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2010

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Topics in String Duality: D-Branes and Geometry

by

Christopher John Beem

A dissertation submitted in partial satisfaction

of the requirements for the degree of

Doctor of Philosophy

in

Physics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Mina Aganagic, Chair

Professor Ori Ganor

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Spring 2010

Topics in String Duality: D-Branes and Geometry

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Abstract

Topics in String Duality: D-Branes and Geometry

by

Christopher John Beem

Doctor of Philosophy in Physics

University of California, Berkeley

Professor Mina Aganagic, Chair

One of the most striking and surprising aspects of string theory is the vast web of string dualities which relates many seemingly distinct string backgrounds. In this dissertation, I explore several novel dualities between systems of D-branes and closed string flux compactifications. I use geometric transitions to analyze a variety of supersymmetry-breaking D-brane constructions. I additionally present an explanation of the Calabi-Yau fourfold geometries which appear in the computation of superpotentials for D-branes on compact Calabi-Yau threefolds.

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Introduction

The interplay of string dualities and D-branes has been an important source of insight into string theory for many years. One of the most interesting aspects of this interplay is that in some scenarios, D-brane configurations are related to closed string backgrounds which can be described in purely geometric terms. Iconic examples of this type of relation are the AdS/CFT correspondence [1], large N Chern-Simons/topological string duality [2], and the description of type IIB orientifolds in terms of F-theory [3]. In this dissertation, I investigate a number of relations of this type.

In chapter one of this thesis, I construct metastable configurations of branes and anti-branes wrapping two-cycles inside local Calabi-Yau manifolds and study their large N duals. These duals are Calabi-Yau manifolds in which the wrapped two-cycles have been replaced by three-cycles with flux through them, and supersymmetry is spontaneously broken. The geometry of the non-supersymmetric vacuum is exactly calculable to all orders in the 't Hooft parameter, and to leading order in $1/N$. The computation utilizes the same matrix model techniques that were used in the supersymmetric context. This provides a novel mechanism for breaking supersymmetry in the context of flux compactifications. These investigations were carried out in collaboration with Mina Aganagic, Jihye Seo, and Cumrun Vafa. They have previously appeared in [4].

The second chapter describes an extension of this work to a more complex class of local Calabi-Yau geometries. With only branes present, the Calabi-Yau manifolds in question give rise to $\mathcal{N} = 2$ ADE quiver theories deformed by superpotential terms. I show that the large N duality conjecture of chapter one reproduces correctly the known qualitative features of the brane/antibrane physics. In the supersymmetric case, the gauge theories have Seiberg dualities, which are represented as flops in the geometry. Moreover, the holographic dual geometry encodes the whole RG flow of the gauge theory. In the non-supersymmetric case, the large N duality predicts that the brane/antibrane theories also enjoy such dualities, and allows one to pick out the good description at a given energy scale. These investigations were carried out in collaboration with Mina Aganagic and Ben Freivogel. They have previously appeared in [5].

In the third chapter, I show that the physics of D-brane theories that exhibit dynamical SUSY breaking due to stringy instanton effects is well captured by geometric transitions, which recast the nonperturbative superpotential as a classical flux superpotential. This allows for simple engineering of Fayet, Polonyi, O’Raifeartaigh, and other canonical models of supersymmetry breaking in which an exponentially small scale of breaking can be understood either as coming from stringy instantons or as arising from the classical dynamics of fluxes. These investigations were carried out in collaboration with Mina Aganagic and Shamit Kachru. They have previously appeared in [6].

In chapter four, geometric transitions are again used to study metastable vacua in string theory and certain confining gauge theories. The gauge theories in question are, again, $\mathcal{N} = 2$ supersymmetric theories deformed to $\mathcal{N} = 1$ by superpotential terms. I first geometrically engineer supersymmetry-breaking vacua by wrapping D5 branes on rigid two-cycles in non-compact Calabi-Yau geometries, such that the central charges of the branes are misaligned. In a limit of slightly misaligned charges, this has a gauge theory description, where supersymmetry is broken by Fayet-Iliopoulos D-terms. Geometric transitions relate these configurations to dual Calabi-Yaus with fluxes, where H_{RR} , H_{NS} , and dJ are all nonvanishing. I argue that the dual geometry can be effectively used to study the resulting non-supersymmetric, confining vacua. These investigations were carried out in collaboration with Mina Aganagic. They have previously appeared in [7].

In the fifth chapter, I study $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theories coupled to an adjoint chiral field with superpotential. I consider the full supersymmetric moduli space of these theories obtained by adding all allowed chiral operators. These include higher-dimensional operators that introduce a field-dependence for the gauge coupling. I show how Feynman diagram/matrix model/string theoretic techniques can all be used to compute the IR glueball superpotential. Moreover, in the limit of turning off the superpotential, this leads to a deformation of $\mathcal{N} = 2$ Seiberg-Witten theory. In the case where the superpotential drives the squared gauge coupling to a negative value, I find that supersymmetry is spontaneously broken, which can be viewed as a novel mechanism for breaking supersymmetry. I propose a new duality between a class of $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theories with field-dependent gauge couplings and a class of $U(N)$ gauge theories where supersymmetry is softly broken by nonzero expectation values for auxiliary fields in spurion superfields. These investigations were carried out in collaboration with Mina Aganagic, Jihye Seo, and Cumrun Vafa. They have previously appeared in [8].

The sixth and final chapter addresses the problem of computing spacetime superpotentials for D-branes wrapping cycles in a compact Calabi-Yau threefold, which is computed by the disk partition function of the open topological string. I use string duality to show that when appropriately formulated, the problem admits a natural geometrization in terms of a non-compact Calabi-Yau fourfold without D-branes. The duality relates the D-brane superpotential to a flux superpotential on the fourfold. This sheds light on several features of superpotential computations appearing in the literature, in particular on the observation that Calabi-Yau fourfold geometry enters the problem. In one of the examples, I show that the geometry of fourfolds also reproduces D-brane superpotentials obtained from matrix factorization methods. These investigations were carried out in collaboration with Mina Aganagic. They have previously appeared in [9].

Acknowledgments

First and foremost, I want to thank my advisor, Mina Aganagic, who has taught me more about doing physics than I ever expected to know, and whose direction and advice have shaped my growth as a physicist in every way. I also want to thank Shamit Kachru, whose enthusiasm and willingness to humor a confused, but curious, undergraduate set me off on this path five years ago. Additionally, doing physics for the past four years has been a great joy because of my brilliant collaborators: Ben Freivogel, Jihye Seo, and Cumrun Vafa, as well as Mina and Shamit. I owe them all endless thanks for all I've learned from them.

I've been very fortunate to work in an environment as enjoyable and vibrant as the Berkeley Center for Theoretical Physics, with all of its short and long term members. In particular, I want to thank Raphael Bousso, Ori Ganor and Petr Hořava for their generosity, kindness, and instruction. I have also benefitted greatly from discussions with many other graduate students at Stanford and Berkeley: Matt Buckley, Daniel Butter, Brian Feldstein, Marat Freytsis, Daniel Green, Jonathan Hsu, Ben Kain, Stefan Leichenauer, Charles Melby-Thompson, David Poland, Surjeet Rajendran, David Starr, and Daniel Stolarski. I owe special thanks to Ben Freivogel and Yu Nakayama for all their help and for putting up with an unreasonable number of my questions. And finally, life in the CTP would not have been the same were it not for my vivacious and insightful office mate, Jeremy Mardon.

Finally, I want to thank my Mom and Dad for their unwavering support, my brother for all of his inspiration over the years (intentional or not), and Grandma Po and Grandpa Doc for their unconditional love and enthusiastic encouragement of all my pursuits.

Chapter 1

Geometric Metastability and Holography

One of the central questions currently facing string theory is how to break supersymmetry in a controllable way. The most obvious ways to break it typically lead to instabilities signaled by the appearance of tachyons in the theory. One would like to find vacua in which supersymmetry is broken, but stability is not lost. It seems difficult (or impossible, at present) to obtain exactly stable non-supersymmetric vacua from string theory. Therefore, the only candidates would appear to be metastable non-supersymmetric vacua. This idea has already been realized in certain models (See [10] for a review and the relevant literature). More recently, the fact that metastable vacua are also generic in ordinary supersymmetric gauge theories [11] has added further motivation for taking this method of breaking supersymmetry seriously within string theory. Potential realizations of such metastable gauge theories have been considered in string theory [12,13,14] (see also [15,16]).

The aim of this chapter is to study an alternative approach to breaking supersymmetry via metastable configurations, as suggested in [17]. In this scenario, we wrap branes and anti-branes on cycles of local Calabi-Yau manifolds, and metastability is a consequence of the Calabi-Yau geometry. In a sense, this is a geometrically induced metastability. The branes and the anti-branes are wrapped over two-cycles which are rigid and separated. In order for the branes to annihilate, they have to move, which costs energy as the relevant minimal two-spheres are rigid. This leads to a potential barrier due to the stretching of the brane and results in a configuration which is metastable. It is particularly interesting to study the same system at large N , where we have a large number of branes and anti-branes. In this case, it is better to use a dual description obtained via a geometric transition in which the two-spheres shrink and are replaced by three-spheres with fluxes through them. The dual theory has $\mathcal{N} = 2$ supersymmetry, which the flux breaks spontaneously. If we have only branes in the original description, then the supersymmetry is broken to an $\mathcal{N} = 1$ subgroup. With only anti-branes present, we expect it to be broken to a *different* $\mathcal{N} = 1$ subgroup, and with both branes *and* anti-branes, the supersymmetry should be completely broken. The vacuum structure can be analyzed from a potential which can be computed

exactly using topological string theory [18] or matrix models [19].

Unlike the cases studied before – involving only branes – with branes and anti-branes present, we expect to find a metastable vacuum which breaks supersymmetry. We will find that this is the case, and moreover this leads to a controllable way of breaking supersymmetry at large N where to all orders in the 't Hooft coupling, but to leading order in the $1/N$ expansion, we can compute the geometry of the vacua and the low energy Lagrangian.

The organization of this chapter is as follows. In section 2 we review the case where we have a single stack of branes and extend it to the case where we have a single stack of anti-branes. In section 3 we discuss the case with more than one S^2 in the geometry, and then specialize to the case in which there are only two. We will explain how if we have only branes or only anti-branes, supersymmetry is not broken, whereas if we have branes wrapped on one S^2 and anti-branes on the other, supersymmetry is spontaneously broken. In section 4 we estimate the decay rate of the metastable vacuum. In section 5 we conclude with open questions and suggestions for future work.

1.1. Branes and Anti-Branes on the Conifold

Let us begin by recalling the physics of N D5 branes on the conifold singularity. We consider type IIB string theory with the branes wrapping the S^2 obtained by resolving the conifold singularity. The local geometry of the Calabi-Yau threefold is a \mathbb{P}^1 with normal bundle given by the sum of two line bundles

$$\mathcal{O}(-1) + \mathcal{O}(-1) \rightarrow \mathbb{P}^1$$

At low energies, the field theory on the space-filling D5 branes is a pure $U(N)$ gauge theory with $\mathcal{N} = 1$ supersymmetry. That the theory has $\mathcal{N} = 1$ supersymmetry follows from the fact that string theory on the background of local Calabi-Yau manifolds preserves $\mathcal{N} = 2$, and the D-branes break half of it. In particular, there are no massless adjoint fields, which follows from the important geometric fact that the \mathbb{P}^1 which the D5 brane wraps is *isolated* (its normal bundle is $\mathcal{O}(-1) + \mathcal{O}(-1)$, which does not have a holomorphic section). In other words, any deformation of the \mathbb{P}^1 in the normal direction increases its volume, and so corresponds to a massive adjoint field, at best. (see Fig. 1.1)

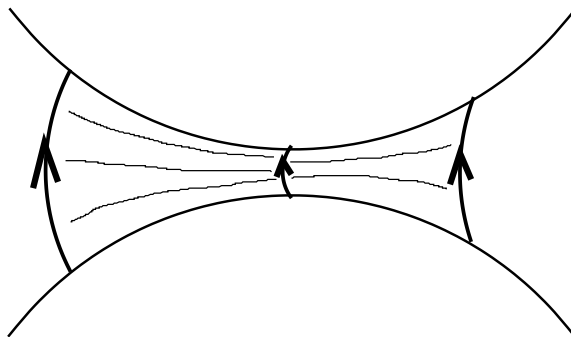


Fig. 1.1. The geometry near a resolved conifold. Moving away in the normal direction, the volume of the wrapped two-cycle (represented here as an S^1) must increase.

The $\mathcal{N} = 1$ pure Yang-Mills theory on the brane is expected to be strongly coupled in the IR, leading to gaugino bilinear condensation, confinement, and a mass gap. There are N massive vacua corresponding to the gaugino superfield getting an expectation value:

$$\langle S \rangle = \Lambda_0^3 \exp\left(-\frac{2\pi i \alpha}{N}\right) \exp\left(\frac{2\pi i k}{N}\right) \quad k = 1, \dots, N \quad (1.1.1)$$

In the future we will suppress the phase factor which distinguishes the N vacua. Above, α is the bare gauge coupling constant defined at the cutoff scale Λ_0

$$\alpha(\Lambda_0) = -\frac{\theta}{2\pi} - i \frac{4\pi}{g_{\text{YM}}^2(\Lambda_0)},$$

and $S = \frac{1}{32\pi^2} \text{Tr} W_\alpha W^\alpha$. Recalling that the gauge coupling in this theory runs as

$$2\pi i \alpha(\Lambda_0) = -\log\left(\frac{\Lambda}{\Lambda_0}\right)^{3N} \quad (1.1.2)$$

where Λ is the strong coupling scale of the theory, we can also write (1.1.1) as

$$\langle S \rangle = \Lambda^3 \quad (1.1.3)$$

The theory has an anomalous axial $U(1)_R$ symmetry which rotates the gauginos according to $\lambda \rightarrow \lambda e^{i\varphi}$, so

$$S \rightarrow S e^{2i\varphi}.$$

The anomaly means that this is a symmetry of the theory only provided the theta angle shifts

$$\theta \rightarrow \theta + 2N\varphi. \quad (1.1.4)$$

Since the theta angle shifts, the tree level superpotential $\mathcal{W}_{\text{tree}} = \alpha S$ is not invariant under the R-symmetry, and in the quantum theory additional terms must be generated to correct for this. Adding the correction terms produces the effective Veneziano-Yankielowicz superpotential [20],

$$\mathcal{W}(S) = \alpha S + \frac{1}{2\pi i} N S (\log(S/\Lambda_0^3) - 1) \quad (1.1.5)$$

whose critical points are (1.1.1).

One can also understand the generation of this superpotential from the viewpoint of the large N holographically dual theory [17]. This configuration of branes wrapping a \mathbb{P}^1 is dual to a closed string theory on the Calabi-Yau manifold obtained by a geometric transition which replaces the wrapped \mathbb{P}^1 with a finite sized \mathbf{S}^3 .

In the dual theory, the 5-branes have disappeared, and in their place there are N units of Ramond-Ramond flux

$$\oint_A H = N$$

through the three-cycle A corresponding to the S^3 . Here

$$H = H^{RR} + \tau H^{NS}$$

where τ is the type IIB dilaton-axion, $\tau = C_0 + \frac{i}{g_s}$. There are also fluxes turned on through the dual B cycle

$$\int_B H = -\alpha.$$

These fluxes generate a superpotential [21,22]:

$$\mathcal{W} = \int H \wedge \Omega$$

where Ω is the holomorphic $(3, 0)$ form on the Calabi-Yau manifold. This can be written in terms of the periods

$$S = \oint_A \Omega, \quad \frac{\partial}{\partial S} \mathcal{F}_0 = \int_B \Omega$$

as

$$\mathcal{W}(S) = \alpha S + N \frac{\partial}{\partial S} \mathcal{F}_0. \quad (1.1.6)$$

Above, \mathcal{F}_0 is the prepotential of the $\mathcal{N} = 2$ $U(1)$ gauge theory which is the low-energy effective theory of type IIB string theory on this geometry before turning on fluxes.

For our case, the B -period was computed in [17]

$$\frac{\partial}{\partial S} \mathcal{F}_0 = \frac{1}{2\pi i} S (\log(S/\Lambda_0^3) - 1)$$

and so (1.1.6) exactly reproduces the effective superpotential (1.1.5) which we derived from gauge theory arguments. Moreover, the superpotential is an F-term which should not depend on the cutoff Λ_0 , so the flux has to run (this agrees with the interpretation of Λ_0 as an IR cutoff in the conifold geometry which regulates the non-compact B -cycle).

$$2\pi i \alpha(\Lambda_0) = -\log\left(\frac{\Lambda}{\Lambda_0}\right)^{3N}$$

In the dual gauge theory, this corresponds to the running of the gauge coupling in the low energy theory. This theory and the duality were studied from a different perspective in [23] (see also [24]).

1.1.1. The anti-brane holography

Now consider replacing the D5 branes with anti-D5 branes wrapping the \mathbb{P}^1 . It is natural that the physics in the presence of the two types of branes should be identical. In particular, in the open string theory, we again expect gaugino condensation, a mass gap and confinement. We conjecture that N antibranes wrapping the \mathbb{P}^1 are also holographically dual to the conifold deformation with flux through the S^3 . In fact, we have no choice but to require this as it is the result of acting by CPT on both sides of the duality. On the open string side, we replace the branes with anti-branes, and on the closed string dual we have N units of flux through the S^3 , but N is now *negative*. In other words, the superpotential is still given by (1.1.5), but with $N < 0$.

At first sight, this implies that there is a critical point as given by (1.1.1), but now with N negative. However, this cannot be right, since S , which is the size of the S^3 , would grow without bound as we go to weak coupling $S \sim \exp(\frac{8\pi^2}{|N|g_{\text{YM}}^2})$. This is clearly unphysical, and moreover the description of the conifold breaks down when S is larger than the cutoff Λ_0^3 in the dual closed string geometry.

To see what is going on, recall that on the open string side the background has $\mathcal{N} = 2$ supersymmetry, and adding D5 branes preserves an $\mathcal{N} = 1$ subset of this, while adding anti-branes preserves an orthogonal $\mathcal{N} = 1$ subset. By holography, we should have the same situation on the other side of the duality. Namely, before turning on flux, the background has $\mathcal{N} = 2$ supersymmetry, and turning on flux should break this to $\mathcal{N} = 1$. But now holography implies that depending on whether we have branes or anti-branes in the dual – so depending on the sign of N – we should have one or the other $\mathcal{N} = 1$ subgroup of the supersymmetry preserved.

It is clear that the superpotential (1.1.5) has been adapted to a superspace in which the manifest $\mathcal{N} = 1$ supersymmetry is the one preserved by branes, and hence is *not* well adapted for the supersymmetry of the anti-branes. Nevertheless, the theory with negative flux should somehow find a way to capture the supersymmetry, as string holography predicts! We will now show that this is indeed the case.

The vacua of the theory are clearly classified by the critical points of the physical *potential* V of the theory:

$$\partial_S V = 0,$$

where

$$V = g^{S\bar{S}} |\partial_S \mathcal{W}|^2.$$

Above, $g_{S\bar{S}}$ is the Kähler metric for S . In this case, the theory has softly broken $\mathcal{N} = 2$ supersymmetry, and so g is given in terms of the prepotential as in the $\mathcal{N} = 2$ case

$$g_{S\bar{S}} = \text{Im}(\tau),$$

where

$$\tau(S) = \partial_S^2 \mathcal{F}_0 = \frac{1}{2\pi i} \log(S/\Lambda_0^3)$$

With superpotential \mathcal{W} as given in (1.1.6), the effective potential becomes

$$V = \frac{2i}{(\tau - \bar{\tau})} |\alpha + N\tau|^2$$

It is easy to see that the critical points are at

$$\partial_S V = -\frac{2i}{(\tau - \bar{\tau})^2} \partial_S^3 \mathcal{F}_0 (\bar{\alpha} + N\bar{\tau}) (\alpha + N\bar{\tau}) = 0.$$

This has two solutions. The first is at

$$\alpha + N\tau = 0 \tag{1.1.7}$$

which solves $\partial\mathcal{W} = 0$, and corresponds to (1.1.1). It is physical when N is positive. The second critical point is at

$$\alpha + N\bar{\tau} = 0. \quad (1.1.8)$$

Note that in this vacuum, $\partial\mathcal{W} \neq 0$. In terms of S , it corresponds to

$$\langle S \rangle = \Lambda_0^3 \exp\left(\frac{2\pi i}{|N|}\bar{\alpha}(\Lambda_0)\right) \quad (1.1.9)$$

and is the physical vacuum when N is *negative* (i.e. where we have $|N|$ anti-branes).

So how can it be that even though $\partial\mathcal{W} \neq 0$, supersymmetry is nevertheless preserved? Toward understanding this, recall that before turning on flux, the closed string theory has $\mathcal{N} = 2$ supersymmetry with one $\mathcal{N} = 2$ $U(1)$ vector multiplet \mathcal{A} . Adding D5 branes in the original theory corresponds to turning on positive flux, which forces this to decompose into two $\mathcal{N} = 1$ supermultiplets: a chiral multiplet \mathcal{S} containing S and its superpartner ψ , and a vector multiplet W_α containing the $U(1)$ gauge field (coming from the four-form potential decomposed in terms of the harmonic three-form on S^3 and a one-form in 4d) and the gaugino λ :

$$\mathcal{A} = (\mathcal{S}, W^\alpha)$$

where

$$\begin{aligned} \mathcal{S} &= S + \theta\psi + \dots \\ W_\alpha &= \lambda_\alpha + \frac{i}{2}(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \dots \end{aligned}$$

The Lagrangian in $\mathcal{N} = 1$ superspace is given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\mathcal{W}$$

where

$$\mathcal{L}_0 = \text{Im} \left(\int d^2\theta d^2\bar{\theta} \bar{\mathcal{S}}_i \frac{\partial \mathcal{F}_0}{\partial \mathcal{S}_i} + \int d^2\theta \frac{1}{2} \frac{\partial^2 \mathcal{F}_0}{\partial \mathcal{S}_i \partial \mathcal{S}_j} W_i^\alpha W_{j\alpha} \right)$$

is the action for an $\mathcal{N} = 2$ supersymmetric theory with prepotential \mathcal{F}_0 , and the superpotential term is

$$\mathcal{L}_\mathcal{W} = \int d^2\theta \mathcal{W}(S) + c.c.$$

where the superpotential $\mathcal{W}(S)$ is given by (1.1.6) and is repeated here for the reader's convenience

$$\mathcal{W}(S) = \alpha S + N \frac{\partial}{\partial S} \mathcal{F}_0.$$

The puzzle is now what happens in the anti-D5 brane case. Since the Lagrangian with flux apparently has only $\mathcal{N} = 1$ supersymmetry preserved by the D5 branes for *any* value of N , positive or negative, how is it possible that the anti-brane preserves a different $\mathcal{N} = 1$? One might guess that despite the flux, the Lagrangian \mathcal{L} actually preserves the full $\mathcal{N} = 2$ supersymmetry. This is too much to hope for. In particular, turning on flux should be holographically dual to adding in the branes, which *does* break half of the

$\mathcal{N} = 2$ supersymmetry of the background. Instead, it turns out that the flux breaks $\mathcal{N} = 2$ supersymmetry in a rather exotic way. Namely, *which* $\mathcal{N} = 1$ is preserved off-shell turns out to be a choice of a “gauge”: we can write the theory in a way which makes either the brane or the antibrane $\mathcal{N} = 1$ supersymmetry manifest, no matter what N is. On shell however we have no such freedom, and only one $\mathcal{N} = 1$ supersymmetry can be preserved. Which one this is depends only on whether the flux is positive or negative, and not on the choice of the $\mathcal{N} = 1$ supersymmetry made manifest by the lagrangian.

To see how all this comes about, let us try to make the spontaneously broken $\mathcal{N} = 2$ supersymmetry manifest. The $\mathcal{N} = 2$ vector multiplet \mathcal{A} is really a chiral multiplet, satisfying the $\mathcal{N} = 2$ chiral constraint $\overline{D}_{i\alpha}\mathcal{A} = 0$,

$$\mathcal{A} = S + \theta^i \Psi_i + \theta^i \theta^j X_{ij} + \frac{1}{2}(\epsilon_{ij} \theta^i \sigma^{\mu\nu} \theta^j) F_{\mu\nu} + \dots \quad (1.1.10)$$

Here $i, j = 1, 2$ are the $SU(2)_R$ indices, and Ψ_i is a doublet of fermions:

$$\Psi = \begin{pmatrix} \psi \\ \lambda \end{pmatrix}.$$

The auxiliary fields X_{ij} of each $\mathcal{N} = 2$ chiral superfield form an $SU(2)_R$ triplet, satisfying a reality constraint

$$\overline{X}^{ij} = \epsilon_{il} \epsilon_{jk} X^{lk} \equiv X_{ij}. \quad (1.1.11)$$

In particular, $X^{11} = \overline{X}^{22} = X_{22}$, and so on.¹

The action \mathcal{L} can be written in terms of $\mathcal{N} = 2$ superfields, where turning on fluxes in the geometry corresponds to giving a vacuum expectation value to some of the $\mathcal{N} = 2$ F-terms [17,25,26]. Our presentation here follows closely [26]. Namely, consider the Lagrangian

$$\text{Im} \left(\int d^2\theta_1 d^2\theta_2 \mathcal{F}_0(\mathcal{A}) \right) + X_{ij} E^{ij} + X^{ij} E_{ij}. \quad (1.1.12)$$

where E^{ij} is the triplet of Fayet-Iliopolous terms, with same properties as X has. Since the X_{ij} transform by total derivatives, the FI term $X_{ij} E^{ij}$ preserves the $\mathcal{N} = 2$ supersymmetry. This will match \mathcal{L} precisely if we set

$$E^{11} = \alpha = \overline{E}^{22}, \quad E^{12} = 0$$

and moreover, we give X_{ij} a non-zero imaginary part

$$X_{ij} \rightarrow X_{ij} + iN_{ij}. \quad (1.1.13)$$

¹ More precisely, \mathcal{A} is a *reduced* $\mathcal{N} = 2$ chiral multiplet, meaning it satisfies an additional constraint: $D^{ij} D_{ij} \mathcal{A} \sim \nabla^2 \overline{\mathcal{A}}$, where $D_{ij} = D_{i\alpha} D_j^\alpha$, and ∇^2 is the standard Laplacian. The reducing constraint says that $\nabla^2 \epsilon_{il} \epsilon_{jk} X^{lk} = \nabla^2 \overline{X}^{ij}$. This implies that we can shift X by a *constant* imaginary part that does not satisfy (1.1.11).

where

$$N_{11} = 0, \quad N_{22} = 2N, \quad N_{12} = 0. \quad (1.1.14)$$

It is easy to see from (1.1.10) that to decompose \mathcal{A} in terms of $\mathcal{N} = 1$ multiplets, we can simply expand it in powers of θ_2

$$\mathcal{A}(\theta_1, \theta_2) = \mathcal{S}(\theta_1) + \theta_2^\alpha W_\alpha(\theta_1) + \theta_2 \theta_2 G(\theta_1) \quad (1.1.15)$$

where the chiral multiplet G is given by (see, for example, [27])²

$$G(y, \theta_1) = \int d^2 \bar{\theta}_1 \bar{\mathcal{S}}(y - i\theta_1 \sigma \bar{\theta}_1, \bar{\theta}_1) = X_{22} + \dots$$

Then, plugging in the vacuum expectation values above in (1.1.12) and integrating over θ^2 , we recover the $\mathcal{N} = 1$ form of \mathcal{L} . More precisely (this will get a nice interpretation later) we recover it, with the addition of a constant term $8\pi N/g_{YM}^2$. Note that by shifting X we have turned on a non-zero F term in the θ^2 direction *off-shell*. This breaks $\mathcal{N} = 2$ supersymmetry, leaving the Lagrangian with only $\mathcal{N} = 1$ supersymmetry along θ^1 direction.

Consider now the vacua of the theory.³ Let us denote the full F -term as

$$\widehat{X}_{ij} = X_{ij} + iN_{ij}.$$

Then it is easy to see that X gets an expectation value,

$$\langle \widehat{X}_{11} \rangle = \frac{2i}{\tau - \bar{\tau}} (\bar{\alpha} + N\bar{\tau}) \quad (1.1.16)$$

and

$$\langle \widehat{X}_{22} \rangle = \frac{2i}{\tau - \bar{\tau}} (\alpha + N\bar{\tau}) \quad (1.1.17)$$

Now, recall that the *physical* vacua depend on whether the flux is positive or negative. In particular, in the brane vacuum, where N is positive, we have $\alpha + N\tau = 0$, so

$$\langle \widehat{X}_{11} \rangle = 0, \quad \langle \widehat{X}_{22} \rangle \neq 0.$$

² Explicitly,

$$\theta_2 \theta_2 G(\theta_1) = X_{22} \theta_2 \theta_2 + \frac{1}{2} (\epsilon_{ij} \theta^i \sigma_{\mu\nu} \theta^j) (\theta^2 \sigma^{\mu\nu} \sigma^\rho \partial_\rho \psi) + \frac{1}{3!} (\epsilon_{ij} \theta^i \sigma_{\mu\nu} \theta^j)^2 \nabla^2 \bar{\mathcal{S}}.$$

See for example [28]

³ Explicitly, in this language, the F-term potential becomes

$$\frac{1}{2i} \tau \widehat{X}_{ij} \widehat{X}^{ij} - \frac{1}{2i} \bar{\tau} \widehat{X}_{ij} \widehat{X}^{ij} + \widehat{X}_{ij} E^{ij} + \widehat{X}^{ij} E_{ij}.$$

where indices are always raised and lowered with ϵ tensor, e.g. $\widehat{X}^{ij} = \epsilon^{ik} \epsilon^{jl} \widehat{X}_{kl}$. Moreover, using reality properties of \widehat{X} , it is easy to see that $\widehat{X}^{ij} = \widehat{X}_{ij} - i(N_{ij} + N^{ij})$, and the result follows.

Now, since the supersymmetry variations of the fermions are

$$\delta\Psi_i = i\widehat{X}_{ij}\epsilon^j + \dots$$

It follows immediately that in the brane vacuum

$$\delta\psi = 0, \quad \delta\lambda \neq 0$$

and λ is the goldstino. The unbroken supersymmetry pairs up S and ψ into a chiral field \mathcal{S} and λ and the gauge field into W_α , as in (1.1.15).

By contrast, in the anti-brane vacuum, $\alpha + N\bar{\tau} = 0$, so

$$\langle\widehat{X}_{11}\rangle \neq 0, \quad \langle\widehat{X}_{22}\rangle = 0.$$

Correspondingly, now ψ is the goldstino:

$$\delta\psi \neq 0, \quad \delta\lambda = 0$$

Now the unbroken supersymmetry corresponds to pairing up S and λ into a chiral field $\widetilde{\mathcal{S}}$, and ψ is the partner of the gauge field in \widetilde{W}_α . In other words, now, it is natural to write \mathcal{A} as

$$\mathcal{A} = (\widetilde{\mathcal{S}}, \widetilde{W}_\alpha)$$

or, more explicitly:

$$\mathcal{A}(\theta_1, \theta_2) = \widetilde{\mathcal{S}}(\theta_2) + \theta_1^\alpha \widetilde{W}_\alpha(\theta_2) + \theta_1\theta_1 \widetilde{G}(\theta_2). \quad (1.1.18)$$

It should now be clear how it comes about that even though the Lagrangian \mathcal{L} has only $\mathcal{N} = 1$ supersymmetry, depending on whether the flux N is positive or negative, we can have *different* $\mathcal{N} = 1$ subgroups preserved on shell. The supersymmetry of the Lagrangian was broken since the flux shifted X as in (1.1.13). More precisely, supersymmetry was broken *only* because this shift of X could not be absorbed in the field redefinition of X , consistent with (1.1.11). Off shell, X is allowed to fluctuate, but because the shift is by iN_{ij} , which is not of the form (1.1.11), its fluctuations could not be absorbed completely and supersymmetry really is broken. However the shift of X by N_{ij} in (1.1.14) is indistinguishable from shifting X by

$$N_{11} = -2N, \quad N_{22} = 0, \quad N_{12} = 0. \quad (1.1.19)$$

since the difference between the two shifts *can* be absorbed into a redefinition of the fields. This would preserve a different $\mathcal{N} = 1$ subgroup of the $\mathcal{N} = 2$ supersymmetry, the one that is natural in the anti-brane theory. Correspondingly, in all cases, $\mathcal{N} = 2$ supersymmetry is broken to $\mathcal{N} = 1$ already at the level of the Lagrangian. However, *which* $\mathcal{N} = 1$ is realized off-shell is a gauge choice.

Next we compute the masses of bosonic and fermionic excitations around the vacua. The relevant terms in the $\mathcal{N} = 2$ Lagrangian are:

$$\int d^4\theta \frac{1}{2} \partial_S^3 \mathcal{F}_0 (\Psi_i \theta^i) (\Psi_j \theta^j) (\widehat{X}_{kl} \theta^k \theta^l)$$

In the present context, we can simplify this as

$$\frac{1}{2} \partial_S^3 \mathcal{F}_0(\Psi_1 \Psi_1) \widehat{X}_{22} + \frac{1}{2} \partial_S^3 \mathcal{F}_0(\Psi_2 \Psi_2) \widehat{X}_{11}$$

This gives fermion masses⁴

$$m_\psi = \frac{2i}{(\tau - \bar{\tau})^2} (\alpha + N\bar{\tau}) \partial_S^3 \mathcal{F}_0$$

$$m_\lambda = \frac{2i}{(\tau - \bar{\tau})^2} (\bar{\alpha} + N\bar{\tau}) \partial_S^3 \mathcal{F}_0.$$

For comparison, the mass of the glueball field S is⁵

$$m_S = \frac{2}{|\tau - \bar{\tau}|^2} (|\alpha + N\tau|^2 + |\alpha + N\bar{\tau}|^2)^{\frac{1}{2}} |\partial_S^3 \mathcal{F}_0|.$$

As a check, note that in the brane vacuum (1.1.7), λ is indeed massless as befits the partner of the gauge field (since the original theory had $\mathcal{N} = 2$ supersymmetry, λ here is in fact a goldstino!). Moreover, it is easy to see that the masses of ψ and S agree, as they should,

$$|m_S| = \frac{1}{2\pi} \frac{N^2}{|S| |\text{Im}(\alpha)|} = |m_\psi|, \quad m_\lambda = 0.$$

Now consider the case of anti-branes where, in the holographic dual, we have negative flux. This corresponds to the vacuum (1.1.8). The gauge field A is still massless and S is massive. From (1.1.8), it follows that in this vacuum it is ψ which is massless, and λ becomes massive. Moreover, the mass of λ is the same as for S .

$$|m_S| = \frac{1}{2\pi} \frac{N^2}{|S| |\text{Im}(\alpha)|} = |m_\lambda|, \quad m_\psi = 0.$$

This is all beautifully consistent with holography!

To summarize, the anti-brane vacuum preserves different supersymmetry than the brane vacuum, and the same is true in the large N dual. In other words the anti-brane/negative flux has oriented the $\mathcal{N} = 1$ supersymmetry differently, as is expected based on the holographic duality.

Let us now try to see if we can understand the anti-brane gluino condensate (1.1.9) directly from the gauge theory. Before adding in branes, the background has $\mathcal{N} = 2$ supersymmetry. The corresponding supercharges are two weyl fermions Q, \tilde{Q} that transform as

⁴ This is the physical mass, with canonically normalized kinetic terms.

⁵ It is important to note here that the kinetic terms for S are fixed for us by the string large N duality, and $\mathcal{N} = 2$ supersymmetry. They differ from the “canonical” kinetic terms of [20] We are writing here the physical masses, in the basis where all the fields have canonical kinetic terms.

a doublet under the $SU(2)_R$ symmetry of the theory. In either vacuum with the branes, only the $U(1)_R$ subgroup of the $SU(2)_R$ symmetry is preserved. If adding in D-branes preserves Q , then anti-D branes will preserve \tilde{Q} . Since Q and \tilde{Q} transform *oppositely* under the $U(1)_R$ symmetry, in going from brane to anti-brane, the chirality of the world-volume fermions flips.

The superpotential $\widetilde{\mathcal{W}}$ that is an honest F term with respect to the supersymmetry preserved by the anti-brane theory, and which reproduces (1.1.9), is

$$\widetilde{\mathcal{W}} = \bar{\alpha} S + \frac{1}{2\pi i} N S(\log(S/\Lambda_0^3) - 1),$$

where S is the anti-brane gaugino condensate, and N is negative. This reflects the fact that, if we keep the background fixed, then θ transforms under the R symmetry still as (1.1.4), but the gauginos transform oppositely from before. The rest is fixed by holography – in other words, α is the good chiral field which has θ as a component. Since it is $\widetilde{\mathcal{W}}$ that is the F-term in this case, it is this, and not \mathcal{W} , that should be independent of scale for the anti-brane, and hence α should run as

$$2\pi i \alpha(\Lambda_0) = -\log\left(\frac{\bar{\Lambda}}{\Lambda_0}\right)^{-3N}$$

Note that this is consistent with turning on the other F-term vev (1.1.19), which directly preserves the anti-brane supersymmetry.

Finally, consider the value of the potential in the minimum. At the brane vacuum, supersymmetry is unbroken, and

$$V^{\text{brane}} = 0$$

At the anti-brane vacuum, with negative N , we have:

$$V^{\text{anti-brane}} = -16\pi \frac{N}{g_{YM}^2} = 16\pi \frac{|N|}{g_{YM}^2}$$

This treats branes and anti-branes asymmetrically. It is natural then to shift the zero of the potential and have

$$V_* = V + 8\pi \frac{N}{g_{YM}^2}$$

which leads to the minimum value

$$V_*^{\text{anti-brane}} = 8\pi \frac{|N|}{g_{YM}^2} = V_*^{\text{brane}}$$

This is equivalent to the inclusion of the constant term which came from the magnetic FI term in the $\mathcal{N} = 2$ theory. Note that this corresponds to the tension of $|N|$ (anti-)branes which is predicted by holography! We thus find a beautiful confirmation for the stringy anti-brane holography.

1.2. Geometric Metastability with Branes and Anti-Branes

There are many ways to naively break supersymmetry in string theory, but they typically lead to instabilities. The simplest example of such a situation is found in type IIB string compactifications on a Calabi-Yau, where one introduces an equal number of $D3$ branes and anti-branes filling the spacetime, but located at points in the Calabi-Yau. Supersymmetry is clearly broken, since branes and anti-branes preserve orthogonal subsets of supersymmetry. This does not typically lead to metastability, however, as one can move branes to where the anti-branes are located, and such a motion would be induced by an attractive force. This is then similar to a familiar problem in classical electrostatics, and is plagued by the same difficulty as one has in finding locally stable equilibrium configurations of static electric charges.

One way to avoid such an obstacle, as was suggested in [17], is instead of considering branes and anti-branes occupying points in the Calabi-Yau, to consider wrapping them over minimal cycles inside the Calabi-Yau. Moreover, we choose the minimal cycles to be isolated. In other words we assume there is no continuous deformations of the minimal cycles. We wrap branes over a subset C_1 of such cycles and we wrap the anti-branes over a distinct subset C_2 . In the case of compact Calabi-Yau, it is necessary that the class $[C_1] + [C_2] = 0$, so that there is no net charge (this condition can be modified if we have orientifold planes). Because the branes and anti-branes cannot move without an increase in energy, these will be a metastable equilibrium configurations (see Fig. 1.2). This is true as long as the cycles C_1 and C_2 are separated by more than a string scale distance, so that the geometric reasoning remains valid (when they are closer there are tachyon modes which cause an instability).

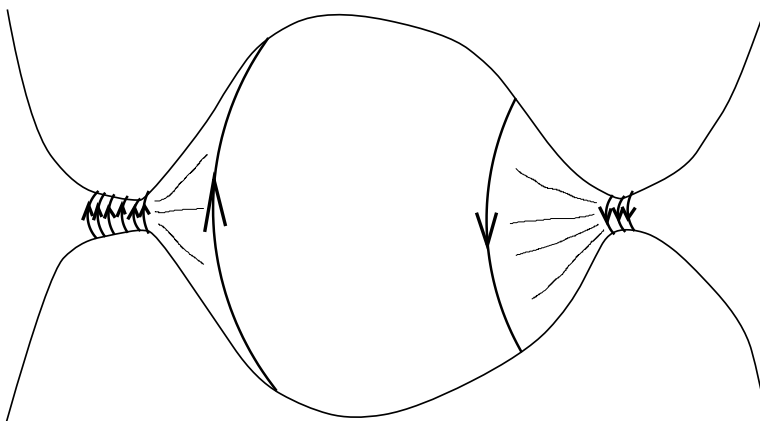


Fig. 1.2. Before antibranes and branes can annihilate one other, they will have to move and meet somewhere, and thus they will have to increase their volume due to the Calabi-Yau geometry.

