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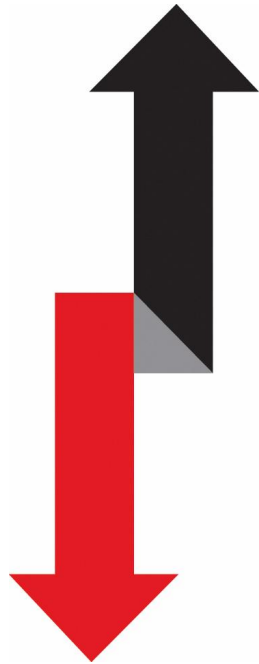
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Time-Varying Risk Premia and Stock Return Autocorrelation

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Abstract

Autocorrelation in stock returns is one important measure of the efficiency of securities markets pricing. Autocorrelation may be a sign of genuine pricing inefficiency: partial price adjustment (PPA), in which trades occur at prices that do not fully reflect the available information. However, autocorrelation may also arise from three other sources: bid-ask bounce (BAB), nonsynchronous trading (NT), and time-varying risk premia (TVRP). TVRP is not an indication of inefficient pricing. It can arise in a securities market equilibrium because the equilibrium returns of the available investments change over time; in particular, the presence of TVRP is entirely compatible with the absence of arbitrage in securities markets. Anderson, Eom, Hahn and Park (2006) provide methods for identifying a portion of the autocorrelation that can only be attributable to PPA and TVRP. This paper provides bounds on TVRP, as a function of the return period, the time horizon over which the autocorrelations are calculated, and the variability of risk premia. We find that the impact of TVRP is negligible in the empirical setting in Anderson, Eom, Hahn and Park (2006), but could be significant in other settings, requiring correction in estimates and hypothesis tests.

KEYWORDS: Stock return autocorrelation, Time-varying risk premia, Efficient Markets Hypothesis

JEL Classification: C10, C81, G14

1 Introduction

Anderson, Eom, Hahn and Park (2006) provide methods to decompose stock return autocorrelation into four components: bid-ask bounce (BAB), the nonsynchronous trading effect (NT), partial price adjustment (PPA), and time-varying risk premia (TVRP). Of these four components, only PPA indicates securities pricing inefficiency, so estimating the magnitude of PPA is of considerable interest. Anderson, Eom, Hahn and Park's methods identify a component of return autocorrelation that can only come from PPA and TVRP, but do not directly distinguish between PPA and TVRP. Thus, TVRP may be viewed as inducing a bias in their measurement of PPA. In this paper, we derive theoretical bounds on the bias induced by TVRP in the measurement of PPA, as a function of the time horizon, return period, and the variability of risk premia. As the reader will see, the derivation involves stochastic calculus and somewhat delicate error estimates.

In the specific empirical context considered in Anderson, Eom, Hahn and Park (daily return autocorrelations calculated over a two-year period, so the return period is one day and the time horizon is two years), we show that the bias induced by TVRP in the measurement of PPA is negligible. However, in other contexts, the contribution of TVRP to return autocorrelation is potentially large enough to matter, so estimates of the role of PPA in return autocorrelation should be adjusted.

It is important to distinguish between time-varying expected rates of return and TVRP. Under the assumption that stock prices follow one of the standard processes in finance (such as a Geometric Itô or Geometric Lévy Process), rejection of the hypothesis that stock return autocorrelation is zero is equivalent to rejection of the hypothesis that the expected rate of return is constant. In other words, if we impose the assumption that the return in each period is composed of an expected rate of return plus a volatility term, where the volatility term is uncorrelated with the returns in disjoint periods, then returns are uncorrelated if and only if the expected rate of return is constant. As noted by Campbell et al. (1997, page 66), the " R^2 of a regression of returns on a constant and its first lag is the square of the slope coefficient, which is simply the first-order autocorrelation." As a consequence, if the first-order autocorrelation coefficient of return is α , the proportion of the variation in return that "is predictable using the preceding day's . . . return" is α^2 . Thus, time-varying expected rates of return and return autocorrelation are simply different faces of a single phenomenon.

However, time-varying expected rates of return and TVRP are distinct phenomena. To illustrate, suppose that stock prices follow Itô Processes of the form $\frac{dS}{S} = \mu dt + \sigma dW$; a similar analysis holds if prices follow other standard processes. The absence of arbitrage is equivalent to the existence of a vector process λ of prices of risk such that $\mu - r = \sigma \lambda'$; here, μ is the vector process of expected rates of return and r is the risk-free rate. The expected rate of return μ will vary as a result of changes in r , σ , and λ , and the resulting variation in μ cannot be exploited by arbitrage; this is the variation attributable to TVRP. Other variation in μ constitutes time-varying expected rates of return, not TVRP; it can be exploited by arbitrage.

