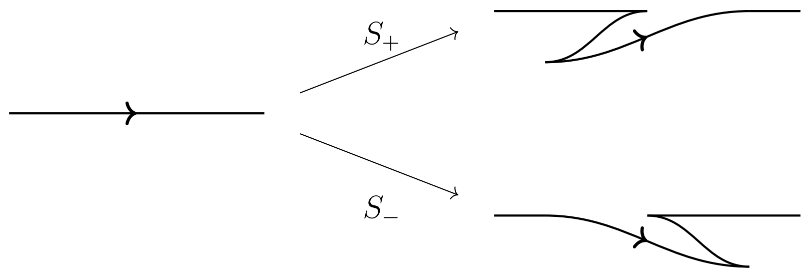


Figure 2.1: Stabilization of the x-axis in the front projection.

We point out that any smooth, closed, immersed curve in $\mathbb{R}^2$ with no vertical tangencies lifts to a unique Legendrian knot in $(\mathbb{R}^3, \xi_{\text{std}})$. Indeed, the y-coordinate of the lift is determined by the equation $y = \frac{dz}{dx}$, ensuring that $dz - y\,dx = 0$ along the lift. All figures depicting Legendrian knots in this thesis are in fact front projections.

The front projection affords us a modification of Legendrian knots in (S^3, ξ_{std}) known as *stabilization*. For now this modification will remain somewhat opaque, but we will discuss a correspondence between stabilization and *bypass attachment* in Section 2.4.1.

Definition. Given an oriented Legendrian knot L in (S^3, ξ_{std}), the *positive stabilization* and *negative stabilization* of L, denoted $S_+(L)$ and $S_-(L)$, respectively, are obtained by modifying the front projection of L as depicted in Figure 2.1.

Our discussion of Legendrian knots ends (for now) with a description of the *classical invariants* of a Legendrian knot in (S^3, ξ_{std}). The first of these is the *Thurston-Bennequin number*.

Definition. Given an oriented Legendrian knot L in a contact manifold (M, ξ), fix a Seifert surface Σ for L. Choose a vector field v along L which is transverse to ξ, and let L' be the pushoff of L in the direction of v. The *Thurston-Bennequin number* $tb(L)$ is the signed intersection number of L' with Σ.

Verifying that this definition is independent of choices is a standard exercise. The second classical invariant is the *rotation number*.

Definition. Given an oriented Legendrian knot L in a contact manifold (M, ξ), fix a Seifert surface Σ for L. Let v be a non-zero vector field on Σ which is tangent to $\xi|_\Sigma$, and let w be a non-zero vector field on L which is tangent to L. The *rotation number* of L is the twisting of w relative to v in ξ.

Again, verification of the definition of the rotation number is a standard exercise.

2.3 Symplectic fillings of contact manifolds

The story that we tell repeatedly in this thesis begins with a given contact manifold and asks for a classification of its *symplectic fillings*. In this section we recall the definition of a symplectic manifold and briefly describe symplectic fillings of contact manifolds.

Definition. A *symplectic structure* or *symplectic form* on a smooth manifold (which may have boundary) is a closed, nondegenerate differential 2-form. A *symplectic manifold* is a pair (W, ω) consisting of a smooth manifold W and a symplectic form ω on W.

In certain conditions, a symplectic manifold-with-boundary will induce a contact structure on its boundary. We say that such a symplectic manifold *fills* its contact boundary.

Definition. Fix a cooriented contact manifold (M, ξ) and suppose that (W, ω) is a symplectic manifold with $\partial W = M$ as oriented manifolds. We say that (W, ω) is

- a *weak symplectic filling* of (M, ξ) if $\omega|_\xi > 0$;
- a *strong symplectic filling* of (M, ξ) if there is a 1-form λ on W such that $\omega = d\lambda$ on some neighborhood of ∂W and such that $\lambda|_{\partial W}$ is a contact form for ξ;
- an *exact filling* of (M, ξ) if there is a 1-form λ on W such that $\omega = d\lambda$ on all of W and such that $\lambda|_{\partial W}$ is a contact form for ξ.

We say that (M, ξ) is *weakly symplectically fillable*, *strongly symplectically fillable*, or *exactly fillable* if it admits a symplectic filling of the corresponding type.

Of course there is no reason to expect a generic contact manifold to admit any type of symplectic filling. Indeed, a classic argument of Eliashberg and Gromov shows that overtwisted contact 3-manifolds admit no symplectic fillings.

Theorem 2.6 ([Eli90],[Gro85])**.** *A weakly symplectically fillable contact 3-manifold is tight.*

Remark. Analogous results hold in higher dimensions, but we will continue to restrict our attention to the case of contact 3-manifolds and 4-dimensional symplectic fillings.

It is clear that an exact filling is a strong symplectic filling, and that a strong symplectic filling is a weak symplectic filling. In fact, we have the following sequence of inclusions:

$$\{\text{exactly fillable}\} \subsetneq \{\text{strongly fillable}\} \subsetneq \{\text{weakly fillable}\} \subsetneq \{\text{tight}\}$$

The first examples of tight contact manifolds admitting no symplectic fillings were given by Etnyre-Honda in [EH02]. Eliashberg produced a weakly-but-not-strongly fillable contact manifold in [Eli96], and Ghiggini constructed a family of strongly-but-not-exactly fillable contact manifolds in [Ghi05].

2.4 Convexity in contact topology

Consider an embedded surface in a contact 3-manifold. Though we cannot hope for this surface to be everywhere tangent to the contact structure, it is often the case that we can recover the contact structure (locally) from the germ it leaves behind on the embedded surface. This germ is known as the *characteristic foliation*.

Definition. Given a contact 3-manifold (M, ξ) and a smoothly embedded surface $\Sigma \subset M$, the *characteristic foliation* Σ_ξ of Σ is the singular foliation induced by ξ, defined so that $\Sigma_\xi(p)$ is the intersection $\xi_p \cap T_p\Sigma$. We say that points $p \in \Sigma$ with $\xi_p = T_p\Sigma$ are *singular points*.

Characteristic foliations can be terribly complicated, but when studying *convex surfaces*, we will see that we can forget much of the information about the characteristic foliation, remembering only its *dividing set*.

Definition. A *contact vector field* on a contact 3-manifold (M, ξ) is a vector field X whose flow preserves ξ. By this we mean that, for any contact form α for ξ, $\mathcal{L}_X\alpha = g\,\alpha$, for some positive smooth function g on M. We say that a smoothly embedded surface $\Sigma \subset M$ is *convex* if there is a contact vector field for (M, ξ) which is transverse to Σ.

We point out that convex surfaces are not too hard to find in dimension 3.

Theorem 2.7 ([Gir91])**.** *Any closed surface in a contact 3-manifold (M, ξ) is C^∞-close to a convex surface.*

Remark. In higher dimensions, the current state of the art is due to Honda-Huang, who give the foundations for convex hypersurface theory in [HH19]. There they show that hypersurfaces in higher-dimensional contact manifolds are C^0-close to convex surfaces; experts generally suspect that the smooth version of the statement is false.

As mentioned above, much of the power of convex surface theory is that we may trade characteristic foliations for dividing sets, which we now define.

Definition. If $\Sigma \subset (M, \xi)$ is convex and X is a contact vector field transverse to Σ, then the *dividing set* of Σ is the multi-curve

$$\Gamma_\Sigma := \{p \in \Sigma \,|\, X(p) \in \xi_p\}.$$

Three important observations about Γ_Σ are

(1) Γ_Σ divides Σ into positive and negative regions: $\Sigma \setminus \Gamma_\Sigma = R_+(\Sigma) \sqcup R_-(\Sigma)$;

(2) Γ_Σ is transverse to the characteristic foliation Σ_ξ of Σ;

(3) Σ admits a volume form ω and a vector field Y such that Y points transversely out of $R_+(\Sigma)$ along Γ_Σ, directs Σ_ξ, and dilates ω in the sense that $\pm\mathcal{L}_Y\omega > 0$ on $R_\pm(\Sigma)$.

These three characteristics determine Γ_Σ up to isotopy, so we will refer to the dividing set Γ_Σ and the regions $R_\pm(\Sigma)$ of a convex surface Σ without reference to a particular contact vector field.

Giroux's Flexibility Theorem explains the power of the dividing set.

Theorem 2.8 ([Gir91]). *Let Σ be a closed convex surface in a contact manifold (M, ξ), as witnessed by a contact vector field X with dividing set Γ_Σ, and consider some singular foliation $\mathcal{F}$ on Σ. Suppose we can put a contact structure ξ' on a neighborhood of Σ such that $\Sigma_{\xi'} = \mathcal{F}$ and such that the dividing set of Σ is isotopic to Γ_Σ. Then there is an isotopy φ_t, $t \in [0,1]$, of Σ in (M, ξ) such that*

(1) *$\varphi_0 = \mathrm{id}$ and $\varphi_t|_{\Gamma_\Sigma} = \mathrm{id}$ for all t;*
(2) *$\varphi_t(\Sigma)$ is transverse to X for all t;*
(3) *$\varphi_1(\Sigma)$ has characteristic foliation $\mathcal{F}$.*

So the dividing set of a convex surface records all of the interesting contact topology in a neighborhood of the surface. We now discuss two important applications of convex surface theory.

2.4.1 Bypasses

A natural and interesting question in convex surface theory proceeds as follows: say we have a 1-parameter family of surfaces in a contact manifold, and that each surface in the family is convex, except at some finite number of distinguished parameters. (This is a C^∞-generic property for 1-parameter families of embedded surfaces.) We would like to know how the characteristic foliation of the surface changes as we pass through these distinguished parameter values. More accurately, we would like to know how the dividing set changes. The fundamental modification to the dividing set is the *bypass attachment*, developed by Honda in [Hon00a].

Definition. If Σ is a convex surface in a contact 3-manifold (M, ξ), then a *bypass* for Σ is an oriented, embedded half-disc D such that

(1) ∂D is the union of two Legendrian arcs α_1, α_2 which intersect at their endpoints;
(2) D intersects Σ transversely along α_1;
(3) D has positive elliptic tangencies at $\alpha_1 \cap \alpha_2$, one negative elliptic tangency on the interior of α_1, and only positive tangencies along α_2, alternating between elliptic and hyperbolic;

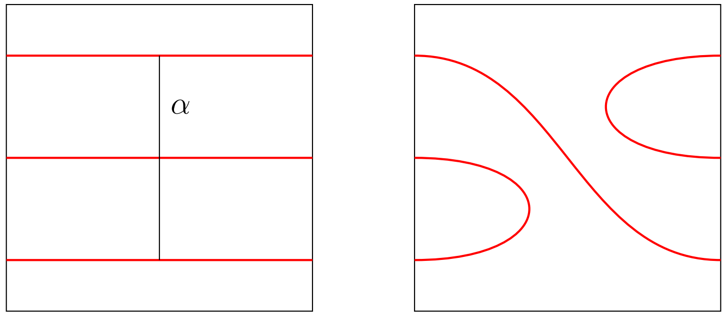


Figure 2.2: On the left, the dividing set Γ_{Σ_0} in a neighborhood the attaching arc α. On the right, the dividing set Γ_{Σ_1}.

(4) α_1 intersects the dividing set Γ_Σ exactly at the elliptic points of α_1.

We will refer to $\alpha_1 \subset D$ as the *attaching arc* for the bypass D, and we say that D *straddles* the component $c \subset \Gamma_\Sigma$ containing the negative elliptic tangency.

Remark. The bypass attachment has been generalized to higher dimensions by Honda-Huang [HH18].

When a bypass D for Σ exists, it is known that there is a neighborhood of $\Sigma \cup D$, diffeomorphic to $\Sigma \times [0, 1]$, such that the surfaces $\Sigma_i = \Sigma \times \{i\}$, $i = 0, 1$, are convex, and such that the dividing set Γ_{Σ_1} is obtained from Γ_{Σ_0} by Honda's *bypass attachment* operation, depicted in Figure 2.2. A bypass which does not change the dividing set (up to isotopy) is said to be *trivial*. The effect of bypass attachment on the dividing set of Σ can also be seen through Giroux's contact handle decompositions. The surface Σ_1 is obtained from Σ_0 by attaching a contact 1-handle and then a contact 2-handle in a topologically canceling manner. A detailed description of this process can be found in [Ozb11, Section 3].

Given a Legendrian arc α_1 in a convex surface Σ, we have no assurances that a bypass for Σ exists along α_1. We can, however, produce bypasses with stabilizations. Namely, if $L \subset (M, \xi)$ is a Legendrian knot, then the symmetric difference of L and $S_\pm(L)$, once oriented appropriately, bounds a bypass disc, with sign corresponding to the sign of the stabilization.

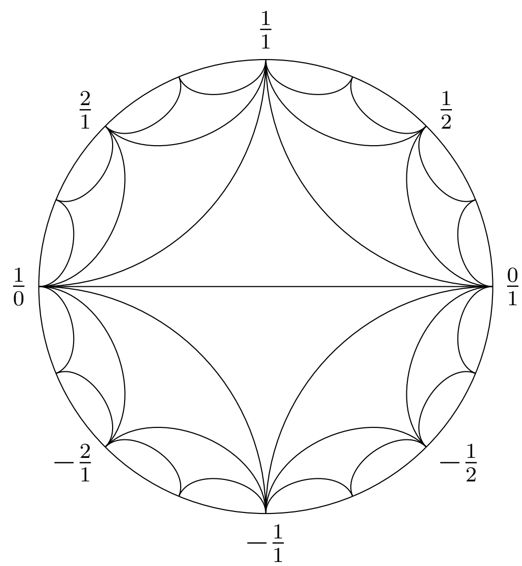


Figure 2.3: The Farey tessellation of the hyperbolic unit disc. The rational numbers p/q and p'/q' are each connected to $(p+p')/(q+q')$ by an arc.

2.4.2 Basic slices

Honda introduced the bypass in his quest to classify tight contact structures. For torus bundles and lens spaces, another fundamental building block of the classification of tight structures is the *basic slice*.

Throughout this thesis, whenever we have a convex torus $T^2 \subset (M, \xi)$, we will assume that the dividing set Γ_{T^2} has two parallel components. Under the identification $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, these components are isotopic to a line of rational slope $s(T^2)$, which we refer to as the *slope* of the dividing set.

Definition. Let $(T^2 \times I, \xi)$ be tight, with convex boundary, and let s_i be the slope of $T^2 \times \{i\}$, $i = 0, 1$. We call $(T^2 \times I, \xi)$ a *basic slice* if

- s_0 and s_1 are connected by an edge of the Farey tessellation;
- for each $t \in [0, 1]$, if $T^2 \times \{t\}$ is convex, then its slope s lies in the interval $[s_1, s_0]$ on the Farey tessellation.

The Farey tessellation is depicted in Figure 2.3, with $[s_1, s_0]$ denoting a counterclockwise arc from s_1 to s_0.

Remark. The rational numbers s_0 and s_1 will be connected by an edge of the Farey tessellation if and only if the minimal integral vectors representing s_0, s_1 form a basis of $\mathbb{Z}^2$.

For our purposes, the most pertinent fact about basic slices is the following: once we fix rational slopes s_0, s_1 which are Farey neighbors, there are precisely two tight contact structures ξ on $T^2 \times I$ making $(T^2 \times I, \xi)$ a basic slice with slopes s_0, s_1, and these structures are distinguished by their relative Euler classes. Indeed, we have $PD(e(\xi, s)) = \pm(0, 1)$ in $\mathbb{Z}^2 \cong H_1(T^2; \mathbb{Z})$.

Basic slices are useful in the classification of tight contact structures because one can decompose a tight thickened torus $(T^2 \times I, \xi)$ into basic slices, each of which admits two tight contact structures. These basic slices naturally cluster into *continued fraction blocks*, within which the basic slices may be shuffled. The standard model for a continued fraction block is a tight structure on $T^2 \times [0, m]$ with boundary slopes $s_m = -1 - m$ and $s_0 = -1$, for some $m \geq 1$. We assume that this tight structure decomposes into m basic slices, with $s_k = -1 - k$ for $0 \leq k \leq m$. There are $m + 1$ tight structures satisfying these requirements, distinguished by their relative Euler classes, which satisfy $PD(e(\xi, s)) = (0, k) \in H_1(T^2; \mathbb{Z})$ for some $k \in \{-m, 2 - m, \ldots, m - 2, m\}$.

Of the $m + 1$ tight contact structures on a continued fraction block $T^2 \times [0, m]$, just two — those with $PD(e(\xi, s)) = \pm(0, m)$ — are universally tight. The remaining tight structures will contain adjacent basic slices of opposite sign, which causes their lifts to the universal cover to be overtwisted. More generally, a tight contact structure ξ on a thickened torus $T^2 \times I$ can have each of its continued fraction blocks be universally tight, while ξ itself is virtually overtwisted. Again, this occurs when the basic slice decomposition of $(T^2 \times I, \xi)$ includes adjacent basic slices of opposite sign. See [Hon00b, Section 4.4.5] for details.

2.5 Liouville hypersurfaces and symplectic handles

Exact symplectic manifolds-with-boundary are also called *Liouville domains*. In addition to filling their contact boundaries, Liouville domains will play an important role for us as the positive and negative regions of a convex surface.

Definition. A *Liouville domain* is a pair (Σ_L, β), where

(1) Σ_L is a smooth, compact manifold-with-boundary;

(2) $d\beta$ is a symplectic form on Σ_L;

(3) the vector field X_β defined by $\iota_{X_\beta} d\beta = \beta$ points transversely out of $\partial\Sigma_L$.

We call X_β the *Liouville vector field* for (Σ_L, β).

Once again, our definition works in any dimension, but we will focus on 2-dimensional Liouville domains.

Definition. Let (M, ξ) be a contact 3-manifold and let (Σ_L, β) be a 2-dimensional Liouville domain. A *Liouville embedding* $i\colon (\Sigma_L, \beta) \hookrightarrow (M, \xi)$ is an embedding for which there exists a contact form λ on (M, ξ) satisfying $i^*\lambda = \beta$. We call the image of a Liouville embedding a *Liouville hypersurface* and denote it by $(\Sigma_L, \beta) \subset (M, \xi)$.

The standard example of a Liouville hypersurface is the positive region of a convex surface. The following result says that these regions are in fact the source of all Liouville hypersurfaces.

Proposition 2.9 ([Avd12, Proposition 7.2])**.** *A hypersurface $\Sigma_L \subset (M, \xi)$ is Liouville if and only if there is a convex hypersurface $\Sigma \subset (M, \xi)$ for which Σ_L is $R_+(\Sigma)$ minus some collar neighborhood of $\partial R_+(\Sigma)$.*

Given a Liouville hypersurface (Σ_L, β), Avdek constructs a symplectic handle $(H_{\Sigma_L}, \omega_\beta)$, and we summarize this construction here. For full details see [Avd12].

The construction begins with a standard neighborhood $\mathcal{N}(\Sigma_L)$ of (Σ_L, β) in (M, ξ). If λ is a contact form for (M, ξ) satisfying $\lambda|_{T\Sigma_L} = \beta$, then there is a neighborhood $N(\Sigma_L) = [-\epsilon, \epsilon] \times \Sigma_L$ with $\lambda|_{N(\Sigma_L)} = dz + \beta$, for some sufficiently small ϵ. This neighborhood will have corners at $\{\pm\epsilon\} \times \partial\Sigma_L$, but an edge-rounding process produces $\mathcal{N}(\Sigma_L)$, a neighborhood of (Σ_L, β) with smooth, convex boundary.

With an abstract copy of this standard neighborhood in hand, consider the symplectic manifold

$$(H_{\Sigma_L}, \omega_\beta) = ([-1,1] \times \mathcal{N}(\Sigma), d\theta \wedge dz + d\beta),$$

where θ and z are the coordinates on $[-1,1]$ and $[-\epsilon, \epsilon]$, respectively. This is the symplectic handle constructed from (Σ_L, β). There is a vector field $V_\beta = z\partial_z + X_\beta$ which points transversely out of ∂H_{Σ_L} along $[-1,1] \times \partial\mathcal{N}(\Sigma_L)$ and whose flow dilates ω_β. This vector field can be perturbed so that it also points into ∂H_{Σ_L} along $\{\pm 1\} \times \mathcal{N}(\Sigma_L)$, making this portion of $\partial H(\Sigma_L)$ concave while $[-1,1] \times \partial\mathcal{N}(\Sigma_L)$ is convex.

Remark. In the special case where $(\Sigma_L, \beta) \cong (T^*S^1, \lambda_{\text{can}})$, we refer to $(H_{\Sigma_L}, \omega_\beta)$ as a (four-dimensional) *round symplectic 1-handle*.

Avdek's construction generalizes the notion of a *Weinstein 1-handle*, which is produced in dimension four by considering the Liouville domain $(\mathbb{D}^2, \frac{1}{2}(p\,dq - q\,dp))$, where $\mathbb{D}^2$ has coordinates p, q. We will also refer in this thesis to (4-dimensional) *Weinstein 2-handles*, which are Liouville domains of the form $(\mathbb{D}^2 \times \mathbb{D}^2, \lambda_2)$, where the Liouville vector field points transversely into the boundary along $\partial\mathbb{D}^2 \times \mathbb{D}^2$, points transversely into the boundary along $\mathbb{D}^2 \times \partial\mathbb{D}^2$, and admits a Morse Lyapunov function.

Let us describe how Avdek attaches the symplectic handle $(H_{\Sigma_L}, \omega_\beta)$ to a strong symplectic filling (W, ω). For this attachment to be possible there must exist a pair of disjoint Liouville embeddings

$$i_1 \colon (\Sigma_L, \beta) \hookrightarrow (M, \xi) \quad \text{and} \quad i_2 \colon (\Sigma_L, \beta) \hookrightarrow (M, \xi),$$

where (M, ξ) is the boundary of (W, ω). These embeddings admit standard neighborhoods $\mathcal{N}(i_1(\Sigma_L))$ and $\mathcal{N}(i_2(\Sigma_L))$, each contactomorphic to $\mathcal{N}(\Sigma_L)$. We form a sort of symplectic-filling-with-corners $W^\square$ by removing $\mathcal{N}(i_1(\Sigma_L))$ and $\mathcal{N}(i_2(\Sigma_L))$ from (W, ω) and attaching H_{Σ_L} along $\{\pm 1\} \times \mathcal{N}(\Sigma_L)$. Because (W, ω) is a strong filling of (M, ξ), there is a Liouville vector field on W pointing out of ∂W. We glue $(H_{\Sigma_L}, \omega_\beta)$ to $(W \setminus (\mathcal{N}(i_1(\Sigma_L)) \cup \mathcal{N}(i_2(\Sigma_L))), \omega)$ in such a way that this vector field agrees with V_β along $\{\pm 1\} \times \mathcal{N}(\Sigma_L)$. The edges of $W^\square$

are then rounded to produce a new symplectic filling (W', ω'). This new filling is the result of attaching the handle $(H_{\Sigma_L}, \omega_\beta)$ to (W, ω) along $i_1(\Sigma_L)$ and $i_2(\Sigma_L)$.

2.6 Contact handles

In this section we briefly describe standard models for 3-dimensional contact handles of index 1 and 2. Our goal is to interpret the bypass attachment operation described in Section 2.4.1 using contact handles. This idea is due to Giroux, and details can be found in [Ozb11].

The models are given as subsets of $(\mathbb{R}^3, \ker \alpha)$, where $\alpha = dz + y\,dx + 2x\,dy$. For some small $\epsilon > 0$ we have

$$H_1 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + z^2 \leq \epsilon,\ y^2 \leq 1\}$$

and

$$H_2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + z^2 \leq 1,\ y^2 \leq \epsilon\}.$$

Notice that $Z = 2x\,\partial_x - y\,\partial_y + z\,\partial_z$ is a contact vector field on $(\mathbb{R}^3, \ker \alpha)$, and is transverse to ∂H_1 and ∂H_2 away from corners. So ∂H_1 and ∂H_2 are convex surfaces, and their dividing sets are given by

$$\partial H_1 \cap \{z = 0\} \quad \text{and} \quad \partial H_2 \cap \{z = 0\}.$$

Our standard model for a contact 1-handle is then $(H_1, \ker \alpha)$, with attaching region $\partial H_1 \cap \{y = \pm 1\}$. The attaching region is that portion of ∂H_1 where Z is inward-pointing. Similarly, $(H_2, \ker \alpha)$ is the standard model for a contact 2-handle, though it is attached using $-Z$, so that $\partial H_2 \cap \{x^2 + z^2 = 1\}$ is the attaching region.

Now suppose that (M, ξ) is a contact 3-manifold-with-boundary whose boundary is convex. We attach a 1-handle to (M, ξ) by identifying the attaching region of our standard model with regular neighborhoods of a pair of points $p, q \in \partial M$. We require this identification to match the dividing sets of the regular neighborhoods with that of the attaching region, so that the contact structure on (M, ξ) matches up with that on the standard model, yielding a new contact manifold. We analogously define contact 2-handle attachment.

In Chapter 3 we will need contact handle attachment to extend the contact form (and not just the contact structure). One circumstance in which this can be accomplished is when (M, ξ) carries the structure of a *sutured contact manifold*, as defined in [CGH11, Section 2]. In this case, there is a neighborhood $U(\Gamma) \subset M$ of the dividing set $\Gamma \subset \partial M$ on which the contact structure admits a standard form, and this standard form will match $(H_1, \ker \alpha)$, allowing us to extend the contact form as desired.

Finally, we remark on the connection to bypasses. Suppose that D is a bypass disc for a convex surface $\Sigma \subset (M, \xi)$. In Section 2.4.1 we described a neighborhood of $\Sigma \cup D$ which is diffeomorphic to $\Sigma \times [0, 1]$, with each surface $\Sigma_i := \Sigma \times \{i\}$ convex, and with the dividing set Γ_{Σ_1} obtained from Γ_{Σ_0} via the bypass attachment operation depicted in Figure 2.2. Consider now a one-sided neighborhood $N(\Sigma) = \Sigma \times [0, \epsilon]$ of Σ. We choose the coordinates on $N(\Sigma)$ so that ξ has the form $\ker(dt + \beta)$, for some 1-form β on Σ. Then the boundary components $\Sigma \times \{0\}$ and $\Sigma \times \{\epsilon\}$ are convex, with dividing sets $\Gamma_\Sigma \times \{0\}$ and $\Gamma_\Sigma \times \{\epsilon\}$. By attaching a contact 1-handle to $N(\Sigma)$ along $\Sigma \times \{\epsilon\}$ and then a contact 2-handle to the resulting neighborhood, we obtain the desired neighborhood $\Sigma \times [0, 1]$.

2.7 Legendrian surgery diagrams

The overarching goal of this thesis is to classify the symplectic fillings of contact 3-manifolds. In this section we introduce the construction which will define many of the contact 3-manifolds of interest to us. This details of this construction can be found in a variety of sources, including [OS13, Chapters 2 & 11].

2.7.1 Dehn surgery

Starting from the most basic closed 3-manifold, S^3, we will use *Dehn surgery* to produce more interesting smooth 3-manifolds. Dehn surgery removes a solid torus $N \subset S^3$ from S^3, and then glues in a new solid torus $S^1 \times D^2$ via an orientation-reversing diffeomorphism

$$\phi\colon \partial(S^1 \times D^2) \to \partial(S^3 \setminus \mathring{N}).$$

The solid torus to be removed from S^3 will be recorded by a choice of knot $K \subset S^3$, with N given by a tubular neighborhood $N(K)$ of K. Additionally, one can show that the result of Dehn surgery is determined up to diffeomorphism by the homology class

$$\phi_*([\{\text{pt}\} \times \partial D^2]) \in H_1(\partial(S^3 \setminus \mathring{N}(K)); \mathbb{Z}).$$

So the data used for Dehn surgery consists of a knot $K \subset S^3$, along with a homology class a in $H_1(\partial(S^3 \setminus \mathring{N}(K)); \mathbb{Z})$.

In fact, we can represent a via a rational number. If we choose an orientation for K, we have canonical choices for a meridian μ and longitude[1] λ of K, which together give a basis for $H_1(\partial(S^3 \setminus \mathring{N}(K)); \mathbb{Z}) \cong \mathbb{Z}^2$. With this basis fixed, there exists a unique pair of relatively prime integers p and q such that $a = p\,\mu + q\,\lambda$. These integers depend on the orientation we have chosen for K: reversing this orientation reverses the signs of μ and λ, and thus of p and q. However, the ratio p/q (which may be infinite, if $q = 0$) is orientation-independent. So in fact the data fed into Dehn surgery consists of a knot $K \subset S^3$ and a (possibly infinite) rational number p/q.

Definition. Given a knot $K \subset S^3$ and a rational number $p/q \in \mathbb{Q} \cup \{\infty\}$, let $S^3_{p/q}(K)$ denote the smooth manifold

$$(S^3 \setminus \mathring{N}(K)) \cup_\phi (S^1 \times D^2),$$

where $\phi\colon \partial(S^1 \times D^2) \to \partial(S^3 \setminus \mathring{N}(K))$ is determined by the fact that $\phi_*([\{\text{pt}\} \times \partial D^2]) = p\,\mu + q\,\lambda$. Here μ and λ are the meridian and longitude classes in $H_1(S^3 \setminus \mathring{N}(K))$ determined by K, with some fixed choice of orientation. We call the smooth manifold $S^3_{p/q}(K)$ the *Dehn surgery along K with slope p/q*.

[1] In general, a knot in a 3-manifold does not necessarily admit a canonical choice of longitude. But all knots in S^3 are null-homologous, and therefore admit canonical framings.

In case we have link L, each of whose components is decorated with a rational number, we can tell this story again. Namely, we simply perform Dehn surgery along all of the components simultaneously. We omit the details of this construction, but refer to the input data (the decorated link) as a *surgery diagram* for the resulting 3-manifold.

Definition. A *surgery diagram* for a smooth 3-manifold M is a link L in S^3, along with a choice of rational number for each component of L, such that Dehn surgery with this input yields a manifold diffeomorphic to M.

A landmark result of Lickorish and Wallace tells us that the construction outlined here produces all reasonably nice smooth 3-manifolds.

Theorem 2.10 ([Lic62, Wal60])**.** *Every closed, oriented 3-manifold can be obtained via integral Dehn surgery on a link in S^3.*

Remark. By *integral* Dehn surgery, we mean that each rational number which appears in the surgery presentation is an integer. Integral Dehn surgery is particularly important, because it is equivalent to 4-dimensional 2-handle attachment. That is, if p/q is an integer, then $S^3_{p/q}(K)$ can be realized as the boundary of $B^4 \cup_{K,p/q} X^4_2$, the 4-manifold obtained by attaching a 2-handle to B^4 along $K \subset S^3 = \partial(B^4)$ with framing p/q.

Remark. Theorem 2.10 tells us that every smooth 3-manifold M admits a surgery diagram, but of course there is no reason that this diagram should be unique. In [Kir78], Kirby gives necessary and sufficient conditions for distinct surgery diagrams to produce diffeomorphic 3-manifolds, but we will not concern ourselves with this question here.

2.7.2 Contact surgery

Because we will be working in the contact setting, we would like a notion of Dehn surgery which produces contact 3-manifolds. Here our starting point is again S^3, this time with the standard tight contact structure ξ_{std}. This construction can be carried out in essentially the same generality as above: given a Legendrian knot $L \subset (S^3, \xi_{\text{std}})$ and a rational number p/q, we can perform (p/q)-surgery (measured with respect to the Thurston-Bennequin framing of K) on S^3 along K, and produce a contact structure on the surgered manifold. However, in this section we give a somewhat less delicate treatment which addresses only (-1)-surgery, as this is the case of most interest to us.

We mentioned in Section 2.7.1 that integral surgery diagrams allow us to see 3-manifolds as boundaries of 4-manifolds: the diagram can be used to construct a 3-manifold via Dehn surgery or a 4-manifold via 2-handle attachment, and the 3-manifold will bound the 4-manifold. In the contact setting, we will only define (-1)-surgery, and will do so via 4-dimensional handle attachment, rather than Dehn surgery. We start with the well-known fact that the boundary of a symplectic filling with a Weinstein 2-handle attached is smoothly equivalent to the result of (-1)-surgery.

Theorem 2.11. *Suppose that (W, ω) is a symplectic 4-manifold which strongly (respectively, exactly) fills its boundary, and that $L \subset \partial W$ is a Legendrian curve with respect to the induced contact structure. Let $W \cup_L X_2^4$ denote the 4-manifold obtained by attaching a 2-handle X_2^4 to W along L with framing -1 with respect to its canonical contact framing. Then ω may be extended to $W \cup_L X_2^4$ in such a way that $W \cup X_2^4$ strongly (respectively, exactly) fills its boundary.*

The upshot of Theorem 2.11 is that we can associate to any Legendrian link $L \subset (S^3, \xi_{\text{std}})$ an exactly fillable contact manifold $(S^3(L), \xi_L)$. This is because $(B^4, \omega_{\text{std}})$ is an exact symplectic filling of (S^3, ξ_{std}), so we may define

$$(S^3(L), \xi_L) := \partial(B^4 \cup_L X_2^4, \omega),$$

where ω is obtained from ω_{std} via Theorem 2.11. As a smooth manifold, $S^3(L)$ is equivalent to the result of surgery along $L \subset S^3$, with the component L_i of L given framing $\text{tb}(L_i)-1$. It is a well-known fact (c.f. [DG04]) that, up to contactomorphism, $(S^3(L), \xi_L)$ is determined by the isotopy class of L.

Definition. A *Legendrian surgery diagram* for a contact 3-manifold (M, ξ) is a Legendrian link $L \subset (S^3, \xi_{\text{std}})$, identified up to Legendrian isotopy, such that (M, ξ) is contactomorphic to $(S^3(L), \xi_L)$.

Of course many contact 3-manifolds do not admit Legendrian surgery diagrams, since a surgered contact manifold $(S^3(L), \xi_L)$ is necessarily fillable. But in this thesis our interest will often be in contact manifolds constructed via this process.

CHAPTER 3

Splitting symplectic fillings

When studying symplectic fillings of contact manifolds, one often wonders whether decompositions which exist for the contact manifold extend to its fillings. For instance, Eliashberg proved the following result.

Theorem 3.1 ([Eli90, CE12])**.** *Suppose that a 3-dimensional contact manifold (M, ξ) is obtained from another contact manifold (M', ξ') via connected sum. Then every symplectic filling of (M, ξ) is obtained by attaching a Weinstein 1-handle to a symplectic filling of (M', ξ').*

So the symplectic fillings of a contact manifold obtained by connected sum are determined by the fillings of the parties to the connected sum. Thus, one may attempt to classify the symplectic fillings of a contact manifold (M, ξ) by identifying an embedded sphere along which (M, ξ) decomposes as a connected sum, and then classifying the symplectic fillings of the contact manifold(s) resulting from this decomposition.

Our goal in this chapter is to produce an analogue of Theorem 3.1 for surfaces of higher genus. That is, if there is a surface of genus $g \geq 1$ along which our contact manifold (M, ξ) decomposes, we would like to extend this decomposition to any filling of (M, ξ). This will allow us to study symplectic fillings of (M, ξ) by instead analyzing fillings of the (hopefully simpler) pieces into which (M, ξ) splits.

We call the surfaces which allow such a decomposition *splitting surfaces*, and will give a precise definition in Section 3.1. The main result of this chapter is then the following.

Theorem 3.2. *Let (M, ξ) be a closed, cooriented 3-dimensional contact manifold and let (W, ω) be a strong (respectively, exact) filling of (M, ξ). If (M, ξ) admits a splitting surface Σ of genus g, then there exists a symplectic manifold (W', ω') such that*

(1) *(W', ω') is a strong (respectively, exact) filling of its boundary (M', ξ');*

(2) *there are Legendrian graphs $\Lambda_1, \Lambda_2 \subset \partial W'$ with standard neighborhoods $N(\Lambda_1), N(\Lambda_2)$ such that*

$$M \simeq \left(\partial W' - \bigcup_{i=1}^{2} \operatorname{int}(N(\Lambda_i)) \right) / (\partial N(\Lambda_1) \sim \partial N(\Lambda_2)),$$

where the boundaries $\partial N(\Lambda_i)$ are glued in such a way that their dividing sets and meridians are identified;

(3) *(W, ω) can be recovered from (W', ω') by attaching a symplectic handle $(H_{R_+(\Sigma)}, \omega_\beta)$ constructed from the positive region of Σ.*

Remark. The genus 1 version of Theorem 3.2 first appeared in Michael Menke's thesis, [Men18], and will be the version we use in later chapters. The generalization to higher genus is the result of joint work with Menke.

Our proof strategy for Theorem 3.2 follows in the tradition of Eliashberg's "filling by holomorphic discs," initiated in [Eli90]. A splitting surface $\Sigma \subset (M, \xi)$ of genus g gives us two surfaces in M with genus 0 and $g + 1$ boundary components, each of which can be lifted to a family of J-holomorphic curves in the symplectization of M. If we have a filling (W, ω) of (M, ξ), these families can be extended to a single 1-dimensional family of J-holomorphic curves in the completion $(\widehat{W}, \widehat{\omega})$, and the geometric conditions on Σ will control the topology of this family. Removing a neighborhood of this family will lead us to the new symplectic manifold (W', ω').

3.1 Splitting surfaces

In this section we define the splitting surfaces which appear in Theorem 3.2. An unsurprising condition on a splitting surface $\Sigma \subset (M, \xi)$ is that Σ should be convex; this ensures that Σ admits a reasonably nice neighborhood in (M, ξ). Next, we want the regions $R_\pm(\Sigma)$ to be diffeomorphic to one another, as these will appear as the ends of a 1-parameter family of J-holomorphic curves. Our remaining conditions are much more technical, though their purpose should become apparent as we pursue the proof strategy outlined above.