Here we will consider non-compact examples of this scenario. This decouples the lower dimensional gravity from the discussion, and moreover, the condition that the net class $[C_1] + [C_2]$ be zero is unnecessary as the flux can go to infinity. In particular, we will consider type IIB strings on a non-compact Calabi-Yau threefold where we can wrap 4d space-time filling $D5$ and anti- $D5$ branes over different isolated S^2 's in the Calabi-Yau. The local geometry of each S^2 will be the resolved conifold reviewed in the previous

section. The only additional point here is that we have more than one such S^2 in the same homology class. Moreover, we will consider the large N limit of such brane/anti-brane systems and find that the holographically dual closed string geometry is the same one as in the supersymmetric case with just branes, except that some of the fluxes will be negative. This leads, on the dual closed string side, to a metastable vacuum with spontaneously broken supersymmetry.

We will first describe general geometries of this kind which support metastable configurations with both D5 and anti-D5 branes. The relevant Calabi-Yau geometries turn out to be the same ones studied in [18], which led to a non-perturbative formulation of the dual geometry in terms of matrix models [19]. The new twist is that we now allow not just branes, but both branes and anti-branes to be present. We will describe the holographically dual flux vacua. We then specialize to the case of just two S^2 's with branes wrapped over one S^2 and anti-branes wrapped over the other, and study it in more detail.

1.2.1. Local multi-critical geometries

Consider a Calabi-Yau manifold given by

$$uv = y^2 + W'(x)^2 \tag{1.2.1}$$

where

$$W'(x) = g \prod_{k=1}^n (x - a_k).$$

If, for a moment, we set $g = 0$, then this is an A_1 ALE singularity at every point in the x plane. One can resolve this by blowing up, and this gives a family of \mathbb{P}^1 , i.e. holomorphic two-spheres, parameterized by x . Let us denote the size of the blown up \mathbb{P}^1 by r . In string theory, this is complexified by the B^{NS} -field⁶. Turning g back on lifts most of the singularities, leaving just n isolated \mathbb{P}^1 's at $x = a_k$. Of course, the S^2 still exists over each point x , but it is not represented by a holomorphic \mathbb{P}^1 . In other words, its area is not minimal in its homology class. We have

$$A(x) = (|r|^2 + |W'|^2)^{1/2}. \tag{1.2.2}$$

where $A(x)$ denotes the area of the minimal S^2 as a function of x . This can be seen by the fact that the equation for each x is an ALE space and the above denotes the area of the smallest S^2 in the ALE geometry for a fixed x . Even though the minimal \mathbb{P}^1 's are now isolated, i.e. at points where $W' = 0$, they are all in the same homology class, the one inherited from the ALE space⁷. Note also that the first term, $|r|^2$, does not depend on x .

⁶ On the world volume of the D5 brane B_{NS} gets mixed up with the RR potential, so more precisely by r/g_s we will mean the complex combination of the kahler modulus with $B^{RR} + \tau B^{NS}$. To not have the dilaton turned on, we will in fact keep only the later, setting the geometric size to zero [23].

⁷ As explained in [21], the parameters g and a_i which enter (1.2.1) and W are *not* dynamical fields, but enter in specifying the theory. This is possible because the Calabi-Yau is non compact.

One can now consider wrapping some number of branes N_k , $k = 1, \dots, n$ on each \mathbb{P}^1 . The case when all N_k 's are positive and all the branes are D5 branes was studied in [18]. In that case, the gauge theory on the branes is an $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theory with an adjoint matter field Φ and superpotential given by

$$\text{Tr } W(\Phi) \tag{1.2.3}$$

where W is the same polynomial whose derivative is given by $W'(x)$ in the defining equation of Calabi-Yau [18,29,30,31]. The eigenvalues of Φ are identified with positions of the D5 brane on the x -plane. Here

$$N = \sum_k N_k$$

and the choice of distribution of D5 branes among the critical points a_k corresponds to the choice of a Higgs branch of this supersymmetric gauge theory where, the gauge group is broken

$$U(N) \rightarrow \prod_{k=1}^n U(N_k)$$

In the low energy limit we have $\mathcal{N} = 1$ Yang-Mills theories $U(N_k)$ with Φ field corresponding to a massive adjoint matter for each. Note that (1.2.3) is consistent with (1.2.2) in the large r limit, because wrapping a brane over \mathbb{P}^1 and considering its energy as a function of x , it is minimized along with the area at points where $W'(x) = 0$. Furthermore, it is clear from this that the effective coupling constant g_{YM} of the four dimensional gauge theory living on the brane is the area of the minimal \mathbb{P}^1 's times $1/g_s$:

$$\frac{1}{g_{\text{YM}}^2} = \frac{|r|}{g_s}$$

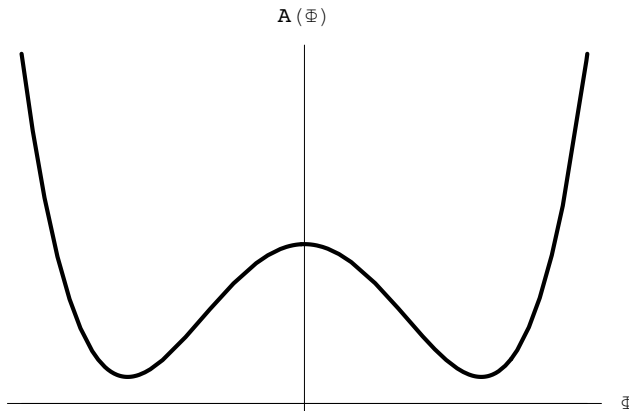


Fig. 1.3. The brane theory with a tree level physical potential $g_s V(x) = |r| + \frac{1}{2r} |dW|^2 \approx A(x)$ is depicted along a real line in the x -plane for the case where W has two critical points. The potential reflects the Calabi-Yau geometry, and the leading term represents the brane tension.

Below, we will study what happens when some of the \mathbb{P}^1 's are wrapped with D5 branes and others with anti-D5 branes. By making the vacua a_i very widely separated,

the branes and the anti-branes should interact very weakly. Therefore, we still expect to have an approximate $\mathcal{N} = 1$ supersymmetric gauge theory for each brane, with gaugino condensation and confinement at low energies, just as discussed in the previous section. However, because supersymmetry is broken and there are lower energy vacua available where some of the branes annihilate, the system should be only metastable. Note that the fact that the \mathbb{P}^1 's are isolated but in the same class is what guarantees metastability. For the branes to annihilate the anti-branes, they have to climb the potential depicted in Fig. 1.3. This should be accurate description of the potential when the branes are far away from each other, where the effect of supersymmetry breaking is small. When the branes and anti-branes are very close together – for example when they are within a string distance – there would be tachyon in the theory and the above potential will not be an accurate description. Nevertheless it is clear that the minimum of a brane and anti-brane system is realized when they annihilate each other. We have thus geometrically engineered a metastable brane configuration which breaks supersymmetry. We will discuss aspects of the open string gauge dynamics of this configuration in section 3.6. As we will discuss in that section, unlike the supersymmetric case, there seems to be no simple field theory with a finite number of degrees of freedom which captures the brane/anti-brane geometry. Of course, one can always discuss it in the context of open string field theory.

We have a control parameter for this metastability, which also controls the amount of supersymmetry breaking, which is related to the separation of the critical points a_i . The farther apart they are, the more stable our system is. We will discuss stability and decay rate issues in section 4. In the next subsection, we study the large N holographic dual for this system.

1.2.2. The large N dual description

The supersymmetric configuration of branes for this geometry was studied in [18], where a large N holographic dual was proposed. The relevant Calabi-Yau geometry was obtained by a geometric transition of (1.2.1) whereby the \mathbb{P}^1 's are blown down and the n resulting conifold singularities at $x = a_k$ are resolved into S^3 's by deformations of the complex structure. It is given by

$$uv = y^2 + W'(x)^2 + f_{n-1}(x), \quad (1.2.4)$$

where $f_{n-1}(x)$ is a degree $n - 1$ polynomial in x . As explained in [18], the geometry is effectively described by a Riemann surface which is a double cover of the x plane, where the two sheets come together along n cuts near $x = a_k$ (where the \mathbb{P}^1 's used to be). The A and B cycles of the Calabi-Yau project to one-cycles on the Riemann surface, as in Fig. 1.4. The geometry is characterized by the periods of the $(3, 0)$ form Ω ,

$$S_k = \oint_{A^k} \Omega, \quad \partial_{S_k} \mathcal{F}_0 = \int_{B_k} \Omega.$$

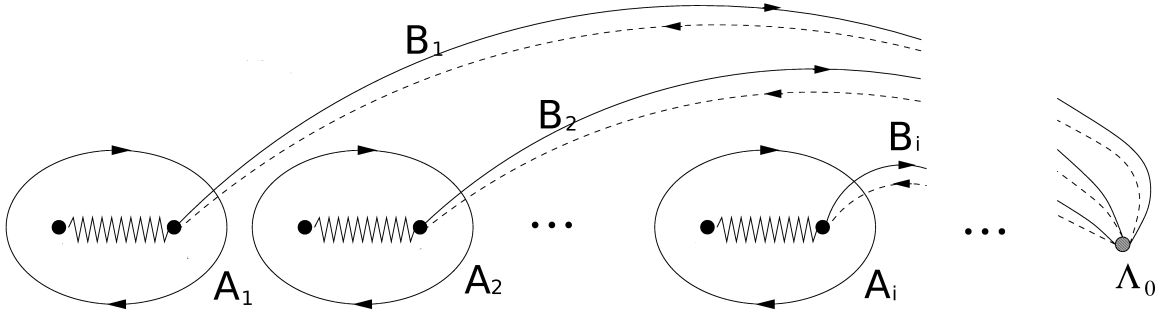


Fig. 1.4. The multi-critical Calabi-Yau geometry projected to the x -plane. The cut structure reflects the branching of y over the x plane for $u = v = 0$.

The complex scalars S_k are the scalar partners of the n $U(1)$ vector multiplets under the $\mathcal{N} = 2$ supersymmetry, and $\mathcal{F}_0(S)$ is the corresponding prepotential. As before, while (1.2.4) depends also on a_i and g , the latter are just fixed parameters.

If, before the transition, all of the \mathbb{P}^1 's were wrapped with a large number of branes, the holographically dual type IIB string theory is given by the (1.2.4) geometry, where the branes from before the transition are replaced by fluxes

$$\oint_{A^k} H = N_k, \quad \int_{B_k} H = -\alpha.$$

We want to conjecture that this duality holds whether or not all the $N_k \geq 0$. In fact, the discussion of the previous section shows that if all the $N_k \leq 0$ the duality should continue to hold by CPT conjugation. Our conjecture is that it also holds as when some N_k are positive and some negative. The flux numbers N_k will be positive or negative depending on whether we had D5 branes or anti-D5 branes wrapping the k 'th \mathbb{P}^1 before the transition. The flux through the B_k cycles will correspond to the bare gauge coupling constant on the D-branes wrapping the corresponding \mathbb{P}^1 , and is the same for all k as the \mathbb{P}^1 's are all in the same homology class. Turning on fluxes generates a superpotential [21]

$$\mathcal{W} = \int H \wedge \Omega,$$

$$\mathcal{W}(S) = \sum_k \alpha S_k + N_k \partial_{S_k} \mathcal{F}_0. \quad (1.2.5)$$

In [18], in the case when all N_k 's are positive, the $\mathcal{N} = 1$ chiral superfield corresponding to S_k is identified with the gaugino condensates of the $SU(N_k)$ subgroup of the $U(N_k)$ gauge group factor, $S_k = \frac{1}{32\pi^2} \text{Tr}_{SU(N_k)} W_\alpha W^\alpha$, before the transition. When we have both branes and anti-branes, as long as they are very far separated, this picture should persist. Namely, the brane theory should still have gaugino condensation and confinement, and S_k 's should still correspond to the gaugino condensates, even though we expect the supersymmetry to be completely broken in this metastable vacuum. Moreover, for each k , there is a remaining massless $U(1)$ gauge field in the dual geometry. It gets identified with

the massless $U(1)$ on the gauge theory side which is left over by the gaugino condensation that confines the $SU(N_k)$ subgroup of the $U(N_k)$ gauge theory.

In the supersymmetric case studied in [18], the coefficients of the polynomial $f(x)$ determining the dual geometry and the sizes of S_i is fixed by the requirement that

$$\partial_{S_k} W(S) = 0$$

and this gives a supersymmetric holographic dual. In the case of interest for us, with mixed fluxes, we do not expect to preserve supersymmetry. Instead we should consider the physical potential $V(S)$ and find the dual geometry by extremizing

$$\partial_{S_k} V(S) = 0$$

which we expect to lead to a metastable vacuum. The effective potential V is given in terms of the special geometry data and the flux quanta:

$$V = g^{S_i \bar{S}_j} \partial_{S_i} \mathcal{W} \overline{\partial_{S_j} \mathcal{W}}$$

where the Kähler metric is given by $g_{i\bar{j}} = \text{Im}(\tau_{ij})$ in terms of the period matrix of the Calabi-Yau

$$\tau_{ij} = \partial_{S_i} \partial_{S_j} \mathcal{F}_0.$$

In terms of τ , we can write

$$V = \left(\frac{2i}{\tau - \bar{\tau}} \right)^{jk} (\alpha_j + \tau_{jj'} N^{j'}) (\bar{\alpha}_k + \bar{\tau}_{kk'} N^{k'}) \quad (1.2.6)$$

where we have all the $\alpha_j = \alpha$. To be explicit, we consider in detail the case where we have only two S^3 's in the dual geometry before turning to the more general case.

1.2.3. More precise statement of the conjecture

Let us make our conjecture precise. We conjecture that the large N limit of brane/anti-brane systems are Calabi-Yaus with fluxes. In particular, the solutions to tree level string equations on the closed string side lead to an all order summation of planar diagrams of the dual brane/anti-brane system (i.e. to all orders in the 't Hooft coupling). We translate this statement to mean that the geometry of the closed string vacuum at tree level is captured by extremizing the physical potential. Moreover the physical potential is characterized by the fluxes (which fix the superpotential) as well as by the Kähler potential; the main ingredient for both of these objects is the special geometry of the Calabi-Yau after transition. This, in turn, is completely fixed by tree level topological string theory, or equivalently by the planar limit of a certain large N matrix model [19]. Here we are using the fact that since $\mathcal{N} = 2$ is softly broken by the flux terms, the special Kähler metric is unaffected at the string tree level. It is quite gratifying to see that topological objects, such as matrix integrals, play a role in determining the geometry of non-supersymmetric string vacua!

Of course the Kähler potential should be modified at higher string loops since. In particular the $1/N$ corrections to our duality should involve such corrections. Note that, in the supersymmetric case studied in [18], there were no $1/N$ corrections to modify the geometry of the vacua. We do expect the situation to be different in the non-supersymmetric case. Note however, from the discussion of section 2, the soft breaking is such that it is ambiguous *which* $\mathcal{N} = 1$ supersymmetry the Lagrangian has. This should constrain what kind of quantum corrections one can have beyond those allowed by a generic soft breaking. This deserves further study.

1.2.4. The case of two S^3 's

For simplicity we start with the case where we have just two S^3 's. Before the transition, there are two shrinking \mathbb{P}^1 's at $x = a_{1,2}$. Let us denote by Δ the distance between them,

$$\Delta = a_1 - a_2.$$

The theory has different vacua depending on the number of branes we put on each \mathbb{P}^1 . The vacua with different brane/antibrane distributions are separated by energy barriers. To overcome these, the branes must first become more massive.

The effective superpotential of the dual geometry, coming from the electric and magnetic FI terms turned on by the fluxes, is

$$\mathcal{W}(S) = \alpha(S_1 + S_2) + N_1 \partial_{S_1} \mathcal{F}_0 + N_2 \partial_{S_2} \mathcal{F}_0$$

The B-periods have been computed explicitly in [18]. We have

$$\begin{aligned} 2\pi i \partial_{S_1} \mathcal{F}_0 = & W(\Lambda_0) - W(a_1) + S_1 \left(\log\left(\frac{S_1}{g\Delta^3}\right) - 1 \right) - 2(S_1 + S_2) \log\left(\frac{\Lambda_0}{\Delta}\right) \\ & + (2S_1^2 - 10S_1S_2 + 5S_2^2)/(g\Delta^3) + \dots \end{aligned} \quad (1.2.7)$$

and

$$\begin{aligned} 2\pi i \partial_{S_2} \mathcal{F}_0 = & W(\Lambda_0) - W(a_2) + S_2 \left(\log\left(\frac{S_2}{g\Delta^3}\right) - 1 \right) - 2(S_1 + S_2) \log\left(\frac{\Lambda_0}{\Delta}\right) \\ & - (5S_1^2 - 10S_1S_2 + 2S_2^2)/(g\Delta^3) + \dots \end{aligned} \quad (1.2.8)$$

where the omitted terms are of order $S_1^{n_1} S_2^{n_2} / (g\Delta^3)^{n_1+n_2-1}$, for $n_1+n_2 > 2$. In the above, Λ_0 is the cutoff used in computing the periods of the non-compact B cycles, and physically corresponds to a high energy cutoff in the theory.

To the leading order (we will justify this a posteriori), we can drop the quadratic terms in the $\frac{S_k}{g\Delta^3}$'s, and higher. To this order,

$$\begin{aligned} 2\pi i \tau_{11} = & 2\pi i \partial_{S_1}^2 \mathcal{F}_0 \approx \log\left(\frac{S_1}{g\Delta^3}\right) - \log\left(\frac{\Lambda_0}{\Delta}\right)^2 \\ 2\pi i \tau_{12} = & 2\pi i \partial_{S_1} \partial_{S_2} \mathcal{F}_0 \approx -\log\left(\frac{\Lambda_0}{\Delta}\right)^2 \\ 2\pi i \tau_{22} = & 2\pi i \partial_{S_2}^2 \mathcal{F}_0 \approx \log\left(\frac{S_2}{g\Delta^3}\right) - \log\left(\frac{\Lambda_0}{\Delta}\right)^2 \end{aligned}$$

In particular, note that at the leading order τ_{12} is independent of the S_i , so we can use τ_{ii} as variables. It follows easily that the minima of the potential are at

$$Re(\alpha) + Re(\tau)_{ij} N^j = 0$$

and

$$Im(\alpha) + Im(\tau)_{ij} |N^j| = 0$$

For example, with branes on the first \mathbb{P}^1 and anti-branes on the second, $N_1 > 0 > N_2$,

$$\langle S_1 \rangle = g\Delta^3 \left(\frac{\Lambda_0}{\Delta}\right)^2 \left(\frac{\overline{\Lambda_0}}{\Delta}\right)^{2|\frac{N_2}{N_1}|} e^{-2\pi i\alpha/|N_1|}, \quad \langle S_2 \rangle = g\Delta^3 \left(\frac{\Lambda_0}{\Delta}\right)^2 \left(\frac{\overline{\Lambda_0}}{\Delta}\right)^{2|\frac{N_1}{N_2}|} e^{2\pi i\overline{\alpha}/|N_2|}. \quad (1.2.9)$$

To see how to interpret this, let us recall the supersymmetric situation when $N_1, N_2 > 0$. There, to the same order, one finds

$$\langle S_1 \rangle = g\Delta^3 \left(\frac{\Lambda_0}{\Delta}\right)^{2(1+|\frac{N_2}{N_1}|)} e^{-2\pi i\alpha/|N_1|}, \quad \langle S_2 \rangle = g\Delta^3 \left(\frac{\Lambda_0}{\Delta}\right)^{2(1+|\frac{N_1}{N_2}|)} e^{-2\pi i\alpha/|N_2|}. \quad (1.2.10)$$

The interpretation of the above structure in the supersymmetric case is as follows. In the IR, the theory flows to a product of two supersymmetric gauge theories with gauge group $U(N_1) \times U(N_2)$, where each $U(N_i)$ factor is characterized by a scale Λ_i ,

$$\langle S_1 \rangle = \Lambda_1^3, \quad \langle S_2 \rangle = \Lambda_2^3. \quad (1.2.11)$$

Let us denote by $\alpha_{1,2}$ the the bare coupling constants of the low energy theory $U(N_1)$ and $U(N_2)$ theories, i.e.

$$2\pi i\alpha_i = -\log\left(\frac{\Lambda_i}{\Lambda_0}\right)^{3|N_i|}, \quad i = 1, 2 \quad (1.2.12)$$

On the one hand, we can simply read them off from (1.2.10) since we can write (1.2.11) as $\langle S_i \rangle = \Lambda_0^3 e^{-2\pi i\alpha_i/|N_i|}$:

$$\begin{aligned} 2\pi i\alpha_1 &= 2\pi i\alpha + \log\left(\frac{\Lambda_0}{m_\Phi}\right)^{|N_1|} - 2\log\left(\frac{\Lambda_0}{m_W}\right)^{|N_2|} \\ 2\pi i\alpha_2 &= 2\pi i\alpha + \log\left(\frac{\Lambda_0}{m_\Phi}\right)^{|N_2|} - 2\log\left(\frac{\Lambda_0}{m_W}\right)^{|N_1|}. \end{aligned} \quad (1.2.13)$$

We could also have obtained this by using the matching relations with the high energy coupling α . Namely, in flowing from high energies we have to take into account the massive fields we are integrating out. The fields being integrated out in this case are the massive adjoint Φ near each vacuum, whose mass goes as $m_\Phi = g\Delta$, and the massive W bosons in the bifundamental representation, whose masses are $m_W = \Delta$. The standard contribution of these heavy fields to the running coupling constant is simply what is written in (1.2.13).

In the non-supersymmetric case, from (1.2.11) and (1.2.12), we can write the gaugino vevs in (1.2.9) very suggestively exactly as in (1.2.13) except that we replace

$$\Lambda_0^{|N_1|+|N_2|} \rightarrow \Lambda_0^{|N_1|} \overline{\Lambda_0}^{|N_2|}, \Lambda_0^{|N_2|} \overline{\Lambda_0}^{|N_1|},$$

and

$$m_W \rightarrow \overline{m}_W$$

leading to

$$\begin{aligned} 2\pi i\alpha_1 &= 2\pi i\alpha + \log\left(\frac{\Lambda_0}{m_\Phi}\right)^{|N_1|} - 2\log\left(\frac{\overline{\Lambda}_0}{\overline{m}_W}\right)^{|N_2|} \\ 2\pi i\alpha_2 &= \overline{2\pi i\alpha} + \log\left(\frac{\Lambda_0}{m_\Phi}\right)^{|N_2|} - 2\log\left(\frac{\overline{\Lambda}_0}{\overline{m}_W}\right)^{|N_1|} \end{aligned} \tag{1.2.14}$$

which gives back (1.2.9). In other words, the leading result is as in the supersymmetric case, except that branes of the opposite type lead to complex conjugate running. In particular, this implies that the real part of the coupling runs as before, but the θ -angle is running differently due to the matter field coming between open strings stretched between the branes and anti-branes. We will discuss potential explanations of this in section 3.6, where we discuss the matter structure in the brane/anti-brane system.

The potential at the critical point is given by⁸

$$V_*^{+-} = \frac{8\pi}{g_{YM}^2} (|N_1| + |N_2|) - \frac{2}{\pi} |N_1||N_2| \log \left| \frac{\Lambda_0}{m_W} \right|^2 \tag{1.2.15}$$

The first term, in the holographic dual, corresponds to the tensions of the branes. The second term should correspond to the Coleman-Weinberg one loop potential which is generated by zero point energies of the fields. This interpretation coincides nicely with the fact that this term is proportional to $|N_1||N_2|$, and thus comes entirely from the $1-2$ sector of open strings with one end on the branes and the other on the anti-branes. The fields in the $1-1$ and $2-2$ sectors with both open string endpoints on the same type of brane do not contribute to this, as those sectors are supersymmetric and the boson and fermion contributions cancel. We shall return to this in section 3.6. For comparison, in the case of where both \mathbb{P}^1 's were wrapped by D5 branes, the potential at the critical point V_*^{++} equals

$$V_*^{++} = \frac{8\pi}{g_{YM}^2} (|N_1| + |N_2|) = V_*^{--}$$

and is the same as for all anti-branes. This comes as no surprise, since the tensions are the same, and the interaction terms cancel since the theory is now truly supersymmetric.

We now consider the masses of bosons and fermions in the brane/anti-brane background. With supersymmetry broken, there is no reason to expect the 4 real bosons of the theory, coming from the fluctuations of $S_{1,2}$ around the vacuum, to be pairwise degenerate. To compute them, we simply expand the potential V to quadratic order. More

⁸ We have shifted the overall zero point energy of V as we did in the previous section, to make the brane and the anti-brane tensions equal. More precisely, we set

$$V \rightarrow V + \frac{8\pi}{g_{YM}^2} (N_1 + N_2).$$

precisely, to compute the physical masses as opposed to the naive hessian of the potential, we have to go to the basis where kinetic terms of the fields are canonical. The computation is straightforward, if somewhat messy. At the end of the day, we get the following expressions.

$$(m_{\pm}(c))^2 = \frac{(a^2 + b^2 + 2abcv) \pm \sqrt{(a^2 + b^2 + 2abcv)^2 - 4a^2b^2(1-v)^2}}{2(1-v)^2}$$

where c takes values $c = \pm 1$, and

$$a \equiv \left| \frac{N_1}{2\pi\Lambda_1^3 \text{Im}\tau_{11}} \right|, \quad b \equiv \left| \frac{N_2}{2\pi\Lambda_2^3 \text{Im}\tau_{22}} \right|$$

$$v \equiv \frac{(\text{Im}\tau_{12})^2}{\text{Im}\tau_{11}\text{Im}\tau_{22}}.$$

Indeed we find that our vacuum is metastable, because all the $m^2 > 0$ (which follows from the above formula and the fact that $v < 1$ in the regime of interest $|S_i/g\Delta^3| < 1$). This is a nice check on our holography conjecture, as the brane/anti-brane construction was clearly metastable. Moreover, we see that there are four real bosons, whose masses are generically non-degenerate, as expected for the spectrum with broken supersymmetry.

Since supersymmetry is completely broken from $\mathcal{N} = 2$ to $\mathcal{N} = 0$, we expect to find 2 massless Weyl fermions, which are the Goldstinos. More precisely, we expect this *only* at the closed string tree level (i.e at the leading order in $1/N$ expansion). Namely, at string tree level, we can think about turning on the fluxes as simply giving them an “expectation value”, which would break the $\mathcal{N} = 2$ supersymmetry spontaneously. However, at higher orders, the Kähler potential should a priori *not* be protected in that breaking is soft, and not spontaneous. This will affect the computation of the mass terms in section 2, which relied on the $\mathcal{N} = 2$ relation between the Kähler potential and the superpotential, and should result in only one massless fermion remaining. Masses of the fermions are computed from the $\mathcal{N} = 2$ Lagrangian, as in section two, but now with two vector multiplets. We again redefine the fields so as to give them canonical kinetic terms. We indeed find two massless fermions, at the order at which we are working at. Since supersymmetry is broken these are interpreted as the Goldstinos (we will give a more general argument in the next subsection). There are also two massive fermions, with masses

$$m_{f1} = \frac{a}{1-v}, \quad m_{f2} = \frac{b}{1-v},$$

Note that v controls the strength of supersymmetry breaking. In particular when $v \rightarrow 0$ the 4 boson masses become pairwise degenerate and agree with the two fermion masses a and b , as expected for a pair of $\mathcal{N} = 1$ chiral multiplets.

The mass splitting between bosons and fermions is a measure of the supersymmetry breaking. In order for supersymmetry breaking to be weak, these splittings have to be small. There are two natural ways to make supersymmetry breaking be small. One way is to take the number of anti-branes to be much smaller than the number of branes. The

other way is to make the branes and anti-branes be very far from each other. We will consider mass splittings in both of these cases.

Adding a small anti-brane charge should be a small perturbation to a system of a large number of branes. In that context, the parameter that measures supersymmetry breaking should be $|N_2/N_1|$, where $|N_2|$ is the number of anti-branes. Let us see how this is reflected in v . We should see that in this limit, v becomes small.

Note that (1.2.9) implies that

$$\left| \frac{S_1}{g\Delta^3} \right|^{|N_1|} = \left| \frac{S_2}{g\Delta^3} \right|^{|N_2|} = \left| \frac{\Lambda}{\Delta} \right|^{2(|N_1|+|N_2|)} = \exp\left(- (1-\delta) \frac{8\pi^2}{g_{\text{YM}}^2}\right)$$

with

$$\delta \equiv \left(\frac{g_{\text{YM}}}{2\pi}\right)^2 (|N_1| + |N_2|) \log \left| \frac{\Lambda_0}{\Delta} \right|.$$

Note that

$$\left| \frac{\delta}{1-\delta} \right| = \left| \frac{\log \Lambda_0/\Delta}{\log \Delta/\Lambda} \right| \quad (1.2.16)$$

For finite non-vanishing $S_i/g\Delta^3$, clearly δ is very close to 1 because of the fact that g_{YM}^2 is very small for large cutoff Λ_0 . Of course, this is compatible with the definition of δ as we recall the running of g_{YM}^2 . Therefore, let us write

$$\delta = 1 - \epsilon$$

We then have (without any approximation)

$$v = \frac{(\text{Im}\tau_{12})^2}{\text{Im}\tau_{11}\text{Im}\tau_{22}} = \frac{(1-\epsilon)^2}{(1+\epsilon^2) + \epsilon\left(\left|\frac{N_2}{N_1}\right| + \left|\frac{N_1}{N_2}\right|\right)} \quad (1.2.17)$$

And in the limit of $\frac{|N_1|}{|N_2|} \rightarrow \infty$ (or similarly when $\frac{|N_2|}{|N_1|} \rightarrow \infty$) we find that $v \rightarrow 0$. This is as expected, since in this limit our expectation is that supersymmetry breaking effects will be small.

In Fig. 1.5, we plot the masses for non-supersymmetric vacua, with fixed $|N_1 N_2|$ but varying $\log \left| \frac{N_2}{N_1} \right|$. They demonstrate how mass splitting vanishes in the extreme limits of the ratio N_1/N_2 , where supersymmetry is restored.

It is also natural to consider the mass splittings in the limit of large separation of branes, where supersymmetry is expected to be restored. We have to be somewhat careful in taking this limit. It turns out that the right limit is when

$$|\Lambda| \ll |\Delta| \ll |\Lambda_0|$$

and where

$$|\Delta/\Lambda| \gg |\Lambda_0/\Delta|$$

which implies from (1.2.16) that $\delta \rightarrow 0$ and from (1.2.17) we have $v \rightarrow 0$ and thus the mass splittings disappear as expected.

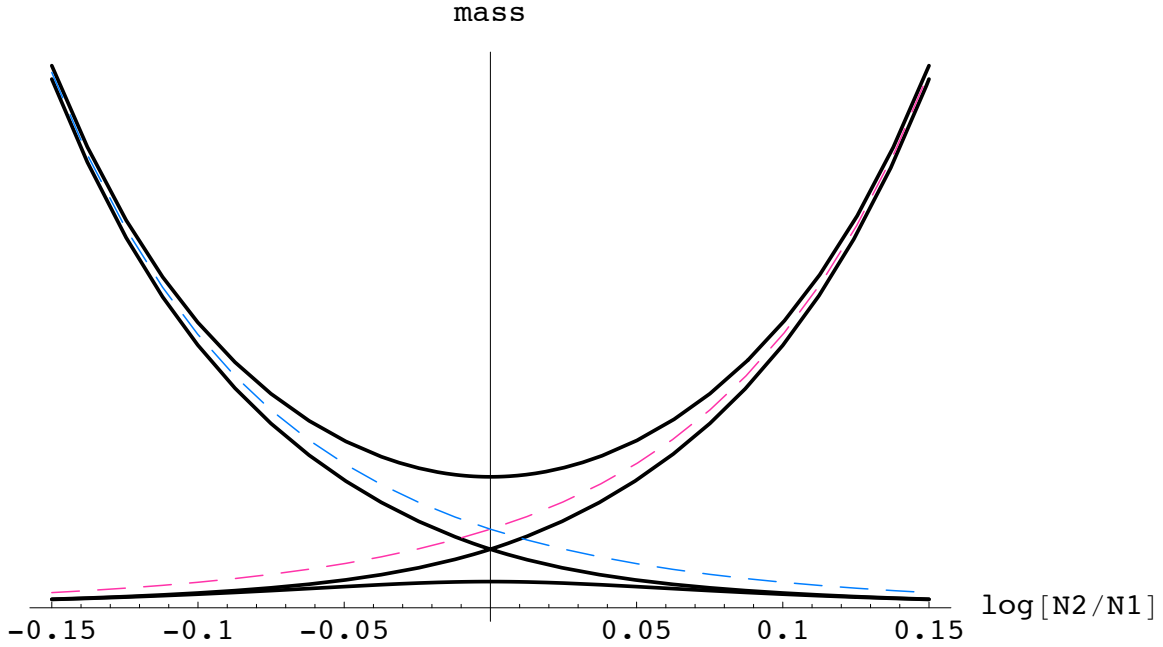


Fig. 1.5. Here we depict the masses of the four bosons as solid lines and the two non-vanishing fermionic masses as dashed lines. The other two fermions, not shown here, are massless Goldstinos. Note that supersymmetry breaking is most pronounced when $|N_1| = |N_2|$. This plot is for fixed $|N_1 N_2|$, as we vary $|N_1/N_2|$.

It is also interesting to write the energy of the vacuum (1.2.15) as a function of v . In particular let ΔE represent the shift in energy from when the branes are infinitely far away from one another. Then we find from (1.2.15) that

$$\frac{\Delta E}{E} = -\frac{\delta}{(\sqrt{\frac{|N_1|}{|N_2|}} + \sqrt{\frac{|N_2|}{|N_1|}})^2}$$

Note that the energy shift goes to zero as $|N_2/N_1| \rightarrow 0, \infty$ as expected. It also goes to zero in the large brane/anti-brane separation, because then $\delta \rightarrow 0$. The sign of the shift is also correct, as one would expect that the attraction between the branes and the anti-branes should would decrease the energy of the system. Note also that for fixed $|N_1 N_2|$, $\Delta E/E$, as well as the mass splittings, is maximal when $N_1 = -N_2$, i.e. when we have equal number of branes and anti-branes, whereas if one of them is much larger than the other, mass splittings as well as $\Delta E/E \rightarrow 0$. This is again consistent with the fact that in the limit with an extreme population imbalance of branes, the supersymmetry breaking goes away.

1.2.5. $n S^3$'s

Let us now generalize our analysis to the case with an arbitrary number of blown up S^3 's. In this case, before the geometric transition there are n shrinking \mathbb{P}^1 's located at $x = a_i, i = 1, 2, \dots, k$. We will denote the distance between them as

$$\Delta_{ij} = a_i - a_j$$

As in the simpler case, for a specified total number of branes, we can classify the vacua of the theory by the distribution of the branes among the \mathbb{P}^1 's, and these vacua will be separated by potential barriers corresponding to the increase in the size of the wrapped \mathbb{P}^1 between critical points.

The effective superpotential of the dual geometry after the geometric transition, in which the branes have been replaced by fluxes, is given by (1.2.4). The B-periods will now take the form

$$2\pi i \partial_{S_i} \mathcal{F}_0 = S_i \log \frac{S_i}{W''(a_i) \Lambda_0^2} + \sum_{j \neq i} S_j \log \left(\frac{\Delta_{ij}^2}{\Lambda_0^2} \right) + \dots$$

where again, Λ_0 is interpreted as a UV cutoff for the theory, and we are omitting terms analogous to the polynomial terms in the example of the last section.

To leading order, the generalization of the 2 S^3 case to the current situation is straightforward. We have

$$2\pi i \tau_{ii} = 2\pi i \partial_{S_i}^2 \mathcal{F}_0 \approx \log \left(\frac{S_1}{W''(a_i) \Lambda_0^2} \right)$$

$$2\pi i \tau_{ij} = 2\pi i \partial_{S_i} \partial_{S_j} \mathcal{F}_0 \approx -\log \left(\frac{\Lambda_0^2}{\Delta_{ij}^2} \right)$$

The potential (3.7) has its critical points where

$$\partial_{S_i} V = \frac{i}{2} \mathcal{F}_{ijl} g^{S_j S'_j} g^{S_k S'_k} (\bar{\alpha}_j + \bar{\tau}_{jj'} N^{j'}) (\alpha_k + \bar{\tau}_{kk'} N^{k'}) = 0 \quad (1.2.18)$$

The solutions which correspond to minima are then determined by

$$Re(\alpha) + Re(\tau)_{ij} N^j = 0$$

and

$$Im(\alpha) + Im(\tau)_{ij} |N^j| = 0$$

Indeed, in the case where all N^i are either positive (negative), these are just the (non-)standard F-flatness conditions of the $\mathcal{N} = 1$ supersymmetric theory, so they will be satisfied exactly. In the non-supersymmetric cases, the conditions will receive perturbative corrections coming from the corrections to the Kähler potential at higher string loops.

The expectation values of the bosons S^i take a natural form in these vacua, being given by

$$\langle S_i \rangle = W''(a_i) \Lambda_0^2 \prod_{j \neq i}^{N_j > 0} \left(\frac{\Lambda_0}{\Delta_{ij}} \right)^{2 \frac{|N_j|}{|N_i|}} \prod_{k \neq i}^{N_k < 0} \left(\frac{\Lambda_0}{\Delta_{ik}} \right)^{2 \frac{|N_k|}{|N_i|}} \exp \left(-\frac{2\pi i \alpha}{|N_i|} \right), \quad N_i > 0$$

$$\langle S_i \rangle = W''(a_i) \Lambda_0^2 \prod_{j \neq i}^{N_j > 0} \left(\frac{\Lambda_0}{\Delta_{ij}} \right)^{2 \frac{|N_j|}{|N_i|}} \prod_{k \neq i}^{N_k < 0} \left(\frac{\Lambda_0}{\Delta_{ik}} \right)^{2 \frac{|N_k|}{|N_i|}} \exp \left(\frac{2\pi i \bar{\alpha}}{|N_i|} \right), \quad N_i < 0$$

The explanation of these expectation values is the natural generalization of the case from the previous section. We characterize each gauge theory in the IR by a scale Λ_i , where $S_i = \Lambda_i^3$. The massive adjoints near the i 'th critical point now have $m_{\Phi_i} = W''(a_i)$, whereas the massive W bosons have mass $m_{W_{ij}} = \Delta_{ij}$. Then matching scales, we find

$$\begin{aligned} \langle S_i^{\text{brane}} \rangle^{|N_i|} &= \Lambda_i^{3|N_i|} = e^{-2\pi i\alpha} \Lambda_0^{2N_{\text{brane}}} \bar{\Lambda}_0^{2N_{\text{antibrane}}} m_{\Phi_i}^{|N_i|} \prod_{j \neq i}^{\text{same}} m_{W_{ij}}^{-2N_j} \prod_{j \neq i}^{\text{opposite}} \overline{m_{W_{ij}}}^{-2N_j} \\ \langle S_i^{\text{antibrane}} \rangle^{|N_i|} &= \Lambda_i^{3|N_i|} = e^{2\pi i\bar{\alpha}} \bar{\Lambda}_0^{2N_{\text{brane}}} \Lambda_0^{2N_{\text{antibrane}}} m_{\Phi_i}^{|N_i|} \prod_{j \neq i}^{\text{same}} m_{W_{ij}}^{-2N_j} \prod_{j \neq i}^{\text{opposite}} \overline{m_{W_{ij}}}^{-2N_j} \end{aligned}$$

Above, N_{brane} and $N_{\text{anti-brane}}$ are the total number of branes and anti-branes, respectively, and the products are over the sets of branes of the same and opposite types as the brane in question. In the supersymmetric case, this reduces to the expected relation between the scales. The non-supersymmetric case is, to leading order, identical except that the branes of opposite type contribute complex conjugate running, which is explained in section 3.6.