Standard autocorrelation tests are designed to test for time-varying expected rates of return, but cannot distinguish TVRP from other forms of time-varying expected rates of return. In particular, if a stock has a run of positive returns, autocorrelation tests will conclude that the stock had a high expected rate of return over that period, but cannot

distinguish whether or not this is the result of a high risk premium. If it is, then it cannot be exploited by arbitrage by informed traders.

If the high expected rate of return is not the result of a high risk premium, then it *can* be exploited by arbitrage. If no one knew the expected rate of return was high, there would be nothing pushing the stock higher, and it would stay relatively stable until the good news underlying the high expected rate of return were announced, at which point the stock price would rise abruptly. If it were widely known that the expected rate of return was high, then many traders would buy the stock, forcing the price to rise abruptly until the future expected rate of return was reduced to the appropriate risk-adjusted level. These abrupt rises in price would be captured econometrically as volatility, and not as autocorrelation. Thus, if we see a string of positive returns establishing statistically significant positive autocorrelation, after eliminating NT and BAB, it can only come from two sources: a period of high risk premia (TVRP), or the strategic decision of a small group of informed traders with positive information to exercise their informational advantage slowly (PPA). Similarly, if we see a string of negative returns establishing statistically significant positive autocorrelation, after eliminating NT and BAB, it can only come from TVRP and PPA.

In some cases, PPA may result in negative autocorrelation. For example, uninformed traders may attempt to exploit the information of informed traders using momentum strategies, which may lead to overshooting and statistically significant negative autocorrelation. As we shall see below, the bias in the Pearson correlation coefficient induced by TVRP is distributed roughly symmetrically around a positive mean. At any given level of significance, the bias induced by TVRP decreases the probability that the Pearson correlation will be significant and negative. Thus, if we find statistically significant negative autocorrelation after eliminating NT and BAB, we have statistically significant confirmation of negative PPA.

The bias resulting from TVRP in the measured autocorrelation depends on the return period, the time horizon over which the autocorrelations are calculated, and the variability of the risk premium over the time horizon:

- The bias becomes larger as the time horizon increases, because the variation of risk premium is larger over longer time horizons.
- The bias becomes larger as the return period increases. Daily returns are much noisier than yearly returns, so the bias coming from variation in mean returns represents a smaller fraction of daily volatility than of volatility over long return periods.
- The effect of a given size bias on hypothesis tests increases as the number of return periods per time horizon increases. For example, suppose we look at daily return autocorrelation over a two-year time horizon, so we have roughly $n = 500$ days and $n - 1 = 499$ daily returns. The standard error in the autocorrelation tests decreases as n increases, so a bias of a given size represents a larger multiple of the standard error as n gets larger.

We find that under plausible assumptions on the variability of risk premia, the bias in daily returns over a two-year time horizon is very small; however, the bias could be substantial in other settings, and point estimates and hypothesis tests need to be corrected in those settings.

2 The Bound on the Bias Induced by TVRP

Definition 2.1 We assume that one year consists of 250 trading days, so that one month is $\frac{250}{12}$ trading days. We consider individual stock and portfolio return autocorrelations, where

- the return period is d trading days; thus, $d = 1$, $d = 5$, and $d = \frac{250}{12}$ correspond to daily, weekly and monthly return autocorrelation;
- time is measured in years;
- the price of a security or portfolio follows the stochastic differential equation

$$\frac{dS}{S} = \mu dt + \sigma dW$$

where W is a standard Brownian Motion and μ and σ are continuous deterministic functions of time; thus, the mean return in the k^{th} return period is $\mu\left(\frac{kd}{250}\right)$; and

- the time horizon over which the autocorrelations are calculated is y years.
- Let $n = 250y/d$ denote the number of return periods in the time horizon. In the analysis that follows, we assume that $n \geq 104$; this corresponds to monthly returns over a period of at least nine years, weekly returns over a period of at least two years, and daily returns over a period of at least five months. This is a mild restriction in practice, since for $n < 104$, the standard errors in return autocorrelation tests are too large to permit effective hypothesis testing.

