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Klein–Maskit combination theorem for Anosov subgroups: free products

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Abstract

We prove a generalization of the classical Klein–Maskit combination theorem, in the free product case, in the setting of Anosov subgroups. Namely, if Γ_A and Γ_B are Anosov subgroups of a semisimple Lie group G of noncompact type, then under suitable topological assumptions, the group generated by Γ_A and Γ_B in G is again Anosov, and is naturally isomorphic to the free product $\Gamma_A * \Gamma_B$. Such a generalization was conjectured in Dey et al. (Math Z 293(1–2):551–578, 2019).

The classical Klein–Maskit combination theorem for Kleinian groups (discrete isometry groups of the hyperbolic space $\mathbb{H}^n$) establishes sufficient conditions for a subgroup $\Gamma < G = \mathrm{Isom}(\mathbb{H}^n)$ generated by two discrete subgroups Γ_A, Γ_B of G to be discrete and isomorphic to the free product $\Gamma_A * \Gamma_B$, see [18–20, 22] as well as Maskit's book [21, Theorem C.2, p. 149] and papers by Ivascu [7], Li, Ohshika and Wang [13]. In this situation, the group Γ is said to be obtained via *Klein combination* of the subgroups Γ_A, Γ_B. Moreover, Maskit gave sufficient conditions for the Maskit combination of two geometrically finite subgroups of Γ to be again geometrically finite and described the limit set of Γ in terms of those of Γ_A, Γ_B. Maskit's condition was formulated in terms of *fundamental domains* of group actions on the boundary sphere S^{n-1} of $\mathbb{H}^n$. Further generalizations of the Klein–Maskit combination theorem in the context of group actions on Gromov-hyperbolic spaces appear in [1, 6, 15–17].

In the last decade, *Anosov subgroups* of higher rank Lie groups have emerged as a higher-rank generalization of *convex cocompact* Kleinian groups. The goal of this article is to prove an analog of Maskit's theorem in the setting of *Anosov subgroups* of semisimple Lie groups G.[1] An earlier form of such an analogue was proven in our paper with Bernhard Leeb, [4], using the *local-to-global principle* for Morse quasigeodesics and extending the earlier work

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[1] We note that Maskit also proved some far-reaching generalizations of the above combination theorem, dealing with the cases of amalgamated free products and HNN extensions. Such a generalization is not an objective of the present paper but is discussed in our subsequent work in [3].

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by Kapovich, Leeb and Porti [9]. However, our earlier work required passage to certain finite index subgroups in the given Anosov subgroups. In the paper [4] we conjectured a sharp form of the combination theorem for Anosov subgroups, which exactly matches Maskit's conditions for Kleinian groups. The main goal of the present paper is to prove this conjecture. Unlike [4], the methods of the present paper are purely dynamical, relying upon another characterization (given in the work of Kapovich, Leeb and Porti [10]) of Anosov subgroups, as *asymptotically embedded* subgroups. The appropriate flag-manifolds $\text{Flag}(\tau_{\text{mod}})$ (quotients of G by suitable parabolic subgroups $P_{\tau_{\text{mod}}} < G$) serve as substitutes to the ideal boundary sphere S^{n-1} of $\mathbb{H}^n$ in Maskit's setting. The subsets $A, B \subset \text{Flag}(\tau_{\text{mod}})$ appearing below act as replacements to the complements of the interiors of fundamental domains of the subgroups Γ_A, Γ_B in S^{n-1} used by Maskit. The *antipodality* condition in the theorem is the appropriate higher-rank analogue of the disjointness condition imposed by Maskit. The main difficulty in the proof of our main theorem is to establish that certain sequences of *nested images* of the sets A, B under *alternating* sequences in the free product $\Gamma_A * \Gamma_B$ have singleton intersections, see Lemma 3.5. (This intersection property allows us to prove both τ_{mod}-regularity of Γ and construct a continuous equivariant embedding of the Gromov boundary of Γ into $\text{Flag}(\tau_{\text{mod}})$.) The intersection property would have been easy if we knew that nontrivial elements of Γ_A, resp. Γ_B, *uniformly contract* on B, resp. A. Such uniform contraction, in general, fails under the assumption of the Klein–Maskit combination theorem even in the case of $\mathbb{H}^n$, see Example 6.5.

Let G be a noncompact real semisimple Lie group with a finite center, $X = G/K$ be the associated symmetric space of noncompact type, σ_{mod} be a model spherical chamber (a model facet) in the Tits building $\partial_{\text{Tits}} X$ of X, $\iota : \sigma_{\text{mod}} \to \sigma_{\text{mod}}$ be the *opposition involution*, and $\tau_{\text{mod}} \subset \sigma_{\text{mod}}$ be an ι-invariant face. We will assume some mild conditions on G, see Sect. 1.3 for more details.

Our main result is:

Theorem A (Combination Theorem) *Suppose that $A, B \subset \text{Flag}(\tau_{\text{mod}})$ are (disjoint) compact sets with nonempty interiors which are* antipodal *(see Definition 1.1) to each other. Let Γ_A and Γ_B be τ_{mod}-Anosov subgroups[2] of G such that all nontrivial elements $\alpha \in \Gamma_A$, $\beta \in \Gamma_B$ satisfy*

$$\alpha(B) \subset \text{int}(A), \quad \beta(A) \subset \text{int}(B).$$

Then:

(i) *The subgroup $\Gamma < G$ generated by Γ_A, Γ_B is naturally isomorphic to the abstract free product $\Gamma_A * \Gamma_B$.*

(ii) *The subgroup $\Gamma < G$ is τ_{mod}-Anosov.*

(iii) *The τ_{mod}-limit set of Γ (see Sect. 1.8) is contained in $A \cup B$.*

This theorem has an immediate generalization to the case of several Anosov subgroups of G that we will discuss in Sect. 6.

To end this introduction, we would like to mention that Danciger, Guéritaud, and Kassel in [2] announced a combination theorem for the free products of *convex cocompact* subgroups of $G = \text{PGL}(n, \mathbf{R})$. See Proposition 12.4 in their paper for a precise statement. Notably, by Theorem 1.15 of their paper, the class of word-hyperbolic convex cocompact subgroups of

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[2] Although we do not state original definition of τ_{mod}-Anosov subgroups (due to Labourie [12]) in this paper, an equivalent characterization of these as τ_{mod}-asymptotically embedded subgroups is discussed in Sect. 1.8. The proof of our main result relies upon this characterization.

G coincides with the class of *projective* Anosov subgroups of G preserving some properly convex domain in $\mathbb{P}(\mathbf{R}^n)$. Therefore, the combination theorem announced in their paper is also closely related to our main result.

Organization of the paper

In Sect. 1, we prove some preliminary results needed for the proof of Theorem A. A crucial result proven in this section is that the *stars in flag-manifolds are connected*, see Sect. 1.4, which is also of independent interest. In Sects. 2 and 3, we show that under a suitable condition, a pair of τ_{mod}-regular subgroups of G generate another one; see Theorem 3.1 for the precise statement. In Sect. 4, we show that, under the hypothesis of Theorem A, the subgroup $\Gamma_A *$ $\Gamma_B < G$ is τ_{mod}-*boundary embedded*. In Sect. 5, we conclude the proof of Theorem A. In the final section, Sect. 6, we discuss some consequences.

1 Preliminary notions and results

The goal of this section is to set up our notation and terminology, and prove some preliminary results which will be used in the later sections.

1.1 Reduced forms

Let Γ_A and Γ_B be two groups and let $\Gamma = \Gamma_A * \Gamma_B$ be their free product. We will regard Γ_A and Γ_B as subgroups of Γ under their natural embeddings in Γ. Each element $\gamma \in \Gamma$ can be written as a unique word of the form

$$\gamma = \gamma_k \cdots \gamma_1, \tag{1.1}$$

where $k \in \mathbf{N}$ depends on γ, such that the following conditions are satisfied:

  (i) Each letter γ_i in the above expression belong to either Γ_A or Γ_B.
  (ii) No two successive letters γ_{i+1}, γ_i in the above expression belong to the same group Γ_A or Γ_B.

The unique expression (1.1) for γ is called the *reduced form* of γ, and the number of nontrivial letters in the right side of (1.1) is called the *relative (word-)length* of γ and denoted by $\mathrm{rl}(\gamma)$.

An element $\gamma' \in \Gamma$ is called a *leftmost* (resp. *rightmost*) *subword* of $\gamma \in \Gamma$ if the reduced form of γ' is obtained by deleting some letters from the right (resp. left) of the reduced form of γ. For example, if $\gamma \in \Gamma$ has the reduced form $\gamma = \alpha\beta\delta$, then the elements α, $\alpha\beta$, and $\alpha\beta\delta$ of Γ are leftmost subwords of γ whereas the elements δ, $\beta\delta$, and $\alpha\beta\delta$ are rightmost subwords of γ.

1.2 Hausdorff distance and convergence

Given a compact metric space (Z, d), we have the Hausdorff distance on the set $\mathcal{C}(Z)$ of nonempty closed subsets of Z. This distance defines the *topology of Hausdorff-convergence* on $\mathcal{C}(Z)$, which coincides with the *Chabauty topology* on $\mathcal{C}(Z)$. Under this topology, a sequence (A_n) in $\mathcal{C}(Z)$ *converges* to a singleton $\{z\} \in \mathcal{C}(Z)$, denoted by

$$A_n \to z,$$

if and only if the diameter of $A_n \cup \{z\}$ goes to zero as $n \to \infty$.

1.3 Geometric background

Let G be a noncompact real semisimple Lie group with a finite center. We will impose some additional assumptions on G given at the end of this subsection. Let $K < G$ be a maximal compact subgroup of G. The *symmetric space* of G is the simply-connected space $X = G/K$, equipped with a G-invariant Riemannian metric. It is a standard fact that such a Riemannian metric has *non-positive curvature*, and moreover, X has no flat de Rham factors. We refer to Eberlein's book, [5], for a detailed discussion of symmetric spaces.

The ideal boundary of X is the set of asymptotic classes of geodesic rays in X. The ideal boundary carries a natural G-invariant spherical building structure, called the *Tits building* of X, and is denoted by $\partial_{\text{Tits}} X$: The apartments in this building are the ideal boundaries of the maximal flats in X. Top-dimensional simplices (*facets*) in $\partial_{\text{Tits}} X$ are called (spherical) *chambers* and codimension one simplices are called *panels*. The group G acts transitively on the sets of apartments and chambers of $\partial_{\text{Tits}} X$. Let $F_{\text{mod}} \subset X$ denote a chosen maximal flat; the ideal boundary of this flat, denoted by a_{mod}, is called the *model apartment*. The action of $G_{a_{\text{mod}}}$, the stabilizer of a_{mod} in G, on a_{mod} factors through a finite group W, called the *Weyl group* associated to G. A chosen fundamental domain for the action $W \curvearrowright a_{\text{mod}}$ is called the *model chamber*, and is denoted by σ_{mod}. The other facets in $\partial_{\text{Tits}} X$ are the G-translates of σ_{mod}.

The reflection about a(ny) point $x \in F_{\text{mod}}$ determines an involution $\text{inv} : a_{\text{mod}} \to a_{\text{mod}}$ which preserves the chambers of a_{mod}. The *longest element* $w_0 \in W$ is the unique element which sends the chamber $\text{inv}(\sigma_{\text{mod}}) \subset a_{\text{mod}}$ to σ_{mod}. The composition $w_0 \circ \text{inv} : \sigma_{\text{mod}} \to \sigma_{\text{mod}}$ is a simplicial map, called the *opposition involution* and denoted by ι.

The stabilizer of σ_{mod} in G is called the *minimal parabolic subgroup*, denoted by $P_{\sigma_{\text{mod}}}$. More generally, the stabilizers of faces $\eta_{\text{mod}} \subset \sigma_{\text{mod}}$ are the *parabolic subgroups* of G, denoted by $P_{\eta_{\text{mod}}}$. Therefore, the space of all simplices in $\partial_{\text{Tits}} X$ of *type*[3] η_{mod} can be identified with the *partial flag variety*

$$\text{Flag}(\eta_{\text{mod}}) := G/P_{\eta_{\text{mod}}}.$$

It is easy to see that $\eta'_{\text{mod}} \subset \eta_{\text{mod}}$ if and only if $P_{\eta_{\text{mod}}} < P_{\eta'_{\text{mod}}}$. The *full flag variety*, $\text{Flag}(\sigma_{\text{mod}}) = G/P_{\sigma_{\text{mod}}}$, is also known as the *Furstenberg boundary* of X.

For a face $\eta_{\text{mod}} \subset \sigma_{\text{mod}}$, it is often convenient to use the notation $\pm \eta_{\text{mod}}$ to denote the pair η_{mod}, $\iota \eta_{\text{mod}}$, respectively. A pair of points $\eta_\pm \in \text{Flag}(\pm \eta_{\text{mod}})$ is called *antipodal* if there exists a complete geodesic line in X, which is forward (resp. backward) asymptotic to an interior point of η_+ (resp. η_-). Equivalently, antipodal simplices in $\partial_{\text{Tits}} X$ are those which are swapped by some Cartan involution of X. To avoid possible confusions, we remark here that pairs of antipodal points in $\partial_{\text{Tits}} X$ are necessarily distinct.