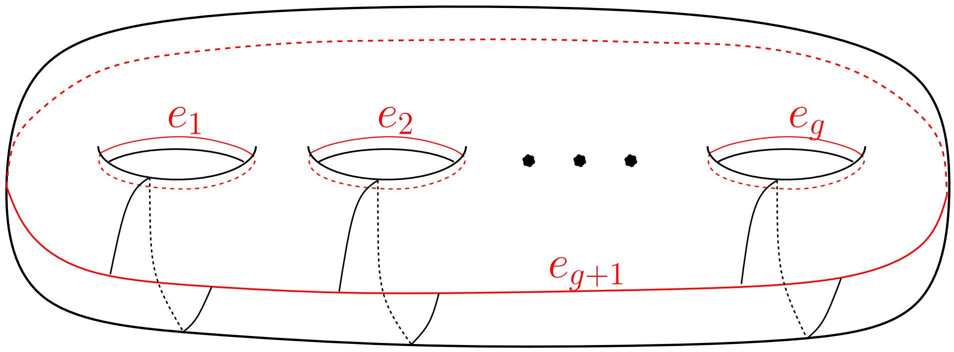


Figure 3.1: A splitting surface Σ_g with dividing curves $e_1, \ldots, e_{g+1}$, each of which is an elliptic orbit with Conley-Zehnder index 1. Some of the attaching arcs are also depicted.

Definition. We call a closed, connected, oriented, convex surface $\Sigma \subset (M, \xi)$ of genus g a *splitting surface* if

(1) the regions $R_\pm(\Sigma)$ are planar, with $g+1$ boundary components $c_1, \ldots, c_{g+1}$;

(2) there exist bypasses $D_1^\pm, \ldots, D_g^\pm \subset (M, \xi)$, attached to Σ along Legendrian arcs $\alpha_1^\pm, \ldots, \alpha_g^\pm$, with $\alpha_i^\pm$ straddling c_i and having its endpoints on c_{g+1};

(3) for $i = 1, \ldots, g$, there is an arc $a_i \subset c_{g+1}$ which contains the endpoints of α_i^+ and α_i^-, and contains no endpoints of $\alpha_j^\pm$ for $j \neq i$;

(4) the bypasses $D_1^+, \ldots, D_g^+$ are attached from one side of Σ and the bypasses $D_1^-, \ldots, D_g^-$ are attached from the other side.

A *mixed torus* is a splitting surface of genus 1.

Because of their appearance throughout the rest of this thesis, we point out that a mixed torus can equivalently be defined to be an embedded convex torus $T \subset (M, \xi)$ admitting a virtually overtwisted neighborhood of the form $T^2 \times [0, 2]$, where T is identified with $T^2 \times \{1\}$ and each of $T^2 \times [0, 1]$ and $T^2 \times [1, 2]$ is a basic slice. The notion of *splitting* (M, ξ) *along* T is described as follows. Let s_i denote the slope of $T^2 \times \{i\}$. The identification of T^2 with $\mathbb{R}^2/\mathbb{Z}^2$ may be normalized so that $s_0 = -1$ and $s_1 = \infty$. With this normalization, Theorem 3.2 will produce a filling of (M', ξ'), the contact manifold obtained from (M, ξ) by splitting along T with slope s, for some integer $0 \leq s \leq s_2 - 1$. We define

$$M' := S_0 \cup_{\psi_0} (M \setminus T) \cup_{\psi_1} S_1,$$

where each S_i is a solid torus and $\psi_i\colon \partial S_i \to T_i$ is chosen so that the image of a meridian in ∂S_i has slope s in T_i. Notice that the dividing set is vertical, and thus s must be an integer. We define ξ' to agree with ξ on $M \setminus T$, and on $S_i \subset M'$, ξ' is the unique tight contact structure determined by the characteristic foliation of ∂S_i.

Having introduced this language for mixed tori, we may state the genus 1 version of Theorem 3.2 somewhat more succinctly.

Theorem 3.3 ([Men18, Theorem 1.1.1])**.** *Let (M, ξ) be a closed, cooriented 3-dimensional contact manifold, and let (W, ω) be a strong (exact) symplectic filling of (M, ξ). If there exists a mixed torus $T^2 \subset (M, \xi)$, with normalized embedding $T^2 \times [0, 2]$, then there exists a (possibly disconnected) symplectic manifold (W', ω') such that:*

- *(W', ω') is a strong (exact) filling of its boundary (M', ξ');*
- *(M', ξ') is the result of splitting (M, ξ) along T with some slope $0 \leq s \leq s_2 - 1$;*
- *(W, ω) can be recovered from (W', ω') by round symplectic 1-handle attachment.*

We also point out the first application of Theorem 3.3, which will be of use to us in later chapters.

Theorem 3.4 ([Men18, Theorem 1.1.3])**.** *Let $L \subset (M, \xi)$ be a Legendrian knot in a contact 3-manifold, and let (M', ξ') be the result of contact surgery on (M, ξ) along $S_+(S_-(L))$. Then every strong (exact) symplectic filling of (M', ξ') may be obtained, up to diffeomorphism, from a strong (exact) symplectic filling of (M, ξ) by attaching a Weinstein 2-handle along $S_+(S_-(L))$.*

3.2 Proof of Theorem 3.2

Throughout this section we take (M, ξ) to be a contact manifold satisfying the hypotheses of Theorem 3.2. Let (W, ω) be a strong filling of (M, ξ) and Σ_g a splitting surface of genus g, with dividing set $\Gamma_{\Sigma_g} = c_1 \cup \cdots \cup c_{g+1}$. There are attaching arcs $\alpha_1^{\pm}, \ldots, \alpha_g^{\pm}$ and associated bypasses $D_1^{\pm}, \ldots, D_g^{\pm}$ as described in the definition of splitting surfaces.

We will denote by $(\widehat{W}, \widehat{\omega})$ the completion of (W, ω), obtained by attaching the positive end $([0, \infty) \times M, d(e^t\alpha))$ of the symplectization of M. We take J to be an almost complex structure on $\widehat{W}$ adapted to the contact form α for (M, ξ). That is, J is translation invariant, $J\xi = \xi$, and $J\partial_t = R_\alpha$, where t is the $[0, \infty)$-coordinate on the symplectization and R_α is the Reeb vector field for α.

We will prove Theorem 3.2 by adapting the proof of [Men18, Theorem 1.1.1]. Specifically, our goal is to use Σ_g to construct a 1-parameter family $\mathcal{S}$ which sweeps out a properly embedded handlebody in $(\widehat{W}, \widehat{\omega})$. Removing this handlebody from (W, ω) will leave us with the desired manifold (W', ω').

Because our proof is adapted from [Men18], many of our lemmas are arbitrary-genus analogues of lemmas found there. Some of these require new proofs, while others, such as the following standardization of the contact form on M, are genus-independent and therefore survive unaltered.

Lemma 3.5 (c.f. [Men18, Lemma 1.3.1])**.** *There is a choice of contact form on a neighborhood of Σ_g such that the components Γ_{Σ_g} are non-degenerate elliptic Reeb orbits of Conley-Zehnder index 1 with respect to the framing induced by Σ_g.*

Denote the Reeb orbits constructed in Lemma 3.5 by $e_1, \ldots, e_{g+1}$, with e_{g+1} containing the endpoints of $\alpha_1^\pm, \ldots, \alpha_g^\pm$ and e_i the dividing curve straddled by $\alpha_i^\pm$. Menke's proof of Lemma 3.5 produces an explicit model for Σ_g with these orbits comprising the dividing set, and this model is depicted in Figure 3.1.

Lemma 3.6. *Let $\Sigma_g \subset (M, \xi)$ be a splitting surface of genus $g > 1$, with dividing set $e_1 \cup \cdots \cup e_{g+1}$ and bypasses $D_1^\pm, \ldots, D_g^\pm$ as described above. There is a one-sided neighborhood*

$$N = N(\Sigma_g \cup D_1^+ \cup \cdots \cup D_g^+)$$

and an extension of the contact form α chosen in Lemma 3.5 to N. This neighborhood contains contact 1-handles N_1^i, contact 2-handles N_2^i, and surfaces with corners $\Sigma_g^{i-1}, \Sigma_{g+1}^i$ of genus g and $(g+1)$, respectively, for $i = 1, \ldots, g$. Moreover,

(1) *the boundary ∂N is given by Σ_g and $\tilde{\Sigma}$, where $\tilde{\Sigma}$ is another convex surface of genus g, with dividing set given by elliptic orbits $\tilde{e}_1, \ldots, \tilde{e}_{g+1}$;*

(2) *$\Sigma_g^0 = \Sigma_g$, and for $i = 1, \ldots, g-1$, Σ_g^i meets $\tilde{\Sigma}$ in the orbits $\tilde{e}_1, \ldots, \tilde{e}_i$, meets Σ_g in the orbits $e_{i+1}, \ldots, e_g$, and has dividing set given by these orbits, along with an elliptic orbit e_{g+1}^{i+1};*

(3) *for $i = 1, \ldots, g$ we have a neighborhood*

$$N(\Sigma_g^{i-1} \cup D_i^+) = N_1^i \cup_{\Sigma_{g+1}^i} N_2^i,$$

with Σ_{g+1}^i containing the orbits $\tilde{e}_1, \ldots, \tilde{e}_{i-1}, e_i, \ldots, e_g, e_{g+1}^{i+1}$, as well as the elliptic orbit $\overline{e}_i$;

(4) *all of the elliptic orbits listed have Conley-Zehnder index 1;*

(5) *the Reeb vector field R_α is positively (negatively) transverse to the positive (negative) region of each of the surfaces listed;*

(6) *there are hyperbolic orbits $h_{g+1}^i, \tilde{h}_i$ in N_1^i and N_2^i, respectively, which have Conley-Zehnder index 0 with respect to Σ_g;*

(7) *if γ is any other Reeb orbit in N and $\bar{\gamma}$ is any of e_i, h_{g+1}^i, or $\tilde{h}_i$, then*

$$\mathcal{A}(\bar{\gamma}) < \mathcal{A}(\overline{e}_j), \mathcal{A}(\tilde{e}_j) \ll \mathcal{A}(\gamma),$$

for all j. In particular, $\mathcal{A}(\gamma)$ is sufficiently large as to prohibit the existence of a pseudoholomorphic curve in the symplectization of M from having γ among its negative ends while its positive ends form a subset of the curves listed.

Proof. Our goal is to extend the neighborhood of Σ_g identified by Lemma 3.5 to a bypass neighborhood; we accomplish this by succesively attaching to our neighborhood pairs of contact 1- and 2-handles, and extending α over each of these handles.

We write $\Sigma_g \times [0, \epsilon]$ for (half of) the neighborhood produced by Lemma 3.5. Within this one-sided neighborhood, we find $N_{1,\epsilon}^1$, a neighborhood obtained by "thickening Σ_g away from e_1." The boundary $\partial N_{1,\epsilon}^1$ has two components, which we call $\Sigma_g \times \{0\}$ and $\Sigma_g \times \{\epsilon\}$, and which meet along the elliptic orbit e_1. Conflating e_{g+1} with a parallel copy in $\Sigma_g \times \{\epsilon\}$, we see that each of e_1 and e_{g+1} is a dividing curve for the convex surface $\Sigma_g \times \{\epsilon\}$, and we

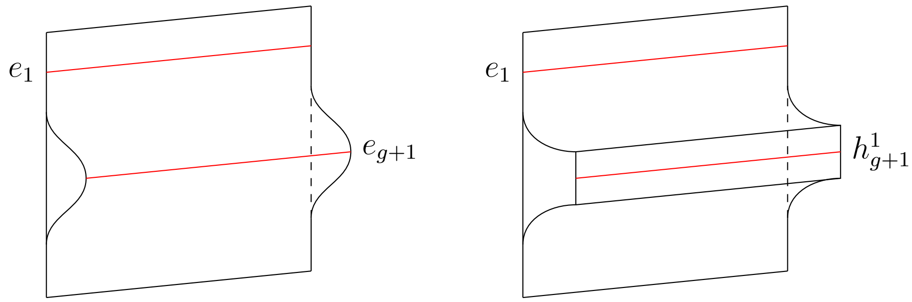


Figure 3.2: We construct $N^1_{1,\epsilon}$ by first thickening Σ_g away from e_1, as on the left. Conflating e_{g+1} with a parallel copy, we perform a convex-to-sutured modification, which introduces the hyperbolic orbit h^1_{g+1}, as seen on the right. Though not drawn, the Reeb orbit e_{g+1} persists on the right, becoming part of the interior of $N^1_{1,\epsilon}$.

know that the bypass D^+_1 has endpoints on e_{g+1}. According to [CGH11, Lemma 4.1], we may modify $N^1_{1,\epsilon}$ by introducing a canceling hyperbolic orbit h^1_{g+1} for e_{g+1}, as depicted in Figure 3.2. Following this modification, we attach a contact 1-handle to $N^1_{1,\epsilon}$ along a pair of points in h^1_{g+1}, extending the contact form α as we go, to yield the neighborhood N^1_1. This neighborhood has boundary components Σ_g and Σ^1_{g+1}, where Σ^1_{g+1} is a surface of genus $g+1$ with dividing curves e_1, $\overline{e}_1$, e^2_{g+1}, and $e_2, \ldots, e_g$. We choose our extension of α so that the actions of $\overline{e}_1$ and e^2_{g+1} are much larger than those of e_1, e^1_{g+1}, and h^1_{g+1}.

We have now produced $N^1_1 \subset N(\Sigma_g \cup D^+_1)$, as depicted in Figure 3.3. Attaching a topologically canceling contact 2-handle to N^1_1 along Σ^1_{g+1} will give $N(\Sigma_g \cup D^+_1)$, allowing us to identify N^1_2. An understanding of the Reeb dynamics of N^1_2 is given by viewing N^1_2 as the result of attaching a contact 1-handle to $N^1_{2,\epsilon} \subset \Sigma_g \times [1-\epsilon, 1] \subset N(\Sigma_g \cup D^+_1)$. We now repeat this process for $i = 2, \ldots, g$ to obtain the neighborhood N. The fact that all other Reeb orbits intersecting N have sufficiently large action as to be irrelevant follows from [Vau15, Theorem 2.1]. □

A schematic of the neighborhood $N(\Sigma^{i-1}_g \cup D^+_i)$ is depicted in Figure 3.3.

As stated above, we will build a 1-parameter family of holomorphic curves in $\widehat{W}$ that will sweep out a handlebody of genus g. The splitting surface Σ_g will help us do this by providing targets $R_\pm(\Sigma_g)$ for which our family can aim at its ends. That is, our 1-parameter family will have its ends in the symplectization part $[0, \infty) \times M$ of $\widehat{W}$, and we want the

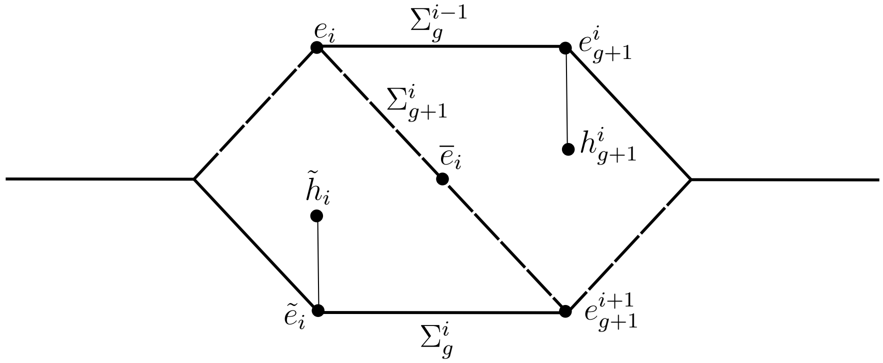


Figure 3.3: Orbits in the neighborhood $N(\Sigma_g^{i-1} \cup D_i^+)$. The heavily shaded curves represent the walls identified in Lemma 3.9. The segments on the far left and right are included in Σ_g^{i-1}, Σ_g^i, and Σ_{g+1}^i. The middle (dashed) segment is included only in Σ_{g+1}^i, while the dashed segments in the upper left and lower right are also included in Σ_g^{i-1} and Σ_g^i, respectively.

projection $\pi\colon [0,\infty) \times M \to M$ to take the ends of our family to the regions $R_\pm(\Sigma_g)$. The first step towards building our 1-parameter family is then to lift $R_\pm(\Sigma_g)$ to embedded holomorphic curves

$$u_\pm\colon S^2 \setminus \{p_1, \dots, p_{g+1}\} \to [0,\infty) \times M.$$

We can obtain these lifts by employing the following strategy: for each $1 \le i \le g+1$ we construct a holomorphic half-cylinder

$$u_i\colon [0,\infty) \times S^1 \to \mathbb{R} \times M$$

which is positively asymptotic to e_i. These half-cylinders project under π to collar neighborhoods of $e_1, \dots, e_{g+1}$ in $R_\pm(\Sigma_g)$, the deletion of which leaves $R'_\pm$, a 2-dimensional Weinstein domain. Our lifting problem is then solved if we can lift $R'_\pm$ to a holomorphic curve in $\mathbb{R} \times M$ and then glue the holomorphic half-cylinders $u_1, \dots, u_{g+1}$ to the boundary. The following lemma, proved in [Men18], allows us to lift $R'_\pm$.

Lemma 3.7 (c.f. [Men18, Lemma 1.3.4])**.** *Let $(B, \beta = -df \circ J)$ be a 2-dimensional Weinstein domain, where $f\colon B \to \mathbb{R}$ is a Morse function such that ∂B is a level set of f, and let $\alpha = dt + \beta$ be a contact form on $[-\epsilon, \epsilon] \times B$, where t is the coordinate on $[-\epsilon, \epsilon]$. Then there is an adapted almost complex structure on $\mathbb{R} \times [-\epsilon, \epsilon] \times B$ such that we can lift B to a holomorphic curve by the map $u(\mathbf{x}) = (f(\mathbf{x}), 0, \mathbf{x})$.*

The construction of the holomorphic half-cylinders $u_1, \ldots, u_{g+1}$ and the gluing of these to our lifts is also carried out in [Men18]; this establishes the following result.

Lemma 3.8 (c.f. [Men18, Lemma 1.3.5]). *There are embedded holomorphic curves*

$$u_\pm \colon S^2 \setminus \{p_1, \ldots, p_{g+1}\} \to [0, \infty) \times M$$

such that

(1) *both are Fredholm regular with index 2 and positively asymptotic to* $e_1, \ldots, e_{g+1}$*;*

(2) *under the projection* $\pi \colon [0, \infty) \times M \to M$ *we have* $\operatorname{im}(\pi \circ u_\pm) = R_\pm(\Sigma_g)$*.*

The same holomorphic half-cylinder strategy is used in [Men18] to prove the next result that we will need. Because Σ_g is a splitting surface, it admits collections of bypasses $\mathbf{D}_+$ and $\mathbf{D}_-$ from opposite sides, and Lemma 3.6 describes the orbits that appear in a neighborhood $N(\Sigma_g \cup \mathbf{D}_+ \cup \mathbf{D}_-)$. Specifically, Lemma 3.6 gives a list of relevant orbits in $N(\Sigma_g \cup \mathbf{D}_+)$, and produces a corresponding list in $N(\Sigma_g \cup \mathbf{D}_-)$. We distinguish the orbits in $N(\Sigma_g \cup \mathbf{D}_-)$ from those in $N(\Sigma_g \cup \mathbf{D}_+)$ with a prime (e.g., $\overline{e}_i'$ instead of $\overline{e}_i$). Some of these orbits are represented diagrammatically in Figure 3.4. In Lemma 3.6, the attachment of the bypass D_i^+ was accomplished by attaching the contact handles N_1^i and N_2^i; we use the handles $(N_2^i)'$ and $(N_1^i)'$ to attach D_i^-. The same approach used to prove Lemma 3.8 produces a collection of holomorphic curves which project to $N(\Sigma_g \cup \mathbf{D}_+ \cup \mathbf{D}_-)$ and will be useful to us in constructing our 1-parameter family.

Lemma 3.9 (c.f. [Men18, Lemma 1.3.6]). *For* $i = 1, \ldots, g$*, there are embedded holomorphic curves*

$$\bar{u}_{\pm,i}, \bar{u}_{\pm,i}' \colon S^2 \setminus \{p_1, \ldots, p_{g+2}\} \to [0, \infty) \times N(\Sigma_g \cup \mathbf{D}_+ \cup \mathbf{D}_-)$$

and

$$\tilde{u}_{\pm,i}, \tilde{u}_{\pm,i}' \colon S^2 \setminus \{p_1, \ldots, p_{g+1}\} \to [0, \infty) \times N(\Sigma_g \cup \mathbf{D}_+ \cup \mathbf{D}_-),$$

all Fredholm regular of index 2, all positively asymptotic to $\tilde{e}_1, \ldots, \tilde{e}_{i-1}, e_{i+1}, \ldots, e_g$*, and additionally*

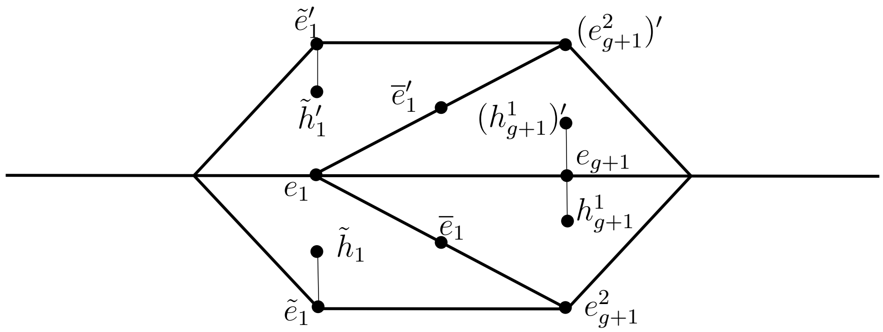


Figure 3.4: Orbits in the neighborhood $N(\Sigma_g \cup D_1^+ \cup D_1^-)$. The heavily shaded curves represent some of the walls identified in Lemma 3.9.

(1) *$\bar{u}_{\pm,i}$ is positively asymptotic to $e_i, \bar{e}_i$, and e^{i+1}_{g+1};*

(2) *$\bar{u}'_{\pm,i}$ is positively asymptotic to $e_i, \bar{e}'_i$, and $(e^{i+1}_{g+1})'$;*

(3) *$\tilde{u}_{\pm,i}$ is positively asymptotic to $\tilde{e}_i$ and e^{i+1}_{g+1};*

(4) *$\tilde{u}'_{\pm,i}$ is positively asymptotic to $\tilde{e}'_i$ and $(e^{i+1}_{g+1})'$.*

Curves with the same asymptotic ends are distinguished by whether their projections to $\Sigma_g \subset M$ agree with $R_+(\Sigma_g)$ or $R_-(\Sigma_g)$.

The holomorphic curves given by Lemma 3.9 serve as "walls" between the contact handles that have been attached to Σ_g, and will be used to enumerate certain holomorphic curves appearing in the symplectization $\mathbb{R} \times M$. Some of these walls are depicted as heavily shaded curves in Figure 3.4.

Let $\mathcal{M}(e_1, \dots, e_{g+1})$ be the index-2 moduli space of curves $u\colon S^2 \setminus \{p_1, \dots, p_{g+1}\} \to \mathbb{R} \times M$ which are positively asymptotic to $e_1, \dots, e_{g+1}$ and homologous to either u_+ or u_-. This space admits an obvious translation action by $\mathbb{R}$, and the following lemma describes the compactification of $\mathcal{M}(e_1, \dots, e_{g+1})/\mathbb{R}$.

Lemma 3.10. *The compactification $\overline{\mathcal{M}(e_1, \dots, e_{g+1})/\mathbb{R}}$ contains a pair of closed intervals $\mathcal{N}_\pm$ such that*

(1) *$\mathcal{N}_\pm$ contains the equivalence class of $u_\pm$;*

(2) *the boundary $\partial \mathcal{N}_\pm$ contains a two-level holomorphic building with top level $v_{1,\pm}$ a cylinder positively asymptotic to e_{g+1} and negatively asymptotic to h^1_{g+1}, and with bottom level $v_{0,\pm}$ positively asymptotic to $e_1, \dots, e_g, h^1_{g+1}$;*

(3) *the other boundary element of* $\partial\mathcal{N}_\pm$ *is a two-level holomorphic building with top level* $v'_{1,\pm}$ *a cylinder positively asymptotic to* e_{g+1} *and negatively asymptotic to* $(h^1_{g+1})'$*, and with bottom level* $v'_{0,\pm}$ *positively asymptotic to* $e_1, \ldots, e_g, (h^1_{g+1})'$*.*

Proof. We assume that $\mathcal{A}(e_1) = \mathcal{A}(e_2) = \cdots = \mathcal{A}(e_{g+1})$; we will use this action information as well as a description of the homology classes of the relevant curves to determine $\partial\mathcal{N}_\pm$. Consider

$$H_1(N(\Sigma_g \cup \mathbf{D}_+ \cup \mathbf{D}_-)) \simeq H_1(\Sigma_g) \simeq \mathbb{Z}^{2g},$$

and notice that we may choose curves $b_1, \ldots, b_g \subset \Sigma_g$ so that $[e_1], \ldots, [e_g], [b_1], \ldots, [b_g]$ forms a basis for $H_1(\Sigma_g)$. Moreover, the curve b_i is chosen so that if the attaching arc $\alpha_i^\pm$ is joined with (a subarc of) the arc a_i identified in the definition of a splitting surface, then the resulting closed curve is homologous to b_i. After orienting the curves $b_1, \ldots, b_g$, we compute the following homology classes[1]

$$[\tilde{e}_i] = [e_i] - \sum_{k=1}^{g-i}[b_{i+k}], \quad [e^i_{g+1}] = [e^{i-1}_{g+1}] + \sum_{k=1}^{g-i}[b_{i+k}], \quad \text{and} \quad [\overline{e}_i] = [b_i], \tag{3.1}$$

where $e^1_{g+1} := e_{g+1}$. The equation on the left is valid for $1 \leq i \leq g$, the right is valid for $2 \leq i \leq g+1$, and we recall that $\tilde{e}_{g+1} = e^{g+1}_{g+1}$. Similarly,

$$[\tilde{e}'_i] = [e_i] + \sum_{k=1}^{g-i}[b_{i+k}], \quad [(e^i_{g+1})'] = [(e^{i-1}_{g+1})'] + \sum_{k=1}^{g-i}[b_{i+k}], \quad \text{and} \quad [\overline{e}'_i] = -[b_i], \tag{3.2}$$

with the same conventions. Of course $[h^1_{g+1}] = [(h^1_{g+1})'] = [e_{g+1}]$, while $[h^i_{g+1}] = [e^i_{g+1}]$ and $[(h^i_{g+1})'] = [(e^i_{g+1})']$ for $2 \leq i \leq g$. We also have $[\tilde{h}_i] = [\tilde{e}_i]$ for $1 \leq i \leq g$. Now suppose we have a $(k+1)$-level holomorphic building $w_k \cup w_{k-1} \cup \cdots \cup w_0$ in $\partial\mathcal{N}_\pm$, with top level w_k and bottom level w_0. Let w_i^+ and w_i^- denote the sets of Reeb orbits to which w_i is positively and negatively asymptotic, respectively. We denote by $\mathcal{A}(w_i^\pm)$ the sum of the α-actions of the Reeb orbits in $w_i^\pm$ and by $[w_i^\pm]$ the sum of their homology classes. Of course we must have $\mathcal{A}(w_i^-) < \mathcal{A}(w_i^+)$ and $[w_i^-] = [w_i^+]$. We also point out that the curves $\overline{u}_{\pm,i}, \overline{u}'_{\pm,i}, \tilde{u}_{\pm,i}$, and $\tilde{u}'_{\pm,i}$

[1] The curves $b_1, \ldots, b_g$ are not canonically oriented, but we fix their orientations according to equation 3.1.

are all disjoint from the curves $u_\pm$ and hence, by the positivity of intersections, from our holomorphic building. In particular, these curves are disjoint from each level w_i. Moreover, the projections of these curves to M remain disjoint, so for each i, the image of $\pi \circ w_i$ is contained in a neighborhood N_1^j or $(N_2^j)'$, for some j.

Now because $u_\pm$ is positively asymptotic to $e_1, e_2, \ldots, e_{g+1}$, we know that

$$w_k^+ \subseteq \{e_1, e_2, \ldots, e_{g+1}\}.$$

We now consider the neighborhoods N_1^j or $(N_2^j)'$ in which $\pi \circ w_k$ might land. First, suppose that $\pi \circ w_k \subset N_1^j$ for some $j > 2$. Because Σ_g^{j-1} meets Σ_g in the curves $e_j, \ldots, e_g$, we have

$$w_k^+ \subset \{e_j, \ldots, e_g\} \quad \text{and} \quad w_k^- \subset \{e_j, \ldots, e_g, \overline{e}_j, e_{g+1}^j, h_{g+1}^j, e_{g+1}^{j+1}\}.$$

The action bounds of Lemma 3.6 allow us to exclude other curves from w_k^-. From equation 3.1 we see that the homological requirement $[w_k^+] = [w_k^-]$ can only be satisfied if we have $w_k^+ = w_k^-$, and this of course violates the action requirement $\mathcal{A}(w_k^+) > \mathcal{A}(w_k^-)$. We conclude that $\pi \circ w_k$ cannot be contained in N_1^j if $j > 1$. A completely analogous argument shows that $\pi \circ w_k$ cannot be contained in $(N_2^j)'$ when $j > 1$.

So $\pi \circ w_k$ is contained in either N_1^1 or $(N_2^1)'$. In the first case we see that

$$w_k^+ \subset \{e_1, \ldots, e_{g+1}\} \quad \text{and} \quad w_k^- \subset \{e_1, \ldots, e_{g+1}, \overline{e}_1, h_{g+1}^1, e_{g+1}^2\},$$

again using the action bounds of Lemma 3.6. The homological requirement $[w_k^+] = [w_k^-]$ then leads us to

$$w_k^- = (w_k^+ \setminus \{e_{g+1}\}) \cup \{h_{g+1}^1\} \quad \text{or} \quad w_k^- = (w_k^+ \setminus \{e_{g+1}\}) \cup \{\overline{e}_1, e_{g+1}^2\}.$$

The latter case is ruled out by part (7) of Lemma 3.6 and the fact that $\mathcal{A}(w_k^-) < \mathcal{A}(w_k^+)$. So if $\pi \circ w_k$ is contained in N_1^1, then $w_k^- = (w_k^+ \setminus \{e_{g+1}\}) \cup \{h_{g+1}^1\}$, and similarly if $\pi \circ w_k$ is contained in $(N_2^j)'$, then $w_k^- = (w_k^+ \setminus \{e_{g+1}\}) \cup \{(h_{g+1}^1)'\}$.

An important observation at this point is that w_k^- contains either h_{g+1}^1 or $(h_{g+1}^1)'$, and thus so does w_{k-1}^+. As with $\pi \circ w_k$, $\pi \circ w_{k-1}$ must be contained in a neighborhood of the form N_1^j or $(N_2^j)'$. Indeed, if $h_{g+1}^1 \in w_{k-1}^+$, then $\pi \circ w_{k-1} \subset N_1^1$ and if $(h_{g+1}^1)' \in w_{k-1}^+$, then $\pi \circ w_{k-1} \subset (N_2^1)'$. We now consider these two cases.

If $\pi \circ w_{k-1}$ is contained in N_1^1, then

$$w_{k-1}^- \subseteq \{e_1, \ldots, e_{g+1}, \overline{e}_1, h_{g+1}^1, e_{g+1}^2\}.$$

Now w_{k-1}^+ must contain h_{g+1}^1, must be homologous to w_{k-1}^-, and must satisfy $\mathcal{A}(w_{k-1}^-) < \mathcal{A}(w_{k-1}^+)$. The first two conditions are satisfied if

$$w_{k-1}^+ = \{e_1, \ldots, e_g, h_{g+1}^1\} \quad \text{and} \quad w_{k-1}^- = \emptyset$$

or if

$$w_{k-1}^+ = \{h_{g+1}^1\} \cup \bar{w} \quad \text{and} \quad w_{k-1}^- = \{\bar{e}_1, e_{g+1}^2\} \cup \bar{w}$$

for some $\bar{w} \subseteq \{e_1, \ldots, e_g\}$. However, the latter case is prohibited by the action bound, so we conclude that $w_{k-1}^- = \emptyset$, meaning that our building has height two. All that remains is to verify that the top level of our building is a cylinder. To see that this is the case, notice that w_{k-1} must be connected, since $w_{k-1}^+ = \{e_1, \ldots, e_g, h_{g+1}^1\}$ and the only null-homologous combination of these positive ends is $e_1 + \cdots + e_g + h_{g+1}^1$. So if w_k has more than one negative end, then the building $w_k \cup w_{k-1}$ has nonzero genus. Of course this is impossible, since all of the curves in $\mathcal{M}(e_1, \ldots, e_{g+1})/\mathbb{R}$ are planar. So w_k is a cylinder with positive end e_{g+1} and negative end h_{g+1}^1, as desired.

If instead the image of $\pi \circ w_{k-1}$ is contained in $(N_2^1)'$, then the same considerations lead us to conclude that w_k is a cylinder with positive end e_{g+1} and negative end $(h_{g+1}^1)'$, and that w_{k-1} is positively asymptotic to $e_1, \ldots, e_g, h_{g+1}^1$, with no negative ends. We thus define $v_{0,\pm} = w_{k-1}$ and $v_{1,\pm} = w_k$ in the case that $\pi \circ w_{k-1}$ is contained in N_1^1 and define $v'_{0,\pm} = w_{k-1}$ and $v'_{1,\pm} = w_k$ in the case that $\pi \circ w_{k-1}$ is contained in $(N_2^1)'$. $\square$

Now let $\mathcal{M}_{\widehat{W}}(e_1, \ldots, e_g, h^1_{g+1})$ be the index-1 moduli space of holomorphic curves in $\widehat{W}$ which are positively asymptotic to $e_1, \ldots, e_g, h^1_{g+1}$ and represent the same homology class as $v_{0,+}$ or $v_{0,-}$, the curves identified (up to translation) in Lemma 3.10. The following lemma will allow us to use this moduli space to interpolate between $v_{0,+}$ and $v_{0,-}$, producing what will serve as the middle part of our 1-parameter family.

Lemma 3.11. *One component of the compactification $\overline{\mathcal{M}_{\widehat{W}}(e_1, \ldots, e_g, h^1_{g+1})}$ is a closed interval I with $\partial I = \{v_{0,+}, v_{0,-}\}$.*

Proof. We denote $\mathcal{M}_{\widehat{W}}(e_1, \ldots, e_g, h^1_{g+1})$ by $\mathcal{M}_{\widehat{W}}$ and investigate the objects that could appear in the boundary of the compactification of $\mathcal{M}_{\widehat{W}}$. Because this is an index-1 family, the compactification will not contain any nodal curves, and the only possible boundary elements are holomorphic buildings in the symplectization end of $\widehat{W}$. Suppose we have such a building, and let

$$w\colon S^2 \setminus \{p_1, \ldots, p_k\} \to \mathbb{R} \times M$$

be its topmost level. As in the proof of Lemma 3.10, the curves $\bar{u}_{\pm,i}, \bar{u}'_{\pm,i}, \tilde{u}_{\pm,i}$, and $\tilde{u}'_{\pm,i}$ are all disjoint from elements of $\mathcal{M}_{\widehat{W}}$ and hence, by the positivity of intersections, from w. So the image of the projection $\pi \circ w$ must be contained in one of the neighborhoods $N^j_1, N^j_2, (N^j_1)', (N^j_2)'$ identified above. We claim that this is only possible if w is positively asymptotic to $e_1, \ldots, e_g, h^1_{g+1}$ and has no negative ends.

We first show that $\pi \circ w$ cannot be contained in a neighborhood of the form N^j_2 or $(N^j_1)'$. To this end, suppose that $\pi \circ w$ is contained in N^j_2. Then

$$w^+ \subset \{e_1, \ldots, e_g\} \quad \text{and} \quad w^- \subset \{e_1, \ldots, e_g, \overline{e}_j, \tilde{e}_j, \tilde{h}_j, e^{j+1}_{g+1}\}.$$

But the homology classes computed in equation 3.1 tell us that curves chosen in this way can only satisfy $[w^+_k] = [w^-_k]$ if in fact $w^+_k = w^-_k$. Of course this violates the inequality $\mathcal{A}(w^+_k) > \mathcal{A}(w^-_k)$, and we see that $\pi \circ w$ cannot be contained in N^j_2 for any j. The same reasoning shows that $\pi \circ w$ also cannot be contained in a neighborhood of the form $(N^j_1)'$.

Just as in the proof of Lemma 3.10, the projection $\pi \circ w$ of the topmost level w cannot be contained in N_1^j or $(N_2^j)'$ if $j > 1$. These leaves two possibilities — either $\pi \circ w$ is contained in N_1^1, or in $(N_2^1)'$ — which we now consider.

Suppose that the image of $\pi \circ w$ is contained in N_1^1, meaning that

$$w^+ \subseteq \{e_1, \dots, e_g, h_{g+1}^1\} \quad \text{and} \quad w^- \subseteq \{e_1, \dots, e_{g+1}, h_{g+1}^1, \bar{e}_1, e_{g+1}^2\}.$$

Again we must have $[w^+] = [w^-]$. Because we could have $[w^+] = 0$, it is possible that w^- is empty, and we have a holomorphic building of height one. Suppose this is not the case. Because $[h_{g+1}^1] = [e_{g+1}^2] + [\overline{e}_1] = [e_{g+1}]$, one homological possibility is that as we move from w^+ to w^- we replace the curve h_{g+1}^1 with e_{g+1} or with e_{g+1}^2 and $\overline{e}_1$. That is, if $w^- \neq \emptyset$, then either $w^+ = w^-$,

$$w^+ = \{h_{g+1}^1\} \cup \overline{w} \quad \text{and} \quad w^- = \{e_{g+1}\} \cup \overline{w}$$

for some $\overline{w} \subseteq \{e_1, \dots, e_g\}$, or

$$w^+ = \{h_{g+1}^1\} \cup \overline{w} \quad \text{and} \quad w^- = \{e_{g+1}^2, \overline{e}_1\} \cup \overline{w}.$$

However, all of these possibilities are prohibited by the action requirement $\mathcal{A}(w^-) < \mathcal{A}(w^+)$. The first possibility obviously violates this requirement, while the second and third do so because $\mathcal{A}(h_{g+1}^1) < \mathcal{A}(e_{g+1}) < \mathcal{A}(e_{g+1}^2) + \mathcal{A}(\overline{e}_1)$. From all of this we conclude that $w^- = \emptyset$ and thus w cannot be the topmost level of a building of height greater than one. The same reasoning shows that if $\pi \circ w$ is contained in $(N_2^1)'$ then $w^- = \emptyset$. So in any case, w^- is empty, and $\pi \circ w$ is contained in either N_1^1 or $(N_2^1)'$. But if $\pi \circ w$ is contained in $(N_2^1)'$, then

$$w^+ \subseteq \{e_1, \dots, e_g\},$$

so we cannot have $[w^+] = 0$. So in fact the image of $\pi \circ w$ lies in N_1^1, and w has no negative ends.