Let

$$\begin{aligned} \mu_k &= \mu\left(\frac{kd}{250}\right) \\ \sigma_k &= \sigma\left(\frac{kd}{250}\right) \\ W_k &= W\left(\frac{kd}{250}\right) \\ \bar{\mu} &= \frac{1}{n-2} \left(\frac{\mu_1}{2} + \sum_{k=2}^{n-2} \mu_k + \frac{\mu_{n-1}}{2} \right) \\ &\simeq \frac{1}{y} \int_0^y \mu(t) dt \\ \Delta W_k &= W_{k+1} - W_k \\ \bar{v} &= \frac{1}{n-2} \left(\frac{\sigma_1 \Delta W_1}{2} + \sum_{k=2}^{n-2} \sigma_k \Delta W_k + \frac{\sigma_{n-1} \Delta W_{n-1}}{2} \right) \\ &\simeq \frac{1}{n-2} \int_0^y \sigma dW \\ r_k &= \frac{\mu_k d}{250} + \sigma_k \Delta W_k \end{aligned}$$

$$\begin{aligned}
\bar{r} &= \frac{1}{n-2} \left(\frac{r_1}{2} + \sum_{k=2}^{n-2} r_k + \frac{r_{n-1}}{2} \right) \\
&= \frac{\bar{\mu}d}{250} + \bar{v} \\
\sigma_r^2 &= \frac{1}{n-2} \left(\frac{(r_1 - \bar{r})^2}{2} + \sum_{k=2}^{n-2} (r_k - \bar{r})^2 + \frac{(r_{n-1} - \bar{r})^2}{2} \right) \\
&= \frac{1}{n-2} \left(\frac{1}{2} \left(\frac{(\mu_1 - \bar{\mu})d}{250} + \sigma_1 \Delta W_1 - \bar{v} \right)^2 + \sum_{k=2}^{n-2} \left(\frac{(\mu_k - \bar{\mu})d}{250} + \sigma_k \Delta W_k - \bar{v} \right)^2 \right. \\
&\quad \left. + \frac{1}{2} \left(\frac{(\mu_{n-1} - \bar{\mu})d}{250} + \sigma_{n-1} \Delta W_{n-1} - \bar{v} \right)^2 \right) \\
&\simeq \frac{1}{y} \int_0^y \left(\frac{(\mu(t) - \bar{\mu})d}{250} \right)^2 dt + \frac{2}{n-2} \int_0^y \left(\frac{(\mu(t) - \bar{\mu})d}{250} \right) \sigma(t) dW - \frac{2\bar{v}}{y} \int_0^y \frac{(\mu(t) - \bar{\mu})d}{250} dt \\
&\quad + \frac{1}{n-2} \left(\frac{\sigma_1^2 (\Delta W_1)^2}{2} + \sum_{k=2}^{n-2} \sigma_k^2 (\Delta W_k)^2 + \frac{\sigma_{n-1}^2 (\Delta W_{n-1})^2}{2} \right) - \bar{v}^2 \\
&\simeq \frac{1}{y} \int_0^y \left(\frac{(\mu(t) - \bar{\mu})d}{250} \right)^2 dt + \frac{2}{n-2} \int_0^y \left(\frac{(\mu(t) - \bar{\mu})d}{250} \right) \sigma(t) dW \\
&\quad + \frac{1}{n-2} \int_0^y \sigma^2(t) dt - \bar{v}^2
\end{aligned}$$

Then the Pearson sample autocorrelation coefficient is given by

$$\begin{aligned}
r_p &= \frac{\sum_{k=1}^{n-2} \left(r_k - \frac{1}{n-2} \sum_{j=1}^{n-2} r_j \right) \left(r_{k+1} - \frac{1}{n-2} \sum_{j=2}^{n-1} r_j \right)}{\sqrt{\sum_{k=1}^{n-2} \left(r_k - \frac{1}{n-2} \sum_{j=1}^{n-2} r_j \right)^2} \sqrt{\sum_{k=2}^{n-1} \left(r_k - \frac{1}{n-2} \sum_{j=2}^{n-1} r_j \right)^2}} \\
&\simeq \frac{\sum_{k=1}^{n-2} (r_k - \bar{r}) (r_{k+1} - \bar{r})}{\sqrt{\sum_{k=1}^{n-2} (r_k - \bar{r})^2} \sqrt{\sum_{k=2}^{n-1} (r_k - \bar{r})^2}} \\
&\simeq \frac{\sum_{k=1}^{n-2} (r_k - \bar{r}) (r_{k+1} - \bar{r})}{(n-2) \sigma_r^2} \\
&= \frac{\sum_{k=1}^{n-2} \left(\frac{(\mu_k - \bar{\mu})d}{250} + \sigma_k \Delta W_k - \bar{v} \right) \left(\frac{(\mu_{k+1} - \bar{\mu})d}{250} + \sigma_{k+1} \Delta W_{k+1} - \bar{v} \right)}{(n-2) \sigma_r^2} \\
&\simeq \frac{1}{\sigma_r^2} \left(\frac{1}{y} \int_0^y \left(\frac{(\mu(t) - \bar{\mu})d}{250} \right)^2 dt + \frac{2}{n-2} \int_0^y \left(\frac{(\mu(t) - \bar{\mu})d}{250} \right) \sigma(t) dW \right. \\
&\quad \left. - \frac{2\bar{v}}{y} \int_0^y \left(\frac{(\mu(t) - \bar{\mu})d}{250} \right) dt + \frac{1}{n-2} \sum_{k=1}^{n-2} \sigma_k \sigma_{k+1} \Delta W_k \Delta W_{k+1} - \bar{v}^2 \right) \\
&= \frac{1}{\sigma_r^2} \left(\frac{1}{y} \int_0^y \left(\frac{(\mu(t) - \bar{\mu})d}{250} \right)^2 dt + \frac{2}{n-2} \int_0^y \left(\frac{(\mu(t) - \bar{\mu})d}{250} \right) \sigma(t) dW \right. \\
&\quad \left. + \frac{1}{n-2} \sum_{k=1}^{n-2} \sigma_k \sigma_{k+1} \Delta W_k \Delta W_{k+1} - \bar{v}^2 \right)
\end{aligned}$$