For $\eta \in \text{Flag}(\eta_{\text{mod}})$, let

$$C(\eta) := \{\eta_- \in \text{Flag}(\iota \eta_{\text{mod}}) \mid \eta_- \text{ is antipodal to } \eta\}. \tag{1.2}$$

This is an open dense cell in $\text{Flag}(\iota \eta_{\text{mod}})$. The complement of $C(\eta)$ in $\text{Flag}(\iota \eta_{\text{mod}})$, denoted by $E(\eta)$, is the *exceptional subvariety* for η.

In the following definition, we assume that $\tau_{\text{mod}} \subset \sigma_{\text{mod}}$ is an ι-invariant face, i.e., $\tau_{\text{mod}} = \iota \tau_{\text{mod}}$.

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[3] A simplex $\eta \in \partial_{\text{Tits}} X$ is called of *type* η_{mod} if there exists $g \in G$ such that $g \eta_{\text{mod}} = \eta$.

Definition 1.1 (*Antipodality*) A subset $\Lambda \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$ is called *antipodal* if every pair of distinct points $\lambda_{\pm} \in \Lambda$ is antipodal. If A and B are subsets of $\mathrm{Flag}(\tau_{\mathrm{mod}})$, then they are called *antipodal to each other* (or A is *antipodal* to B) if for all points $a \in A$, $b \in B$, a and b are antipodal.

Throughout, we impose the following mild assumptions on G:

(i) The group G is commensurable with the full isometry group of X.
(ii) The Tits building of X is *thick*, i.e., every *panel* in $\partial_{\mathrm{Tits}}X$ is a face of at least three different chambers.

These are standing assumptions in the papers [8, 10, 11] by Kapovich, Leeb, and Porti we rely upon in this work. Moreover, we also refer to these papers for more details on the notions introduced in this section.

1.4 Flag varieties

Let $\eta_{\mathrm{mod}} \subset \sigma_{\mathrm{mod}}$ be a face. Consider the canonical G-equivariant algebraic morphism

$$\pi_{\eta_{\mathrm{mod}}} : \mathrm{Flag}(\sigma_{\mathrm{mod}}) \to \mathrm{Flag}(\eta_{\mathrm{mod}}). \tag{1.3}$$

The fiber of $\pi_{\eta_{\mathrm{mod}}}$ over any point $\eta \in \mathrm{Flag}(\eta_{\mathrm{mod}})$ is called the *star* of η, denoted by $\mathrm{St}_{\mathrm{Fu}}(\eta)$, and is the smooth subvariety of $\mathrm{Flag}(\sigma_{\mathrm{mod}})$ consisting of all chambers σ which contain η as a face. These fibers are diffeomorphic to the quotients $P_{\eta_{\mathrm{mod}}}/P_{\sigma_{\mathrm{mod}}}$.

Lemma 1.2 *Let η_{mod} and τ_{mod} be any faces of σ_{mod}. Then:*

(i) *Fibers of the map $\pi_{\eta_{\mathrm{mod}}}$ in (1.3) are connected.*
(ii) *The projection of $\mathrm{St}_{\mathrm{Fu}}(\eta)$ to $\mathrm{Flag}(\tau_{\mathrm{mod}})$ is connected.*

Proof (i) Any two chambers $\sigma_1, \sigma_2 \in \mathrm{St}_{\mathrm{Fu}}(\eta)$ lie in a common apartment a in the spherical building $\partial_{\mathrm{Tits}}X$. Then, there exists a gallery[4] in a consisting of chambers in the star of η and connecting σ_1 and σ_2. Hence, it suffices to prove that σ_1, σ_2 lie in the same component of $\mathrm{St}_{\mathrm{Fu}}(\eta)$, provided that σ_1, σ_2 share a panel τ and lie in a common spherical apartment $a \subset \partial_{\mathrm{Tits}}X$. Let w denote an element of the Weyl group W_a of the apartment a fixing τ (pointwise) and swapping the chambers σ_1, σ_2. (The element w comes from an isometry $g_w \in G$ preserving the apartment a.)

Let $\hat{\tau} \subset a$ be the unique simplex of type $\iota\tau_{\mathrm{mod}}$ opposite to τ. Let $f \subset X$ denote a flat of dimension one less than the rank of X such that $\partial_{\infty} f \subset a$ and $\tau, \hat{\tau}$ are both contained in $\partial_{\infty} f$. The parallel set $P(f) \subset X$ of the flat f splits isometrically as the product of a rank 1 symmetric subspace $Y \subset X$ and f and the apartment a is contained in the ideal boundary of $P(f)$. The isometry group of $P(f)$ fixes (pointwise) the ideal boundary of f and, hence, the simplex τ. We connect a generic point $\xi \in \tau$ to its antipode $\hat{\xi} \in \hat{\tau}$ by a geodesic γ_1 on a passing through the interior of σ_1. Then the geodesic $w(\gamma_1) = \gamma_2$ also connects $\xi, \hat{\xi}$ and passes through the interior of σ_2. There exists a (unique) point $\zeta_1 \in \gamma_1$ which lies in the ideal boundary of $\partial_{\infty} Y \cap a$. Hence, $w(\zeta_1) = \zeta_2$ is in the ideal boundary of Y as well. The connected component of the isometry group of Y acts transitively on the ideal boundary of Y (since Y has rank 1). If follows that there is a 1-parameter family h_t, $t \in [1, 2]$, of isometries of X preserving Y and fixing $\partial_{\infty} f$ pointwise such that $h_1 = \mathrm{id}$ and $h_2(\zeta_1) = \zeta_2$. Thus,

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[4] A *gallery* is a finite sequence of chambers such that every two consecutive chambers in the sequence are adjacent, i.e., share a *panel* (a codimension one face).

$h_1(\gamma_1) = \gamma_2$ and $h(\sigma_1) = \sigma_2$. We obtain a 1-parameter family $\sigma_t = h_t(\sigma_1)$, $t \in [1, 2]$ of chambers in $\mathrm{St}_{\mathrm{Fu}}(\tau)$, connecting σ_1, σ_2. Thus, σ_1, σ_2 are in the same component of $\mathrm{St}_{\mathrm{Fu}}(\eta)$. (ii) This follows immediately from part (i). □

Lemma 1.3 *Unless the projection of* $\mathrm{St}_{\mathrm{Fu}}(\eta)$ *to* $\mathrm{Flag}(\tau_{\mathrm{mod}})$ *is a singleton, it is not contained in the antipodal set of any simplex* $\tau_- \in \mathrm{Flag}(\iota\tau_{\mathrm{mod}})$.

Note that the condition that this projection is *not* a singleton amounts to saying that τ_{mod} is not contained in η_{mod}.

Proof of Lemma 1.3 The projection of $\mathrm{St}_{\mathrm{Fu}}(\eta)$ to $\mathrm{Flag}(\tau_{\mathrm{mod}})$ consists of all simplices τ of type τ_{mod} contained in the star of η. Consider one of these simplices, τ_+, antipodal to a simplex τ_- (of type $\iota\tau_{\mathrm{mod}}$). Then there exists an apartment a in the spherical building of the symmetric space X, containing τ_- and the chamber σ which, in turn contains τ_+ and η. Since τ_+ is not contained in η, the stabilizer of η in the Weyl group W of a does not fix τ_+, there exists $w \in W$ fixing η but not τ_+. Thus, $w(\tau_+) = \tau$ also has type τ_{mod} and is contained in the star of η in a. However, an apartment cannot contain two simplices of the same type both antipodal to the same simplex. Thus, the simplex τ cannot be antipodal to τ_-. □

Corollary 1.4 *Suppose that* η *is a simplex in the Tits building* $\partial_{\mathrm{Tits}} X$ *of type* η_{mod} *which is not a face of* τ_{mod}. *Then for any* $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$, $C(\tau)$ *cannot contain the projection* $\pi_{\tau_{\mathrm{mod}}}(\mathrm{St}_{\mathrm{Fu}}(\eta))$ *for any simplex* η *of type* η_{mod}.

In particular, if A *and* B *are nonempty sets in* $\mathrm{Flag}(\tau_{\mathrm{mod}})$ *which are antipodal to each other, then neither* A *nor* B *can contain any stars as described above.*

1.5 Contraction and regularity

Let $\tau_{\mathrm{mod}} \subset \sigma_{\mathrm{mod}}$ be a face (which is not necessarily ι-invariant). We equip

$$\bar{X}^{\tau_{\mathrm{mod}}} := X \sqcup \mathrm{Flag}(\tau_{\mathrm{mod}}).$$

with the topology of *flag-convergence*; see [11, Definition 3.90]. A sequence (g_n) in G is said to be flag-converging to $\tau_+ \in \mathrm{Flag}(\tau_{\mathrm{mod}})$, denoted by $g_n \xrightarrow{\mathrm{flag}} \tau_+$, if the orbit-sequence $(g_n x)$ flag-converges to τ_+ in $\bar{X}^{\tau_{\mathrm{mod}}}$. This notion is independent of the choice of $x \in X$. In particular, if a sequence (g_n) converges to τ_+ then for $h \in G$, the sequence of compositions $(g_n h)$ also flag-converges to τ_+. At the same time, the sequence of compositions (hg_n) flag-converges to $h(\tau_+)$.

Remark 1.5 We do not define the notion of a τ_{mod}-*regular sequence* in this paper. We refer our reader to [8, Definition 4.7]. We note only that a sequence is *not* τ_{mod}-regular if and only if it contains a subsequence, which has no accumulation points in $\mathrm{Flag}(\tau_{\mathrm{mod}})$. Note that if a sequence (g_n) in G is τ_{mod}-regular, then for any $h \in G$ the sequence (hg_n) is also τ_{mod}-regular. Moreover, a sequence (g_n) in G is τ_{mod}-regular if and only if (g_n^{-1}) is $\iota\tau_{\mathrm{mod}}$-regular. In particular, if τ_{mod} is ι-invariant, then (g_n) is τ_{mod}-regular if and only if the inverse sequence (g_n^{-1}) is τ_{mod}-regular; see the fourth paragraph of [8, page 2560].

Proposition 1.6 *Let* (g_n) *be a sequence in* G, *and let* $\tau_\pm \in \mathrm{Flag}(\pm\tau_{\mathrm{mod}})$. *The following are equivalent:*

(i) *As* $n \to \infty$, $g_n|_{C(\tau_-)} \to \tau_+$ *uniformly on compacts.*
(ii) *As* $n \to \infty$, $g_n^{-1}|_{C(\tau_+)} \to \tau_-$ *uniformly on compacts.*

(iii) *The sequence (g_n) (equivalently, (g_n^{-1})) is τ_{mod}-regular ($\iota\tau_{\mathrm{mod}}$-regular), and $g_n^{\pm 1} \xrightarrow{\text{flag}} \tau_{\pm}$ as $n \to \infty$.*

Proof For (i) $\iff$ (ii), see [8, Lemma 4.4]. For the direction (iii) $\implies$ (i), see [8, Lemma 4.18]. We now show that (i) and (ii) together imply (iii). For (i) implies (g_n) is τ_{mod}-regular (similarly, (ii) implies (g_n^{-1}) is τ_{mod}-regular), see [8, Proposition 4.16]. It remains to show that $g_n^{\pm} \xrightarrow{\text{flag}} \tau_{\pm}$. If $\hat{\tau}_+ \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ is any flag-accumulation point for the sequence (g_n), then there exists a subsequence (g_{n_k}) of (g_n) such that $g_{n_k} \xrightarrow{\text{flag}} \hat{\tau}_+$. Applying the direction (iii) $\implies$ (i) and after extraction of (g_{n_k}), there exists $\hat{\tau}_- \in \mathrm{Flag}(\iota\tau_{\mathrm{mod}})$ such that $g_{n_k}|_{C(\hat{\tau}_-)} \to \hat{\tau}_+$. By the uniqueness of the attractor, see [8, Lemma 4.6], we get that $\tau_+ = \hat{\tau}_+$. Hence, $g_n \xrightarrow{\text{flag}} \tau_+$. Similarly, applying the direction (iii) $\implies$ (ii), it follows that $g_n^{-1} \xrightarrow{\text{flag}} \tau_-$. $\square$

1.6 Pure sequences

In this section, our discussion concerns η_{mod}-*pure* sequences in G, where $\eta_{\mathrm{mod}} \subset \sigma_{\mathrm{mod}}$ is a face. We do not give a definition of η_{mod}-pure sequences here as it requires some lengthier discussion, but we refer our readers to [8, Definition 4.7].

Remark 1.7 The condition of being η_{mod}-pure is stronger than being η_{mod}-regular. Furthermore, if a divergent[5] sequence (g_n) in G is *not* τ_{mod}-regular, then it contains a subsequence which is η_{mod}-pure for some face η_{mod} of σ_{mod} which does not contain τ_{mod}. See [8] for more details.

The accumulation dynamics on the Furstenburg boundary of η_{mod}-pure sequences is well-understood by [8, Prop. 9.5]; we record their result in the proposition below. For $\eta \in \mathrm{Flag}(\eta_{\mathrm{mod}})$, let $C_{\mathrm{Fu}}(\eta) := \pi_{\eta_{\mathrm{mod}}}^{-1}(C(\eta))$. See (1.3) for the definition of the map $\pi_{\eta_{\mathrm{mod}}}$.