So w is a holomorphic curve in the symplectization end of $\widehat{W}$ positively asymptotic to $e_1, \dots, e_g, h_{g+1}^1$. In Lemma 3.10 we showed that there are precisely two such curves — $v_{0,+}$

and $v_{0,-}$ — so w must be one of these two. We conclude that

$$\partial\overline{\mathcal{M}_{\widehat{W}}(e_1,\ldots,e_g,h^1_{g+1})} = \{v_{0,+}, v_{0,-}\}.$$

So $\overline{\mathcal{M}_{\widehat{W}}(e_1,\ldots,e_g,h^1_{g+1})}$ contains the desired component I. □

Lemma 3.12. *There is a 1-parameter family*

$$\mathcal{S} = \{u_t \colon S^2 \setminus \{p_1,\ldots,p_{g+1}\} \to \widehat{W} \mid du_t \circ j = J \circ du_t\}_{t\in\mathbb{R}}$$

of embedded pseudoholomorphic curves in $(\widehat{W},\widehat{\omega})$ such that

(1) *for $t \gg 0$, the images of u_t and u_{-t} are contained in the symplectization part of $\widehat{W}$;*

(2) *for $t \gg 0$, the image of $\pi \circ u_{\pm t}$ is $R_\pm(\Sigma_g)$, where $\pi\colon [0,\infty)\times M \to M$ is the obvious projection;*

(3) *the images of u_{t_1} and u_{t_2} are disjoint whenever $t_1 \neq t_2$.*

Proof. Consider the interval I given by Lemma 3.11. We take this interval to be the "middle part" of $\mathcal{S}$ and for $t \gg 0$ we take $u_{\pm t}$ to be $v_{0,\pm}$, translated by $t+c$ in the symplectization end $[0,\infty)\times M$, where c is some constant. Property (1) follows immediately. Because $v_{0,\pm}$ is positively asymptotic to h^1_{g+1} and not e_{g+1}, we must isotope Σ_g to ensure that $R_\pm(\Sigma_g) = \mathrm{im}(\pi\circ v_{0,\pm})$ and thus satisfy property (2). Finally, notice that if $t_1 \neq t_2$ are large then the images of u_{t_1} and u_{t_2} are disjoint; the positivity of intersections and the homotopy invariance of the intersection number tells us that in fact u_{t_1} and u_{t_2} are disjoint for any $t_1 \neq t_2$. □

Lemma 3.13. *The map $\iota\colon \mathbb{R}\times(S^2\setminus\{p_1,\ldots,p_{g+1}\}) \to \widehat{W}$ defined by*

$$\iota(t,x) := u_t(x),$$

with u_t as identified in Lemma 3.12, is an embedding of a genus-g handlebody into $\widehat{W}$.

Proof. For an arbitrary $t\in\mathbb{R}$ the curve u_t is an embedding and thus each curve $u_{t'}$, for t' near t, can be thought of as a section of the normal bundle N_{u_t}. We can compute the first

Chern number of this bundle according to

$$c_1(N_{u_t}) = c_1(u_t^*T\widehat{W}) - \chi(S^2 \setminus \{p_1, \dots, p_{g+1}\}) = c_1(u_t^*T\widehat{W}) + g - 1,$$

but first we must compute $c_1(u_t^*T\widehat{W})$. For this we appeal to [Wen10a, Equation 1.1], which says that

$$2c_1(u_t^*T\widehat{W}) = \mathrm{ind}(u_t) + \chi(S^2 \setminus \{p_1, \dots, p_{g+1}\}) - \mu_{CZ}(u_t),$$

where the last term is a signed count of the Conley-Zehnder indices of the orbits to which u_t is asymptotic. Then

$$2c_1(u_t^*T\widehat{W}) = 1 + (1-g) - g = 2 - 2g,$$

so $c_1(u_t) = 1 - g$ and it follows that $c_1(N_{u_t}) = 0$. So sections of N_{u_t} are zero-free, meaning that ι is an embedding. □

The stage is now set for the construction of (W', ω'), the symplectic manifold promised by Theorem 3.2. This construction proceeds exactly as in [Men18], with small changes to the statements of the lemmas found there. The strategy is to remove from W the handlebody $H \subset \widehat{W}$ embedded by ι in Lemma 3.13. This is done in stages. First W is enlarged to $W_R := W \cup ([0,R] \times M)$, with R chosen large enough that there exists some $T \gg 0$ such that $\mathrm{im}(u_{\pm T})$ is contained in $\widehat{W} \setminus W$, with $\mathrm{im}(\pi \circ u_{\pm T})$ equal to $R_\pm(\Sigma_g)$ minus a small collar neighborhood. From W_R we remove $\tilde{N}(\Gamma_{\Sigma_g})$, a small tubular neighborhood of $\{R\} \times \Gamma_{\Sigma_g}$, leaving us with $W_R' := W_R - \tilde{N}(\Gamma_{\Sigma_g})$. This allows us to decompose $\partial W_R'$ into its horizontal part

$$\partial_h W_R' = \partial W_R' - \partial W_R \simeq \bigsqcup_{i=1}^{g+1} (S^1 \times D^2) \tag{3.3}$$

and its vertical part $\partial_v W_R' = \partial W_R' - \partial_h W_R'$. Note that the deletion of $\tilde{N}(\Gamma_{\Sigma_g})$ from W_R removes small collar neighborhoods from $\{R\} \times R_\pm$, leaving us with $\{R\} \times R_\pm'$. We now begin modifying H in preparation for its removal from W_R'.

Lemma 3.14 (c.f. [Men18, Lemma 1.3.10])**.** *There exists an embedding* $\Sigma_L \times [-T, T] \hookrightarrow W_R'$

so that

(1) Σ_L *is a compact surface with genus 0 and* $g+1$ *boundary components;*

(2) $\Sigma_L \times \{\pm T\} = \{R\} \times R'_\pm$*;*

(3) *using the identification given in equation 3.3 we have*

$$\partial\Sigma_L \times \{t\} = \bigsqcup_{i=1}^{g+1}(S^1 \times \gamma(t)) \subset \partial_h W'_R$$

for $t \in [-T, T]$*, where* $\gamma(t)$ *is the straight arc from* $(-1,0)$ *to* $(1,0)$ *in* D^2*.*

We denote the embedded copy of $\Sigma_L \times [-T, T]$ by $H' \subset W'_R$ and endow it with the obvious coordinates (x, t). The following two results are proven in [Men18] and allow us to cut W'_R along H' to obtain a symplectic manifold (W', ω') that strongly fills its boundary.

Lemma 3.15 (c.f. [Men18, Lemma 1.3.11])**.** *Let* $B = [-T, T] \times [-\epsilon, \epsilon]$ *with coordinates* (t, w)*. After slight adjustments of* H' *and* W'_R*, there exists a neighborhood* $N(H') \simeq H' \times [-\epsilon, \epsilon] \subset W'_R$ *and a 1-form* $\lambda = \lambda_B + \lambda_{\Sigma_L}$ *on* $N(H')$ *such that*

(1) $\Sigma_L \times \{\pm T\} \times [-\epsilon, \epsilon] \subset \partial_v W'_R$ *and* $(\partial\Sigma_L) \times B \subset \partial_h W'_R$*;*

(2) λ_{Σ_L} *is the Liouville form for* $R'_\pm$*;*

(3) $\lambda_B = t\, dw$*;*

(4) $d\lambda$ *is the symplectic form on* W'_R*;*

(5) λ *agrees with the Liouville form on* W'_R *near* $\partial W'_R$*.*

Lemma 3.16 (c.f. [Men18, Lemma 1.3.12])**.** *There exists a modification*

$$\lambda' = \lambda + d(tw) = 2t\ dw + w\ dt + \lambda_{\Sigma_L},$$

whose Liouville vector field $Z' = 2t\partial_t - w\partial_w + Z_{\Sigma_L}$ *points into* $N(H')$ *along* $w = \pm\epsilon$*.*

At last we define $W' := W'_R - N(H')$ and $\omega' := d\lambda'$ and from Lemma 3.16 we conclude that (W', ω') strongly fills its boundary. In case our original symplectic filling was exact we ask the same of (W', ω'). Once again we may appeal to [Men18], where the proof of the following lemma is genus-independent.

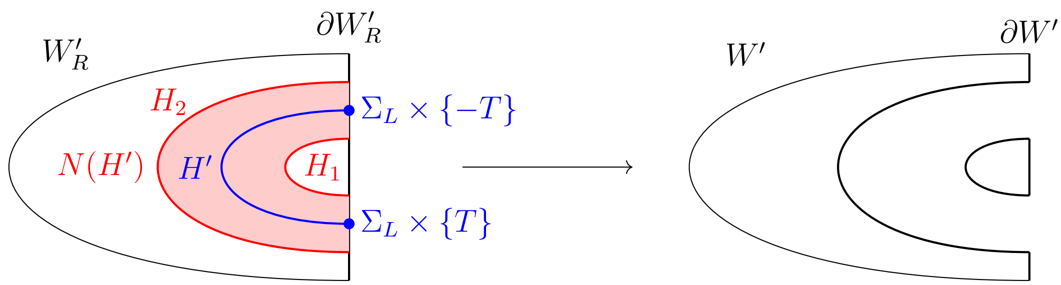


Figure 3.5: The removal of $N(H')$ from W'_R. On the right, $\partial W'$ has two connected components.

Lemma 3.17 (c.f. [Men18, Lemma 1.3.13]). *If $(W, \omega = d\beta)$ is an exact filling, then there exists a 1-parameter family of Liouville forms β_τ, $\tau \in [0,1]$, on W'_R such that $\beta_0 = \beta$ and $\beta_1 = \lambda'$ on $N(H') \cap \{-\epsilon/2 \leq w \leq \epsilon/2\}$.*

Let us give an informal summary of the relationship between $\partial W'$ and M. The first step in constructing W' was to consider W_R, whose boundary is contactomorphic to M. From W_R we deleted a neighborhood of the dividing set of Σ_g. This provided a decomposition of $\partial W'_R$ into its horizontal and vertical parts, but the overall effect on ∂W_R was trivial. The last step in our construction — deleting $N(H')$ from W'_R — made the most substantive changes to the boundary. We first identified H', a handlebody in W'_R which picked out for us two copies of Σ_L in $\partial W'_R$. Namely, H' distinguished the Liouville hypersurfaces $\Sigma_L \times \{\pm T\} = \{R\} \times R'_\pm$. Then $N(H')$ is a neighborhood of H', part of whose boundary lies in $\partial W'_R$. The part of $\partial N(H')$ lying in the interior of W'_R consists of two disjoint copies of H', and the part lying in $\partial W'_R$ includes $\Sigma_L \times \{\pm T\}$. So deleting $N(H')$ from W'_R cuts $\partial W'_R$ open along the Liouville hypersurfaces $\Sigma_L \times \{\pm T\}$ and glues in two handlebodies modeled on H'. This process is depicted in Figure 3.5.

All that remains is to use symplectic handle attachment to recover W from W'. To this end we observe that the neighborhood $(N(H'), d\lambda')$ we have removed from W'_R is precisely the abstract symplectic handle $(H_{\Sigma_L}, \omega_{\lambda_{\Sigma_L}})$ constructed from the Liouville domain $(\Sigma_L, \lambda_{\Sigma_L})$. That is, we have obtained W' from W by removing a symplectic handle, and thus may recover W by reattaching said handle as described in Section 2.5.

CHAPTER 4

Virtually overtwisted lens spaces

In this chapter we use Theorem 3.3 to classify the symplectic fillings of virtually overtwisted contact structures on *lens spaces* up to diffeomorphism. We will define lens spaces via their surgery diagrams, and recall the classification of tight contact structures on lens spaces due to Giroux and Honda. The strong symplectic fillings of universally tight lens spaces were classified by Lisca in [Lis08], and the main result of this chapter, Theorem 4.2, reduces the analogous classification for virtually overtwisted structures to the universally tight case.

From the perspective of Dehn surgery, lens spaces give a very simple class of smooth 3-manifolds.

Definition. For any relatively prime integers $p > q \geq 1$, the *lens space* $L(p,q)$ is the smooth manifold obtained via Dehn surgery on the unknot in S^3, with framing $-p/q$.

With an eye towards Legendrian surgery diagrams, we prefer to present $L(p,q)$ as the result of integral surgery on a link in S^3. First, let us write

$$-\frac{p}{q} = [a_0, a_1, \ldots, a_n] := a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cdots \cfrac{1}{a_n}}},$$

for some uniquely determined integers $a_0, a_1, \ldots, a_n \leq -2$. Then it is well-known that the surgery diagram in Figure 4.1 produces the lens space $L(p,q)$. One obtains this diagram from the $(-p/q)$-framed unknot via a sequence of *slam dunks*.

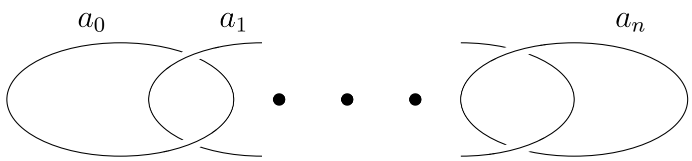


Figure 4.1: Handlebody diagram for a filling of $L(p, q)$. We produce a contact structure on $L(p, q)$ by putting each of the unknots in Legendrian position and stabilizing appropriately.

4.1 Tight contact structures on lens spaces

From the surgery diagram for $L(p, q)$ found in Figure 4.1 we can produce contact structures on $L(p, q)$. We do so by putting the link $L = L_0 \cup L_1 \cup \cdots \cup L_n$ into Legendrian position in $(S^3, \xi_{\mathrm{std}})$, with $\mathrm{tb}(L_i) - 1 = a_i$. In $(S^3, \xi_{\mathrm{std}})$, the unknot is Legendrian simple, and thus its Legendrian realizations are determined up to isotopy by their Thurston-Bennequin invariants and rotation numbers. With the Thurston-Bennequin invariant fixed at $\mathrm{tb}(L_i) = a_i + 1$, there are $|a_i + 1|$ possibilities for the rotation number:

$$\mathrm{r}(L_i) \in \{a_i + 2, a_i + 4, \ldots, a_i + 2|a_i + 1|\}.$$

In this way, we produce $|(a_0 + 1)(a_1 + 1) \cdots (a_n + 1)|$ distinct Legendrian surgery diagrams for $L(p, q)$. For instance, the nine diagrams associated to $L(24, 7)$ are seen in Figure 4.2. We make the important observation that all of the resulting contact structures are, by construction, symplectically fillable.

This construction of contact structures on $L(p, q)$ leaves open the questions of whether structures corresponding to distinct Legendrian surgery diagrams are in fact distinct, and whether there are any other fillable contact structures on $L(p, q)$. These questions were answered independently by Giroux and Honda.

Theorem 4.1 ([Gir00],[Hon00a, Theorem 2.1])**.** *There are exactly $|(a_0+1)(a_1+1) \cdots (a_n+1)|$ tight contact structures on $L(p, q)$, up to isotopy, all obtained via the surgery diagrams described above.*

Remark. Because symplectically fillable contact structures are tight, Theorem 4.1 tells us that $L(p, q)$ admits no fillable structures other than those we have constructed.

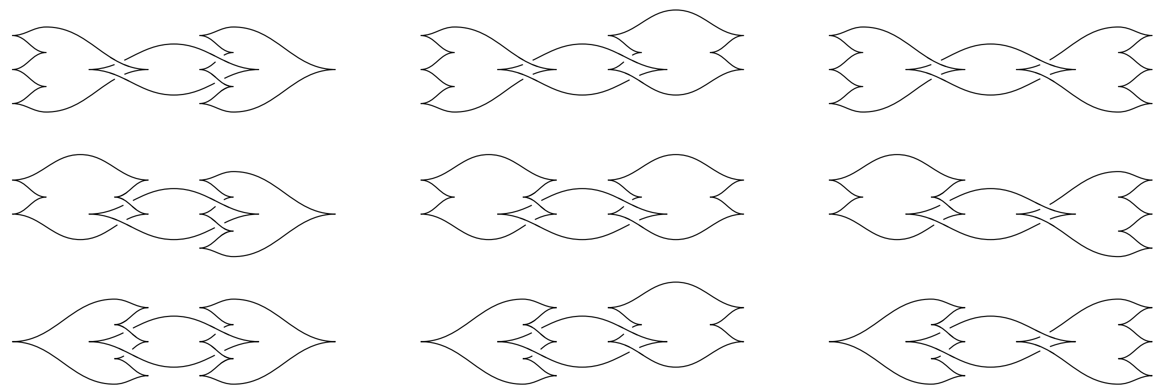


Figure 4.2: Legendrian surgery diagrams for $L(24, 7)$

Now that we are discussing the contact structures on $L(p, q)$ in terms of their Legendrian surgery diagrams, it will be convenient to identify these diagrams by their stabilizations. Namely, we obtain a Legendrian surgery diagram for $L(p, q)$ by first drawing a chain of Legendrian unknots, each with $\mathrm{tb}(L_i) = -1$. We then perform $|a_i + 2|$ stabilizations on L_i, and the rotation number of L_i will be given by the difference between the number of positive stabilizations and the number of negative stabilizations.

We point out that Honda also proved in [Hon00a] that the universally tight contact structures on $L(p, q)$ are precisely those which have Legendrian surgery diagrams featuring stabilizations of just one sign. Any Legendrian surgery diagram for $L(p, q)$ which features both positive and negative stabilizations will produce a virtually overtwisted contact structure. For example, the top-left and bottom-right Legendrian surgery diagrams in Figure 4.2 produce universally tight contact structures on $L(24, 7)$. We refer to the structure whose Legendrian surgery diagram has only positive stabilizations as the *standard contact structure* on $L(p, q)$.

4.2 History of the filling problem

Having produced a large class of contact manifolds with relatively simple topology, we naturally wonder whether or not the symplectic fillings of these contact manifolds may be classified. The earliest results in this direction were obtained before the work of Giroux and Honda. For instance, Eliashberg showed in [Eli90] that the standard tight contact structure on S^3 has a unique minimal symplectic filling, given by B^4 with its standard symplectic structure. Next, McDuff showed in [McD91] that the standard contact structure on $L(p,1)$ has a unique minimal symplectic filling, provided $p \neq 4$. For $L(4,1)$ she showed that the standard contact structure admits exactly two such fillings. Because the standard structure is related by orientation-reversing diffeomorphism to the structure obtained with only negative stabilizations, McDuff's classification settles the filling problem for both universally tight contact structures on $L(p,1)$, $p > 2$. (The lens space $L(2,1) \cong \mathbb{R}P^3$ admits just one tight contact structure, up to isotopy.)

Arguably the most substantial progress in the classification of symplectic fillings of lens spaces was made by Lisca in [Lis08]. For any relatively prime integers $p > q \geq 1$, Lisca constructs a finite list of minimal symplectic fillings of $(L(p,q), \xi_{\text{std}})$, and shows that in fact any minimal symplectic filling of $(L(p,q), \xi_{\text{std}})$ is diffeomorphic to one of these. As noted above, ξ_{std} is universally tight and, up to isotopy, $-\xi_{\text{std}}$ is the only other universally tight contact structure on $L(p,q)$. (If $q = p-1$, $-\xi_{\text{std}}$ is isotopic to ξ_{std}.) Lisca's classification then settles the symplectic filling problem for all universally tight lens spaces.

Following Lisca's work, the symplectic filling problem remained open for virtually overtwisted lens spaces. The first result in this direction was given by Plamenevskaya and van Horn-Morris in [PV10], where it is shown that any virtually overtwisted contact structure on $L(p,1)$, $p > 2$, has a unique (up to symplectic deformation equivalence) minimal symplectic filling. As a result, the symplectic filling problem has been solved for all contact structures on $L(p,1)$. The techniques used by Plamenevskaya and van Horn-Morris — developed by Wendl in [Wen10b] — were pushed further by Kaloti in [Kal13] to solve the symplectic filling problem for lens spaces of the form $L(pm+1, m)$. Further progress in the virtually overtwisted case was made by Fossati in [Fos19] and [Fos20], using the work of Wendl as well as Theorem 3.3.

4.3 Fillings of virtually overtwisted contact structures

4.3.1 The classification statement

In this section, we complete the classification of symplectic fillings of tight contact structures on lens spaces, up to diffeomorphism. In particular, we reduce the classification problem for a given virtually overtwisted lens space to the same problem on a corresponding collection of universally tight lens spaces.

Theorem 4.2. *Let ξ be a virtually overtwisted tight contact structure on the lens space $L(p, q)$, with $p > q \geq 1$ and $(p, q) = 1$. Then every strong (respectively, exact) symplectic filling of $(L(p, q), \xi)$ is obtained by attaching a sequence of Weinstein 2-handles to a strong (respectively, exact) symplectic filling of a connected sum of universally tight lens spaces.*

Remark. This result appeared in [CL20], joint work with Y. Li, and independently in [ER20], work of Etnyre-Roy. Etnyre-Roy more thoroughly examine the consequences of this classification result.

As noted, the virtually overtwisted contact structures on $L(p, q)$ are those whose Legendrian surgery diagrams feature at least one stabilization of each sign. If any given knot K in the diagram has been stabilized both positively and negatively, then we may immediately apply Theorem 3.4 to conclude that all fillings of $(L(p, q), \xi)$ result from attaching a Weinstein 2-handle to $(L(p', q'), \xi')\#(L(p'', q''), \xi'')$, the connected sum that remains when K is removed from the diagram. Thus the work of proving Theorem 4.2 is reduced to the case where each knot in the Legendrian surgery diagram for $(L(p, q), \xi)$ features stabilizations of only one sign, but for which these signs do not all agree. In such a case we are still able to find a mixed torus (as Menke does when proving Theorem 3.4), but the contact manifold $\partial(W', \omega')$ which results from applying Theorem 3.3 to a filling (W, ω) of $(L(p, q), \xi)$ is not uniquely determined. We will obtain Theorem 4.2 by enumerating the possibilities.

4.3.2 Proof of Theorem 4.2

For the duration of this chapter, we consider a fixed lens space $L(p,q)$, with virtually overtwisted contact structure ξ obtained from a Legendrian surgery diagram as described above. Namely,

$$-\frac{p}{q} = [a_0, a_1, \ldots, a_n]$$

for uniquely determined integers $a_i \leq -2$, and the stabilizations applied to the knots in Figure 4.1 do not all have the same sign. We will prove Theorem 4.2 by showing that every strong or exact symplectic filling of $(L(p,q),\xi)$ can be obtained by attaching a Weinstein 2-handle to a filling of a connected sum of the form

$$(L(p',q'),\xi')\#(L(p'',q''),\xi''),$$

obtained by deleting a single knot from the diagram describing $(L(p,q),\xi)$. Beginning with an arbitrary filling of $(L(p,q),\xi)$, this decomposition may be inductively applied (in conjunction with Theorem 3.1) until we have a symplectic filling of a connected sum of the form

$$\mathop{\#}_{i=1}^{\ell}(L(p_i,q_i),\xi_i),$$

where each $(L(p_i,q_i),\xi_i)$ is a universally tight lens space. To produce a complete list of the fillings of $(L(p,q),\xi)$, we consider the fillings of all connected sums of this form which may result from $(L(p,q),\xi)$.

We noted above that the case where one of the knots in Figure 4.1 has been stabilized both positively and negatively is easily dispatched. We now focus on the case where no knots have been stabilized both positively and negatively. In this case we may identify knots K_+ and K_-, each of which has been stabilized at least once, with all stabilizations being positive or negative, respectively. Moreover, we may choose K_+ and K_- to be adjacent, in that none of the knots between them have been stabilized. Finally, our argument loses no generality by assuming that K_+ is to the right of K_- in Figure 4.1.

We now define

$$-\frac{p'}{q'} = [a_0, \ldots, a_{n-1}, a_n + 1], \tag{4.1}$$

where we identify $[a_0, \ldots, a_{n-1}, a_n+1]$ with $[a_0, \ldots, a_{n-2}, a_{n-1}+1]$ if $a_n = -2$. Now [Hon00a, Section 4.6] allows us to write $L(p, q) = V_0 \cup_A V_1$, where V_0 and V_1 are solid tori with a map $A\colon \partial V_0 \to \partial V_1$, the dividing curves of ∂V_0 are vertical, and the dividing curves of ∂V_1 have slope $-p'/q'$. Moreover, we may decompose V_1 as

$$V_1 = N \cup (V_1 \setminus N),$$

with $V_1 \setminus N \cong T^2 \times I$, such that $s_0 = -1$ and $s_1 = -p'/q'$. Here we denote by s_i the slope of the dividing curves of $T^2 \times \{i\}$, for $i = 0, 1$.

The thickened torus $T^2 \times I$ has a basic slice decomposition which we now describe. Let

$$0 \leq i_1 < i_2 < \cdots < i_\ell \leq n$$

be the indices for which $a_{i_j} \leq -3$. Then $T^2 \times I$ decomposes into ℓ *continued fraction blocks*, with a total of

$$|(a_{i_1} + 2)(a_{i_2} + 2) \cdots (a_{i_\ell} + 2)|$$

basic slices. The basic slices in each continued fraction block will all be of a single sign, and the continued fraction blocks corresponding to K_+ and K_- will be adjacent, and of opposite sign. We immediately see that the boundary convex torus T sitting between the continued fraction blocks associated to K_+ and K_- is a mixed torus, sandwiched between basic slices $S_+ \subset K_+$ and $S_- \subset K_-$ of opposite sign.

Let $-p'_1/q'_1$ be the slope of the dividing curves on T, and let $-p'_2/q'_2$, $-p'_0/q'_0$ be the opposite slopes of S_+, S_-, respectively. We would like to normalize this neighborhood of T. After observing that

$$q'_1 p'_2 - p'_1 q'_2 = 1 \quad \text{and} \quad q'_0 p'_1 - p'_0 q'_1 = 1,$$

we see that applying the transformation

$$\begin{pmatrix} 1 & 0 \\ p_2'q_0' - q_2'p_0' - 1 & 1 \end{pmatrix} \begin{pmatrix} -p_1' & -q_1' \\ p_2' & q_2' \end{pmatrix} \in SL(2, \mathbb{Z})$$

leaves us with the slopes

$$s_0 = -1, \quad s_1 = \infty, \quad s_2 = p_2'q_0' - q_2'p_0' - 1.$$

According to Theorem 3.3, applying the symplectic JSJ decomposition to a filling of $(L(p,q),\xi)$ will produce a filling of (M',ξ'), obtained from $(L(p,q),\xi)$ by splitting open along T with slope $0 \le s \le p_2'q_0' - q_2'p_0' - 2$.

We now claim that $p_2'q_0' - q_2'p_0' - 2 = m + 1$, where m is the number of unstabilized knots between K_+ and K_- in Figure 4.1. According to [Hon00a, Lemma 4.12], the slopes of the basic slice decomposition of $T^2 \times I$ are obtained by incrementing the last entry of the continued fraction expansion of $-p'/q'$ (as in 4.1) until we have -1. In particular, we may write

$$-\frac{p_2'}{q_2'} = [a_0, a_1, \ldots, a_k, \overbrace{-2, \ldots, -2}^{m+1}]$$

for some $k < n$ with $a_k \le -3$ and then see that

$$-\frac{p_1'}{q_1'} = [a_0, a_1, \ldots, a_k + 1]$$

and

$$-\frac{p_0'}{q_0'} = [a_0, a_1, \ldots, a_k + 2], \text{ if } a_k \le -4 \quad \text{or} \quad -\frac{p_0'}{q_0'} = [a_0, a_1, \ldots, a_{k-1} + 1], \text{ if } a_k = -3.$$

We can now verify our claim inductively. If $a_k \le -4$ we have

$$[a_k + 2] = -\frac{a_k + 2}{-1} \quad \text{and} \quad [a_k, \overbrace{-2, \ldots, -2}^{m+1}] = -\frac{(m+2)a_n + (m+1)}{-(m+2)}$$

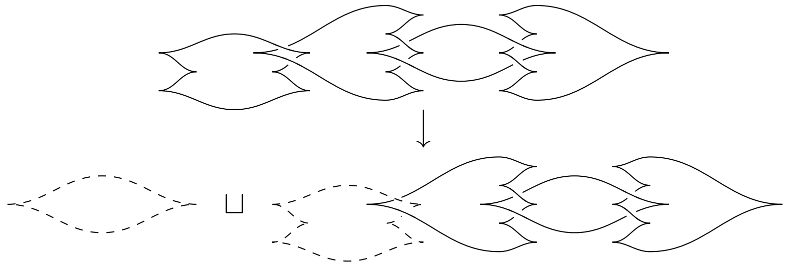


Figure 4.3: Every filling of the top lens space $L(89, 24)$ with the given contact structure is obtained by attaching a round symplectic 1-handle to a filling of the disjoint union $S^3 \sqcup L(24, 7)$ below; the round 1-handle is attached along the dashed knots. Fillings of $L(24, 7)$ can be further decomposed as seen in Figure 4.4.

and observe that

$$(-1)((m+2)a_k + (m+1)) - (-(m+2))(a_k + 2) = m + 3.$$

If we instead have $a_k = -3$, then

$$[a_{k-1} + 1] = -\frac{a_{k-1} + 1}{-1} \quad \text{and} \quad [a_{k-1}, -3, \overbrace{-2, \ldots, -2}^{m+1}] = -\frac{(2m+5)a_{k-1} + (m+2)}{-(2m+5)},$$

so

$$(-1)((2m+5)a_{k-1} + (m+2)) - (-(2m+5))(a_{k-1} + 1) = m + 3.$$

In either case, we may now apply the following inductive step. If a/b and a'/b' satisfy $ab' - a'b = m + 3$, then

$$[r, a/b] = \frac{ar - b}{a} \quad \text{and} \quad [r, a'/b'] = \frac{a'r - b'}{a'}$$

satisfy

$$(ar - b)a' - a(a'r - b') = ab' - ba' = m + 3.$$

This proves our claim, so we see that every filling of $(L(p, q), \xi)$ is obtained by attaching a round symplectic 1-handle to a filling of a contact manifold which is obtained from $(L(p, q), \xi)$ by splitting along T with slope $0 \leq s \leq m + 1$.

It is now straightforward to check that splitting along T with slope $s = 0$ produces a disjoint union of lens spaces, obtained from Figure 4.1 by deleting K_+ and realizing the two resulting chains of unknots in separate diagrams. Attaching a round symplectic 1-handle to this disjoint union corresponds to first attaching a Weinstein 1-handle — which produces the connected sum these lens spaces — and then attaching a Weinstein 2-handle along K_+. Similarly, splitting $(L(p,q),\xi)$ along T with slope $s = m+1$ corresponds to deleting the knot K_-. Each intermediate slope corresponds to deleting an unstabilized knot between K_+ and K_-. In any case, we see — as claimed above — that every filling of $(L(p,q),\xi)$ can be obtained by attaching a Weinstein 2-handle to a symplectic filling of a connected sum of lens spaces which is obtained by erasing a single knot from Figure 4.1. If the constituent lens spaces in this connected sum are virtually overtwisted, we may repeat this process until we have a connected sum of universally tight lens spaces. This proves Theorem 4.2.

Theorem 4.2 and its proof provide a recipe for classifying the fillings of a virtually overtwisted lens space $(L(p,q),\xi)$. Given a depiction of the lens space as in Figure 4.1, we can produce a tree whose leaves are disjoint unions of universally tight lens spaces, and every filling of $(L(p,q),\xi)$ can be obtained by attaching a specified sequence of round symplectic 1-handles to a filling of one of these disjoint unions. An example of such a tree is given in Figure 4.3. The root of our tree is $(L(p,q),\xi)$, and we move to a new level of the tree by applying the decomposition described in this section. If the mixed torus leading to the decomposition comes from a knot which has been stabilized both positively and negatively, we have a single branch. If the mixed torus is associated to a pair K_+, K_- of adjacent knots with opposite signs, then we have $m+2$ branches, where m is the number of unstabilized knots between K_+ and K_-.

We observe that this argument recovers Fossati's classification of fillings for virtually overtwisted structures on lens spaces which result from contact surgery on the Hopf link ([Fos19, Theorem 1]). Consider $-p/q = [a_1, a_2]$, for some $a_1, a_2 \leq -2$, and let ξ_{vot} be a virtually overtwisted contact structure on $L(p,q)$. Our decomposition tells us that every filling of $(L(p,q),\xi_{\text{vot}})$ is obtained by a specified Weinstein 2-handle attachment to a filling of either $L(-a_1,1)$ or $L(-a_2,1)$, with a particular (not necessarily virtually overtwisted) contact structure. With the exception of a universally tight structure on $L(4,1)$, each lens

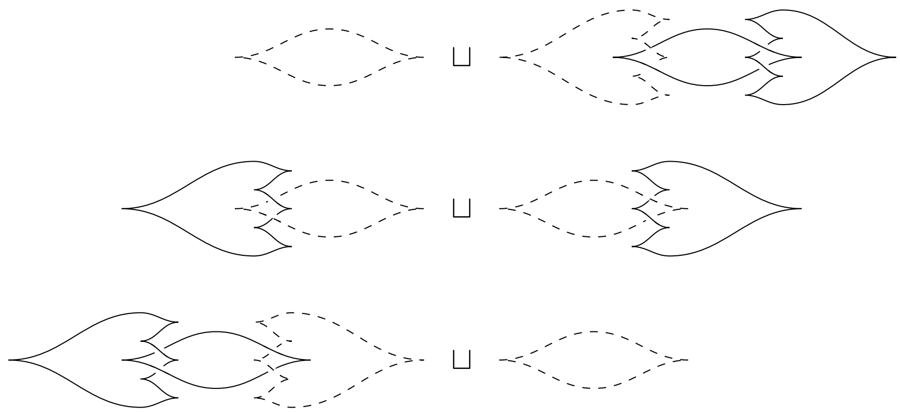


Figure 4.4: Applying the JSJ decomposition to a filling of $L(24,7)$ with the contact structure seen in Figure 4.3 yields a filling of one of the three disjoint unions seen here. We recover a filling of $L(24,7)$ by attaching a round symplectic 1-handle along the dashed knots.

space $L(-a_i, 1)$ has a unique exact filling. Moreover, we see from our decomposition that attaching a Weinstein 2-handle to such a standard filling in the manner prescribed will always yield the standard filling of $(L(p,q), \xi_{\mathrm{vot}})$. So we have the following corollary.

Corollary 4.3 (c.f. [Fos19, Theorem 1])**.** *Let $(L(p,q), \xi_{\mathrm{vot}})$ be a virtually overtwisted lens space, with $-p/q = [a_1, a_2]$, for some $a_1, a_2 \leq -2$. Then $(L(p,q), \xi_{\mathrm{vot}})$ has*

- *a unique exact filling, up to diffeomorphism, if $a_1 \neq -4$ and $a_2 \neq -4$, or if at least one of a_1, a_2 is -4 and the corresponding knot has been stabilized both positively and negatively;*
- *precisely two exact fillings, up to diffeomorphism, if at least one of a_1, a_2 is -4, and the corresponding knot has stabilizations of a single sign.*

CHAPTER 5

Virtually overtwisted torus bundles

In this chapter we reduce the problem of classifying strong symplectic fillings of virtually overtwisted contact structures on torus bundles to the same problem for tight lens spaces. The tight contact structures on torus bundles which fiber over the circle were completely classified by Honda in [Hon00b], and thus a natural next step is to ask geography and botany questions about their symplectic fillings — which of these contact structures is fillable, and what are their fillings? By construction, convex tori are abundant in torus bundles; in case our contact structure is virtually overtwisted, we find a mixed torus which is isotopic to a fiber of the bundle, and use Theorem 3.3 to decompose our contact manifold and its fillings.

5.1 The classification

Given an element $A \in SL(2, \mathbb{Z})$, we define the mapping torus

$$M_A := (\mathbb{R}^2/\mathbb{Z}^2 \times I)/\sim,$$

where $(\mathbf{x}, 1) \sim (A\mathbf{x}, 0)$. We call M_A the *torus bundle with monodromy A*, and notice that the diffeomorphism type of M_A depends only on the conjugacy class of A in $SL(2, \mathbb{Z})$. We will say that M_A is

- *elliptic* if $|\operatorname{tr} A| < 2$;
- *parabolic* if $|\operatorname{tr} A| = 2$;
- *hyperbolic* if $|\operatorname{tr} A| > 2$.

This trichotomy will be used in the statement of our classification result, as will generators

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

of $SL(2,\mathbb{Z})$. It is known (c.f. [Hon00b, Lemma 2.1]) that every conjugacy class of $SL(2,\mathbb{Z})$ can be represented by one of

- $A = \pm S$;
- $A = \pm T^{-1}S, \pm(T^{-1}S)^2$;
- $A = \pm T^n$, $n \in \mathbb{Z}$;
- $A = \pm T^{r_0}ST^{r_1}S \cdots T^{r_k}S$, $r_i \leq -2$, $r_0 < -2$.

In the last case, the choice of conjugacy class representative is not unique, but in all cases the $\pm$ sign is unique. We will refer to the monodromy A as *positive* or *negative* depending on sign of its conjugacy class representative(s).