Similarly to the $2 S^3$ case, the gauge coupling constant in each $U(N_k)$ factor will run as

$$2\pi i\alpha_i = 2\pi i\alpha + \log\left(\frac{\Lambda_0}{m_{\Phi_i}}\right)^{|N_i|} - 2 \sum_{j \neq i}^{\text{same}} \log\left(\frac{\Lambda_0}{m_{W_{ij}}}\right)^{|N_j|} - 2 \sum_{j \neq i}^{\text{opposite}} \log\left(\frac{\bar{\Lambda}_0}{\overline{m_{W_{ij}}}}\right)^{|N_j|},$$

for a brane, ($N_i > 0$). For antibrane, ($N_i < 0$) and the running is

$$2\pi i\alpha_i = \overline{2\pi i\alpha} + \log\left(\frac{\Lambda_0}{m_{\Phi_i}}\right)^{|N_i|} - 2 \sum_{j \neq i}^{\text{same}} \log\left(\frac{\Lambda_0}{m_{W_{ij}}}\right)^{|N_j|} - 2 \sum_{j \neq i}^{\text{opposite}} \log\left(\frac{\bar{\Lambda}_0}{\overline{m_{W_{ij}}}}\right)^{|N_j|}.$$

This again has a simple form corresponding to integrating out massive adjoints and W bosons. Moreover, the fields corresponding to strings with both endpoints on branes, or on the anti-branes, contribute to running of the gauge coupling as in the supersymmetric case. The fields coming from strings with one end on the brane and the other on the anti-brane, again give remarkably simple contributions, almost as in the supersymmetric case: the only apparent difference being in how they couple to the θ angle. We will come back to this later.

One could in principle compute the masses of the bosons and fermions in a generic one of these vacua. We will not repeat this here. Instead, let us just try to understand the *massless* fermions, or the structure of supersymmetry breaking in this theory. The results of section 2 are easy to generalize, to find the fermion mass matrices⁹

⁹ Note that the mass matrices are now being expressed not in the basis in which the kinetic terms take the canonical form, but in the basis in which the Lagrangian was originally expressed. This will not affect our analysis.

$$m_{\psi^i \psi^j} = -\frac{i}{2} g^{S_l \bar{S}_k} \mathcal{F}_{ijl} (\alpha_k + \bar{\tau}_{kk'} N^{k'})$$

$$m_{\lambda^i \lambda^j} = -\frac{i}{2} g^{S_l \bar{S}_k} \mathcal{F}_{ijl} (\bar{\alpha}_k + \bar{\tau}_{kk'} N^{k'}).$$

In the above, the repeated indices are summed over, and we have denoted

$$\mathcal{F}_{ijk} = \frac{\partial^3 \mathcal{F}_0}{\partial S_i \partial S_j \partial S_k}.$$

At the order to which we are working (corresponding to one loop on the field theory side), \mathcal{F}_{ijk} is diagonal. The masses then imply that in vacua where the standard F-term

$$\partial_{S_i} \mathcal{W} = \alpha_i + \tau_{ij} N^j$$

fails to vanish, a zero eigenvalue is generated for the ψ mass matrix, while when the non-standard F-term

$$\partial_{S_i} \widetilde{\mathcal{W}} = \bar{\alpha}_i + \tau_{ij} N^j$$

fails to vanish, a zero eigenvalue is generated for the λ mass matrix.

More generally, note that the *exact* conditions (at leading order in $1/N$) for critical points of the potential (1.2.18) can be re-written in terms of the fermion mass matrices as

$$m_{\psi^i \psi^j} g^{S_j S_k} (\bar{\alpha}_k + \bar{\tau}_{kk'} N^{k'}) = 0 \quad (1.2.19)$$

$$m_{\lambda^i \lambda^j} g^{S_j S_k} (\alpha_k + \bar{\tau}_{kk'} N^{k'}) = 0. \quad (1.2.20)$$

In the case when supersymmetry is unbroken, and all the \mathbf{P}^1 's are wrapped by branes, we will find both of these equations to be truly exact, and satisfied trivially: the λ mass matrix vanishes identically, so (1.2.20) is trivial, and (1.2.19) is satisfied without constraining $m_{\psi^i \psi^j}$ directly since the vanishing of $(\alpha_k + \bar{\tau}_{kk'} N^{k'})$ is the F-flatness condition. A similar story holds in the anti-brane vacuum.

In the case with both branes and antibranes, the equations are *not* trivial. Instead, they say that both the ψ - and the λ -mass matrices have at least one zero eigenvalue. At the string tree level (i.e. at the leading order in $1/N$ expansion), we expect them to have *exactly* one zero eigenvalue each. To this order, we can think about turning on the fluxes as simply giving them an “expectation value”, which would break the $\mathcal{N} = 2$ supersymmetry spontaneously. However, at higher orders, the Kähler potential is *not* protected in that the symmetry breaking is soft, but not spontaneous. Consequently, the form of the equations (1.2.19), (1.2.20) should presumably receive corrections, and only one massless fermion should remain, corresponding to breaking $\mathcal{N} = 1$ to $\mathcal{N} = 0$.

The leading-order potential in these vacua is given by

$$V_* = \frac{8\pi}{g_{\text{YM}}^2} \left(\sum_i |N_i| \right) - \left(\sum_{i,j}^{N_i > 0, N_j < 0} \frac{2}{\pi} |N_i| |N_j| \log \left| \frac{\Lambda_0}{\Delta_{ij}} \right| \right)$$

This takes the natural form of a brane-tension contribution for each brane in the system, plus interaction terms between the brane/anti-brane pairs. If we define

$$\delta_{ij} \equiv \left(\frac{g_{\text{YM}}}{2\pi}\right)^2 \left(\sum_k |N_k|\right) \log \left| \frac{\Lambda_0}{\Delta_{ij}} \right|.$$

Then in terms of these parameters δ_{ij} , the physical potential takes the simpler form of

$$V_* = \frac{8\pi}{g_{\text{YM}}^2} \left(\sum_i |N_i| - \frac{\sum_{i,j}^{N_i>0, N_j<0} \delta_{ij} |N_i| |N_j|}{\sum_i |N_i|} \right)$$

so that the energy shift is

$$\frac{\Delta E}{E} = \frac{-\sum_{i,j}^{N_i>0, N_j<0} \delta_{ij} |N_i| |N_j|}{(\sum_i |N_i|)^2}$$

Once again, the forces among the (anti-)branes cancel. Moreover, the branes and the anti-branes attract, which lowers the energy of the vacuum, below the supersymmetric one. This energy shift will effectively vanish only when there is an extreme imbalance between number of the branes and antibranes, just as in the 2 S^3 case.

1.2.6. Brane/anti-brane gauge system

It is natural to ask whether we can find a simple description of the non-supersymmetric theory corresponding to the brane/anti-brane system. This certainly exists in the full open string field theory. However, one may wonder whether one can construct a field theory version of this which maintains only a *finite* number of degrees of freedom. In fact, it is not clear if this should be possible, as we will now explain.

Consider a simpler situation where we have a stack of N parallel D3 branes. This gives an $\mathcal{N} = 4$ supersymmetric $U(N)$ gauge theory. Now consider instead $(N + k)$ D3 branes and k anti-D3 branes separated by a distance a . This is clearly unstable, and will decay back to a system with N D3 branes. Can one describe this in a field theory setup? There have been attempts [32] along these lines (in connection with Sen's conjecture [33]), however no complete finite truncation seems to exist. This system is similar to ours. In particular, consider the case with 2 S^3 's, with a total of N D5 branes. We know that we can have various supersymmetric vacua of this theory in which

$$U(N) \rightarrow U(N_1) \times U(N - N_1)$$

where we have N_1 branes wrap one \mathbb{P}^1 and $N - N_1$ wrap the other. However we also know that the theory with N D5 branes can come from a metastable vacuum with $(N + k)$ D5 branes wrap around one \mathbb{P}^1 and k anti-D5 branes around the other. This theory would have a $U(N + k) \times U(k)$ gauge symmetry. Clearly we need more degrees of freedom than are present in the $U(N)$ theory in order to describe this theory. In fact, since k can be arbitrarily large, we cannot have a single system with a finite number of degrees of

freedom describing all such metastable critical points. Of course, one could imagine that there could be a theory with a finite number of fields which describes all such configurations up to a given maximum k , but this does not seem very natural. It is thus reasonable to expect that string field theory would be needed to fully describe this system.

Nevertheless we have seen, to leading order, a very simple running of the two gauge groups in the non-supersymmetric case, and it would be natural to ask if we can explain this from the dual open string theory. In this dual theory, where we have some branes wrapping the first \mathbb{P}^1 and some anti-branes the other, we have three different types of sectors: the 1-1, 2-2 and the 1-2 open string sectors. The 1-1 and the 2-2 subsectors are supersymmetric and give rise to the description of the $\mathcal{N} = 1$ supersymmetric $U(N_i)$ theories for $i = 1, 2$, coupled to the adjoint matter fields Φ which are massive. This part can be inherited from the supersymmetric case, and in particular explains the fact that the running of the coupling constant of the $U(N_i)$ from the massive field Φ is the same as in the supersymmetric case. The difference between our case and the supersymmetric case comes from the 1-2 subsector. In the NS sector we have the usual tachyon mode whose mass squared is shifted to a positive value, as long as Δ is sufficiently larger than the string scale, due to the stretching of the open string between the far separated \mathbb{P}^1 's. We also have the usual oscillator modes. In the Ramond sector ground state the only difference between the supersymmetric case and our case is that the fermions, which in the supersymmetric case is the gaugino partner of the massive W_{12} boson, has the opposite chirality. The fact that they have the opposite chirality explains the fact that the θ term of the gauge group runs with an opposite sign compared to the supersymmetric case. In explaining the fact that the norm runs the same way as the supersymmetric case, the contributions of the fermions is clear, but it is not obvious why the NS sector contribution should have led to the same kind of running. It would be interesting to explain this directly from the open string theory annulus computation in the NS sector.

It is easy to generalize the discussion above to the $n S^3$ case. In fact just as in the $2 S^3$ case, the only subsectors which are non-supersymmetric are the $i - j$ sectors where i is from a brane and j is from an anti-brane cut. This agrees with the results obtained in the general case, where supersymmetry breaking contributions come, to leading order, precisely from these sectors.

1.3. Decay Rates

In the previous sections, we constructed non-supersymmetric vacua of string theory corresponding to wrapping D5 branes and anti-D5 branes on the two-cycles of the a local Calabi-Yau. Here, branes and anti-branes are wrapping rigid two-cycles which are in the same homology class, but are widely separated, and there are potential barriers between them of tunable height. As we emphasized, the crucial aspect of this is that the vacua obtained in this way can be long lived, and thus are different from brane-anti brane systems in flat space, or systems of branes and anti-branes probing the Calabi-Yau manifold, which have been considered in the literature. Moreover, when the numbers of branes on each two-cycle is large, this has a holographic dual in terms of non-supersymmetric flux vacua. By duality, we also expect the corresponding flux vacua to be metastable, despite breaking supersymmetry. In this section we will explore the stability of these vacua in more detail.

In both open and closed string language, the theory starts out in a non-supersymmetric vacuum. Since there are lower energy states available, the non-supersymmetric vacuum is a false vacuum. As long as the theory is weakly coupled throughout the process, as is the case here, the decay can be understood in the semi-classical approximation. This does not depend on the details of the theory, and we will review some aspects of the beautiful analysis in [34]. Afterwards, we apply this to the case at hand.

The false vacuum decays by nucleating a bubble of true vacuum by an instanton process. The rate of decay Γ is given in terms of the action of the relevant instanton as

$$\Gamma \sim \exp(-S_I).$$

The instanton action S_I is the action of the euclidean bounce solution, which interpolates between the true vacuum inside the bubble and the false vacuum outside of the bubble¹⁰. Assuming that the dominant instanton is a spherical bubble of radius R , the euclidean instanton action is given by [34]

$$S_I = -\frac{\pi^2}{2}R^4\Delta V + 2\pi^2R^3S_D$$

The first term is the contribution to the action from inside the bubble. Here ΔV is the difference in the energy density between the true and the false vacuum, and this is multiplied by the volume of the bubble (a four-sphere of radius R). The second term is the contribution to the action from the domain wall that interpolates between the inside and the outside on the bubble, assuming the domain wall is thin. There, S_D is the tension of the domain wall, and $2\pi^2R^3$ is its surface area (the area of a three-sphere of radius R). The radius R of the bubble is determined by energetics: The bubble can form when the gain in energy compensates for the mass of the domain wall that is created. The energy cost to create a bubble of radius R is

$$E = -\frac{4\pi}{3}R^3\Delta V + 4\pi R^2S_D$$

where the first term comes from lowering the energy of the vacuum inside the bubble, and the last term is the energy cost due to surface tension of the domain wall. We can create a bubble at no energy cost of radius

$$R_* = 3\frac{S_D}{\Delta V}.$$

Bubbles created with radii smaller than this recollapse. The bubbles created at $R = R_*$ expand indefinitely, as this is energetically favored. At any rate, the relevant instanton action is $S_I(R_*)$, or

$$S_I = \frac{27\pi}{2} \frac{S_D^4}{(\Delta V)^3}.$$

¹⁰ The coefficient of proportionality comes from the one loop amplitude in the instanton background, and as long as S_I is large, its actual value does not matter.

Correspondingly, the vacuum is longer lived the higher the tension of the domain wall needed to create it, and the lower the energy splitting between the true and the false vacuum.

With this in hand, let us consider the decays of the vacua we found. When the number of branes wrapping each \mathbb{P}^1 is small, the open string picture is appropriate. Consider, for simplicity, the case where the superpotential has only two critical points

$$W'(x) = g(x - a_1)(x - a_2),$$

and we have only a single brane at $x = a_1$ and a single anti-brane at $x = a_2$. The effective potential that either of these branes sees separately is given simply by the size of the 2-sphere it wraps

$$A(x) = (|W'(x)|^2 + |r|^2)^{1/2} \tag{1.3.1}$$

where $|r|$ is the tension of the brane wrapping the minimal \mathbb{P}^1 (see section 3). The branes are, of course, also charged, so there is an electric field flux tube (of the six-form potential) running between the brane and the anti-brane, along a three-cycle corresponding to an S^2 being swept from $x = a_1$ to $x = a_2$. Despite the flux tube, the system is (meta)stable if the potential barrier from (1.3.1) is high enough. The decay process corresponds, for example, to having the D5-brane tunnel under the energy barrier from the vacuum at $x = a_1$ to the vacuum $x = a_2$, to annihilate with the anti-D5-brane there. More precisely, what happens is that a bubble of true vacuum is created in $R^{3,1}$, inside of which the branes are annihilated. The boundary of the bubble is a domain wall which, in the thin wall approximation, corresponds to a D5 brane wrapping the S^2 boundary of the bubble in R^3 and the three-cycle in the Calabi-Yau where the flux tube was. The tension of the domain wall is the the size of the three-cycle that the D5 brane wraps in the Calabi-Yau times the tension of the 5-brane

$$S_D = \frac{1}{g_s} \int_{a_1}^{a_2} A(x) dx \approx \frac{1}{g_s} |W(a_2) - W(a_1)| = \frac{1}{3g_s} |g\Delta^3|$$

where the second term in (1.3.1) gives vanishingly small contribution¹¹ in the limit $\Delta \gg r$. The difference in the cosmological constants of the two vacua corresponds, to the leading order, to the difference in tensions between the original configuration, with $V_i = 2|r|/g_s$ and the final one where the branes have annihilated $V_f = 0$, so the gain in energy is simply

$$\Delta V = 2 \frac{|r|}{g_s} = \frac{2}{g_{YM}^2}$$

When the number of branes wrapping the \mathbb{P}^1 's is large, we use the dual geometry from after the transition, with fluxes. Consider, for example, the case when N_1 branes wraps the first \mathbb{P}^1 and $|N_2|$ anti-branes the second, with $|N_{1,2}|$ large, and satisfying

$$N_1 > 0 > N_2.$$

¹¹ For us, the mass of the brane comes from the B-field only, and the brane is heavy only because the string coupling is weak.

Then, the C cycle that runs between the two cuts, $[C] = [B_1] - [B_2]$ is a compact flux tube, with $|N_2|$ units of H^{RR} flux running through it. To decay, one kills off the flux lines one by one, by creating an D5 brane wrapping a zero size \mathbb{P}^1 where the first \mathbb{P}^1 used to be before the transition. This brane grows and eats up the flux line until it vanishes again at the first cut. After all the flux has decayed, this leaves us with $N_1 + N_2$ units of flux through the first S^3 . The tension of the corresponding domain wall is

$$S_D = \frac{|N_2|}{g_s} \int_C |\Omega|$$

where

$$\oint_C \Omega = \int_{B_1} \Omega - \int_{B_2} \Omega \approx W(a_2) - W(a_1).$$

where we omit terms that are exponentially suppressed in the vacuum.¹² In this case, the energy of the false vacuum is, to the leading order, $V_i = (|N_1| + |N_2|)|r|/g_s$. After the flux has decayed, we have $V_f = (|N_1| - |N_2|)|r|/g_s$ coming from the flux through first S^3 only. The difference is just

$$\Delta V = V_i - V_f \approx 2|N_2| \frac{|r|}{g_s} = \frac{2|N_2|}{g_{YM}^2}$$

We see that both the domain wall tension *and* the difference in vacuum energies before and after are equal to what we found in the open string picture, albeit rescaled by the amount of brane charge which disappears! This is, of course, not surprising because the domain wall relevant in the closed string case is the same one as that relevant in the open string case, just before the large N transition.

We can now put everything together to compute the action of the instanton corresponding to nucleating a bubble of true vacuum:

$$S_I \approx \frac{\pi}{48} \frac{|N_2|}{g_s} \frac{|(g\Delta^3)^4|}{|r|^3}.$$

This leads to the decay rate

$$\Gamma \sim \exp(-S_I).$$

Note that the relevant instantons come from D-branes wrapping cycles, and thus it is natural for the instanton action to depend on g_s in the way the D-brane action does.

This exactly reproduces the results one would have expected from our discussion in section 3. There are two effects controlling the stability of the system. Namely, one is the height of the potential barrier, which is controlled by here by $g\Delta^3$ and which enters the action of the domain wall. The branes and anti-branes become more separated the larger the Δ , and their interactions weaker. In addition, increasing the height of the potential

¹² More precisely, we should evaluate the corrections to this at the values of $S_{1,2}$ corresponding to being somewhere in the middle between the true and the false vacuum. Either way, the omitted terms are suppressed by, at least, $\frac{\Lambda_{1,2}}{\Delta}$, and we can neglect them.

barrier by making Δ larger, overcomes the electrostatic attraction of the branes. The other relevant parameter is the relative difference in energies of the true and the false vacuum, which depends on the ratio of brane to anti-brane numbers $|N_2/N_1|$. When this is very small, the vacuum with both the branes and the anti-branes is nearly degenerate with the vacuum with just the branes, and correspondingly, the decay time gets longer as $\Delta V/V$ gets smaller.

1.4. Open Questions

In this chapter, we have seen how a large N system of branes which geometrically realizes metastability by wrapping cycles of a non-compact CY can be dual to a flux compactification which breaks supersymmetry. We provided evidence for this duality by studying the limit where the cycles of the Calabi-Yau are far separated.

There are a number of open questions which remain to be studied: It would be nice to write down a non-supersymmetric gauge theory with finite number of degrees of freedom which describes this brane configuration. It is not clear that this should be possible, but it would be interesting to settle this question conclusively one way or the other. This is also important for other applications in string theory where dynamics of brane/anti-brane systems is relevant.

Another issue which would be important to study is the phase structure of our system. Since we have broken supersymmetry, we have no holomorphic control over the geometry of the solutions. It would be interesting to see what replaces this, and how one should think of the global phase structure of these non-supersymmetric solutions. Since the full potential is characterized by the $\mathcal{N} = 2$ prepotential and the flux data, which in turn are characterized by integrable data (matrix model integrals of the superpotential), one still expects that the critical points of the potential should be characterized in a nice way. Understanding this structure would be very important. For example, it would tell us what happens when the separation of the wrapped cycles is small where one expects to lose metastability due to the open string tachyons stretching between branes and anti-branes.

One other issue involves the $1/N$ corrections to the holographic dual theory. In this chapter we have focused on the leading order in the $1/N$ expansion, i.e. at the string tree level. How about subleading corrections? This is likely to be difficult to address, as one expects non-supersymmetric corrections to string tree level to be difficult to compute. The fact that we have a metastable system suggests that we would not want to push this question to an exact computation, as we know the system is ultimately going to decay to a stable system. However, we would potentially be interested in studying the regions of phase space where metastability is just being lost, such as when the cycles are close to one another.

Finally, perhaps the most important question is about the embedding of our mechanism for inducing metastable flux compactification vacua into a *compact* Calabi-Yau geometry (see [35]). At first sight, it may be unclear whether there would be any obstacles to a compact embedding of our story. There is at least an example similar to what we are expecting in the compact case. In [36], among the solutions studied, it was found that flipping the signs of some fluxes leads to apparently metastable non-supersymmetric

vacua. It would be interesting to see if this connects to the mechanism introduced in this chapter, where the existence of metastability for flux vacua is a priori expected. Moreover, from our results it is natural to expect that most flux vacua *do* have metastable non-supersymmetric solutions. For instance, in the non-compact case we have studied, only 2 out of 2^n choice of signs for the n fluxes were non-supersymmetric. This suggests that in the study of flux compactifications, the most natural way to break supersymmetry is simply studying metastable vacua of that theory, without the necessity of introducing any additional anti-branes into the system. It would be very interesting to study this further.

Chapter 2

Metastable Quivers and Holography

Geometric transitions have proven to be a powerful means of studying the dynamics of supersymmetric D-branes. String theory relates these transitions to large N dualities, where before the transition, at small 't Hooft coupling, one has D-branes wrapping cycles in the geometry, and after the transition, at large 't Hooft coupling, the system is represented by a different geometry, with branes replaced by fluxes. The AdS/CFT correspondence can be thought of in this way. Geometric transitions are particularly powerful when the D-branes in question wrap cycles in a Calabi-Yau manifold. Then, the topological string can be used to study the dual geometry exactly to all orders in the 't Hooft coupling. In [4] it was conjectured that topological strings and large N dualities can also be used to study non-supersymmetric, metastable configurations of branes in Calabi-Yau manifolds, that confine at low energies. This conjecture was considered in greater detail in [37,38]. String theory realizations of metastable, supersymmetry breaking vacua have appeared in [16,17,39-46]. The gauge theoretic mechanism of [11] has further been explored in string theory in [12,13,14,47-51].

In this chapter we study D5 brane/anti-D5 brane systems in IIB on non-compact, Calabi-Yau manifolds that are ADE type ALE space fibrations over a plane. These generalize the case of the A_1 ALE space studied in detail in [4,37,38]. The ALE space is fibered over the complex plane in such a way that at isolated points, the two-cycles inherited from the ALE space have minimal area. These minimal two-cycles are associated to positive roots of the corresponding ADE Lie algebra. Wrapping these with branes and antibranes is equivalent to considering only branes, but allowing both positive and *negative* roots to appear, corresponding to two different orientations of the S^2 's. The system can be metastable since the branes wrap *isolated* minimal two-cycles, and the cost in energy for the branes to move, due to the tensions of the branes, can overwhelm the Coulomb/gravitational attraction between them.

The geometries in question have geometric transitions in which the sizes of the minimal

S^2 's go to zero, and the singularities are resolved instead by finite sized S^3 's. The conjecture of [4] is that at large N , the S^2 's disappear along with the branes and antibranes and are replaced by S^3 's with positive and negative fluxes, the sign depending on the charge of the replaced branes. As in the supersymmetric case (see [18,31,52]), the dual gravity theory has $\mathcal{N} = 2$ supersymmetry softly broken to $\mathcal{N} = 1$ by the fluxes. The only difference is that now some of the fluxes are negative. On-shell, the positive and the negative fluxes preserve different halves of the original supersymmetry, and with both present, the $\mathcal{N} = 2$ supersymmetry is completely broken in the vacuum (see [53] for discussion of a similar supersymmetry breaking mechanism and its phenomenological features in the context of heterotic M-theory). The topological string computes not only the superpotential, but also the Kähler potential.¹³ We show that the Calabi-Yau's with fluxes obtained in this way are indeed metastable, as expected by holography. In particular, for widely separated branes, the supersymmetry breaking can be made arbitrarily weak.¹⁴ In fact, we can use the gravity dual to learn about the physics of branes and antibranes. We find that at one-loop, the interaction between the branes depends on the topological data of the Calabi-Yau in a simple way. Namely, for every brane/antibrane pair, so for every positive root e_+ and negative root e_- , we find that the branes and the antibranes attract if the inner product

$$e_+ \cdot e_-$$

is *positive*. They repel if it is negative, and do not interact at all if it is zero. In the A_k type ALE spaces, this result is already known from the direct open string computation [54,55], so this is a simple but nice test of the conjecture for these geometries. Moreover, we show that certain aspects of these systems are universal. We find that generically, just like in [37], metastability is lost when the 't Hooft coupling becomes sufficiently large. Moreover, once stability is lost, the system appears to roll down toward a vacuum in which domain walls interpolating between different values of the fluxes become light. We also present some special cases where the non-supersymmetric brane/antibrane systems are exactly stable. In these cases, there are no supersymmetric vacua to which the system can decay.

When all the branes are D5 branes and supersymmetry is preserved, the low energy theory geometrically realizes [31,52] a 4d $\mathcal{N} = 2$ supersymmetric quiver gauge theory with a superpotential for the world-volume adjoints which breaks $\mathcal{N} = 2$ to $\mathcal{N} = 1$. These theories are known to have Seiberg-like dualities [56] in which the dual theories flow to the same IR fixed point, and where different descriptions are more weakly coupled, and hence preferred, at different energy scales. The Seiberg dualities are realized in the geometry in a beautiful way [52]. The ADE fibered Calabi-Yau geometries used to engineer the

¹³ While the superpotential is exact, the Kähler potential is not. Corrections to the Kähler potential coming from warping, present when the Calabi-Yau is compact, have been investigated in [46].

¹⁴ The natural measure of supersymmetry breaking in this case is the mass splitting between the bosons and their superpartners. For a compact Calabi-Yau, the scale of supersymmetry breaking is set by the mass of the gravitino, which is of the order of the cosmological constant. In our case, gravity is not dynamical, and the mass splittings of the dynamical fields are tunable [4].

gauge theories have *intrinsic* ambiguities in how one resolves the singularities by blowing up S^2 's. The different possible resolutions are related by flops that shrink some two-cycles, and blow up others. The flops act nontrivially on the brane charges, and hence on the ranks of the gauge groups. The flop of a two-cycle $S_{i_0}^2$ corresponds to a Weyl reflection about the corresponding root of the Lie algebra. On the simple roots e_i , this acts by

$$S_i^2 \rightarrow \tilde{S}_i^2 = S_i^2 - (e_i \cdot e_{i_0}) S_{i_0}^2.$$

Brane charge conservation then implies that the net brane charges transform satisfying

$$\sum_i N_i S_i^2 = \sum_i \tilde{N}_i \tilde{S}_i^2. \quad (2.0.1)$$

Moreover, from the dual gravity solution one can reconstruct the whole RG flow of the gauge theory. The sizes of the wrapped two-cycles encode the gauge couplings, and one can read off how these vary over the geometry, and correspondingly, what is the weakly coupled description at a given scale. Near the S^3 's, close to where the branes were prior to the transition, corresponds to long distances in the gauge theory. There, the S^2 's have shrunk, corresponding to the fact that in the deep IR the gauge theories confine. As one goes to higher energies, the gauge couplings may simply become weaker, and the corresponding S^2 's larger, in which case the same theory will describe physics at all energy scales. Sometimes, however, some of the gauge couplings grow stronger, and the areas of the S^2 's eventually become negative. Then, to keep the couplings positive, the geometry must undergo flop transitions.¹⁵ This rearranges the brane charges and corresponds to replacing the original description at low energies by a different one at high energies. Moreover, the flops of the S^2 's were found to coincide exactly with Seiberg dualities of the supersymmetric gauge theories.

In the non-supersymmetric case, there is generally no limit in which these brane constructions reduce to field theories with a finite number of degrees of freedom. Thus there are no gauge theory predictions to guide us. However, the string theory still has intrinsic ambiguities in how the singularities are resolved. This is *exactly* the same as in the supersymmetric case, except that now not all N_i in (2.0.1) need be positive. Moreover, we can use holography to follow the varying sizes of two-cycles over the geometry, and find that indeed in some cases they can undergo flops in going from the IR to the UV. When this happens, descriptions in terms of different brane/antibrane configurations are more natural at different energy scales, and one can smoothly interpolate between them. This is to be contrasted with, say, the A_1 case, where regardless of whether one considers just branes or branes and antibranes, it is only one description that is ever really weakly coupled, and the fact that another exists is purely formal.

The chapter is organized as follows. In section 2 we introduce the metastable D5 brane/anti-D5 brane configurations, focusing on A_k singularities, and review the conjecture of [4] applied to this setting. In section 3 we study in detail the A_2 case with a

¹⁵ It is important, and one can verify this, that this happens in a completely *smooth* way in the geometry, as the gauge coupling going to infinity corresponds to zero Kähler volume of the two-cycle, while the physical size of the two-cycle is finite everywhere away from the S^3 's.

quadratic superpotential. In section 4 we consider general ADE type geometries. In section 5 we discuss Seiberg-like dualities of these theories. In section 6 we study a very simple, exactly solvable case. In appendices A and B, we present the matrix model computation of the prepotential for A_2 ALE space fibration, as well as the direct computation from the geometry. To our knowledge, these computations have not been done before, and the agreement provides a direct check of the Dijkgraaf-Vafa conjecture for these geometries. Moreover, our methods extend easily to the other A_n cases. In appendix 2.C, we collect some formulas useful in studying the metastability of our solutions in section 3.

2.1. Quiver Branes and Antibranes

Consider a Calabi-Yau which is an A_k type ALE space,

$$x^2 + y^2 + \prod_{i=1}^{k+1} (z - z_i(t)) = 0, \quad (2.1.1)$$

fibered over the t plane. Here, $z_i(t)$ are polynomials in t . Viewed as a family of ALE spaces parameterized by t , there are k vanishing two-cycles,

$$S_i^2, \quad i = 1, \dots, k \quad (2.1.2)$$

that deform the the singularities of (2.1.1). In the fiber over each point t in the base, the two-cycle in the class S_i^2 has holomorphic area given by

$$\int_{S_{i,t}^2} \omega^{2,0} = z_i(t) - z_{i+1}(t). \quad (2.1.3)$$

where $\omega^{2,0}$ is the reduction of the holomorphic three-form Ω on the fiber. The only singularities are at points where $x = y = 0$ and

$$z_i(t) = z_j(t), \quad i \neq j \quad (2.1.4)$$

for some i and j . At these points, the area of one of the two-cycles inherited from the ALE space goes to zero.

These singularities can be smoothed out by blowing up the two-cycles, i.e., by changing the Kähler structure of the Calabi-Yau to give them all nonvanishing area.¹⁶ The homology classes of the vanishing cycles (2.1.4) then correspond to positive roots of the A_k Lie algebra (see e.g. [31]).¹⁷ In this case, the k simple, positive roots e_i correspond to the generators of the second homology group. These are the classes of the S_i^2 mentioned above which

¹⁶ As we will review later, the blowup is not unique, as not all the Kähler areas of the cycles in (2.1.2) need to be positive for the space to be smooth. Instead, there are different possible blowups which differ by flops.

¹⁷ The negative roots correspond to two-cycles of the opposite orientation.

resolve the singularities where $z_i(t) = z_{i+1}(t)$. We denote the complexified Kähler areas of the simple roots by

$$r_i = \int_{S_i^2} k + iB^{NS},$$

where k is the Kähler form. In most of our applications, we will take the real part of r_i to vanish. The string theory background is nonsingular as long as the imaginary parts do not also vanish. They are positive, per definition, since we have taken the S_i^2 to correspond to positive roots. In classical geometry, the r_i are independent of t . Quantum mechanically, in the presence of branes, one finds that they are not.

There are also positive, non-simple roots $e_I = \sum_{i=j}^l e_i$, for $l > j$ where $z_{l+1}(t) = z_j(t)$. The two-cycle that resolves the singularity is given by

$$S_I^2 = \sum_{i=j}^l S_i^2$$

in homology. Its complexified Kähler area is given as a sum of Kähler areas of simple roots

$$r_I = \sum_{i=j}^l r_i.$$

The total area $A(t)$ of a two-cycle S_I^2 at a fixed t receives contributions from both Kähler and holomorphic areas:

$$A_I(t) = \sqrt{|r_I|^2 + |W_I'(t)|^2}. \quad (2.1.5)$$

The functions W_I' capture the holomorphic volumes of two-cycles, and are related to the geometry by

$$W_I(t) = \sum_{i=j}^k W_i(t),$$

$$W_i(t) = \int (z_i(t) - z_{i+1}(t)) dt. \quad (2.1.6)$$

These will reappear as superpotentials in matrix models which govern the open and closed topological string theory on these geometries.

For each positive root I there may be more than one solution to (2.1.4). We will label these with an additional index p when denoting the corresponding two-cycles, $S_{I,p}^2$. For each solution there is an isolated, minimal area S^2 , but they are all in the same homology class, labeled by the root. They have minimal area because (2.1.5) is minimized at those points in the t plane where $W_I'(t)$ vanishes. These, in turn, correspond to solutions of (2.1.4).

We will consider wrapping branes in the homology class

$$\sum_{I,p} M_{I,p} S_{I,p}^2,$$

with I running over *all* positive roots, and p over the corresponding critical points. We get branes or antibranes on $S_{I,p}^2$ depending on whether the charge $M_{I,p}$ is positive or negative.¹⁸ We will study what happens when we wrap branes on some of the minimal S^2 's and antibranes on others.

The brane/antibrane system is not supersymmetric. If we had branes wrapping all of the S^2 's, they would have each preserved the same half of the original $\mathcal{N} = 2$ supersymmetry. However, with some of the branes replaced by antibranes, some stacks preserve the opposite half of the original supersymmetry, and so globally, supersymmetry is completely broken. The system can still be metastable. As in flat space, there can be attractive Coulomb/gravitational forces between the branes and the antibranes. For them to annihilate, however, they have to leave the minimal two-cycles that they wrap. In doing so, the area of the wrapped two-cycle increases, as can be seen from (2.1.5), and this costs energy due to the tension of the branes. At sufficiently weak coupling, the Coulomb and gravitational interactions should be negligible compared to the tension forces – the former are a one-loop effect in the open string theory, while the latter are present already at tree-level – so the system should indeed be metastable. For this to be possible, it is crucial that the parameters of the background, *i.e.* the Kähler moduli r_i and the complex structure moduli that enter into the $W_i(t)$, are all non-normalizable, and so can be tuned at will.

While this theory is hard to study directly in the open string language, it was conjectured in [4] to have a holographic dual which gives an excellent description when the number of branes is large.

2.1.1. Supersymmetric large N duality

Here we review the case where only branes are wrapped on the minimal S^2 's, and so supersymmetry is preserved. Denoting the net brane charge in the class S_i^2 by N_i , this geometrically engineers an $\mathcal{N} = 2$ supersymmetric $\prod_{i=1}^k U(N_i)$ quiver gauge theory in four dimensions, deformed to $\mathcal{N} = 1$ by the presence of a superpotential. The corresponding quiver diagram is the same as the Dynkin diagram of the A_k Lie algebra. The k nodes correspond to the k gauge groups, and the links between them to bifundamental hypermultiplets coming from the lowest lying string modes at the intersections of the S^2 's in the ALE space. The superpotential for the adjoint valued chiral field Φ_i , which breaks the supersymmetry to $\mathcal{N} = 1$, is

$$W_i(\Phi_i), \quad i = 1, \dots, k$$

where $W_i(t)$ is given in (2.1.6). The chiral field Φ_i describes the position of the branes on the t plane. As shown in [31], the gauge theory has many supersymmetric vacua, corresponding to all possible ways of distributing the branes on the S^2 's,

$$\sum_{i=1}^k N_i S_i^2 = \sum_{I,p} M_{p,I} S_I^2,$$

¹⁸ We could have instead declared all the $M_{I,p}$ to be positive, and summed instead over positive and negative roots.

where I labels the positive roots and p the critical points associated with a given root. This breaks the gauge symmetry as

$$\prod_i U(N_i) \rightarrow \prod_{p,I} U(M_{p,I}). \quad (2.1.7)$$

At low energies the branes are isolated and the theory is a pure $\mathcal{N} = 1$ gauge theory with gauge group (2.1.7). The $SU(M_{I,p})$ subgroups of the $U(M_{I,p})$ gauge groups experience confinement and gaugino condensation.

This theory has a holographic, large N dual where branes are replaced by fluxes. The large N duality is a geometric transition which replaces (2.1.1) with a dual geometry

$$x^2 + y^2 + \prod_{i=1}^{k+1} (z - z_i(t)) = f_{r-1}(t)z^{k-1} + f_{2r-1}(t)z^{k-2} + \dots + f_{kr-1}(t), \quad (2.1.8)$$

where $f_n(t)$ are polynomials of degree n , with r being the highest of the degrees of $z_i(t)$. The geometric transition replaces each of the $S_{I,p}^2$'s by a three-sphere, which will be denoted $A_{I,p}$, with $M_{I,p}$ units of Ramond-Ramond flux through it,

$$\int_{A_{I,p}} H^{RR} + \tau H^{NS} = M_{I,p}.$$

In addition, there is flux through the non-compact dual cycles $B_{I,p}$,

$$\int_{B_{I,p}} H^{RR} + \tau H^{NS} = -\alpha_I,$$

where τ is the IIB axion-dilaton $\tau = a + \frac{i}{g_s}$. These cycles arise by fibering $S_{I,p}^2$ over the t plane, with the two-cycles vanishing at the branch cuts where the S^3 's open up. The nonzero H flux through the B -type cycles means that

$$\int_{S_{I,p}^2} B^{RR} + \tau B^{NS}$$

varies over the t plane. In the gauge theory, this combination determines the complexified gauge coupling. Since

$$\frac{4\pi}{g_i^2} = \frac{1}{g_s} \int_{S_i^2} B^{NS}, \quad \frac{\theta_i}{2\pi} = \int_{S_i^2} B^{RR} + a B^{NS},$$

one naturally identifies α_i with the gauge coupling of the $U(N_i)$, $\mathcal{N} = 2$ theory at a high scale¹⁹

$$\alpha_i = -\frac{\theta_i}{2\pi} - \frac{4\pi i}{g_i^2}.$$

¹⁹ For the large N dual to be an honest Calabi-Yau, as opposed to a generalized one, we will work with $\int_{S_i^2} k = 0$.

For each positive root I , we then define α_I as

$$\alpha_I = \sum_{i=j}^k \alpha_i$$

Turning on fluxes gives rise to an effective superpotential [21]

$$\mathcal{W}_{\text{eff}} = \int_{CY} (H^{RR} + \tau H^{NS}) \wedge \Omega.$$

Using the special geometry relations

$$\int_{A_{I,p}} \Omega = S_{I,p}, \quad \int_{B_{I,p}} \Omega = \partial_{S_{I,p}} \mathcal{F}_0,$$

the effective superpotential can be written as

$$\mathcal{W}_{\text{eff}} = \sum_{I,p} \alpha_I S_{I,p} + M_{I,p} \partial_{S_{I,p}} \mathcal{F}_0. \quad (2.1.9)$$

Here, $S_{I,p}$ gets identified with the value of the gaugino bilinear of the $U(M_{I,p})$ gauge group factor on the open string side. The effective superpotential (2.1.9) can be computed directly in the gauge theory. Alternatively, it can be shown [19,57] that the relevant computation reduces to computing planar diagrams in a gauged matrix model given by the zero-dimensional path integral

$$\frac{1}{\prod_{i=1}^k \text{vol } U(N_i)} \int \prod_{i=1}^k d\Phi_i dQ_{i,i+1} dQ_{i+1,i} \exp\left(-\frac{1}{g_s} \text{Tr} \mathcal{W}(\Phi, Q)\right)$$

where

$$\text{Tr} \mathcal{W}(\Phi, Q) = \sum_{i=1}^r \text{Tr} W(\Phi_i) + \text{Tr} (Q_{i+1,i} \Phi_i Q_{i,i+1} - Q_{i,i+1} \Phi_{i+1} Q_{i+1,i}).$$

The critical points of the matrix model superpotential correspond to the supersymmetric vacua of the gauge theory. The prepotential $\mathcal{F}_0(S_{I,p})$ that enters the superpotential (2.1.9) is the planar free energy of the matrix model [19,58,59,60], expanded about a critical point where the gauge group is broken as in (2.1.7). More precisely, we have

$$2\pi i \mathcal{F}_0(S) = \mathcal{F}_0^{np}(S) + \sum_{\{h_a\}} \mathcal{F}_{0,\{h_a\}} \prod_a S_a^{h_a}$$

where $\mathcal{F}_{0,\{h_a\}} \prod_a (M_a g_s)^{h_a}$ is the contribution to the planar free energy coming from diagrams with h_a boundaries carrying the index of the $U(M_a)$ factor of the unbroken gauge group. Here a represents a pair of indices,

$$a = (I, p),$$

and we've denoted $S_a = M_a g_s$. The “nonperturbative” contribution, $\mathcal{F}_0^{np}(S)$, to the matrix model amplitude comes from the volume of the gauge group (2.1.7) that is *unbroken* in the vacuum at hand [58,60], and is the prepotential of the leading order conifold singularity corresponding to the shrinking S^3 , which is universal. We will explain how to compute the matrix integrals in appendix 2.A. The supersymmetric vacua of the theory are then given by the critical points of the superpotential \mathcal{W}_{eff} .