Proposition 1.8 *Suppose that a sequence (g_n) in G is η_{mod}-pure and η_{mod}-contracting[6] for $\eta_{\pm} \in \mathrm{Flag}(\pm\eta_{\mathrm{mod}})$. Then, after extraction,*

$$g_n|_{C_{\mathrm{Fu}}(\eta_-)} \to \phi \quad \text{uniformly on compacts},$$

where the map

$$\phi : C_{\mathrm{Fu}}(\eta_-) \to \mathrm{St}_{\mathrm{Fu}}(\eta_+)$$

is an open (in manifold topology) algebraic map. Moreover, for every $\hat{\eta} \in C(\eta_-)$, the restriction $\phi|_{\mathrm{St}_{\mathrm{Fu}}(\hat{\eta})}$ is given by the restriction of an element of $g \in G$, and hence is an algebraic isomorphism.

Suppose that we have a η_{mod}-pure sequence (g_n) in G, such that

$$g_n^{\pm 1} \xrightarrow{\text{flag}} \eta_{\pm} \in \mathrm{Flag}(\pm\eta_{\mathrm{mod}}),$$

and which satisfies the the conclusion of the above proposition, i.e., $g_n|_{C_{\mathrm{Fu}}(\eta_-)} \to \phi$ uniformly on compacts as $n \to \infty$. Consider a point $\eta \in \mathrm{Flag}(\eta_{\mathrm{mod}})$ antipodal to η_-. Then, $\mathrm{St}_{\mathrm{Fu}}(\eta) \subset$

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[5] That is, the sequence (g_n) has no accumulation points in G.

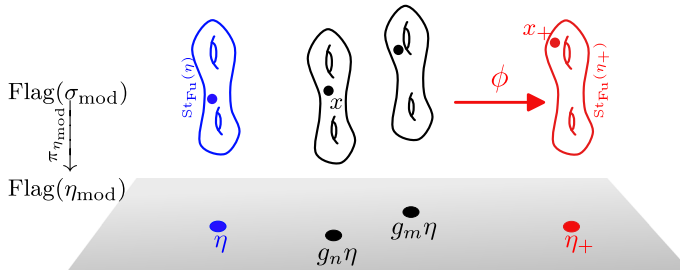[6] That is, $g_n|_{C(\eta_-)} \to \eta_+$ uniformly on compacts.

Fig. 1 The algebraic map ϕ

$C_{\mathrm{Fu}}(\eta_-)$. The proposition above implies that, for large n, $g_n : \mathrm{St}_{\mathrm{Fu}}(\eta) \to \mathrm{Flag}(\sigma_{\mathrm{mod}})$ *approximates* the algebraic isomorphism $\phi : \mathrm{St}_{\mathrm{Fu}}(\eta) \to \mathrm{St}_{\mathrm{Fu}}(\eta_+)$.

After fixing a distance function d on $\mathrm{Flag}(\sigma_{\mathrm{mod}})$ which is compatible with the manifold topology, we consider the quantity

$$D_{m,n} := \max_{y \in \mathrm{St}_{\mathrm{Fu}}(\eta)} d(g_m y, g_n y) = \max_{x \in \mathrm{St}_{\mathrm{Fu}}(g_n \eta)} d(g_m g_n^{-1} x, x).$$

See Fig. 1 for an illustration.

Lemma 1.9 $D_{m,n} \to 0$ *as* $m, n \to \infty$.

Proof Suppose that, on the contrary, there exist sequences of natural numbers (m_k), and (n_k) such that $m_k, n_k \to \infty$ as $k \to \infty$, for which the following holds: For all $k \in \mathbf{N}$, there exists $x_k \in \mathrm{St}_{\mathrm{Fu}}(g_{n_k} \eta)$ such that

$$d(g_{m_k} g_{n_k}^{-1}(x_k), x_k) \geq \epsilon > 0. \tag{1.4}$$

Note that the sequence (x_k) accumulates in $\mathrm{St}_{\mathrm{Fu}}(\eta_+)$, and $y_k := g_{n_k}^{-1}(x_k) \in \mathrm{St}_{\mathrm{Fu}}(\eta)$. Passing to a subsequence of the sequence (n_k), we may assume that the sequence $x_k \to x_+ \in \mathrm{St}_{\mathrm{Fu}}(\eta_+)$, and $y_k \to y_0 \in \mathrm{St}_{\mathrm{Fu}}(\eta)$, as $k \to \infty$.

We will show that $\phi(y_0) = \lim_{m \to \infty} g_m y_0 = x_+$. Consider the compact set $A = \{y_i \mid i \in \mathbf{N}\} \cup \{y_0\} \subset \mathrm{St}_{\mathrm{Fu}}(\eta)$. Then,

$$g_m|_A \to \phi|_A, \quad \text{as } m \to \infty.$$

Since the convergence is uniform, $g_m y_i$ converges to $\phi(y_0)$ whenever $i \to \infty$ and $m \to \infty$. Now, since $x_k = g_{n_k} y_k$, and $\lim_{k \to \infty} n_k \to \infty$, we obtain that

$$\phi(y_0) = \lim_{i,m \to \infty} g_m y_i$$
$$= \lim_{k \to \infty} g_{n_k} y_k = \lim_{k \to \infty} x_k = x_+.$$

In particular, we get $g_{m_k} g_{n_k}^{-1}(x_k) = g_{m_k} y_k \to x_+$ as $k \to \infty$. Since also $x_k \to x_+$, we must have

$$\lim_{k \to \infty} d(g_{m_k} g_{n_k}^{-1}(x_k), x_k) = 0.$$

This contradicts (1.4). □

1.7 A regularity criterion

Let $\tau_{\mathrm{mod}} \subset \sigma_{\mathrm{mod}}$ be a face, which is not assumed to be ι-invariant. We equip $\mathrm{Flag}(\tau_{\mathrm{mod}})$ with a distance function d which is compatible with its manifold topology. Note that since $\mathrm{Flag}(\tau_{\mathrm{mod}})$ is compact, the notion of the convergence $A_n \to \tau$ defined in Sect. 1.2, where (A_n) is a sequence of nonempty compact subsets of $\mathrm{Flag}(\tau_{\mathrm{mod}})$ and $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$, does not depend on the choice of such a distance function d.

Lemma 1.10 *Let (g_n) be a sequence in G. If there exists a compact subset $A \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$ with nonempty interior and a point $\tau_+ \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ such that $g_n A \to \tau_+$, then (g_n) is τ_{mod}-regular and (g_n) flag-converges to τ_+.*

Proof Since $g_n A \to \tau_+$ as $n \to \infty$ and A has nonempty interior, it is clear that the sequence (g_n) is divergent in G. Therefore, if (g_n) is not τ_{mod}-regular, then, after extraction, (g_n) is η_{mod}-pure for some $\eta_{\mathrm{mod}} \subset \sigma_{\mathrm{mod}}$ which does not contain τ_{mod}; see Remark 1.7. After further extraction, there exists $\eta_\pm \in \mathrm{Flag}(\pm\eta_{\mathrm{mod}})$ such that (g_n) is η_{mod} contracting for $\eta_\pm$. Let $\phi : C_{\mathrm{Fu}}(\eta_-) \to \mathrm{St}_{\mathrm{Fu}}(\eta_+)$ be the surjective algebraic map that we obtain from Proposition 1.8. Since $\widetilde{A} := \pi_{\tau_{\mathrm{mod}}}^{-1}(A)$ is Zariski dense (since it has nonempty interior), $\phi(\widetilde{A})$ is also Zariski dense in $\mathrm{St}_{\mathrm{Fu}}(\eta_+)$. Now, under the hypothesis $g_n A \to \tau_+$, we obtain that

$$g_n \widetilde{A} \text{ accumulates on } \mathrm{St}_{\mathrm{Fu}}(\tau_+), \quad \text{as } n \to \infty.$$

Hence, we must have $\mathrm{St}_{\mathrm{Fu}}(\tau_+) \supset \mathrm{St}_{\mathrm{Fu}}(\eta_+)$, which leads to $\tau_{\mathrm{mod}} \subset \eta_{\mathrm{mod}}$. This is a contradiction.

The convergence $g_n \xrightarrow{\text{flag}} \tau_+$ follows from Proposition 1.6. □

Corollary 1.11 *Let (g_n) be a sequence in G. If there exists a compact subset $A \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$ with nonempty interior such that*

$$\lim_{n \to \infty} \mathrm{diam}\, g_n A = 0,$$

then (g_n) is a τ_{mod}-regular sequence.

Proof If $\mathrm{diam}\, g_n A \to 0$ as $n \to \infty$, then, after extraction, $g_n A \to \tau_+$ for some $\tau_+ \in \mathrm{Flag}(\tau_{\mathrm{mod}})$. Then, Proposition 1.10 implies that (g_n) is τ_{mod}-regular. □

1.8 Regular and Anosov subgroups

In this paper, we consider the following classes of discrete subgroups of G.

Let $\tau_{\mathrm{mod}} \subset \sigma_{\mathrm{mod}}$ be an ι-invariant face. A discrete subgroup $\Gamma < G$ is called τ_{mod}-*regular* if all sequences of distinct elements (γ_n) in Γ are τ_{mod}-regular. If Γ is such a subgroup, then the τ_{mod}-*limit set*

$$\Lambda_{\tau_{\mathrm{mod}}}(\Gamma) \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$$

consists of all points $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ such that there exists a sequence (γ_n) in Γ for which $\gamma_n \xrightarrow{\text{flag}} \tau$. This is a Γ-invariant closed subset of $\mathrm{Flag}(\tau_{\mathrm{mod}})$.

The class of τ_{mod}-*Anosov subgroups* of G, a notion originally introduced by Labourie [12], is a special class of regular subgroups that generalizes the *convex cocompact* Kleinian groups into higher rank.

Remark 1.12 There are several equivalent characterizations of τ_{mod}-Anosov subgroups. For our purpose, we use the following characterization of τ_{mod}-Anosov subgroups as τ_{mod}-*asymptotically embedded* subgroups, which was introduced by Kapovich, Leeb, and Porti in [10]. See [10, Theorem 5.47] for the equivalence between these two (and several other) definitions.

Definition 1.13 (*Asymptotically embedded*) A subgroup $\Gamma < G$ is called τ_{mod}-*asymptotically embedded* if it is τ_{mod}-regular and satisfies the following conditions:

(i) Γ as an abstract group is word-hyperbolic.
(ii) The τ_{mod}-flag limit set $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ is antipodal.
(iii) There exists a Γ-equivariant homeomorphism

$$\xi : \partial_\infty \Gamma \to \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$$

from the Gromov boundary of Γ onto its τ_{mod}-flag limit set, which continuously extends the orbit map $o_x : \Gamma \to \Gamma x \subset X$.

Note that the antipodality of $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ and injectivity of ξ is equivalent to the condition that the map ξ is *antipodal*, i.e. it sends distinct elements of $\partial_\infty \Gamma$ to antipodal elements of $\mathrm{Flag}(\tau_{\mathrm{mod}})$.

2 Discreteness

In what follows, τ_{mod} will always denote an ι-invariant face of σ_{mod}.

Definition 2.1 Let Γ_A and Γ_B be a pair of discrete subgroups of G. A pair (A, B) of disjoint compact subsets of $\mathrm{Flag}(\tau_{\mathrm{mod}})$ is called a *ping-pong pair* for (Γ_A, Γ_B) if the following conditions are satisfied:

(i) A and B have nonempty interiors.
(ii) For all nontrivial elements $\alpha \in \Gamma_A$ and $\beta \in \Gamma_B$, we have that $\alpha B \subset \mathrm{int}\, A$ and $\beta A \subset \mathrm{int}\, B$.

Let Γ_A and Γ_B are discrete subgroups of G such that (Γ_A, Γ_B) admits a ping-pong pair (A, B) in $\mathrm{Flag}(\tau_{\mathrm{mod}})$. The classical *ping-pong argument* of Felix Klein (cf. [23]), shows that the subgroup $\Gamma := \langle \Gamma_A, \Gamma_B \rangle$ generated by Γ_A and Γ_B in G is *naturally* isomorphic to the free product $\Gamma_A * \Gamma_B$:

Lemma 2.2 (Ping-pong) *Let Γ_A and Γ_B be two discrete subgroups of G such that (Γ_A, Γ_B) admits a ping-pong pair (A, B) in $\mathrm{Flag}(\tau_{\mathrm{mod}})$. Then, the natural homomorphism $\phi : \Gamma_A * \Gamma_B \to G$ extending the embeddings $\Gamma_A \to G$, $\Gamma_B \to G$, is injective.*

Proof In the free product $\Gamma_A * \Gamma_B$, pick any nontrivial element γ.

Suppose that the rightmost letter of the reduced form of γ is in Γ_A and the leftmost letter of the reduced form of γ is in Γ_B (see Sect. 1.1). Then, $\phi(\gamma)A \subset \mathrm{int}\, A$. Hence, $\phi(\gamma) \neq 1 \in G$. Similarly if the rightmost/leftmost letters of the reduced form of γ are both in Γ_A, then $\phi(\gamma)B \subset \mathrm{int}\, A$. Hence, $\phi(\gamma) \neq 1 \in G$ as well. The verification of the non-triviality of $\phi(\gamma)$ in the other two possibilities on the location of rightmost/leftmost letters in the reduced form of γ is similar; we leave the details to the reader. □

In what follows, we reserve the notation Γ to denote the subgroup of G generated by Γ_A and Γ_B,

$$\Gamma := \langle \Gamma_A, \Gamma_B \rangle \cong \Gamma_A * \Gamma_B.$$

Next, we impose the τ_{mod}-regularity condition for the groups Γ_A and Γ_B. Recall the notion of the τ_{mod}-flag limit set from Sect. 1.8. The lemma below shows that the limit sets of Γ_A and Γ_B are contained in A and B, respectively.