$$\simeq \frac{A + Z}{B + Z}$$

where

$$\begin{aligned} A &= \frac{1}{n-2} \sum_{k=1}^{n-2} \sigma_k \sigma_{k+1} \Delta W_k \Delta W_{k+1} - \bar{v}^2 \\ B &= \frac{1}{n-2} \int_0^y \sigma(t)^2 dt - \bar{v}^2 \\ Z &= Z_1 + Z_2 \\ Z_1 &= \frac{1}{y} \int_0^y \left(\frac{(\mu(t) - \bar{\mu})d}{250} \right)^2 dt \\ Z_2 &= \frac{2}{n-2} \int_0^y \left(\frac{(\mu(t) - \bar{\mu})d}{250} \right) \sigma(t) dW \end{aligned}$$

In the absence of TVRP, we would have $\mu(t) = \bar{\mu}$ for all t , hence Z_1 and Z_2 would be identically zero. In the presence of TVRP, Z_1 and Z_2 induce a bias into our measurement of PPA. Notice that neither Z_1 nor Z_2 depends on the rate at which μ changes, only on the distribution of μ and (in the case of Z_2) the correlation between μ and σ . Z_1 is a positive constant, the variance of the mean return per return period (day, week, month etc.). Z_2 is normally distributed with mean zero and standard deviation

$$\sigma_Z = \frac{2}{n-2} \sqrt{\int_0^y \left(\frac{(\mu(t) - \bar{\mu})d}{250} \right)^2 \sigma^2(t) dt}$$

Moreover, the conditional distribution of Z_2 , conditional on A and B , is asymptotically normal.

Anderson, Eom, Hahn and Park (2006) make two types of tests. The first type compute portfolio return autocorrelations and test the hypothesis that it is less than or equal to zero. In those cases where the hypothesis is rejected (portfolios of small and medium stocks), the rejection is overwhelming and it is easy to see that the small bias induced by TVRP, i.e. by Z cannot make any difference in those results. The second type compute a large number of return autocorrelations, and count the number in which the hypothesis that the return is less than or equal to zero is rejected at the one-sided 2.5% level. The effect of the bias on these tests is more delicate, and we need to carefully estimate the effect of the bias on the expected number of rejections.

The Pearson test compares $\frac{\sqrt{n-2}r_p}{\sqrt{1-r_p^2}}$ to the standard normal. We have

$$\begin{aligned} \sqrt{n-2} \frac{r_p}{\sqrt{1-r_p^2}} &= \sqrt{n-2} \frac{\frac{A+Z}{B+Z}}{\sqrt{1 - \frac{(A+Z)^2}{(B+Z)^2}}} \\ &= \sqrt{n-2} \frac{A+Z}{\sqrt{(B+Z)^2 - (A+Z)^2}} \end{aligned}$$

Let

$$g_{AB}(Z) = \sqrt{n-2} \frac{A+Z}{\sqrt{(B+Z)^2 - (A+Z)^2}}$$

and let

$$h_{AB}(Z) = \sqrt{\frac{n-2}{B^2-A^2}} \left(A + \frac{B}{B+A} Z \right)$$

be the first-order Taylor approximation to g_{AB} at $Z = 0$.

The correct test would measure only the autocorrelation coming from PPA. The actual test measures the autocorrelation coming from PPA and TVRP. Since $Z_1 \geq 0$, and $|A| \leq B$, the presence of Z_1 leads the actual test to reject the null hypothesis in situations in which the correct test would fail to reject; in any situation in which the correct test rejects, the actual test will also reject. Thus, the presence of Z_1 increases the expected number of rejections.