2.1.2. Non-supersymmetric large N duality

Now consider replacing some of the branes with antibranes while keeping the background fixed. The charge of the branes, as measured at infinity, is computed by the RR flux through the S^3 that surrounds the branes. In the large N dual geometry, the S^3 surrounding the wrapped $S^2_{I,p}$ is just the cycle $A_{I,p}$. Replacing the branes with antibranes on some of the S^2 's then has the effect of changing the signs of the corresponding $M_{I,p}$'s. Moreover, supersymmetry is now broken, so the vacua of the theory will appear as critical points of the physical potential

$$V = G^{S_a \bar{S}_b} \partial_{S_a} \mathcal{W}_{\text{eff}} \overline{\partial_{S_b} \mathcal{W}_{\text{eff}}} + V_0. \quad (2.1.10)$$

The superpotential \mathcal{W}_{eff} is still given by (2.1.9), and G is the Kähler metric of the $\mathcal{N} = 2$ theory,

$$G_{a\bar{b}} = \text{Im}(\tau)_{a\bar{b}}$$

where

$$\tau_{ab} = \partial_{S_a} \partial_{S_b} \mathcal{F}_0$$

and a, b stand for pairs of indices (I, p) . In the absence of gravity, we are free to add a constant, V_0 , to the potential,²⁰ which we will take to be

$$V_0 = \sum_{I,p} \frac{M_{I,p}}{g_I^2}. \quad (2.1.11)$$

A priori, V_0 can be either positive or negative, depending on the charges. However, we will see that in all the vacua where the theory is weakly coupled, the leading contribution to the effective potential at the critical point will turn out to be just the tensions of all the branes, which is strictly positive.

2.2. A Simple Example

We now specialize to an A_2 quiver theory with quadratic superpotential. The geometry which engineers this theory is given by (2.1.1), with

$$z_1(t) = -m_1(t - a_1), \quad z_2(t) = 0, \quad z_3(t) = m_2(t - a_2).$$

There are three singular critical points (2.1.4) (assuming generic m_i) corresponding to

$$t = a_i, \quad i = 1, 2, 3$$

where $a_3 = (m_1 a_1 + m_2 a_2)/(m_1 + m_2)$. Blowing up to recover a smooth Calabi-Yau, the singular points are replaced by three positive area S^2 's,

$$S^2_1, S^2_2, S^2_3$$

²⁰ This simply adds a constant to the Lagrangian, having nothing to do with supersymmetry, or its breaking.

with one homological relation among them,

$$S_3^2 = S_1^2 + S_2^2. \quad (2.2.1)$$

$S_{1,2}^2$ then correspond to the two simple roots of the A_2 Lie algebra, $e_{1,2}$, and S_3^2 is the one non-simple positive root, $e_1 + e_2$. Now consider wrapping branes on the three minimal two-cycles so that the total wrapped cycle C is given by

$$C = M_1 S_1^2 + M_2 S_2^2 + M_3 S_3^2.$$

If some, but not all, of the M_I are negative, supersymmetry is broken. As was explained in the previous section, as long as the branes are widely separated, this system should be perturbatively stable.

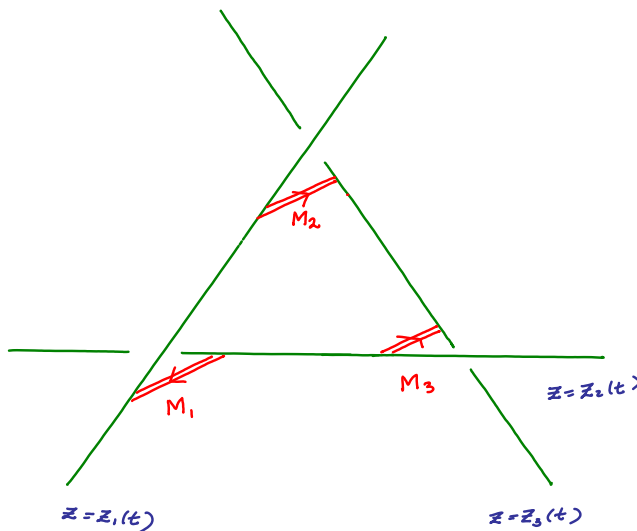


Fig. 2.1. The figure corresponds to the A_2 singularity in the z - t plane with quadratic “superpotential”. There are three conifold singularities at $z_i = z_j$ which can be blown up by three S^2 ’s, spanning two homology classes. Wrapping M_1 anti-D5 branes on S_1^2 and $M_{2,3}$ D5 branes on $S_{2,3}^2$, we can engineer a metastable vacuum. The orientations of the branes are indicated by arrows.

Non-perturbatively, we expect the branes to be able to tunnel to a lower energy state. The minimum energy configuration that this system can achieve depends on the net brane charges in the homology classes S_1^2 and S_2^2 , given by $N_1 = M_1 + M_3$ and $N_2 = M_2 + M_3$. When N_1 and N_2 have the same sign, the system can tunnel to a supersymmetric vacuum with new charges

$$M_I \rightarrow M'_I$$

where all the M'_I share the same sign, and the net charges $N_1 = M'_1 + M'_3$ and $N_2 = M'_2 + M'_3$ are unchanged. All the supersymmetric vacua are degenerate in energy, but for the metastable, non-supersymmetric vacua, the decay rates will depend on the M'_I . Alternatively, if one of the $N_{1,2}$ is positive and the other is negative, the lowest energy configuration is necessarily not supersymmetric. In this way we get a stable, non-supersymmetric state which has nowhere to which it can decay.

In the remainder of this section, we will study these systems using the large N dual geometry with fluxes.

2.2.1. The large N dual

The large N dual geometry in this case is given by

$$x^2 + y^2 + z(z - m_1(t - a_1))(z + m_2(t - a_2)) = cz + dt + e. \quad (2.2.2)$$

The three S^2 's at the critical points have been replaced by three S^3 's, A_I , whose sizes are related to the coefficients c, d, e above. There are also three non-compact, dual three-cycles B_I . The geometry of the Calabi-Yau is closely related to the geometry of the Riemann surface obtained by setting $x = y = 0$ in (2.1.8). The Riemann surface can be viewed as a triple cover of the t plane, by writing (2.2.2) as

$$0 = (z - z'_1(t))(z - z'_2(t))(z - z'_3(t))$$

where $z'_i(t)$ correspond to the $z_i(t)$ which are deformed in going from (2.1.1) to (2.2.2). In particular, the holomorphic three-form Ω of the Calabi-Yau manifold descends to a one-form on the Riemann surface, as can be seen by writing

$$\Omega = \omega^{2,0} \wedge dt$$

and integrating $\omega^{2,0}$ over the S^2 fibers, as in (2.1.3). The A and B cycles then project to one-cycles on the Riemann surface. The three sheets are glued together over branch cuts which open up at $t = a_I$. We have

$$S_I = \frac{1}{2\pi i} \int_{a_I^-}^{a_I^+} (z'_J(t) - z'_K(t)) dt, \quad \partial_{S_I} \mathcal{F}_0 = \frac{1}{2\pi i} \int_{a_I^+}^{\Lambda_0} (z'_J(t) - z'_K(t)) dt$$

for cyclic permutations of distinct I, J and K . This allows one to compute the prepotential \mathcal{F}_0 by direct integration (see appendix 2.B). Alternatively, by the conjecture of [58], the same prepotential can be computed from the corresponding matrix model. The gauge fixing of the matrix model is somewhat involved, and we have relegated it to appendix 2.A, but the end result is very simple. The field content consists of:

- a. Three sets of adjoints Φ_{ii} of $U(M_i)$, which describe the fluctuations of the branes around the three S^2 's.
- b. A pair of bifundamental matter fields Q_{12}, \tilde{Q}_{21} , coming from the 12 strings.
- c. Anticommuting bosonic ghosts, B_{13}, C_{31} and B_{32}, C_{23} , representing the 23 and 31 strings.

Note that physical bifundamental matter from S^2 's with positive intersection corresponds to commuting bosonic bifundamentals in the matrix model, whereas W bosons between S^2 's with negative intersection in the physical theory correspond to bosonic ghosts, similarly to what happened in [60].

The effective superpotential for these fields is

$$\begin{aligned} \mathcal{W}_{\text{eff}} = & \frac{1}{2} m_1 \text{Tr} \Phi_{11}^2 + \frac{1}{2} m_2 \text{Tr} \Phi_{22}^2 + \frac{1}{2} m_3 \text{Tr} \Phi_{33}^2 \\ & + a_{12} \text{Tr} Q_{12} \tilde{Q}_{21} + a_{23} \text{Tr} B_{32} C_{23} + a_{31} \text{Tr} B_{13} C_{31} \\ & + \text{Tr}(B_{32} \Phi_{22} C_{23} - C_{23} \Phi_{33} B_{32}) + \text{Tr}(B_{13} \Phi_{33} C_{31} - C_{31} \Phi_{11} B_{13}) \\ & + \text{Tr}(\tilde{Q}_{21} \Phi_{11} Q_{12} - Q_{12} \Phi_{22} \tilde{Q}_{21}) \end{aligned}$$

where $a_{ij} = a_i - a_j$. From this we can read off the propagators

$$\langle \Phi_{ii} \Phi_{ii} \rangle = \frac{1}{m_i}, \quad \langle Q_{12} \tilde{Q}_{21} \rangle = \frac{1}{a_{12}}$$

and

$$\langle B_{23} C_{32} \rangle = -\frac{1}{a_{23}}, \quad \langle B_{31} C_{13} \rangle = -\frac{1}{a_{31}},$$

as well as the vertices.

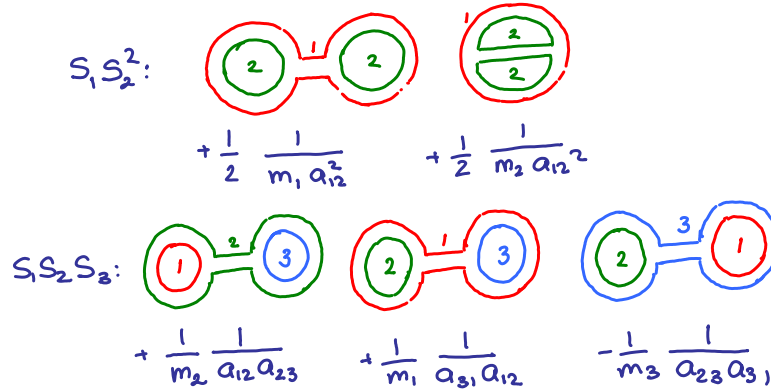


Fig. 2.2. Some of the two-loop Feynman graphs of the matrix model path integral, which are computing the prepotential \mathcal{F}_0 . The path integral is expanded about a vacuum corresponding to distributing branes on the three nodes. Here, the boundaries on node one are colored red, on node two are green and on node three are blue.

Keeping only those contractions of color indices that correspond to planar diagrams, and carefully keeping track of the signs associated with fermion loops, we find:

$$\begin{aligned} 2\pi i \mathcal{F}_0(S_i) &= \frac{1}{2} S_1^2 \left(\log\left(\frac{S_1}{m_1 \Lambda_0^2}\right) - \frac{3}{2} \right) + \frac{1}{2} S_2^2 \left(\log\left(\frac{S_2}{m_2 \Lambda_0^2}\right) - \frac{3}{2} \right) + \frac{1}{2} S_3^2 \left(\log\left(\frac{S_3}{m_3 \Lambda_0^2}\right) - \frac{3}{2} \right) \\ &\quad - \log\left(\frac{a_{12}}{\Lambda_0}\right) S_1 S_2 + \log\left(\frac{a_{31}}{\Lambda_0}\right) S_1 S_3 + \log\left(\frac{a_{23}}{\Lambda_0}\right) S_2 S_3 \\ &\quad + \frac{1}{2\Delta^3} (S_1^2 S_2 + S_2^2 S_1 + S_3^2 S_1 + S_3^2 S_2 - S_1^2 S_3 - S_2^2 S_3 - 6S_1 S_2 S_3) + \dots \end{aligned}$$

where

$$\Delta^3 = \frac{m_1 m_2}{m_3} a_{12}^2, \quad m_3 = m_1 + m_2. \quad (2.2.3)$$

The terms quadratic in the S_i 's correspond to one-loop terms in the matrix model, the cubic terms to two-loop terms, and so on. The fact that the matrix model result agrees with the direct computation from the geometry is a nice direct check of the Dijkgraaf-Vafa conjecture for quiver theories. The large N limit of quiver matrix models was previously studied using large N saddle point techniques in [31,52,61,62].

Consider now the critical points of the potential (2.1.10),

$$\partial_{S_I} V = 0.$$

The full potential is very complicated, but at weak 't Hooft coupling (we will show this is consistent *a posteriori*) it should be sufficient to keep only the leading terms in the expansion of \mathcal{F}_0 in powers of S/Δ^3 . These correspond to keeping only the one-loop terms in the matrix model. In this approximation, the physical vacua of the potential (2.1.10) correspond to solutions of

$$\alpha_I + \sum_{M_J > 0} \tau_{IJ} M^J + \sum_{M_J < 0} \bar{\tau}_{IJ} M^J = 0. \quad (2.2.4)$$

To be more precise, there *are* more solutions with other sign choices for $\pm M_J$, but only *this* choice leads to $\text{Im}(\tau)$ being positive definite. Since $\text{Im}(\tau)$ is also the metric on the moduli space, only this solution is physical.

Depending on how we choose to distribute the branes, there are two distinct classes of non-supersymmetric vacua which can be constructed in this way. We will discuss both of them presently.

2.2.2. $M_1 < 0, M_{2,3} > 0$

In this case, the critical points of the potential correspond to

$$\begin{aligned} \bar{S}_1^{|M_1|} &= \left(\overline{\Lambda_0^2 m_1}\right)^{|M_1|} \left(\frac{a_{12}}{\Lambda_0}\right)^{|M_2|} \left(\frac{a_{31}}{\Lambda_0}\right)^{-|M_3|} \exp(-2\pi i \alpha_1) \\ S_2^{|M_2|} &= \left(\Lambda_0^2 m_2\right)^{|M_2|} \left(\frac{\bar{a}_{12}}{\Lambda_0}\right)^{|M_1|} \left(\frac{a_{23}}{\Lambda_0}\right)^{-|M_3|} \exp(-2\pi i \alpha_2) \quad . \\ S_3^{|M_3|} &= \left(\Lambda_0^2 m_3\right)^{|M_3|} \left(\frac{a_{23}}{\Lambda_0}\right)^{-|M_2|} \left(\frac{\bar{a}_{31}}{\Lambda_0}\right)^{-|M_1|} \exp(-2\pi i \alpha_3) \end{aligned}$$

The S_i are identified with the gaugino condensates of the low energy, $U(M_1) \times U(M_2) \times U(M_3)$ gauge theory. The gaugino condensates are the order parameters of the low energy physics and as such should not depend on the cutoff Λ_0 . Let's then introduce three new confinement scales, Λ_i , defined as

$$S_i = \Lambda_i^3.$$

In fact, only two of these are independent. As a consequence of homology relation (2.2.1), the gauge couplings satisfy $\alpha_1 + \alpha_2 = \alpha_3$, which implies that

$$\left(\frac{\bar{\Lambda}_1}{\Delta}\right)^{3|M_1|} \left(\frac{\Lambda_2}{\Delta}\right)^{3|M_2|} = \left(\frac{\Lambda_3}{\Delta}\right)^{3|M_3|},$$

where Δ is given in (2.2.3). Requiring that the scales Λ_i do not depend on the cutoff scale, we can read off how the gauge couplings run with Λ_0 ,

$$\begin{aligned} g_1^{-2}(\Lambda_0) &= -\log\left(\frac{\bar{\Lambda}_1^3}{\Lambda_0^2 m_1}\right)^{|M_1|} - \log\left(\frac{\Lambda_0}{a_{12}}\right)^{|M_2|} - \log\left(\frac{\Lambda_0}{a_{13}}\right)^{-|M_3|} \\ g_2^{-2}(\Lambda_0) &= -\log\left(\frac{\Lambda_2^3}{\Lambda_0^2 m_2}\right)^{|M_2|} - \log\left(\frac{\bar{\Lambda}_0}{a_{12}}\right)^{|M_1|} - \log\left(\frac{\Lambda_0}{a_{23}}\right)^{-|M_3|}. \end{aligned} \quad (2.2.5)$$

As was noticed in [4], this kind of running of the gauge couplings and relation between strong coupling scales is very similar to what occurs in the *supersymmetric* gauge theory (as studied in [52]) obtained by wrapping M_i branes of the same kind on the three S^2 's. The only difference is that branes and antibranes lead to complex conjugate running, as if the spectrum of the theory remained the same, apart from the chirality of the fermions on the brane and the antibrane getting flipped. This is natural, as the branes and the antibranes have opposite GSO projections, so indeed a different chirality fermion is kept. In addition, the open string RR sectors with one boundary on branes and the other on antibranes has opposite chirality kept as well, and this is reflected in the above formulas.

To this order, the value of the potential at the critical point is

$$V_* = \sum_I \frac{|M_I|}{g_I^2} - \frac{1}{2\pi} |M_1| |M_2| \log\left(\left|\frac{a_{12}}{\Lambda_0}\right|\right) + \frac{1}{2\pi} |M_1| |M_3| \log\left(\left|\frac{a_{13}}{\Lambda_0}\right|\right)$$

The first terms are just due to the tensions of the branes. The remaining terms are due to the Coulomb and gravitational interactions of the branes, which come from the one-loop interaction in the open string theory. There is no force between the M_2 branes wrapping S_2^2 and the M_3 branes on S_3^2 , since $M_{2,3}$ are both positive, so the open strings stretching between them should be supersymmetric. On the other hand, the M_1 antibranes on S_1^2 should interact with the $M_{2,3}$ branes as the Coulomb and gravitational interactions should no longer cancel. This is exactly what one sees above. The M_1 antibranes on S_1^2 attract the M_3 branes on S_3^2 , while they repel the branes on S_2^2 . We will see in the next section that more generally, branes and antibranes wrapping two-cycles with negative intersection numbers (in the ALE space) attract, and those wrapping two-cycles with positive intersection numbers repel. Since²¹

$$e_1 \cdot e_2 = 1, \quad e_1 \cdot e_3 = -1,$$

this is exactly what we see here.

2.2.3. $M_{1,2} > 0, M_3 < 0$

With only the non-simple root wrapped by antibranes, the critical points of the potential now correspond to

$$\begin{aligned} S_1^{|M_1|} &= \left(\Lambda_0^2 m_1\right)^{|M_1|} \left(\frac{a_{12}}{\Lambda_0}\right)^{|M_2|} \left(\frac{\overline{a_{31}}}{\Lambda_0}\right)^{-|M_3|} \exp(-2\pi i \alpha_1) \\ S_2^{|M_2|} &= \left(\Lambda_0^2 m_2\right)^{|M_2|} \left(\frac{a_{12}}{\Lambda_0}\right)^{|M_1|} \left(\frac{\overline{a_{23}}}{\Lambda_0}\right)^{-|M_3|} \exp(-2\pi i \alpha_2) \\ \overline{S_3}^{|M_3|} &= \left(\overline{\Lambda_0^2 m_3}\right)^{|M_3|} \left(\frac{a_{23}}{\Lambda_0}\right)^{-|M_2|} \left(\frac{a_{31}}{\Lambda_0}\right)^{-|M_1|} \exp(-2\pi i \alpha_3) \end{aligned}$$

²¹ The second relation is due to the self intersection numbers of S_1^2 and S_2^2 being -2 .

In this case, the Kähler parameters $\alpha_{1,2}$ run as

$$g_1^{-2}(\Lambda_0) = -\log\left(\frac{\overline{\Lambda_1^3}}{\Lambda_0^2 m_1}\right)^{|M_1|} - \log\left(\frac{\Lambda_0}{a_{12}}\right)^{|M_2|} - \log\left(\frac{\Lambda_0}{a_{13}}\right)^{-|M_3|}$$

$$g_2^{-2}(\Lambda_0) = -\log\left(\frac{\Lambda_2^3}{\Lambda_0^2 m_2}\right)^{|M_2|} - \log\left(\frac{\overline{\Lambda_0}}{a_{12}}\right)^{|M_1|} - \log\left(\frac{\Lambda_0}{a_{23}}\right)^{-|M_3|}$$

where

$$\left(\frac{\Lambda_1}{\Delta}\right)^{3|M_1|} \left(\frac{\Lambda_2}{\Delta}\right)^{3|M_2|} = \left(\frac{\overline{\Lambda_3}}{\Delta}\right)^{3|M_3|}.$$

This follows the same pattern as seen in [4] and in the previous subsection. The branes and antibranes give complex conjugate running, as do the strings stretching between them.

The value of potential at the critical point is, to this order,

$$V_* = \sum_I \frac{|M_I|}{g_I^2} + \frac{1}{2\pi} |M_1| |M_3| \log\left(\left|\frac{a_{13}}{\Lambda_0}\right|\right) + \frac{1}{2\pi} |M_2| |M_3| \log\left(\left|\frac{a_{23}}{\Lambda_0}\right|\right).$$

Again, the first terms are universal, coming from the brane tensions. The remaining terms are the one-loop interaction terms. There is no force between the M_1 branes wrapping S_1^2 and the M_2 branes on S_2^2 , since now both $M_{1,2}$ have the same sign. The M_3 antibranes on S_3^2 attract both M_1 branes on S_1^2 and the M_2 branes on S_2^2 , since, in the ALE space

$$e_1 \cdot e_3 = e_2 \cdot e_3 = -1.$$

In the next subsection, we will show that both of these brane/antibrane systems are perturbatively stable for large separations.

2.2.4. Metastability

The system of branes and antibranes engineered above should be perturbatively stable when the branes are weakly interacting – in particular, at weak 't Hooft coupling. The open/closed string duality implies that the dual closed string vacuum should be metastable as well. In this subsection, we will show that this indeed is the case. Moreover, following [37], we will show that perturbative stability is lost as we increase the 't Hooft coupling. While some details of this section will be specific to the A_2 case discussed above, the general aspects of the analysis will be valid for any of the ADE fibrations discussed in the next section.

To begin with, we note that the equations of motion, derived from the potential (2.1.10), are

$$\partial_k V = \frac{-1}{2i} \mathcal{F}_{kef} G^{ae} G^{bf} (\alpha_a + M^c \bar{\tau}_{ac}) (\bar{\alpha}_b + M^d \bar{\tau}_{bd}) = 0, \quad (2.2.6)$$

and moreover, the elements of the Hessian are

$$\begin{aligned} \partial_p \partial_q V &= G^{ia} G^{bj} i \mathcal{F}_{abpq} (\alpha_i + M^k \bar{\tau}_{ki}) (\bar{\alpha}_j + M^r \bar{\tau}_{rj}) \\ &\quad + 2G^{ia} G^{bc} G^{dj} i \mathcal{F}_{abp} i \mathcal{F}_{cdq} (\alpha_i + M^k \bar{\tau}_{ki}) (\bar{\alpha}_j + M^r \bar{\tau}_{rj}) \\ \partial_q \partial_{\bar{q}} V &= -G^{ia} G^{bc} G^{dj} i \mathcal{F}_{abp} i \bar{\mathcal{F}}_{cdq} (\alpha_i + M^k \bar{\tau}_{ki}) (\bar{\alpha}_j + M^r \tau_{rj}) \\ &\quad - G^{ic} G^{da} G^{bj} i \mathcal{F}_{abp} i \bar{\mathcal{F}}_{cdq} (\alpha_i + M^k \tau_{ki}) (\bar{\alpha}_j + M^r \bar{\tau}_{rj}) \end{aligned} \quad (2.2.7)$$

where we have denoted $\partial_c \tau_{ab} = \mathcal{F}_{abc}$, and similarly for higher derivatives of τ .

In the limit where all the 't Hooft couplings $g_i^2 N_i$ are very small, the sizes of the dual three-cycles $S_a = \Lambda_a^3$ are small compared to the separations between them, so we can keep only the leading terms in the expansion of \mathcal{F}_0 in powers of S , i.e., the one-loop terms in the matrix model. At one-loop, the third and fourth derivatives of the prepotential are nonzero only if all of the derivatives are with respect to the same variable. Expanding about the physical solution to this order,

$$\alpha_a + \sum_{M_b > 0} \tau_{ab} M_b + \sum_{M_b < 0} \bar{\tau}_{ab} M_b = 0. \quad (2.2.8)$$

The nonvanishing elements of the Hessian are

$$\begin{aligned} \partial_i \partial_j V &= \frac{1}{4\pi^2} \frac{|M^i M^j|}{S_i S_j} G^{ij} & i, j \text{ opposite type} \\ \partial_i \partial_{\bar{j}} V &= \frac{1}{4\pi^2} \frac{|M^i M^{\bar{j}}|}{S_i \bar{S}_j} G^{ij} & i, j \text{ same type} \end{aligned} \quad (2.2.9)$$

where the ‘type’ of an index refers to whether it corresponds to branes or antibranes.

To get a measure of supersymmetry breaking, consider the fermion bilinear couplings. Before turning on fluxes, the theory has $\mathcal{N} = 2$ supersymmetry, and the choice of superpotential (2.1.6) breaks this explicitly to $\mathcal{N} = 1$. For each three-cycle, we get a chiral multiplet (S_i, ψ_i) and a vector multiplet (A_i, λ_i) where ψ_i, λ_i are a pair of Weyl fermions. It is easy to work out [4] that the coefficients of the nonvanishing fermion bilinears are

$$\begin{aligned} m_{\psi^a \psi^b} &= \frac{1}{2} G^{cd} (\alpha_d + M^e \bar{\tau}_{de}) \mathcal{F}_{abc}, \\ m_{\lambda^a \lambda^b} &= \frac{1}{2} G^{cd} (\bar{\alpha}_d + M^e \bar{\tau}_{de}) \mathcal{F}_{abc}, \end{aligned}$$

and evaluating this in the vacuum we find

$$\begin{aligned} m_{\psi^a \psi^b} &= -\frac{1}{4\pi} \frac{M_a + |M_a|}{S_a} \delta_{ab}, \\ m_{\lambda^a \lambda^b} &= -\frac{1}{4\pi} \frac{M_a - |M_a|}{S_a} \delta_{ab}. \end{aligned}$$

Bose-Fermi degeneracy is restored in the limit where we take

$$(G_{ij})^2 / G_{ii} G_{jj} \ll 1, \quad i, j \text{ opposite type.}$$

In this limit we get a decoupled system of branes and antibranes except that for nodes wrapped with branes, the S_a get paired up with ψ 's, and for nodes wrapped with antibranes they pair with λ 's, corresponding to a different half of $\mathcal{N} = 2$ supersymmetry

being preserved in the two cases.²² This is the limit of extremely weak 't Hooft coupling, and the sizes of the cuts are the smallest scale in the problem by far

$$\frac{\Lambda_i}{\Delta} \ll \frac{a_{ij}}{\Lambda_0}, \quad \frac{\Delta}{\Lambda_0} < 1. \quad (2.2.10)$$

In this limit the Hessian is manifestly positive definite. In fact the Hessian is positive definite as long as the one-loop approximation is valid. To see this note that the determinant of the Hessian is, up to a constant, given by

$$\text{Det}(\partial^2 V) \sim \left(\frac{1}{\text{Det } G} \prod_{i=1}^n \left| \frac{M^i}{S_i} \right|^2 \right)^2. \quad (2.2.11)$$

It is never zero while the metric remains positive definite, so a negative eigenvalue can never appear. Thus, one can conclude that as long as all the moduli are in the regime where the 't Hooft couplings are small enough for the one-loop approximation to be valid, the system will remain stable to small perturbations.

Let's now find how the solutions are affected by the inclusion of higher order corrections. At two loops, an exact analysis of stability becomes difficult in practice. However, in various limits one can recover systems which can be understood quite well. For simplicity, we will assume that the α_i are all pure imaginary, and all the parameters a_{ij} and Λ_0 are purely real. Then there are solutions where the S_i are real. In appendix 2.B, we show that in this case, upon including the two-loop terms, the determinant of the Hessian becomes

$$(\text{Det } G^{ab})^2 \left(\prod_c \frac{|M^c|}{i\mathcal{F}_{ccc}} \right)^4 \text{Det} \left(\delta_{cb} + G_{cb} \frac{i\mathcal{F}_{bbbb}\delta^b}{i\mathcal{F}_{bbb}i\mathcal{F}_{ccc}|M^c|} \right) \text{Det} \left(\delta_{cb} - G_{cb} \frac{i\mathcal{F}_{bbbb}\delta^b}{i\mathcal{F}_{bbb}i\mathcal{F}_{ccc}|M^c|} \right) \quad (2.2.12)$$

where

$$\delta^k = \frac{1}{2|M^k|\mathcal{F}_{kkk}} \mathcal{F}_{kab} (-|M^a||M^b| + M^a M^b) \quad (2.2.13)$$

and δ_{cb} is the Kronecker delta. The first two terms in (2.2.12) never vanish, since the metric has to remain positive definite, so we need only analyze the last two determinants. We can plug in the one-loop values for the various derivatives of the prepotential, and in doing so obtain

$$\text{Det} \left(\delta_{ab} \pm 2\pi G_{ab} \frac{S^a}{\Delta^3} \frac{x^b}{|M^a|} \right) = 0 \quad (2.2.14)$$

with either choice of sign. Above, we have rewritten eq. (2.2.13) as

$$\delta^a = \frac{S^a}{\Delta^3} x^a. \quad (2.2.15)$$

This is a convenient rewriting because S/Δ^3 is the parameter controlling the loop expansion, and x^a is simply a number which depends on the N^i but no other parameters.

²² The kinetic terms of both bosons and fermions are computed with the same metric G_{ab} .

Consider the case where, for some i , a given S_i^3 grows much larger than the other two. We can think of this as increasing the *effective* 't Hooft coupling for that node, or more precisely, increasing

$$\left(\frac{\Lambda_{M_i}}{\Delta}\right)^3 = \exp\left(-\frac{1}{|M_i|g_{i,eff}^2(\Delta)}\right).$$

Recall that the two-loop equations of motion for *real* S_i , are given by

$$g_{i,eff}^{-2}(\Delta) = -|M_i|\log\left(\left|\frac{S_i}{\Delta^3}\right|\right) + G_{ik}\delta^k \quad (2.2.16)$$

where

$$\begin{aligned} g_{1,eff}^{-2}(\Delta) &= g_1^{-2}(\Lambda_0) - |M_1|(L_{12} + L_{13}) + |M_2|L_{12} - |M_3|L_{13} \\ g_{2,eff}^{-2}(\Delta) &= g_2^{-2}(\Lambda_0) - |M_2|(L_{12} + L_{23}) + |M_1|L_{12} - |M_3|L_{13} \\ g_{3,eff}^{-2}(\Delta) &= g_3^{-2}(\Lambda_0) - |M_3|(L_{13} + L_{23}) - |M_1|L_{13} - |M_2|L_{23}. \end{aligned}$$

Here we've adopted the notation $L_{ij} = \log \frac{\Lambda_0}{a_{ij}}$ and the δ^k are as defined in (2.2.13). Note that in each case, two of the equations can be solved straight off. It is the remaining equations which provide interesting behavior and can result in a loss of stability. Correspondingly, the vanishing of the Hessian determinant in (2.2.12) is then equivalent to the vanishing of its ii entry (where we have assumed a vacuum at real S):

$$1 \pm G_{ii} \frac{S_i}{\Delta^3} \frac{x_i}{|M_i|} = 0. \quad (2.2.17)$$

We will see that we can approximate

$$G_{ii} = -\log\left(\left|\frac{S_i}{\Delta^3}\right|\right) + L_i \sim L_i$$

where we have defined

$$L_i = L_{ij} + L_{ik}, \quad i \neq j \neq k,$$

so this provides the following conditions:

$$\pm 1 = L_i \frac{x_i}{|M_i|} \frac{S_i}{\Delta^3}. \quad (2.2.18)$$

The above equation, taken with positive sign, is equivalent to the condition for stability being lost by setting the determinant of the gradient matrix of the equations to zero. The equation with minus sign comes from losing stability in imaginary direction. Correspondingly, the equation of motion for the one node with growing 't Hooft coupling becomes

$$g_{i,eff}^{-2} = -|M_i|\log \frac{S_i}{\Delta^3} + L_i x_i \frac{S_i}{\Delta^3}. \quad (2.2.19)$$

One of the equations (2.2.18) must be solved in conjunction with (2.2.19) if stability is to be lost.

The sign of x_i can vary depending on the specifics of the charges. In all the cases, as the effective 't Hooft coupling increases, solutions move to larger values of S_i . For sufficiently large values, in the absence of some special tuning of the charges, (2.2.18) will be satisfied for one of the two signs. The only question then is whether the S_i can get large enough, or whether a critical value above which the equation of motion can no longer be solved is reached *before* an instability sets in. In the equation above, if x_i is negative, then there will be no such critical value, and S_i can continue to grow unbounded. Correspondingly, a large enough value of the 't Hooft coupling can always be reached where (2.2.18) is satisfied with negative sign. Alternatively, if the coefficient x_i is positive, there will be a critical value for L_i at which the right hand side of the equation takes a minimum value. This occurs at $(S_{i,*}/\Delta^3) = \frac{|M_i|}{x_i L_i}$, which is precisely (2.2.18) with positive left hand side. So, for *any* value of x_i an instability develops at finite effective 't Hooft coupling corresponding to

$$\frac{S_{i,*}}{\Delta^3} = \frac{|M_i|}{|x_i|L_i},$$

or more precisely, at

$$|M_i|g_{i,eff}^2(\Delta) = \log^{-1}\left(\frac{|M_i|}{x_i L_i}\right).$$

This critical value of the effective 't Hooft coupling can be achieved by increasing the number of branes on that node, or, in case of nodes one and two, by letting the corresponding *bare* 't Hooft coupling increase. This is true as long as supersymmetry is broken and the corresponding two-loop correction is nonvanishing, i.e. as long as $x_i \neq 0$. It is reasonable to suspect that in the degenerate case, where charges conspire to set x_i to zero even with broken supersymmetry, the instability would set in at three loops.

It is natural to ask the fate of the system once metastability is lost. It should be the case [37] that it rolls to another a critical point corresponding to shrinking the one compact B -type cycle, $B_1 + B_2 - B_3$. To describe this point in the moduli space, introduce a new basis of periods in which this shrinking B -cycle becomes one of the A periods:

$$\begin{aligned} \oint_{A'_1} H &= M_1 + M_3, & \oint_{A'_2} H &= M_2 + M_3, & \oint_{A'_3} H &= 0, \\ \int_{B'_1} H &= \alpha_1, & \int_{B'_2} H &= \alpha_2, & \int_{B'_3} H &= M_3, \end{aligned}$$

where $H = H^{RR} + \tau H^{NS}$. In particular, there is no flux through the new cycle A'_3 . In fact, by setting $M_1 = M_2 = -M_3 = M$, there is no flux through any of the A' cycles.²³ For $S'_i = \int_{A'_i} \Omega$ sufficiently large that we can ignore the light D3 branes wrapping this cycle,

$$\tau'_{ii} \sim \frac{1}{2\pi i} \log \frac{S'_i}{\Delta^3}, \quad \tau'_{i \neq j} \sim \text{const},$$

²³ In the more general case, the system should be attracted to a point where only A'_3 shrinks.

it is easy to see that the system has an effective potential that would attract it to the point where the $S'_i = 0$ and the cycles shrink:

$$V_{eff} \sim V_0 + \sum_i \left| \frac{c_i}{\log \left| \frac{S'_i}{\Delta^{\frac{1}{3}}} \right|} \right|^2$$

where $c_i \sim \int_{B'_i} H$. By incorporating the light D3 branes wrapping the flux-less, shrinking cycles, the system would undergo a geometric transition to a non-Kähler manifold [63]. There, the cycle shrinks and a new two-cycle opens up, corresponding to condensing a D3 brane hypermultiplet. However, this two-cycle becomes the boundary of a compact three-cycle B' which get punctured in the transition that shrinks the A' cycles. A manifold where such a two-cycle has nonzero volume is automatically non-Kähler, but it is supersymmetric. As we will review shortly, the shrinking cycle A'_3 is also the cycle wrapped by the D5 brane domain walls that mediate the nonperturbative decay of the metastable flux vacua. The loss of metastability seems to be correlated with existence of a point in the moduli space where the domain walls become light and presumably fluxes can annihilate classically (this also happened in the A_1 model studied in [37]). In particular, in the last section of this chapter, we will provide two examples of a system where the corresponding points in the complex structure moduli space are *absent*, but which are exactly stable perturbatively even though they are non-supersymmetric (one of them will be stable non-perturbatively as well). It must be added, as discussed in [37], that it is far from clear whether the light domain walls can be ignored, and so whether the system truly rolls down to a supersymmetric vacuum. A more detailed analysis of the physics at this critical point is beyond the scope of this chapter.

It was suggested in [37] that the loss of stability might be related to the difference in the value of V_* between the starting vacuum and a vacuum to which it might tunnel becoming small, and thus the point where Coulomb attraction starts to dominate in a subset of branes. In the more complicated geometries at hand, it seems that such a simple statement does not carry over. This can be seen by noting that, for certain configurations of brane charges in our case, an instability can be induced without having any effect on the ΔV_* between vacua connected by tunneling events. We are led to conclude that the loss of stability is a strong coupling effect in the non-supersymmetric system, which has no simple explanation in terms of our open string intuition. This should have perhaps been clear, in that the point to which the system apparently rolls has no straightforward explanation in terms of brane annihilation.

2.2.5. Decay rates

We now study the decays of the brane/antibrane systems of the previous section. This closely parallels the analysis of [4]. We have shown that when the branes and antibranes are sufficiently well-separated, the system is perturbatively stable. Non-perturbatively, the system can tunnel to lower energy vacua, if they are available. In this case, the available vacua are constrained by charge conservation – any two vacua with the same net charges

$$N_1 = M_1 + M_3, \quad N_2 = M_2 + M_3$$

are connected by finite energy barriers. The false vacuum decay proceeds by the nucleation of a bubble of lower energy vacuum.

The decay process is easy to understand in the closed string language. The vacua are labeled by the fluxes through the three S^3 's

$$\int_{A_I} H^{RR} = M_I. \quad I = 1, 2, 3$$

Since RR three-form fluxes jump in going from the false vacuum to the true vacuum, the domain walls that interpolate between the vacua are D5 branes. Over a D5 brane wrapping a compact three-cycle C in the Calabi-Yau, the fluxes jump by an amount

$$\Delta M_I = \#(C \cap A_I)$$

In the present case, it is easy to see that there is only one compact three-cycle C that intersects the A -cycles,

$$C = B_1 + B_2 - B_3.$$

So, across a D5 brane wrapping C , the fluxes through $A_{1,2}$ decrease by one unit, and the flux through A_3 increases by one unit. Note that this is consistent with charge conservation for the branes. In fact, the domain walls in the open and the closed picture are essentially the same. In the open string language, the domain wall is also a D5 brane, but in this case it wraps a three-chain obtained by pushing C through the geometric transition. The three-chain has boundaries on the minimal S^2 's, and facilitates the homology relation (2.2.1) between the two-cycles.