Lemma 2.3 *Let* $\Gamma_A, \Gamma_B, A,$ *and* B *be as in Lemma* 2.2. *Assume that* Γ_A *(resp.* Γ_B) *is* τ_{mod}-*regular. Then,*

$$\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A) \subset A \quad (resp.\ \Lambda_{\tau_{\mathrm{mod}}}(\Gamma_B) \subset B).$$

Proof Let us verify that $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A) \subset A$. The verification of $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_B) \subset B$ is similar.

Pick arbitrary point $\tau_+ \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A)$. There exists a sequence (α_n) in Γ_A such that $\alpha_n^{\pm 1} \xrightarrow{\mathrm{flag}} \tau_\pm$, where τ_- is some point in the limit set $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A)$. Since B has nonempty interior, and $C(\tau_-)$ is an open dense subset of $\mathrm{Flag}(\tau_{\mathrm{mod}})$, $B \cap C(\tau_-) \neq \emptyset$. Pick any point $\hat{\tau} \in B \cap C(\tau_-)$. Then, by Proposition 1.6, $\lim_{n \to \infty} \alpha_n \hat{\tau} = \tau_+$. On the other hand, for all $n \in \mathbf{N}$, $\alpha_n \hat{\tau} \in \alpha_n(B) \subset A$, as long as $\alpha_n \neq 1$. By the compactness of A, we get $\tau_+ \in A$. Hence, $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A) \subset A$. □

Now, we state and prove the main result of this section.

Proposition 2.4 (Discreteness) *Let* Γ_A *and* Γ_B *be* τ_{mod}-*regular subgroups of* G *such that* (Γ_A, Γ_B) *admits a ping-pong pair* (A, B) *in* $\mathrm{Flag}(\tau_{\mathrm{mod}})$. *If* A *is antipodal to* B, *then the group* $\Gamma := \langle \Gamma_A, \Gamma_B \rangle \cong \Gamma_A * \Gamma_B$ *is a discrete subgroup of* G.

Proof Suppose, to the contrary, that Γ is not discrete. Then, there exists a sequence $(\gamma_n)_{n \in \mathbf{N}}$ of distinct elements of Γ such that $\gamma_n \to 1 \in G$. After passing to a subsequence, we may assume that

(i) either, for all $n \in \mathbf{N}$, the rightmost letter in the reduced form of γ_n (see Sect. 1.1) is in Γ_A,

(ii) or, for all $n \in \mathbf{N}$, the rightmost letter in the reduced form of γ_n is in Γ_B.

Since the two cases differ by relabelling, we make the additional assumption that (i) holds. By the assumption $\gamma_n \to 1$, we have

$$\gamma_n B \to B, \tag{2.1}$$

i.e., the Hausdorff distance between $\gamma_n B$ and B converges to zero as $n \to \infty$.

Since for all $b \in B$, $\lim_{n \to \infty} \gamma_n b = b$, we must have the following: For all large $n \in \mathbf{N}$, the (nontrivial) leftmost letter of γ_n is $\beta_n \in \Gamma_B$. In particular, for all large $n \in \mathbf{N}$,

$$\beta_n^{-1} \gamma_n B \subset A. \tag{2.2}$$

Case 1 Suppose that the sequence (β_n) is bounded in G. After extraction, we may assume that $\beta_n \equiv \beta \in \Gamma_B \setminus \{1\}$, i.e., (β_n) is a constant sequence. By (2.2), for all large $n \in \mathbf{N}$,

$$\gamma_n B \subset \beta A \subset \mathrm{int}\, B \subset B.$$

In particular, any accumulation point in $\mathrm{Flag}(\tau_{\mathrm{mod}})$ of the sequence of subsets $(\gamma_n B)$ is contained in βA, which is properly contained in B. This contradicts (2.1).

Case 2 Suppose now that the leftmost letter sequence (β_n) associated to (γ_n) is unbounded in G. Since the sequence (β_n) comes from Γ_B, and Γ_B is τ_{mod}-regular, after extraction of the sequence, we obtain that there exist points $b_\pm \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma_B)$ such that

$$\beta_n|_{C(b_-)} \to b_+, \quad \text{as } n \to \infty,$$

uniformly on compact subsets. Recall that, by Lemma 2.3, $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_B) \subset B$.

Since A and B are antipodal to each other and $b_- \in B$ (see Lemma 2.3), we obtain that $A \subset C(b_-)$; cf. (1.2). Then, by Proposition 1.6, $\beta_n A \to b_+$ as $n \to \infty$. Moreover, by (2.2),

$$\gamma_n B \subset \beta_n A, \quad \forall n \gg 1.$$

Hence, we also have that $\gamma_n B \to b_+$ as $n \to \infty$. This contradicts (2.1).

Combining the above two cases, we complete the proof of this proposition.   □

3 Regularity

The main result of this section stated below shows that the subgroup Γ in the conclusion of Theorem A is τ_{mod}-regular.

Theorem 3.1 (Regularity) *Let Γ_A and Γ_B be τ_{mod}-regular subgroups of G. Suppose that (Γ_A, Γ_B) admits a ping-pong pair (A, B) in $\mathrm{Flag}(\tau_{\mathrm{mod}})$. If A is antipodal to B, then $\Gamma := \langle \Gamma_A, \Gamma_B \rangle < G$ is a τ_{mod}-regular subgroup.*

In special the case $G = \mathrm{Isom}(\mathbb{H}^n)$, the regularity of subgroups of G is equivalent to discreteness. Recall that, for general semisimple Lie groups G, we have already established the discreteness of Γ in the conclusion of the above result; see Proposition 2.4. However, if G is of higher rank (e.g., $G = \mathrm{PSL}(3, \mathbf{R})$), then regularity of subgroups is a much stronger property than just discreteness. Therefore, Theorem 3.1 is nontrivial when G is of higher rank.

Before discussing the proof of Theorem 3.1, we introduce the following definition:

Definition 3.2 A sequence of distinct elements of G is τ_{mod}-*irregular* if it contains no τ_{mod}-regular subsequences.

Note that a discrete subgroup $\Gamma < G$ is τ_{mod}-regular if and only if it contains no τ_{mod}-irregular sequences.

Theorem 3.1 is a key step in the proof of our main theorem (Theorem A) and the proof of it occupies the rest of this section. Here we lay out our plan for the proof of Theorem 3.1. The main technical step in the proof is to show that a certain kind of sequences in Γ (the resulting subgroup in the conclusion of Theorem 3.1), which are called *special*, are τ_{mod}-regular; see Lemma 3.4. The proof of this lemma relies upon several results established in Sects. 1.4, 1.6 and 1.7. An immediate consequence of this lemma is that *alternating* sequences in Γ are τ_{mod}-regular, which is then used (with the help of Proposition 1.6) to show that certain sequences of nested images of the subsets A, B under alternating sequences have singleton intersections; see Lemma 3.5 for a precise statement. With this knowledge, we finish the proof of Theorem 3.1 by contradiction: Assuming Γ is not τ_{mod}-regular, we may obtain a τ_{mod}-irregular sequence (γ_n') in Γ. Using the τ_{mod}-irregularity property, we are able to find a subsequence (γ_n) of (γ_n') and a special sequence (ω_n) in Γ such that for all $n \in \mathbf{N}$, ω_n is a rightmost subword of γ_n. Then, with the help of Lemma 3.5, we can easily extract a

subsequence of (γ_n^{-1}) under which the images of some (fixed) compact subset of $\mathrm{Flag}(\tau_{\mathrm{mod}})$ with nonempty interior converge to a point, which would imply (by Proposition 1.10) that this subsequence (hence, its inverse sequence, which is a subsequence of the original sequence (γ_n')) is τ_{mod}-regular; see Lemma 3.6. This would be a contradiction to the assumption of τ_{mod}-irregularity of (γ_n').

We now discuss our proof in detail:

Proof of Proposition 3.1 By Proposition 2.4, we know that Γ is discrete. Thus, we have to show that Γ does not contain any τ_{mod}-irregular sequences consisting of distinct elements. We first show that such sequences in Γ cannot have uniformly bounded relative length (see Sect. 1.1 for the definition). ☐

Lemma 3.3 *If a sequence of distinct elements of Γ has uniformly bounded relative length, then it is τ_{mod}-regular.*

Proof Let (γ_n) be such a sequence. After passing to a subsequence, we can (and will) assume that there exists $N \in \mathbf{N}$ such that, for all $n \in \mathbf{N}$, $\mathrm{rl}(\gamma_n) = N$. If $N = 1$, then clearly the sequence lies in $\Gamma_A \cup \Gamma_B$, and hence it is τ_{mod}-regular (since both subgroups Γ_A, Γ_B of G are assumed to be τ_{mod}-regular).

We prove the claim by an induction on N. Suppose that any sequence in Γ whose relative length is constant and $< N$ is τ_{mod}-regular.

Let (γ_n) be a sequence such that, for all $n \in \mathbf{N}$, $\mathrm{rl}(\gamma_n) = N$. Deleting the leftmost letter $\delta_n \in \Gamma_A \cup \Gamma_B$ from the reduced form of each element in (γ_n), we obtain a new sequence (γ_n'), i.e., for each $n \in \mathbf{N}$, $\gamma_n = \delta_n \gamma_n'$. We consider two possibilities. ☐

Case 1 Suppose that $\{\delta_n \mid n \in \mathbf{N}\}$ is finite. In this case, using the induction hypothesis (γ_n') and, consequently (γ_n), is τ_{mod}-regular (see the discussion in Sect. 1.5).

Case 2 Thus, we now assume that $\{\delta_n \mid n \in \mathbf{N}\} \subset \Gamma_A \cup \Gamma_B$ infinite. Then, after passing to a subsequence of (γ_n), (δ_n) is τ_{mod}-regular and, either (δ_n) is a sequence in Γ_A, or a sequence in Γ_B. It suffices to consider the former case, i.e., (δ_n) is a sequence in Γ_A (the other case is obtained by relabelling). Moreover, after further extraction of (γ_n), we also assume that the rightmost letters of the reduced form of all the elements of (γ_n') come from either Γ_A or Γ_B; suppose that all the rightmost letters are in Γ_A (again, the other case when all the rightmost letters are in Γ_B is similar, and we skip that case). Since (δ_n) is τ_{mod}-regular, after passing to a subsequence, $\delta_n^{\pm 1} \xrightarrow{\text{flag}} d_{\pm} \in A$. Since $B \subset C(d_-)$, we obtain

$$\gamma_n(B) \subset \delta_n(B) \to d_+, \quad \text{as } n \to \infty.$$

Using Proposition 1.10, the above shows that (γ_n) is τ_{mod}-regular. ☐

By the above lemma, our study of τ_{mod}-irregular sequences (γ_n) in Γ is reduced to the case when (γ_n) has unbounded relative length. In particular, if (γ_n) is τ_{mod}-irregular, then after passing to a subsequence, we further assume that

$$\mathrm{rl}(\gamma_n) < \mathrm{rl}(\gamma_{n+1}), \quad \forall n \in \mathbf{N}. \tag{3.1}$$

We next study the τ_{mod}-regularity property of a certain kind of sequences which we call *special*: A sequence (γ_n) in Γ is called *special* if it satisfies (3.1), and, for all $n \in \mathbf{N}$, γ_n is a rightmost subword of γ_{n+1} (see Sect. 1.1 for the definition).

Lemma 3.4 *Special sequences in Γ are τ_{mod}-regular.*

Proof Let (γ_n) be a special sequence. Suppose that the rightmost and leftmost letters of the reduced form of each element γ_n in the sequence are in Γ_A (there are three more possibilities, and the analysis in each case is similar). After extraction, we reduce to one of the following two cases: □

Case 1 Assume that the leftmost letter sequence (α_n) of (γ_n) is unbounded. In this case, after passing to a subsequence of (γ_n), we assume that the leftmost letter sequence (α_n) of (γ_n) flag-converges:

$$\exists a_\pm \in A \text{ such that } \alpha_n^{\pm 1} \xrightarrow{\text{flag}} a_\pm, \quad \text{as } n \to \infty.$$

Since the B is antipodal to a_-, we deduce that

$$\gamma_n(B) \subset \alpha_n(B) \to a_+, \quad \text{as } n \to \infty.$$

Applying Proposition 1.10, we get that (γ_n) is τ_{mod}-regular.

Case 2 Next, we look at the complementary case: For all $n \in \mathbf{N}$, the leftmost letters of the reduced form of all γ_n come from a finite subset $S \subset \Gamma_A$.