$|Z_2|$ is typically larger than Z_1 . However, we shall see that Z_2 induces a smaller bias in the expected number of rejections because Z_2 can be either positive or negative. The presence of Z_2 leads the actual test to reject in some cases in which the correct test does not reject, and to fail to reject in some cases in which the correct test does reject. The symmetry of Z_2 will imply that the two effects very nearly cancel.

Noting that A and B are random variables, let $Y = \frac{\sqrt{n-2}A}{\sqrt{B^2-A^2}}$ be the random variable $g_{AB}(0) = h_{AB}(0)$. Let N denote the cumulative distribution function of the standard normal. The probability that Z changes an insignificant value to a significant value, using the critical value α , is the probability that $Y = g_{AB}(0) < \alpha$ and $g_{AB}(Z) \geq \alpha$. Since g_{AB} is increasing in Z , this probability only depends on the value Z_α for which $g_{AB}(Z_\alpha) = \alpha \simeq 1.96$. Recall that we assumed that $n \geq 104$, so $\sqrt{n-2} > 10$.

$$\frac{g_{AB}(Z_\alpha)}{\sqrt{n-2}} \simeq \frac{1.96}{\sqrt{n-2}} < 0.2$$

It is easy to check numerically that for $0 \leq A \leq 0.2B$ and $0 \leq Z \leq 10B$, g_{AB} is concave in Z . If the presence of Z changes an insignificant value to significant, then $Z \geq 0$ and $Y \leq \alpha \simeq 1.96$; since $n \geq 104$, this implies that $A \leq 0.2B$. We assume the following:

$$|Z_2| \leq 4\sigma_Z, g_{AB}(0) < \alpha, g_{AB}(Z) \geq \alpha \Rightarrow A \geq 0, Z \leq 10B \quad (1)$$

and the presence of Z changes an insignificant value to significant, then $A \geq 0$. Equation (1) thus implies that $g_{AB}(Z_\alpha) \leq h_{AB}(Z_\alpha)$. We believe that Equation (1) is satisfied in every reasonable empirical situation. We shall show in Section 3 how to check it in a given situation, and find that is overwhelmingly satisfied in the specific empirical situation of Anderson, Eom, Hahn and Park (2006).

The probability that Z changes an insignificant value to a significant value is bounded above by

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}\sigma_Z} \int_0^\infty e^{-z^2/2\sigma_Z^2} \left(N(\alpha) - N\left(\alpha - \frac{\sqrt{n-2}B}{\sqrt{B^2-A^2}(B+A)} (Z_1 + z)\right) \right) dz \\ & \leq 1 - N(4) + \frac{1}{\sqrt{2\pi}\sigma_Z} \int_0^{4\sigma_Z} e^{-z^2/2\sigma_Z^2} (N(\alpha) - N(\alpha - \gamma_1 (Z_1 + z))) dz \end{aligned}$$

where γ_1 is the value of $\frac{\sqrt{n-2}B}{\sqrt{B^2-A^2}(B+A)}$ corresponding to $Y = \alpha - \gamma_1 (Z_1 + 4\sigma_Z)$. Although it is hard to solve for γ_1 exactly, we can estimate it using $0 \leq A \leq 0.2B$ as follows:

$$\frac{\partial}{\partial A} \left(\frac{\sqrt{n-2}B}{\sqrt{B^2-A^2}(B+A)} \right)$$

$$\begin{aligned}
&= \sqrt{n-2}B \left(\frac{\partial}{\partial A} \left((B^2 - A^2)^{-1/2} (B+A)^{-1} \right) \right) \\
&= \sqrt{n-2}B \left(A (B^2 - A^2)^{-3/2} (B+A)^{-1} - (B^2 - A^2)^{-1/2} (B+A)^{-2} \right) \\
&\geq -\sqrt{n-2}B (B^2 - A^2)^{-1/2} (B+A)^{-2} \\
&\geq -\frac{1.0207\sqrt{n-2}}{B^2}
\end{aligned}$$

It follows that

$$\gamma_1 \leq \gamma_0 + \frac{1.0207\sqrt{n-2}}{B^2} (Z_1 + 4\sigma_Z)$$

where γ_0 is the value of $\frac{\sqrt{n-2}B}{\sqrt{B^2-A^2}(B+A)}$ corresponding to $Y = \alpha$; for $\alpha = 1.96$, we have

$$\begin{aligned}
A &= \frac{1.96B}{\sqrt{n-2+1.96^2}} \\
\gamma_0 &= \frac{\sqrt{n-2}B}{\sqrt{B^2-A^2}(B+A)} \\
&= \frac{\sqrt{n-2}}{B} \frac{1}{\sqrt{1 - \frac{1.96^2}{n-2+1.96^2}} \left(1 + \frac{1.96}{\sqrt{n-2+1.96^2}}\right)} \\
&\leq \frac{\sqrt{n-2}}{B} \frac{1}{\sqrt{1 - \frac{1.96^2}{n-2+1.96^2}}} \\
&\leq \frac{1.019\sqrt{n-2}}{B}
\end{aligned}$$