Ours is not the first study of symplectic fillings for contact torus bundles. In [DG01], Ding-Geiges showed that every torus bundle admits an infinite family of weakly-but-not-strongly symplectically fillable contact structures, all of which are universally tight. In [BO14], Bhupal-Ozbagci show that for certain parabolic and hyperbolic torus bundles, there are precisely two isotopy classes of Stein fillable contact structures, and along the way they construct a Stein filling for every tight contact structure on a torus bundle with positive hyperbolic monodromy. Golla-Lisca, in [GL15], construct Stein fillable structures on a large family of torus bundles, with many of these structures being universally tight. One of these constructions provides a Stein filling for each virtually overtwisted tight contact structure on a torus bundle with negative hyperbolic monodromy. In [DL18], Ding-Li consider the question of strong symplectic fillability for some contact structures on negative parabolic and negative hyperbolic torus bundles, and construct a Stein filling for the negative parabolic torus bundle with monodromy $-T^n$, $n \leq -1$.

In the present chapter, we use Theorem 3.3 to obtain a classification result for fillings of virtually overtwisted contact structures on torus bundles. Combined with (non-)existence results previously established in [BO14], [GL15], and [DL18], our main result is the following.

Theorem 5.1. *Let M be a 3-dimensional torus bundle, ξ a virtually overtwisted tight contact structure on M.*

(*A*) *If M is an elliptic torus bundle, then (M, ξ) is not strongly symplectically fillable.*

(*B*) *If M is the positive parabolic torus bundle with monodromy T^n $(n \geq 2)$, then (M, ξ) is not strongly symplectically fillable. If M is any other parabolic torus bundle, then (M, ξ) admits a unique strong filling up to symplectic deformation equivalence and blowup.*

(C) *If M is a hyperbolic torus bundle, then there is a nonempty, finite list*

$$(L(p_1, q_1), \xi_1), \ldots, (L(p_m, q_m), \xi_m)$$

of tight lens spaces such that every strong (*exact*) *symplectic filling of (M, ξ) can be obtained from a strong* (*exact*) *symplectic filling of $(L(p_i, q_i), \xi_i)$, for some $1 \leq i \leq m$, by attaching a round symplectic 1-handle. In particular, (M, ξ) is fillable.*

Remark.

(1) As noted above, some of the existence statements in Theorem 5.1 are not new. Indeed, (A) is weaker than [EH02, Theorem 1.1], which says that virtually overtwisted structures on elliptic torus bundles are not even *weakly* symplectically fillable. For the parabolic torus bundles, Ding-Li established the existence of a symplectic filling for the unique virtually overtwisted torus bundle with monodromy $-T^n$, $n \leq -1$, in [DL18]. The existence statement for hyperbolic torus bundles was established in [BO14] and [GL15].

(2) The proof of Theorem 5.1 produces explicit attaching data for the round symplectic 1-handles that appear in (C). Namely, we identify Legendrian knots L^i_-, L^i_+ in $(L(p_i, q_i), \xi_i)$ so that attaching a round symplectic 1-handle to a filling of $(L(p_i, q_i), \xi_i)$ along standard neighborhoods of L^i_- and L^i_+ yields a filling of the torus bundle (M, ξ),

for each $1 \leq i \leq m$. Moreover, for $1 \leq i \leq m-1$, we obtain $(L(p_{i+1}, q_{i+1}), \xi_{i+1})$ from $(L(p_i, q_i), \xi_i)$ by performing $(+1)$-surgery along L^i_+ and (-1)-surgery along L^i_-.

(3) While the tight contact structure ξ is virtually overtwisted, the structures $\xi_1, \ldots, \xi_m$ could be either universally tight or virtually overtwisted.

In Section 5.3.3 we will give an explicit construction of the list of lens spaces specified by (C) of Theorem 5.1. We will see that if M has monodromy

$$A = \pm T^{r_0} S T^{r_1} S \cdots T^{r_k} S, \tag{5.1}$$

with $r_0 \leq -3$ and $r_i \leq -2$ for $1 \leq i \leq k$, then each lens space in the list has the diffeomorphism type of $L(p, q)$, where

$$-\frac{p}{q} = [r_k, \ldots, r_1] \quad \text{or} \quad -\frac{p}{q} = [r_{j-1}, r_{j-2}, \ldots, r_0, r_k, r_{k-1}, \ldots, r_{j+1}]$$

for some $1 \leq j \leq k$. By $[a_0, \ldots, a_k]$ we mean

$$[a_0, \ldots, a_k] := a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cdots \cfrac{1}{a_k}}},$$

as in Chapter 4.

This list of lens spaces (along with their contact structures) can be encoded diagramatically as follows. Per [BO14] and [GL15], the standard filling of a tight torus bundle with monodromy given by (5.1) is obtained from that of $(S^1 \times S^2, \xi_{std})$ by attaching Weinstein 2-handles along the link $\Lambda \subset S^1 \times S^2$ depicted in Figure 5.1. (The knots in this link are stabilized according to the contact structure ξ.) To each $1 \leq i \leq k$ we associate a tight lens space (L_i, ξ_i), with a filling obtained from Figure 5.1 by erasing the 1-handle and the unknot K_i.

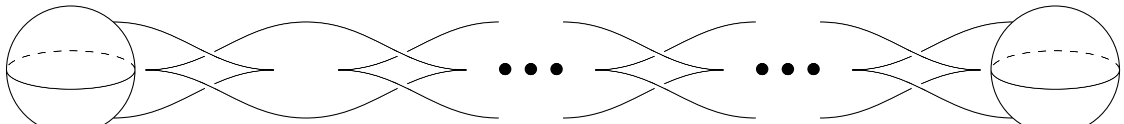


Figure 5.1: The natural filling of (M, ξ), which has positive monodromy with coefficients $(r_0, r_1, \ldots, r_k)$, is obtained by attaching Weinstein 2-handles to the unique filling of $(S^1 \times S^2, \xi_{std})$ along the Legendrian knots $K_0, \ldots, K_k \subset (S^1 \times S^2, \xi_{std})$. For each $0 \leq i \leq k$, the knot K_i is stabilized to ensure that $\mathrm{tb}(K_i) = r_i + 1$, and that $\mathrm{r}(K_i)$ is determined by the contact structure ξ. If the monodromy of M is negative, we modify Λ by requiring $lk(K_0, K_k) = -1$.

Now let

$$0 = i_1 < i_2 < \cdots < i_m \leq k$$

be precisely those indices for which $r_{i_j} \leq -3$. Then (M, ξ) decomposes into m continued fraction blocks, with a continued fraction block associated to each i_j. Because ξ is virtually overtwisted, (M, ξ) admits a mixed torus $T \subset (M, \xi)$, and we use T to build a list $\mathcal{L}$ of tight lens spaces from whose fillings the fillings of (M, ξ) will be obtained. If T is interior to the continued fraction block associated to i_j, for some $1 \leq j \leq m$, then $\mathcal{L}$ has just one element — the lens space (L_{i_j}, ξ_{i_j}). Otherwise, T sits at the intersection of two continued fraction blocks, say, associated to i_j and i_{j+1} for some $1 \leq j \leq m$. (Here $i_{m+1} = i_1$.) In this case

$$\mathcal{L} = ((L_{i_j}, \xi_{i_j}), (L_{i_j+1}, \xi_{i_j+1}), \ldots, (L_{i_{j+1}}, \xi_{i_{j+1}})).$$

In words, each lens space in $\mathcal{L}$ is obtained by deleting the 1-handle and an unknot from Figure 5.1. For each continued fraction block that T meets, $\mathcal{L}$ contains the corresponding lens space, as well as the intermediate lens spaces obtained by deleting (-2)-framed unknots. Our result says that every filling of (M, ξ) is obtained from a filling of some $(L_i, \xi_i) \in \mathcal{L}$ by attaching a round 1-handle. For the standard filling of (L_i, ξ_i), obtained from Figure 5.1 as described, this round 1-handle is attached along knots $K_i', K_i'' \subset (L_i, \xi_i)$ obtained from Figure 5.1 by replacing each of the 3-balls with a cusp. See Figure 5.2.

If $\mathrm{tb}(K_i) \neq -1$, there remains the ambiguity of how we distribute the stabilizations of K_i among K_i' and K_i'', but this is resolved by the mixed torus we are using. Namely,

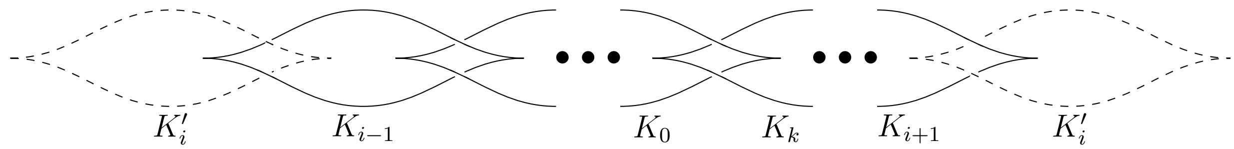


Figure 5.2: A diagram for the standard filling of (L_i, ξ_i), obtained from Figure 5.1 by deleting the 1-handle and the knot K_i. Weinstein 2-handles are attached to (B^4, ω_{std}) along $K_{i-1}, \dots, K_{i+1} \subset (S^3, \xi_{std})$, but not along the dashed knots K_i', K_i''. These are obtained from the knot K_i in Figure 5.1 by replacing the 3-balls with cusps, with the stabilizations on K_i' and K_i'' determined by our choice of mixed torus. Attaching a round symplectic 1-handle to this filling along K_i', K_i'' yields the standard filling of (M, ξ).

the continued fraction block associated to K_i decomposes into basic slices, each of which we think of as either positive or negative, and the stabilizations of K_i correspond to these basic slices. The stabilizations of K_i are then distributed among K_i' and K_i'' according to the number of basic slices on either side of our mixed torus. In particular, if the mixed torus lies at the boundary of two continued fraction blocks, then all of the stabilizations of K_i will lie on one of K_i' or K_i'', with the other being a Legendrian unknot with Thurston-Bennequin number -1. In all cases, attaching a round symplectic 1-handle along K_i' and K_i'' corresponds to attaching a Weinstein 1-handle along $p_i' \in K_i'$ and $p_i'' \in K_i''$, followed by a Weinstein 2-handle attachment along K_i. The resulting filling of (M, ξ) is the standard filling depicted in Figure 5.1.

For each torus bundle (M, ξ) that we consider, our classification will follow the same basic recipe. We begin by identifying a fiber-isotopic mixed torus $\Sigma \subset (M, \xi)$. To identify this torus, we realize M as the result of identifying the boundary components of $T^2 \times I$ via some monodromy A. In each case, the contact structure ξ on M lifts to a (perhaps non-unique) tight contact structure on $T^2 \times I$, which we also denote by ξ. We then use the fact that (M, ξ) is virtually overtwisted to find basic slices of opposite sign on either side of $T^2 \times \{0\}$ (perhaps after a contact isotopy), making $T^2 \times \{0\}$ a mixed torus.

Having identified a mixed torus in (M, ξ), we suppose that (W, ω) is an exact filling of (M, ξ) and turn to Theorem 3.3. This produces (W', ω'), an exact filling of a contact manifold (M', ξ'), and tells us how to obtain (W, ω) from (W', ω'), as well as providing a relationship between (M, ξ) and (M', ξ'). Specifically, because our mixed torus is the fiber

$T^2 \times \{0\}$, the decomposition theorem allows us to write

$$M' = S_0 \cup (T^2 \times I) \cup S_1$$

for some identifications $\partial S_i \to T^2 \times \{i\}$, where S_0 and S_1 are solid tori. That is, M' is a lens space. Moreover, Theorem 3.3 tells us that we can recover M from M' by removing the interiors of S_0 and S_1 and identifying the dividing curves and meridians of the resulting boundary components. At the level of fillings, this corresponds to attaching a round symplectic 1-handle to a filling of the lens space along the cores of the solid tori — for the lens space $(L(p_i, q_i), \xi_i)$ produced by Theorem 5.1, these core curves are the Legendrian knots L^i_- and L^i_+.

The dividing curves of ∂S_0 and ∂S_1 have slopes s_0 and s_1, respectively, determined by the contact structure on $T^2 \times I$. If the meridians of S_0 and S_1 are denoted μ_0, μ_1, then $A\mu_1 = \mu_0$. These meridians are not *a priori* determined by Theorem 3.3, but by investigating different choices for the meridian μ_1, we are in many cases able to either completely determine (M', ξ'), or able to show that no exact filling of (M, ξ) exists[1].

Our primary technique for ruling out possible meridians μ_1 involves an analysis of the slopes of μ_0, μ_1, Γ_0, and Γ_1, where Γ_i is the dividing set of $T^2 \times \{i\}$. Because (M', ξ') is fillable, ξ' must be a tight contact structure. Now consider a family of convex tori in M', beginning with the boundary of a tubular neighborhood of the core of $S_1 \subset M'$, passing through the fibers of $T^2 \times I \subset M'$, and tending towards the boundary of a tubular neighborhood of the core of $S_0 \subset M'$. Each torus in this family has a pair of dividing curves, and the slopes of these curves will vary from the slope of μ_1 to s_1 to s_0 to the slope of μ_0 as we traverse the family. Because ξ' is tight, the total angle through which these dividing curves pass cannot exceed π. We will rule out many possibilities for the meridian μ_1 by showing that these choices would cause this last condition to be violated.

[1]As noted, the material in this chapter first appeared in [Chr21]. At the time of writing that paper, the portion of Theorem 3.3 which limits the contact manifolds which may result from applying the symplectic JSJ decomposition had not yet been developed. Now that this statement has been added, the analysis contained in this chapter could be abbreviated; we leave the analysis intact, as it may help to illuminate the slope analysis in the proof of Theorem 3.3.

Type	Monodromy	# of VOT structures
Elliptic ($\|\mathrm{tr}\ A\| < 2$)	$A = S$ $A = (T^{-1}S)^2$	1 2
Parabolic ($\|\mathrm{tr}\ A\| = 2$)	$A = T^2$ $A = T^n, n > 2$ $A = T^n, n \leq -2$ $A = -T^n, n < 0$	1 2 $\|n-1\| - 2$ 1
Hyperbolic ($\|\mathrm{tr}\ A\| > 2$) ($r_0 \leq -3, r_i \leq -2$)	$A = T^{r_0}ST^{r_1}S \cdots T^{r_k}S$ $A = -T^{r_0}ST^{r_1}S \cdots T^{r_k}S$	$\|(r_0+1)\cdots(r_k+1)\| - 2$ $\|(r_0+1)\cdots(r_k+1)\|$

Table 5.1: Virtually overtwisted contact structures on torus bundles over S^1

5.2 Tight contact structures on torus bundles

The contact manifolds of central interest in this chapter are *torus bundles over* S^1. These are smooth 3-manifolds whose tight contact structures were classified by Honda in [Hon00b]. The tight contact structures on the torus bundle M_A are given in terms of the monodromy A, and are each identified as being either universally tight or virtually overtwisted.

Honda's classification is contained in Table 5.1, omitting torus bundles which admit only universally tight contact structures. We continue to use the generators S and T of $SL(2,\mathbb{Z})$ identified in Section 5.1.

Honda constructs each of the contact structures figured in Table 5.1 by thinking of (M_A, ξ) as a thickened torus $(T^2 \times I, \xi)$, with the ends $T^2 \times \partial I$ identified by the monodromy A. (We will thus rely on the discussion in Section 2.4.2 of tight contact structures on $T^2 \times I$.) When the monodromy A is negative (in the sense described above), this identification will induce a change of sign, leading some universally tight contact structures on $T^2 \times I$ to induce virtually overtwisted contact structures on M_A. See [Hon00b] for full details.

5.3 Proof of Theorem 5.1

In this section we use Theorem 3.3 to prove Theorem 5.1. We treat the elliptic, parabolic, and hyperbolic cases separately.

5.3.1 Elliptic torus bundles

There are two elliptic torus bundles which admit a virtually overtwisted contact structure — one with monodromy $A = S$, and the other with monodromy $A = (T^{-1}S)^2$. In both cases we obtain a virtually overtwisted structure by starting with a minimally twisting tight structure on $T^2 \times I$ with boundary slopes $s_1 = 0$ and $s_0 = \infty$ and then passing to the torus bundle M_A. There are two such structures on $T^2 \times I$, and they become contact isotopic on M_A when $A = S$. When $A = (T^{-1}S)^2$, the structures remain distinct.

So there are three virtually overtwisted, elliptic torus bundles. None admit a strong symplectic filling.

Proposition 5.2. *Let (M, ξ) be a virtually overtwisted, elliptic torus bundle. Then (M, ξ) is not strongly symplectically fillable.*

Proof. As stated above, there are three contact manifolds (M, ξ) which satisfy the hypotheses of this proposition, and all three are obtained from a tight $(T^2 \times I, \xi)$ with boundary slopes $s_1 = 0$, $s_0 = \infty$. We consider the case $A = S$; the case $A = (T^{-1}S)^2$ is similar.

Our first claim is that the image of the fiber $T^2 \times \{0\}$ in (M, ξ) is a mixed torus, and our proof of this fact mimics Honda's proof of the fact that ξ is an overtwisted contact structure. (c.f. [Hon00a, Section 4]). In particular, we begin with a tight contact structure ξ on $T^2 \times [0, 1]$ with boundary slopes $s_1 = 0$ and $s_0 = \infty$. Because these slopes are Farey neighbors, $(T^2 \times [0, 1], \xi)$ is a basic slice, and there are precisely two possibilities for ξ, distinguished by their relative Euler classes. As in [Hon00a, Section 4.7], we compute these relative Euler classes to be

$$PD(e(\xi, s)) = \pm(v_1 - v_0) = \pm((1, 0) - (0, 1)) = \pm(1, -1) \in H_1(T^2; \mathbb{Z}),$$

where v_0 and v_1 are minimal vectors in $\mathbb{Z}^2$ representing the slopes s_0 and s_1, respectively. Now we may decompose $T^2 \times [0, 1]$ into a pair of basic slices

$$(N_1 := T^2 \times [0, 1/2], \xi_1 := \xi|_{N_1}) \quad \text{and} \quad (N_2 := T^2 \times [1/2, 1], \xi_2 := \xi|_{N_2})$$

with boundary slope $s_{1/2} = 1$, since 1 is a Farey neighbor of both 0 and ∞. The relative Euler classes of these basic slices are then

$$PD(e(\xi_1, s)) = \pm(v_{1/2} - v_0) \quad \text{and} \quad PD(e(\xi_2, s)) = \pm(v_1 - v_{1/2}),$$

where $v_{1/2} = (1, 1)$. Because these basic slices glue to form the basic slice $(T^2 \times [0, 1], \xi)$, their signs agree.

Now because the ends of $T^2 \times [0, 1]$ are identified to form M, we may also view N_2 as $T^2 \times [-1/2, 0]$ by applying the monodromy A. Applying the monodromy alters the relative Euler class:

$$PD(e(\xi_2, s)) = \pm A(v_1 - v_{1/2}) = \pm(Av_1 - Av_{1/2}) = \pm(-v_0 - (v_1 - v_0)) = \mp v_1.$$

Here we are using the fact that $A = S$ is a rotation by $-\pi/2$. So the basic slice $(T^2 \times [-1/2, 0], \xi_{T^2\times[-1/2,0]})$ has a sign which is necessarily opposite that of the basic slice $(T^2 \times [0, 1/2], \xi_{T^2\times[0,1/2]})$, making the fiber $T^2 \times \{0\}$ sandwiched between them a mixed torus.

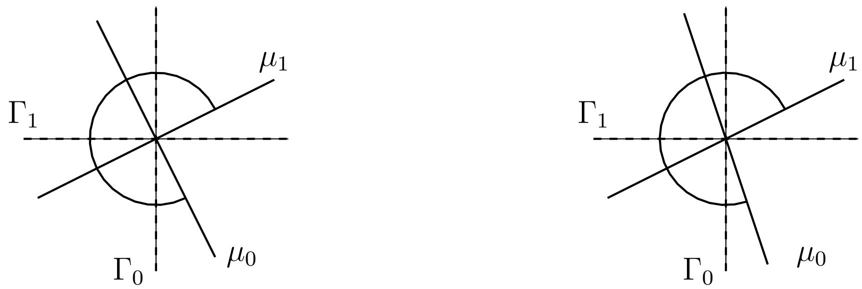
In [Hon00a], Honda uses this relative Euler class computation to show that (M, ξ) is virtually overtwisted. An analogous computation for the monodromy $A = (T^{-1}S)^2$ allows us to distinguish between the two virtually overtwisted tight contact structures on this torus bundle, and to see that the fiber $T^2 \times \{0\}$ is a mixed torus in each of them.

As outlined in our strategy, we now suppose that (M, ξ) is fillable, with strong symplectic filling (W, ω), using the notation from Section 5.1. Theorem 3.3, applied to (W, ω) and $T^2 \times \{0\} \subset (M, \xi)$, produces a strong filling (W', ω') of (M', ξ'), with

$$M' \simeq S_0 \cup (T^2 \times I) \cup S_1.$$

Because $s_1 = 0$ and $s_0 = \infty$, the dividing sets Γ_i of ∂S_i can be represented by

$$\Gamma_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$



(a) $A = S$. (b) $A = (T^{-1}S)^2$.

Figure 5.3: Slope analysis for virtually overtwisted, elliptic torus bundles.

The shortest integer vector representing the meridian μ_1 must form an integral basis with Γ_1 for $\mathbb{Z}^2$, and we have $\mu_0 = A\mu_1$. So we have representatives

$$\mu_1 = \begin{pmatrix} m \\ 1 \end{pmatrix} \quad \text{and} \quad \mu_0 = \begin{pmatrix} 1 \\ -m \end{pmatrix}$$

for some $m \in \mathbb{Z}$.

Notice that if $m \geq 0$, then a counterclockwise rotation from μ_1 to Γ_1 passes through an angle of at least $\pi/2$, and a counterclockwise rotation from Γ_1 to Γ_0 takes us through an angle of precisely $\pi/2$. The rotation from Γ_0 to μ_0 is then non-trivial, and we see that our dividing curves pass through an angle in excess of π as we move from the core of S_1 to $T^2 \times I$, and on to the core of S_0. The same rotation occurs when $m \leq 0$. In fact, this rotation is always $3\pi/2$ — regardless of the value of m — as can be seen in Figure 5.3a.

The case $A = (T^{-1}S)^2$ is not much different. The dividing curves will rotate through an angle of $3\pi/2 - \theta(m)$, where $\theta(m)$ is the acute angle between $(1, m)^T$ and $(1, -m-1)^T$. In particular, the dividing curves rotate through an angle greater than π. The case $m = 2$ is depicted in Figure 5.3b.

In either case, we see that (M', ξ') contains a thickened torus with non-minimal twisting which may be completed to a solid torus S' strictly containing S_1. The non-minimal twisting means that $(S', \xi|_{S'})$ is overtwisted (c.f. [Hon00a, Section 2.3]), and thus so is (M', ξ'). Note that this is the case no matter the value of m. But of course this contradicts the fillability of (M', ξ'), so we conclude that no strong symplectic filling (W, ω) of (M, ξ) exists. □

5.3.2 Parabolic torus bundles

Topologically, the parabolic torus bundles admitting virtually overtwisted contact structures are all circle bundles, with base either the torus or the Klein bottle. The virtually overtwisted structures on these manifolds then fall into three families.

For $n \geq 2$, the monodromy $A = T^n$ produces a circle bundle over T^2 with Euler number n. This bundle admits a unique virtually overtwisted contact structure if $n = 2$, and admits two such structures if $n > 2$. These structures are exceptional in that they are the only virtually overtwisted contact structures on any torus bundles which fail to be minimally twisting. We will show that these structures are not strongly symplectically fillable.

Proposition 5.3. *Fix $n \geq 2$ and let M_n be the torus bundle with monodromy $A = T^n$. Let ξ be a virtually overtwisted contact structure on M. Then (M_n, ξ) is not strongly symplectically fillable.*

Proof. Per the classification in [Hon00a], there are two tight contact structures on $T^2 \times I$ with boundary slopes $s_0 = s_1 = 0$, and these pass to tight contact structures on M_n. In [Hon00b], Honda shows that these structures are distinct when $n > 2$, and for $n \geq 2$ give all virtually overtwisted contact structures on M_n. Let ξ be one of these structures on $T^2 \times I$, passing to ξ (via notational abuse) on M.

Because $(T^2 \times I, \xi)$ is not minimally twisting, the dividing curves rotate through an angle of π as we move from $T^2 \times \{0\}$ to $T^2 \times \{1\}$, and thus all slopes are achieved by some boundary-parallel torus. We take $T^2 \times \{1/2\}$ with slope $s_{1/2} = \infty$. Because ∞ is connected to 0 by an edge of the Farey tessellation, each of $T^2 \times [0, 1/2]$ and $T^2 \times [1/2, 1]$ is a basic slice, and the fact that their union $T^2 \times I$ is not universally tight makes $T^2 \times \{1/2\}$ a mixed torus.

Now suppose that (M_n, ξ) admits a strong symplectic filling (W, ω), and let (M', ξ') be the strongly symplectically fillable manifold that results from applying Theorem 3.3 to (W, ω), using the mixed torus $T^2 \times \{1/2\}$. In order to split (M_n, ξ) open along $T^2 \times \{1/2\}$, we think of this torus bundle as the result of identifying the ends of $(T^2 \times [-1/2, 1/2], \xi)$ via the monodromy. Then, as above, M' results from gluing solid tori $S_{-1/2}$ and $S_{1/2}$ to the boundary components of $T^2 \times [-1/2, 1/2]$, and we are left to determine the meridians

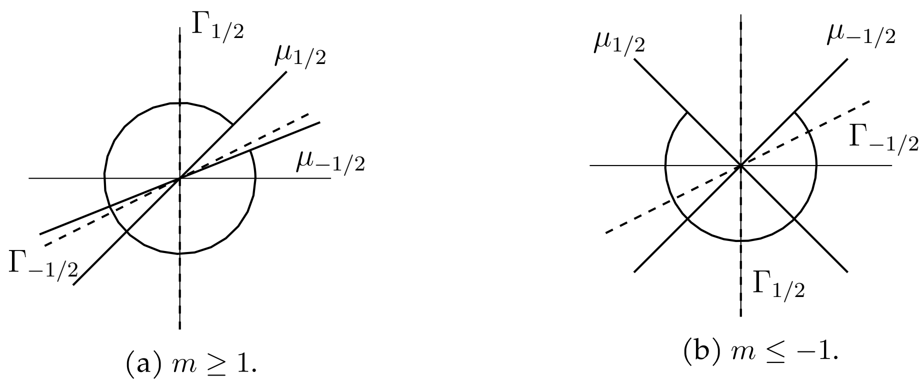


Figure 5.4: Slope analysis for a VOT torus bundle with monodromy T^n, $n \geq 2$.

$\mu_{-1/2}, \mu_{1/2}$. Notice that the dividing curves $\Gamma_{-1/2}$ and $\Gamma_{1/2}$ have slopes $s_{-1/2} = 1/n$ and $s_{1/2} = \infty$, respectively. Because the shortest integer vectors representing $\Gamma_{1/2}$ and $\mu_{1/2}$ must form an integer basis for $\mathbb{R}^2/\mathbb{Z}^2$, we may represent $\mu_{1/2}$ by the vector $(1, m)^T$, for some $m \in \mathbb{Z}$.

Suppose we have $m = 0$, so that $\mu_{1/2} = (1, 0)^T$. Then $\mu_{-1/2} = A(1, 0)^T = (1, 0)$, so the counterclockwise rotation from $\mu_{1/2}$ to $\Gamma_{1/2}$ to $\Gamma_{-1/2}$ to $\mu_{-1/2}$ takes us through an angle of 2π, and we find that (M', ξ') is overtwisted. So $m \neq 0$.

Next, suppose $m \geq 1$, meaning that $\mu_{-1/2}$ is represented by $A(1, m)^T = (1 + mn, m)$. Because $n \geq 2$, we have

$$0 < \frac{m}{1 + mn} < \frac{1}{n} < m < \infty,$$

which is to say that

$$0 < \mathrm{slope}(\mu_{-1/2}) < \mathrm{slope}(\Gamma_{-1/2}) < \mathrm{slope}(\mu_{1/2}) < \mathrm{slope}(\Gamma_{1/2}).$$

In this case, the dividing curves rotate through an angle in excess of $3\pi/2$, and again (M', ξ') is overtwisted. So $m < 0$.

But if $m < 0$ then we find that

$$\mathrm{slope}(\mu_{1/2}) < 0 < \mathrm{slope}(\Gamma_{-1/2}) < \mathrm{slope}(\mu_{-1/2}) < \mathrm{slope}(\Gamma_{1/2}),$$

since $m < 0 < 1/n < m/(1+mn)$. In this case, counterclockwise rotation from $\mu_{1/2}$ to $\Gamma_{1/2}$ to $\Gamma_{-1/2}$ to $\mu_{-1/2}$ takes us through an angle of $3\pi/2$, for any choice of $m < 0$ and $n \geq 2$, and again we find that (M', ξ') is overtwisted. Because there are no suitable choices for the meridian $\mu_{1/2}$, we must conclude that (M_n, ξ) does not admit a strong filling (W, ω). The overtwistedness when $m \neq 0$ is seen in Figure 5.4. □

When $n \leq -2$, the monodromy $A = T^n$ will again give us a circle bundle over T^2 with Euler number n, but in this case we have $|n-1| - 2$ virtually overtwisted structures. These bundles admit a unique strong filling, up to symplectic deformation equivalence and blowup, for any virtually overtwisted structure. As a corollary, the exact fillings of these bundles are unique up to symplectomorphism.

Proposition 5.4. *Fix $n \leq -2$ and let M_n be the torus bundle with monodromy $A = T^n$. Let ξ be a virtually overtwisted contact structure on M_n. Then (M_n, ξ) admits a unique strong filling, up to symplectic deformation equivalence and blowup.*

Proof. Consider a tight contact structure ξ' on $T^2 \times I$ with boundary slopes $s_0 = \frac{-1}{1-n}$ and $s_1 = -1$. Honda showed in [Hon00a] that there are $1-n$ such structures, distinguished by the number k of positive basic slices they have in a continued fraction block. Concretely, each such structure ξ' admits a decomposition into $-n$ basic slices

$$\left(T^2 \times \left[0, \frac{1}{n}\right]\right) \cup \left(T^2 \times \left[\frac{1}{n}, \frac{2}{n}\right]\right) \cup \cdots \cup \left(T^2 \times \left[\frac{n-1}{n}, 1\right]\right), \tag{5.2}$$

and we can identify ξ' by counting the number of these basic slices which are positive. These structures remain tight and distinct when we pass to M_n, and if k is neither 0 nor $1-n$, the resulting structure is virtually overtwisted. If $k = 0$ or $k = 1-n$, the basic slices above all have the same sign, and the structure on M_n is universally tight.

Let ξ be a virtually overtwisted structure on M_n, induced by a structure ξ' on $T^2 \times I$. Because ξ is virtually overtwisted, the basic slices in the decomposition (5.2) do not all have the same sign. In particular, we may shuffle the basic slices so that that the first and last basic slices have opposite sign, ensuring that the image of $T^2 \times \{0\}$ in (M_n, ξ) is a mixed torus.

Now suppose that (W, ω) is an exact filling of (M_n, ξ) and let (W', ω') be the symplectic manifold-with-boundary produced by splitting (W, ω) open along our mixed torus. This manifold exactly fills its boundary (M', ξ'), and we may write

$$M' = S_0 \cup (T^2 \times I) \cup S_1,$$

where the boundary slopes of $T^2 \times I$ are $s_0 = \frac{-1}{1-n}$ and $s_1 = -1$. Denote by Γ_i and μ_i the dividing curves and meridian, respectively, of ∂S_i, for $i = 0, 1$. Then Γ_0 and Γ_1 are represented by the vectors $(n-1, 1)^T$ and $(1, -1)^T$, respectively. Because the shortest integer vectors representing μ_1 and Γ_1 must form an integral basis for $\mathbb{Z}^2$, we may represent μ_1 by $(m, k)^T$, with $m \geq 0$ and $k = \pm 1 - m$. It follows that μ_0 is represented by

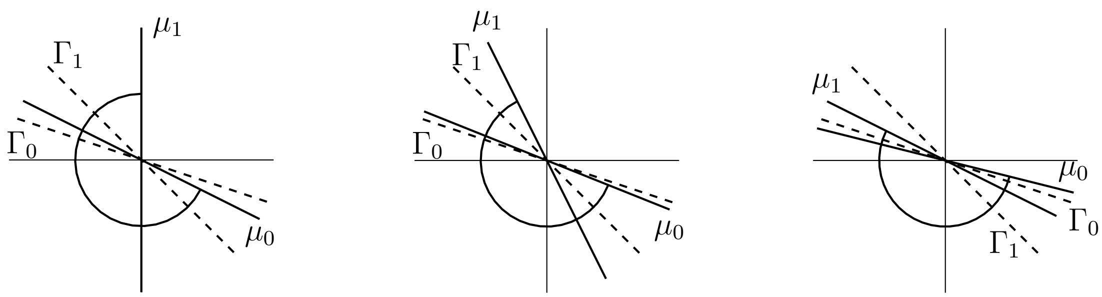
$$A \begin{pmatrix} m \\ \pm 1 - m \end{pmatrix} = \begin{pmatrix} \pm n - m(n-1) \\ \pm 1 - m \end{pmatrix}.$$

We now begin ruling out candidate values for m.

First, m must be positive. Indeed, if $m = 0$ then the slopes of μ_1 and μ_0 are given by ∞ and $1/n$, respectively. It follows that as we move a convex torus from the core of S_1 to $T^2 \times \{1\}$, then to $T^2 \times \{0\}$, and finally on to the core of S_0, the dividing curve slopes will range from ∞ to -1 to $\frac{-1}{1-n}$ and on to $1/n$, rotating through an angle greater than π, as indicated in Figure 5.5a. This would mean that (M', ξ') is overtwisted, violating its fillability.

Next, we must have $k = 1 - m$. To this end, notice that for $n \leq -2$ and $m \geq 1$ we have

$$\frac{-1-m}{m} < -1 < \frac{-1-m}{-n-m(n-1)} < \frac{-1}{1-n}.$$



(a) Rotation in the case $m = 0$. (b) The case $k = -1 - m$. (c) The case $m \geq 2$, $k = 1 - m$.

Figure 5.5: Slope analysis for the monodromy $A = -T^n$, $n \leq -1$ or $A = T^n, n \leq -2$.

This tells us that if $k = -1 - m$, then

$$\text{slope}(\mu_1) < \text{slope}(\Gamma_1) < \text{slope}(\mu_0) < \text{slope}(\Gamma_0),$$

and again the dividing curves pass through too great an angle for (M', ξ') to be tight. This rotation is seen in Figure 5.5b. So $k = 1 - m$ and μ_1 is represented by a vector of the form $(m, 1 - m)^T$, with $m \geq 1$.

Finally we show that $m = 1$. If $m \geq 2$ then we have

$$-1 < \frac{1-m}{m} < \frac{-1}{1-n} < \frac{1-m}{n - m(n-1)},$$

so

$$\text{slope}(\Gamma_1) < \text{slope}(\mu_1) < \text{slope}(\Gamma_0) < \text{slope}(\mu_0),$$

and yet again the counterclockwise rotation from μ_1 to Γ_1 to Γ_0 to μ_0 takes us through an angle in excess of π. This rotation can be seen in Figure 5.5c.

So the meridians are given by $\mu_1 = (1, 0)^T$ and $\mu_0 = (1, 0)^T$, meaning that $M' \simeq S^2 \times S^1$. This determines ξ', since $S^2 \times S^1$ has a unique tight contact structure ξ_{std}, and in fact determines (W', ω'), since $(S^2 \times S^1, \xi_{\text{std}})$ has a unique strong filling up to symplectic deformation equivalence and blowup. Because we have a recipe for reconstructing (W, ω) from (W', ω') — namely, by attaching a round symplectic 1-handle to (W', ω') along S_0 and S_1 — we see that this filling is uniquely determined by (M_n, ξ). $\square$

Finally, for $n \leq -1$ we consider the monodromy $A = -T^n$, which produces a non-orientable circle bundle over the Klein bottle. This bundle admits a unique virtually overtwisted contact structure, and in [DL18], Ding and Li constructed a Stein filling for this structure. We show that Ding-Li's filling is the unique exact filling of this torus bundle, up to symplectomorphism. Indeed, theirs is the only *strong* filling, up to symplectic deformation equivalence and blowup.

Proposition 5.5. *Fix $n \leq -1$ and let (M_n, ξ) be the virtually overtwisted torus bundle with monodromy $A = -T^n$. Then (M_n, ξ) admits a unique exact filling, up to symplectic deformation equivalence and blowup.*

Proof. As mentioned, Ding and Li construct a Stein filling of (M_n, ξ) in [DL18]. We use the mixed torus approach to show that no other exact fillings exist. Honda showed in [Hon00b] that the $1-n$ distinct tight contact structures on $T^2 \times I$ with boundary slopes $s_0 = \frac{-1}{1-n}$ and $s_1 = -1$ all descend to ξ on M_n. Each of these structures on $T^2 \times I$ is divided into $-n$ basic slices

$$\left(T^2 \times \left[0, \frac{1}{n}\right]\right) \cup \left(T^2 \times \left[\frac{1}{n}, \frac{2}{n}\right]\right) \cup \cdots \cup \left(T^2 \times \left[\frac{n-1}{n}, 1\right]\right),$$

and the contact structure on $T^2 \times I$ is determined by the number of positive basic slices in this decomposition. If $n \leq -2$, then there are $-n \geq 2$ basic slices, and we can choose a structure on $T^2 \times I$ for which the basic slices $T^2 \times [0, \frac{1}{n}]$ and $T^2 \times [\frac{n-1}{n}, 1]$ are positive. The change of sign induced by A will then cause the image of $T^2 \times \{0\}$ in (M_n, ξ) to sit between basic slices of opposite sign — that is, $T^2 \times \{0\}$ will be a mixed torus.