The decay rate Γ is given in terms of the action S_{inst} of the relevant instanton.

$$\Gamma \sim \exp(-S_{inst})$$

Since the Calabi-Yau we have been considering is non-compact, we can neglect gravity, and the instanton action is given by

$$S_{inst} = \frac{27\pi}{2} \frac{S_D^4}{(\Delta V_*)^3}$$

where S_D is the tension of the domain wall, and ΔV_* is the change in the vacuum energy across the domain wall. While this formula was derived in [34] in a scalar field theory, it is governed by energetics, and does not depend on the details of the theory as long as the semi-classical approximation is applicable.

In the present case, the tension of the domain wall is bounded below by

$$S_D = \frac{1}{g_s} \int_C \Omega, \tag{2.2.20}$$

since the $\int_C \Omega$ computes the lower bound on the volume of any three-cycle in this class, and the classical geometry is valid to the leading order in $1/N$, the order to which we

are working. The tension of the domain wall is thus the same as the tension of a domain wall interpolating between the supersymmetric vacua, and to leading order (open-string tree-level) this is given by the difference between the tree-level superpotentials (2.1.6)

$$\int_C \Omega \sim W_3(a_3) - W_1(a_1) - W_2(a_2) = \frac{1}{2} \Delta^3,$$

where Δ^3 is defined in (2.2.3). This is just the “holomorphic area” of the triangle in figure 2.1. The area is large as long as all the brane separations are large, and as long as this is so, it is independent of the fluxes on the two sides of the domain wall.

At the same time, the difference in the potential energy between the initial and the final states is given by the classical brane tensions,

$$\Delta V = V_i - V_f = \sum_I (|M_I| - |M'_I|) / g_I^2.$$

The fate of the vacuum depends on the net charges. If $N_{1,2}$ are both positive, then the true vacuum is supersymmetric. Moreover, there is a landscape of degenerate such vacua, corresponding to all possible ways of distributing branes consistent with charge conservation such that M'_I are all positive. Starting with, say, $(M_1, M_2, M_3) = (N_1 + k, N_2 + k, -k)$, where $k > 0$, this can decay to $(N_1, N_2, 0)$ since

$$\Delta V = V_i - V_f = 2 \frac{k |r_3|}{g_s},$$

corresponding to k branes on S_3^2 getting annihilated, where r_3 is the Kähler area of S_3^2 . The decay is highly suppressed as long as string coupling g_s is weak and the separation between the branes is large. The action of the domain wall is k times that of (2.2.20),²⁴ so

$$S_{inst} = \frac{27\pi}{32} \frac{k}{g_s} \frac{|\Delta|^{12}}{|r_3|^3} = \frac{27\pi}{32} \frac{|g_3|^3}{g_s^4} k |\Delta|^{12}. \quad (2.2.21)$$

The instanton action (2.2.21) depends on the cutoff scale Λ_0 due to the running of the gauge coupling $g_3^{-2}(\Lambda_0)$. The dependence on Λ_0 implies [45] that (2.2.21) should be interpreted as the rate of decay corresponding to fluxes decaying in the portion of the Calabi-Yau bounded by Λ_0 .

If instead we take say $N_1 > 0 > N_2$, then the lowest energy state corresponds to N_1 branes on node 1, N_2 antibranes on node 2, with node 3 unoccupied. This is the case at least for those values of parameters corresponding to the system being weakly coupled. In this regime, this particular configuration gives an example of an exactly stable, non-supersymmetric vacuum in string theory – there is no other vacuum with the same charges that has lower energy. Moreover, as we will discuss in section 6, for some special values of the parameters $m_{1,2}$ the system is exactly solvable, and can be shown to be *exactly* stable even when the branes and the antibranes are close to each other.

²⁴ All quantities being measured in string units.

2.3. Generalizations

Consider now other ADE fibrations over the complex plane. As in (2.1.1) we start with the deformations of two-complex-dimensional ALE singularities:

$$\begin{aligned}
 A_k &: x^2 + y^2 + z^{k+1} = 0 \\
 D_r &: x^2 + y^2 z + z^{r-1} = 0 \\
 E_6 &: x^2 + y^3 + z^4 = 0 \\
 E_7 &: x^2 + y^3 + yz^3 = 0 \\
 E_8 &: x^2 + y^3 + z^5 = 0
 \end{aligned}$$

and fiber these over the complex t plane, allowing the coefficients parameterizing the deformations to be t dependent. The requisite deformations of the singularities are canonical (see [31] and references therein). For example, the deformation of the D_r singularity is

$$x^2 + y^2 z + z^{-1} \left(\prod_{i=1}^r (z - z_i^2) - \prod_{i=1}^r z_i^2 \right) + 2 \prod_{i=1}^r z_i y.$$

In fibering this over the t plane, the z_i become polynomials $z_i(t)$ in t .²⁵ After deformation, at a generic point in the t plane, the ALE space is smooth, with singularities resolved by a set of r independent two-cycle classes

$$S_i^2, \quad i = 1, \dots, r$$

where r is the rank of the corresponding Lie algebra. The two-cycle classes intersect according to the ADE Dynkin diagram of the singularity:

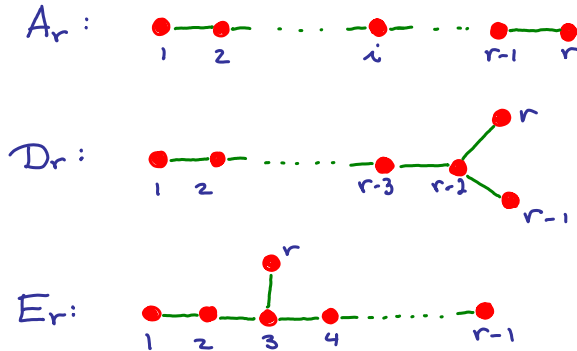


Fig. 2.3. Dynkin diagrams of the ADE Lie algebras. Every node corresponds to a simple root and to a two-cycle class of self intersection -2 in the ALE space. The nodes that are linked correspond to two-cycles which intersect with intersection number $+1$.

²⁵ This is the so called “non-monodromic” fibration. The case where the z_i are instead multi-valued functions of t corresponds to the “monodromic” fibration [31].

The deformations can be characterized by “superpotentials”,

$$W'_i(t) = \int_{S^2_{i,t}} \omega^{2,0},$$

which compute the holomorphic volumes of the two-cycles at fixed t . For each positive root e_I , which can be expanded in terms of simple roots e_i as

$$e_I = \sum_I n_I^i e_i$$

for some positive integers n_I^i , one gets a zero-sized, primitive two-cycle at points in the t -plane where

$$W'_I(t) = \sum_i n_I^i W'_i(t) = 0. \quad (2.3.1)$$

Blowing up the singularities supplies a minimal area to the two-cycles at solutions of (2.3.1),

$$t = a_{I,p},$$

where I labels the positive root and p runs over all the solutions to (2.3.1) for that root.

As shown in [31] and references therein, the normal bundles to the minimal, holomorphic S^2 's obtained in this way are always $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and correspondingly the S^2 's are isolated.²⁶ This implies that when branes or antibranes are wrapped on the S^2 's, there is an energy cost to moving them off. Moreover, the parameters that enter into defining the W_i , as well as the Kähler classes of the S^2 's, are all nondynamical in the Calabi-Yau. As a consequence, if we wrap branes and antibranes on minimal S^2 's, the non-supersymmetric system obtained is metastable, at least in the regime of parameters where the S^2 's are well separated.

The ALE fibrations have geometric transitions in which each minimal S^2 is replaced by a minimal S^3 . A key point here is that none of the two-cycles have compact, dual four-cycles, so the transitions are all locally conifold transitions. The one-loop prepotential \mathcal{F}_0 for all these singularities was computed in [52], and is given by

$$2\pi i \mathcal{F}_0(S) = \frac{1}{2} \sum_b S_b^2 \left(\log \left(\frac{S_b}{W'_I(a_b) \Lambda_0^2} \right) - \frac{3}{2} \right) + \frac{1}{2} \sum_{b \neq c} e_{I(b)} \cdot e_{J(c)} S_b S_c \log \left(\frac{a_{bc}}{\Lambda_0} \right), \quad (2.3.2)$$

where the sum is over all critical points $b = (I, p)$, and $I(b) = I$ denotes the root I to which the critical point labeled by b corresponds. We are neglecting cubic and higher order terms

²⁶ In [31] the authors also considered the monodromic ADE fibrations, where the two-cycles of the ALE space undergo monodromies around paths in the t plane. In this case, the novelty is that the S^2 's can appear with normal bundles $\mathcal{O} \oplus \mathcal{O}(-2)$ or $\mathcal{O}(-1) \oplus \mathcal{O}(3)$. Wrapping branes and antibranes on these cycles is not going to give rise to new metastable vacua, since there will be massless deformations moving the branes off of the S^2 's. It would be interesting to check this explicitly in the large N dual.

in the $S_{I,p}$, which are related to higher loop corrections in the open string theory. Above, $W_I(t)$ is the superpotential corresponding to the root e_I , and $e_I \cdot e_J$ is the inner product of two positive, though not necessarily simple, roots. Geometrically, the inner product is the same as *minus* the intersection number of the corresponding two-cycles classes in the ALE space.

Consider wrapping M_b branes or antibranes on the minimal S^2 's labeled by $b = (I, p)$. We will take all the roots to be positive, so we get branes or antibranes depending on whether M_b is positive or negative. The effective superpotential for the dual, closed-string theory is given by (2.1.9). From this and the corresponding effective potential (2.1.10), we compute the expectation values for S_b in the metastable vacuum to be

$$\begin{aligned} \overline{S_b}^{|M_b|} &= \left(\overline{\Lambda_0^2 W_I''(a_b)} \right)^{|M_b|} \prod_{b \neq c}^{M_c < 0} \left(\frac{\overline{a_{bc}}}{\Lambda_0} \right)^{|M_c|} \prod_c^{M_c > 0} \left(\frac{a_{bc}}{\Lambda_0} \right)^{|M_c|} \exp(-2\pi i \alpha_{I(b)}), & M_b < 0 \\ S_b^{|M_b|} &= \left(\Lambda_0^2 W_I''(a_b) \right)^{|M_b|} \prod_{b \neq c}^{M_c > 0} \left(\frac{a_{bc}}{\Lambda_0} \right)^{|M_c|} \prod_c^{M_c < 0} \left(\frac{\overline{a_{bc}}}{\Lambda_0} \right)^{|M_c|} \exp(-2\pi i \alpha_{I(b)}). & M_b > 0 \end{aligned}$$

The value of the effective potential at the critical point is given by

$$V_* = \sum_b \frac{|M_b|}{g_{I(b)}^2} + \sum_{b,c}^{M_b > 0 > M_c} \frac{1}{2\pi} e_{I(b)} \cdot e_{J(c)} \log \left(\left| \frac{a_{bc}}{\Lambda_0} \right| \right).$$

The first term in the potential is just the contribution of the tensions of all the branes and antibranes. The second term comes from the Coulomb and gravitational interactions between branes, which is a one-loop effect in the open string theory. As expected, at this order only the brane/antibrane interactions affect the potential energy. The open strings stretching between a pair of (anti)branes, are supersymmetric, and the (anti)branes do not interact. The interactions between branes and antibranes depend on

$$e_I \cdot e_J$$

which is *minus* the intersection number – in the ALE space – of the two-cycle classes wrapped by the branes. The branes and antibranes attract if the two-cycles they wrap have *negative* intersection, while they repel if the intersection number is *positive*, and do not interact at all if the two-cycles do not intersect.

For example, consider the A_k quiver case, and a set of branes and antibranes wrapping the two-cycles obtained by blowing up the singularities at

$$z_i(t) = z_j(t), \quad z_m(t) = z_n(t)$$

where $i < j$ and $m < n$. The branes do not interact unless i or j coincide with either m or n . The branes attract if $i = m$ or $j = n$, in which case the intersection is either -1 or -2 , depending on whether one or both of the above conditions are satisfied. This is precisely the case when the branes and antibranes can at least partially annihilate. If $j = m$ or

$i = n$, then the two-cycles have intersection $+1$, and the branes repel. In this case, the presence of branes and antibranes should break supersymmetry, but there is a topological obstruction to the branes annihilating, even partially. In fact, in the A_k type ALE spaces, this result is known from the direct, open string computation [54,55]. The fact that the direct computation agrees with the results presented here is a nice test of the conjecture of [4].

2.4. A Non-Supersymmetric Seiberg Duality

In the supersymmetric case, with all M_I positive, the engineered quiver gauge theories have Seiberg-like dualities. In string theory, as explained in [52], the duality comes from an *intrinsic* ambiguity in how we resolve the ADE singularities to formulate the brane theory.²⁷ The different resolutions are related by flops of the S^2 's under which the charges of the branes, and hence the ranks of the gauge groups, transform in nontrivial ways. The RG flows, which are manifest in the large N dual description, force some of the S^2 's to shrink and others to grow, making one description preferred over the others at a given energy scale. In this section, we argue that Seiberg dualities of this sort persist even when some of the branes are changed to antibranes and supersymmetry is broken.

2.4.1. Flops as Seiberg dualities

For a fixed set of brane charges, one can associate different Calabi-Yau geometries. There is not a unique way to blow up the singularity where an S^2 shrinks, and the different blowups are related by flops that shrink some two-cycles and grow others. Instead of giving a two-cycle class S_i^2 a positive Kähler volume

$$r_i = \int_{S_i^2} B^{NS}$$

we can give it a *negative* volume, instead. This can be thought of as replacing the two-cycle class by one of the opposite orientation

$$S_i^2 \rightarrow \tilde{S}_i^2 = -S_i^2.$$

The flop of a simple root S_i^2 acts on the other roots as a Weyl reflection which permutes the positive roots

$$S_j^2 \rightarrow \tilde{S}_j^2 = S_j^2 - (e_j \cdot e_i) S_i^2. \quad (2.4.1)$$

The net brane charges change in the process, but in a way consistent with charge conservation

$$\sum_i N_i S_i^2 = \sum_i \tilde{N}_i \tilde{S}_i^2. \quad (2.4.2)$$

²⁷ The idea that Seiberg dualities have a geometric interpretation in string theory goes back a long while, see for example [64-68]. The fact that these dualities arise dynamically in string theory has for the first time been manifested in [23,52].

We can follow how the number of branes wrapping the minimal two-cycles change in this process. If i is the simple root that gets flopped,²⁸ then $M_{i,p}$ goes to $\widetilde{M}_{i,p} = -M_{i,p}$ and for other roots labeled by $J \neq i$

$$M_{J,p} = \widetilde{M}_{w(J),p} \quad (2.4.3)$$

where $w(J)$ is the image of J under the Weyl group action.

The size of the wrapped S^2 is proportional to the inverse gauge coupling for the theory on the wrapping branes,

$$g_i^{-2}(t) \propto \frac{1}{g_s} \int_{S_{i,t}^2} B_{NS}, \quad (2.4.4)$$

so the flop (2.4.1) transforms the gauge couplings according to

$$g_j^{-2} \rightarrow \widetilde{g}_j^{-2} = g_j^{-2} - (e_j \cdot e_i) g_i^{-2}. \quad (2.4.5)$$

Generally, there is one preferred description for which the gauge couplings are all positive. In the geometry, we have the freedom to choose the sizes of the two-cycles $S_{i,t}^2$ at some fixed high scale, but the rest of their profile is determined by the one-loop running of the couplings (2.2.5) throughout the geometry and by the brane charges. The most invariant way of doing this is to specify the scales Λ_i at which the couplings (2.4.4) become strong. We can then follow, using holography, the way the B -fields vary over the geometry as one goes from near where the S^3 's are minimal, which corresponds to low energies in the brane theory, to longer distances, far from where the branes were located, which corresponds to going to higher energies. The S^2 's have finite size and shrink or grow depending on whether the gauge coupling is increasing or decreasing. We will see that as we vary the strong coupling scales of the theory, we can smoothly interpolate between the two dual descriptions. Here it is crucial that the gauge coupling going through zero is a *smooth* process in the geometry: while the Kähler volume of the two-cycle vanishes as one goes through a flop, the *physical* volume, given by (2.1.5), remains finite. Moreover, we can read off from the geometry which description is the more appropriate one at a given scale.

2.4.2. The A_2 example

For illustration, we return to the example of the A_2 quiver studied in section 3. To begin with, for a given set of charges M_i , we take the couplings g_i^{-2} of the theory to be *weak* at the scale Δ set by the “superpotential”. This is the characteristic scale of the open-string ALE geometry. Then S_i/Δ^3 is small in the vacuum, and the weak coupling expansion is valid. From (2.2.5), we can deduce the one-loop running of the couplings with energy scale $\mu = t$

$$\mu \frac{d}{d\mu} g_1^{-2}(\mu) = (2|M_1| + |M_3| - |M_2|), \quad \mu \frac{d}{d\mu} g_2^{-2}(\mu) = (2|M_2| + |M_3| - |M_1|). \quad (2.4.6)$$

²⁸ Flopping non-simple roots can be thought of in terms of a sequence of simple node flops, as this generates the full Weyl group.

Suppose now, for example

$$2|M_1| + |M_3| \leq |M_2|, \quad (2.4.7)$$

so then at high enough energies, $g_1^{-2}(\mu)$ will become negative, meaning that the size of $S_{i,t}^2$ has become negative. To keep the size of all the S^2 's positive, at large enough t , the geometry undergoes a flop of S_1^2 that sends

$$\begin{aligned} S_1^2 &\rightarrow \tilde{S}_1^2 = -S_1^2 \\ S_2^2 &\rightarrow \tilde{S}_2^2 = S_2^2 + S_1^2, \end{aligned} \quad (2.4.8)$$

and correspondingly,

$$\tilde{N}_1 = N_2 - N_1, \quad \tilde{N}_2 = N_2, \quad (2.4.9)$$

while

$$\tilde{M}_1 = -M_1, \quad \tilde{M}_2 = M_3, \quad \tilde{M}_3 = M_2. \quad (2.4.10)$$

Recall the supersymmetric case first. The supersymmetric case with $M_1 = 0$ was studied in detail in [52]. It corresponds to a vacuum of a low energy $U(N_1) \times U(N_2)$, $\mathcal{N} = 2$ theory where the superpotential breaks the gauge group to $U(M_2) \times U(M_3)$. The formulas (2.4.6) are in fact the same as in the supersymmetric case, when all the M_i are positive – the beta functions simply depend on the absolute values of the charges. If (2.4.7) is satisfied, the $U(N_1)$ factor is not asymptotically free, and the coupling grows strong at high energies. There, the theory is better described in terms of its Seiberg dual, the asymptotically free $U(\tilde{N}_1) \times U(\tilde{N}_2)$ theory, broken to $U(\tilde{M}_2) \times U(\tilde{M}_3)$ by the superpotential.²⁹ The vacua at hand, which are visible semi-classically in the $U(N_1) \times U(N_2)$ theory, are harder to observe in the $U(\tilde{N}_1) \times U(\tilde{N}_2)$ theory, which is strongly coupled at the scale of the superpotential. But, the duality predicts that they are there. In particular, we can smoothly vary the strong coupling scale Λ_{N_1} of the original theory from (i) $\Lambda_{N_1} < \Delta < \mu$, where the description at scale μ is better in terms of the original $U(N_1) \times U(N_2)$ theory, to (ii) $\Delta < \mu < \Lambda_{N_1}$, where the description is better in terms of the dual $U(\tilde{N}_1) \times U(\tilde{N}_2)$ theory.

For the dual description of a theory to exist, it is necessary, but not sufficient (as emphasized in [12]), that the brane charges at infinity of the Calabi-Yau be the same in both descriptions. In addition, the gauge couplings must run in a consistent way. In this supersymmetric A_2 quiver, this is essentially true automatically, but let's review it anyway with the non-supersymmetric case in mind. On the one hand, (2.4.5) implies that the under the flop, the couplings transform as

$$\begin{aligned} g_1^{-2}(\mu) &\rightarrow \tilde{g}_1^{-2}(\mu) = -g_1^{-2}(\mu) \\ g_2^{-2}(\mu) &\rightarrow \tilde{g}_2^{-2}(\mu) = g_1^{-2}(\mu) + g_2^{-2}(\mu). \end{aligned} \quad (2.4.11)$$

²⁹ The superpotential of the dual theory is not the same as in the original. As explained in [52], we can think of the flop as permuting the $z'_i(t)$, in this case exchanging $z'_1(t)$ with $z'_2(t)$, which affects the superpotential as $W_1(\Phi_1) \rightarrow -W_1(\Phi_1)$, and $W_2(\Phi_2) \rightarrow W_1(\Phi_2) + W_2(\Phi_2)$.

On the other hand, from (2.2.5) we know how the couplings \tilde{g}_i^{-2} corresponding to charges \tilde{M}_i run with scale μ . The nontrivial fact is that these two are consistent – the flop simply exchanges $\tilde{M}_2 = M_3$ and $\tilde{M}_3 = M_2$, and this is consistent with (2.4.11).

Now consider the non-supersymmetric case. Let's still take $M_1 = 0$, but now with $M_2 > 0 > M_3$, such that (2.4.7) is satisfied. It is still the case that if we go to high enough energies, i.e. large enough μ , the gauge coupling g_1^{-2} will become negative, and the corresponding S_1^2 will undergo a flop. We can change the basis of two-cycles as in (2.4.5) and (2.4.8) so that the couplings are all positive, and then the charges transform according to (2.4.10). Moreover, just as in the supersymmetric theory, after the flop the gauge couplings run exactly as they should given the new charges \tilde{M}_i , which are again obtained by exchanging node two and three. Moreover, by varying the scale Λ_{N_1} where g_1^{-2} becomes strong, we can smoothly go over from one description to the other, just as in the supersymmetric case. For example, in the A_2 case we have a non-supersymmetric duality relating a $U(|N_1|) \times U(N_2)$ theory, where the rank $N_1 = M_3$ is *negative* and $N_2 = M_2 + M_3$ positive, which is a better description at low energies, to a $U(\tilde{N}_1) \times U(\tilde{N}_2)$ theory with positive ranks $\tilde{N}_1 = N_2 - N_1 = M_2$ and $\tilde{N}_2 = N_2 = M_2 + M_3$, which is a better description at high energies.

More generally, one can see that this will be the case in any of the ADE examples of the previous section. This is true regardless of whether all $M_{I,p}$ are positive and supersymmetry is unbroken, or they have different signs and supersymmetry is broken. In the case where supersymmetry is broken, we have no gauge theory predictions to guide us, but it is still natural to *conjecture* the corresponding non-supersymmetric dualities based on holography. Whenever the charges are such that in going from low to high energies a root ends up being dualized

$$S_{i,p}^2 \rightarrow -S_{i,p}^2,$$

there should be a non-supersymmetric duality relating a brane/antibrane system which is a better description at low energies to the one that is a better description at high energies, with charges transforming as in (2.4.2) and (2.4.3). The theories are dual in the sense that they flow to the same theory in the IR, and moreover, there is no sharp phase transition in going from one description to the other. This can be seen from the fact that by varying the strong coupling scales of the theory, one can *smoothly* interpolate between one description and the other being preferred at a given energy scale μ . We don't expect these to correspond to gauge theory dualities (in the sense of theories with a finite number of degrees of freedom and a separation of scales), but we do expect them to be string theory dualities.

2.4.3. Dualizing an occupied root

When an occupied node gets dualized, negative ranks $M < 0$ will appear. This is true even in the supersymmetric case. It is natural to wonder whether this is related to the appearance of non-supersymmetric vacua in a supersymmetric gauge theory. Conversely, starting with a non-supersymmetric vacuum at high energies, one may find that the good description at low energies involves all the charges being positive. We propose that when an occupied node gets dualized, there is essentially only one description which is ever

really weakly coupled. In particular, “negative rank” gauge groups can appear formally but never at weak coupling. Moreover, while the supersymmetric gauge theories *can* have non-supersymmetric vacua, the phenomenon at hand is unrelated to that. This is in tune with the interpretation given in [52].

Consider the A_2 theory in the supersymmetric case, $\widetilde{M}_{1,2,3} > 0$, with both gauge groups $U(\widetilde{N}_{1,2})$ being asymptotically free. The $U(\widetilde{N}_1) \times U(\widetilde{N}_2)$ theory gives a good description at low energies, for

$$\Lambda_{\widetilde{N}_1} \ll \Delta$$

where Δ is the characteristic scale of the ALE space, and $\Lambda_{\widetilde{N}_1}$ is the strong coupling scale of the $U(\widetilde{N}_1)$ theory. Now consider adiabatically increasing the strong coupling scale until

$$\Lambda_{\widetilde{N}_1} \geq \Delta.$$

Then the $U(N_1) \times U(N_2)$ description appears to be better at low energies, with N 's related as in (2.4.11). Namely, from (2.4.11) we can read off that the strong coupling scales match up as $\Lambda_{N_1} = \Lambda_{\widetilde{N}_1}$, so at least formally this corresponds to a more weakly coupled, IR free $U(N_1)$ theory. However, after dualizing node 1, its charge becomes negative

$$\widetilde{M}_1 = -M_1.$$

How is the negative rank $M_1 < 0$ consistent with the theory having a supersymmetric vacuum?

The dual theory clearly cannot be a weakly coupled theory. A weakly coupled theory of branes and antibranes breaks supersymmetry, whereas the solution at hand is supersymmetric. Instead, as we increase Λ_{N_1} and follow what happens to the supersymmetric solution, the scale Λ_{M_1} associated with gaugino condensation on node 1 increases as well, $\Delta < \Lambda_{M_1} \sim \Lambda_{N_1}$, and we find that at all energy scales below Λ_{M_1} we have a strongly coupled theory, without a simple gauge theoretic description. The holographic dual theory of course *does* have a weakly coupled vacuum with charges $M_1 < 0$, $M_{2,3} > 0$, which breaks supersymmetry. However, the gauge couplings in this vacuum run at high energies in a different way than in the supersymmetric $U(N_1) \times U(N_2)$ gauge theory. As emphasized in [12], this means we *cannot* interpret this non-supersymmetric vacuum as a metastable state of the supersymmetric gauge theory.

We could alternatively start with a weakly coupled, non-supersymmetric A_2 theory with $M_1 < 0$, $M_{2,3} > 0$. If (2.4.7) is not satisfied, the theory is asymptotically free. Increasing the strong coupling scale Λ_{N_1} of this theory until $\Lambda_{N_1} \sim \Delta$, the theory becomes strongly coupled, and one is tempted to dualize it to a theory with $\widetilde{M}_i > 0$ at lower energies. However, from the vacuum solutions in section 3, we can read off that, just as in the supersymmetric case, this implies that the scale Λ_{M_1} of the gaugino condensate of node 1 becomes larger than the scale Δ , and no weakly coupled description exists. What is new in the non-supersymmetric case is that, as we have seen in section 3, increasing the strong coupling scale Λ_{M_1} to near Δ causes the system to lose stability.

Nevertheless, we can formally extend the conjectured Seiberg dualities to all the supersymmetric and non-supersymmetric vacua even when the node that gets dualized is *occupied*, except that the dual description is, in one way or another, always strongly coupled.

2.5. A Very Simple Case

Let's now go back to the A_2 case studied in section 3 and suppose that two of the masses are equal and opposite $m_1 = -m_2 = -m$, so³⁰

$$z_1(t) = 0, \quad z_2(t) = -mt, \quad z_3(t) = -m(t - a). \quad (2.5.1)$$

It is easy to see from (2.1.4) that there are now only two critical points at $t = 0$ and $t = a$, which get replaced by S_1^2 and S_3^2 . The third intersection point, which corresponds to the simple root S_2^2 , is absent here, and so is the minimal area two-cycle corresponding to it. We study this as a special case since now the prepotential \mathcal{F}_0 can be given in closed form, so the theory can be solved exactly. This follows easily either by direct computation from the geometry, or from the corresponding matrix model (see appendix 2.A). The large N dual geometry corresponds to the two S^2 's being replaced by two S^3 's:

$$x^2 + y^2 + z(z + mt)(z + m(t - a)) = s_1(z + ma) + s_3(z + m(t - a)).$$

The *exact* prepotential is given by

$$2\pi i \mathcal{F}_0(S) = \frac{1}{2} S_1^2 \left(\log\left(\frac{S_1}{m\Lambda_0^2}\right) - \frac{3}{2} \right) + \frac{1}{2} S_3^2 \left(\log\left(\frac{S_3}{m\Lambda_0^2}\right) - \frac{3}{2} \right) + S_1 S_3 \log\left(\frac{a}{\Lambda_0}\right). \quad (2.5.2)$$

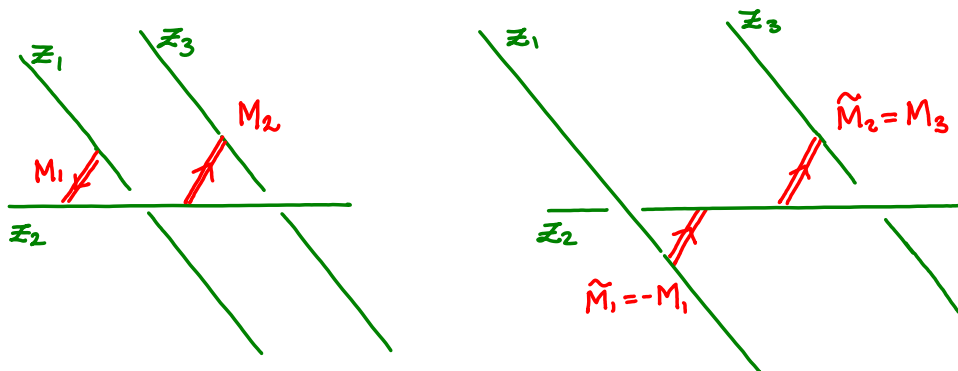


Fig. 2.4. There are only two minimal S^2 's in the A_2 geometry with $m_1 = -m_2$. The figure on the left corresponds to the first blowup discussed in the text, with two minimal S^2 's of intersection number +1 in the ALE space wrapped by M_1 anti-D5 branes and M_3 D5 branes. The figure on the right is the flop of this.

We can now consider wrapping, say, M_1 antibranes on S_1^2 and M_3 branes on S_3^2 . We get an exact vacuum solution at

$$\begin{aligned} \bar{S}_1^{|M_1|} &= \left(\overline{\Lambda_0^2 m}\right)^{|M_1|} \left(\frac{a}{\Lambda_0}\right)^{-|M_3|} \exp(-2\pi i \alpha_1), \\ S_3^{|M_3|} &= \left(\Lambda_0^2 m\right)^{|M_3|} \left(\frac{a}{\Lambda_0}\right)^{-|M_1|} \exp(-2\pi i \alpha_3), \end{aligned}$$

³⁰ More precisely, relative to the notation of that section, we've performed a flop here that exchanges z_1 and z_2 .

where the potential between the branes is given by

$$V_* = \frac{|M_1|}{g_1^2} + \frac{|M_3|}{g_3^2} + \frac{1}{2\pi}|M_1||M_3|\log\left(\left|\frac{a}{\Lambda_0}\right|\right).$$

Using an analysis identical to that in [4], it follows that the solution is *always* stable, at least in perturbation theory. Borrowing results from [4], the masses of the four bosons corresponding to fluctuations of $S_{1,3}$ are given by

$$(m_{\pm}(c))^2 = \frac{(a^2 + b^2 + 2abcv) \pm \sqrt{(a^2 + b^2 + 2abcv)^2 - 4a^2b^2(1-v)^2}}{2(1-v)^2} \quad (2.5.3)$$

and the masses of the corresponding fermions are

$$|m_{\psi_1}| = \frac{a}{1-v}, \quad |m_{\psi_2}| = \frac{b}{1-v} \quad (2.5.4)$$

where c takes values $c = \pm 1$, and

$$a = \left| \frac{M_1}{2\pi\Lambda_1^3 \text{Im}\tau_{11}} \right|, \quad b = \left| \frac{M_3}{2\pi\Lambda_3^3 \text{Im}\tau_{33}} \right|. \quad (2.5.5)$$

The parameter controlling the strength of supersymmetry breaking v is defined by

$$v = \frac{(\text{Im}\tau_{13})^2}{\text{Im}\tau_{11}\text{Im}\tau_{33}}.$$

That v controls the supersymmetry breaking can be seen here from the fact that at $v = 0$, the masses of the four real bosons become degenerate in pairs, and match up with the fermion masses [4]. The masses of bosons are strictly positive since the metric on the moduli space $\text{Im}\tau$ is positive definite, which implies

$$1 > v \geq 0 \\ \Lambda_{N_{1,2}} \ll a$$

where $\Lambda_{N_{1,2}}$ is the scale at which the gauge coupling $g_{1,2}^{-2}$ becomes strong.³¹

The fact that the system is stable perturbatively is at first sight surprising, since from the open string description one would expect that for sufficiently small a an instability develops, ultimately related to the tachyon that appears when the brane separation is below the string scale. In particular, we expect the instability to occur when the coupling on the branes becomes strong enough that the Coulomb attraction overcomes the tension effects from the branes. However, it is easy to see that there is *no* stable solution for small a . As we decrease a , the solution reaches the boundary of the moduli space,

$$\Lambda_0 \exp\left(-\frac{1}{g_{1,3}^2 |M_{3,1}|}\right) < a,$$

³¹ From the solution, one can read off, e.g., $g_1^{-2} = -(2|M_1| + |M_3|)\log\left(\frac{\Lambda_{N_1}}{\Lambda_0}\right)$.

where $\text{Im}\tau$ is positive definite, *before* the instability can develop.³² Namely, if we view Λ_0 as a cutoff on how much energy one has available, then for a stable solution to exist at fixed coupling, the branes have to be separated by more than $\sim \Lambda_0$, and said minimum separation increases as one moves towards stronger coupling. The couplings, however, do run with energy, becoming weaker at higher Λ_0 , and because of that the lower bound on a actually *decreases* with energy. Alternatively, as we will discuss in the next subsection, there is a lower bound on how small $|a|$ can get, set by the strong coupling scales $\Lambda_{N_{1,2}}$ of the brane theory. When this bound is violated, the dual gravity solution disappears.

The fact that the system is perturbatively stable should be related to the fact that in this case there is no compact B cycle. Namely, in section 3 we have seen that when perturbative stability is lost, the system rolls down to a new minimum corresponding to shrinking a compact B -cycle without flux through it. In this case, such a compact B -cycle is absent, so the system has no vacuum it can roll away to, and correspondingly it remains perturbatively stable.

The theory has another vacuum with the same charges, which can have lower energy. This vacuum is not a purely closed string vacuum, but it involves branes. Consider, for example, the case with $M_1 = -M_3 = -M$. In this case, the brane/antibrane system should be exactly stable for large enough separation a . However, when a becomes small enough, it should be energetically favorable to decay to a system with simply M branes on S_2^2 , which is allowed by charge conservation. This should be the case whenever

$$A(S_2^2) \leq A(S_1^2) + A(S_3^2)$$

where the areas on the right hand side refer to those of the minimal S^2 's at the critical points of $W_1'(t) = z_1(t) - z_2(t)$ and $W_3'(t) = z_1(t) - z_3(t)$,

$$A(S_1^2) = |r_1|, \quad A(S_3^2) = |r_1| + |r_2|.$$

In the class of S_2^2 , there is no holomorphic two-cycle, as $W_2'(t) = z_2(t) - z_3(t) = -ma$ never vanishes, so

$$A(S_2^2) = \sqrt{|r_2|^2 + |ma|^2}. \tag{2.5.6}$$

Clearly, when a is sufficiently small, the configuration with M branes on S_2^2 should correspond to the ground state of the system. If instead $M_{1,3}$ are generic, we end up with a vacuum with intersecting branes, studied recently in [50]. Here one has additional massless matter coming from open strings at intersection of the branes, and correspondingly there is no gaugino condensation and no closed string dual. As a result, the methods based on holography we use here have nothing to say about this vacuum.

³² Since $\text{Im}(\tau)$ is a symmetric real matrix of rank two, a necessary condition for the eigenvalues to be positive is that the diagonal entries are positive. The equation we are writing corresponds to the positivity of the diagonal entries of $\text{Im}(\tau)$ evaluated at the critical point. For weak gauge coupling, this is also the sufficient condition.

2.5.1. A stable non-supersymmetric vacuum

Consider now the flop of the simple A_2 singularity of the previous sub-section, where z_1 and z_2 get exchanged,

$$\tilde{z}_1(t) = -mt, \quad \tilde{z}_2(t) = 0, \quad \tilde{z}_3(t) = -m(t - a),$$

and where

$$S_1^2 \rightarrow \tilde{S}_1^2 = -S_1^2.$$

We now wrap $\tilde{M}_1 < 0$ antibrans on \tilde{S}_1^2 and $\tilde{M}_2 > 0$ branes on \tilde{S}_2^2 . In this case, one would expect the system to have a stable, non-supersymmetric vacuum for *any* separation between the branes. This is the case because the system has nowhere to which it can decay. Suppose we wrap one antibrane on \tilde{S}_1^2 and one brane on \tilde{S}_2^2 . If a cycle C exists such that

$$C = -\tilde{S}_1^2 + \tilde{S}_2^2 \tag{2.5.7}$$

then the brane/antibrane system can decay to a brane on C . In the present case, such a C does not exist. The reason for that is the following. On the one hand, all the curves in this geometry come from the ALE space fibration, and moreover all the S^2 's in the ALE space have self intersection number -2 . On the other hand, because the intersection number of \tilde{S}_1^2 and \tilde{S}_2^2 is $+1$, (2.5.7) would imply that the self intersection of C is -6 . So, the requisite C cannot exist. The vacuum is, in fact, both perturbatively and nonperturbatively stable; we will see that the holographic dual theory has no perturbative instabilities for *any* separation between the branes.

Because the z 's have been exchanged and the geometry is now different; we get a new prepotential $\tilde{\mathcal{F}}_0$ and effective superpotential

$$\mathcal{W}_{\text{eff}} = \sum_{i=1,2} \tilde{\alpha}_i \tilde{S}_i + \tilde{M}_i \partial_{\tilde{S}_i} \tilde{\mathcal{F}}_0(\tilde{S}) \tag{2.5.8}$$

where

$$2\pi i \mathcal{F}_0(\tilde{S}) = \frac{1}{2} \tilde{S}_1^2 \left(\log\left(\frac{\tilde{S}_1}{(-m)\Lambda_0^2}\right) - \frac{3}{2} \right) + \frac{1}{2} \tilde{S}_2^2 \left(\log\left(\frac{\tilde{S}_2}{m\Lambda_0^2}\right) - \frac{3}{2} \right) - \tilde{S}_1 \tilde{S}_2 \log\left(\frac{a}{\Lambda_0}\right).$$

Alternatively, we should be able to work with the old geometry and prepotential (2.5.2), but adjust the charges and the couplings consistently with the flop. The charges and the couplings of the two configurations are related by

$$\tilde{M}_1 = -M_1, \quad \tilde{M}_2 = M_3, \tag{2.5.9}$$

where $M_{1,3}$ are now both positive, and

$$\tilde{g}_1^{-2} = -g_1^{-2}, \quad \tilde{g}_2^{-2} = g_3^{-2}, \tag{2.5.10}$$

where g_1^{-2} is now negative. The effective superpotential is

$$\mathcal{W}_{\text{eff}} = \sum_{i=1,3} \alpha_i S_i + M_i \partial_{S_i} \mathcal{F}_0(S), \quad (2.5.11)$$

in terms of the old prepotential (2.5.2). Indeed, the two are related by $\mathcal{F}_0(S_1, S_3) = \tilde{\mathcal{F}}_0(\tilde{S}_1, \tilde{S}_2)$ and a simple change of variables

$$\tilde{S}_1 = -S_1, \quad \tilde{S}_2 = S_3,$$

leaves the superpotential invariant. The critical points of the potential associated to (2.5.11) with these charges are

$$\begin{aligned} \tilde{S}_1^{|\tilde{M}_1|} &= \left(\overline{\Lambda_0^2 m}\right)^{|\tilde{M}_1|} \left(\frac{a}{\Lambda_0}\right)^{|\tilde{M}_2|} \exp(-2\pi i \alpha_1) \\ \tilde{S}_2^{|\tilde{M}_2|} &= \left(\Lambda_0^2 m\right)^{|\tilde{M}_2|} \left(\frac{\overline{a}}{\Lambda_0}\right)^{|\tilde{M}_1|} \exp(-2\pi i \alpha_2) \end{aligned}$$

with effective potential at the critical point

$$V_* = \frac{|\tilde{M}_1|}{\tilde{g}_1^2} + \frac{|\tilde{M}_2|}{\tilde{g}_2^2} - \frac{1}{2\pi} |\tilde{M}_1| |\tilde{M}_2| \log\left(\left|\frac{a}{\Lambda_0}\right|\right).$$

The masses of the bosons in this vacuum are again given by (2.5.3)(2.5.5) with the obvious substitution of variables. Just as in the previous subsection, the masses are positive in any of these vacua. Moreover, because there are no two-loop corrections to the prepotential, as we have seen in section 3, the vacuum is stable as long as the metric remains positive definite. In the previous section, we expected an instability for small enough a , and found that the perturbatively stable non-supersymmetric solution escapes to the boundary of the moduli space (defined as the region where $\text{Im}\tau$ is positive definite) when this becomes the case. In this case, we do not expect any instability for any a , as there is nothing for the vacuum to decay to. Indeed, we find that $\text{Im}\tau$ is now positive definite for any $a \neq 0$.