For the specific empirical setting considered in Anderson, Eom, Hahn and Park (2006), $n = 500$: for $n \geq 500$, we can do slightly better:

$$\begin{aligned}
\gamma_1 &\leq \gamma_0 + \frac{0.9950\sqrt{n-2}}{B^2} (Z_1 + 4\sigma_Z) \\
\gamma_0 &\leq \frac{.9231\sqrt{n-2}}{B}
\end{aligned}$$

Similarly, the probability that Z changes an insignificant value to a significant value is bounded below by

$$\frac{1}{\sqrt{2\pi}\sigma_Z} \int_0^{4\sigma_Z} e^{-z^2/2\sigma_Z^2} (N(\alpha + \gamma_2(z - Z_1)) - N(\alpha)) dz$$

where

$$\gamma_2 \geq \gamma_0 - \frac{1.04318\sqrt{n-2}}{B^2} (4\sigma_Z - Z_1)$$

Thus,

$$0 \leq \gamma_1 - \gamma_2 \leq \frac{8.26\sqrt{n-2}\sigma_Z}{B^2}$$

For $n \geq 500$, 1.04318 improves to 1.009 and 8.26 improves to 8.02.

Using the second order Taylor Expansion for the normal cumulative distribution function, the increase in the probability of rejection resulting from Z is bounded above by

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi}\sigma_Z} \int_0^{4\sigma_Z} e^{-z^2/2\sigma_Z^2} (2N(\alpha) - N(\alpha - \gamma_1(Z_1 + z)) - N(\alpha + \gamma_2(z - Z_1))) dz \\
& \quad + 1 - N(4) \\
& \leq \frac{1}{\sqrt{2\pi}\sigma_Z} \int_0^\infty e^{-z^2/2\sigma_Z^2} (N'(\alpha)(\gamma_1 + \gamma_2)Z_1 + (\gamma_1 - \gamma_2)z) dz \\
& \quad - \frac{1}{\sqrt{2\pi}\sigma_Z} \int_0^\infty e^{-z^2/2\sigma_Z^2} \frac{N''(\alpha)}{2} ((\gamma_1(Z_1 + z))^2 + (\gamma_2(z - Z_1))^2) dz \\
& \quad + \frac{1}{\sqrt{2\pi}\sigma_Z} \int_0^{4\sigma_Z} e^{-z^2/2\sigma_Z^2} \frac{N'''(\xi(z))}{6} (\gamma_1(Z_1 + z))^3 dz + 10^{-4} \\
& \quad \text{for some measurable function } \xi : [0, \infty) \rightarrow \mathbf{R} \\
& \leq N'(\alpha) \left(\frac{\gamma_1 + \gamma_2}{2} Z_1 + \frac{(\gamma_1 - \gamma_2)\sigma_Z}{\sqrt{2\pi}} \right) \\
& \quad - \frac{N''(\alpha)}{2} \left(\frac{\gamma_1^2 + \gamma_2^2}{2} Z_1^2 + \frac{2\sigma_Z}{\sqrt{2\pi}} (\gamma_1^2 - \gamma_2^2) Z_1 + \frac{\gamma_1^2 + \gamma_2^2}{2} \sigma_Z^2 \right) \\
& \quad + \frac{N'''(\xi(z))\gamma_1^3}{6} \left(\frac{Z_1^3}{2} + \frac{3Z_1^2\sigma_Z}{\sqrt{2\pi}} + \frac{3Z_1\sigma_Z^2}{2} + \frac{2\sigma_Z^3}{\sqrt{2\pi}} \right) + 10^{-4} \\
& \leq \frac{e^{-\alpha^2/2}}{\sqrt{2\pi}} \left(\frac{\gamma_1 + \gamma_2}{2} Z_1 + \frac{(\gamma_1 - \gamma_2)\sigma_Z}{\sqrt{2\pi}} \right) \\
& \quad + \frac{\alpha e^{-\alpha^2/2}}{2\sqrt{2\pi}} \left(\frac{\gamma_1^2 + \gamma_2^2}{2} (Z_1^2 + \sigma_Z^2) + \frac{2\sigma_Z}{\sqrt{2\pi}} (\gamma_1^2 - \gamma_2^2) Z_1 \right) \\
& \quad + \frac{e^{-3/2}\gamma_1^3}{3\sqrt{2\pi}} \left(\frac{Z_1^3}{2} + \frac{3Z_1^2\sigma_Z}{\sqrt{2\pi}} + \frac{3Z_1\sigma_Z^2}{2} + \frac{2\sigma_Z^3}{\sqrt{2\pi}} \right) + 10^{-4}
\end{aligned}$$