In case $n = -1$, either of the contact structures on $T^2 \times I$ with $s_1 = -1$ and $s_0 = -1/(1-n) = -1/2$ is a basic slice. There is some $t_0 \in I$ so that the convex torus $T^2 \times \{t_0\}$ has slope $s_{t_0} = -2/3$, so we may further divide $T^2 \times I$ into a pair of basic slices

$$T^2 \times I = \left(T^2 \times [0, t_0]\right) \cup \left(T^2 \times [t_0, 1]\right),$$

each with the same sign. As before, the change of sign produced by the monodromy allows us to realize $T^2 \times \{0\}$ as a mixed torus, sitting between the basic slices $T^2 \times [0, t_0]$ and $T^2 \times [t_0 - 1, 0]$ of opposite sign. In any case, $T^2 \times \{0\}$ is a mixed torus.

Once again we suppose that (W, ω) is an exact filling of (M_n, ξ) and let (M', ξ') be the exactly fillable contact manifold produced by Theorem 3.3. We write

$$M' = S_0 \cup (T^2 \times I) \cup S_1$$

and let Γ_i, μ_i denote the dividing curves and meridian of ∂S_i, for $i = 0, 1$. Because $s_1 = -1$, Γ_0 is represented by $(1, -1)^T \in T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The shortest integer vectors representing Γ_1 and μ_1 form an integral basis for $\mathbb{Z}^2$, so $\mu_1 = (m, k)^T$, where $k = \pm 1 - m$ and $m \geq 0$. It follows that μ_0 is represented by

$$A \begin{pmatrix} m \\ \pm 1 - m \end{pmatrix} = - \begin{pmatrix} \pm n - m(n-1) \\ \pm 1 - m \end{pmatrix}.$$

Our first claim is that $m \geq 1$. If $m = 0$, then the meridians μ_1 and μ_0 are represented by $(0, \pm 1)^T$ and $(\pm n, \pm 1)$, respectively. Thus the counterclockwise rotation from μ_1 to Γ_1 to Γ_0 to μ_0 takes us from a slope of ∞ to a slope of -1 to a slope of $\frac{-1}{1-n}$, and then on to a slope of $1/n$. In particular, the dividing curves of tori in (M', ξ') rotate through an angle greater than π, and (M', ξ') is overtwisted. The case $m = 0$ is depicted in Figure 5.5a.

Next we claim that $k = 1 - m$. Notice that for $m \geq 1$ and $n \leq -1$,

$$\frac{-1-m}{m} < -1 < \frac{1+m}{n + m(n-1)} < \frac{-1}{1-n} < 0.$$

So if $k = -1 - m$, then

$$\text{slope}(\mu_1) < \text{slope}(\Gamma_1) < \text{slope}(\mu_0) < \text{slope}(\Gamma_0) < 0.$$

As before, this tells us that the contact planes of (M', ξ') will rotate through an angle in excess of π, making (M', ξ') overtwisted. The case $k = -1 - m$ is depicted in Figure 5.5b.

Finally, we claim that $m = 1$. If $m \geq 2$ and $n \leq -1$, then

$$-1 < \frac{1-m}{m} \leq \frac{-1}{1-n} < \frac{1-m}{n-m(n-1)} < 0,$$

which is to say

$$\mathrm{slope}(\Gamma_1) < \mathrm{slope}(\mu_1) \leq \mathrm{slope}(\Gamma_0) < \mathrm{slope}(\mu_0) < 0,$$

since $k = 1 - m$. Once again, this causes (M', ξ') to be overtwisted. See Figure 5.5c.

At last we see that $m = 1$ and $k = 0$, so that μ_1 is represented by $(1, 0)^T$. Then μ_0 is represented by $A(1, 0)^T = (-1, 0)^T$, and $M' \simeq S^2 \times S^1$. But $S^2 \times S^1$ admits a unique tight contact structure, and this structure has a unique strong symplectic filling up to symplectic deformation and blowup. That is, the output (W', ω') of Theorem 3.3 is independent of the filling (W, ω) of (M_n, ξ) with which we start. Because the decomposition recovers (W, ω) from (W', ω'), the filling (W, ω) is unique. □

5.3.3 Hyperbolic torus bundles

Hyperbolic torus bundles represent the generic case for torus bundles, where the monodromy A has $|\operatorname{tr}(A)| > 2$. The monodromy has the form

$$A = \pm T^{r_0} S T^{r_1} S \cdots T^{r_k} S,$$

where $r_0 \leq -3$ and $r_i \leq -2$ for $1 \leq i \leq k$ and S, T are the generators of $SL(2, \mathbb{Z})$ identified in Section 5.1. Honda showed in [Hon00b] that the torus bundle with this monodromy admits $|(r_0 + 1) \cdots (r_k + 1)|$ minimally twisting tight contact structures. If the monodromy is positive, then two of these structures are universally tight; otherwise they are all virtually overtwisted.

We will find it convenient to change our monodromy by a conjugation, and to relabel our coefficients. Given $a \leq -3$ and $\tau \geq 0$, set

$$C_{a,\tau} = T^{a+1} S (T^{-2} S)^{\tau} T^{-1}.$$

We determine the monodromy A by choosing integers $a_0, \dots, a_k \leq -3$ and $\tau_0, \dots, \tau_k \geq 0$ and setting

$$A = \pm C_{a_0,\tau_0} \cdots C_{a_k,\tau_k} = \pm T^{a_0+1} S (T^{-2} S)^{\tau_0} \cdots T^{a_k} S (T^{-2} S)^{\tau_k} T^{-1}.$$

Then $T^{-1}AT$ has the form identified above. Notice that with this notation the count of tight contact structures is $|(a_0+1)\cdots(a_k+1)|$. The primary benefit to this new notation is that we may easily identify a basic slice decomposition of (M, ξ). Namely, our decomposition will have $|a_0 + \cdots + a_k + 2(k+1)|$ basic slices, divided into $k+1$ continued fraction blocks.

With this notation established, we may state more explicitly our result for hyperbolic torus bundles.

Proposition 5.6. *Let M be a hyperbolic torus bundle and let ξ be a virtually overtwisted tight contact structure on M. Then there is a nonempty, finite list $(L_1, \xi_1), \dots, (L_m, \xi_m)$ of tight lens spaces and a corresponding list $L^1_\pm, \dots, L^m_\pm$ of Legendrian knots $L^i_\pm \subset (L_i, \xi_i)$ such that every strong (exact) symplectic filling of (M, ξ) can be obtained from a strong (exact) symplectic filling of (L_i, ξ_i), for some $1 \leq i \leq m$, by attaching a round symplectic 1-handle along $L^i_\pm$. Moreover,*

(1) *if (M, ξ) has a virtually overtwisted continued fraction block, we have $m = 1$;*
(2) *if the monodromy of M has coefficients $a_0, a_1, \dots, a_k$ and $\tau_0, \dots, \tau_k$, then we have $m \leq 2 + \max\{\tau_i\}$.*

Remark. The lists $(L_1, \xi_1), \dots, (L_m, \xi_m)$ and $L^1_\pm, \dots, L^m_\pm$ are determined by the choice of a mixed torus in (M, ξ). We will construct this list for each mixed torus in (M, ξ), and prove the last two statements of the proposition by showing that (M, ξ) admits a mixed torus leading to a list of the desired length.

As in the previous cases, our strategy is to identify a mixed torus in (M, ξ), and then to determine which lens spaces may result from cutting (M, ξ) open along this torus and gluing on solid tori. We then use Theorem 3.3 to conclude that all fillings of (M, ξ) result from fillings of these lens spaces via round symplectic 1-handle attachment. Notice that Theorem 3.3 specifies the attaching regions for these lens spaces. We will follow Honda

[Hon00b] in identifying M with a quotient of $T^2 \times I$, where $T^2 \times \{1\}$ has slope $s_1 = \infty$. The basic slice decomposition of $T^2 \times I$ will then have $k+1$ continued fraction blocks, with a block corresponding to each coefficient a_i. The continued fraction block associated to a_i will consist of $|a_i + 2|$ basic slices, and will have $|a_i + 1|$ tight contact structures, distinguished by the number of basic slices in the block which have positive Euler characteristic.

Because there are $|a_i + 1|$ tight contact structures on the continued fraction block associated with a_i, we see that there are $|(a_0+1)\cdots(a_k+1)|$ tight contact structures on $T^2 \times I$ with the desired boundary slopes. Honda shows in [Hon00b] that these pass to distinct tight contact structures on M. Exactly two of the structures on $T^2 \times I$ are universally tight — those two for which every basic slice has the same sign. If our monodromy is positive, these two structures remain universally tight when we pass to M; if A is negative, these structures become virtually overtwisted.

Our basic slice decomposition of M shows us that there are $\ell = |a_0 + \cdots + a_k + 2(k+1)|$ tori which appear along the boundary of a basic slice. If (M, ξ) is virtually overtwisted, then at least one of these ℓ tori sits between basic slices of opposite sign, and is thus a mixed torus. We choose such a torus $T^2 \times \{t_0\}$ and call it T^2.

Now A has an oriented eigenbasis $\{v_1, v_2\}$ with associated eigenvalues $\lambda_1 > 1$ and $0 < \lambda_2 < 1$. These vectors are necessarily irrational, and we denote their slopes by

$$\Lambda^s := \mathrm{slope}(v_1) \quad \text{and} \quad \Lambda^u := \mathrm{slope}(v_2).$$

These are the *stable* and *unstable slopes*, respectively. On the Farey tessellation, the slopes corresponding to the dividing sets of fibers of our $T^2 \times I$ are located in the counterclockwise arc connecting Λ^u to Λ^s. In particular, $\mathrm{slope}(T^2)$ and $\mathrm{slope}(AT^2)$ are in this sector.

The slopes $\mathrm{slope}(T^2)$ and $\mathrm{slope}(AT^2)$ will play the roles played by s_1 and s_0, respectively, in previous cases. Namely, the lens space (M', ξ') that results from Theorem 3.3 will have the form

$$S_{t_0} \cup (T^2 \times [t_0, t_0+1]) \cup S_{t_0+1},$$

with boundary slopes $s_{t_0+1} = \text{slope}(T^2)$ and $s_{t_0} = \text{slope}(AT^2)$. We now identify the possible slopes for the meridian μ of S_{t_0+1}. Certainly $s(\mu)$ must be connected to s_{t_0+1} by an edge on the Farey tessellation, since the shortest integral vectors representing these slopes form an integral basis for $\mathbb{Z}^2$. Our next claim is that $s(\mu)$ must not lie in the same sector of the Farey tessellation as $T^2 \times I$.

Lemma 5.7. *On the Farey tessellation, $s(\mu)$ lies in the counterclockwise arc connecting Λ^s to Λ^u.*

Proof. As in the previous cases, rotating from $s(\mu)$ to s_{t_0+1} to s_{t_0} to $s(A\mu)$ must not take us through an angle in excess of π, lest (M', ξ') be overtwisted. On the Farey tessellation, this means that the counterclockwise path connecting these slopes (in this order) must not overlap itself. Now if $s(\mu)$ lies on the counterclockwise arc $[s(T^2), \Lambda^s]$ between $s(T^2)$ and Λ^s, then $s(A\mu)$ lies on $[s(\mu), \Lambda^s]$. But then the arc $[s(\mu), s(T^2)]$ contains $s(A\mu)$, meaning that the rotation described above is through an angle greater than π, and (M', ξ') is overtwisted. On the other hand, if $s(\mu)$ lies on $[\Lambda^u, s(T^2)]$, then $s(A\mu)$ is contained in $[s(\mu), s(AT^2)]$. Again this means that our path of slopes overlaps itself, and (M', ξ') is overtwisted. We conclude that $s(\mu)$ lies on $[\Lambda^s, \Lambda^u]$. □

With Lemma 5.7 in hand, we quickly obtain the finite list $(L_1, \xi_1), \ldots, (L_m, \xi_m)$ guaranteed by Proposition 5.6. Indeed, suppose we have a sequence $(s(\mu_i))$ of meridian slopes connected to $s(T^2)$ on the Farey tessellation (as all candidate meridian slopes must be). Then this sequence converges to $s(T^2)$, and thus only finitely many of the slopes are contained in $[\Lambda^s, \Lambda^u]$, since $s(T^2)$ is not contained in this interval. So there are only finitely many possible meridians μ for S_{t_0+1}, and hence only finitely many lens spaces to which M' could be diffeomorphic. Notice that the contact structure ξ' on M' is determined by ξ, so we now know that we have a finite list of tight lens spaces. Moreover, each of these lens spaces has a pair $L^i_\pm \subset (L_i, \xi_i)$ of distinguished Legendrian knots, along which we attach symplectic 1-handles to produce fillings of (M, ξ). Each of these is constructed along with its lens space as the core curves of S_{t_0} and S_{t_0+1}, respectively. We will obtain parts (1) and (2) of Proposition 5.6 by examining the slopes which appear in a continued fraction block.

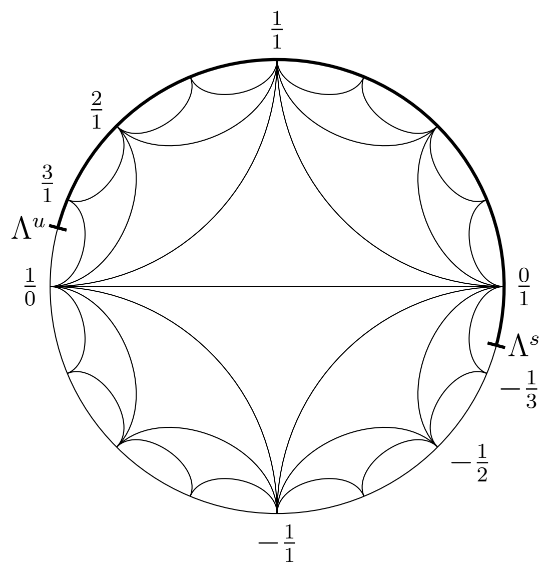


Figure 5.6: The interval $[\Lambda^s, \Lambda^u]$ for $A = C_{-5,2}$. Notice that $s_1 = \infty$ is connected to $0, 1, 2, 3$, while 0 is the only element of $[\Lambda^s, \Lambda^u]$ to which either of -1 or $-1/2$ is connected.

Lemma 5.8. *Choose $a \leq -3$ and $\tau \geq 0$, and let $A = C_{a,\tau}$, with stable and unstable slopes Λ^s and Λ^u. Let $(T^2 \times I, \xi)$ be tight and have boundary slopes s_1 and s_0 corresponding to $(0,1)^T$ and $A(0,1)^T$, respectively, and let $T^2 \subset T^2 \times I$ be a boundary component for a basic slice. Then*

(1) *if T^2 is not a boundary torus, there is exactly one slope s connected to $s(T^2)$ in the arc $[\Lambda^s, \Lambda^u]$ on the Farey tessellation;*

(2) *if $T^2 = T^2 \times \{1\}$, there are $\tau + 2$ slopes connected to $s(T^2)$ in the arc $[\Lambda^s, \Lambda^u]$.*

Proof. First, note that the boundary torus $T^2 \times \{1\}$ has slope ∞, and that $(T^2 \times I, \xi)$ contains $|a+2|$ basic slices. In particular, there are $|a+3|$ non-boundary tori which lie between basic slices, and these have slopes

$$-1, -1/2, \ldots, 1/(a+3).$$

We will prove this lemma by showing that 0 is the only value in $[\Lambda^s, \Lambda^u]$ that is connected to any of these slopes on the Farey tessellation, and by counting the values in $[\Lambda^s, \Lambda^u]$ which are connected to ∞. Note that expanding A gives

$$A = \begin{pmatrix} -\tau - (\tau+1)(1+a) & a+2 \\ -(\tau+1) & 1 \end{pmatrix},$$

and thus $T^2 \times \{0\}$ has slope $\frac{1}{a+2}$. In particular, since $[\Lambda^s, \Lambda^u] \subset [s_0, s_1]$, we have $\frac{1}{a+2} < \Lambda^s$. Similarly, because

$$A^{-1} = \begin{pmatrix} 1 & -a-2 \\ \tau+1 & -\tau - (\tau+1)(1+a) \end{pmatrix},$$

$A^{-1}(0,1)^T$ has slope $1 + \tau + \frac{1}{|a+2|} > \Lambda^u$. At the same time,

$$A \begin{pmatrix} 1 \\ \Lambda^u \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \Lambda^u \end{pmatrix}$$

for some $0 < \lambda < 1$. An explicit calculation shows that

$$\Lambda^u = \frac{1}{2}(1 + \tau + \sqrt{(1+\tau)^2 + 4(1+\tau)/|a+2|}) > 1 + \tau.$$

So $1 + \tau < \Lambda^u < 1 + \tau + \frac{1}{|a+2|} < 2 + \tau$. Similar reasoning shows that $\frac{1}{a+2} < \Lambda^s < 0$. On the Farey tessellation, ∞ is connected only to integers, and we see that $[\Lambda^s, \Lambda^u]$ contains precisely $\tau + 2$ integers: $0, 1, \ldots, \tau, 1 + \tau$. So s_1 is connected to $\tau + 2$ slopes in $[\Lambda^s, \Lambda^u]$, proving part (2). We also notice that each of the slopes $-1, -1/2, \ldots, 1/(a+3)$ is connected to 0 on the Farey tessellation, and, with the exception of -1, is connected only to rational numbers $p/q < 1/(a+2)$. In particular, 0 is the only value in $[\Lambda^s, \Lambda^u]$ to which any of the slopes $-1, -1/2, \ldots, 1/(a+3)$ is connected. See Figure 5.6. This proves part (1). □

While the precise slopes identified by Lemma 5.8 change when we compose to obtain

$$A = \pm C_{a_0,\tau_0} \cdots C_{a_k,\tau_k},$$

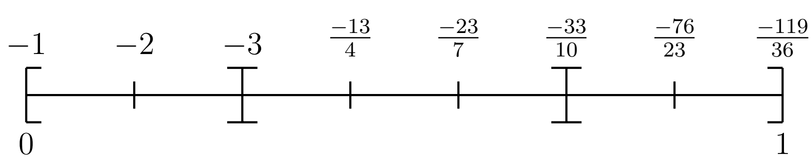
the counts do not. Indeed, each C_{a_i,τ_i} is an element of $SL(2,\mathbb{Z})$ and will therefore preserve connections and order on the Farey tessellation. So while the finite list of slopes $s_1 = \infty, -1, -1/2, \ldots, 1/(a+3), 1/(a+2) = s_0$ and the interval $[\Lambda^s, \Lambda^u]$ will change after composition, the connections between the former and elements of the latter will be unaltered.

In particular, if (M, ξ) has a virtually overtwisted continued fraction block, then there is a mixed torus T^2 which is interior to this block. According to Lemma 5.8, applying Theorem 3.3 along this mixed torus leaves us with precisely one possible meridian μ for S_{t_0+1}. Another possibility is for (M, ξ) to be virtually overtwisted, but to have continued fraction blocks which are all universally tight. In this case we find a mixed torus T^2 which lies between two continued fraction blocks. Lemma 5.8 tells us that $s(T^2)$ is connected to at most $2 + \max\{\tau_i\}$ distinct slopes in $[\Lambda^s, \Lambda^u]$, and thus there are at most $2 + \max\{\tau_i\}$ distinct lens spaces which may result from applying Theorem 3.3 along T^2. This completes the proof of Proposition 5.6.

Example 5.9. Let M be the positive hyperbolic torus bundle with coefficients $(a_0, a_1, a_2) = (-4, -5, -4)$ and $(\tau_0, \tau_1, \tau_2) = (0, 2, 0)$. That is, M has monodromy

$$A = T^{-4}ST^{-5}ST^{-2}ST^{-2}ST^{-4}S = \begin{pmatrix} 119 & -83 \\ -43 & 30 \end{pmatrix}.$$

According to the classification of tight contact structures on torus bundles ([Hon00b]), the tight contact structures on M are in one-to-one correspondence with the tight contact structures on $T^2 \times I$ which have boundary slopes $s_0 = -1$ and $s_1 = [-4, -2, -2, -5, -3] = -119/36$. Each such tight structure decomposes into seven basic slices, distributed among three continued fraction blocks, visualized as follows:



The long tick marks at slopes $-1, -3, -33/10$, and $-119/36$ indicate divisions between continued fraction blocks. The tori $T^2 \times \{0\}$ and $T^2 \times \{1\}$ (whose dividing sets have slopes

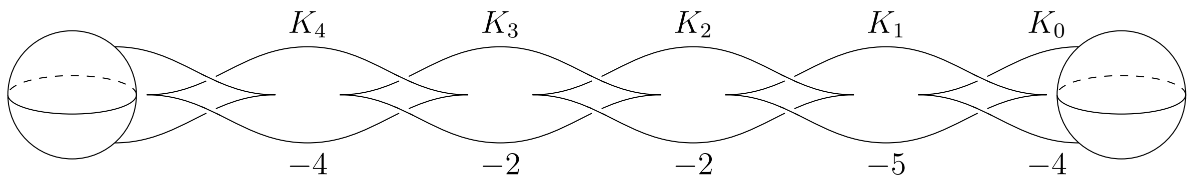


Figure 5.7: The standard filling of any tight structure on the torus bundle with positive monodromy coefficients $(r_0, r_1, r_2, r_3, r_4) = (-4, -5, -2, -2, -4)$ is obtained from Legendrian surgery along the above link in $(S^1 \times S^2, \xi_{std})$. Each knot K_i is stabilized so that its Thurston-Bennequin number exceeds its framing by 1, and the rotation numbers are determined by the tight structure ξ on M. The lens spaces produced by Theorem 5.1 may be obtained by erasing the 1-handle and an unknot, leaving a link in (S^3, ξ_{std}).

-1 and $-119/36$, respectively) will be identified by the monodromy A.

We now determine a tight structure on M by decorating each basic slice with a sign, indicating the sign of the relative Euler class of the tight structure when restricted to this basic slice. Because basic slices may be shuffled within a given continued fraction block, there are $3 \cdot 4 \cdot 3 = 36$ tight contact structures on M. As observed by Bhupal-Ozbagci in [BO14], each of these structures admits a Stein filling, depicted in Figure 5.7.

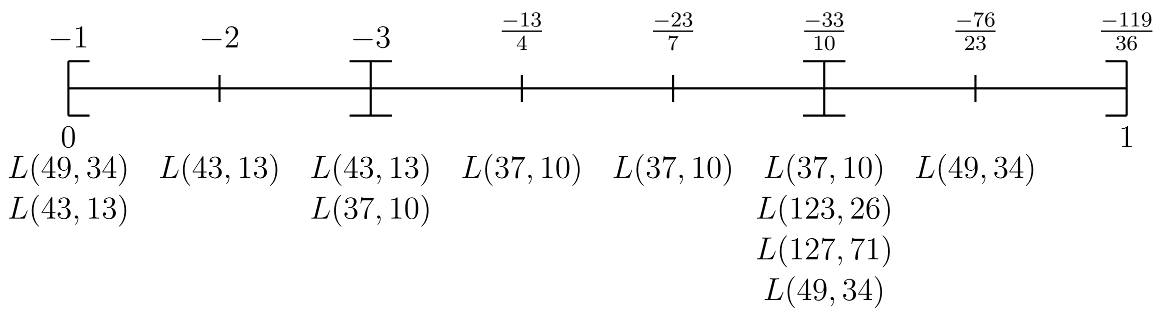
Each of the three continued fraction blocks in M has a corresponding tight lens space. Stein fillings of these lens spaces may be obtained from Figure 5.7 by erasing the 1-handle along with one of the unknots K_0, K_1, or K_4. So the lens spaces associated to the continued fraction blocks have topological types $L(43, 13)$, $L(37, 10)$, and $L(49, 34)$, respectively. The tight structure on the resulting lens space is determined by Figure 5.7. If a given continued fraction block in (M, ξ) is virtually overtwisted, then (M, ξ) admits a mixed torus interior to this continued fraction block, and each exact filling of (M, ξ) may be obtained from an exact filling of the corresponding lens space by round 1-handle attachment.

There are eight tight contact structures on M which are virtually overtwisted, but for which all three continued fraction blocks are universally tight. In these cases, (M, ξ) admits a mixed torus which lies at the boundary of two continued fraction blocks, and the list of lens spaces produced by Theorem 5.1 consists of the lens spaces corresponding to these two continued fraction blocks, as well as all lens spaces *between* these. These extra lens spaces are obtained from Figure 5.7 by deleting the round 1-handle and an unknot with a -2-framing. For instance, if we have a mixed torus of slope $-33/10$ — sitting at the boundary of the second and third continued fraction blocks — then the list of lens spaces produced by Theorem 5.1 has topological type

$$(L(37,10), L(123,26), L(127,71), L(49,34)).$$

A mixed torus between the first and second or third and first continued fraction blocks will yield a list of exactly two lens spaces, as there are no intermediate lens spaces in these cases.

The following diagram summarizes this example:



If ξ is a virtually overtwisted tight contact structure on M, then there must be a mixed torus at one of the seven slopes above. (Because $T^2 \times \{1\}$ is identified with $T^2 \times \{0\}$, we need not consider this torus separately.) Below each slope we see the diffeomorphism types of the lens spaces produced by Theorem 5.1 when a mixed torus occurs with this slope. As mentioned above, the tight structures on these lens spaces will be determined by ξ, as depicted in Figure 5.7.

5.4 Distinct decompositions of fillings

Even with the classification of fillings of lens spaces completed in Chapter 4, the only immediate consequence of Theorem 5.1 is a recipe which is guaranteed to construct all fillings of such a torus bundle, but these constructions need not be unique. Indeed, if $(L(p_1, q_1), \xi_1), \ldots, (L(p_m, q_m), \xi_m)$ is the list of lens spaces provided by Theorem 5.1 for some hyperbolic torus bundle (M, ξ), it is possible that there are fillings (W_i, ω_i) of $(L(p_i, q_i), \xi_i)$ for $i = 1, \dots, m$, each of which yield the same filling (W, ω) of (M, ξ) after round 1-handle attachment.

To fully explicate such overcounting of the fillings of (M, ξ) will require a detailed understanding of the fillings of $(L(p_1, q_1), \xi_1), \dots, (L(p_m, q_m), \xi_m)$. The proof of Theorem 5.1 establishes a relationship among the lens spaces $(L(p_1, q_1), \xi_1), \dots, (L(p_m, q_m), \xi_m)$ that allows us to identify one source of overcounting. In particular, the construction associates to each lens space $(L(p_i, q_i), \xi_i)$ a pair of distinguished Legendrian knots $L^i_-, L^i_+ \subset (L(p_i, q_i), \xi_i)$ which arise as the core curves of the solid tori which are glued onto $T^2 \times \{0\}$ and $T^2 \times \{1\}$, respectively, after (M, ξ) is cut open along its mixed torus. For $1 \leq i \leq m-1$, we obtain the lens space $(L(p_{i+1}, q_{i+1}), \xi_{i+1})$ by simultaneously performing $(+1)$-surgery along L^i_+ and (-1)-surgery along L^i_-.

This relationship between the lens spaces gives us an algorithm for building fillings of $(L(p_{i+1}, q_{i+1}), \xi_{i+1})$ from fillings of $(L(p_i, q_i), \xi_i)$. If (W_i, ω_i) is a strong symplectic filling of $(L(p_i, q_i), \xi_i)$ in which L^i_+ bounds a Lagrangian disc, then we may remove a neighborhood of the disc from (W_i, ω_i) to obtain a symplectic manifold strongly filling its boundary (c.f. [CET21, Theorem 3.1]), and the effect on the boundary is to perform $(+1)$-surgery along L^i_+. We may then attach a Weinstein 2-handle to this new filling along L^i_-, and at the boundary this has the effect of performing a (-1)-surgery along L^i_-. The result is (W_{i+1}, ω_{i+1}), a strong symplectic filling of $(L(p_{i+1}, q_{i+1}), \xi_{i+1})$, and Proposition 5.10 will tell us that this filling leads to the same filling of (M, ξ) as does (W_i, ω_i) after round 1-handle attachment.

To make the statement of Proposition 5.10 less cumbersome, we establish some notation. Suppose we have a strong symplectic filling (X, ω) of a contact manifold (M, ξ), with Legendrian knots $L_0, L_1 \subset (M, \xi)$. Moreover, suppose that L_0 is the boundary of a Lagrangian disc $D \subset (X, \omega)$ which meets ∂X transversely. As alluded to above, we may identify a neighborhood $N \subset X$ of D with a neighborhood of the cocore of a Weinstein 2-handle, and remove this neighborhood to obtain a new symplectic filling $(\overline{X}, \overline{\omega})$. (The existence of such a neighborhood is proven in Theorem 3.1 of [CET21].) With (X, ω), D, L_0, and L_1 understood, we denote by (X', ω') the strong symplectic filling which results from attaching a Weinstein 2-handle to $(\overline{X}, \overline{\omega})$ along L_1. Each of L_0 and L_1 has a natural Legendrian pushoff in $\partial(X', \omega')$, given by the boundary of the core and cocore of the associated 2-handle, respectively, which we denote L_0' and L_1'. With this notation in hand, we have the following result.

Proposition 5.10. *Let (X, ω) and (X', ω') be strong symplectic fillings related by the construction above. Let (W, ω_W) be the strong symplectic filling obtained by attaching a round symplectic 1-handle to (X, ω) along L_0, L_1, and let (W', ω_W') be the analogous filling for (X', ω'). Then (W, ω) and (W', ω_W') are symplectomorphic fillings of (M, ξ).*

Our strategy for proving this result is to realize round symplectic 1-handle attachment as a sequence of Weinstein handle attachments, and then to reorder the Weinstein handles. The decomposition of a round 1-handle into Weinstein handles is explained in great generality in [Avd12, Section 7.2], but the precise statement we need here is the following.

Lemma 5.11. *Fix a strong symplectic filling (X, ω) of a contact 3-manifold (M, ξ), and identify Legendrian knots $L_0, L_1 \subset (M, \xi)$. Consider the following symplectic fillings:*

1. *(X_r, ω_r), obtained by attaching a round symplectic 1-handle to (X, ω) along the knots $L_0, L_1 \subset (M, \xi)$;*
2. *(X_w, ω_w), obtained by attaching a Weinstein 1-handle to (X, ω) along points $p_i \in L_i$, $i = 0, 1$, and then attaching a Weinstein 2-handle to the resulting filling along the knot L obtained by surgering L_0 and L_1 along p_0 and p_1.*

Then (X_r, ω_r) is symplectomorphic to (X_w, ω_w).

Before providing a proof of Lemma 5.11, let us say what we mean by *surgering L_0 and L_1 along p_0 and p_1*. A 4-dimensional Weinstein 1-handle H_1 admits a Lagrangian submanifold-with-boundary $\Lambda \subset H_1$ to which the Liouville vector field on H_1 is tangent, and such that this vector field gives Λ the structure of a 2-dimensional Weinstein 1-handle. We then have Legendrian submanifolds $\partial_{in}\Lambda \subset \partial_{in}H_1$ and $\partial_{out}\Lambda \subset \partial_{out}H_1$. Now we attach H_1 to (X, ω) along p_0, p_1 by choosing a contactomorphism from the attaching region $\partial_{in}H_1$ to a neighborhood of the points p_0, p_1. This contactomorphism can be chosen so that

$\partial_{in}\Lambda \subset \partial_{in}H_1$ is mapped to arcs a_0, a_1 of the Legendrians L_0, L_1. In the boundary of the symplectic filling that results, we will have a Legendrian knot

$$L = (L_0 \setminus a_0) \cup \partial_{out}\Lambda \cup (L_1 \setminus a_1),$$

and it is this knot that we consider to be the result of surgering L_0 and L_1 along p_0 and p_1.

Proof of Lemma 5.11. Once we identify our round symplectic 1-handle with a symplectic handle in the sense of Avdek, this fact follows from the discussion in [Avd12, Section 7.2]. We will present Avdek's argument in our particular case.

As mentioned in Chapter 2, Avdek defines an abstract round symplectic 1-handle as follows. Let $(\Sigma, \beta) = (D^*S^1, \lambda_{can})$ be the unit disc bundle in (T^*S^1, λ_{can}), and consider the contact manifold

$$(N(\Sigma) = [-\epsilon, \epsilon] \times \Sigma, \alpha = dz + \beta), \tag{5.3}$$

where z is the coordinate on $[-\epsilon, \epsilon]$. Avdek rounds the edges of $N(\Sigma)$ to obtain $\mathcal{N}(\Sigma)$, and then defines the symplectic manifold

$$(H_\Sigma, \omega_\beta) = ([-1, 1] \times \mathcal{N}(\Sigma), d\theta \wedge dz + d\beta), \tag{5.4}$$

where θ is the coordinate on $[-1, 1]$. After more edge-rounding, this is an abstract copy of a round symplectic 1-handle, to be attached along the ends $\{\pm 1\} \times \mathcal{N}(\Sigma)$. In particular, we have the Liouville form

$$\lambda_\Sigma = -\theta\, dz - 2z\, d\theta + \beta,$$

and $(H_\Sigma, \lambda_\Sigma)$ carries a Liouville vector field Z which points into H_Σ along $\{\pm 1\} \times \mathcal{N}(\Sigma)$ and out of H_Σ along $[-1, 1] \times \partial\mathcal{N}(\Sigma)$.

Now (Σ, β) has an obvious handle decomposition as a Weinstein domain, given by attaching a Weinstein 1-handle to a Weinstein 0-handle, and yielding the filtration

$$(\mathbb{D}^2, \lambda_{std}) = (\Sigma_0, \beta_0) \subset (\Sigma, \beta)$$

of (Σ, β). By carrying out the constructions of (5.3) and (5.4) for (Σ_0, β_0), we obtain filtrations

$$(\mathbb{D}^3, \alpha_{std}) = (\mathcal{N}(\Sigma_0), \alpha_0) \subset (\mathcal{N}(\Sigma), \alpha) \quad \text{and} \quad (H_{\Sigma_0}, \lambda_{\Sigma_0}) \subset (H_\Sigma, \lambda_\Sigma),$$

and it is clear that $(H_{\Sigma_0}, \lambda_{\Sigma_0})$ is symplectomorphic to a Weinstein 1-handle. It remains to verify that we obtain $(H_\Sigma, \lambda_\Sigma)$ from $(H_{\Sigma_0}, \lambda_{\Sigma_0})$ by attaching a Weinstein 2-handle.

To this end, we identify a Legendrian ribbon (Σ^i, β^i) of $L_i \subset (M, \xi)$ with (Σ, β), using the filtration of (Σ, β) to define $(\Sigma^i_0, \beta^i_0) \subset (\Sigma^i, \beta^i)$, for $i = 0, 1$. Now the Liouville hypersurfaces (Σ^0, β^0) and (Σ^1, β^1) admit standard neighborhoods $N(\Sigma^0), N(\Sigma^1) \subset (M, \xi)$ along which a round symplectic 1-handle may be attached to (X, ω) to yield (X_r, ω_r). On the other hand, let (X_1, ω_1) denote the result of attaching the Weinstein 1-handle $(H_{\Sigma_0}, \lambda_{\Sigma_0})$ to (X, ω) along $N(\Sigma^0_0)$ and $N(\Sigma^1_0)$. By definition, we obtain (X_w, ω_w) from (X_1, ω_1) by attaching a Weinstein 2-handle, but the filtration $(H_{\Sigma_0}, \lambda_{\Sigma_0}) \subset (H_\Sigma, \lambda_\Sigma)$ above allows us to view (X_1, ω_1) as living inside of (X_r, ω_r). We will use this perspective to see (X_r, ω_r) as the result of attaching a Weinstein 2-handle to (X_1, ω_1) and thus obtain the desired symplectomorphism.

For $i = 0, 1$, let $\Lambda_i \subset L_i \subset (\Sigma^i, \beta^i)$ be the core disc of the 1-handle attached to (Σ^i_0, β^i_0) to yield (Σ^i, β^i). This is a Legendrian chord in the boundary of (X_1, ω_1), and we identify Λ_i inside of $N(\Sigma^i)$ as

$$\Lambda_i = \{z = 0\} \times \Lambda_i \subset [-\epsilon, \epsilon] \times \Sigma^i = N(\Sigma^i).$$

At the same time, consider the disc

$$\tilde{\Lambda} = [-1, 1] \times \Lambda \subset (H_\Sigma \setminus \operatorname{int} H_{\Sigma_0}),$$

where $\Lambda \subset (\Sigma, \beta)$ is the analogous chord in Σ. Viewing (X_1, ω_1) as a subset of (X_r, ω_r), we see that $(X_1, \omega_1) \cap \tilde{\Lambda} = \Lambda_0 \sqcup \Lambda_1$, and that $\tilde{\Lambda}$ represents the core disc of the Weinstein 2-handle $(H_\Sigma \setminus \operatorname{int} H_{\Sigma_0}, \lambda_\Sigma)$. The latter statement follows from the fact that Λ is the core disc of the Weinstein 1-handle $(\Sigma \setminus \Sigma_0, \beta)$, meaning that β vanishes along Λ, and thus λ_Σ

vanishes along $\tilde{\Lambda}$. Up to smoothing, this disc has boundary

$$\partial\tilde{\Lambda} = \Lambda_0 \cup ([-1,1] \times \partial\Lambda) \cup \Lambda_1 \subset (X_1, \omega_1),$$

which is the result in (X_1, ω_1) of surgering L_0 to L_1. So (X_r, ω_r) is obtained from (X_1, ω_1) by attaching a Weinstein 2-handle along this surgered knot, as desired, and thus round 1-handle attachment may be realized as Weinstein 1-handle attachment followed by Weinstein 2-handle attachment. □

We are now prepared to prove Proposition 5.10.

Proof of Proposition 5.10. We continue using the notation established before the statement of Proposition 5.10. In particular, we have a symplectic filling $(\overline{X}, \overline{\omega})$ with Legendrian knots $\overline{L}_0, \overline{L}_1 \subset \partial(\overline{X}, \overline{\omega})$ such that (X, ω) and (X', ω') are obtained from $(\overline{X}, \overline{\omega})$ by attaching a Weinstein 2-handle along $\overline{L}_0$ and $\overline{L}_1$, respectively.