The vacuum is stable perturbatively and nonperturbatively – there simply are no lower energy states with the same charges available to which this can decay. So, this gives an example of an exactly stable, non-supersymmetric vacuum in string theory, albeit without four dimensional gravity.³³ Moreover, since in this case there are no tachyons in the brane/antibrane system, this should have a consistent limit where we decouple gravity and stringy modes, and are left with a pure, non-supersymmetric, confining gauge theory, with a large N dual description. This is currently under investigation [69].

³³ This fact has been noted in [55].

Appendix 2.A. Matrix Model Computation

Using large N duality in the B model topological string [58], the prepotential \mathcal{F}_0 of the Calabi-Yau manifolds studied in this chapter can be computed using a matrix model describing branes on the geometry before the transition. The same matrix model [19] captures the dynamics of the glueball fields S in the $\mathcal{N} = 1$ supersymmetric gauge theory in space-time, dual to the Calabi-Yau with fluxes in the physical superstring theory. In this appendix, we use these matrix model/gauge theory techniques to compute the prepotential for Calabi-Yau manifolds which are A_2 fibrations with quadratic superpotentials, as studied in sections 3 and 6. To our knowledge, this computation has not previously been carried out.

The matrix model is a $U(N_1) \times U(N_2)$ quiver with Hermitian matrices Φ_1 and Φ_2 which transform in the adjoint of the respective gauge groups, and bifundamentals Q and \tilde{Q} which correspond to the bifundamental hypermultiplets coming from 12 and 21 strings. The relevant matrix integral is then given by

$$Z = \frac{1}{\text{vol}(U(N_1) \times U(N_2))} \int d\Phi_1 d\Phi_2 dQ d\tilde{Q} \exp\left(\frac{1}{g_s} \text{Tr} \mathcal{W}(\Phi_1, \Phi_2, Q, \tilde{Q})\right)$$

where \mathcal{W} is the superpotential of the corresponding $\mathcal{N} = 1$ quiver gauge theory, given by

$$\mathcal{W} = \text{Tr} W_1(\Phi_1) + \text{Tr} W_2(\Phi_2) + \text{Tr}(\tilde{Q}\Phi_1 Q) - \text{Tr}(Q\Phi_2\tilde{Q}) \quad (2.A.1)$$

with

$$\text{Tr} W_1(\Phi_1) = -\frac{m_1}{2} \text{Tr}(\Phi_1 - a_1 \text{id}_{N_1})^2, \quad \text{Tr} W_2(\Phi_2) = -\frac{m_2}{2} \text{Tr}(\Phi_2 - a_2 \text{id}_{N_2})^2.$$

The saddle points of the integral correspond to breaking the gauge group as

$$U(N_1) \times U(N_2) \rightarrow U(M_1) \times U(M_2) \times U(M_3) \quad (2.A.2)$$

where

$$N_1 = M_1 + M_3, \quad N_2 = M_2 + M_3,$$

by taking as expectation values of the adjoints and bifundamentals to be

$$\Phi_{1,*} = \begin{pmatrix} a_1 \text{id}_{M_1} & 0 \\ 0 & a_3 \text{id}_{M_3} \end{pmatrix}, \quad \Phi_{2,*} = \begin{pmatrix} a_2 \text{id}_{M_2} & 0 \\ 0 & a_3 \text{id}_{M_3} \end{pmatrix}$$

where $a_3 = (m_1 a_1 + m_2 a_2)/(m_1 + m_2)$, and

$$(Q\tilde{Q})_* = \begin{pmatrix} 0 & 0 \\ 0 & -W'_1(a_3) \text{id}_{M_3} \end{pmatrix}, \quad (\tilde{Q}Q)_* = \begin{pmatrix} 0 & 0 \\ 0 & W'_2(a_3) \text{id}_{M_3} \end{pmatrix},$$

where $-W'_1(a_3) = m_1(a_1 - a_3) = W'_2(a_3)$.

Now let's consider the Feynman graph expansion about this vacuum. The end result is a very simple path integral. However, to get there, we need to properly implement the

gauge fixing (2.A.2), and this is somewhat laborious. It is best done in two steps. First, consider fixing the gauge that simply reduces $U(N_{1,2})$ to $U(M_{1,2}) \times U(M_3)$. This follows [60] directly. Let

$$\Phi_1 = \begin{pmatrix} \Phi_{11}^1 & \Phi_{13}^1 \\ \Phi_{31}^1 & \Phi_{33}^1 \end{pmatrix}.$$

To set the $M_1 \times M_3$ block in Φ_1 to zero

$$F_1 = \Phi_{13}^1 = 0$$

we insert the identity into the path integral in the form

$$\text{id} = \int d\Lambda \delta(F_1) \text{Det}\left(\frac{\delta F_1}{\delta \Lambda}\right),$$

where the integral is over those gauge transformations not in $U(M_1) \times U(M_3)$. The determinant can be expressed in terms of two pairs of ghosts, B_{13}, B_{31} and C_{31}, C_{13} , which are anticommuting bosons, as

$$\begin{aligned} \text{Det}\left(\frac{\delta F_1}{\delta \Lambda}\right) &= \int dB_{13}dC_{31}dB_{31}dC_{13} \exp\left(\frac{1}{g_s} \text{Tr}(B_{13}\Phi_{33}^1 C_{31} - C_{31}\Phi_{11}^1 B_{13})\right) \\ &\quad \exp\left(\frac{1}{g_s} \text{Tr}(B_{31}\Phi_{11}^1 C_{13} - C_{13}\Phi_{33}^1 B_{31})\right). \end{aligned}$$

By an identical argument, we can gauge fix the second gauge group factor

$$U(N_2) \rightarrow U(M_1) \times U(M_3)$$

to set the $M_2 \times M_3$ block of Φ_2 to zero. We do this by again inserting the identity into the path integral, but now with the determinant replaced by

$$\begin{aligned} \text{Det}\left(\frac{\delta F_2}{\delta \Lambda}\right) &= \int dB_{23}dC_{32}dB_{32}dC_{23} \exp\left(\frac{1}{g_s} \text{Tr}(B_{23}\Phi_{33}^2 C_{32} - C_{32}\Phi_{22}^2 B_{23})\right) \\ &\quad \exp\left(\frac{1}{g_s} \text{Tr}(B_{32}\Phi_{22}^2 C_{23} - C_{23}\Phi_{33}^2 B_{32})\right). \end{aligned}$$

Finally, since the vacuum will break the two copies of $U(M_3)$ to a single copy, we need to gauge fix that as well. To do this, we will fix a gauge

$$F_3 = Q_{33} - q \text{id} = 0$$

where Q_{33} refers to the 33 block of Q , and *integrate* over q . This is invariant under the diagonal $U(M_3)$ only. To implement this, insert the identity in the path integral, written as

$$\text{id} = \int d\Lambda_{33} \oint \frac{dq}{q} \delta(Q_{33} - q \text{id}) q^{M_3^2}.$$

The above is the identity since

$$\text{Det}\left(\frac{\delta F_3}{\delta \Lambda_{33}}\right) = q^{M_3^2},$$

and we have taken the q -integral to be around $q = 0$. Inserting this, we can integrate out Q_{33} , and \tilde{Q}_{33} . The Q_{33} integral sets it to equal q . The \tilde{Q}_{33} integral is a delta function setting

$$\Phi_{33}^1 = \Phi_{33}^2, \tag{2.A.3}$$

but there is a left over factor of $q^{-M_3^2}$ from the Jacobian of $\delta(q(\Phi_{33}^1 - \Phi_{33}^2))$. Integrating over q gives simply 1.

The remaining fields include a pair of regular bosons Q_{13}, \tilde{Q}_{31} in the bifundamental representation of $U(M_1) \times U(M_3)$ and a pair of ghosts C_{13}, B_{31} , with *exactly* the same interactions. Consequently, we can integrate them out *exactly* and their contribution is simply 1. This also happens for Q_{32}, \tilde{Q}_{23} and B_{23}, C_{32} , which also cancel out. We are left with the spectrum presented in section 3 which very naturally describes branes with open strings stretching between them.

2.A.1. A special case

In the special case when $m_2 = -m_1 = m$, the matrix integral is one-loop exact. To begin with, the effective superpotential is given by (2.A.1) with

$$\text{Tr}W_1(\Phi_1) = -\frac{m}{2}\text{Tr}(\Phi_1)^2, \quad \text{Tr}W_2(\Phi_2) = \frac{m}{2}\text{Tr}(\Phi_2 - a \text{id}_{N_2 \times N_2})^2.$$

The theory now has only one vacuum, where Φ_1 and Q, \tilde{Q} vanish, and

$$\Phi_2 = a \text{id}_{N_2 \times N_2}.$$

Expanding about this vacuum, the superpotential can be re-written as

$$\mathcal{W}_{\text{eff}} = -\frac{m}{2}\text{Tr} \Phi_1^2 + \frac{m}{2}\text{Tr} \Phi_2^2 - a\text{Tr} Q\tilde{Q} + \text{Tr}(\tilde{Q}\Phi_1 Q - Q\Phi_2\tilde{Q}).$$

If we now redefine

$$\tilde{\Phi}_1 = \Phi_1 + \frac{1}{m}Q\tilde{Q}, \quad \tilde{\Phi}_2 = \Phi_2 + \frac{1}{m}\tilde{Q}Q,$$

the superpotential becomes quadratic in all variables, and the planar free energy is given by the exact expression:

$$\mathcal{F}_0 = \frac{S_1^2}{2} \left(\log \frac{S_1}{m\Lambda_0^2} - \frac{3}{2} \right) + \frac{S_2^2}{2} \left(\log \frac{S_2}{(-m)\Lambda_0^2} - \frac{3}{2} \right) - S_1 S_2 \log \frac{a}{\Lambda_0}$$

There are higher genus corrections to this result, but they all come from the volume of the $U(N)$ gauge groups, and receive no perturbative corrections.

Appendix 2.B. Geometrical Calculation of the Prepotential

One can derive the same prepotential by direct integration. We only sketch the computation here. The equation for the geometry (2.2.2) can be rewritten

$$\begin{aligned} x^2 + y^2 + z(z - m_1(t - a_1))(z + m_2(t - a_2)) = \\ - s_1 m_1(z + m_2(t - a_2)) - s_2 m_2(z - m_1(t - a_1)) - s_3 m_3 z. \end{aligned} \quad (2.B.1)$$

Here s_i are deformation parameters. This is a convenient rewriting of (2.2.2) because we will find that the periods of the compact cycles are given by $S_i = s_i + \mathcal{O}(S^2)$. As mentioned in the main text, the holomorphic three-form Ω of the Calabi-Yau descends to a one-form defined on the Riemann surface obtained by setting $x = y = 0$ in (2.B.1). The equation for the Riemann surface is thus

$$-1 = \frac{m_1 s_1}{z(z - m_1(t - a_1))} + \frac{m_2 s_2}{z(z + m_2(t - a_2))} + \frac{m_3 s_3}{(z - m_1(t - a_1))(z + m_2(t - a_2))}. \quad (2.B.2)$$

The one-form can be taken to be $\omega = zdt - tdz$. The one-form is only defined up to a total derivative; a total derivative changes only the periods of the non-compact cycles, and our choice avoids quadratic divergences in the non-compact periods. These divergences would not contribute to physical quantities in any case. The equation for the Riemann surface is a cubic equation for $z(t)$, so the Riemann surface has three sheets, which are glued together along branch cuts. The compact periods are given by integrals around the cuts, while the non-compact periods are given by integrals from the cuts out to a cutoff, which we take to be $t = \Lambda_0$.

It is convenient to make the change of variables

$$u = \frac{-t + a_1 + z/m_1}{a_{21}} \quad v = -z \frac{m_3}{a_{21} m_1 m_2} \quad (2.B.3)$$

where $a_{21} = a_2 - a_1$. In the new variables, the equation for the Riemann surface takes the simple form

$$1 = \frac{s_1}{\Delta^3} \frac{1}{uv} - \frac{s_2}{\Delta^3} \frac{1}{v(u+v+1)} + \frac{s_3}{\Delta^3} \frac{1}{u(u+v+1)} \quad (2.B.4)$$

with $\Delta^3 = (a_2 - a_1)^2 m_1 m_2 / m_3$ as in the main text. The change of variables is symplectic up to an overall factor, so in the new variables the one-form becomes

$$\omega = \Delta^3 (udv - vdu). \quad (2.B.5)$$

The change of variables makes it clear that we can think of the problem as having one dimensionful scale Δ , and three dimensionless quantities, s_i/Δ^3 , which we will take to be small. There are many other dimensionless quantities in the problem, such as m_i/m_j , but they do not appear in the rescaled equations so they will not appear in the periods, with one small caveat. While the equation for the Riemann surface and the one-form only depend on Δ and s_i/Δ^3 , the cutoff is defined in terms of the original variables, $t = \Lambda_0$, so the cutoff dependent contributions to the periods can depend on the other parameters.

We sketch how to compute one compact period and one non-compact period. Though it is not manifest in our equations, the problem has a complete permutation symmetry among $(s_1, s_2, -s_3)$, so this is actually sufficient. One compact cycle (call it S_1) is related to the region in the geometry where u and v are small, so that to a first approximation

$$1 \approx \frac{s_1}{\Delta^3} \frac{1}{uv}. \quad (2.B.6)$$

We expand (2.B.4) for small u, v to get

$$uv = \frac{s_1}{\Delta^3} - \frac{s_2}{\Delta^3} u(1-u-v) + \frac{s_3}{\Delta^3} v(1-u-v) + \dots \quad (2.B.7)$$

This will be sufficient for the order to which we are working, and the equation is quadratic. We could solve for $u(v)$ or $v(u)$ in this regime; we would find a branch cut and integrate the one-form around it. Equivalently, we can do a two dimensional integral

$$S_1 = \Delta^3 \int du \wedge dv \quad (2.B.8)$$

over the region bounded by the Riemann surface (this is Stokes' Theorem). One can derive a general formula for the integral over a region bounded by a quadratic equation by changing coordinates so that it is the integral over the interior of a circle. In this case, the result is

$$S_1 = s_1 + \frac{1}{\Delta^3} (s_1 s_2 - s_1 s_3 - s_2 s_3) + \mathcal{O}\left(\frac{s^3}{\Delta^6}\right). \quad (2.B.9)$$

The permutation symmetry of the problem then determines the other compact periods.

Now we compute the integral over the cycle dual to S_1 . The contour should satisfy $uv \approx s_1/\Delta^3$ and go to infinity. Also, the contour must intersect the compact one-cycle in a point. A contour which satisfies these criteria is to take u, v to be real and positive (this choice works as long as the s_i are real and positive, but the result will be general). We will need two different perturbative expansions to do this integral: one for small u and the other for small v . Since we have $uv \approx s_1/\Delta^3$, we will need a "small u " expansion which is valid up to $u \sim \sqrt{s_1/\Delta^3}$, and similarly for the small v expansion.

To expand for small v , we first multiply (2.B.4) through by v to get

$$v = \frac{s_1}{\Delta^3} \frac{1}{u} - \frac{s_2}{\Delta^3} \frac{1}{1+u+v} + \frac{s_3}{\Delta^3} \frac{v}{u(1+u+v)}$$

We now solve perturbatively for $v(u)$, using the fact that throughout the regime of interest $v \ll 1+u$. The largest that $v/(1+u)$ gets in this regime is

$$\frac{v}{1+u} < \sqrt{\frac{s_1}{\Delta^3}}.$$

To zeroth order in $v/(1+u)$,

$$v^{(0)} = \frac{s_1}{\Delta^3} \frac{1}{u} - \frac{s_2}{\Delta^3} \frac{1}{1+u}. \quad (2.B.10)$$

To first order,

$$v^{(1)} = \frac{s_1}{\Delta^3} \frac{1}{u} - \frac{s_2}{\Delta^3} \frac{1}{1+u+v^{(0)}} + \frac{s_3}{\Delta^3} \frac{v^{(0)}}{u(1+u)},$$

which upon expanding becomes

$$v^{(1)} = \frac{s_1}{\Delta^3} \frac{1}{u} - \frac{s_2}{\Delta^3} \left(\frac{1}{1+u} - \frac{v^{(0)}}{(1+u)^2} \right) + \frac{s_3}{\Delta^3} \frac{v^{(0)}}{u(1+u)}. \quad (2.B.11)$$

We need keep one more order in the perturbative expansion in order to get the prepotential to the desired order:

$$v^{(2)} = \frac{s_1}{\Delta^3} \frac{1}{u} - \frac{s_2}{\Delta^3} \frac{1}{1+u+v^{(1)}} + \frac{s_3}{\Delta^3} \frac{v^{(1)}}{u(1+u+v^{(0)})}$$

which upon expanding becomes

$$v^{(2)} = \frac{s_1}{\Delta^3} \frac{1}{u} - \frac{s_2}{\Delta^3} \left[\frac{1}{1+u} - \frac{v^{(1)}}{(1+u)^2} + \frac{(v^{(0)})^2}{(1+u)^3} \right] + \frac{s_3}{\Delta^3} \left[\frac{v^{(1)}}{u(1+u)} - \frac{(v^{(0)})^2}{(1+u)^2} \right] \quad (2.B.12)$$

Note that using (2.B.10)(2.B.11), this is an explicit equation for $v(u)$. We could similarly expand to find $u(v)$ in the regime of small u , but actually we can save ourselves the computation by noting that the equation for the Riemann surface is invariant under $u \leftrightarrow v$, $s_2 \leftrightarrow -s_3$. We are now in a position to perform the integral of the one-form ω over the contour. We use the approximation (2.B.12) for the part of the integral where v is small, and the corresponding formula for $u(v)$ for the part of the integral where u is small. We can choose to go over from one approximation to another at a point $u_{\min} = v_{\min}$. Such a point will be approximately $u_{\min} = \sqrt{s_1/\Delta^3}$, but we need a more precise formula. By setting $u = v$ in the equation for the Riemann surface, and perturbing around $u_{\min} = \sqrt{s_1/\Delta^3}$, we find

$$u_{\min}^2 = \frac{s_1}{\Delta^3} - \frac{s_2 - s_3}{\Delta^3} \sqrt{\frac{s_1}{\Delta^3}} + \frac{(s_2 - s_3)^2}{2\Delta^6} + \frac{2s_1(s_2 - s_3)}{\Delta^6} + \dots$$

We spare the reader the details of the integration. The result is cutoff dependent, and we assume that the cutoff is sufficiently large so that we can drop contributions which depend inversely on the cutoff. After doing the integral, we rewrite the result in terms of the compact periods S_i using (2.B.9). The result is:

$$\begin{aligned} \partial_{S_1} \mathcal{F}_0 &= (S_1 - S_2) \log u_{\max} + (S_1 + S_3) \log v_{\max} - (S_1 \log \frac{S_1}{\Delta^3} - S_1) \\ &\quad - \frac{1}{\Delta^3} \left(\frac{1}{2} S_2^2 + \frac{1}{2} S_3^2 + S_1 S_2 - S_1 S_3 - 3S_2 S_3 \right) + \mathcal{O} \left(\frac{S^3}{\Delta^6} \right). \end{aligned}$$

Here u_{\max} and v_{\max} are cutoffs at large u, v . Since our cutoff is $t = \Lambda_0$, we can solve for u_{\max}, v_{\max} . When u is large, v is small, since $uv \approx S_1/\Delta^3$. Looking back at the change of variables (2.B.3), we find

$$u_{\max} = \frac{\Lambda_0}{a_{21}} \quad v_{\max} = \frac{\Lambda_0 m_3}{a_{21} m_2} = \frac{\Lambda_0}{a_{31}}$$

Again, the other non-compact periods are determined by symmetry. It is now a simple matter to find \mathcal{F}_0 :

$$\begin{aligned}
2\pi i \mathcal{F}_0 &= \frac{1}{2} S_1^2 \log \frac{\Lambda_0^2}{a_{21} a_{31}} + \frac{1}{2} S_2^2 \log \frac{\Lambda_0^2}{a_{21} a_{23}} + \frac{1}{2} S_3^2 \log \frac{\Lambda_0^2}{a_{31} a_{23}} \\
&\quad - S_1 S_2 \log \frac{\Lambda_0}{a_{21}} + S_1 S_3 \log \frac{\Lambda_0}{a_{31}} + S_2 S_3 \log \frac{\Lambda_0}{a_{23}} \\
&\quad - \frac{1}{2} S_1^2 \left(\log \frac{S_1}{\Delta^3} - \frac{3}{2} \right) - \frac{1}{2} S_2^2 \left(\log \frac{S_2}{\Delta^3} - \frac{3}{2} \right) - \frac{1}{2} S_3^2 \left(\log \frac{S_3}{\Delta^3} - \frac{3}{2} \right) \\
&\quad - \frac{1}{2\Delta^3} (S_1 S_2^2 + S_1^2 S_2 + S_1 S_3^2 - S_1^2 S_3 + S_2 S_3^2 - S_2^2 S_3 - 6S_1 S_2 S_3) + \mathcal{O} \left(\frac{S^4}{\Delta^6} \right).
\end{aligned}$$

This result agrees with the matrix model computation of appendix 2.A. Recall that we dropped terms which depend inversely on the cutoff. More precisely, we dropped contributions to the non-compact period of the form $S_i |a_{12}| / \Lambda_0$. This is necessary in order to match the result of the matrix model computation. In particular, in order to justify keeping the corrections we do keep, we require

$$\frac{S_i}{\Delta^3} \gg \frac{|a_{12}|}{\Lambda_0}. \tag{2.B.13}$$

Appendix 2.C. The Hessian at Two Loops

The equations required to analyze stability simplify if we introduce the notation

$$u^a \equiv i G^{ab} \alpha_b.$$

Since we are taking the α_i to be pure imaginary, u^a will be real and positive. Furthermore, since we are taking τ_{ab} to be pure imaginary, we can replace it with the metric, $\tau_{ab} = i G_{ab}$. Then the equation of motion (2.2.6) takes the simple form

$$\frac{1}{2} i \mathcal{F}_{kab} (u^a u^b - M^a M^b) = 0. \tag{2.C.1}$$

At one-loop, the third derivative of the prepotential is nonzero only if all of the derivatives are with respect to the same variable, so at one-loop the solutions are $u^a = \pm M^a$. As discussed earlier, the physically relevant solutions are

$$u^a = |M^a|. \tag{2.C.2}$$

This is just a rewriting of the one-loop solutions (2.2.8) in terms of the new notation.

At two loops, we can find the solution by perturbing around the one-loop result. Let $u^a = |M^a| + \delta^a$. We find that

$$\delta^k = \frac{1}{2|M^k| \mathcal{F}_{kkk}} \mathcal{F}_{kab} (-|M^a| |M^b| + M^a M^b). \tag{2.C.3}$$

Having solved the equations of motion at two loops, we proceed to the Hessian, providing less detail. Assuming the same reality conditions, the matrices of second derivatives are given by

$$\partial_k \partial_l V = \partial_{\bar{k}} \partial_{\bar{l}} V = \frac{1}{2} (i\mathcal{F}_{abkl} + i\mathcal{F}_{cak} i\mathcal{F}_{dbl} G^{cd}) (u^a u^b - M^a M^b), \quad (2.C.4)$$

$$\partial_k \partial_{\bar{l}} V = \partial_{\bar{k}} \partial_l V = \frac{1}{2} i\mathcal{F}_{cak} i\mathcal{F}_{dbl} G^{cd} (u^a u^b + M^a M^b). \quad (2.C.5)$$

The relations between the different mixed partial derivatives arise because we are perturbing about a real solution.

At two loops, taking four derivatives of the prepotential gives zero unless all of the derivatives are with respect to the same variable, so the first term in (2.C.4) can be simplified as

$$i\mathcal{F}_{abkl} (u^a u^b - M^a M^b) = \delta_{kl} i\mathcal{F}_{kkkk} (u^k u^k - M^k M^k) = 2\delta_{kl} i\mathcal{F}_{kkkk} |M^k| \delta^k. \quad (2.C.6)$$

Though it is not obvious at this stage, the other terms on the right hand side can be approximated by their one-loop value in the regime of interest. This is very useful because, as mentioned previously, at one-loop the third derivatives of the prepotential vanish unless all indices are the same. With these simplifications, the nonzero second derivatives become

$$\begin{aligned} (\partial_a + \partial_{\bar{a}})(\partial_b + \partial_{\bar{b}})V = \\ \sum_c 2i\mathcal{F}_{aaa} i\mathcal{F}_{ccc} G^{ac} |M^a| |M^c| \left(\delta_{cb} + G_{cb} \frac{i\mathcal{F}_{bbbb} \delta^b}{i\mathcal{F}_{bbb} i\mathcal{F}_{ccc} |M^c|} \right), \end{aligned} \quad (2.C.7)$$

$$\begin{aligned} (\partial_a - \partial_{\bar{a}})(-\partial_b + \partial_{\bar{b}})V = \\ \sum_c 2i\mathcal{F}_{aaa} i\mathcal{F}_{ccc} G^{ac} |M^a| |M^c| \left(\delta_{cb} - G_{cb} \frac{i\mathcal{F}_{bbbb} \delta^b}{i\mathcal{F}_{bbb} i\mathcal{F}_{ccc} |M^c|} \right). \end{aligned} \quad (2.C.8)$$

In these equations, no indices are implicitly summed over.

In order to analyze the loss of perturbative stability, we compute the determinant of the Hessian. Since the eigenvalues remain real, in order to go from a stable solution to an unstable one, an eigenvalue should pass through zero. We therefore analyze where the determinant is equal to zero. Up to possible constant factors, the determinant is given by

$$(\text{Det } G^{ab})^2 \left(\prod_c \frac{|M^c|}{i\mathcal{F}_{ccc}} \right)^4 \text{Det} \left(\delta_{cb} + G_{cb} \frac{\mathcal{F}_{bbbb} \delta^b}{i\mathcal{F}_{bbb} \mathcal{F}_{ccc} |M^c|} \right) \text{Det} \left(\delta_{cb} - G_{cb} \frac{\mathcal{F}_{bbbb} \delta^b}{i\mathcal{F}_{bbb} \mathcal{F}_{ccc} |M^c|} \right) \quad (2.C.9)$$

and so in order to vanish, one of the last two determinants must go to zero.

Chapter 3

Geometric Transitions and Dynamical Supersymmetry Breaking

It is of significant interest to find simple examples of dynamical supersymmetry breaking in string theory. One class of examples, where stringy D-instanton effects play a starring role, was described in [70]. These models exhibit “retrofitting” of the classic SUSY breaking theories (Fayet, Polonyi and O’Raifeartaigh) [71] without incorporating any non-trivial gauge dynamics. Instead, stringy instantons [72,16] automatically implement the exponentially small scale of SUSY breaking in theories with only Abelian gauge fields. A related idea using disc instantons instead of D-instantons appears in [73]. These models are simpler in many ways than their existing field theory analogues [74].

In this chapter, we show that these results (and many generalizations) admit a clear and computationally powerful understanding using geometric transition techniques [17] (see also [23,24]). Such techniques are well known to translate quantum computations of superpotential interactions in nontrivial gauge theories to classical geometric computations of flux-induced superpotentials [21]. They are most powerful when the theories in question exhibit a mass gap. While the classic models we study *do* manifest light degrees of freedom (and hence do not admit a complete description in terms of geometry and fluxes), we find that a mixed description involving small numbers of D-branes in a flux background – which arises after a geometric transition from a system of branes at a singularity – nicely captures the relevant physics of supersymmetry breaking.³⁴ In the original theory without flux, the SUSY breaking effects are generated by D-instantons either in $U(1)$ gauge factors or on unoccupied, but orientifolded, nodes of the quiver gauge theory (analogous to those studied in [70,75,43]). Both are in some sense “stringy” effects. Simple generalizations involve more familiar transitions on nodes with large N gauge groups.

³⁴ For an application of geometric transitions to the study of supersymmetry breaking in the context of brane/antibrane systems, see [4,37,38,46,5].

The geometric transition techniques we apply have two advantages over the description using stringy instantons in a background without fluxes. First, they allow for a classical computation of the relevant superpotential instead of requiring a nontrivial instanton calculation. Second, they incorporate higher order corrections (due to multi-instanton effects in the original description) which had not been previously calculated.

The organization of this chapter is as follows. In section 2, we remind the reader of the relevant background about geometric transitions. In section 3, we discuss the geometries we will use to formulate our theories of dynamical supersymmetry breaking. In sections 4-6, we give elementary examples that yield Fayet, Polonyi, and O’Raifeartaigh models that break SUSY at exponentially low scales. In section 7, we present a single geometry that unifies the three models, reducing to them in various limits. In section 8, we provide a more general, exact analysis of the existence of these kinds of susy-breaking effects. In section 9, we give a few other examples of simple DSB theories (related to recent or well-known literature in the area). Finally, in section 10, we extend the technology to orientifold models, in particular recovering models which are closely related to the specific examples of [70].

3.1. Geometric Transitions

Computing nonperturbative corrections in string theory, even to holomorphic quantities such as a superpotential, is in general very difficult. A surprising recent development [17,19] is that in some cases – namely for massive theories – these nonperturbative effects can be determined by perturbative means in a dual language.³⁵

Consider, for example, N D5 branes in type IIB string theory wrapping an isolated, rigid \mathbb{P}^1 in a local Calabi-Yau manifold. In the presence of D5 branes, D1 brane instantons wrapping the \mathbb{P}^1 generate a superpotential for its Kähler modulus.³⁶ The instanton effects are proportional to

$$\exp\left(-\frac{t}{Ng_s}\right)$$

where $t = \int_{S^2} (B^{NS} + ig_s B^{RR})$. For general N , these D1 brane instantons are gauge theory instantons. More precisely, they are the fractional $U(N)$ instantons of the low energy $\mathcal{N} = 1$ $U(N)$ gauge theory on the D5 brane. However, on the basis of zero-mode counting, one expects that stringy instanton effects are present even for a single D5 brane.

In the absence of D5 branes, the theory has $\mathcal{N} = 2$ supersymmetry, and the Kähler moduli space is unlifted. In that case, the local Calabi-Yau with a rigid \mathbb{P}^1 is known to have another phase where the S^2 has shrunk to zero size and has been replaced by a finite S^3 . The two branches meet at $t = 0$, where there is a singularity at which D3 branes wrapping the S^3 become massless.

What happens to this phase transition in the presence of D5 branes? Classically, we can still connect the S^2 to the S^3 side by a geometric transition. The only difference is

³⁵ For a two-dimensional example, see [76].

³⁶ This is a slight misnomer, since t is a parameter, and not a dynamical field for a non-compact Calabi-Yau.

that to account for the D5 brane charge, we need there to be N units of RR flux through the S^3 ,

$$\int_{S^3} H^{RR} = N.$$

Quantum mechanically the effect is more dramatic. In the presence of D5 branes, there is no sharp phase transition at all between the S^2 and the S^3 sides; the interpolation between them is completely *smooth*. As a consequence, the two sides of the transition provide *dual* descriptions of the same physics. Since the theory is massive now, the interpolation occurs by varying the coupling constants of the theory. The fact that the singularity where the S^3 shrinks to zero size is eliminated is consistent with the fact that D3 branes wrapping an S^3 with RR flux through it are infinitely massive. The most direct proof of the absence of a phase transition is in the context of M-theory on a G_2 holonomy manifold [77,78,79]. This is related to the present transition by mirror symmetry and an M-theory lift. In M-theory, the transition is analogous to a perturbative flop transition of type IIA string theory at the conifold, except that in M-theory, the classical geometry gets corrected by M2 brane instantons instead of worldsheet instantons [77]. The argument that the two sides are connected smoothly is analogous to Witten's argument for the absence of a sharp phase transition in IIA [80]. In both cases, the presence of instantons is crucial for the singularities in the interior of the classical moduli space to be eliminated.

The fact that the two sides of the transition are connected smoothly implies that the superpotentials should be the same on both sides. The instanton-generated superpotential has a dual description on the S^3 side as a *perturbative* superpotential generated by fluxes. The flux superpotential

$$\mathcal{W} = \int H \wedge \Omega$$

is perturbative, given by

$$\mathcal{W} = \frac{t}{g_s} S + N \partial_S \mathcal{F}_0 \tag{3.1.1}$$

where $\mathcal{F}_0(S)$ is the prepotential of the Calabi-Yau, and

$$S = \int_{S^3} \Omega.$$

The first term in (3.1.1) comes from the running of the gauge coupling, t/g_s , which implies that there is a nonzero H^{NS} flux through a three-chain on the S^2 side. This three-chain becomes the non-compact three-cycle dual to the S^3 after the transition. Near the conifold point,

$$\partial_S \mathcal{F}_0 = S \left(\log \left(\frac{S}{\Delta^3} \right) - 1 \right) + \dots$$

where the omitted terms are a model dependent power series in S , and Δ is a high scale at which t is defined. Integrating out S in favor of t , the superpotential \mathcal{W} becomes

$$\mathcal{W}_{inst} = -\Delta^3 \exp\left(-\frac{t}{Ng_s}\right) + \dots$$

up to two- and higher-order instanton terms that depend on the power series in $\mathcal{F}_0(S)$. The duality should persist even in the presence of other branes and fluxes, as long as the S^2 that the branes wrap remains isolated, and the geometry near the branes is unaffected. As we will discuss in section 10, this can also be extended to D5 branes wrapping \mathbb{P}^1 's in Calabi-Yau orientifolds.

3.2. The Theories

To construct the models in question, we will consider type IIB on non-compact Calabi-Yau threefolds which are A_r ADE type ALE spaces fibered over the complex plane $\mathbb{C}[x]$. These are described as hypersurfaces in \mathbb{C}^4 as follows:

$$uv = \prod_{i=1}^{r+1} (z - z_i(x)). \quad (3.2.1)$$

This geometry is singular at points where $u, v = 0$ and $z_i(x) = z_j(x) = z$. At these points, there are \mathbb{P}^1 's of vanishing size which can be blown up by deforming the Kähler parameters of the Calabi-Yau. There are r two-cycle classes, which we will denote

$$S_i^2.$$

These correspond to the blow-ups of the singularities at $z_i = z_{i+1}$, $i = 1, \dots, r$. It is upon these \mathbb{P}^1 's that we wrap D5 branes to engineer our gauge theories.

The theory on the branes can be thought of as an $\mathcal{N} = 2$ theory, corresponding to D5 branes wrapping two-cycles of the ALE space, which is then deformed to an $\mathcal{N} = 1$ theory by superpotentials for the adjoints. For the branes on S_i^2 , this superpotential is denoted $W_i(\Phi_i)$. The adjoints Φ_i describe the positions of the branes in the x -direction, and the superpotential arises because the ALE space is fibered nontrivially over the x -plane. The superpotential can be computed by integrating [81,30]

$$W = \int_C \Omega$$

over a three-chain C with one boundary as the wrapped S^2 . In this particular geometry, it takes an extra simple form (the details of the computation appear in appendix 3.A),

$$W_i(x) = \int (z_i(x) - z_{i+1}(x)) dx. \quad (3.2.2)$$

In addition to the adjoints, for each intersecting pair of two-cycles S_i^2, S_{i+1}^2 there is a bifundamental hypermultiplet at the intersection, consisting of chiral multiplets $Q_{i,i+1}$ and $Q_{i+1,i}$, with a superpotential interaction inherited from the $\mathcal{N} = 2$ theory,

$$\text{Tr}(Q_{i,i+1} \Phi_{i+1} Q_{i+1,i} - Q_{i,i+1} Q_{i+1,i} \Phi_i).$$

Classically, the vacua of the theory correspond to the different ways of distributing branes on the minimal \mathbb{P}^1 's in the geometry [31]. When one of the nodes is massive, the instantons corresponding to D1 branes wrapping the S^2 can be summed up in the dual geometry after a geometric transition. As explained in [70], and as we will see in the next section, this can trigger supersymmetry breaking in the rest of the system.

As an aside, we note that the systems we are studying are a slight generalization of those described in [75,70]. Those geometries are related to the family of geometries studied here, but correspond to particular points in the parameter space where the adjoint masses have been taken to be large and the branes and/or O-planes have been taken to coincide in the x -plane. In addition, we allow the possibility of $U(1)$ (or in some cases higher-rank) gauge groups on the transitioning node, whereas in [75,70] the instanton effects were associated with nodes that were only occupied by O-planes. Nevertheless, we will find the same qualitative physics as in [70] in this broader class of theories.

3.3. The Fayet Model

We now turn to a specific geometry which will engineer the Fayet model at low energies. This is an A_3 geometry, and (3.2.1) can be written explicitly as

$$uv = (z - mx)(z + mx)(z - m(x - 2a)). \quad (3.3.1)$$

After blowing up, we wrap M branes each on S_1^2 at $z_1(x) = z_2(x)$ and S_2^2 at $z_2(x) = z_3(x)$, and one brane on S_3^2 at $z_3(x) = z_4(x)$. The tree-level superpotential (3.2.2) is now given by

$$W = \sum_{i=1}^3 W_i(\Phi_i) + \text{Tr}(Q_{12}\Phi_2Q_{21} - Q_{21}\Phi_1Q_{12}) + \text{Tr}(Q_{23}\Phi_3Q_{32} - Q_{32}\Phi_2Q_{23}) \quad (3.3.2)$$

where

$$W_1(\Phi_1) = m\Phi_1^2, \quad W_2(\Phi_2) = -m\Phi_2^2, \quad W_3(\Phi_3) = m(\Phi_3 - a)^2.$$

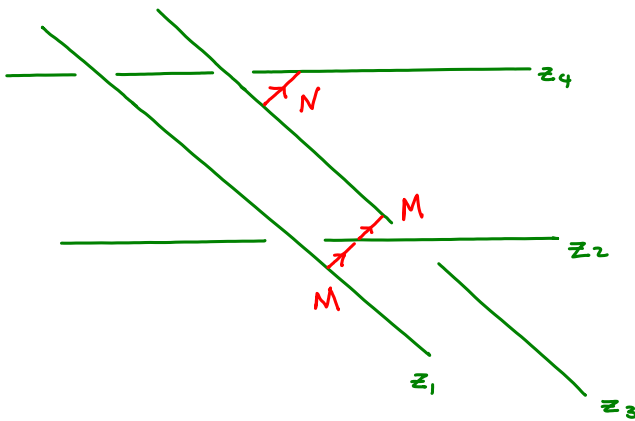


Fig. 3.1. The A_3 geometry, used for retrofitting the Fayet model, *before* the geometric transition. The red lines represent the \mathbb{P}^1 's, wrapped by D5 branes. The third node does not intersect the other two and is massive. The geometry after the transition sums up the corresponding instantons. For $N = 1$ branes on S_3^2 , the instantons are stringy. For $N > 1$, these are fractional instantons associated with gaugino condensation in the pure $U(N)$ $\mathcal{N} = 1$ gauge theory on that node.

The branes on nodes one and two intersect, since both of the corresponding \mathbb{P}^1 's are at $x = 0$. However, the third node, and the single brane on it, is isolated at $x = a$, and the theory living on it is massive. Correspondingly, the instanton effects due to D-instantons wrapping the third node can be summed up in a dual geometry where we trade S_3^2 for a three-cycle S^3 with one unit of flux through it,

$$\int_{S^3} H^{RR} = 1 .$$

The geometry after the transition is described by the deformed equation

$$uv = (z - mx)(z + mx)((z - mx)(z + m(x - 2a)) - s), \quad (3.3.3)$$

where the size of the S^3

$$\int_{S^3} \Omega = S$$

is given by $S = s/m$. It is fixed to be exponentially small by the flux superpotential, as we shall see shortly. The third brane is gone now, and so are the fields Q_{23} , Q_{32} and Φ_3 . The effective superpotential can now be written to leading order in S as

$$W_{eff} = W_1(\Phi_1) + \widetilde{W}_2(\Phi_2, S) + \text{Tr}(Q_{12}\Phi_2Q_{21} - Q_{21}\Phi_1Q_{12}) + \mathcal{W}_{flux}(S).$$

In this geometry, the exact flux superpotential is

$$\mathcal{W}_{flux} = \frac{t}{g_s} S + S \left(\log \frac{S}{\Delta^3} - 1 \right)$$

without any polynomial corrections in S . It is crucial here that the superpotential for Φ_2 has changed, due to the change in the geometry, to $\widetilde{W}_2(\Phi_2)$, where

$$\widetilde{W}_2(x) = \int (z_2(x) - \widetilde{z}_3(x)) dx,$$

while the superpotential for Φ_1 is unaffected. We have defined

$$(z - \widetilde{z}_3(x))(z - \widetilde{z}_4(x)) = (z - z_3(x))(z - z_4(x)) - s$$

with $\widetilde{z}_3(x)$ being the branch which asymptotically looks like $z_3(x)$ at large values of x . In other words,

$$\widetilde{W}_2(x) = \int_{\Delta}^x \left(-m(x' + a) - \sqrt{m^2(x' - a)^2 + s} \right) dx'.$$

This superpotential sums up the instanton effects due to Euclidean branes wrapping node three.