3 An Example

In this section, we present an example showing how to apply the estimates just developed in a particular empirical setting. Anderson, Eom, Hahn and Park (2006) considered the specific situation of daily return autocorrelations of NYSE stocks over a two-year time horizon, so we have $d = 1$, $n = 500$.

We must first determine an upper bound on the plausible magnitude of the variation of the risk premium of NYSE stocks and portfolios over a two-year time horizon. Stocks and portfolios could conceivably have expected rates of return below the risk-free rate, but only if they were negatively correlated with undiversifiable risks and thus provided insurance against those risks. The portfolios considered in Anderson, Eom, Hahn and Park (2006) are stratified by firm size, but they are otherwise diversified. The portfolios are positively correlated with two important undiversifiable risks (the market as a whole and aggregate income), so it is implausible that investors would hold the portfolios if they had an expected rate of return below the risk-free rate. To maintain equilibrium, stock prices would have to

fall to raise the future expected rate of return sufficiently to induce stockholders to retain their holdings. On the other hand, if the expected rate of return of the portfolio exceeded the risk-free rate by 15% per annum, investors would surely choose to substantially increase their stockholdings: taking the volatility into account, there is very little chance of a substantial decline in stock prices and a very good chance of a substantial gain.

For U.S. stocks, it is natural to take the three-month Treasury Bill Rate as the risk-free rate. Anderson, Eom, Hahn and Park (2006) divided a ten-year data period (1993-2002) into five two-year subperiods, and computed autocorrelations over these two-year time horizons. The variations (max-min) in the three-month Treasury Bill rates for their five two-year subperiods are as follows: 3.34% (6.39%-3.05%) (93-94), 1.28% (6.40%-5.12%) (95-96), 0.86% (5.83%-4.97%) (97-98), 1.99% (6.84%-4.85%) (99-00), and 4.96% (6.27%-1.31%) (01-02). The average of the subperiod variations is 2.49%. Thus, TVRP should induce variation in the expected return of the portfolios of no more than 18% per annum over a two-year period.

In the case of an individual stock, the expected rate of return should reflect the risk premia of the factors underlying its pricing. Some stocks may have low—even negative—risk premia, while others may have large risk premia. However, the correlation of any given stock with the main risk factors should be relatively stable over time periods of a year or two.¹ Thus, TVRP should induce variation in the expected returns of the individual stocks of no more than 18% per annum over a two-year period.

Assuming that μ is distributed uniformly over an interval of length $18\% = .18$ per annum, $Z_1 = 4.320 \times 10^{-8}$. Assuming that μ is uniformly distributed over an interval of length .18 per annum and σ is constant, $\sigma_Z = 1.180 \times 10^{-6}\sigma$.

Notice that A and B are quadratic in σ (i.e. if we double the function $\sigma(t)$ at all times, then A and B are quadrupled), while Z_1 is independent of σ and Z_2 is linear in σ . Thus, the bias induced in r_p by Z is maximized when σ is minimized. The returns on individual stocks in a portfolio are more volatile than the returns of the portfolio, and the returns of smaller stocks are more volatile than the returns of larger stocks. We take the volatility of the S&P 500 index as a lower bound on the volatility of the individual stocks in our analysis. For the S&P 500 index, the average value of σ_r , over our five two-year subperiods, is .01041. Assuming σ is constant, we obtain the estimate $\sigma = \sqrt{249} \ln(1.01041) \simeq .16341$. With probability $2 \times (1 - N(4)) > 1 - 2 \times 10^{-4}$, $|B - (\sigma_r)^2| = |Z| \leq 4.320 \times 10^{-8} + 4 \times 1.18 \times 10^{-6}\sigma = 8.145 \times 10^{-7}$, so $B = (1.084 \pm .008) \times 10^{-4}$; since the bias is maximized when B is minimized, we assume $B = 1.076 \times 10^{-4}$, $\sigma = .1600$, and $\sigma_Z = 1.888 \times 10^{-7}$. $\gamma_2 \leq \gamma_0 = \frac{.9231\sqrt{498}}{B} = 1.915 \times 10^5$, $\gamma_1 \leq \gamma_0 + \frac{.9950\sqrt{498}}{B^2} (Z_1 + 4\sigma_z) = 1.930 \times 10^5$, $0 \leq \gamma_1 - \gamma_2 \leq \frac{8.02\sqrt{498}\sigma_z}{B^2} = 0.0364 \times 10^5$, $0 \leq \gamma_1^2 - \gamma_2^2 = (\gamma_1 - \gamma_2)(\gamma_1 + \gamma_2) \leq 1.400 \times 10^9$.