Note that, since (γ_n) is special, for each $n \in \mathbf{N}$, the rightmost letter of $\gamma_{n+1}\gamma_n^{-1}$ is in Γ_B, and

$$\gamma_{n+1}\gamma_n^{-1}\widetilde{A} \subset S\widetilde{B} \subset \text{int } \widetilde{A}, \tag{3.2}$$

where

$$\widetilde{A} = \bigcup_{a \in A} \text{St}_{\text{Fu}}(a), \quad \widetilde{B} = \bigcup_{b \in B} \text{St}_{\text{Fu}}(b).$$

Note that since $S\widetilde{B}$ is a union of a finite number of compact sets, the subset $S\widetilde{B}$ is compact. We observe that $(\widetilde{A}, \widetilde{B})$ is a ping-pong pair for (Γ_A, Γ_B) in Flag(σ_{mod}).

Suppose that (γ_n) is not τ_{mod}-regular. After extraction, the sequence (γ_n) is η_{mod}-pure, and $\gamma_i^{\pm 1} \xrightarrow{\text{flag}} \eta_\pm$, for some $\eta_\pm \in \text{Flag}(\pm\eta_{\text{mod}})$, where $\eta_{\text{mod}} \subset \sigma_{\text{mod}}$ is a face such that

$$\tau_{\text{mod}} \not\subset \eta_{\text{mod}}, \tag{3.3}$$

see Remark 1.7. Recall from Sect. 1.6 that, after further extraction of (γ_n), there exists a surjective algebraic map $\phi : C_{\text{Fu}}(\eta_-) \to \text{St}_{\text{Fu}}(\eta_+)$ such that $\gamma_k|_{C_{\text{Fu}}(\eta_-)} \to \phi$ uniformly on compacts. See Proposition 1.8.

We pick a point $z \in C_{\text{Fu}}(\eta_-) \cap \widetilde{A}$.[7] Then, z is contained in a subvariety $\text{St}_{\text{Fu}}(\eta)$, for some (unique) $\eta \in C(\eta_-)$. Since, for all $n \in \mathbf{N}$, $\gamma_n(z) \in S\widetilde{B}$, and $S\widetilde{B} \subset \text{int } \widetilde{A}$, it follows that $\text{St}_{\text{Fu}}(\gamma_n\eta)$ intersects the interior of $\widetilde{A}$.

Let us fix a background distance function d on Flag(σ_{mod}) which is compatible with the manifold topology. By Lemma 1.9,

$$D_{n+1,n} = \max_{x \in \text{St}_{\text{Fu}}(\gamma_n\eta)} d(\gamma_{n+1}\gamma_n^{-1}x, x) \to 0, \quad \text{as } n \to \infty. \tag{3.4}$$

Next, we observe that, for all $n \in \mathbf{N}$, $\text{St}_{\text{Fu}}(\gamma_n\eta) \not\subset \widetilde{A}$: For if $\text{St}_{\text{Fu}}(\gamma_n\eta) \subset \widetilde{A}$, then we must have that $\pi_{\tau_{\text{mod}}}(\text{St}_{\text{Fu}}(\gamma_n\eta)) \subset A$, where $\pi_{\tau_{\text{mod}}}$ is the projection map given by (1.3). In this case, in view of (3.3), we obtain a contradiction with Lemma 1.4.

Since $\text{St}_{\text{Fu}}(\gamma_n\eta)$ is connected (see Lemma 1.2), the preceding paragraph ensures that the intersections $\text{St}_{\text{Fu}}(\gamma_n\eta) \cap \partial\widetilde{A}$ are nonempty, where $\partial\widetilde{A}$ denotes the frontier of the subset $\widetilde{A}$

---

[7] Since $\widetilde{A}$ has nonempty interior, $C_{\text{Fu}}(\eta_-) \cap \widetilde{A} \neq \emptyset$.

of Flag(σ_{mod}). For each $n \in \mathbf{N}$, choose $x_n \in \text{St}_{\text{Fu}}(\gamma_n \eta) \cap \partial \widetilde{A}$. By (3.4) and the triangle inequality, we have

$$\lim_{n \to \infty} d(\gamma_{n+1} \gamma_n^{-1} x_n, \partial \widetilde{A}) = 0. \tag{3.5}$$

However, since $x_n \in \widetilde{A}$, by (3.2), the quantity $d(\gamma_{n+1} \gamma_n^{-1} x_n, \partial \widetilde{A})$ must be uniformly (over n) bounded below by a positive number. This contradicts (3.5).

Combining the above two cases, we complete the proof of the lemma. □

A sequence (ω_n) in Γ is called *alternating* if, there exist sequences (α_n) in $\Gamma_A \setminus \{1\}$ and (β_n) in $\Gamma_B \setminus \{1\}$ such that

$$
\begin{aligned}
\text{Type A} \quad &: \quad \text{Either, } \forall n \in \mathbf{N}, \ \omega_n = \alpha_1 \beta_1 \ldots \alpha_{n-1} \beta_{n-1} \alpha_n, \\
\text{Type B} \quad &: \quad \text{Or, } \forall n \in \mathbf{N}, \ \omega_n = \beta_1 \alpha_1 \ldots \beta_n \alpha_n.
\end{aligned} \tag{3.6}
$$

Applying ω_n to B, we obtain a sequence of compact subsets $(\omega_n B)$ of Flag(τ_{mod}). Note that, for all $n \in \mathbf{N}$, $\omega_{n+1} B = \omega_n \beta_n (\alpha_{n+1} B) \subset \omega_n (\beta_n A) \subset \omega_n (B)$; cf. item (ii) in Definition 2.1. Thus, we obtain a nested sequence of compact subsets

$$\omega_1 B \supset \omega_2 B \supset \cdots . \tag{3.7}$$

Moreover, by definition of being alternating, for all $n \in \mathbf{N}$, ω_n is a leftmost subword of ω_{n+1}, and therefore, (ω_n^{-1}) is special.

Lemma 3.5 *If (ω_n) is alternating, then (ω_n) is τ_{mod}-regular and intersection*

$$\bigcap_{i \in \mathbf{N}} \omega_n B$$

is singleton. In particular, as $n \to \infty$, $\omega_n B$ converges to a point in Flag(τ_{mod}).

Proof We first assume that (ω_n) is of type A.

The first claim about τ_{mod}-regularity follows directly by the observation that (ω_n^{-1}) is special, and then by applying Lemma 3.4 to conclude that (ω_n^{-1}) is τ_{mod}-regular. Hence, (ω_n) is also τ_{mod}-regular, see Remark 1.5.

By τ_{mod}-regularity, there exists a subsequence, (ω_{n_k}) of (ω_n) and $\tau_{\pm} \in$ Flag(τ_{mod}) such that $\omega_{n_k}^{\pm 1} \xrightarrow{\text{flag}} \tau_{\pm}$. By Proposition 1.6, as $k \to \infty$,

$$\omega_{n_k}^{\pm 1}|_{C(\tau_{\mp})} \to \tau_{\pm}, \quad \text{uniformly on compacts.}$$

Because of the special form of ω_n given in (3.6), we observe that $\tau_{\pm} \in A$: For if $\hat{\tau} \in B$ is any point antipodal to the pair $\tau_{\pm}$, then, for all $k \in \mathbf{N}$, $\omega_{n_k} \hat{\tau} \in \omega_{n_k} B \subset A$, where the last inclusion follows from the fact that (ω_n) is of type A. In particular, $\omega_{n_k} \hat{\tau} \in A$. Furthermore, since $\omega_{n_k} \hat{\tau} \to \tau_+$ and A is compact, we obtain that $\tau_+ \in A$. On the other hand, since we also have $\omega_{n_k}^{-1} B \subset A$, applying a similar argument, we get that $\tau_- \in A$ as well.

In particular, $B \subset C(\tau_-)$. Hence, $\omega_{n_k} B \to \tau_+$, as $k \to \infty$. The lemma then follows by the nesting in (3.7).

Next, we consider the complementary case that (ω_n) is of type B. However, this case follows from the previous one by observing that $(\beta_1^{-1} \omega_n)$ is of type A, where $\beta_1 \in \Gamma_B$ is the common leftmost letter of the reduced form of the elements in (ω_n). □

Now we finish the proof of Theorem 3.1. Suppose, to the contrary, that (γ_n') is a τ_{mod}-irregular sequence in Γ. We can (and will) assume that (γ_n') satisfies (3.1), cf. Lemma 3.3 above.

We construct two sequences (ω_n) and (γ_n) in Γ, such that (ω_n) is special, (γ_n) is a subsequence of (γ'_n), and for each $n \in \mathbf{N}$, ω_n is a rightmost subword of γ_n: After passing to a subsequence of (γ'_n), we assume that the rightmost letters α_n of the reduced form of the elements of (γ'_n) are all in either Γ_A or Γ_B. We assume that $\alpha_n \in \Gamma_A$ for all $n \in \mathbf{N}$; the other possibility is analyzed by relabelling. Note that the rightmost letter sequence (α_n) must be bounded in G because, otherwise, by Case 1 in the proof of Lemma 3.4, (γ'^{-1}_n), and hence (γ'_n), would contain a τ_{mod}-regular sequence. Thus, (γ'_n) has a subsequence (γ'_{n_i}) such that the rightmost letters of the reduced form of γ'_{n_i}'s are all the same, say all these are equal to $\alpha_1 \in \Gamma_A$. Set $\omega_1 := \alpha_1$ and $\gamma_1 := \gamma'_{n_1}$. Applying a similar argument (to the sequence $(\gamma'_{n_i}\alpha_1^{-1})_{i \in \mathbf{N}}$), we obtain a further subsequence $(\gamma'_{n_{i_k}})$ of (γ'_{n_i}) such that second letters from the right in the reduced form of $\gamma'_{n_{i_k}}$'s are all the same, say all these are equal to $\beta_1 \in \Gamma_B$. Set $\omega_2 := \beta_1\alpha_1$ and $\gamma_2 := \gamma'_{n_{i_1}}$. Proceeding inductively, we construct a special sequence (ω_n) in Γ and a subsequence of (γ_n) of (γ'_n) such that

$$\omega_n \text{ is an rightmost subword of } \gamma_n, \quad \forall n \in \mathbf{N}.$$

In particular, for all $n \in \mathbf{N}$, $\gamma_n = \delta_n\omega_n$, where $\delta_n = \gamma_n\omega_n^{-1}$ is a leftmost subword of γ_n. After an extraction of (γ_n), we can assume that the leftmost letters of the reduced form of the elements of (δ_n) all come from the same group, say Γ_A. Moreover, by the construction, the sequence $(\omega_{2n+1}^{-1})_{n \in \mathbf{N}}$ is type A alternating (see (3.6)).

Lemma 3.6 (γ_{2n+1}) is a τ_{mod}-regular sequence.

Proof We will prove the equivalent statement: (γ_{2n+1}^{-1}) is a τ_{mod}-regular subsequence, see Remark 1.5.

We observe that γ_{2n+1}^{-1} has the following form:

$$\gamma_{2n+1}^{-1} = \omega_{2n+1}^{-1}\delta_{2n+1}^{-1} = \underbrace{\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_{n+1}^{-1}}_{\omega_{2n+1}^{-1}}\underbrace{\beta_1^{(2n+1)}\alpha_1^{(2n+1)}\cdots\alpha_q^{(2n+1)}}_{\delta_{2n+1}^{-1}}$$

Therefore,

$$\gamma_{2n+1}^{-1}B \subset \omega_{2n+1}^{-1}B.$$

Since $(\omega_{2n+1}^{-1})_{n \in \mathbf{N}}$ is type A alternating, by Lemma 3.5, $\omega_{2n+1}^{-1}B$ converges to a point $\tau_+ \in \mathrm{Flag}(\tau_{\mathrm{mod}})$. Therefore, by the above inclusion, we also obtain

$$\gamma_{2n+1}^{-1}B \to \tau_+, \quad \text{as } n \to \infty.$$

By Proposition 1.10, (γ_{2n+1}^{-1}) is τ_{mod}-regular subsequence. $\square$

Hence, the sequence (γ'_n) contains a τ_{mod}-regular subsequence, namely (γ_{2n+1}). This is a contradiction!

4 Boundary embedding

In this section, we work under the hypothesis of Theorem A, i.e., we assume that Γ_A and Γ_B are τ_{mod}-Anosov subgroups admitting a ping-pong pair (A, B) in $\mathrm{Flag}(\tau_{\mathrm{mod}})$ such that A and B are antipodal to each other. By Lemma 2.3, we know that $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A) \subset A$,

and $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_B) \subset B$. Moreover, since τ_{mod}-Anosov subgroups of G are τ_{mod}-regular (see Sect. 1.8), by Theorem 3.1, $\Gamma := \langle \Gamma_A, \Gamma_B \rangle \cong \Gamma_A * \Gamma_B$ is a τ_{mod}-regular subgroup of G.

Anosov subgroups are intrinsically word-hyperbolic (see Definition 1.13). Hence, the free product $\Gamma_A * \Gamma_B$ is also word-hyperbolic. We fix a finite generating set $S = S_A \cup S_B$ for $\Gamma_A * \Gamma_B$, where S_A and S_B generate Γ_A and Γ_B, respectively. Using this choice of S we define the Cayley graph of $\Gamma_A * \Gamma_B$. We equip $\Gamma_A * \Gamma_B$ with the corresponding word-metric.