Now since Lemma 5.11 tells us that round symplectic 1-handle attachment consists of a Weinstein 1-handle attachment followed by a Weinstein 2-handle attachment, we see that (W, ω) and (W', ω') are each obtained from $(\overline{X}, \overline{\omega})$ by a sequence of Weinstein handle attachments. In particular, we obtain (W, ω) by attaching a Weinstein 2-handle along $\overline{L}_0$, then attaching a Weinstein 1-handle along points in $L_0, L_1 \subset \partial(X, \omega)$ — knots which are Legendrian pushoffs of $\overline{L}_0, \overline{L}_1$ — and then attaching a Weinstein 2-handle along the resulting surgered knot. By considering $\overline{L}_1$ instead of $\overline{L}_0$, we obtain (W', ω'). In either case, we may reorder our handle attachments so that the round symplectic 1-handle is attached first to produce a filling $(\overline{X}', \overline{\omega}')$, leaving us to attach a Weinstein 2-handle along either $\overline{L}_0'$ or $\overline{L}_1'$ — these being Legendrian pushoffs of the corresponding knots in $(\overline{X}, \overline{\omega})$. But since we have attached a round symplectic 1-handle along $\overline{L}_0, \overline{L}_1$, their pushoffs are Legendrian isotopic in $\partial(\overline{X}', \overline{\omega}')$, with the isotopy being witnessed by a cylinder in the boundary of the round 1-handle. So (W, ω) and (W', ω') are obtained from $(\overline{X}, \overline{\omega})$ by the same sequence of Weinstein handle attachments, and thus are symplectomorphic. □

The practical upshot of Proposition 5.10 is this: for $1 \leq i \leq m-1$, the lens space

$(L(p_i, q_i), \xi_i)$ produced by Theorem 5.1 comes with distinguished Legendrian knots $L^i_-, L^i_+ \subset L(p_i, q_i)$ which are used to produce $(L(p_{i+1}, q_{i+1}), \xi_{i+1})$ as described above. Proposition 5.10 says that if we have a filling of $(L(p_i, q_i), \xi_i)$ in which L^i_+ bounds a Lagrangian disc, then the filling of (M, ξ) produced by round 1-handle attachment is also obtained from the corresponding filling of $(L(p_{i+1}, q_{i+1}), \xi_{i+1})$. In particular, if we are using Theorem 5.1 to tabulate the fillings of (M, ξ), then any filling of $(L(p_i, q_i), \xi_i)$ in which L^i_+ bounds a Lagrangian disc may be ignored, as there is a filling of $(L(p_{i+1}, q_{i+1}), \xi_{i+1})$ which will produce the same filling of (M, ξ). Concretely, we have the following corollary.

Corollary 5.12. *Let (M, ξ) be a virtually overtwisted hyperbolic torus bundle, and let*

$$(L(p_1, q_1), \xi_1), \ldots, (L(p_m, q_m), \xi_m)$$

be the list of tight lens spaces produced by Theorem 5.1, with distinguished Legendrian knots $L^1_-, L^1_+, \ldots, L^m_-, L^m_+$. For $1 \leq i \leq m-1$, if (W_i, ω_i) is a strong symplectic filling of $(L(p_i, q_i), \xi_i)$ in which L^i_+ bounds a Lagrangian disc, then there is a strong filling (W_{i+1}, ω_{i+1}) of $(L(p_{i+1}, q_{i+1}), \xi_{i+1})$ such that (W_i, ω_i) and (W_{i+1}, ω_{i+1}) yield symplectomorphic fillings of (M, ξ) after round symplectic 1-handle attachment along L^i_-, L^i_+ or L^{i+1}_-, L^{i+1}_+.

Corollary 5.12 provides just one answer to the following question: under what conditions do fillings (W_i, ω_i) and (W_{i+1}, ω_{i+1}) of $(L(p_i, q_i), \xi_i)$ and $(L(p_{i+1}, q_{i+1}), \xi_{i+1})$, respectively, yield the same filling of (M, ξ) after round symplectic 1-handle attachment? A full answer to this question will complete the classification of fillings for virtually overtwisted contact structures on torus bundles.

CHAPTER 6

Surgeries on Legendrian negative cables

In this chapter we aim to classify symplectic fillings for spaces obtained from $(S^3, \xi_{\mathrm{std}})$ via contact surgery along certain Legendrian knots. Theorem 1.1.3 of [Men18] is the first instance of such a result, showing that these surgeries have unique fillings (up to diffeomorphism) when the Legendrian knot has been stabilized both positively and negatively. In this section we study fillings in the case that our knot is a Legendrian negative cable of a Legendrian with stabilizations of opposite sign.

6.1 The result

First defined in [Ng01], a thorough study of Legendrian satellite knots can be found in [EV18], some notation of which we now recall. We consider a contact manifold (V, ξ_V) defined by $V = D^2_{y,z} \times S^1_\theta$, $\xi_V = \ker(dz - y\,d\theta)$. Any Legendrian knot $L \subset (S^3, \xi_{\mathrm{std}})$ has a neighborhood $\nu(L)$ which is contactomorphic to (V, ξ_V), and given any Legendrian knot $Q \subset V$, we denote by $Q(L) \subset \nu(L)$ the image of Q under this contactomorphism. We pay special attention to the case where $Q \subset V$ is a Legendrian (p, q)-torus knot, for some coprime p, q with $q > 0$, in which case we call $Q(L)$ a *Legendrian cable* of L. We point out that if Q is a (p, q)-torus knot and $\mathcal{K}$ is the knot type of L, then the knot type of Q is $\mathcal{K}_{p+q\,\mathrm{tb}(L),q}$, that of a smooth $(p+q\,\mathrm{tb}(L), q)$-cable of L. The reason for this is that the contactomorphism between $\nu(L)$ and V identifies the product framing on V with the contact framing on $\nu(L)$; see [EV18, Section 5] for more details.

We will also need a particular embedding of (V, ξ_V) into itself. Notice that the core C of V is a Legendrian curve, and that $(V, \xi_V) = \nu(C)$ is a standard neighborhood of C. We may stabilize C to obtain $S_+(C) \subset V$ and identify a standard neighborhood $\nu(S_+(C)) \subset V$ of the stabilization. We have a contactomorphism between $\nu(C) = (V, \xi_V)$ and $\nu(S_+(C)) \subset (V, \xi_V)$, giving us an embedding $\zeta\colon (V, \xi_V) \hookrightarrow (V, \xi_V)$.

Next we point out that a Legendrian knot $Q \subset V$ which is smoothly a (p, q)-torus knot can be used to determine a tight contact structure ξ_Q on $L(q^2, pq - 1)$. The construction is as follows: let $S \cong (D^2 \times S^1, \xi_{\text{std}})$ be a tight solid torus, glued to V in such a way that $V \cup S \cong (S^2 \times S^1, \xi_{\text{std}})$. Then $(L(q^2, pq - 1), \xi_Q)$ is the result of Legendrian surgery on $(S^2 \times S^1, \xi_{\text{std}})$ along $\zeta(Q)$.

Our final preparation before stating the result of this chapter is to explain how we may represent a Legendrian knot $L \subset (S^3, \xi_{\text{std}})$ as a knot in $(L(q^2, pq - 1), \xi_Q)$. First, consider the knot $K = \{\text{pt}\} \times S^1$ in $(S^2 \times S^1, \xi_{\text{std}})$; we take K to be disjoint from $Q \subset S^2 \times S^1$. By performing the contact connected sum $(S^2 \times S^1, \xi_{\text{std}}) \# (S^3, \xi_{\text{std}})$ along points $x \in K$ and $y \in L$, we obtain $K \# L$ as a Legendrian knot in $(S^2 \times S^1, \xi_{\text{std}})$. Finally, we perform Legendrian surgery on $(S^2 \times S^1, \xi_{\text{std}})$ along $\zeta(Q)$, and $K \# L$ passes to a Legendrian knot in $(L(q^2, pq - 1), \xi_Q)$. Abusing notation, we call this Legendrian knot L.

We are now prepared to state our result.

Theorem 6.1. *Let $L \subset (S^3, \xi_{\text{std}})$ be a Legendrian knot with smooth knot type $\mathcal{K}$, and let $Q(S_+S_-(L))$ be a Legendrian negative cable of $S_+S_-(L)$, the smooth knot type of which is $\mathcal{K}_{p,q}$. Suppose that the Thurston-Bennequin number of $Q(S_+S_-(L))$ is maximal among such Legendrian knots, and let (M, ξ) be the contact manifold which results from Legendrian surgery on (S^3, ξ_{std}) along $Q(S_+S_-(L))$. Then every strong symplectic filling of (M, ξ) may be obtained, up to diffeomorphism, by attaching a Weinstein 2-handle to a strong symplectic filling of $(L(q^2, pq - 1), \xi_Q)$ along $S_-(L) \subset L(q^2, pq - 1)$.*

Remark.

(1) Because $Q(S_+S_-(L))$ has the smooth knot type $\mathcal{K}_{p,q}$, this knot is a Legendrian $(p + q(2 - \text{tb}(L)), q)$-cable of $S_+S_-(L)$. Since $Q(S_+S_-(L))$ is a Legendrian negative cable, $p < q(\text{tb}(L) - 2)$.

(2) According to [EV18, Theorem 5.16], the Thurston-Bennequin number of $Q(S_+S_-(L))$

is pq, and thus M is the result of $(pq-1)$-surgery along $\mathcal{K}_{p,q}$. By [Gor83, Corollary 7.3], this surgery is diffeomorphic to $(pq-1)/q^2$-surgery along $\mathcal{K}$.

6.2 Proof of Theorem 6.1

Throughout this section we let $L \subset (S^3, \xi_{\text{std}})$ be a Legendrian knot with smooth knot type $\mathcal{K}$, and let $Q(S_+S_-(L))$ be a Legendrian negative cable of $S_+S_-(L)$ with smooth knot type $\mathcal{K}_{p,q}$, $p < q(\text{tb}(L)-2)$. We suppose that the Thurston-Bennequin number of $Q(S_+S_-(L))$ is maximal among such knots, and we let (M,ξ) be the contact manifold obtained by Legendrian surgery along $Q(S_+S_-(L))$.

We may use the stabilizations on $S_+S_-(L)$ to identify a mixed torus in (M,ξ). In particular, let $N(S_-(L)) \subset (S^3, \xi_{\text{std}})$ be a standard neighborhood of $S_-(L)$. We let V_1 be the solid torus obtained from this neighborhood via Legendrian surgery along $Q(S_+S_-(L))$, and let $V_2 = S^3 \setminus N(S_-(L))$. Then $M = V_1 \cup V_2$, and we claim that the common boundary $\partial V_1 = \partial N(S_-(L)) = \partial V_2$ is a mixed torus. Indeed, consider the three convex tori $\partial N(L)$, $\partial N(S_-(L))$, and $\partial N(S_+S_-(L))$. The tori $\partial N(L)$ and $\partial N(S_-(L))$ cobound a negative basic slice in M. The tori $\partial N(S_-(L))$ and $\partial N(S_+S_-(L))$ cobound a positive basic slice in $N(S_-(L))$, but this may not survive to a basic slice in M, since $Q(S_+S_-(L))$ may not be disjoint from $\partial N(S_+S_-(L))$. However, we can subdivide this basic slice to find a boundary parallel convex torus cobounding a positive basic slice with $\partial N(S_-(L))$ (c.f. [EH01, Lemma 3.15]). So $\partial N(S_-(L))$ sits between basic slices of opposite sign, and is therefore a mixed torus.

Now because of our assumptions that $p < q(\text{tb}(L)-2)$ and that the Thurston-Bennequin number of $Q(S_+S_-(L))$ is maximal, [EV18, Theorem 5.16] tells us that $\text{tb}(Q(S_+S_-(L)) = pq$. So the Legendrian surgery used to produce V_1 from $N(S_-(L))$ is smoothly $(pq-1)$-surgery. According to Lemmas 7.2 and 7.3 of [Gor83], V_1 is then a solid torus $D^2 \times S^1$ whose meridional curves have slope $(pq-1)/q^2$ in the coordinates of $\partial N(S_-(L))$ given by the

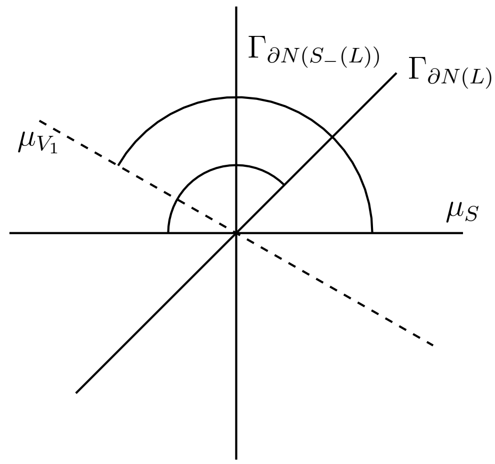


Figure 6.1: If the slope of μ_S were negative, then $V_1 \cup S$ would be overtwisted; if the slope were positive, then $V_2 \cup S$ would be overtwisted. So μ_S is horizontal.

meridian μ and the preferred longitude λ. We now apply

$$\begin{pmatrix} 1 & 0 \\ 1 - \operatorname{tb}(L) & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$$

to the coordinates of $\partial N(S_-(L))$. In the original coordinates, the dividing curves of $\partial N(L)$ and $\partial N(S_-(L))$ had slopes $1/\operatorname{tb}(L)$ and $1/(\operatorname{tb}(L) - 1)$, respectively. In our new coordinates we find that $\Gamma_{\partial N(L)}$ has slope 1, $\Gamma_{\partial N(S_-(L))}$ is vertical, and the meridional slope of μ_{V_1} is represented by the vector $(pq - 1 + q^2(1 - \operatorname{tb}(L)), q^2)$ in $\mathbb{Z}^2$.

Having made these preparations, we now suppose that (W, ω) is a strong symplectic filling of (M, ξ). Applying Theorem 3.3 to this filling yields (W', ω'), a strong symplectic filling of its boundary (M', ξ'), which we may write as

$$M' = M_1 \sqcup M_2 := (V_1 \cup S) \sqcup (V_2 \cup S),$$

for some identifications $\partial S \to \partial V_i$, where S is a solid torus. The gluing maps $\partial S \to \partial V_i$ identify dividing curves, but the meridian μ_S of S could in principle take any number of values. Our first observation is that, because Γ_{V_i} is vertical, $\mu_S = (1, m) \in \mathbb{Z}^2$ for some $m \in \mathbb{Z}$. Next, the fact that (M', ξ') is fillable means that each of M_1 and M_2 is tight. On M_1, we see that as we move from the core of S to ∂V_1 and then towards the core of V_1, the contact planes rotate from the slope of μ_S towards that of Γ_{V_1}, and finally towards the slope of μ_{V_1}. Because of our assumption that $p < q(\mathrm{tb}(L) - 2)$, we find that $-1 < \mu_{V_1} < 0$. Tightness demands that the total rotation of the contact planes is through an angle smaller than π, meaning that $m \geq 0$. See Figure 6.1. On M_2 we see that the contact planes rotate counterclockwise from 1, the slope of $\Gamma_{\partial N(L)}$, to the slope of $\Gamma_{\partial N(S_-(L))}$, and finally to the slope m of μ_S. Because this rotation must be smaller than π, we see that $m \leq 0$. So we conclude that $m = 0$.

Because the solid torus S is attached with slope $m = 0$, we find that $M_1 = L(q^2, pq - 1)$ and $M_2 \cup S = S^3$. Moreover, we see from the definition of M_1 that M_1 results from surgery on $(S^2 \times S^1, \xi_{\mathrm{std}})$ along $\zeta(Q)$, as described in Section 6.1. So $M_1 \cong (L(q^2, pq - 1), \xi_Q)$; on S^3 we have the unique tight contact structure ξ_{std}. Now Theorem 3.3 tells us that we recover (W, ω) from (W', ω') by attaching a round symplectic 1-handle along the cores of the two copies of S — one in M_1 and the other in M_2. In M_1 this core is given by K, the image of $\{\mathrm{pt}\} \times S^1 \subset (S^2 \times S^1, \xi_{\mathrm{std}})$ after performing surgery along $\zeta(Q) \subset S^2 \times S^1$. In M_2 the core is given by $S_-(L)$. We attach the round symplectic 1-handle by first attaching a Weinstein 1-handle along a pair of points $x \in K$ and $y \in S_-(L)$, and then attaching a Weinstein 2-handle along the resulting knot $K\#S_-(L)$. That is, we obtain (W, ω) from (W', ω') by attaching a Weinstein 2-handle to

$$(L(q^2, pq - 1), \xi_Q)\#(S^3, \xi_{\mathrm{std}}) \cong (L(q^2, pq - 1), \xi_Q)$$

along $K\#S_-(L) \cong S_-(L)$, proving Theorem 6.1.

CHAPTER 7

Seifert fibered spaces

In this section we apply Theorem 3.3 to large classes of contact structures on spaces which are Seifert fibered over S^2, with at least three singular fibers. Our results reduce the classification of fillings of these spaces to the classification problem for lens spaces — a problem which is settled by Theorem 4.2. We will first consider Seifert fibered spaces whose Euler number e_0 is non-negative, and then consider spaces with $e_0 \leq -3$. Here the *Euler number* of a Seifert fibered space $M(r_1, \ldots, r_n)$ over S^2 is defined to be $e_0 := \Sigma \lfloor r_i \rfloor$. Starkston [Sta15] and Choi-Park [CP19] have previously studied fillings of small Seifert fibered spaces satisfying $e_0 \leq -3$, but we consider a distinct collection of contact structures on these spaces.

On small Seifert fibered spaces — those with precisely three singular fibers — the contact structures satisfying $e_0 \geq 0$ or $e_0 \leq -3$ have been classified by Ghiggini-Lisca-Stipsicz [GLS06] and Wu [Wu04], and we will see that Theorem 3.3 applies to a great many of these structures. For Seifert fibered spaces over S^2 with more than three singular fibers, the tight structures have not been fully classified, but we can construct large classes of tight structures for which Theorem 3.3 applies.

7.1 The results

We present our results in two sections, depending on the Euler number e_0.

7.1.1 The case $e_0 \geq 0$

We now consider a Seifert fibered space over S^2 with $n \geq 3$ singular fibers. Choose coprime integers $q_i, p_i > 0$, $i = 1, \ldots, n$, which satisfy $q_i < p_i$ for $i = 1, \ldots, n-1$. We may construct a tight contact structure ξ on $M = M(\frac{q_1}{p_1}, \cdots, \frac{q_n}{p_n})$ by realizing M as the boundary of a Stein domain with handlebody description as in Figure 7.1. Here

$$-\frac{p_i}{q_i} = [a_0^i, a_1^i, \ldots, a_{l_i}^i] \quad \text{for } i = 1, \ldots, n, \tag{7.1}$$

for some uniquely determined integers

$$a_0^n \leq -1 \quad \text{and} \quad a_0^1, \ldots, a_0^{n-1}, a_1^i, \ldots, a_{l_i}^i \leq -2.$$

We obtain a Stein structure on the handlebody in Figure 7.1 by putting each unknot in Legendrian position with clockwise orientation and stabilizing until the framing coefficient becomes -1 with respect to the contact framing. It is possible for distinct choices of stabilizations to lead to the same tight contact structure on M — that is, there are equivalence relations among the handlebody diagrams. For small Seifert fibered spaces, Ghiggini-Lisca-Stipsicz show in [GLS06] that there are precisely

$$\left| \left(\prod_{i=1}^{3} (a_0^i + 1) - \prod_{i=1}^{3} a_0^i \right) \prod_{i=1}^{3} \prod_{j=1}^{l_i} (a_j^i + 1) \right|$$

positive tight contact structures on M, up to isotopy. If M has four singular fibers, Medetoğullari shows in [Med10] that the number of distinct Stein fillable contact structures is between

$$\left| \left(\prod_{i=1}^{4} (a_0^i + 1) - \prod_{i=1}^{4} a_0^i \right) \prod_{i=1}^{4} \prod_{j=1}^{l_i} (a_j^i + 1) \right| \quad \text{and} \quad 2 \left| \left(\prod_{i=1}^{4} (a_0^i + 1) - \prod_{i=1}^{4} a_0^i \right) \prod_{i=1}^{4} \prod_{j=1}^{l_i} (a_j^i + 1) \right|.$$

Generally, if $n \geq 4$, then M contains incompressible tori and therefore admits infinitely many tight contact structures according to work of Colin [Col01a, Col01b] and Honda-Kazez-Matić [HKM02].

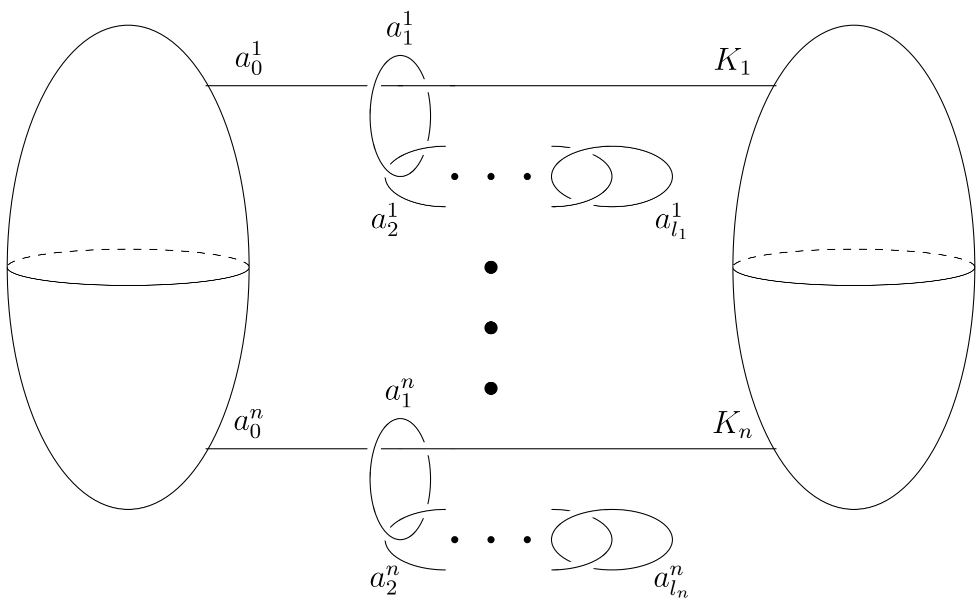


Figure 7.1: Handlebody decomposition of a Stein filling of $M = M(\frac{q_1}{p_1}, \cdots, \frac{q_n}{p_n})$. A contact structure is produced on M by putting each of the knots in Legendrian position and stabilizing appropriately.

Our first result for spaces M which are Seifert fibered over S^2 applies to tight contact structures which are *thoroughly mixed*, a notion we will define precisely in Section 7.2. The definition is designed so that each singular fiber of M admits a tubular neighborhood whose boundary is a mixed torus. Applying Theorem 3.3 to a filling of (M, ξ) will then leave us with a boundary connected sum of fillings of lens spaces. In Section 7.2 we will also define *lightly mixed* tight contact structures, in which all but two singular fibers admit mixed tori. With these provisional definitions, we may state our results.

Theorem 7.1. *Let ξ be a tight contact structure on the Seifert fibered space $M = M(\frac{q_1}{p_1}, \cdots, \frac{q_n}{p_n})$, for some $n \geq 3$ and coprime positive integers q_i, p_i with $q_i < p_i$ for $1 \leq i \leq n-1$ and $p_i \geq 2$ for $1 \leq i \leq n$. If ξ is thoroughly mixed, then there are tight contact structures ξ_i on $L(q_i, -p_i)$ for $i = 1, \ldots, n$ and Legendrian knots $L_i^- \subset (L(q_i, -p_i), \xi_i)$, $L_i^+ \subset (L(q_{i+1}, -p_{i+1}), \xi_{i+1})$ for $i = 1, \ldots, n-1$ such that every strong symplectic filling of (M, ξ) is obtained from a disjoint union of strong fillings of $(L(q_1, -p_1), \xi_1), \ldots, (L(q_n, -p_n), \xi_n)$ by attaching a round symplectic 1-handle along $L_i^\pm$, for $i = 1, \ldots, n-1$.*

Several families of tight lens spaces are known to have unique exact fillings, and from these we obtain families of tight Seifert fibered spaces with unique exact fillings.

Corollary 7.2. *Let ξ be a thoroughly mixed tight contact structure on $M = M\left(\frac{q_1}{p_1}, \cdots, \frac{q_n}{p_n}\right)$, with $q_i < p_i$ for $1 \leq i \leq n-1$ and $p_i \geq 2$ and $\gcd(q_i, p_i) = 1$ for $1 \leq i \leq n$. If any of the following conditions hold, then (M, ξ) admits a unique exact symplectic filling, up to diffeomorphism:*

(*a*) $q_i \in \{1, 2, 3\}$;

(*b*) *for some $b_0^{(i)} - 2 > b_1^{(i)} \geq 2$ and $m_1, \ldots, m_{n-1} \geq 2$, $m_n \geq 1$, we have*

$$\frac{q_i}{p_i} = \frac{b_0^{(i)} b_1^{(i)} + 1}{m_i(b_0^{(i)} b_1^{(i)} + 1) - b_1^{(i)}}$$

for $i = 1, \ldots, n$;

(*c*) *for some $b_0^{(i)}, b_1^{(i)} \geq 5$ and $m_1, \ldots, m_{n-1} \geq 2$, $m_n \geq 1$, we have*

$$\frac{q_i}{p_i} = \frac{b_0^{(i)} b_1^{(i)} - 1}{m_i(b_0^{(i)} b_1^{(i)} - 1) - b_1^{(i)}}$$

for $i = 1, \ldots, n$.

Proof. Theorem 7.1 provides a recipe for constructing any exact filling of (M, ξ) from fillings of $(L(q_i, -p_i), \xi_i)$, so this is simply a matter of observing that if any of these conditions hold, then $(L(q_i, -p_i), \xi_i)$ is uniquely fillable. If condition (a) holds, then either $L(q_i, -p_i) = L(q_i, 1)$ or $L(q_i, -p_i) = L(3, 2)$. In either case, the tight contact structures on $L(q_i, -p_i)$ are all uniquely fillable by work of Eliashberg [Eli90], McDuff [McD91], and Plamenevskaya–Van Horn-Morris [PV10]. When condition (b) holds, we are considering tight contact structures on $L(b_0^{(i)} b_1^{(i)} + 1, b_1^{(i)})$. Universally tight structures on such a lens space were shown to be uniquely fillable by Lisca [Lis08]. In particular, we have $p = b_0^{(i)} b_1^{(i)} + 1$ and $q = b_1^{(i)}$, so

$$\frac{p}{p-q} = \frac{b_0^{(i)} b_1^{(i)} + 1}{b_0^{(i)} b_1^{(i)} + 1 - b_1^{(i)}} = [2, \ldots, 2, b_1^{(i)} + 1],$$

where the number of copies of 2 at the start of this continued fraction is $b_0^{(i)}$. In Lisca's notation, it follows that the unique exact symplectic filling of $L(p, q)$ is $W_{p,q}((1, 2, \ldots, 2, 1))$. The virtually overtwisted structures on $L(b_0^{(i)} b_1^{(i)} + 1, b_1^{(i)})$ are uniquely fillable according to work of Kaloti [Kal13, Theorem 1.10]. Similarly, condition (c) produces lens spaces of the form $L(b_0^{(i)}b_1^{(i)}-1, b_1^{(i)})$, the fillings of which are known to be unique by work of Lisca [Lis08] in the universally tight case and Fossati [Fos19, Theorem 1] in the virtually overtwisted case. The relevant continued fraction for applying Lisca's work to these lens spaces is

$$\frac{b_0^{(i)}b_1^{(i)}-1}{b_0^{(i)}b_1^{(i)}-1-b_1^{(i)}} = [2, \ldots, 2, 3, 2, \ldots, 2],$$

which begins with $b_0^{(i)}-2$ copies of 2 and ends with $b_1^{(i)}-2$ copies. Once again, the unique exact filling is given by $W_{p,q}((1, 2, \ldots, 2, 1))$ in Lisca's notation. The observation that

$$\frac{b_0^{(i)}b_1^{(i)}-1}{b_1^{(i)}} = [b_0^{(i)}, b_1^{(i)}]$$

allows us to apply Fossati's result. □

Strong symplectic fillings of lightly mixed contact structures on Seifert fibered spaces may also be decomposed into lens space fillings, though one of the lens spaces will have a slightly more complicated expression.

Theorem 7.3. *Let ξ be a tight contact structure on the Seifert fibered space $M = M(\frac{q_1}{p_1}, \cdots, \frac{q_n}{p_n})$, for some coprime positive integers q_i, p_i with $q_i < p_i$ for $1 \leq i \leq n-1$ and $p_i \geq 2$ for $1 \leq i \leq n$. Let each $-p_i/q_i$ have continued fraction as above. Suppose that ξ is lightly mixed about K_i and K_j, and let*

$$-\frac{p'}{q'} = [a_{l_i}^i, \ldots, a_1^i, a_0^i + a_0^j, a_1^j, \ldots, a_{l_j}^j].$$

Then there exist (1) a tight contact structure ξ_k on $L(q_k, -p_k)$, for each $k \neq i, j$; (2) a tight contact structure ζ' on $L(p', q')$; (3) Legendrian knots K_k' in $\#_{k\neq i,j}(L(q_k, -p_k), \xi_k)\#(L(p', q'), \zeta')$ for $k \neq i, j$, such that every strong (exact) symplectic filling of (M, ξ) is obtained from a strong (exact) symplectic filling of

$$\underset{k\neq i,j}{\#}(L(q_k, -p_k), \xi_k)\#(L(p', q'), \zeta')$$

by attaching a symplectic 2-handle along each K_k', $k \neq i, j$.

As with Theorem 7.1, we may use Theorem 7.3 and the classification of strong fillings of some lens spaces to identify families of tight Seifert fibered spaces whose strong fillings we may classify. One example of such a family is given by the following corollary.

Corollary 7.4. *Choose* $p_1, p_2, p_3 \geq 2$*. If* ξ *is a tight contact structure on* $M = M(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3})$ *which is lightly mixed about* K_{i-1} *and* K_{i+1} *— where subscripts are labeled modulo 3 — and* $p_{i-1} + p_{i+1} \neq 4$*, then* (M, ξ) *admits a unique exact filling, up to diffeomorphism.*

Proof. Applying Theorem 7.3 to such a filling leaves us with a filling of $S^3 \# L(p_{i-1} + p_{i+1}, 1)$, with some tight contact structure. By work of Eliashberg [Eli90] this filling must be the boundary connected sum of a filling of (S^3, ξ_{std}) with a filling of $(L(p_{i-1} + p_{i+1}, 1), \zeta)$. Because $p_{i-1} + p_{i+1}$ is not equal to 4, results of McDuff [McD91] and Plamenevskaya–Van Horn-Morris [PV10] show that $(L(p_{i-1} + p_{i+1}, 1), \zeta)$ is uniquely fillable, as is the case for (S^3, ξ_{std}). So (M, ξ) is uniquely fillable. □

We will see in Section 7.2 that there are precisely six tight contact structures on $M(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3})$ which are neither lightly nor thoroughly mixed, and we will show that each of these structures is universally tight. According to Corollaries 7.2 and 7.4, these six are the only tight structures on $M(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3})$ which we cannot conclude have unique exact fillings if, say, $p_1, p_2, p_3 \geq 3$.

Corollary 7.5. *Choose integers* $p_1, p_2, p_3 \geq 2$*, no two of which sum to 4. If* ξ *is a virtually overtwisted tight contact structure on* $M = M(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3})$*, then* (M, ξ) *admits a unique exact filling* (W, ω)*, up to diffeomorphism. Moreover,* W *is simply connected, and has* $H_2(W) = \mathbb{Z}^2$*.*

Proof. The uniqueness of the filling (W, ω) follows from Corollaries 7.2 and 7.4. To see that W is simply connected and has $H_2(W) = \mathbb{Z}^2$, consider the handlebody diagram for W given by Figure 7.1. This diagram consists of a single 1-handle, with three 2-handles attached along parallel knots K_1, K_2, K_3 which pass over the 1-handle. We may handleslide K_2 and K_3 over K_1 and then cancel K_1 with the 1-handle to obtain a handlebody diagram for W which consists of two 2-handles attached to a 0-handle. Such a handlebody is simply connected, with $H_2(W) = \mathbb{Z}^2$. □

Wrapping up loose ends, we have the following result.

Theorem 7.6. *Let ξ be a tight contact structure on a small Seifert fibered space M, with surgery diagram as in Figure 7.1. If any of the horizontal links have both positive and negative stabilizations, then every strong symplectic filling of (M, ξ) can be obtained from a disjoint union of a filling of a universally tight small Seifert fibered space, along with fillings of universally tight lens spaces, by attaching a sequence of round symplectic 1-handles.*

7.1.2 The case $e_0 \leq -3$

Finally, we discuss Seifert fibered spaces over S^2 with Euler number $e_0 \leq -3$. In particular, we consider $M = M(-\frac{q_1}{p_1}, \cdots, -\frac{q_n}{p_n})$ with $p_i \geq 2$, $q_i \geq 1$, and $(p_i, q_i) = 1$ for $i = 1, \ldots, n$. The Euler number is then given by

$$e_0 = \sum_{i=1}^{n} \left\lfloor -\frac{q_i}{p_i} \right\rfloor \leq -n.$$

We have continued fraction expansions

$$-\frac{q_i}{p_i} = [a_0^i, \ldots, a_{l_i}^i],$$

for some uniquely determined integers satisfying $a_0^i = -(\lfloor \frac{q_i}{p_i} \rfloor + 1)$ and $a_j^i \leq -2$ for $j \geq 1$. Then M admits a surgery diagram as in Figure 7.2. Notice that $e_0 = \Sigma_{i=1}^n a_0^i$.

We may construct contact structures on M by putting the knots in Figure 7.2 into Legendrian position and stabilizing until the framing coefficient becomes -1 with respect to the contact framing. We see that there are

$$\left| (e_0 + 1) \prod_{i=1}^{n} \prod_{j=1}^{\ell_i} (a_j^i + 1) \right|$$

choices for these stabilizations, and in case we have a small Seifert fibered space, Wu shows in [Wu04] that each such choice leads to a distinct contact structure up to isotopy, and indeed all tight contact structures on M can be constructed in this way.

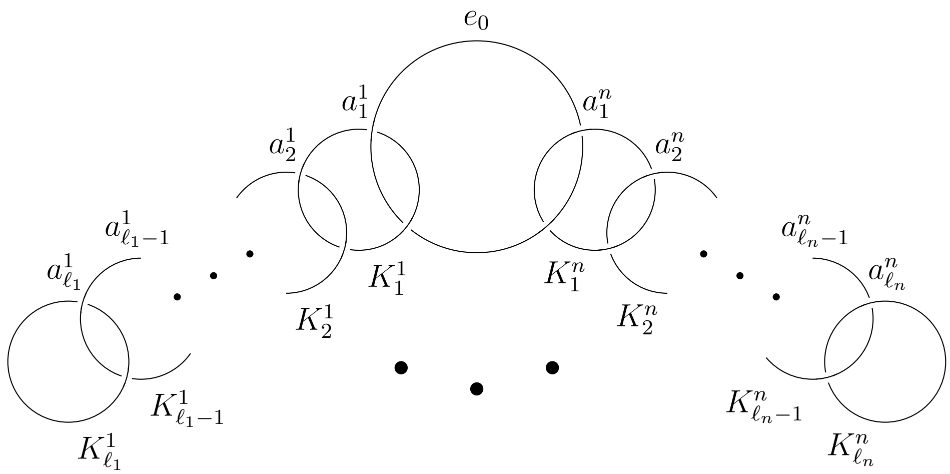


Figure 7.2: A surgery diagram for $M(-\frac{q_1}{p_1}, \cdots, -\frac{q_n}{p_n})$, $e_0 \leq -3$.

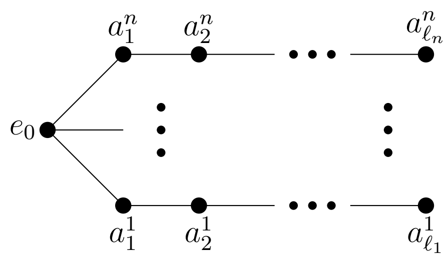


Figure 7.3: The plumbing graph associated to the surgery diagram in Figure 7.2.

Definition. Let $M = M(-\frac{q_1}{p_1}, \cdots, -\frac{q_n}{p_n})$ be as above, and construct a tight contact structure ξ on M by putting the knots of Figure 7.2 into Legendrian position. We say that (M, ξ) is *centrally mixed* if the central knot of Figure 7.2 is stabilized both positively and negatively.

Remark. Notice that if (M, ξ) is centrally mixed, then $e_0 \leq -4$.

Our Seifert fibered space M admits a canonical contact structure as the boundary of a plumbing 4-manifold. The 4-manifold is a plumbing of disc bundles of 2-spheres, with plumbing graph as in Figure 7.3. Each node of the graph corresponds to a symplectic 2-sphere, and each edge represents an orthogonal intersection between them; in this way we produce a symplectic structure on the plumbing 4-manifold, and a canonical contact structure on its boundary M. The fillings of this canonical contact structure were studied by Starkston [Sta15] and Choi-Park [CP19], each under some additional assumptions on M. Starkston provided topological restrictions on the strong symplectic fillings of *dually positive*

Seifert fibered spaces over S^2. In some cases, these restrictions produce classifications up to diffeomorphism of minimal strong symplectic fillings. Choi-Park classified all minimal symplectic fillings of small Seifert 3-manifolds $M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3})$, with $e_0 \leq -4$ and with the canonical contact structure. An infinite family of Seifert fibered spaces with canonical contact structure and $e_0 = -3$ also saw their fillings classified by Schönenberger in [Sch07, Theorem 4.4].