Before the transition, the vacuum was at $\Phi_2 = 0$. At the end of the day, we expect it to be perturbed by exponentially small terms of order S , so the relevant part of the superpotential is

$$\widetilde{W}_2(\Phi_2) = -m \text{Tr} \Phi_2^2 - \frac{1}{2} S \text{Tr} \log \frac{a - \Phi_2}{\Delta} + \dots, \quad (3.3.4)$$

where we've omitted terms of order S^2 and higher and dropped an irrelevant constant. We comment on the form of these corrections in appendix 3.B.

The theories on nodes one and two are asymptotically free. If the fields S and $\Phi_{1,2}$ have very large masses, we can integrate them out and keep only the light degrees of freedom. Keeping only the leading instanton corrections, the relevant F-terms are

$$\begin{aligned} F_{\Phi_1} &= 2m\Phi_1 - Q_{12}Q_{21} \\ F_{\Phi_2} &= -2m\Phi_2 + Q_{21}Q_{12} + \frac{S}{2(a - \Phi_2)} \\ F_S &= t/g_s + \log S/\Delta^3 - \frac{1}{2}\text{Tr} \log(a - \Phi_2)/\Delta \end{aligned} \tag{3.3.5}$$

Setting these to zero, we obtain

$$S_* = \Delta^3 \exp\left(-\frac{\tilde{t}}{g_s}\right) + \dots$$

where

$$\tilde{t} = t - \frac{1}{2}Mg_s \log(a/\Delta)$$

and

$$\begin{aligned} \Phi_{1,*} &= -\frac{1}{2m}Q_{12}Q_{21} \\ \Phi_{2,*} &= \frac{1}{2m}Q_{21}Q_{12} + \frac{1}{4ma}S_* + \dots \end{aligned} \tag{3.3.6}$$

The omitted terms are higher order in $Q_{21}Q_{12}/ma$ and $\exp(-\frac{\tilde{t}}{g_s})$. The low energy, effective superpotential is

$$W_{eff} = \frac{1}{m}\text{Tr}(Q_{12}Q_{21}Q_{12}Q_{21}) - \frac{S_*}{4ma}\text{Tr} Q_{12}Q_{21} + \dots,$$

where we have neglected corrections to the quartic coupling, and the higher order couplings of Q 's, all of which are exponentially suppressed. As shown in [70], in the presence of a generic FI term for the off-diagonal $U(1)$ under which Q_{12} and Q_{21} are charged,

$$D = Q_{12}Q_{12}^\dagger - Q_{21}^\dagger Q_{21} - r,$$

the exponentially small mass for Q will trigger F-term supersymmetry breaking with an exponentially low scale; we can set $Q_{12,*} = \sqrt{r}$, and then

$$F_{Q_{21}} \sim \frac{\sqrt{r}}{4ma}S_* .$$

Geometrically, turning on the FI term corresponds to choosing the central charges of the branes on the two nodes to be misaligned. Combined with the fact that nodes one and two have become massive with an exponentially low mass, this provides an extremely simple

mechanism of breaking supersymmetry at a low scale. The non-supersymmetric vacuum we found classically is reliable as long as the scale of supersymmetry breaking is far above the strong coupling scales of the $U(M) \times U(M)$ gauge theory. If we take N branes on the massive node instead of one, the story is the same, apart from the fact that the flux increases, and correspondingly the vacuum value of S changes to $S_* \sim \Delta^3 \exp(-\tilde{t}/Ng_s)$. In this case, however, the instantons that trigger supersymmetry breaking are the fractional $U(N)$ instantons.

3.4. The Polonyi Model

In this section we construct the Polonyi model with an exponentially small linear superpotential term for a chiral superfield Φ . This will turn out to be somewhat more subtle, and the existence of the (meta)stable vacuum will depend sensitively on the Kähler potential. We describe specific cases where we know that the relevant Kähler potential does yield a stable vacuum in section 7.

Consider an A_2 geometry given by

$$uv = (z - mx)(z - mx)(z + m(x - 2a)) \quad (3.4.1)$$

which has one D5 brane wrapped on the S_1^2 blown up at $z_1(x) = z_2(x)$, and one D5 brane wrapped on the S_2^2 blown up at $z_2(x) = z_3(x)$. This system has a tree-level superpotential

$$W = W_1(\Phi_1) + W_2(\Phi_2) + Q_{12}\Phi_2Q_{21} - Q_{21}\Phi_1Q_{12}, \quad (3.4.2)$$

where

$$W_1(\Phi_1) = 0, \quad W_2(\Phi_2) = m(\Phi_2 - a)^2.$$

There is a classical moduli space of vacua parameterized by the expectation value of Φ_1 and where $Q_{12,*} = Q_{21,*} = 0$, and $\Phi_{2,*} = a$.

At a generic point in the moduli space, away from $\Phi_1 = a$, the theory on the branes wrapping S_2^2 is massive. Then, the instanton effects associated with D1 branes wrapping this node can be summed up by a geometric transition that replaces S_2^2 by an S^3 with one unit of flux through it. This deforms the Calabi-Yau geometry to

$$uv = (z - mx)((z - mx)(z + m(x - 2a)) - s),$$

which now has an S^3 of size

$$\int_{S^3} \Omega = S$$

where $S = s/m$. With this deformation, the superpotential for node one is altered as well:

$$\widetilde{W}_1(x) = \int (-m(a - x) + \sqrt{m^2(a - x)^2 + s})dx.$$

The effective superpotential after the transition is simply

$$W_{eff} = \widetilde{W}_1(\Phi_1, S) + \mathcal{W}_{flux}(S)$$

where the flux superpotential assumes the simple form

$$\mathcal{W}_{flux}(S) = \frac{t}{g_s} S + S(\log S/\Delta^3 - 1).$$

Note that there is no supersymmetric vacuum, since $F_{\Phi_1} \neq 0$ always.

Suppose that at a point in the moduli space, say at $\Phi_1 = 0$, the Kähler potential takes the form

$$K = |\Phi_1|^2 + c|\Phi_1|^4 + \dots$$

where the higher order terms are suppressed by a characteristic mass scale (which we set to one). Then, provided

$$|ca^2| \gg 1, \quad c < 0,$$

the theory has a non-supersymmetric vacuum at

$$\Phi_{1,*} = \frac{1}{ca^*} \tag{3.4.3}$$

which breaks SUSY at an exponentially low scale. This can be seen as follows. Expanded about small Φ_1 , the superpotential \widetilde{W}_1 takes the form

$$\widetilde{W}_1(\Phi_1) = -\frac{S}{2} \log(a - \Phi_1)/\Delta + \dots$$

where the subleading terms are suppressed by additional powers of S , but are otherwise regular at the origin of Φ_1 space. Integrating out S first, by solving its F-term constraint, we find

$$S_* = \Delta^3 \exp(-\tilde{t}/g_s) + \dots$$

where

$$\tilde{t} = t - \frac{1}{2}g_s \log(a/\Delta)$$

and the subleading terms are of order Φ_1/a , which will turn out to be small in the vacuum. For large \tilde{t} , S is generically very massive, so integrating it out is justified.

The potential for Φ_1 now becomes

$$V_{eff}(\Phi_1) = \frac{1}{1 + c|\Phi_1|^2} \frac{|S_*|^2}{|a - \Phi_1|^2} + \dots$$

It is easy to see that, up to corrections of order $1/|a^2c|$ and $S_*/(ma^2)$, this has a non-supersymmetric vacuum at (3.4.3) where Φ_1 has a mass squared of order

$$-c \left| \frac{S_*}{a} \right|^2.$$

This is positive, and the vacuum is (meta)stable, as long as $c < 0$. Note that we could have obtained the Polonyi model as a limit of the Fayet model in which we turn on a very large FI term for the off-diagonal $U(1)$ of nodes one and two. In this case, the stability of the Fayet model for a generic (effectively canonical) Kählerpotential guarantees that the Polonyi model obtained from it is stable. In fact, as we will review in section 7, one can show this directly by computing the relevant correction to the Kähler potential arising from loops of massive gauge bosons [70].

3.5. The O’Raifeartaigh Model

To represent the third simple classic class of SUSY breaking models, we engineer an O’Raifeartaigh model. Consider the A_3 fibration with

$$z_1(x) = mx, \quad z_2(x) = mx, \quad z_3(x) = mx, \quad z_4(x) = -m(x - 2a). \quad (3.5.1)$$

The defining equation of the non-compact Calabi-Yau is then

$$uv = (z - mx)(z - mx)(z - mx)(z + m(x - 2a)) \quad (3.5.2)$$

and we wrap a single D5 brane on each of $S_{1,2,3}^2$. The adjoints Φ_1 and Φ_2 are massless, while Φ_3 obtains a mass from its superpotential,

$$W_3(x) = \int (z_3(x) - z_4(x)) dx \quad (3.5.3)$$

which gives

$$W_3(\Phi_3) = m(\Phi_3 - a)^2.$$

Of course, there are also quarks Q_{12}, Q_{21} and Q_{23}, Q_{32} . They couple via superpotential couplings

$$Q_{12}\Phi_1 Q_{21} - Q_{12}\Phi_2 Q_{21} + Q_{23}\Phi_2 Q_{32} - Q_{23}\Phi_3 Q_{32}. \quad (3.5.4)$$

Because Φ_3 is locked at a , for generic values of Φ_2 , Q_{23} and Q_{32} are massive. Then node three is entirely massive, and we can perform a geometric transition.

The resulting theory has a new glueball superfield S , and effective superpotential

$$\mathcal{W}_{eff} = Q_{12}\Phi_1 Q_{21} - Q_{12}\Phi_2 Q_{21} - \frac{1}{2}S \log(a - \Phi_2)/\Delta + S(\log(S/\Delta^3) - 1) + \frac{t}{g_s}S + \dots \quad (3.5.5)$$

Integrating out the S field yields (at leading order)

$$S_* = \Delta^3 e^{-\tilde{t}/g_s} \quad (3.5.6)$$

where

$$\tilde{t} = t + \frac{1}{2}g_s \log(a/\Delta).$$

Plugging this into the superpotential yields

$$\mathcal{W}_{eff} = Q_{12}\Phi_1 Q_{21} - Q_{12}\Phi_2 Q_{21} - \frac{1}{2}S_* \Phi_2/a + \dots \quad (3.5.7)$$

The omitted terms are suppressed by more powers of Φ_2/a . We recognize (3.5.7) as the superpotential for an O’Raifeartaigh model, very similar to the one considered in [70]. We see that setting $F_{\Phi_1} = F_{\Phi_2} = 0$ is impossible, so one obtains F-term supersymmetry breaking, with a small scale set by $\Delta^3 \exp(-t/3g_s)$.

The stability of the non-supersymmetric vacuum again depends on the form of (technically) irrelevant corrections to the Kähler potential. As in the case of the Polonyi model, corrections which yield a stable vacuum can be arranged by embedding the model in a slightly larger theory. We now turn to a general analysis of one such larger theory.

3.6. A Master Geometry

It is possible to construct one configuration of branes on an A_4 geometry which, in appropriate limits, can be made to reduce to any of the three simple models discussed in the previous sections. The geometry is described by the defining equation

$$uv = (z - mx)(z - mx)(z + mx)(z - mx)(z + m(x - 2a)) \quad (3.6.1)$$

where we wrap N branes on nodes one, two and three, and a single brane on node four, leading to a superpotential given by

$$W_{master} = \sum_{i=1}^4 W_i(\Phi_i) + \sum_{i=1}^3 \text{Tr}(Q_{i,i+1}\Phi_{i+1}Q_{i+1,i} - Q_{i+1,i}\Phi_iQ_{i,i+1}). \quad (3.6.2)$$

The superpotentials for the adjoints are given by

$$W_1(\Phi_1) = 0, \quad W_2(\Phi_2) = -m\text{Tr}(\Phi_2^2), \quad W_3(\Phi_3) = m\text{Tr}(\Phi_3^2), \quad W_4(\Phi_4) = -m(\Phi_4 - a)^2.$$

In the interest of simplicity, we will set $N = 1$ in this section. The non-Abelian generalization is immediate, since all the nodes are asymptotically free (for large adjoint masses). As long as the scale of supersymmetry breaking driven by the geometric transition is high enough, we can ignore the non-Abelian gauge dynamics on the other nodes.

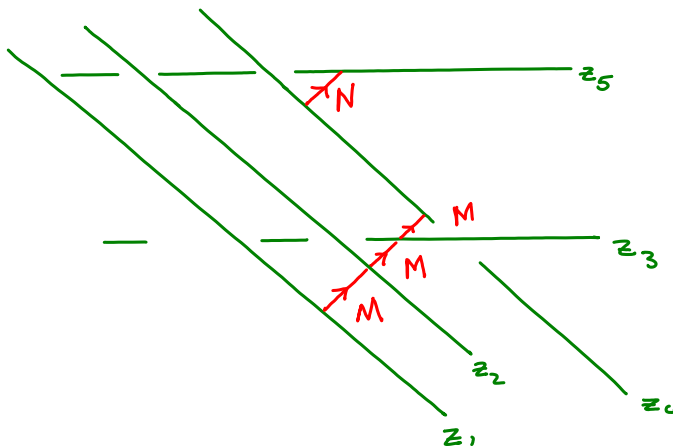


Fig. 3.2. The master A_4 geometry that gives rise to Fayet, Polonyi and O’Raifeartaigh models by turning on suitable FI terms. The stringy instantons associated with the massive fourth node generate the nonperturbative superpotential that triggers dynamical supersymmetry breaking in the rest of the theory.

The master theory has a metastable, non-supersymmetric vacuum for generic, nonzero FI terms. We can recover all three of the models discussed above by introducing large Fayet-Iliopoulos terms for certain pairs of quarks, so we expect that these will have non-supersymmetric vacua as well. This approach to obtaining the canonical models is particularly useful in the case of Polonyi and O’Raifeartaigh models, for which we needed

to assume a particular sign for the subleading correction to the Kähler potential. By obtaining the theories from the master theory, we can compute the leading corrections to the Kähler potential directly and show that they are of the type required to stabilize the susy-breaking vacua.

To see that the master theory has a metastable, non-supersymmetric vacuum, we can proceed as in the Fayet model. Node four is massive, and the corresponding nonperturbative superpotential can be computed in the geometry after a transition. The effective superpotential after performing the transition and integrating out the massive adjoints $\Phi_{2,3}$ is then

$$\mathcal{W}_{eff} = Q_{12}Q_{21}\Phi_1 + \frac{S_*}{4ma}(Q_{23}Q_{32} + \dots)$$

where we have omitted quartic and higher order terms in the Q 's which do not affect the status of the vacuum. With generic FI terms setting

$$|Q_{12}|^2 - |Q_{21}|^2 = r_2, \quad |Q_{23}|^2 - |Q_{32}|^2 = r_3 ,$$

this is easily seen to have an isolated vacuum which breaks supersymmetry.

We will now show that we can recover all of the the three models studied so far in particular regimes of large FI terms.

3.6.1. O'Raifeartaigh

We can recover the O'Raifeartaigh construction by turning on a large FI term for Q_{23} and Q_{32} – that is, for the U(1) under which these are the only charged quarks. This generates a D-term

$$D_{O'R} = |Q_{23}|^2 - |Q_{32}|^2 - r_3. \quad (3.6.3)$$

Taking $r_3 \gg 0$, this requires that Q_{23} acquire a large expectation value. Additionally, there is an F-term for Q_{32}

$$F_{Q_{32}} = Q_{23}(\Phi_3 - \Phi_2) \quad (3.6.4)$$

which, in light of the D-term constraint, will set Φ_2 equal to Φ_3 . The superpotential then becomes just the O'Raifeartaigh superpotential of the previous section (with certain indices renamed),

$$W_{O'R} = m(\Phi_4 - a)^2 + Q_{12}Q_{21}(\Phi_2 - \Phi_1) + Q_{24}Q_{42}(\Phi_4 - \Phi_2) . \quad (3.6.5)$$

By performing a geometric transition on the massive node, we recover the superpotential (3.5.5).

3.6.2. Fayet

Alternatively, we could have turned on a large FI term for Q_{12} and Q_{21} , generating a D-term

$$D_{Fayet} = |Q_{12}|^2 - |Q_{21}|^2 - r_2. \quad (3.6.6)$$

In conjunction with the F-term for Q_{21} , by the same process as in the O’Raifeartaigh model, Φ_1 is set equal to Φ_2 . This time, the remaining superpotential is given by

$$W_{Fayet} = m\Phi_2^2 - m\Phi_3^2 + m(\Phi_4 - a)^2 + Q_{23}Q_{32}(\Phi_3 - \Phi_2) + \dots \quad (3.6.7)$$

which is precisely the superpotential associated with the Fayet geometry (3.3.1). Performing a geometric transition on S_4^2 , we recover the Fayet model as discussed in section 4.

3.6.3. Polonyi

From the Fayet model above, before the geometric transition, we can turn on another D-term for the quarks Q_{23} and Q_{32} which, along with the F-term for Q_{32} sets $\Phi_2 = \Phi_3$. The superpotential becomes

$$W = -m\Phi_3^2 + m(\Phi_4 - a)^2 + Q_{34}Q_{43}(\Phi_4 - \Phi_3)$$

which reproduces the Polonyi model of section 5. Again, performing the geometric transition on S_4^2 results in the actual Polonyi model.

3.6.4. The Kähler potential

The O’Raifeartaigh and Polonyi models have flat directions at tree level. As we discussed for, e.g., the Polonyi model, the existence of a stable SUSY-breaking vacuum depends on the sign of the leading quartic correction to the Kähler potential. When we obtain the model as a suitable limit of our master model as above, we can compute this correction and verify explicitly that the vacuum is stable. Let us go through this in some detail. In fact, for simplicity, we will focus on obtaining a stable Polonyi model as a limit of a Fayet model [70].

After the geometric transition in the Fayet model, the effective theory is characterized by a superpotential

$$W = \frac{S_*}{ma} Q_{23}Q_{32} + \dots \quad (3.6.8)$$

and a D-term

$$D = |Q_{32}|^2 - |Q_{23}|^2 - r_3 . \quad (3.6.9)$$

Here r_3 is the FI term for the $U(1)$ under which only Q_{23} and Q_{32} carry a charge. We can expand this theory about the vev $Q_{23} = \sqrt{r_3}$. Renaming

$$X = Q_{32} ,$$

the effective theory then has

$$W = \frac{S_*}{ma} \sqrt{r_3} X . \quad (3.6.10)$$

To find the Kähler potential for X , we should integrate out the massive $U(1)$ gauge multiplet. What happens to the potential contribution from the D-term of (3.6.9)? As

explained in [82], in the theory with the $U(1)$ gauge field, gauge invariance relates D-term and F-term vevs at any critical point of the scalar potential. When one integrates out the $U(1)$ gauge field, there is a universal quartic correction to the Kähler potential which (using the relation) precisely reproduces the potential contribution from the D-term. For the theory in question, the quartic correction to the Kähler potential for X is just

$$\Delta K = -\frac{g_{U(1)}^2}{M_{U(1)}^2} (X^\dagger X)^2. \quad (3.6.11)$$

Here $M_{U(1)}$ is the mass of the $U(1)$ gauge boson, $M_{U(1)} \sim g_{U(1)}\sqrt{r_3}$. The result is a quartic correction to K

$$\Delta K = -\frac{1}{r_3} (X^\dagger X)^2. \quad (3.6.12)$$

So in the notation of section 5,

$$c = -\frac{1}{r_3}$$

and the sign $c < 0$ results in a stable vacuum, as expected. Plugging in the F-term $F_X \sim \frac{S_*}{ma}\sqrt{r_3}$, (3.6.12) gives X a mass

$$m_X \sim \frac{S_*}{ma},$$

in agreement with what it was in the full Fayet model. Note that while one would obtain other quartic couplings in K after integrating out the $U(1)$ gauge boson, they don't play any role. They involve powers of the heavy field Q_{34} , and since $F_{Q_{34}} \ll F_X$, cross-couplings of the form $Q_{34}^\dagger Q_{34} X^\dagger X$ in K do not appreciably correct the estimate obtained above for the mass of X .

3.7. Generalization

We now present a very general argument for the existence of supersymmetry-breaking effects in a class of stringy quiver gauge theories which includes those just discussed. Suppose we have such an A_r quiver theory in which the last node is isolated and undergoes a transition. Note that this is the case in the master geometry considered in the previous section.

In this case, the transition deforms the geometry to the following:

$$uv = \left(\prod_{i=1}^{r-1} (z - z_i(x)) \right) ((z - z_r(x))(z - z_{r+1}(x)) - s)$$

in which case the superpotential for the branes on the second-to-last node becomes

$$\widetilde{W}_{r-1}(\Phi_{r-1}) = \int dx (\widetilde{z}_r(x) - z_{r-1}(x))$$

where $\tilde{z}_r(x)$ is the solution to the equation

$$(z - z_r(x))(z - z_{r+1}(x)) = s \quad (3.7.1)$$

which asymptotically approaches $z_r(x)$. We can rewrite the superpotential as a correction to the pre-transition superpotential as

$$\widetilde{W}_{r-1}(\Phi_{r-1}) = \int dx (\tilde{z}_r(x) - z_r(x)) + \mathcal{W}_{r-1}(\Phi_{r-1})$$

and the F-term for Φ_{r-1} and the remaining adjoints are then given by

$$\begin{aligned} F_{\Phi_{r-1}} &= W'_{r-1}(\Phi_{r-1}) + (\tilde{z}_r(\Phi) - z_r(\Phi)) + Q_{r-1,r} Q_{r,r-1} \\ F_{\Phi_i} &= W'_i(\Phi_i) + Q_{i-1,i} Q_{i,i-1} - Q_{i,i+1} Q_{i+1,i} \end{aligned} \quad (3.7.2)$$

which we can combine to obtain the constraint

$$\sum_i^{r-1} W'_i(\Phi_i) = z_r(\Phi_{r-1}) - \tilde{z}_r(\Phi_{r-1}). \quad (3.7.3)$$

Note that the right hand side here cannot vanish for any value of Φ_{r-1} since $z_r(x)$ can never solve (3.7.1), the solution to which defines $\tilde{z}_r(x)$

If we now consider turning on generic FI terms for the $U(1)$ gauge groups, the D-term constraints will require that, say, the $Q_{i,i+1}$'s acquire vevs while the $Q_{i+1,i}$'s get fixed at zero. The F-terms for the $Q_{i+1,i}$'s will then in turn require

$$\Phi_i = \Phi_j$$

for all i, j . When the brane superpotentials for the first $r - 1$ nodes are of the form

$$W_i(\Phi_i) = \epsilon_i m \Phi_i^2, \quad i = 1, \dots, r - 1,$$

where $\epsilon_i = 0 \pm 1$, the left hand side of (3.7.3) vanishes, while the right hand side is strictly nonzero. It is exponentially small, as long as the last node was isolated,

$$W_r(\Phi_r) = m(\Phi_r - a)^2,$$

before the transition. This generically triggers low-scale susy breaking.

In terms of the classic models discussed in this chapter, one can immediately see that the susy breaking in the Fayet model and in the master geometry can be explained by the above analysis. In the case of the Polonyi and O'Raiartaigh models, it is even simpler, since the left hand side of (3.7.3) vanishes *identically* for those models. One could conduct a similar analysis for configurations with more complicated superpotentials and nongeneric F-terms on a case-by-case basis. What we see is that often the susy-breaking effects caused by the geometric transition can be understood at an exact level.

3.8. SUSY Breaking by the Rank Condition

In this section, we present models which break supersymmetry due to the “rank condition.” This class of models is very similar to those arising in studies of metastable vacua in SUSY QCD [11]. However, we work directly with the analogue of the magnetic dual variables, and the small scale of SUSY breaking is guaranteed by retrofitting [71].

Consider the A_3 fibration with

$$z_1(x) = mx, \quad z_2(x) = -mx, \quad z_3(x) = -mx, \quad z_4(x) = -m(x - 2a) . \quad (3.8.1)$$

Then the defining equation is

$$uv = (z - mx)(z + mx)(z + mx)(z + m(x - 2a)) . \quad (3.8.2)$$

We choose to wrap $N_f - N_c$ D5 branes on S_1^2 , N_f D5 branes on S_2^2 , and a single D5 on S_3^2 . The tree level superpotential is

$$\mathcal{W} = \sum_{i=1}^3 W_i(\Phi_i) + \sum_{i=1}^2 (Q_{i,i+1} \Phi_{i+1} Q_{i+1,i} - Q_{i+1,i} \Phi_i Q_{i,i+1}), \quad (3.8.3)$$

where

$$W_1(\Phi_1) = m \text{Tr}(\Phi_1)^2, \quad W_2(\Phi_2) = 0, \quad W_3(\Phi_3) = -m(\Phi_3 - a)^2$$

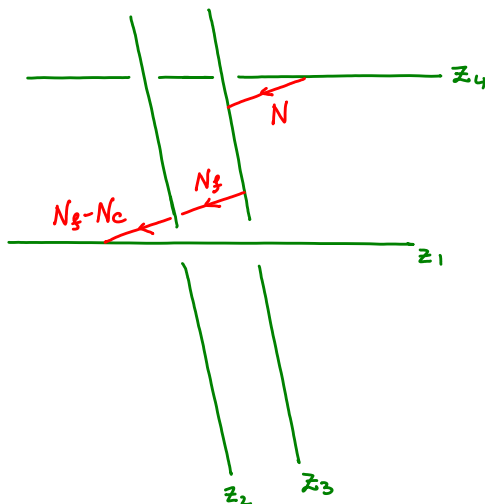


Fig. 3.3. The (magnetic) A_3 geometry that retrofits the ISS model.

Now, we replace the third ($U(1)$) node with an S^3 with flux, and integrate out Φ_1 trivially (we can take the mass to be very large). The result is a superpotential

$$\mathcal{W} = S(\log(S/\Delta^3) - 1) + \frac{t}{g_s} S - \frac{1}{2} S \text{Tr} \log(a - \Phi_2)/\Delta - Q_{12} \Phi_2 Q_{21} + \dots \quad (3.8.4)$$

where the omitted terms are suppressed by additional powers of S . Integrating out S in a Taylor expansion about $\Phi_2 = 0$ produces a theory with superpotential

$$\mathcal{W} = S_* \text{Tr} \Phi_2 / a - \text{Tr} Q_{12} \Phi_2 Q_{21} + \dots \quad (3.8.5)$$

where

$$S_* = \Delta^3 \exp(-\tilde{t}/g_s) , \quad (3.8.6)$$

and $\tilde{t} = t - N_f \frac{1}{2} g_s \log(a/\Delta)$. Computing F_{Φ_2} , we see that the contribution from the first term in (3.8.5) has rank N_f , while the contribution from the second term has maximal rank $N_f - N_c < N_f$. The two cannot cancel, and so SUSY is broken. However, due to the small coefficient of the $\text{Tr} \Phi_2$ term, the breaking occurs at an exponentially small scale.

This model resembles the theories analyzed in [11] (for $N_c + 1 \leq N_f < \frac{3}{2} N_c$) and in section 4 of [50]. One difference is that the origin of the small parameter is dynamically explained. The discussion of corrections due to gauging of the $U(N_f)$ factor (which is a global group in [11]) is identical to that in [50] up to a change of notation, and we will not repeat it here. For large a , the higher order corrections to (3.8.5) (which are suppressed by powers of Φ_2/a) should not destabilize the vacuum at the origin, described in [11,50].

We could also replace the $U(1)$ at node 3 with a $U(N)$ gauge group, still in the same geometry. Then, in (3.8.4), the coefficient of the $S \log S$ term is changed to N . The only effect, after a geometric transition at node three, is the replacement $e^{-t/g_s} \rightarrow e^{-t/g_s N}$ in (3.8.6). This model, where the node which undergoes the geometric transition has non-Abelian gauge dynamics, is a literal example of the retrofitting constructions of [71]. The field Φ_2 appears in the gauge coupling function of the $U(N)$ gauge group at node three because it controls the masses of the quarks Q_{23} and Q_{32} which are charged under $U(N)$. At energies below the quark mass, the $U(N)$ is a pure $\mathcal{N} = 1$ gauge theory and produces a gaugino condensation contribution Λ_N^3 in the superpotential. The standard result for matching the dynamical scale of the low-energy, pure $U(N)$ theory to the scale Λ_{N,N_f} of the higher energy theory with N_f quark flavors with mass matrix \tilde{m} is³⁷

$$\Lambda_N^{3N} = \Lambda_{N,N_f}^{3N-N_f} \det \tilde{m} . \quad (3.8.7)$$

Identifying S with the gaugino condensate [17]

$$S \sim \text{tr}(W_\alpha^2) = \Lambda_N^3 ,$$

and identifying the mass matrix $\tilde{m} = a - \Phi_2$, we predict

$$S^N = \Lambda_{N,N_f}^{3N-N_f} \det(a - \Phi_2) . \quad (3.8.8)$$

This is precisely what carefully integrating S out of (3.8.4) produces, with $\Lambda_{N,N_f}^{3N-N_f} = \Delta^{3N-N_f} e^{-t/g_s}$. So, in our model with $N > 1$, the small $\text{Tr}(\Phi_2)$ term in (3.8.5) can really be thought of as arising from the presence of Φ_2 in the gauge coupling function for the $U(N)$ factor.

³⁷ Here, we are assuming the adjoints are very massive, $m \rightarrow \infty$, and are just matching the QCD theories with quark flavors.

3.9. Orientifold Models

In the presence of orientifold 5-planes, we expect D1 brane instantons wrapping two-cycles that map to themselves to contribute to the superpotential. The D1 brane instanton contributions should again be computable using a geometric transition that shrinks the S^2 and replaces it with an S^3 . Geometric transitions with orientifolds have been studied, e.g., in [83,84].

After the transition, we generally get two different contributions to the superpotential. First, charge conservation for the D5/O5 brane that disappears after the transition requires a flux through the S^3 equal to the amount of brane charge,

$$\mathcal{W}_{flux} = \frac{t}{g_s} S + N_{D5/O5} \partial_S \mathcal{F}_0.$$

Second, there can be additional O5 planes that survive as the fixed points of the holomorphic involution after the transition. The O5 planes, just like D5 branes, generate a superpotential [85]

$$\mathcal{W}_{O5} = \int_{\Sigma} \Omega,$$

where the integral is over a three-chain with a boundary on the orientifold plane. The contributions to the superpotential due to O5 planes and RR flux of the orientifold planes are both computed by topological string RP^2 diagrams. The contribution of physical brane charge comes from sphere diagrams.

In this way, geometric transitions can be used to sum up the instanton-generated superpotentials in orientifold models. In analogy to our discussion in the previous sections, this can be used to understand models of dynamical supersymmetry breaking. We will discuss the Fayet model in detail; other models can be seen to follow in naturally.

3.9.1. The Fayet model

Consider orientifolding the theory from section 3 by combining worldsheet orientation reversal with an involution I of the Calabi-Yau manifold. For this to preserve the same supersymmetry as the D5 branes, the holomorphic involution I of the Calabi-Yau has to preserve the holomorphic three-form $\Omega = \frac{du}{u} dz dx = -\frac{dv}{v} dz dx$.

An example of such an involution is one that takes

$$x \rightarrow -x$$

and

$$u \rightarrow v, \quad v \rightarrow u$$

A simple, Fayet-type model built on this orientifold is an A_5 geometry that is roughly a doubling of that in section 4,

$$uv = (z - mx)^2 (z + mx)^2 (z - m(x - 2a))(z + m(x - 2a)).$$

We will blow this up according to the ordering

$$\begin{aligned} z_1(x) &= mx, & z_2(x) &= -m(x - 2a), & z_3(x) &= mx, \\ z_4(x) &= -mx, & z_5(x) &= +m(x + 2a), & z_6(x) &= -mx. \end{aligned}$$

It can be shown that the orientifold projection ends up mapping

$$S_i^2 \rightarrow S_{6-i}^2,$$

fixing S_3^2 . Consider wrapping M branes on S_i^2 for $i = 1, 2$, and their mirror images, and $2N$ branes on S_3^2 . With a particular choice of orientifold projection, the gauge group on the branes is going to be

$$U(M) \times U(M) \times Sp(N).$$

Since the orientifold flips the sign of x , on the fixed node, S_3^2 , it converts Φ_3 to an adjoint of $Sp(N)$. (Having chosen that the orientifold sends x to minus itself, the action on the rest of the coordinates is fixed by asking that it preserve the same SUSY as the D5 branes *and* that it remain a symmetry after blowing up.) In the model at hand, the tree-level superpotential is

$$\mathcal{W} = \sum_{i=1}^3 W_i(\Phi_i) + \text{Tr}(Q_{12}\Phi_2Q_{21} - Q_{21}\Phi_1Q_{12}) + \text{Tr}(Q_{23}\Phi_3Q_{32} - Q_{32}\Phi_2Q_{23}).$$

where

$$W_1(\Phi_1) = m\text{Tr}(\Phi_1 - a)^2, \quad W_2(\Phi_2) = -m\text{Tr}(\Phi_2 - a)^2, \quad W_3(\Phi_3) = m\text{Tr}\Phi_3^2.$$

Note that, even though the \mathbb{P}^1 is fixed by the orientifold action, it is not fixed pointwise. This means there is no $O5^+$ plane charge on it. Instead, there are two *non-compact* orientifold 5-planes. This model is T-dual [86] to the O6 plane models of [75].

After the geometric transition that shrinks node three and replaces it with an S^3 ,

$$S_3^2 \rightarrow S^3,$$

the geometry becomes

$$uv = (z - mx)(z + mx)(z - m(x - 2a))^2(z + m(x - 2a))^2((z - mx)(z + mx) - s),$$

where

$$\int_{S^3} \Omega = S$$

with $S = s/m$. Since the orientation reversal acted freely on the S_3^2 , there are only N units of D5 flux through the S^3 ,

$$\int_{S^3} H^{RR} = N,$$

which gives rise to a superpotential

$$\mathcal{W}_{flux} = \frac{t}{2g_s} S + NS(\log \frac{S}{\Delta^3} - 1).$$

The overall factor of 1/2 comes from the fact that both the charge on the S^2 and its size have been cut in half by the orientifold action. Above, $t = \int_{S^2} k + ig_s B^{RR}$ is the combination of Kähler moduli that survives the orientifold projection. In addition, the two non-compact O5⁺ planes get pushed through the transition. Because the space still needs two blowups to be smooth, to give a precise description of the O5 planes would require using a geometry covered with 4 patches. At the end of the day, effectively, the O5 planes correspond to non-compact curves over the two points on the Riemann surface

$$(z - \tilde{z}_3(x))(z - \tilde{z}_4(x)) = ((z - mx)(z + mx) - s) = 0$$

located at $x = 0$ and the corresponding values of z , $z_{\pm}(0)$. They generate a superpotential

$$\mathcal{W}_{O5^+} = \int^{z_-(0)} (\tilde{z}_3 - \tilde{z}_4) dx + \int^{z_+(0)} (\tilde{z}_3 - \tilde{z}_4) dx.$$

One can show that the contribution of the O5 planes is

$$\mathcal{W}_{O5^+} = +S(\log \frac{S}{\Delta^3} - 1).$$

The fact that the RP^2 contribution is proportional to that of the sphere is not an accident. It has been shown generally that the contribution of O5 planes in these classes of models is $\pm \partial_S \mathcal{F}_{S^2}$ [87,84]. This means that the O5 planes and the fluxes add up to $N + 1$ units of an “effective” flux on the S^3 .

After the transition, the branes on node three have disappeared, and with them, the fields Φ_3 and Q_{23}, Q_{32} . In addition, the deformation of the geometry induces a deformation of the superpotential for node two,

$$\tilde{W}_2(x) = \int (z_2(x) - \tilde{z}_3(x)) dx,$$

where one picks for \tilde{z}_3 the root that asymptotes to $+mx$. This deformed superpotential is then

$$\tilde{W}_2(x) = \int (-m(x - 2a) - \sqrt{(mx)^2 + s}) dx,$$

which, when expanded near the vacuum at $x = a$, gives

$$\tilde{W}_2(\Phi_2) = -\text{Tr } m(\Phi_2 - a)^2 - \frac{1}{2} S \text{Tr } \log(\Phi_2/\Delta) + \dots$$

The full effective superpotential that sums up the instantons is thus

$$\mathcal{W}_{eff} = W_1(\Phi_1) + \tilde{W}_2(\Phi_2, S) + \text{Tr}(Q_{12}\Phi_2Q_{21} - Q_{21}\Phi_1Q_{12}) + \mathcal{W}_{flux} + \mathcal{W}_{O5}$$

Up to an overall shift of both $\Phi_{1,2}$ by a , this is the same model as in section 3.

We expect a transition here even when $N = 0$, and there are no D5 branes on the S^2 . The transition for $Sp(0)$ is analogous to the transition that occurs for the $U(1)$ gauge theory of a single D-brane on the S^2 . In both cases, the smooth joining of the S^2 and the S^3 phases is due to instantons that correct the geometry. In the orientifold case at hand, it is important to note that, while there is no flux through the S^3 , the D3 brane wrapping it is absent: the orientifold projection projects out [88] the $\mathcal{N} = 1$ $U(1)$ vector multiplet associated with the S^3 , and with it the D3 brane charged under it.

Picking the other orientifold projection, the $Sp(N)$ gauge group gets replaced by $SO(2N)$, with Φ_3 becoming the corresponding adjoint. In this case, much of the story remains the same, except that the RP^2 contribution becomes

$$\mathcal{W}_{O5^-} = -S(\log \frac{S}{\Delta^3} - 1).$$

This means that the $O5^-$ planes and the fluxes add up to $N - 1$ units of an “effective” flux on the S^3 . This is negative or zero for $N \leq 1$. Naively, the negative effective flux breaks supersymmetry after the transition. This is clearly impossible. It has been argued in [84] that the correct interpretation of this is that in fact $SO(2)$, $SO(1)$ and $SO(0)$ cases do not undergo geometric transitions. This has to correspond to the statement that, in these cases, there are no D1 brane instantons on node three, and so the classical picture is *exact*. This translates to the statement that in these cases, S should not be extremized, but rather set to zero identically in the effective superpotential,

$$\mathcal{W}_{eff} = \mathcal{W}_{eff}|_{S=0}.$$

Note that with the SO projection on the space-filling branes, a D-instanton wrapping the same node enjoys an Sp projection. As discussed in [89,75], in this situation, direct zero-mode counting also suggests that the instanton should *not* correct the superpotential. There are more than two fermion zero modes coming from the Ramond sector of strings stretching from the instanton to itself. This is in accord with the results of [84]. In contrast, when one has an Sp projection on the space-filling branes, the instanton receives an SO projection, and the instanton with $SO(1)$ worldvolume gauge group has the correct zero-mode count to contribute. The presence of instanton effects for this projection (and their absence without it), was also confirmed by direct studies of the renormalization group cascade ending in the appropriate geometry in [75].