The goal of this section is to construct a Γ-equivariant antipodal embedding

$$\xi : \partial_\infty \Gamma \to A \cup B \subset \mathrm{Flag}(\tau_{\mathrm{mod}}),$$

such that

$$\xi_A = \xi|_{\partial_\infty \Gamma_A} \quad \text{and} \quad \xi_B = \xi|_{\partial_\infty \Gamma_B}, \tag{4.1}$$

where the homeomorphisms $\xi_A : \partial_\infty \Gamma_A \to \Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A)$ and $\xi_B : \partial_\infty \Gamma_B \to \Lambda_{\tau_{\mathrm{mod}}}(\Gamma_B)$ are the asymptotic embeddings for Γ_A and Γ_B, respectively.

We first recall that there exists a Γ-invariant decomposition

$$\partial_\infty \Gamma = \Gamma(\partial_\infty \Gamma_A \sqcup \partial_\infty \Gamma_B) \sqcup \partial_\infty T$$

where T is the Bass–Serre tree of the free product $\Gamma_A * \Gamma_B$, see e.g. [14]. For the notational convenience, we introduce the following notation:

$$\partial_{\mathrm{I}} \Gamma := \Gamma(\partial_\infty \Gamma_A \sqcup \partial_\infty \Gamma_B) \quad \text{and} \quad \partial_{\mathrm{II}} \Gamma := \partial_\infty T.$$

We will first define the map ξ separately on these two disjoint subsets $\partial_{\mathrm{I}} \Gamma$ and $\partial_{\mathrm{II}} \Gamma$. Then we will piece them together and check that ξ is an equivariant antipodal embedding.

4.1 Construction of ξ

We define a map

$$\xi : \partial_{\mathrm{I}} \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}}) \tag{4.2}$$

as follows: Let $\varepsilon \in \partial_{\mathrm{I}} \Gamma$ be arbitrary. There exists some $\gamma \in \Gamma$ such that $\gamma^{-1} \varepsilon \in \partial_\infty \Gamma_A \sqcup \partial_\infty \Gamma_B$. Suppose, for instance, that[8] $\gamma^{-1} \varepsilon \in \partial_\infty \Gamma_A$. Define

$$\xi(\varepsilon) = \gamma\, \xi_A(\gamma^{-1} \varepsilon).$$

To check well-definedness, we note that such a γ is unique up to right multiplications by elements of Γ_A. Thus, if $\gamma_1 \in \Gamma$ is another choice such that $\gamma_1^{-1} \varepsilon \in \partial_{\mathrm{I}} \Gamma$, then $\alpha = \gamma^{-1} \gamma_1 \in \Gamma_A$. Hence, $\gamma_1\, \xi_A(\gamma_1^{-1} \varepsilon) = \gamma\alpha\, \xi_A(\alpha^{-1}\gamma^{-1}\varepsilon) = \gamma\, \xi_A(\alpha\alpha^{-1}\gamma^{-1}\varepsilon) = \gamma\, \xi_A(\gamma^{-1}\varepsilon)$. Moreover, by the construction, the map $\xi : \partial_{\mathrm{I}} \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ is Γ-equivariant and satisfies (4.1).

Proposition 4.1 *Let $\varepsilon \in \partial_{\mathrm{I}} \Gamma$, and let (γ_n) be any sequence in $\Gamma_A * \Gamma_B$ such that $\gamma_n \xrightarrow{Cay} \varepsilon$ in the compactified (by adding the Gromov boundary) Cayley graph of $\Gamma_A * \Gamma_B$. Then, viewing (γ_n) as a sequence in G, $\gamma_n \xrightarrow{flag} \xi(\varepsilon)$.*

Moreover, the image $\xi(\partial_{\mathrm{I}} \Gamma)$ is contained in $A \cup B$.

Proof Recall that by Lemma 2.3, $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A) \subset A$ and $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_B) \subset B$.

For the first part, it is enough to assume that $\varepsilon \in \partial_\infty \Gamma_A \sqcup \partial_\infty \Gamma_B$. Suppose that $\varepsilon \in \partial_\infty \Gamma_A$ (the other case when $\varepsilon \in \partial_\infty \Gamma_B$ is treated similarly). Since $\gamma_n \xrightarrow{Cay} \varepsilon \in \partial_\infty \Gamma_A$, possibly

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[8] The other possibility that $\gamma^{-1} \varepsilon \in \partial_\infty \Gamma_B$ can be treated by relabelling.
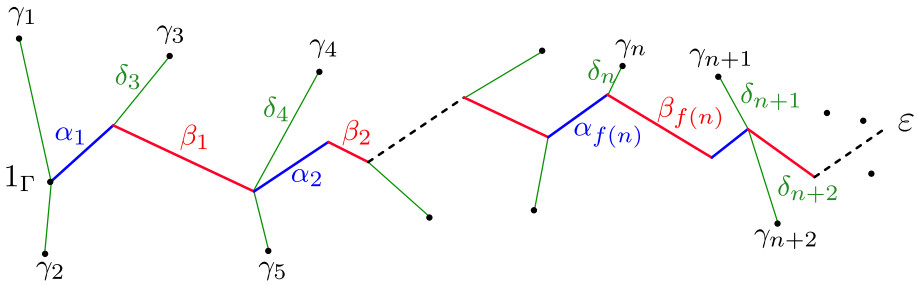
Fig. 2 Fellow traveling ray

after disregarding the first few terms, the leftmost letter sequence (α_n) of (γ_n) comes from Γ_A and $\alpha_n \xrightarrow{\text{Cay}} \varepsilon$. Since $\Gamma_A < G$ is τ_{mod}-regular and B is antipodal relative to $\Lambda_{\tau_{\text{mod}}}(\Gamma_A)$, $\alpha_n(B) \to \{\xi(\varepsilon)\}$ in $\text{Flag}(\tau_{\text{mod}})$. Now, Proposition 1.10 implies that $\gamma_n \xrightarrow{\text{flag}} \xi(\varepsilon)$; see Case 2 in the proof of Lemma 3.3 for a similar argument.

We prove the second part by induction on the relative lengths of elements $\gamma \in \Gamma = \Gamma_A * \Gamma_B$. If, say, $\varepsilon \in \partial_\infty \Gamma_A$ and $\gamma \in \Gamma_A \cup \Gamma_B \setminus \{1\}$, then either $\gamma(\varepsilon) \in \Lambda_{\tau_{\text{mod}}}(\Gamma_A) \subset A$ (when $\gamma \in \Gamma_A$) or $\gamma(\varepsilon) \in B$ (since $\gamma(A) \subset B$ in this case). Inductively, assume that for each element $\gamma \in \Gamma$ of the relative length $n-1$ and of the reduced form $\alpha_1\beta_1 \cdots$, then $\gamma(\varepsilon) \in A$, while if γ is of the reduced form $\beta_1\alpha_2 \cdots$, then $\gamma(\varepsilon) \in B$. Consider an element $\gamma = \alpha_1\beta_1 \cdots$ of the relative length n. Then, by the induction assumption,

$$e' = \beta_1\alpha_2 \cdots (\varepsilon) \in B.$$

Then $\gamma(\varepsilon) = \alpha_1(e') \in A$ by the assumption of Theorem A. Similarly, if $\gamma = \beta_1\alpha_2 \cdots$, then $\gamma(\varepsilon) \in B$.   □

To define a map $\xi : \partial_{\text{II}}\Gamma \to \text{Flag}(\tau_{\text{mod}})$, we use the following result.

Proposition 4.2 *Let $\varepsilon \in \partial_{\text{II}}\Gamma$, and let (γ_n) be any sequence in $\Gamma_A * \Gamma_B$ such that $\gamma_n \xrightarrow{\text{Cay}} \varepsilon$ in the compactified Cayley graph of $\Gamma_A * \Gamma_B$. Then, there exists $\tau_+ \in \text{Flag}(\tau_{\text{mod}})$ depending only on ε such that in $\text{Flag}(\tau_{\text{mod}})$, $\gamma_n \xrightarrow{\text{flag}} \tau_+$.*

Proof Consider a sequence (ω_n) in $\Gamma_A * \Gamma_B$ such that $\omega_n \xrightarrow{\text{Cay}} \varepsilon$ and which is *alternating*, see (3.6). We remark that the alternating sequence lies on the geodesic ray ℓ_ε in the Cayley graph of $\Gamma_A * \Gamma_B$ that emanates at the identity element and is asymptotic to ε. Such an alternating sequence is *uniquely* determined by $\varepsilon \in \partial_{\text{II}}\Gamma$.   □

Let (γ_n) be a sequence as in the statement of the proposition. Since $\gamma_n \xrightarrow{\text{Cay}} \varepsilon$, the sequence (γ_n) "fellow travels" the geodesic ray ℓ_ε. See Fig. 2. More precisely, there exists a sequence (δ_n) in $\Gamma_A * \Gamma_B$ such that the following hold:

(i) There exists a nondecreasing function $f : \mathbf{N} \to \mathbf{N}$ such that $\lim_{n\to\infty} f(n) = \infty$ and $n_0 \in \mathbf{N}$ such that, for all $n \geq n_0$, $\gamma_n = \omega_{f(n)}\delta_n$.

(ii) For $n \geq n_0$, if δ_n is nontrivial and the leftmost letter of the reduced form of δ_n is in Γ_A (resp. Γ_B), then the rightmost letter of the reduced form of $\omega_{f(n)}$ is in Γ_B (resp. Γ_A).

We further assume that the sequence (ω_n) is of type A in (3.6) (type B sequences can be analyzed in a similar way). So, for $n \geq n_0$, γ_n has the following form

$$\gamma_n = \underbrace{\alpha_1 \beta_1 \cdots \beta_{f(n)-1} \alpha_{f(n)}}_{\omega_{f(n)}} \underbrace{\beta_1^{(n)} \alpha_1^{(n)} \cdots}_{\delta_n}.$$

Next, we split the sequence (γ_n) into two disjoint subsequences: The first (resp. second) subsequence (γ_n') (resp. (γ_n'')) consists of all elements of (γ_n) whose rightmost letters in their reduced forms are in Γ_A (resp. Γ_B). Let

$$\tau_+ := \bigcap_{i \in \mathbf{N}} \omega_n B = \lim_{n \to \infty} \omega_n B \tag{4.3}$$

(cf. Lemma 3.5). Then, applying Lemma 3.5, we get

$$\gamma_n'(B) \xrightarrow{\text{flag}} \tau_+, \quad \gamma_n''(A) \xrightarrow{\text{flag}} \tau_+, \quad \text{as } n \to \infty,$$

and then Proposition 1.10 implies $\gamma_n' \xrightarrow{\text{flag}} \tau_+$ and $\gamma_n'' \xrightarrow{\text{flag}} \tau_+$. Compare with the proof of Lemma 3.6. Hence, $\gamma_n \to \tau_+$. $\qquad\square$

Using Proposition 4.2, we define a map

$$\xi : \partial_{\mathrm{II}} \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}}), \quad \varepsilon \mapsto \tau_+, \tag{4.4}$$

where τ_+ is the unique point in $\mathrm{Flag}(\tau_{\mathrm{mod}})$ corresponding to ε obtained by the proposition. This map is Γ-equivariant: Let $\gamma \in \Gamma$ and $\varepsilon \in \partial_{\mathrm{II}} \Gamma$. If (ω_n) is an alternating sequence converging to ε, then the sequence $(\gamma \omega_n)$ converges to $\gamma \varepsilon$. By Proposition 4.2 and the definition of τ_+ in (4.3),

$$\xi(\gamma \varepsilon) = \lim_{n \to \infty} (\gamma \omega_n) B = \gamma \left(\lim_{n \to \infty} \omega_n B \right) = \gamma \, \xi(\varepsilon).$$

Corollary 4.3 *The piecewise-defined map* $\xi : \partial_\infty \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ *from* (4.2) *and* (4.4) *is a* Γ-*equivariant map satisfying* (4.1).

4.2 ξ is antipodal

In this subsection, we show that the map ξ in Corollary 4.3 is antipodal: That is, if $\varepsilon, \hat{\varepsilon} \in \partial_\infty \Gamma$ are distinct points, then $\xi(\varepsilon)$ is antipodal to $\xi(\hat{\varepsilon})$ in $\mathrm{Flag}(\tau_{\mathrm{mod}})$. In the proof, we repeatedly use the fact that

$$\text{the action } G \curvearrowright \mathrm{Flag}(\tau_{\mathrm{mod}}) \text{ preserves antipodality.} \tag{4.5}$$

The discussion naturally divides into three mutually exclusive cases.

Case 1 Suppose that ε and $\hat{\varepsilon}$ both lie in $\partial_{\mathrm{I}} \Gamma$. By (4.5), without loss of generality, we assume that $\varepsilon \in \partial_\infty \Gamma_A \sqcup \partial_\infty \Gamma_B$. If $\hat{\varepsilon}$ is also in $\partial_\infty \Gamma_A \sqcup \partial_\infty \Gamma_B$, then clearly $\xi(\varepsilon)$ is antipodal to $\xi(\hat{\varepsilon})$ because $\xi_A \sqcup \xi_B : \partial_\infty \Gamma_A \sqcup \partial_\infty \Gamma_B \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ is an antipodal embedding. Thus, we assume that $\hat{\varepsilon} \notin \partial_\infty \Gamma_A \sqcup \partial_\infty \Gamma_B$. Suppose further that $\varepsilon \in \partial_\infty \Gamma_A$ (the other case $\varepsilon \in \partial_\infty \Gamma_B$ can be analyzed similarly). Pick some $\gamma \in \Gamma$ of *least* relative length such that $\gamma \hat{\varepsilon}' = \hat{\varepsilon}$ for some $\hat{\varepsilon}' \in \partial_\infty \Gamma_A \sqcup \partial_\infty \Gamma_B$.