We point out that the canonical contact structure is not centrally mixed. Indeed, the Legendrian surgery diagram for the canonical contact structure has all of its stabilizations of a single sign.

Proposition 7.7. *Let $M = M(-\frac{q_1}{p_1}, \cdots, -\frac{q_n}{p_n})$ have canonical contact structure ξ as described above, and Legendrian surgery diagram as in Figure 7.2. All the stabilizations in the Legendrian surgery diagram are of a single sign.*

Proof. As described above, there is a symplectic 4-manifold (W, ω), obtained by plumbing disc bundles of 2-spheres, which fills (M, ξ). For each symplectic sphere S_i^j, the adjunction formula takes the form

$$\langle c_1(W), [S_i^j] \rangle = [S_i^j] \cdot [S_i^j] + 2.$$

At the same time, S_i^j corresponds to surgery along the Legendrian knot K_i^j in Figure 7.2, and thus

$$[S_i^j] \cdot [S_i^j] = \mathrm{fr}(K_i^j) = \mathrm{tb}(K_i^j) - 1.$$

So $\langle c_1(W), [S_i^j] \rangle = \mathrm{tb}(K_i^j) + 1$. Finally, [Gom98, Proposition 2.3] allows us to compute the rotation number of K_i^j:

$$\mathrm{r}(K_i^j) = \langle c_1(W), [S_i^j] \rangle = \mathrm{tb}(K_i^j) + 1.$$

This rotation number can only be obtained by taking every stabilization to be negative. Note that taking all stabilizations to be positive gives a contactomorphic (though not isotopic) contact structure. □

A straightforward consequence of the definition of centrally mixed and Theorem 1.1.3 of [Men18] is the following.

Proposition 7.8. *Let $M = M(-\frac{q_1}{p_1}, \ldots, -\frac{q_n}{p_n})$, with $p_i \geq 2$, $q_i \geq 1$, with $n \geq 3$. If ξ is a centrally mixed tight contact structure on M, then every strong (respectively, exact) symplectic filling of (M, ξ) may be obtained from a strong (respectively, exact) symplectic filling of*

$$\mathop{\#}_{i=1}^{n} (L(p_i', q_i'), \xi_i)$$

by attaching a symplectic 2-handle in a specified manner, where $-\frac{p_i'}{q_i'} = [a_1^i, \ldots, a_{l_i}^i]$ and ξ_i is a tight contact structure determined by ξ.

More generally, we have the following result for any contact structures constructed from Figure 7.2.

Theorem 7.9. *Let $M = M(-\frac{q_1}{p_1}, \cdots, -\frac{q_n}{p_n})$ be as above, and let ξ be a tight contact structure on M obtained by putting the knots of Figure 7.2 into Legendrian position. Then every strong symplectic filling of (M, ξ) can be obtained by attaching a sequence of round symplectic 1-handles to a disjoint union of fillings of universally tight lens spaces and a Seifert fibered space with canonical contact structure.*

This result will be made more precise through a sequence of propositions in Section 7.3.2. We point out that, together with Lisca's classification of fillings for universally tight lens spaces and Choi-Park's classification of fillings for the canonical contact structure on $M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3})$, $e_0 \leq -4$, Theorem 7.9 allows us to classify the minimal symplectic fillings of any contact structure on $M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3})$, $e_0 \leq -4$.

7.2 Mixed contact structures on Seifert fibered spaces

In this section we define what it means for a tight contact structure ξ on a Seifert fibered space $M = M(\frac{q_1}{p_1}, \cdots, \frac{q_n}{p_n})$ to be thoroughly or lightly mixed, and we identify the universally tight contact structures on small Seifert fibered spaces. As in Section 7.1, we take $n \geq 3$,

$q_i, p_i > 0$ coprime, and assume that $q_i < p_i$ for $i = 1, \ldots, n-1$. We also have continued fraction expansions as in (7.1), and we denote by $e_0 = \lfloor \frac{q_n}{p_n} \rfloor$ the Euler number of M.

To accommodate for the fact that we may have $q_n > p_n$, we introduce auxiliary coefficients $b_0^n, \ldots, b_{l_n'}^n$ defined by

$$-\frac{p_n}{q_n} = [a_0^n, a_1^n, ..., a_{l_n}^n] = [-1, -2, ..., -2, b_0^n - 1, b_1^n, ..., b_{l_n'}^n],$$

where $l_n' = l_n - e_0$, so that these continued fractions each have length $l_n + 1$.

The thoroughly mixed tight contact structures will be those which result from a particular construction. We let Σ be a planar surface with n boundary components, and write

$$-\partial(\Sigma \times S^1) = T_1 + T_2 + \cdots + T_n$$

for the torus boundary components of $\Sigma \times S^1$. Now let ξ be an S^1-invariant, virtually overtwisted tight contact structure on $\Sigma \times S^1$ such that

(1) each T_i is a minimal convex torus, with dividing curves of slope -1 for $i < n$ and slope $-e_0 - 1$ for $i = n$;
(2) adjacent to each T_i is a positive basic slice L_i, with $\partial L_i = T_i - T_i'$;
(3) each T_i' is a minimal convex torus, with dividing curves of slope ∞.

Such a contact structure exists by [Hon00b, Section 5].

For each $i = 1, \ldots, n-1$, we will attach $-2 - a_0^i$ basic slices to $(\Sigma \times S^1, \xi)$, with slopes

$$-1, -\frac{1}{2}, -\frac{1}{3}, \ldots, \frac{1}{a_0^i + 1},$$

starting at T_i. Similarly, we attach $-2 - b_0^n$ basic slices, starting from T_n, with slopes

$$-e_0 - 1, -e_0 - \frac{1}{2}, \ldots, -e_0 + \frac{1}{b_0^n + 1}.$$

For $i = 1, \ldots, n$, we call the boundary of the outermost basic slice T_i''. Finally, we let V_i be a solid torus and choose a tight contact structure on V_i such that ∂V_i is minimal, convex, and

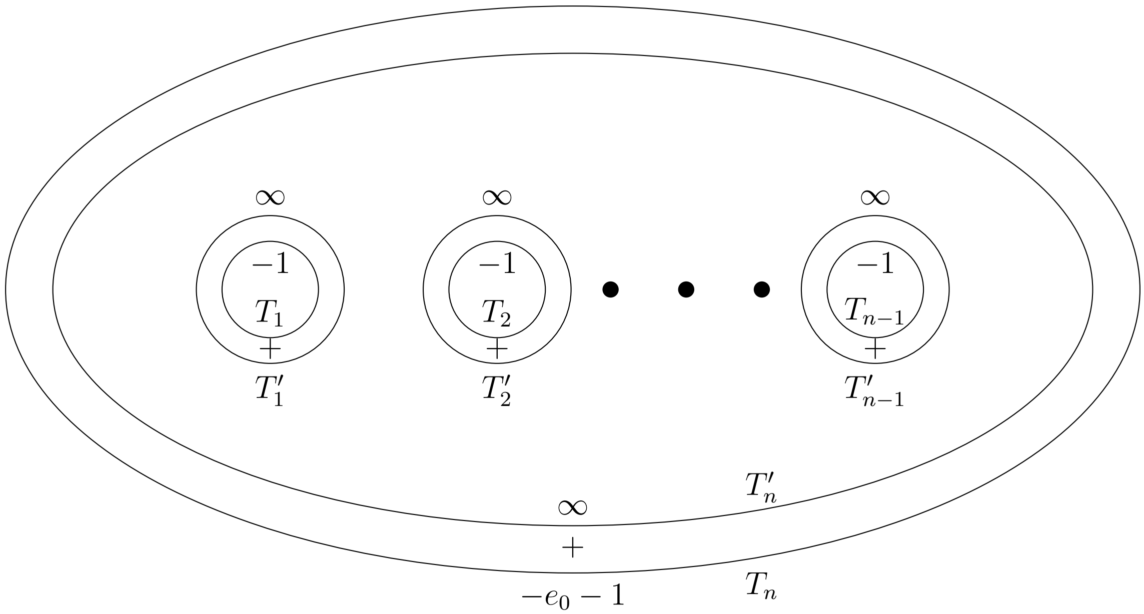


Figure 7.4: The first layer of basic slices attached to $\Sigma \times S^1$.

has dividing curves of slope

$$[a^i_{l_i}, a^i_{l_i-1}, \ldots, a^i_2, a^i_1 + 1]$$

for $1 \leq i \leq n-1$, and slope

$$[b^n_{l'_n}, b^n_{l'_n-1}, \ldots, b^n_2, b^n_1 + 1]$$

for $i = n$. Notice that there are $|\prod_{j=1}^{l_i}(a^i_j + 1)|$ (respectively, $|\prod_{j=1}^{l'_n}(b^n_j + 1)|$) such tight structures on V_i, per Honda's classification [Hon00a]. We then attach each V_i to $\Sigma \times S^1$ by identifying the dividing curves and meridians of ∂V_i with those of T''_i. The result is a tight contact structure on M, and we call any structure resulting from this construction *thoroughly mixed*. Note that this construction is not unique. For instance, we may shuffle the order in which we attach basic slices within a given continued fraction block without changing our contact structure. But the important feature is that by ensuring that the innermost basic slice around each boundary component is positive, we may find n mixed tori.

Lemma 7.10. *In a thoroughly mixed tight contact structure, each torus T'_i, $1 \leq i \leq n$, is a mixed torus with vertical dividing curves.*

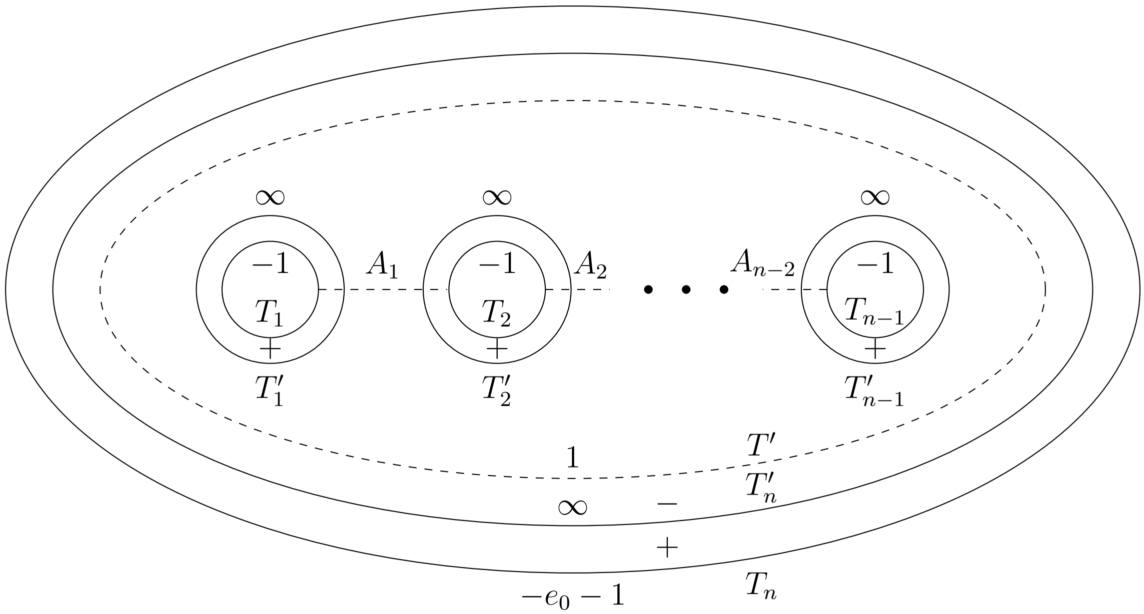


Figure 7.5: Each torus T_i' is mixed.

Proof. We show that T_n' is mixed; the other tori are similar. In $\Sigma \times S^1$, consider a collection $A_1, \ldots, A_{n-2}$ of vertical annuli as in Figure 7.5, with A_i connecting T_i to T_{i+1}. Each annulus will have parallel horizontal dividing curves, and we consider the neighborhood

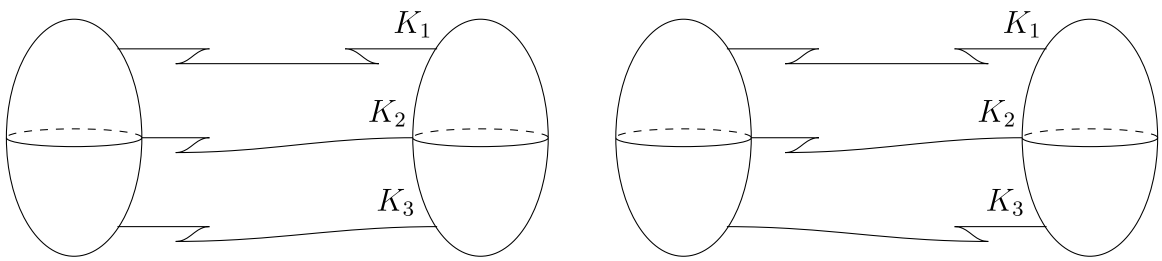
$$N = N(T_1 \cup \cdots \cup T_{n-1} \cup A_1 \cup \cdots \cup A_{n-2}),$$

whose boundary is given by $\partial N = T_1 \cup \cdots \cup T_{n-1} \cup T'$. Here T' has dividing curves of slope 1, measured in the coordinates of T_n. Because each of the basic slices L_i is positive, the toric annulus

$$(\Sigma \times S^1) \setminus (N \cup L_n)$$

is a negative basic slice with boundary slopes ∞ and 1. So T_n' is sandwiched between basic slices of opposite sign whose slopes are $-e_0 - 1, \infty$, and 1, meaning that T_n' is a mixed torus. □

There are also tight contact structures on Seifert fibered spaces which are not thoroughly mixed, but for which some of the tori T_i (not T_i') are mixed tori. Such structures are most easily identified using the construction in Figure 7.1.



(a) A thoroughly mixed contact structure. (b) A lightly mixed contact structure.

Figure 7.6: Contact structures on $M(\frac{1}{3}, \frac{1}{2}, \frac{1}{2})$.

Definition. Let ξ be a tight contact structure on $M(\frac{q_1}{p_1}, \cdots, \frac{q_n}{p_n})$. We will call ξ *lightly mixed* if ξ is not thoroughly mixed, but admits a Stein filling as in Figure 7.1 for which at least $n-2$ of $K_1, \ldots, K_n$ have been stabilized both positively and negatively. We say that ξ is *lightly mixed about* K_i *and* K_j to indicate that ξ admits a Stein filling for which each of $K_1, \ldots, K_n$ except K_i and K_j have been stabilized positively and negatively.

Remark. Every thoroughly mixed tight contact structure can be realized as in Figure 7.1 as well, and the condition of being thoroughly mixed may be stated in terms of stabilizations. In case $e_0 = 0$, ξ is thoroughly mixed if each of $K_1, \ldots, K_n$ has been stabilized positively (or, equivalently, if each has been stabilized negatively). For $e_0 > 0$, K_n has no stabilizations, so ξ is thoroughly mixed if each of $K_1, \ldots, K_{n-1}$ has been stabilized positively and the nearest stabilized unknot adjacent to K_n has also been stabilized positively (or, equivalently, each of these stabilizations is negative). See Figure 7.6.

Consider the tight contact structures on a small Seifert manifold $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$, with $p_i, q_i > 0$ chosen as above, so that $e_0 \geq 0$. According to [GLS06], each of these can be represented as in Figure 7.1. Let K_3' be the nearest unknot adjacent to K_3 which has been stabilized — meaning that $K_3' = K_3$ if $e_0 = 0$. If each of K_1, K_2, K_3' has been stabilized positively at least once (or, according to the classification in [GLS06], if each has been stabilized negatively at least once), then the tight contact structure is thoroughly mixed. On the other hand, if one of K_1, K_2, K_3 has been stabilized both positively and negatively while the other two have stabilizations of a single sign (the signs on the two knots being opposite), then the tight structure is lightly mixed. This leaves precisely $6|\Pi_{i=1}^{3}\Pi_{j=1}^{l_i}(a_j^i + 1)|$

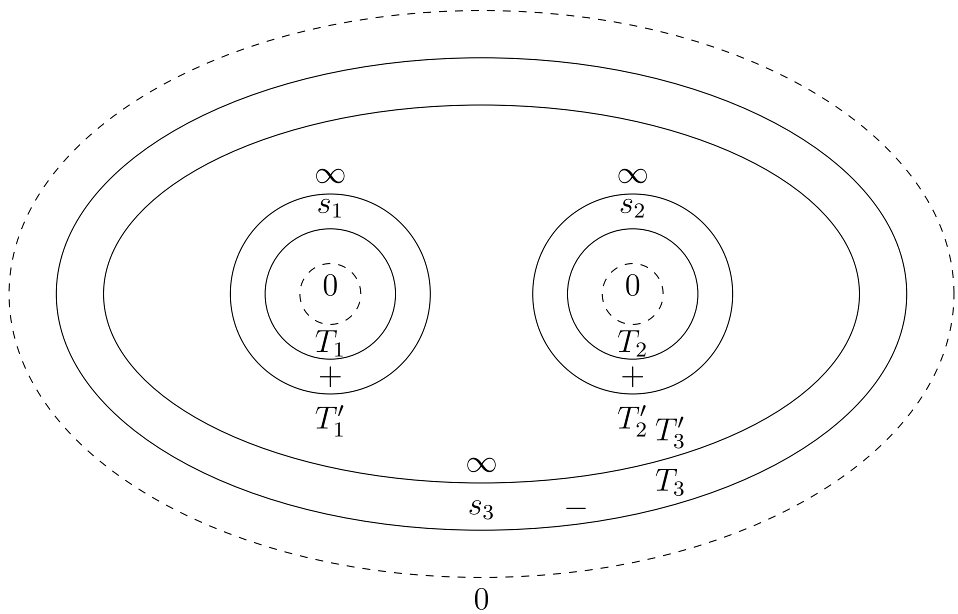


Figure 7.7: The surface $\Sigma \times S^1$ sits inside of a Seifert fibered space $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ which is neither lightly nor thoroughly mixed. This surface may be extended to $\tilde{\Sigma} \times S^1$, whose boundary components have horizontal dividing curves.

tight contact structures on M which are neither lightly nor thoroughly mixed. In these structures, each of K_1, K_2, K_3' has all of its stabilizations of a single sign, but the three knots do not all use the same sign. If the stabilizations of adjacent knots in Figure 7.1 always match, then the following lemma says that we have a universally tight contact structure; note that there are precisely six such structures.

Lemma 7.11. *Let ξ be a tight contact structure on $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ for some $0 < q_i, p_i$, with $q_i < p_i$ for $i = 1, 2$, which is neither lightly mixed nor thoroughly mixed. If each of the horizontal links in Figure 7.1 has stabilizations of only one sign, then ξ is universally tight.*

Proof. Notice that the Euler number of M satisfies $e_0 \geq 0$. Per the classification of tight contact structures due to Wu [Wu04] and Ghiggini-Lisca-Stipsicz [GLS06] on such Seifert fibered spaces, we may write

$$M = M\left(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3}\right) \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1 \cup V_2 \cup V_3),$$

where each V_i is a soid torus, $-\partial(\Sigma \times S^1) = T_1 + T_2 + T_3$, and $\varphi_i \colon \partial V_i \to T_i$ is an orientation-preserving diffeomorphism. Moreover, we may take s_i, the slope of the dividing curves of $T_i = \partial V_i$ in the coordinates of T_i, to satisfy

$$\frac{1}{a_0^i+1} < s_i < -\frac{q_i}{p_i} \text{ for } i=1,2, \quad \text{and} \quad -e_0 + \frac{1}{b_0^3+1} < s_3 < -\frac{q_3}{p_3}.$$

Here a_0^i and b_0^3 are as above. In particular, we have $s_1, s_2 \in (-1,0)$ and $s_3 < 0$.

Continuing to follow [Wu04, Section 3.3], we may thicken each V_i to a solid torus V_i' such that $T_i' := \partial V_i'$ is a minimal convex torus with vertical dividing curves when measured in the coordinates of T_i. Now $V_i' \setminus V_i$ is a toric annulus bounded by T_i and T_i' which we may factor into basic slices. Because ξ fails to be thoroughly or lightly mixed, all of the basic slices between T_i and T_i' must have the same sign, but the signs for $i = 1,2,3$ are not all the same. For instance, Figure 7.7 depicts a case where the basic slices between T_i and T_i' are positive for $i = 1,2$, but negative for $i = 3$. Now consider attaching basic slices of matching sign to each T_i until we obtain $\tilde{\Sigma} \times S^1$, whose boundary components all have horizontal dividing curves. According to [Hon00b, Lemma 5.1], $\tilde{\Sigma} \times S^1$ is universally tight; because $\tilde{\Sigma} \times S^1$ contains $\Sigma \times S^1$, we see that $\Sigma \times S^1$ is also universally tight. Each solid torus V_i is universally tight because the stabilizations used to produce the tight structure on V_i are all of one sign. Moreover, the homomorphism

$$i_* \colon \pi_1(\Sigma \times S^1) \to \pi_1(M)$$

induced by inclusion is a surjection. We conclude that M is universally tight. □

Following this observation, Corollary 7.5 follows from Corollaries 7.2 and 7.4.

Finally, we consider the remaining tight contact structures on $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ — those which are neither thoroughly nor lightly mixed, and to which Lemma 7.11 does not apply. All such contact structures are virtually overtwisted.

Lemma 7.12. *Let ξ be a tight contact structure on $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$, with surgery diagram as in Figure 7.1. If any of the horizontal links have both positive and negative stabilizations, then ξ is virtually overtwisted.*

The proof of Lemma 7.12 will make use of the following topological fact about small Seifert fibered spaces.

Lemma 7.13. *Any small Seifert fibered space M admits a finite sheeted cover $\tilde{M}$ such that the induced Seifert fibration on $\tilde{M}$ has no exceptional fibers.*

Proof. In case the fundamental group $\pi_1(M)$ is infinite, this follows from [Bri07, Lemma 2.4.22], so we focus on the case where $\pi_1(M)$ is finite. The universal cover of a small Seifert fibered space with finite fundamental group is S^3, so we have a diagram

$$\begin{array}{ccc} S^3 & \xrightarrow{p} & M \\ \big\downarrow{\scriptstyle\tilde{\pi}} & & \big\downarrow{\scriptstyle\pi} \\ \tilde{B} & \xrightarrow{\overline{p}} & B \end{array},$$

where B is S^2 with three cone points, $\pi\colon M \to B$ is the Seifert fibration on M, $p\colon S^3 \to M$ is the covering map, and $\tilde{\pi}\colon S^3 \to \tilde{B}$ is the induced Seifert fibration on S^3. Because $\pi_1(M)$ is finite, we have $B = S^2(a,b,c)$ for some $(a,b,c) \in \{(2,2,n),(2,3,3),(2,3,4),(2,3,5)\}$. That is, M is a platonic Seifert fibered space. The map $\overline{p}\colon \tilde{B} \to B$ is an orbifold covering map. We notice that since B has positive orbifold characteristic, the same is true of $\tilde{B}$, and also that $\tilde{B}$ has at most two cone points, since $\tilde{B}$ is the base of a Seifert fibration of S^3. So $\overline{p}$ is a positive orbifold covering map of the form $S^2(a',b') \to S^2(a,b,c)$; such maps are classified by [Boy18, Proposition 5.5], from which we conclude that $\tilde{B} = S^2(d,d)$ for some $d \geq 1$.

At the same time, we use [GL18, Proposition 5.2] to write the Seifert fibration $\tilde{\pi}$ as

$$M(0;(\alpha_1,\beta_1),(\alpha_2,\beta_2)),$$

for some natural numbers $\alpha_1 \geq \alpha_2$ and integers β_1, β_2 satisfying $0 \leq \beta_1 < \alpha_1$ and $\alpha_1\beta_2 + \beta_1\alpha_2 = 1$. The base of this Seifert fibration is $S^2(\alpha_1,\alpha_2)$, so we conclude that $\alpha_1 = \alpha_2 = d \geq 1$. But this means that $d\beta_2 + \beta_1 d = 1$, so we must have $d = 1$. We conclude that $\tilde{B} = S^2(1,1)$ has no cone points, and thus $\tilde{\pi}$ has no exceptional fibers. □

Proof of Lemma 7.12. Let us decompose $M := M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ as in the proof of Lemma 7.11, writing

$$M = (\Sigma' \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1' \cup V_2' \cup V_3'),$$

where $-\partial(\Sigma' \times S^1) = T_1' + T_2' + T_3'$, and the dividing curves on each T_i' have slope ∞. For $i = 1, 2, 3$, we may express the orientation-preserving diffeomorphism $\varphi_i \colon \partial V_i \to T_i$ via

$$\varphi_i = \begin{pmatrix} p_i & -u_i \\ -q_i & v_i \end{pmatrix},$$

for some u_i, v_i satisfying $p_i v_i - q_i u_i = 1$. In the coordinates of $\partial V_i'$, the dividing curves thus have slope represented by

$$\varphi_i^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} v_i & u_i \\ q_i & p_i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} u_i \\ p_i \end{pmatrix}.$$

So V_i' is a solid torus whose boundary has dividing curves of slope p_i/u_i, for $i = 1, 2, 3$. If V_i' is virtually overtwisted, then lifting $\xi|_{V_i'}$ via the p_i-fold cover $\tilde{V}_i' \to V_i'$ produces an overtwisted contact structure on $\tilde{V}_i'$. (See, for example, [Etn, Exercise 6.45].) Now Lemma 7.13 allows us to construct a finite-sheeted cover $p \colon \tilde{M} \to M$ such that V_i' lifts to several copies of $\tilde{V}_i'$, for $i = 1, 2, 3$. Because (M, ξ) has a horizontal link with both positive and negative stabilizations, at least one of V_1', V_2', and V_3' is virtually overtwisted, and thus a lift of this solid torus in $\tilde{M}$ is overtwisted. We conclude that $(\tilde{M}, p^*\xi)$ is overtwisted, and thus that (M, ξ) is virtually overtwisted. □

Lemma 7.12 applies to any contact structure which is neither thoroughly nor lightly mixed, and to which Lemma 7.11 does not apply. Lemma 7.12 also applies to all lightly mixed contact structures and, if $e_0 > 0$, all but two thoroughly mixed contact structures. If $e_0 = 0$ and $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ does not have $q_i = p_i - 1$ for $i = 1, 2, 3$, then each horizontal link in Figure 7.1 has more than one stabilization, and the classification of tight contact structures allows us to change the sign of one stabilization on each horizontal link to ensure that Lemma 7.12 applies to at least one of these links. On the other hand, if $q_i = p_i - 1$ for $i = 1, 2, 3$, then each horizontal link has exactly one stabilization, so Lemma 7.12 does

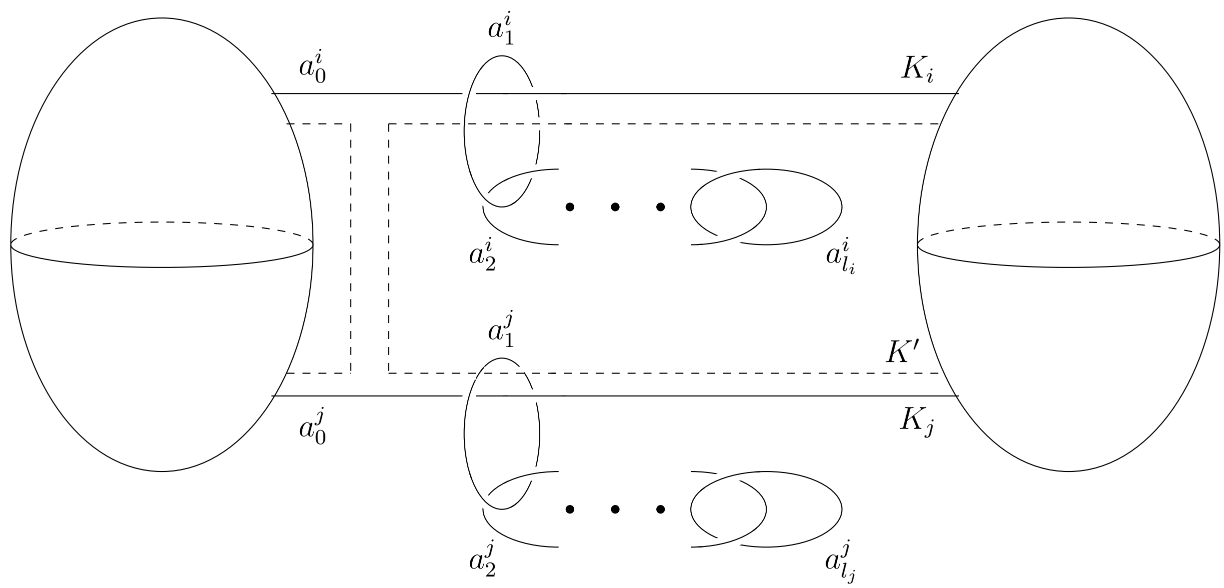


Figure 7.8: In the handlebody diagram for (M_{n-2}, ζ_{n-2}), both K_i and K_j pass over the 1-handle. To realize (M_{n-2}, ζ_{n-2}) as a lens space, we slide K_j over K_i to produce K', which has framing $a_0^i + a_0^j$, and then cancel K_i with the 1-handle.

not apply. Altogether, we see that if $e_0 > 0$ there are at most 8 universally tight contact structures on $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$, while if $e_0 = 0$, there are at most 7 universally tight contact structures. If $e_0 = 0$ and $q_i \neq p_i - 1$ for some $i = 1, 2, 3$, then there are precisely 6 universally tight contact structures on $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$.

7.3 Proofs

As we did when stating the results, we divide our proofs according to the Euler number e_0 of our Seifert fibered space.

7.3.1 The case $e_0 \geq 0$

Theorem 7.3 is a straightforward consequence of Theorem 1.1.3 of [Men18] and the definition of lightly mixed contact structures, so we prove this result first.

Suppose that $M = M(\frac{q_1}{p_1}, \cdots, \frac{q_n}{p_n})$ is a Seifert fibered space, for some $n \geq 3$ and coprime positive integers q_i, p_i. If ξ is a lightly mixed tight contact structure on M, then we may realize (M,ξ) as the boundary of a Stein handlebody as in Figure 7.1, with $n-2$ of the horizontal knots $K_1,\dots,K_n$ having been stabilized both positively and negatively. Without loss of generality, we may assume that each of $K_1,\dots,K_{n-2}$ has been stabilized both positively and negatively. Notice that (M,ξ) is obtained from the contact manifold

$$(L(q_1,-p_1),\xi_1)\#(M_1:=M(\frac{q_2}{p_2},\cdots,\frac{q_n}{p_n}),\zeta_1)$$

by Legendrian surgery along K_1. Here we are using the fact that if $-p_i/q_i=[a_0^i,a_1^i,\dots,a_{l_i}^i]$, then

$$\frac{q_i}{p_i+a_0^iq_i}=[a_1^i,a_2^i,\dots,a_{l_i}^i].$$

The contact structures ξ_1 and ζ_1 are the obvious ones, obtained from the Stein handlebody diagram in Figure 7.1 by erasing K_1. According to [Men18, Theorem 1.1.3], every strong symplectic filling of (M,ξ) is obtained from a strong filling of $(L(q_1,-p_1),\xi_1)\#(M_1,\zeta_1)$ by attaching a symplectic 2-handle along K_1. In the language of round handles, we have Legendrian knots $L_1^-\subset(L(q_1,-p_1),\xi_1)$ and $L_1^+\subset(M_1,\zeta_1)$ along which we may attach a round symplectic 1-handle to a filling of $(L(q_1,-p_1),\xi_1)\sqcup(M_1,\zeta_1)$.

We have presented (M_1,ζ_1) as the boundary of the Stein handlebody depicted in Figure 7.1, with the chain of knots with framings $a_0^1,a_1^1,\dots,a_{l_1}^1$ deleted. By its construction, (M_1,ζ_1) is lightly mixed, with $K_2,\dots,K_{n-2}$ having been stabilized both positively and negatively. We may thus repeat the above procedure to decompose a filling of (M,ξ) into a filling of

$$(L(q_1,-p_1),\xi_1)\sqcup(L(q_2,-p_2),\xi_2)\sqcup(M_2,\zeta_2).$$

We continue this procedure until we are left with

$$(L(q_1,-p_1),\xi_1)\sqcup\cdots\sqcup(L(q_{n-2},-p_{n-2}),\xi_{n-2})\sqcup(M_{n-2},\zeta_{n-2}),$$

where (M_{n-2},ζ_{n-2}) is as in Figure 7.8: there are two horizontal knots, neither of which has stabilizations of both signs. This is a Seifert fibered space over S^2, and thus a lens space. Indeed, after sliding K_2 over K_1, we may cancel the 2-handle attached along K_1 with the

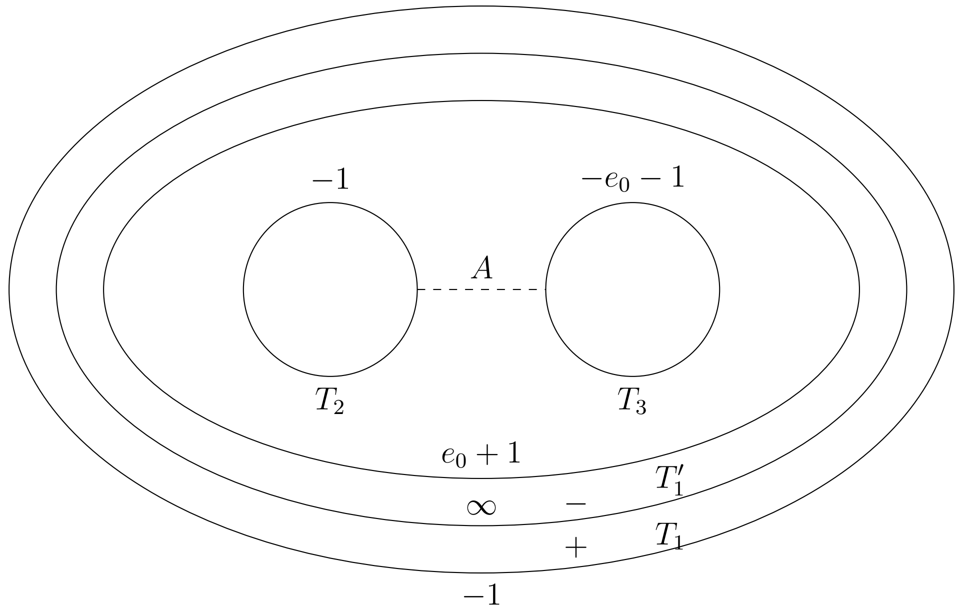


Figure 7.9: If $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ is thoroughly mixed, then T_1' is a mixed torus.

1-handle. We are left with a chain of unknots whose framings are given by

$$a^{i-1}_{l_{i-1}}, \ldots, a^{i-1}_1, a^{i-1}_0 + a^{i+1}_0, a^{i+1}_1, \ldots, a^{i+1}_{l_{i+1}},$$

and thus $M_{n-2} \cong L(p', q')$, where

$$-\frac{p'}{q'} = [a^{i-1}_{l_{i-1}}, \ldots, a^{i-1}_1, a^{i-1}_0 + a^{i+1}_0, a^{i+1}_1, \ldots, a^{i+1}_{l_{i+1}}].$$

See Figure 7.8. This proves Theorem 7.3.

There are some thoroughly mixed contact structures for which $n-1$ of the knots $K_1, \ldots, K_n$ have been stabilized both positively and negatively. For these, the proof of Theorem 7.1 proceeds as did the proof of Theorem 7.3. But the condition of being thoroughly mixed is more relaxed than this, and we will in fact use Theorem 3.3 directly in our proof, rather than Theorem 3.4.

Our argument proceeds by induction on the number n of singular fibers. Consider first the case where $n = 3$. Then, as depicted in Figure 7.9, we have a mixed torus T_1' with vertical dividing curves, sandwiched between basic slices whose other boundary tori have dividing curves of slope -1 and $e_0 + 1$, respectively. Theorem 3.3 would have us split $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ open along this torus and attach a solid torus to each of the resulting pieces. Because the dividing curves of T_1' are vertical, the meridian $\mu(S)$ of the solid torus S must have slope $m \in \mathbb{Z}$. In fact, Theorem 3.3 tells us that we must have $0 \leq m \leq e_0$, since the slopes adjacent to our mixed torus are -1 and $e_0 + 1$.

Now one of the two closed contact manifolds is $L_1 = S \cup_{T_1'} V_1'$, a gluing of two solid tori. The meridian of V_1' has slope q_1/p_1, with $0 < q_1 < p_1$. We may consider a family of tori $T^2 \times [0,1]$ in L_1 such that $T^2 \times \{0\} \subset V_1'$ has dividing curves with slope q_1/p_1, $T^2 \times \{1/2\} = T_1'$, and $T^2 \times \{1\} \subset S$ has dividing curves of slope m. As the dividing curves rotate counterclockwise from q_1/p_1 to ∞ to m, they must not rotate through an angle in excess of π, since L_1 is fillable and thus tight. This restriction is only satisfied when $m = 0$. So we conclude that $m = 0$ and $L_1 = L(q_1, -p_1)$. See Figure 7.10.

The other closed contact manifold produced by our application of Theorem 3.3 is obtained from M by deleting the neighborhood V_1' of a singular fiber and replacing it with the solid torus S, glued in with horizontal meridians. The result is $M(\frac{0}{1}, \frac{q_2}{p_2}, \frac{q_3}{p_3}) = M(\frac{q_2}{p_2}, \frac{q_3}{p_3})$. We may now apply Theorem 3.3 to this Seifert fibered space (which is in fact a lens space) along the mixed torus T_2'. Arguing as before, we find that the solid torus which is glued in at this stage must have horizontal dividing curves. The result of this decomposition is a disjoint union of fillings of some contact structures on

$$L(q_2, -p_2) \quad \text{and} \quad M\left(\frac{0}{1}, \frac{q_3}{p_3}\right) = L(q_3, -p_3).$$

Altogether, we have decomposed a filling of M with a thoroughly mixed tight contact structure into a disjoint union of fillings of $L(q_i, -p_i)$, $i = 1, 2, 3$, with some tight contact structures, and Theorem 3.3 provides the Legendrian knots described in Theorem 7.1. This establishes the base case of our induction.