¹The most likely reason for a major change in the correlation of a stock with the risk factors is diversification into a new line of business, or the sale or spin-off of a line of business. In order to significantly change the correlations, the divested or acquired line of business would have to be a reasonably large fraction of the business of the firm as a whole, and such events are likely to affect relatively few firms over time horizons of a year or two. To further control for changes in the factor loadings of individual stocks, Anderson, Eom, Hahn and Park (2006) eliminated in each two-year time horizon any firm for which the number of outstanding shares changes by more than 10%. This was intended to eliminate most firms that acquired substantial new business lines through acquisition, and many of those that divested substantial business lines through sale or spin-off.

We need to check that Equation (1) is satisfied. Assuming that $Z_2 \leq 4\sigma_Z$,

$$\begin{aligned}
\frac{Z}{B} &\leq \frac{Z_1 + 4\sigma_Z}{B} \\
&= \frac{7.984 \times 10^{-7}}{1.076 \times 10^{-4}} \\
&< 10^{-3} \ll 10 \\
g_{AB}(Z) > 0, A < 0 &\Rightarrow g_{AB}(0) < 0, g_{AB}(Z) \geq \alpha \\
&\Rightarrow g'_{0B}(0) \geq \frac{\alpha}{Z_1 + 4\sigma_Z} = 2.455 \times 10^8 \\
g'_{0B}(0) &= \frac{\sqrt{n-2}}{B} = \frac{\sqrt{498}}{1.076 \times 10^{-4}} = 2.074 \times 10^5 \ll 10^8
\end{aligned}$$

Thus, Equation (1) is satisfied.

Then the increase in the probability of rejection resulting from Z is at most

$$\begin{aligned}
&\frac{e^{-\alpha^2/2}}{\sqrt{2\pi}} \left(\frac{\gamma_1 + \gamma_2}{2} Z_1 + \frac{(\gamma_1 - \gamma_2)\sigma_Z}{\sqrt{2\pi}} \right) \\
&\quad + \frac{\alpha e^{-\alpha^2/2}}{2\sqrt{2\pi}} \left(\frac{\gamma_1^2 + \gamma_2^2}{2} (Z_1^2 + \sigma_Z^2) + \frac{2\sigma_Z}{\sqrt{2\pi}} (\gamma_1^2 - \gamma_2^2) Z_1 \right) \\
&\quad + \frac{e^{-3/2}\gamma_1^3}{3\sqrt{2\pi}} \left(\frac{Z_1^3}{2} + \frac{3Z_1^2\sigma_Z}{\sqrt{2\pi}} + \frac{3Z_1\sigma_Z^2}{2} + \frac{2\sigma_Z^3}{\sqrt{2\pi}} \right) + 3 \times 10^{-4} \\
&= .0584 (8.305 \times 10^{-3} + 2.742 \times 10^{-4}) \\
&\quad + .0573 (1.386 \times 10^{-3} + 9.111 \times 10^{-6}) \\
&\quad + 2.133 \times 10^{14} (4.031 \times 10^{-23} + 4.217 \times 10^{-22} \\
&\quad \quad + 2.310 \times 10^{-21} + 5.370 \times 10^{-21}) \\
&\quad + 3 \times 10^{-4} \\
&= 5.010 \times 10^{-4} + 0.795 \times 10^{-4} + 0.017 \times 10^{-4} + 3 \times 10^{-4} \\
&= 8.762 \times 10^{-4}
\end{aligned}$$

Thus, the bias induced by TVRP increases the probability of rejection from .025 by less than .001 to .026, so the expected number of rejections in 100 autocorrelations increases by at most .1 from 2.5 to 2.6. The tests of individual stock autocorrelations and the tests using SPDRs in Anderson, Eom, Hahn and Park (2006) are all based on comparing the number of rejections to 2.5. Changing 2.5 to 2.6 to adjust for the bias increases the p -values slightly but makes no qualitative change in those findings.

References

- [1] Anderson, Robert M., Kyong Shik Eom, Sang Buhm Hahn and Jong-Ho Park, 2006. "Stock Return Autocorrelation is Not Spurious," preprint.

- [2] Campbell, J., Lo, A., MacKinlay, C., 1997. The Econometrics of Financial Markets. Princeton University Press, Princeton, NJ.