Suppose that $\hat{\varepsilon}' \in \partial_\infty \Gamma_A$. Then, by the assumption that γ has the least relative length, the rightmost letter of the reduced form of γ is in $\Gamma_B \setminus \{1\}$. If the leftmost letter of the reduced form of γ is some element $\alpha \in \Gamma_A \setminus \{1\}$, then we must have $\mathrm{rl}(\gamma) \geq 2$. In this case, the second letter

from the left of the reduced form of γ is some element $\beta \in \Gamma_B$, which is now the leftmost letter of the reduced form of $\alpha^{-1}\gamma$. Thus, $\alpha^{-1}\xi(\hat{\varepsilon}) = \alpha^{-1}\xi(\gamma\hat{\varepsilon}') = \alpha^{-1}\gamma\xi(\hat{\varepsilon}') \in \beta A \subset B$ whereas $\alpha^{-1}\xi(\varepsilon) \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A) \subset A$, Hence, $\alpha^{-1}\xi(\hat{\varepsilon})$ and $\alpha^{-1}\xi(\varepsilon)$ are antipodal, and consequently, by (4.5), $\xi(\hat{\varepsilon})$ and $\xi(\varepsilon)$ are antipodal. Similarly, if the leftmost letter of the reduced form of γ is in $\Gamma_B \setminus \{1\}$, then we must have $\xi(\hat{\varepsilon}) \in B$, and since $\xi(\varepsilon) \in A$, $\xi(\hat{\varepsilon})$ and $\xi(\varepsilon)$ are antipodal.

Now we assume that $\hat{\varepsilon}' \in \partial_\infty \Gamma_B$. Since γ has the least relative length, the rightmost letter of the reduced form of γ lies in $\Gamma_A \setminus \{1\}$. If the leftmost letter in the reduced form of γ is some element $\alpha \in \Gamma_A \setminus \{1\}$, then one may check that $\alpha^{-1}\xi(\hat{\varepsilon}) = \alpha^{-1}\gamma\xi(\hat{\varepsilon}') \in B$, which is antipodal to $\alpha^{-1}\xi(\varepsilon) \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma_A) \subset A$. Thus, by (4.5), $\xi(\hat{\varepsilon})$ and $\xi(\varepsilon)$ are antipodal. Else, if the leftmost letter in the reduced form of γ is some element $\beta \in \Gamma_B \setminus \{1\}$, then $\mathrm{rl}(\gamma) \geq 2$ (since the rightmost letter lies in $\Gamma_A \setminus \{1\}$). In this case, by a similar reasoning as in the preceding paragraph, it follows that $\beta^{-1}\xi(\hat{\varepsilon}) \in A$ is antipodal to $\beta^{-1}\xi(\varepsilon) \in \beta^{-1}A \subset B$, which implies (again, by (4.5)) that $\xi(\hat{\varepsilon})$ and $\xi(\varepsilon)$ are antipodal.

Case 2 Suppose that ε and $\hat{\varepsilon}$ both lie in $\partial_{\mathrm{II}}\Gamma$. Then ε (resp. $\hat{\varepsilon}$) determines[9] an alternating sequence (ω_n) (resp. $(\hat{\omega}_n)$) in Γ. If the sequences (ω_n) and $(\hat{\omega}_n)$ are of different types (see (3.6)), then clearly, $\xi(\varepsilon)$ and $\xi(\hat{\varepsilon})$ lie in different sets A and B. Hence, $\xi(\varepsilon)$ and $\xi(\hat{\varepsilon})$ are antipodal.

Otherwise, the sequences (ω_n) and $(\hat{\omega}_n)$ have the same type, say type A. Let $N \in \mathbf{N}$ be the least number for which $\omega_N \neq \hat{\omega}_N$. Clearly, the sequences $(\omega_N^{-1}\omega_n)_{n \geq N+1}$ and $(\omega_N^{-1}\hat{\omega}_n)_{n \geq N+1}$ are still alternating, but of *different* types. Therefore, by the previous paragraph, $\omega_N^{-1}\xi(\varepsilon) = \xi(\omega_N^{-1}\varepsilon)$ and $\omega_N^{-1}\xi(\hat{\varepsilon}) = \xi(\omega_N^{-1}\hat{\varepsilon})$ are antipodal. Hence, by (4.5), $\xi(\varepsilon)$ and $\xi(\hat{\varepsilon})$ are antipodal.

Case 3 Suppose that $\varepsilon \in \partial_{\mathrm{I}}\Gamma$ and $\hat{\varepsilon} \in \partial_{\mathrm{II}}\Gamma$. By (4.5), without loss of generality, we assume that $\varepsilon \in \partial_\infty\Gamma_A \sqcup \partial_\infty\Gamma_B$. Suppose further that $\varepsilon \in \partial_\infty\Gamma_A$; the other case $\varepsilon \in \partial_\infty\Gamma_B$ can be analyzed by relabelling. Let (ω_n) be the alternating sequence in Γ determined by $\hat{\varepsilon}$. If (ω_n) is of type B, then $\xi(\hat{\varepsilon}) \in B$ and, hence, it is antipodal to $\xi(\varepsilon)$. Otherwise, if (ω_n) is of type A, then $\omega_1 \in \Gamma_A \setminus \{1\}$. In this case, by a similar argument as in Case 2, $\omega_1^{-1}\xi(\varepsilon)$ is antipodal to $\omega_1^{-1}\xi(\hat{\varepsilon})$ which, in conjunction with (4.5), implies that $\xi(\varepsilon)$ is antipodal to $\xi(\hat{\varepsilon})$.

Combining the above cases, we obtain the following:

Proposition 4.4 *The map $\xi : \partial_\infty\Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ in Corollary 4.3 is antipodal. In particular, ξ is injective.*

4.3 ξ is an embedding

Finally, we prove that the map $\xi : \partial_\infty\Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ in Corollary 4.3 is an embedding whose image lies in $A \cup B$. Since ξ is injective (see Proposition 4.4), domain of ξ is a compact space, and the codomain is Hausdorff, it is enough to show that ξ is continuous. It suffices to prove that for each point $\varepsilon \in \partial_\infty\Gamma$ and every sequence $\gamma_n \in \Gamma$ converging to ε, the sequence (γ_n) in G flag-converges to $\xi(\varepsilon)$. This, however, is the content of Lemma 4.1 and Proposition 4.2.

In order to prove that $\xi(\partial_\infty\Gamma) \subset A \cup B$, we observe that $\partial_{\mathrm{I}}\Gamma$ is dense in $\partial_\infty\Gamma$ and that $\xi(\partial_{\mathrm{I}}\Gamma) \subset A \cup B$ (see the second part of Lemma 4.1). Thus, continuity of ξ and the fact that $A \cup B$ is closed, implies that $\xi(\partial_\infty\Gamma) \subset A \cup B$.

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[9] See the beginning of the proof of Proposition 4.2.

5 Conclusion of the proof of Theorem A

Recall that a τ_{mod}-regular subgroup Γ of G, which is word-hyperbolic as an abstract group, is τ_{mod}-asymptotically embedded (see Definition 1.13) if there exists an equivariant antipodal homeomorphism $f : \partial_\infty \Gamma \to \Lambda_{\tau_{\mathrm{mod}}}$, the flag-limit set of Γ in $\mathrm{Flag}(\tau_{\mathrm{mod}})$, which extends continuously the orbit map

$$o_x : \Gamma \to \Gamma x \subset X$$

to the symmetric space X, where continuity is understood with respect to the topology of flag-convergence. The equivariant map f is said to be the *asymptotic embedding* of Γ.

Proposition 5.1 *Under the assumption of Theorem A, the subgroup Γ of G generated by Γ_A and Γ_B is τ_{mod}-asymptotically embedded with the asymptotic embedding given by the map ξ whose image lies in $A \cup B$.*

Proof We already know that the subgroup $\Gamma < G$ is τ_{mod}-regular. See Theorem 3.1.

The discussion in Sect. 4.3 shows that $\xi : \partial_\infty \Gamma \to \mathrm{Flag}(\tau_{\mathrm{mod}})$ is a continuous antipodal map whose image is contained in the flag-limit set of Γ in $\mathrm{Flag}(\tau_{\mathrm{mod}})$ and which continuously extends the orbit map o_x. The fact that the map ξ is onto $\Lambda_{\tau_{\mathrm{mod}}}$ follows from the fact that every sequence in Γ subconverges to a point $\varepsilon \in \partial_\infty \Gamma$. Let (γ_n) denote that convergent subsequence. Then, as we noted above, $o_x(\gamma_n)$ converges to $\xi(\varepsilon)$, i.e. the flag-limit of the sequence $\gamma_n \in \Gamma < G$ is $\xi(\varepsilon) \in \xi(\partial_\infty \Gamma)$. □

This concludes the proof of the Theorem A.

6 Consequences of Theorem A

Below we present some applications of the main theorem of the paper.

6.1 Schottky subgroups

A subgroup Γ of G is said to be a τ_{mod}-*Schottky* subgroup if it is free and τ_{mod}-Anosov. Starting with a subset of pairwise antipodal points $\{a_\pm, b_\pm\}$ in $\mathrm{Flag}(\tau_{\mathrm{mod}})$, let $\alpha, \beta \in G$ be *axial* isometries of X which preserves some τ_{mod}-regular geodesic lines $L_{a_\pm}, L_{b_\pm} \subset X$, respectively, such that $L_{a_\pm}$ (resp. $L_{b_\pm}$) is forward/backward asymptotic to $a_\pm$ (resp. $b_\pm$). It was shown in [9, Section 7.6] that, after passing to sufficiently large powers of α and β, the subgroup $\langle \alpha^m, \beta^n \rangle, m, n \gg 1$, is a τ_{mod}-Schottky subgroup of G, and is naturally isomorphic to the two-generator free group. Below, as a consequence of Theorem A, we show that under suitable conditions, $\langle \alpha, \beta^n \rangle$, $n \gg 1$, is a τ_{mod}-Schottky subgroup of G, i.e., under these conditions, we need to take a large power of only *one* of the generators. In particular, this construction produces a large family of 2-generated τ_{mod}-Schottky subgroups such that one of the generators can be assumed to have arbitrarily small *translation length* in X. We remark that the constructions of such groups do not follow from the results of [9] or [4].

Let $\{a_\pm, b_\pm\}$, α, and β be as above. We will assume that there exists a one-parameter closed subgroup H_α of G containing the element α and such that all elements of H_α preserve the geodesic $L_{a_\pm}$. (This holds, for instance, if α is a *transvection* along $L_{a_\pm}$.)

Lemma 6.1 *All but finitely many elements of $\hat{\alpha} \in H_\alpha$ satisfy the following: The subset $\{b_+, b_-\}$ is antipodal relative to $\hat{\alpha}\{b_+, b_-\}$ in $\mathrm{Flag}(\tau_{\mathrm{mod}})$.*

Proof Let $f : \mathbf{R} \to H_\alpha < G$ be the isomorphism normalized by $f(1) = \alpha$. Under this identification, we have an analytic action $\mathbf{R} \curvearrowright \mathrm{Flag}(\tau_{\mathrm{mod}})^2$ given by $r \cdot (\tau_+, \tau_-) = (f(r)\tau_+, f(r)\tau_-)$. In particular, if $C \subset \mathrm{Flag}(\tau_{\mathrm{mod}})^2$ is an analytic subset and (τ_+, τ_-) is an arbitrary point, then $E_C(\tau_+, \tau_-) := \{r \in \mathbf{R} \mid r \cdot (\tau_+, \tau_-) \in C\}$ is an analytic subset of $\mathbf{R}$.

Recall the notion of exceptional subvariety $E(\tau) \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$ for τ from Sect. 1.3. Setting

$$C = (E(b_\pm) \times \mathrm{Flag}(\tau_{\mathrm{mod}})) \cup (\mathrm{Flag}(\tau_{\mathrm{mod}}) \times E(b_\pm)),$$

we get that $E_C(b_+, b_-)$ is an analytic subset of $\mathbf{R}$. Furthermore, $(\tau_+, \tau_-) \in \mathrm{Flag}(\tau_{\mathrm{mod}})^2$ lies in the complement of C if and only if $\{\tau_+, \tau_-\}$ is antipodal relative to $\{b_+, b_-\}$. Since

$$\lim_{r \to \pm\infty} r \cdot (b_+, b_-) = (a_\pm, a_\pm)$$

and $\{a_+, a_-\}$ is antipodal relative to $\{b_+, b_-\}$, $E_C(b_+, b_-)$ must be a compact subset of $\mathbf{R}$. Hence, $E_C(b_+, b_-)$ is finite. $\qquad\square$

Now we return to the construction of τ_{mod}-Schottky subgroups of G. Suppose that the axial isometries α and β of X are chosen so that:

1. As before, the forward/backward asymptotic points of L_α and L_β in $\mathrm{Flag}(\tau_{\mathrm{mod}})$ are pairwise antipodal points $a_\pm$ and $b_\pm$, where L_α (resp. L_β) is a τ_{mod}-regular line in X preserved by α (resp. β).
2. For all $k \in \mathbf{Z} \setminus \{0\}$, $\alpha^k\{b_+, b_-\}$ is antipodal relative to $\{b_+, b_-\}$.