For the inductive step, the analysis above proceeds as before. Splitting a filling of

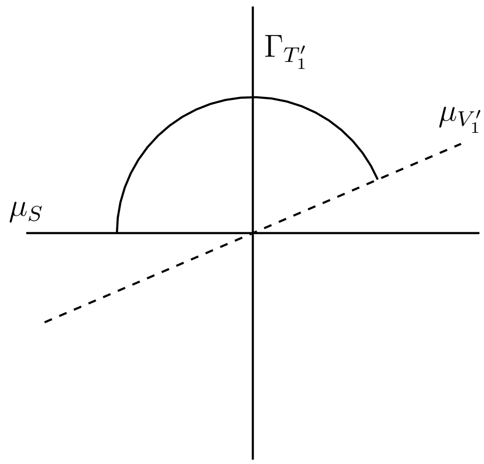


Figure 7.10: Because $0 < q_1/p_1 < 1$, we must have $m = 0$.

$M = M(\frac{q_1}{p_1}, \ldots, \frac{q_n}{p_n})$ open along the mixed torus T'_1 produces symplectic fillings of

$$L(q_1, -p_1) \quad \text{and} \quad M\left(\frac{0}{1}, \frac{q_2}{p_2}, \ldots, \frac{q_n}{p_n}\right).$$

The latter is a thoroughly mixed Seifert fibered space with $n-1$ singular fibers, for which we assume that Theorem 7.1 holds, and thus the decomposition may continue until we have a disjoint union of filling of $L(q_i, -p_i)$, for $i = 1, \ldots, n$. This proves Theorem 7.1.

At last, we address fillings of those contact structures on small Seifert fibered spaces which have at least one horizontal link with both positive and negative stabilizations — these structures are considered in Lemma 7.12. In this case, each of K_1, K_2, and K'_3 has stabilizations of a single sign, but these signs do not all agree. Here, as above, K'_3 is the nearest unknot adjacent to K_3 which has been stabilized, meaning that $K'_3 = K_3$ if $e_0 = 0$. For $i = 1, 2, 3$, we let $\widehat{K}_i$ denote the nearest knot adjacent to K_i with a stabilization of a different sign from those on K_i (or K'_3). By our assumption, at least one $\widehat{K}_i$ exists. Let us write

$$M = M\left(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3}\right) \cong (\Sigma \times S^1) \cup \left(\bigcup_{i=1}^{3}\bigcup_{j=1}^{l_i} L_{i,j}\right) \cup \left(\bigcup_{i=1}^{3} V_i\right), \tag{7.2}$$

where each V_i is a solid torus, $-\partial(\Sigma \times S^1) = T_1 + T_2 + T_3$, and each $L_{i,j} \cong T^2 \times I$ is a continued fraction block corresponding to a knot in the surgery diagram for (M, ξ). Specifically, let L_{i,j_i} be the continued fraction block corresponding to $\widehat{K}_i$, for $i = 1, 2, 3$.

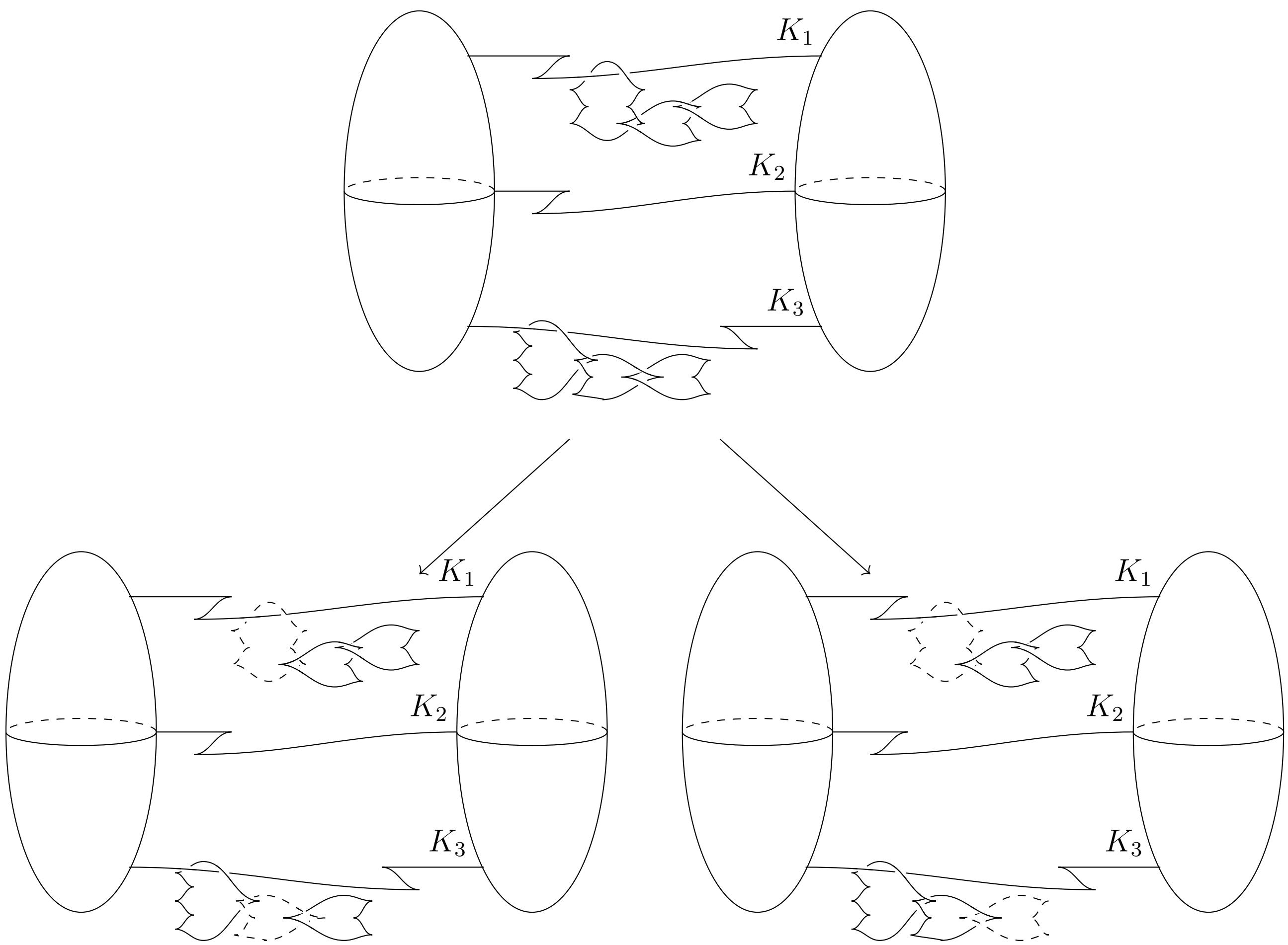


Figure 7.11: Decomposing a filling of a contact structure which is neither thoroughly nor lightly mixed. The result is a filling of a disjoint union of a universally tight small Seifert fibered space and some universally tight lens spaces.

Then the boundary torus $\widehat{T}_i$ between L_{i,j_i} and L_{i,j_i-1} is a mixed torus, and each continued fraction block preceding L_{i,j_i} has basic slices of a single sign, matching the stabilizations of K_i.

Notice that simultaneously splitting (M, ξ) along the mixed tori $\widehat{T}_1, \widehat{T}_2,$ and $\widehat{T}_3$ yields a disjoint union of a universally tight small Seifert fibered space and three lens spaces, independent of the slopes which are used to perform this splitting. It follows that by applying the symplectic JSJ decomposition to a strong symplectic filling of (M, ξ), we may obtain this filling from a disjoint union of a strong filling of a universally tight small Seifert fibered space with strong fillings of three lens spaces. By applying Theorem 4.2 to the three lens space fillings, we prove Theorem 7.6.

Observe that for contact structures which are thoroughly or lightly mixed, the conclusion of Theorem 7.6 follows from Theorems 4.2, 7.1, and 7.3. So, with the small number of exceptions pointed out at the conclusion of Section 7.2, we have reduced the problem of classifying strong symplectic fillings for small Seifert fibered spaces to the same problem for universally tight lens spaces and for universally tight small Seifert fibered spaces. See Figure 7.11.

7.3.2 The case $e_0 \leq -3$

Throughout this section, we will consider a Seifert fibered space $M = M(-\frac{q_1}{p_1}, \cdots, -\frac{q_n}{p_n})$ with $p_i \geq 2$, $q_i \geq 1$, and $(p_i, q_i) = 1$ for $i = 1, \ldots, n$. Every tight contact structure ξ on M that we consider will be constructed by putting the knots of Figure 7.2 into Legendrian position and stabilizing appropriately.

In case ξ is centrally mixed — meaning that the central knot of Figure 7.2 is stabilized both positively and negatively — Proposition 7.8 tells us that the strong symplectic fillings of (M, ξ) are obtained by attaching a sequence of round symplectic 1-handles to a disjoint union of fillings of lens spaces. If ξ is not centrally mixed, then the central knot has stabilizations which are either all positive or all negative; notice that, since $e_0 \leq -n \leq -3$, the central knot must have at least one stabilization. The following proposition considers the case in which the stabilizations of the central knot are all of a single sign.

Proposition 7.14. *Let (M,ξ) be as above, with the central knot in Figure 7.2 having stabilizations which are all of a single sign. Then every strong symplectic filling of (M,ξ) may be obtained by attaching round symplectic 1-handles to a disjoint union of fillings of lens spaces and a canonical Seifert fibered space (M',ξ'). Moreover, (M',ξ') admits a Legendrian surgery diagram as in Figure 7.2, with each leg of the diagram having stabilizations of a single sign.*

Proof. If the Legendrian surgery diagram for (M,ξ) is such that no leg has both positive and negative stabilizations, then we have nothing to do — (M',ξ') is simply (M,ξ). Otherwise, we lose no generality by assuming that the first leg of Figure 7.2 has both positive and negative stabilizations. We will reduce to a case where the first leg does not have both positive and negative stabilizations; by applying this argument to each leg of the diagram, we obtain the desired result.

Note that if the first leg of our diagram contains a knot which is stabilized both positively and negatively, then we may apply Theorem 3.4 to this knot, amputating from the diagram this knot and all those below it in the leg. Thus we assume that each knot in the first leg of our diagram has stabilizations of a single sign. We then have knots K_i^1 and K_j^1, $1 \leq i < j \leq \ell_1$, which have stabilizations of opposite signs, and are such that any knot K_k^1, with $i < k < j$, has no stabilizations. Let us assume that i and j are minimal among such indices. (Here we are using the notation established in Figure 7.2.) We will identify a mixed torus in (M,ξ) associated to this mismatch of signs.

To this end, we decompose (M,ξ) as

$$M \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \cdots \cup \varphi_n)} (V_1 \cup \cdots \cup V_n),$$

where

- Σ is a planar surface such that $-\partial(\Sigma \times S^1) = T_1 + \cdots + T_n$;
- each T_i is a minimal convex torus with dividing curves of slope $\lfloor \frac{q_i}{p_i} \rfloor$;

- each V_i is a solid torus, and ∂V_i has dividing curves of slope $-\frac{q_i - \lfloor \frac{q_i}{p_i} \rfloor p_i}{q_i' - \lfloor \frac{q_i}{p_i} \rfloor p_i'}$, where $p_i \geq p_i' > 0$, $q_i \geq q_i' > 0$, and $p_i q_i' - q_i p_i' = 1$;

- the gluing maps $\varphi_i \colon \partial V_i \to T_i$ are defined by

$$\varphi_i = \begin{pmatrix} p_i & p_i' \\ q_i & q_i' \end{pmatrix}.$$

The solid torus $(V_1, \xi|_{V_1})$ may be further decomposed by peeling off basic slices until we are left with a solid torus whose boundary has dividing curves of slope -1. In this decomposition, we have a continued fraction block of basic slices for each knot $K_1^1, \ldots, K_{\ell_1}^1$ which has been stabilized. In particular, the knots K_i^1 and K_j^1 correspond to adjacent continued fraction blocks. By assumption, these continued fraction blocks are universally tight and of opposite sign, and thus their common boundary T is a mixed torus. We may normalize the slope of the dividing curves on T to be ∞, and on the opposite boundary of the negative basic slice to be -1. As in previous iterations of this argument, the slope s on the opposite boundary of the positive basic slice will depend on the number of unstabilized knots which exist between K_i^1 and K_j^1. In particular, s will be two more than the number of these knots.

We now apply Theorem 3.3 to a filling of (M, ξ) along T. There are s contact manifolds which might be produced by this decomposition, and these correspond diagrammatically to deleting either K_i^1, K_j^1, or one of the intermediate knots from Figure 7.2. In any case, the contact manifold is a disjoint union of a lens space (whose surgery diagram is given by the link below the deleted knot) and a Seifert fibered space (M', ξ'). The stabilizations in the first leg of the Legendrian surgery diagram for (M', ξ') — of which there may be none — all have the same sign as those of K_i^1. In particular, the first leg does not have both positive and negative stabilizations. By applying this theorem to each leg with both positive and negative stabilizations, we reduce to the case of lens spaces and Seifert fibered spaces each of whose legs has stabilizations of a single sign. □

Proposition 7.15. *Let (M, ξ) be as above, with the central knot in Figure 7.2 having stabilizations*

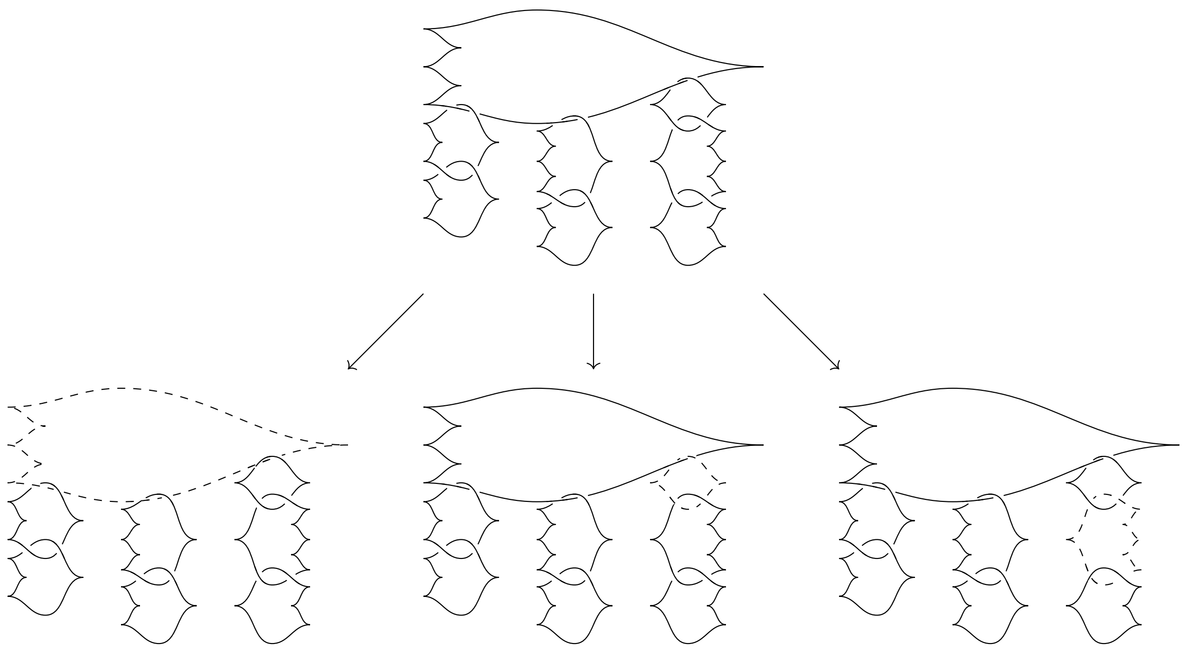


Figure 7.12: In this example, there are three contact manifolds which might result from the JSJ decomposition for symplectic fillings. Notice that each consists of universally tight lens spaces, and possibly a Seifert fibered space with canonical contact structure.

which are all of a single sign, and with each leg having stabilizations of a single sign. Then every strong symplectic filling of (M, ξ) *may be obtained by attaching round symplectic 1-handles to a disjoint union of fillings of lens spaces and a canonical Seifert fibered space.*

Proof. We lose no generality by assuming that the stabilizations of the central knot of (M, ξ) are all positive — as noted above, the central knot must have at least one stabilization. If the ith leg of the Legendrian surgery diagram for (M, ξ) has negative stabilizations, we will identify a mixed torus which allows us to amputate this leg. Now consider decomposing (M, ξ) as

$$M \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \cdots \cup \varphi_n)} (V_1 \cup \cdots \cup V_n),$$

as in the proof of Proposition 7.14. Namely, $-\partial(\Sigma \times S^1) = T_1 + \cdots + T_n$, where each T_i is a minimal convex torus with dividing curves of slope $\lfloor \frac{q_i}{p_i} \rfloor$. We claim that there is a positive basic slice adjacent to T_i in $\Sigma \times S^1$. For ease of notation, let us assume that $i = n$. Then we have a collection $A_1, \ldots, A_{n-2}$ of vertical annuli in $\Sigma \times S^1$, with A_i connecting T_i to T_{i+1} (c.f. the proof of Lemma 7.10). Each annulus will have parallel, horizontal dividing curves, and we consider the neighborhood

$$N = N(T_1 \cup \cdots \cup T_{n-1} \cup A_1 \cup \cdots \cup A_{n-2}),$$

the boundary of which is $\partial N = T_1 \cup \cdots \cup T_{n-1} \cup T$. The minimal convex torus T has dividing curves of slope $-(n-2) - \Sigma_{i=1}^{n-1} \lfloor \frac{q_i}{p_i} \rfloor$, and thus the toric annulus $(\Sigma \times S^1) \setminus N$ is a continued fraction block with boundary slopes $-(n-2) - \Sigma_{i=1}^{n-1} \lfloor \frac{q_i}{p_i} \rfloor$ and $\lfloor \frac{q_n}{p_n} \rfloor$. This continued fraction block consists of

$$\left\lfloor \frac{q_n}{p_n} \right\rfloor - \left(-(n-2) - \sum_{i=1}^{n-1} \left\lfloor \frac{q_n}{p_n} \right\rfloor \right) = (n-2) + \sum_{i=1}^{n} \left\lfloor \frac{q_n}{p_n} \right\rfloor = |e_0 + 2|$$

basic slices, each of which is positive, since the stabilizations of the central knot are all positive. In particular, we have a positive basic slice adjacent to T_n whose opposite slope is $\lfloor \frac{q_n}{p_n} \rfloor - 1$, measured in the coordinates of T_n. We may normalize via the map

$$\begin{pmatrix} \lfloor \frac{q_n}{p_n} \rfloor & -1 \\ 1 - 2\lfloor \frac{q_n}{p_n} \rfloor & 2 \end{pmatrix}$$

to obtain a positive basic slice with slopes -1 and ∞. The same holds for any $1 \leq i \leq n$.

Finally, because the ith leg of our Legendrian surgery diagram has a negative stabilization, we identify a negative basic slice in the solid torus V_i which is adjacent to ∂V_i. After gluing via φ_i, we see that $T_i = \partial V_i$ is a mixed torus. The opposite slope s of the basic slice in V_i will depend as usual on the number of unstabilized knots (if any) which lie between the central knot and the first stabilized knot of the ith leg. In particular, s will be two more than the number of such knots. There are then s possible results of applying the symplectic JSJ decomposition along the mixed torus T_i, and these correspond to deleting either the central knot of our surgery diagram, the first stabilized knot of the ith leg, or an intermediate, unstabilized knot. Deleting the central knot leaves us with a connected sum of lens spaces, while deleting a knot contained in the ith leg leaves us with a disjoint union of a lens space and a Seifert fibered space whose ith leg has no stabilizations.

Clearly the above argument may be applied to each leg with negative stabilizations (still assuming that the central knot is stabilized positively), allowing us to reduce to the case where all stabilizations in our surgery diagram have the same sign. □

From Propositions 7.14 and 7.15 we see that the problem of classifying strong symplectic fillings for Seifert fibered spaces as in Figure 7.2 is reduced to the same problem for lens spaces, and for canonical Seifert fibered spaces. Per above results, the lens spaces may be further reduced to universally tight lens spaces, and thus Theorem 7.9 is established. The results of this chapter provide us with the usual diagrammatic calculus for reducing the classification of fillings problem; see Figure 7.12 for an example.

CHAPTER 8

Circle bundles

Our final application of Theorem 3.3 completes the classification of strong symplectic fillings for virtually overtwisted tight contact structures on circle bundles over closed surfaces. We let $\pi\colon M \to \Sigma$ be a circle bundle over a closed Riemann surface Σ of genus g, and we let ξ be a tight contact structure on M. Honda [Hon00b] defines the *twisting number* $t(S^1) \leq 0$ of ξ to be the maximum non-positive twisting number achieved by a closed Legendrian curve in M which is isotopic to the S^1-fiber. Here the twisting number is measured relative to the fibration framing, and is defined to be zero if M admits a fiber-isotopic Legendrian curve with positive twisting number. In [Hon00a] and [Hon00b], Honda classifies the tight contact structures on M, and in this chapter we classify the exact symplectic fillings of M, provided ξ is virtually overtwisted and $t(S^1) < 0$.

8.1 The result

Theorem 8.1. *Let $M \to \Sigma$ be a circle bundle over a closed Riemann surface of genus $g > 1$, and let ξ be a virtually overtwisted tight contact structure on M with $t(S^1) < 0$. Then (M, ξ) admits a unique exact symplectic filling, up to diffeomorphism.*

Remark. The only circle bundles over S^2 which admit virtually overtwisted contact structures have the form $L(|e|, 1)$, where $e \leq -2$ is the Euler number of the circle bundle. Any virtually overtwisted contact structure on such a lens space is uniquely exactly fillable, per Plamenevskaya–Van Horn-Morris [PV10, Theorem 1.2], so the conclusion still holds. In the $g = 1$ case we have a circle bundle over T^2, which can also be realized as a parabolic torus bundle over S^1. This case is covered in Chapter 5, where we see that if $e \leq -2$, the conclusion continues to hold, but for $e \geq 2$ there are virtually overtwisted structures which admit no strong symplectic fillings.

The only virtually overtwisted circle bundles not addressed by Theorem 8.1, Theorem 4.2 (specifically the special case covered by [PV10]), or Theorem 5.1 are those with $g > 1$ and $t(S^1) = 0$. In [LS02, LS03] Lisca-Stipsicz verify a conjecture of Honda, which says that these structures are not symplectically semi-fillable, and thus are not symplectically fillable. Altogether, we have the following corollary.

Corollary 8.2. *Let $M \to \Sigma$ be a circle bundle over a closed Riemann surface, with virtually overtwisted tight contact structure ξ, and let $t(S^1) \leq 0$ be the twisting number. If $t(S^1) = 0$, then (M, ξ) does not admit a strong symplectic filling; if $t(S^1) < 0$, then (M, ξ) admits a unique exact symplectic filling, up to diffeomorphism.*

8.2 Proof of Theorem 8.1

We borrow the notation of [Hon00b, Part 2] here, letting $\pi\colon M \to \Sigma$ be an oriented circle bundle over a closed, oriented surface Σ with genus g. Once we have fixed a contact structure on M, Honda defines the *twisting number* $t(S^1)$ to be the maximum non-positive twisting number among all closed Legendrian curves in M isotopic to the S^1-fiber, relative to the fibration framing. The twisting number is taken to be zero if M admits a fiber-isotopic Legendrian curve with positive twisting number. We denote by e the Euler number of the bundle $\pi\colon M \to \Sigma$.

If $2g - 2 > e$, Honda shows that there are $(2g-1) - e$ tight contact structures on M with $t(S^1) = -1$; of these, exactly two are universally tight. There are no virtually overtwisted contact structures on M with $t(S^1) < -1$. There are some exceptional cases of virtually overtwisted contact structures on circle bundles with $t(S^1) = 0$, but these are not subject to Theorem 8.1. Instead, these exceptional cases are treated by Lisca-Stipsicz [LS02].

Theorem 8.1 follows immediately from Honda's description of these virtually overtwisted contact structures, as well as Theorem 1.1.3 of [Men18]. Namely, Honda constructs

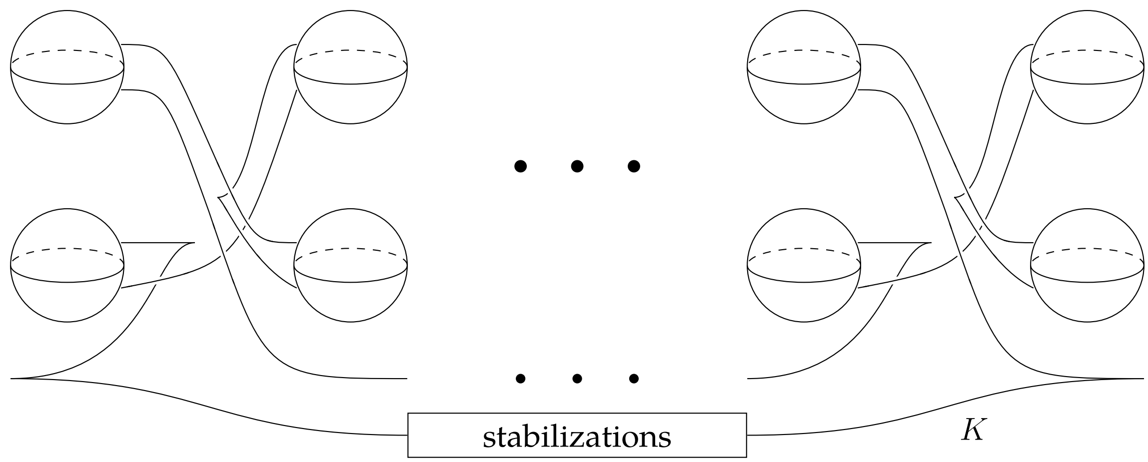


Figure 8.1: Stein handlebody diagrams for filling the tight contact structures on a circle bundle $\pi\colon M \to \Sigma$ with $t(S^1) = -1$. The diagram has $2g$ 1-handles, and the knot K has $2g-2-e$ stabilizations in the marked region.

each of the $(2g-1)-e$ tight contact structures on M by performing Legendrian surgery on a knot K in $(\#^{2g}(S^1 \times S^2), \xi_{\rm std})$ which has been stabilized $(2g-2)-e$ times. Here $\xi_{\rm std}$ is the unique-up-to-isotopy tight contact structure on $\#^{2g}(S^1 \times S^2)$. The universally tight structures on M are precisely those for which all of these stabilizations have the same sign, while each virtually overtwisted contact structure ξ_{vot} results from surgery along a knot which has been stabilized both positively and negatively. According to [Men18, Theorem 1.1.3], every exact filling of (M, ξ_{vot}) is therefore obtained from such a filling of $(\#^{2g}(S^1 \times S^2), \xi_{\rm std})$ by attaching a symplectic 2-handle along K. But $(\#^{2g}(S^1 \times S^2), \xi_{\rm std})$ has a unique exact filling up to diffeomorphism, and thus the same is true of (M, ξ_{vot}). This proves Theorem 8.1.

[Avd12] Russell Avdek. "Liouville hypersurfaces and connect sum cobordisms." *arXiv preprint arXiv:1204.3145*, 2012.

[BEM15] Matthew Strom Borman, Yakov Eliashberg, and Emmy Murphy. "Existence and classification of overtwisted contact structures in all dimensions." *Acta Math.*, **215**(2):281–361, 2015.

[BO14] Mohan Bhupal and Burak Ozbagci. "Canonical contact structures on some singularity links." *Bull. Lond. Math. Soc.*, **46**(3):576–586, 2014.

[Boy18] Keegan Boyle. "On the virtual cosmetic surgery conjecture." *New York J. Math.*, **24**:870–896, 2018.

[Bri07] Matthew G Brin. "Seifert Fibered Spaces: Notes for a course given in the Spring of 1993." *arXiv preprint arXiv:0711.1346*, 2007.

[CE12] Kai Cieliebak and Yakov Eliashberg. *From Stein to Weinstein and Back: Symplectic Geometry of Affine Complex Manifolds*, volume 59. Amer. Math. Soc., 2012.

[CET21] James Conway, John B Etnyre, and Bülent Tosun. "Symplectic fillings, contact surgeries, and Lagrangian disks." *Internat. Math. Res. Notices*, **2021**(8):6020–6050, 2021.

[CGH11] Vincent Colin, Paolo Ghiggini, Ko Honda, and Michael Hutchings. "Sutures and contact homology I." *Geom. Topol.*, **15**(3):1749–1842, 2011.

[Chr21] Austin Christian. "On symplectic fillings of virtually overtwisted torus bundles." *Algebr. Geom. Topol.*, **21**(1):469–505, 2021.

[CL20] Austin Christian and Youlin Li. "Some applications of Menke's JSJ decomposition for symplectic fillings." *arXiv preprint arXiv:2006.16825*, 2020.

[CM19] Austin Christian and Michael Menke. "Splitting symplectic fillings." *arXiv preprint arXiv:1909.00420*, 2019.

[CMP19] Roger Casals, Emmy Murphy, and Francisco Presas. "Geometric criteria for overtwistedness." *J. Amer. Math. Soc.*, **32**(2):563–604, 2019.

[Col01a] Vincent Colin. "Sur la torsion des structures de contact tendues." In *Ann. Sci. École Norm. Sup.*, volume 34, pp. 267–286, 2001.

[Col01b] Vincent Colin. "Une infinité de structures de contact tendues sur les variétés toroïdales." *Comment. Math. Helv.*, **76**(2):353–372, 2001.

[CP19] Hakho Choi and Jongil Park. "On symplectic fillings of small Seifert 3-manifolds." *arXiv preprint arXiv:1904.04955*, 2019.

[DG01] Fan Ding and Hansjorg Geiges. "Symplectic fillability of tight contact structures on torus bundles." *Algebr. Geom. Topol.*, **1**(1):153–172, 2001.

[DG04] Fan Ding and Hansjörg Geiges. "A Legendrian surgery presentation of contact 3-manifolds." In *Math. Proc. Cambridge Philos. Soc.*, volume 136, pp. 583–598. Cambridge University Press, 2004.

[DL18] Fan Ding and Youlin Li. "Strong symplectic fillability of contact torus bundles." *Geom. Dedicata*, **195**(1):403–415, 2018.

[EH01] John B Etnyre and Ko Honda. "Knots and contact geometry I: torus knots and the figure eight knot." *J. Symplectic Geom.*, **1**(1):63–120, 2001.

[EH02] John B Etnyre and Ko Honda. "Tight contact structures with no symplectic fillings." *Invent. Math.*, **148**(3):609–626, 2002.

[Eli89] Yakov Eliashberg. "Classification of overtwisted contact structures on 3-manifolds." *Invent. Math.*, **98**(3):623–637, 1989.

[Eli90] Yakov Eliashberg. "Filling by holomorphic discs and its applications." *London Math. Soc. Lecture Note Ser., Geometry of Low-Dimensional Manifolds*, **2**(151):45–67, 1990.

[Eli96] Yakov Eliashberg. "Unique holomorphically fillable contact structure on the 3-torus." *Int. Math. Res. Not. IMRN*, **1996**(2):77–82, 1996.

[ER20] John B Etnyre and Agniva Roy. "Symplectic fillings and cobordisms of lens spaces." *arXiv preprint arXiv:2006.16687*, 2020.

[Etn] John B Etnyre. "Convex surfaces in contact geometry: class notes.".

[EV18] John Etnyre and Vera Vértesi. "Legendrian satellites." *Int. Math. Res. Not. IMRN*, **2018**(23):7241–7304, 2018.

[Fos19] Edoardo Fossati. "Contact surgery on the Hopf link: classification of fillings." *arXiv preprint arXiv:1905.13026*, 2019.

[Fos20] Edoardo Fossati. "Topological constraints for Stein fillings of tight structures on lens spaces." *J. Symplectic Geom.*, **18**(6):1475–1514, 2020.

[Ghi05] Paolo Ghiggini. "Strongly fillable contact 3–manifolds without Stein fillings." *Geom. Topol.*, **9**(3):1677–1687, 2005.

[Gir91] Emmanuel Giroux. "Convexité en topologie de contact." *Comment. Math. Helv.*, **66**(1):637–677, 1991.

[Gir00] Emmanuel Giroux. "Structures de contact en dimension trois et bifurcations des feuilletages de surfaces." *Invent. Math.*, **141**(3):615–689, 2000.

[GL15] Marco Golla and Paolo Lisca. "On Stein fillings of contact torus bundles." *Bull. Lond. Math. Soc.*, **48**(1):19–37, 2015.

[GL18] Hansjörg Geiges and Christian Lange. "Seifert fibrations of lens spaces." In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 88, pp. 1–22. Springer, 2018.

[GLS06] Paolo Ghiggini, Paolo Lisca, and András I Stipsicz. "Classification of Tight Contact Structures on Small Seifert 3-Manifolds with $e_0 \geq 0$." *Proc. Amer. Math. Soc.*, pp. 909–916, 2006.

[Gom98] Robert E Gompf. "Handlebody construction of Stein surfaces." *Ann. of Math.*, pp. 619–693, 1998.

[Gor83] C McA Gordon. "Dehn surgery and satellite knots." *Trans. Amer. Math. Soc.*, **275**(2):687–708, 1983.

[Gro85] Mikhael Gromov. "Pseudoholomorphic curves in symplectic manifolds." *Invent. Math.*, **82**(2):307–347, 1985.

[Hem87] John Hempel. "Residual finiteness for 3-manifolds." *Combinatorial group theory and topology (Alta, Utah, 1984)*, **111**:379–396, 1987.

[HH18] Ko Honda and Yang Huang. "Bypass attachments in higher-dimensional contact topology." *arXiv preprint arXiv:1803.09142*, 2018.

[HH19] Ko Honda and Yang Huang. "Convex hypersurface theory in contact topology." *arXiv preprint arXiv:1907.06025*, 2019.

[HKM02] Ko Honda, William H Kazez, and Gordana Matić. "Convex decomposition theory." *Internat. Math. Res. Notices*, **2002**(2):55–88, 2002.

[Hon00a] Ko Honda. "On the classification of tight contact structures I." *Geom. Topol.*, **4**(1):309–368, 2000.

[Hon00b] Ko Honda. "On the classification of tight contact structures II." *J. Differential Geom.*, **55**(1):83–143, 2000.

[Kal13] Amey Kaloti. "Stein fillings of planar open books." *arXiv preprint arXiv:1311.0208*, 2013.

[Kir78] Robion Kirby. "A calculus for framed links in S^3." *Invent. Math.*, **45**(1):35–56, 1978.

[Lic62] R. Lickorish. "A representation of orientable combinatorial 3-manifolds." *Ann. Math.*, (76):531–540, 1962.

[Lis08] Paolo Lisca. "On symplectic fillings of lens spaces." *Trans. Amer. Math. Soc.*, **360**(2):765–799, 2008.

[LS02] Paolo Lisca and András I Stipsicz. "Tight, not semi–fillable contact circle bundles." *arXiv preprint math/0211429*, 2002.

[LS03] Paolo Lisca and András I Stipsicz. "An infinite family of tight, not semi-fillable contact three-manifolds." *Geom. Topol.*, **7**(2):1055–1073, 2003.

[Lut77] Robert Lutz. "Structures de contact sur les fibrés principaux en cercles de dimension trois." In *Ann. Inst. Fourier (Grenoble)*, volume 27, pp. 1–15, 1977.

[Mar71] Jean Martinet. "Formes de contact sur les variétés de dimension 3." In *Proceedings of Liverpool Singularities Symposium II*, pp. 142–163. Springer, 1971.

[McD91] Dusa McDuff. "Symplectic manifolds with contact type boundaries." *Invent. Math.*, **103**(1):651–671, 1991.

[Med10] Elıf Medetoğullari. *On the tight contact structures on Seifert fibered 3-manifolds with 4 singular fibers*. PhD thesis, Middle East Technical University, 2010.

[Men18] Michael A Menke. *Some Results on Fillings in Contact Geometry*. PhD thesis, UCLA, 2018.

[Ng01] Lenhard Ng. *Invariants of Legendrian links*. PhD thesis, Massachusetts Institute of Technology, 2001.

[OS13] Burak Ozbagci and András Stipsicz. *Surgery on Contact 3-Manifolds and Stein Surfaces*, volume 13. Springer Science & Business Media, 2013.

[Ozb11] Burak Ozbagci. "Contact handle decompositions." *Topology Appl.*, **158**(5):718–727, 2011.

[PV10] Olga Plamenevskaya and Jeremy Van Horn-Morris. "Planar open books, monodromy factorizations and symplectic fillings." *Geom. Topol.*, **14**(4):2077–2101, 2010.

[Sch07] Stephan Schönenberger. "Determining symplectic fillings from planar open books." *J. Symplectic Geom.*, **5**(1):19–41, 2007.

[Sta15] Laura Starkston. "Symplectic fillings of Seifert fibered spaces." *Trans. Amer. Math. Soc.*, **367**(8):5971–6016, 2015.

[Vau15] Anne Vaugon. "Reeb periodic orbits after a bypass attachment." *Ergodic Theory Dynam. Systems*, **35**(2):615–672, 2015.

[Wal60] Andrew H Wallace. "Modifications and cobounding manifolds." *Canad. J. Math.*, **12**:503–528, 1960.

[Wen10a] Chris Wendl. "Automatic transversality and orbifolds of punctured holomorphic curves in dimension four." *Comment. Math. Helv.*, **85**:347–407, 2010.

[Wen10b] Chris Wendl. "Strongly fillable contact manifolds and J-holomorphic foliations." *Duke Math. J.*, **151**(3):337–384, 2010.

[Wu04] Hao Wu. *Tight contact structures on small Seifert spaces*. PhD thesis, Massachusetts Institute of Technology, 2004.