Note that Lemma 6.1 guarantees that all but at most countably many elements of H_α satisfy the second condition above. In particular, one can choose α to have an arbitrarily small translation length in X.

We have:

Proposition 6.2 (Schottky subgroups) *For all large n, the subgroup $\langle \alpha, \beta^n \rangle$ of G is τ_{mod}-Anosov, and is naturally isomorphic to the free group of rank two.*

Proof We observe that $A_1 = \{\alpha^k b_\pm \mid k \in \mathbf{Z}\setminus\{0\}\} \cup \{a_\pm\}$ and $B_1 = \{b_\pm\}$ are compact sets, which are antipodal relative to each other (by the condition 2 above). Moreover, by our choice of A_1, $\alpha^k(B) \subset A$ for all $k \neq 0$. There exist subsets A and B of $\mathrm{Flag}(\tau_{\mathrm{mod}})$ such that A and B are antipodal relative to each other, and $\mathrm{int}\, A \supset A_1$, $\mathrm{int}\, B \supset B_1$, see [4, Lemma 4.24]. Moreover, for all large $k \in \mathbf{N}$, $\alpha^{\pm k} B \subset \mathrm{int}\, A$. Making B smaller we can assure that $\alpha^k B \subset \mathrm{int}\, A$, for all $k \in \mathbf{Z} \setminus \{0\}$.

On the other hand,

$$\lim_{k \to \pm\infty} \alpha^k(A) = \{b_\pm\}.$$

Hence, for all sufficiently large $n \in \mathbf{N}$, $\beta^{\pm n} A \subset \mathrm{int}\, B$. Applying the Theorem A to $\Gamma_A = \langle \alpha \rangle$, $\Gamma_B = \langle \beta^n \rangle$, $n \gg 1$, and the subsets A and B as above, we finish the proof. $\qquad\square$

6.2 A general version of the Theorem A involving several Anosov subgroups

Our second application of the main theorem is its generalization to the case of several Anosov subgroups:

Corollary 6.3 *Suppose that the subgroups $\Gamma_1, \ldots, \Gamma_n$ of G are τ_{mod}-Anosov, $A_1, \ldots, A_n \subset$ Flag(τ_{mod}) are pairwise antipodal compact subsets with nonempty interiors such that for every nontrivial element $\gamma_i \in \Gamma_i$,*

$$\gamma_i(A_1 \cup \cdots \cup A_n \setminus A_i) \subset \mathrm{int}\, A_i, \tag{6.1}$$

$i = 1, \ldots, n$. Then:

(i) *The subgroup $\Gamma < G$ generated by $\Gamma_1, \ldots, \Gamma_n$ is naturally isomorphic to the free product $\Gamma_1 * \Gamma_2 * \cdots * \Gamma_n$.*
(ii) *The subgroup $\Gamma < G$ is τ_{mod}-Anosov.*
(iii) *The τ_{mod}-limit set of Γ is contained in $A_1 \cup \cdots \cup A_n$.*

Proof The proof is by induction on n. Set $\Gamma_A := \langle \Gamma_1, \ldots, \Gamma_{k-1} \rangle$, $\Gamma_B = \Gamma_k$,

$$A := A_1 \cup \cdots \cup A_{k-1}, \quad B := A_k, \quad k \leq n.$$

We assume that Γ_A is τ_{mod}-Anosov, naturally isomorphic to $\Gamma_1 * \Gamma_2 * \cdots * \Gamma_{k-1}$, and that each nontrivial element $\alpha \in \Gamma_A$ satisfies

$$\alpha(B) \subset \mathrm{int}\, A$$

(it is clear that every nontrivial element $\beta \in \Gamma_B$ satisfies $\beta(A) \subset \mathrm{int}\, B$). Then Theorem A implies that $\Gamma^k := \langle \Gamma_A, \Gamma_B \rangle$ is naturally isomorphic to the free product $\Gamma_A * \Gamma_B \cong \Gamma_1 * \Gamma_2 * \cdots * \Gamma_k$ and is τ_{mod}-Anosov. In order to prove that each nontrivial element $\gamma \in \Gamma^k$ satisfies $\gamma(A_j) \subset A \cup B$, $j = k + 1, \ldots, n$, one argues by induction on the relative length of $\gamma \in \Gamma_A * \Gamma_B$, similarly to the proof of Lemma 4.1. We leave the details to the reader. □

Remark 6.4 We show that the main result (Theorem 1.3) of our paper with Bernhard Leeb [4], follows from the Corollary 6.3: If $\Gamma_1, \ldots, \Gamma_n$ are *pairwise antipodal*,[10] residually finite, τ_{mod}-Anosov subgroups, then consider compact neighborhoods $A_1, \ldots, A_n$ of the limit sets of $\Gamma_1, \ldots, \Gamma_n$, respectively, such that A_i and A_j, for all $i \neq j$, are pairwise antipodal (cf. [4, Lemma 4.24]). For each i, the subset of Γ_i not satisfying (6.1) is finite. This can be seen as follows: Suppose, to the contrary, that there exists an infinite sequence (γ_k) of distinct elements in Γ_i such that, for all $k \in \mathbf{N}$, $\gamma_k(A_1 \cup \cdots \cup A_n \setminus A_i) \not\subset \mathrm{int}\, A_i$. In particular, the Hausdorff distance (with respect to a metric on Flag(τ_{mod}) compatible with the manifold topology) between $\gamma_k(A_1 \cup \cdots \cup A_n \setminus A_i)$ and any point in $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma_i)$ is uniformly bounded below by a positive number. However, passing to a subsequence, we can ensure that $\gamma_k \xrightarrow{\text{flag}} \tau_+ \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma_i) \subset \mathrm{int}\, A_i$. Hence, $\gamma_k(A_1 \cup \cdots \cup A_n \setminus A_i) \to \tau_+$, a contradiction.

Thus, by the residual finiteness assumption, for each i, we pass to a finite index subgroup $\Gamma_i' < \Gamma_i$ such that (6.1) is satisfied for the collections $\Gamma_1', \ldots, \Gamma_n'$ and $A_1, \ldots, A_n$. Applying Corollary 6.3 to these collections, we obtain that the subgroup $\langle \Gamma_1', \ldots, \Gamma_n' \rangle$ of G is τ_{mod}-Anosov and naturally isomorphic to the abstract free product $\Gamma_1' * \cdots * \Gamma_n'$, which is the conclusion of [4, Theorem 1.3].
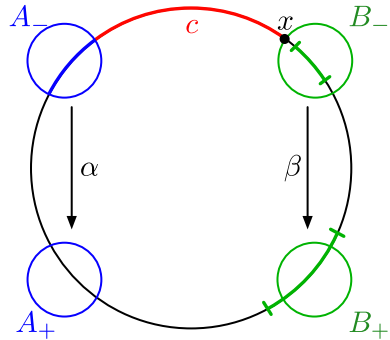
6.3 Another example of a Schottky group

In this section we give an example (claimed in the introduction) of a Schottky group failing the *uniform contraction* property but otherwise satisfying the assumptions of the Combination Theorem.

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[10] That is, the τ_{mod}-limit sets of Γ_i and Γ_j, for all $i \neq j$, are antipodal relative to each other.

Example 6.5 Consider $G = \mathrm{PSL}(2, \mathbf{R})$, the group of orientation-preserving isometries of the hyperbolic plane (in the unit disk model), acting on the unit circle S^1. Take two pairwise disjoint closed round disks $B_\pm$, $A_\pm$ whose boundary circles have the same radii and meet S^1 orthogonally. Take $\alpha, \beta \in G$ such that α sends A_+ to the complement (in the extended complex plane) of A_- and β sends B_+ to the complement of B_-. See Fig. 3. The elements α, β can be chosen so that their restrictions to the boundary circles of A_-, B_+ respectively are isometries of the angular metrics on the respective circles. For instance, one can take α (respectively, β) the composition of the inversion in ∂A_- (respectively, ∂B_-) and a reflection in a line passing through the center of the unit disk. Then the subgroups $\Gamma_A = \langle \alpha \rangle$, $\Gamma_B = \langle \beta \rangle$ and the subsets $A = A_- \cup A_+$, $B = B_- \cup B_+$ satisfy the assumptions of the Combination Theorem in the extended complex plane. It remains to modify A, B to new subsets A', B', still satisfying the assumptions of the Combination Theorem (for the G-action on S^1), so that β is no longer is a strict expansion on B'. We let c denote the (compact) arc on S^1 connecting a point of ∂A_- to a point of ∂B_- and, otherwise, disjoint from $A \cup B$. (Such an arc always exists if we are willing to replace α with α^{-1}.) See again Fig. 3. Now set $A' := (A_+ \cap S^1) \cup (A_- \cap S^1) \cup c$. We also replace $B_- \cap S^1$ with a slightly smaller compact subarc $b_- \subset B_- \cap S^1$ and $B_+ \cap S^1$ with a slightly larger compact subarc $b_+ \subset B_+ \cap S^1$, so that

$$\beta^{\pm 1}(A') \subset b_+ \cup b_-.$$

Lastly, set $B' = b_- \cup b_+$. Then the quadruple Γ_A, Γ_B, A', B' satisfies the assumptions of the Combination Theorem, but β is not expanding on B, specifically, the expansion fails at the intersection point x of c and the circle ∂B_-.

References

1. Baker, M., Cooper, D.: A combination theorem for convex hyperbolic manifolds, with applications to surfaces in 3-manifolds. J. Topol. **1**(3), 603–642 (2008)
2. Danciger, J., Guéritaud, F., Kassel, F.: Convex cocompact actions in real projective geometry. arXiv:1704.08711
3. Dey, S., Kapovich, M.: Klein–Maskit combination theorem for Anosov subgroups: amalgams. arXiv:2301.02354
4. Dey, S., Kapovich, M., Leeb, B.: A combination theorem for Anosov subgroups. Math. Z. **293**(1–2), 551–578 (2019)

5. Eberlein, P.B.: Geometry of Nonpositively Curved Manifolds. Chicago Lectures in Mathematics. University of Chicago Press, Chicago (1996)
6. Gitik, R.: Ping-pong on negatively curved groups. J. Algebra **217**(1), 65–72 (1999)
7. Ivasku, D.: On Klein–Maskit combination theorem. In: Romanian–Finnish Seminar on Complex Analysis, vol. 743, pp. 115–124. Springer Lecture Notes in Mathematics (1976)
8. Kapovich, M., Leeb, B.: Finsler bordifications of symmetric and certain locally symmetric spaces. Geom. Topol. **22**(5), 2533–2646 (2018)
9. Kapovich, M., Leeb, B., Porti, J.: Morse actions of discrete groups on symmetric space. arXiv:1403.7671
10. Kapovich, M., Leeb, B., Porti, J.: Anosov subgroups: dynamical and geometric characterizations. Eur. J. Math. **3**(4), 808–898 (2017)
11. Kapovich, M., Leeb, B., Porti, J.: A Morse lemma for quasigeodesics in symmetric spaces and Euclidean buildings. Geom. Topol. **22**(7), 3827–3923 (2018)
12. Labourie, F.: Anosov flows, surface groups and curves in projective space. Invent. Math. **165**(1), 51–114 (2006)
13. Li, L., Ohshika, K., Wang, X.: On Klein–Maskit combination theorem in space. I. Osaka J. Math. **46**, 1097–1141 (2009)
14. Martin, A., Swiatkowski, J.: Infinitely-ended hyperbolic groups with homeomorphic Gromov boundaries. J. Group Theory **18**, 273–289 (2015)
15. Martinez Pedroza, E.: Combination of quasiconvex subgroups in relatively hyperbolic groups. Ph.D. Thesis, The University of Oklahoma. ProQuest LLC, Ann Arbor (2008)
16. Martínez-Pedroza, E.: Combination of quasiconvex subgroups of relatively hyperbolic groups. Groups Geom. Dyn. **3**(2), 317–342 (2009)
17. Martínez-Pedroza, E., Sisto, A.: Virtual amalgamation of relatively quasiconvex subgroups. Algebr. Geom. Topol. **12**(4), 1993–2002 (2012)
18. Maskit, B.: On Klein's combination theorem. Trans. Am. Math. Soc. **120**, 499–509 (1965)
19. Maskit, B.: On Klein's combination theorem. II. Trans. Am. Math. Soc. **131**, 32–39 (1968)
20. Maskit, B.: On Klein's combination theorem. III. In: Advances in the Theory of Riemann Surfaces (Proc. Conf., Stony Brook, N.Y., 1969), Ann. of Math. Studies, vol. 66, pp. 297–316. Princeton Univ. Press, Princeton (1971)
21. Maskit, B.: Kleinian groups. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 287. Springer, Berlin (1988)
22. Maskit, B.: On Klein's combination theorem. IV. Trans. Am. Math. Soc. **336**(1), 265–294 (1993)
23. Tits, J.: Free subgroups in linear groups. J. Algebra **20**, 250–270 (1972)