Appendix 3.A. Brane Superpotentials

We can compute the superpotential $W(\Phi)$ as function of the wrapped two-cycles Σ by using the superpotential [81,30]

$$W = \int_C \Omega$$

where C is a three-chain with one boundary being Σ and the other being a reference two-cycle Σ_0 in the same homology class. It is easy to show [81] that the critical points of the

superpotential are holomorphic curves. We will evaluate it for the geometries at hand. We can write the holomorphic three-form of the non-compact Calabi-Yau in the usual way,

$$\Omega = \frac{dv \wedge dz \wedge dx}{\frac{dF}{du}} = \frac{dv}{v} \wedge dz \wedge dx. \quad (3.A.1)$$

Now for fixed values of x and z , the equation for the CY threefold becomes $uv = \text{const}$, which is the equation for a cylinder. By shifting the definition of u or v by a phase, we can insist that the constant is purely real, and then by writing $u = x + iy$, $v = x - iy$, the equation can be reformulated as two real equations in terms of the real (x_R, y_R) and imaginary (x_I, y_I) parts of x and y .

$$x_R^2 + y_R^2 = C + x_I^2 + y_I^2, \quad x_R x_I = y_R y_I. \quad (3.A.2)$$

The first of these can be solved for any given values of x_I and y_I to give an S^1 . The second equation restricts the possible values which we choose for x_I and y_I to a one-dimensional curve in the (x_I, y_I) plane, and so we have the topology of $S^1 \times \mathbf{R}$, where the size of the S^1 degenerates at the points where $z = z_i(x)$ for any i . By simultaneously shifting the phases of u and v according to

$$\begin{aligned} u &\rightarrow e^{i\theta} u \\ v &\rightarrow e^{-i\theta} v \end{aligned}$$

the equation for the cylinder remains unchanged, and we simply rotate about the S^1 factor. We can thus integrate Ω around the circle and obtain

$$\int_{S^1} \Omega = dz \wedge dx$$

up to an overall constant. Now the \mathbb{P}^1 's on which we are wrapping the D5 branes are the product of the S^1 just discussed and an interval in the z direction between values where the S^1 fiber degenerates. Thus, for a given \mathbb{P}^1 class in which the vanishing S^1 occurs for $z_i(x)$ and $z_j(x)$, we can integrate $dz \wedge dx$ over the interval in the z -plane and obtain

$$\int_{S^1 \times I_{ij}} \Omega = (z_i(x) - z_j(x)) dx.$$

The superpotential for the D-branes then becomes a superpotential for the location of the branes on the t -plane. Defining an arbitrary reference point t_* , we then have

$$W(x) = \int_{t_*}^t (z_i(x) - z_j(x)) dx. \quad (3.A.3)$$

Of course, the contribution to the superpotential coming from the limit of integration at t_* is just an arbitrary constant and is not physically relevant. Thus we write (3.A.3) instead as the indefinite integral

$$W(x) = \int (z_i(x) - z_j(x)) dx. \quad (3.A.4)$$

Appendix 3.B. Multi-Instanton Contributions

In this appendix we demonstrate the computation of multi-instanton corrections to the superpotential using the Polonyi model of section 5 as an example. All the information about these corrections is contained in the deformed superpotential for Φ ,

$$\widetilde{W}(x) = \int \left(m(x-a) - \sqrt{m^2(x-a)^2 + mS} \right) dx \quad (3.B.1)$$

along with the flux superpotential³⁸

$$\mathcal{W}_{flux} = \frac{t}{g_s} S + S \left(\log \frac{S}{\Delta^3} - 1 \right), \quad (3.B.2)$$

where the scale Δ is determined by the one-loop contributions to the matrix model free energy. The models considered in this chapter are particularly convenient since the purely quadratic superpotential for the massive adjoint at the transition node guarantees that the flux superpotential will be exact at one-loop order in the associated matrix model [19].

Extremizing the flux superpotential and expanding in powers of the instanton action

$$S_{inst} \sim \exp(-t/Ng_s),$$

we can determine multi-instanton contributions to a given superpotential term. Summing up the series contributing to a given Φ^k term in will correspond to computing corrections to a fixed, explicit disc diagram, and so we might expect these series to exhibit some integrality properties.

We first expand the deformed superpotential $\mathcal{W}_1(\Phi)$ as a power series in the glueball superfield S ,

$$\widetilde{W}(\Phi) = \int \left(m(x-a) - m(x-a) \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n(2n-2)!}{2^{2n-1} (n!)^2} y^n \right) \right) dx \quad (3.B.3)$$

where the expansion parameter y can also be expanded as a power series in x ,

$$y = \frac{S}{m(x-a)^2} = \frac{S}{ma^2} \left(1 + \sum_{n=1}^{\infty} (n+1) (-1)^n \left(\frac{x}{a} \right)^n \right). \quad (3.B.4)$$

We can integrate (3.B.3) term by term to obtain an expansion of the effective superpotential in powers of Φ . However, it will be useful to represent this schematically

$$W_1(\Phi) = c_1 \text{Tr} \Phi + c_2 \text{Tr} \Phi^2 + \dots \quad c_i = \sum_{n=1}^{\infty} c_i^{(n)} S^n$$

³⁸ In the case of the Polonyi model these two terms constitute the entire superpotential. In the more general case, however, there will be more fields with superpotential terms, but it will remain the case that only these two contributions play a role in determining instanton corrections.

where the coefficients c_i are themselves written as power series in S . Extremizing the superpotential with respect to S gives an equation for the values of S

$$\log \frac{S}{\Delta^3} = -\frac{t}{g_s} - \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} n c_i^{(n)} S^{n-1} \text{Tr} \Phi^i \quad (3.B.5)$$

which can be solved perturbatively in powers of S_{inst} . Reinserting the resulting values into the original superpotential then allows us to read off the instanton-corrected superpotential of the low energy theory up to any given number of instantons. Below we display the linear and quadratic terms at the four-instanton level:

$$\mathcal{W}_{eff} = \mu \text{Tr} \Phi + m \text{Tr} \Phi^2$$

where

$$\begin{aligned} \mu &= -\frac{1}{2} \frac{\Delta^3}{a} e^{-\frac{t}{g_s}} + \frac{1}{8} \frac{\Delta^6}{ma^3} e^{-\frac{2t}{g_s}} - \frac{1}{16} \frac{\Delta^9}{m^2 a^5} e^{-\frac{3t}{g_s}} + \frac{5}{128} \frac{\Delta^{12}}{m^3 a^7} e^{-\frac{4t}{g_s}} + \dots \\ m &= -\frac{3}{8} \frac{\Delta^3}{a^2} e^{-\frac{t}{g_s}} + \frac{5}{16} \frac{\Delta^6}{ma^4} e^{-\frac{2t}{g_s}} - \frac{9}{32} \frac{\Delta^9}{m^2 a^6} e^{-\frac{3t}{g_s}} + \frac{67}{256} \frac{\Delta^{12}}{m^3 a^8} e^{-\frac{4t}{g_s}} + \dots \end{aligned} \quad (3.B.6)$$

It may be interesting to see if there is some way to relate these to the exact formulae for multicovers derived in the resolution of the singularity in hypermultiplet moduli space when a two-cycle shrinks in IIB string theory, given (up to mirror symmetry) in [90].

Chapter 4

Geometric Transitions and D-Term Supersymmetry Breaking

In the interest of finding controllable, realistic string vacua, it is important to find simple and tractable mechanisms of breaking supersymmetry in string theory. A powerful method which has been put forward in [4] consists of geometrically engineering metastable vacua with D-branes wrapping cycles in a Calabi-Yau manifold, and using geometric transitions and topological string techniques to analyze them. In [4], metastable vacua were engineered by wrapping D5 branes and anti-D5 branes on rigid two-cycles in a Calabi-Yau. The non-supersymmetric vacuum obtained in this fashion was argued to have a simple closed-string dual description, in which branes and antibranes are replaced by fluxes. In [6], geometric transitions were used to study the physics of D-brane theories that break supersymmetry dynamically. Namely, the authors showed that the instanton-generated superpotential that triggers supersymmetry breaking can be computed by classical means in a dual geometry, where some of the branes are replaced by fluxes (for an alternative approach see [75]).

In this chapter, we propose another way of using geometric transitions to study supersymmetry breaking. As in [4], we consider D5 branes wrapping rigid cycles in a non-compact Calabi-Yau X . If $b_2(X) > 1$, supersymmetry can be broken by choosing the complexified Kähler moduli so as to misalign the central charges of the branes,

$$Z_i = \int_{S_i^2} J + iB_{NS}.$$

Since the two-cycles wrapped by the D5 branes are rigid, any deformation of the branes costs energy, and the system is guaranteed to be metastable.³⁹ In the extreme case of

³⁹ Supersymmetry breaking by the misalignment of central charges of D5 branes wrapped on rigid curves was also studied in [91] in the context of compact Calabi-Yau manifolds.

anti-aligned central charges, we recover the brane/antibrane configurations of [4,5]. For slightly misaligned central charges, the system has a gauge theory description in terms of an $\mathcal{N} = 2$ quiver theory deformed to $\mathcal{N} = 1$ by superpotential terms [18,31], and with Fayet-Iliopoulos D-terms turned on [92,93,94].⁴⁰ The D-terms trigger spontaneous supersymmetry breaking in the gauge theory.

We argue that the dynamics of this system is effectively captured by a dual Calabi-Yau with all branes replaced by fluxes. Turning on generic Kähler moduli on the open-string side has a simple interpretation in the dual low-energy effective theory as turning on a more generic set of FI parameters than hitherto considered in this context, but which are allowed by the $\mathcal{N} = 2$ supersymmetry of the background. On shell, this breaks some or all of the $\mathcal{N} = 2$ supersymmetry. Geometrically, this corresponds to not only turning on H_{NS} and H_{RR} fluxes on the Calabi-Yau, but also allowing for $dJ \neq 0$ [96,97].⁴¹ Moreover, we show that the Calabi-Yau geometries with these fluxes turned on have non-supersymmetric, metastable vacua, as expected by construction in the open-string theory.

The chapter is organized as follows. In section two, we review the physics of D5 branes on a single conifold and the dual geometry after the transition, paying close attention to the effect of Fayet-Iliopoulos terms. In section three, we consider the case of an A_2 geometry where misaligned central charges lead to supersymmetry breaking. We provide evidence that the dual geometry correctly captures the physics of the non-supersymmetric brane system. We show that the results are consistent with expectations from the gauge theory, to the extent that these are available. We also comment on the relation of this work to [6], and point out some possible future directions. In an appendix, we lay out the general case for larger quiver theories. We show that in the limit of large separations between nodes, the theory has metastable, non-supersymmetric vacua in all cases where they are expected.

4.1. The Conifold

In this section, we consider N D5 branes on the resolved conifold. We will first review the open-string theory on the branes, and then discuss the dual closed-string description. Our discussion will be more general than the canonical treatment in that we will consider the case where the D5 branes possess an arbitrary central charge.

4.1.1. The D-brane theory

To begin with, let us recall the well-known physics of N D5 branes wrapping the S^2 tip of the resolved conifold. This geometry can be represented as a hypersurface in $\mathbb{C}^4[u, v, z, t]$,

$$uv = z(z - mt).$$

⁴⁰ Some geometric aspects of supersymmetry breaking by F-terms in this context were recently discussed in [95].

⁴¹ For another example of supersymmetry breaking by turning on H_{NS} , H_{RR} and dJ fluxes, see [98].

The geometry has a singularity at the origin of \mathbb{C}^4 which can be repaired by blowing up a rigid \mathbb{P}^1 . This gives the \mathbb{P}^1 a complexified Kähler class

$$Z = \int_{S^2} (J + iB_{NS}) = j + i b_{NS}. \quad (4.1.1)$$

The theory on the D5 branes at vanishing j reduces in the field theory limit to a $d = 4$, $\mathcal{N} = 1$, $U(N)$ gauge theory with an adjoint-valued chiral superfield of mass m . The bare gauge coupling is given by

$$\frac{4\pi}{g_{YM}^2} = \frac{b_{NS}}{g_s} \quad (4.1.2)$$

for positive b_{NS} . In string theory, the tension of the branes generates an energy density related to the four dimensional gauge coupling by

$$V_* = \frac{2N}{g_{YM}^2} = N \frac{b_{NS}}{2\pi g_s}. \quad (4.1.3)$$

Turning on a small, nonzero j can be viewed as a deformation of this theory by a Fayet-Iliopoulos parameter for the $U(1)$ center of the gauge group [92,93]. This deforms the Lagrangian by

$$\Delta\mathcal{L} = \sqrt{2}\xi \text{Tr}D, \quad (4.1.4)$$

where D is the auxiliary field in the $\mathcal{N} = 1$ vector multiplet and

$$\xi = \frac{j}{4\pi g_s}, \quad (4.1.5)$$

where the factor of g_s comes from the disk amplitude. This deformation (4.1.4) breaks the $\mathcal{N} = 1$ supersymmetry which was linearly realized at $j = 0$. In particular, turning on j increases the energy of the vacuum. Integrating out D from the theory by completing the square in the auxiliary field Lagrangian,

$$\mathcal{L}_D = \frac{1}{2g_{YM}^2} \text{Tr}D^2 + \sqrt{2}\xi \text{Tr}D,$$

raises the vacuum energy to

$$V_* = N \frac{b_{NS}}{2\pi g_s} \left(1 + \frac{1}{2} \frac{j^2}{b_{NS}^2} \right). \quad (4.1.6)$$

Supersymmetry is *not* broken, however. At nonzero j , a different $\mathcal{N} = 1$ supersymmetry is preserved⁴² – one that was realized nonlinearly at vanishing j [99,100]. Which

⁴² This is true even in the field theory limit, despite the presence of the constant FI term. Namely, a second, nonlinearly realized supersymmetry is present in the gauge theory as long as there is only a constant energy density [99]. We thank A. Strominger for explaining this to us.

subgroup of the background $\mathcal{N} = 2$ supersymmetry is preserved by the branes is determined by Z in (4.1.1), the BPS central charge in the extended supersymmetry algebra.⁴³ For any Z , the open-string theory on the branes has an alternative description which is manifestly $\mathcal{N} = 1$ supersymmetric, with *vanishing* FI term and bare gauge coupling related to the magnitude of the central charge [31],

$$\frac{1}{\tilde{g}_{YM}^2} = \frac{\sqrt{b_{NS}^2 + j^2}}{4\pi g_s}. \quad (4.1.7)$$

Geometrically, this is just the quantum volume of the resolving \mathbb{P}^1 . As such, the central charge also determines the exact tension of a single D5 brane at nonzero j ,

$$V_* = N \frac{\sqrt{b_{NS}^2 + j^2}}{2\pi g_s}. \quad (4.1.8)$$

For small Kähler parameter,

$$j \ll b_{NS},$$

this agrees with the vacuum energy in the field theory limit (4.1.6).

For any j , the theory is massive; it is expected to exhibit confinement and gaugino condensation at low energies, leaving an effective $U(1)$ gauge theory in terms of the center of the original $U(N)$ gauge group. We will show next that the strongly coupled theory has a simple description for *any* value of Z in terms of a large N dual geometry with fluxes.

4.1.2. The geometric transition at general Z

We will now discuss the large N dual geometry for general values of the central charge Z . Special cases (either vanishing j or vanishing b_{NS}) have been considered in the literature, but the present, expanded discussion is, to our knowledge, new.⁴⁴ We will see that the dual geometry exactly reproduces the expected D5 brane physics. From the perspective of the low-energy effective action, the consideration of general central charge corresponds to turning on a more general set of $\mathcal{N} = 2$ FI terms than previously considered in this context. Geometrically, this will lead us to consider generalized Calabi-Yau manifolds, for which dJ is nonvanishing in addition to having H_{NS} and H_{RR} fluxes turned on. This will provide a local description of the physics for each set of branes in the more general supersymmetry-breaking cases of section three and four.

To begin, let us recall the large N dual description of the D5 brane theory at vanishing j . This is given in terms of closed-string theory on the deformed conifold geometry,

$$uv = z(z - mt) + s. \quad (4.1.9)$$

⁴³ Strictly speaking, the central charge of N branes is NZ . In this chapter, we will always take the number of branes N to be positive, so that we interpolate between branes and antibranes by varying Z .

⁴⁴ See related discussion in [31].

This is related to the open-string geometry by a geometric transition which shrinks the \mathbb{P}^1 and replaces it with an S^3 of nonzero size,

$$S = \int_A \Omega,$$

where A is the three-cycle corresponding to the new S^3 , and the period of the holomorphic three-form over A is related to the parameters of the geometry by $S = s/m$. The D5 branes have disappeared and been replaced by N units of Ramond-Ramond flux through the S^3 ,

$$\int_A H^{RR} = N. \quad (4.1.10)$$

There are Ramond-Ramond and Neveu-Schwarz fluxes through the dual, non-compact B -cycle as well,

$$\alpha = \int_B^{\Lambda_0} (H_{RR} + iH_{NS}/g_s) = b_{RR} + ib_{NS}/g_s, \quad (4.1.11)$$

which corresponds to the complexified gauge coupling of the open-string theory,

$$\alpha = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}.$$

The B -cycle is cut off at the scale, Λ_0 , at which α is measured.⁴⁵ The dependence of α on the IR cutoff in the geometry corresponds to its renormalization group running in the open-string theory.

If it were not for the fluxes, the theory would have $\mathcal{N} = 2$ supersymmetry, with S being the lowest component of an $\mathcal{N} = 2$ $U(1)$ vector multiplet. That theory is completely described by specifying the prepotential, $\mathcal{F}_0(S)$, which can be determined by a classical geometry computation,

$$\int_B \Omega = \frac{\partial}{\partial S} \mathcal{F}_0.$$

The presence of nonzero fluxes introduces electric and magnetic Fayet-Iliopoulos terms in the low-energy theory for the $U(1)$ vector multiplet and its magnetic dual [22,17,25,26]. The effect of the fluxes (4.1.10),(4.1.11) can also be described in the language of $\mathcal{N} = 1$ superspace as turning on a superpotential for the $\mathcal{N} = 1$ chiral superfield with S as its scalar component,

$$\mathcal{W}(S) = \int_X \Omega \wedge (H^{RR} + iH^{NS}/g_s).$$

For the background in question, this takes the form

$$\mathcal{W}(S) = \alpha S - N \frac{\partial}{\partial S} \mathcal{F}_0. \quad (4.1.12)$$

⁴⁵ For simplicity, the IIB axion is set to zero in this chapter.

In terms of the parameters of the D-brane theory, S is identified with the vev of the gaugino condensate. One way to see this is by comparing the superpotentials on the two sides of the duality. The αS superpotential on the closed string side corresponds to the classical superpotential term $\frac{\alpha}{4} \text{Tr} W_\alpha W^\alpha$ on the gauge theory side.

What does the FI term deformation of the D-brane theory correspond to in the closed-string theory? To begin with, let us address this question from the perspective of the low-energy effective action. We know that the $U(1)$ gauge field after the transition coincides [18,101] with the $U(1)$ gauge field that is left over after the $SU(N)$ factor of the gauge group confines. This suggests that we should simply *identify* Fayet-Iliopoulos D-terms on the two sides. More precisely, the Lagrangian of the theory after the transition can be written in terms of $\mathcal{N} = 1$ superfields,

$$\mathcal{S} = S + \sqrt{2}\theta\psi + \theta\theta F, \quad (4.1.13)$$

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha D + \frac{i}{2}(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}, \quad (4.1.14)$$

as an $\mathcal{N} = 2$ action deformed to $\mathcal{N} = 1$ by the superpotential (4.1.12),

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left(\int d^2\theta d^2\bar{\theta} \bar{\mathcal{S}} \frac{\partial \mathcal{F}_0}{\partial \mathcal{S}} + \int d^2\theta \frac{1}{2} \frac{\partial^2 \mathcal{F}_0}{\partial \mathcal{S}^2} W^\alpha W_\alpha + 2 \int d^2\theta \mathcal{W}(\mathcal{S}) \right). \quad (4.1.15)$$

The Fayet-Iliopoulos deformation (4.1.4) should produce an additional term in this Lagrangian,

$$\Delta \mathcal{L} = \frac{j}{2\sqrt{2}\pi g_s} D, \quad (4.1.16)$$

corresponding to an FI term as in (4.1.5). Note that on the D-brane side, the center of mass $U(1)$ corresponds to $1/N$ times the identity matrix in $U(N)$, so that the normalization of (4.1.16) precisely matches (4.1.4).

We now show that the deformation (4.1.16) leads to exactly the physics that we expect on the basis of large N duality. After turning on the FI term, the effective potential of the theory becomes

$$V = \frac{1}{4\pi} G^{S\bar{S}} (|\partial_S \mathcal{W}|^2 + |j/g_s|^2) + \text{const.} \quad (4.1.17)$$

where

$$G_{S\bar{S}} = \frac{1}{2i}(\tau - \bar{\tau}), \quad \tau = \frac{\partial^2}{\partial \mathcal{S}^2} \mathcal{F}_0.$$

We have also shifted the potential by an (arbitrary) constant, which we choose to be the tension of the branes at vanishing j ,

$$\text{const.} = N \frac{b_{NS}}{2\pi g_s},$$

for convenience. We can then rewrite (4.1.17) as

$$\begin{aligned} V &= \frac{i}{2\pi(\tau - \bar{\tau})} \left(|\alpha - N\tau|^2 + |j/g_s|^2 \right) + \text{const.} \\ &= \frac{i}{2\pi(\tau - \bar{\tau})} |\tilde{\alpha} - N\tau|^2 + \text{const.} \end{aligned} \quad (4.1.18)$$

where

$$\tilde{\alpha} = b_{RR} + \frac{i}{g_s} \sqrt{b_{NS}^2 + j^2}$$

and the constant has shifted. As expected from the D-brane picture, the effective potential of the theory with the FI term turned on and with gauge coupling (4.1.2) is the same as that of the theory without the FI term and with gauge coupling (4.1.7).

In this simple example, the prepotential is known to be given exactly by

$$2\pi i \mathcal{F}_0(S) = \frac{1}{2} S^2 \left(\log\left(\frac{S}{\Lambda_0^2 m}\right) - \frac{3}{2} \right).$$

The vacuum of the theory is determined by the minimum of (4.1.18), which occurs at

$$\tilde{\alpha} - N\tau = 0, \tag{4.1.19}$$

or, in terms of the expectation value of the gaugino bilinear, at

$$S_* = m\Lambda_0^2 \exp(2\pi i \tilde{\alpha}/N). \tag{4.1.20}$$

Finally, we note that the energy in the vacuum (4.1.20) is larger than that in the $j = 0$ vacuum by the constant that enters (4.1.18), giving

$$V_* = N \frac{\sqrt{b_{NS}^2 + j^2}}{2\pi g_s}, \tag{4.1.21}$$

which is precisely the tension of the brane after turning on j . This is a strong indication that we have identified parameters correctly on the two sides of the duality.

It is easy to see that in the vacuum, neither the F-term

$$F = \partial_S \mathcal{W},$$

nor the D-term vanishes. Nevertheless, as will now show, this new vacuum preserves half of the $\mathcal{N} = 2$ supersymmetry of the theory we started with, though not the one manifest in the action as written. Defining the $SU(2)_R$ doublet of fermions

$$\Psi = \begin{pmatrix} \psi \\ \lambda \end{pmatrix},$$

the relevant part of the supersymmetry transformations of the $\mathcal{N} = 2$ theory are

$$\delta\Psi^i = X^{ij} \epsilon_j$$

where X is a matrix of F- and D-terms, shifted by an imaginary part due to the presence of a “magnetic” FI term (see, for example, [4] and references therein)

$$X = \frac{i}{\sqrt{2}} \begin{pmatrix} -Y_1 - iY_2 + N & Y_3 \\ Y_3 & Y_1 - iY_2 + N \end{pmatrix} \tag{4.1.22}$$

where the $\mathcal{N} = 2$ auxiliary fields are identified with the auxiliary F-term of \mathcal{S} in (4.1.13) and the D-term of the gauge field in (4.1.14) according to

$$(Y_1 + iY_2) = 2iF \quad Y_3 = \sqrt{2}D.$$

Note that the triplet $\vec{Y} = (Y_1, Y_2, Y_3)$ transform like a vector of the $SU(2)_R$ symmetry of the $\mathcal{N} = 2$ theory. In the vacuum (4.1.19)

$$X = \frac{iN}{\sqrt{2(b_{NS}^2 + j^2)}} \begin{pmatrix} b_{NS} - \sqrt{b_{NS}^2 + j^2} & j \\ j & -b_{NS} - \sqrt{b_{NS}^2 + j^2} \end{pmatrix}.$$

The supersymmetry manifest in (4.1.15) corresponds to ϵ_1 , and it is clearly broken in the vacuum for nonvanishing j , since neither the F- nor the D-term vanish. However, the determinant of X vanishes, and so there is a zero eigenvector corresponding to a preserved supersymmetry.

So far, we have identified turning on j with turning on an FI term in the low-energy effective action. It is natural to ask what this corresponds to geometrically in the Calabi-Yau manifold. In [22,17] (following [63,102]) it was shown that turning on a subset of the FI terms of the low-energy $\mathcal{N} = 2$ theory arising from IIB compactified on a Calabi-Yau manifold corresponds to turning on H_{NS} and H_{RR} fluxes in the geometry. This is what we used in (4.1.12). The question of what corresponds to introducing the full set of FI terms allowed by $\mathcal{N} = 2$ supersymmetry was studied, for example, in [96,97]. To make the $SU(2)_R$ symmetry of the theory manifest, we can write the triplet of the $\mathcal{N} = 2$ FI terms as

$$E = \frac{i}{\sqrt{2}} \begin{pmatrix} -E_1 - iE_2 & E_3 \\ E_3 & E_1 - iE_2 \end{pmatrix}$$

where (E_1, E_2, E_3) transform as a vector under $SU(2)_R$ and enter the action as

$$\frac{1}{4\pi} \text{Re}(\text{Tr} X \bar{E})$$

These are given in terms of ten dimensional quantities by⁴⁶

$$E_1 = \int_B H_{NS}/g_s, \quad E_2 = \int_B H_{RR}, \quad E_3 = \int_B dJ/g_s.$$

Note that this agrees precisely with what we have just derived using large N duality. Just as the bare gauge coupling $\int_{S^2} B_{NS}/g_s = b_{NS}/g_s$ gets mapped to $\int_B H_{NS}/g_s$ after the transition due to running of the coupling, $SU(2)_R$ covariance of the theory demands that turning on $\int_{S^2} J/g_s = j/g_s$ before the transition get mapped to turning on $\int_B dJ/g_s$ after

⁴⁶ This follows from equation (3.53) of [97] up to an $SU(2)_R$ rotation and specializing to a local Calabi-Yau. More precisely, to derive this statement one needs to look at the the transformations of the $\mathcal{N} = 2$ gauginos, not the gravitino as in [97], but these are closely related. See, for example, [102].

the transition. Moreover, we saw in this section that the latter coupling gets identified as a Fayet-Iliopoulos D-term for the $U(1)$ gauge field on the gravity side. This exactly matches the result of [96,97], since E_3 is the Fayet-Iliopoulos D-term parameter. It is encouraging to note that [96,97] reach this conclusion via arguments completely orthogonal to our own. Finally, we observe that an $SO(2) \subset SU(2)_R$ rotation can be used to set the Fayet-Iliopoulos D-term $E_3 = j/g_s$ to zero, at the expense of replacing $E_1 = b_{NS}/g_s$ by $E_1 = \sqrt{b_{NS}^2 + j^2}/g_s$, and this directly reproduces (4.1.18).

4.2. An A_2 Fibration and the Geometric Engineering of a Metastable Vacuum

By wrapping D5 branes on rigid \mathbb{P}^1 's in more general geometries with $b_2(X) > 1$, we can engineer vacua which are guaranteed to be massive and break supersymmetry by choosing the central charges of the branes to be misaligned. Since the D-brane theories experience confinement and gaugino condensation at low energies, we expect to be able to study the dynamics of these vacua in the dual geometries where the branes are replaced by fluxes.

In this section, we will consider the simple example of an A_2 singularity fibered over the complex plane $\mathbb{C}[t]$. This is described as a hypersurface in \mathbb{C}^4 ,

$$uv = z(z - mt)(z - m(t - a)). \quad (4.2.1)$$

This geometry has two singular points at $u, v, z = 0$ and $t = 0, a$. The singularities are isolated, and blowing them up replaces each with a rigid \mathbb{P}^1 . The two \mathbb{P}^1 's are independent in homology, and the local geometry near each of them is the same as that studied in the previous section.

Consider now wrapping N_1 D5 branes on the \mathbb{P}^1 at $t = 0$ and N_2 branes on the \mathbb{P}^1 at $t = a$. If the central charges of the branes,

$$Z_i = \int_{S^2_i} J + iB_{NS} = j_i + ib_{NS,i} \quad (4.2.2)$$

are aligned (e.g., if the Kähler parameters j_i both vanish), the theory on the branes has $\mathcal{N} = 1$ supersymmetry. At sufficiently low energies, it reduces to a $U(N_1) \times U(N_2)$ gauge theory with a bifundamental hypermultiplet Q, \tilde{Q} , a pair of adjoint-valued chiral fields $\Phi_{1,2}$ and a superpotential given by

$$W = \frac{m}{2} \text{Tr} \Phi_1^2 - \frac{m}{2} \text{Tr} \Phi_2^2 - a \text{Tr} Q \tilde{Q} + \text{Tr}(Q \Phi_1 \tilde{Q} - Q \tilde{Q} \Phi_2). \quad (4.2.3)$$

For a small relative phase of the central charges, e.g., if the theory at vanishing Kähler parameters is deformed by j_i by

$$j_i/b_{NS,i} \ll 1, \quad (4.2.4)$$

we expect this to have a pure gauge theory description at low energies in terms of the supersymmetric theory with Fayet-Iliopoulos terms for the two $U(1)$'s.

Misaligning the central charges such that

$$Z_1 \neq c_{12}Z_2, \quad (4.2.5)$$

for any positive, real constant c_{12} , should break all the supersymmetries of the background. Nevertheless, for large enough m and a , the vacuum should be stable. Since the theory is massive, we expect it to exhibit confinement at very low energies, with broken supersymmetry. Nevertheless, as we will now argue, the dynamics of the theory can be studied effectively for any j_i in the dual geometry, where the branes have been replaced by fluxes.

4.2.1. Large N dual geometry

The Calabi-Yau (4.2.1) has a geometric transition which replaces the two \mathbb{P}^1 's by two S^3 's,

$$S_i^2 \quad \rightarrow \quad S_i^3 \quad i = 1, 2.$$

The complex structure of the geometry after the transition is encoded in its description as a hypersurface,

$$uv = z(z - mt)(z - m(t - a)) + ct + d, \quad (4.2.6)$$

where c, d are related to the periods, $S_{1,2}$, of the three-cycles, $A_{1,2}$, corresponding to the two S^3 's,

$$S_i = \int_{A_i} \Omega, \quad \frac{\partial}{\partial S_i} \mathcal{F}_0 = \int_{B_i} \Omega.$$

As before, B_i are the non-compact three-cycles dual to A_i , and \mathcal{F}_0 is the prepotential of the $\mathcal{N} = 2$ theory. The prepotential in this geometry is again given by an exact formula,

$$2\pi i \mathcal{F}_0 = \frac{1}{2} S_1^2 \left(\log\left(\frac{S_1}{\Lambda_0^2 m}\right) - \frac{3}{2} \right) + \frac{1}{2} S_2^2 \left(\log\left(\frac{S_2}{\Lambda_0^2 m}\right) - \frac{3}{2} \right) - S_1 S_2 \log\left(\frac{a}{\Lambda_0}\right).$$

The theory with N_i D5 branes on the \mathbb{P}_i^1 before the transition is dual to a theory with N_i units of RR flux through S_i^3 after the transition:

$$\int_{A_i} H_{RR} = N_i.$$

There are additional fluxes turned on through the non-compact, dual B -cycles,

$$\alpha_i = \int_{B_i} (H_{RR} + iH_{NS}/g_s) = b_{RR,i} + ib_{NS,i}/g_s,$$

corresponding to running gauge couplings, and

$$\int_{B_i} dJ/g_s = j_i/g_s$$

corresponding to Fayet-Iliopoulos terms. The fluxes generate a superpotential,

$$\mathcal{W} = \int_X \Omega \wedge (H_{RR} + iH_{NS}/g_s)$$

or

$$\mathcal{W} = \sum_i \alpha_i S_i - N_i \frac{\partial}{\partial S_i} \mathcal{F}_0,$$

and Fayet-Iliopoulos D-terms,

$$\Delta \mathcal{L} = \sum_i \frac{j_i}{2\sqrt{2\pi g_s}} D_i,$$

where D_i are auxiliary fields in the two $\mathcal{N} = 1$ vector multiplets.

Large N duality predicts that for misaligned central charges (4.2.5), the fluxes should break all supersymmetries, and moreover, that the non-supersymmetric vacuum should be metastable. We now show that this indeed the case. The tree-level effective potential of the theory is

$$V = \frac{1}{4\pi} G^{i\bar{k}} \left(\partial_i \mathcal{W} \overline{\partial_k \mathcal{W}} + j_i j_k / g_s^2 \right) + const,$$

where

$$G_{i\bar{k}} = \frac{1}{2i} (\tau - \bar{\tau})_{ik} \quad \tau_{ik} = \frac{\partial^2}{\partial S_i \partial S_k} \mathcal{F}_0,$$

and the Kähler metric is determined by the off-shell $\mathcal{N} = 2$ supersymmetry of the background. We have shifted the zero of the potential energy by the tension of the branes at vanishing j_i ,

$$const. = \sum_{i=1,2} N_i \frac{b_{NS,i}}{2\pi g_s}.$$

In the case at hand,

$$\tau_{11} = \frac{1}{2\pi i} \log\left(\frac{S_1}{\Lambda_0^2 m}\right), \quad \tau_{22} = \frac{1}{2\pi i} \log\left(\frac{S_2}{\Lambda_0^2 (-m)}\right),$$

whereas τ_{12} is a constant⁴⁷ independent of the S_i ,

$$\tau_{12} = -\frac{1}{2\pi i} \log(a/\Lambda_0).$$

It is straightforward to see that the critical points of the potential correspond to solutions

⁴⁷ For convenience, we will take τ_{12} to be purely imaginary.

of the following equations

$$\begin{aligned} \text{Re}(\alpha_i) + \text{Re}(\tau_{ik})N^k &= 0 \\ G^{ji}G^{jk} (Im(\alpha_i)Im(\alpha_k) + j_i j_k / g_s^2) &= (N^j)^2. \end{aligned}$$

The first equation fixes the phase of S_i 's, and the second their magnitude. Consider the case where the two nodes are widely separated, namely, where the sizes S_i of the two S^3 's are much smaller than the separation a between them. In this limit, the equations of motion can be easily solved to obtain

$$\begin{aligned} S_{1,*}^{N_1} &= (\Lambda_0^2 m)^{N_1} \left(\frac{a}{\Lambda_0}\right)^{N_2 \cos \theta_{12}} \exp(2\pi i \tilde{\alpha}_1) + \dots \\ S_{2,*}^{N_2} &= (-\Lambda_0^2 m)^{N_2} \left(\frac{a}{\Lambda_0}\right)^{N_1 \cos \theta_{12}} \exp(2\pi i \tilde{\alpha}_2) + \dots \end{aligned} \quad (4.2.7)$$

where θ_{ij} is the relative phase between the central charges Z_i and Z_j . We can see that in the limit where the Z_i are aligned, this reduces to the simple case without FI terms where the effective gauge coupling has been replaced with the parameter $\tilde{\alpha}_i$. The case of anti-aligned central charges was studied in [4,5]. The weak coupling limit of two widely separated nodes, in which our approximation is justified, corresponds to

$$S_{i,*} \ll a < \Lambda_0. \quad (4.2.8)$$

$S_{i,*}$'s should be identified with the vev's of gaugino condensates on the branes, and are the order parameters of the theory. This is the case even in the presence of FI terms, as explained in the previous section. For small FI terms, this relies only on the off-shell $\mathcal{N} = 1$ supersymmetry of the theories on both sides of the duality and a comparison of superpotentials. In [4] it was conjectured that this also holds in the brane/antibrane case, where the central charges are anti-aligned and supersymmetry is maximally broken. It is natural, then, that the above limit should correspond to the theory being weakly coupled at the scale of the superpotential (4.2.3).

In the same limit, the vacuum energy is given by

$$V_* = N_1 \frac{\sqrt{b_{NS,1}^2 + j_1^2}}{2\pi g_s} + N_2 \frac{\sqrt{b_{NS,2}^2 + j_2^2}}{2\pi g_s} + \frac{1}{4\pi^2} N_1 N_2 \log\left(\frac{a}{\Lambda_0}\right) (1 - \cos \theta_{12}) + \dots \quad (4.2.9)$$

Note that in the limit of aligned central charges, the potential energy is simply the brane tension. This is in fact true *exactly*, and is related to the fact (which we will demonstrate later on) that in this case supersymmetry is preserved. For any other value of the angle, there is an additional attraction. In the extreme case, when we increase θ_{12} from zero to π , we end up with a brane/antibrane system on the flopped geometry. We can view this as varying one of the Z_i 's until the B_{NS} field through that cycle goes to *minus* itself. This is a flop, and by comparing to [5], it follows that the solution we found above for $\theta_{12} = \pi$ precisely corresponds to a brane/antibrane system in the flopped geometry.

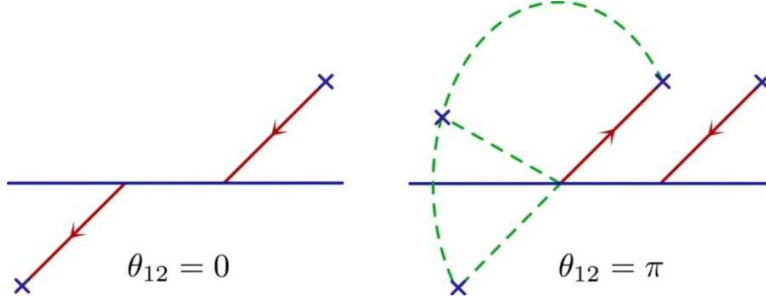


Fig. 4.1. The A_2 geometry from in the text, drawn in the T -dual NS5 brane picture. The D5 branes map to D4 branes and appear as red lines. The NS5 branes are drawn as blue lines/points. At $\theta_{12} = 0$ the system is supersymmetric. For any other value of θ_{12} , supersymmetry is broken. Varying θ_{12} continuously from zero to π produces a geometry which is related to the original A_2 geometry by a flop.

To see that supersymmetry is broken by the vacuum at nonvanishing θ_{12} , we write the action (4.1.15) in an $\mathcal{N} = 2$ invariant way in terms of $\mathcal{N} = 2$ chiral multiplets \mathcal{A}_i consisting of $\mathcal{N} = 1$ chiral multiplets \mathcal{S}_i , and \mathcal{W}_i^α ,

$$\mathcal{A}_i = (\mathcal{S}_i, W_i^\alpha)$$

or

$$\mathcal{A}_i = S_i + \theta^a \Psi_{a,i} + \theta^a \theta^b X_{ab,i} + \frac{1}{2} \epsilon_{ab} (\theta^a \sigma^{\mu\nu} \theta^b) F_{\mu\nu} + \dots$$

The appropriate $\mathcal{N} = 2$ Lagrangian is given by

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left(\int d^4\theta d^4x \mathcal{F}_0(\mathcal{A}_i) \right) + \frac{1}{4\pi} \text{Re}(X_i^{ab} \bar{E}_{ab}^i)$$

where X_i^{ab} is defined as in (4.1.22). Then, the relevant supersymmetry variations of the fermions are given by

$$\delta_\epsilon \Psi_i^a = X_i^{ab} \epsilon_b, \quad (4.2.10)$$

and at the extrema of the effective potential,

$$X_i = \frac{i}{\sqrt{2}} \begin{pmatrix} N^i + G^{ik} \text{Im}(\alpha_k) & -G^{ik} j_k / g_s \\ -G^{ik} j_k / g_s & N^i - G^{ik} \text{Im}(\alpha_k) \end{pmatrix}. \quad (4.2.11)$$

The equations of motion imply that the determinants of both $X_{1,2}$ vanish. For each node, then, X_i has one zero eigenvalue. It can be shown (see the general discussion of appendix 4.A), that (4.2.10) and (4.2.11) imply that a global supersymmetry is preserved if and only if the central charges (4.2.2) are aligned, i.e., if a positive real constant c_{12} exists such that

$$Z_1 = c_{12} Z_2.$$

This is exactly as expected from the open-string picture, and provides a nice test of the large N duality conjecture for general central charges.

Now we will show that the vacuum is indeed metastable. Consider the masses of bosonic fluctuations about the non-supersymmetric vacuum. As was found to be the case in [5], the Hessian of the scalar potential can be block diagonalized. After changing variables to bring the kinetic terms into their canonical form, the eigenvalues become

$$\begin{aligned}
M_{\phi_{1,2}}^2 &= \frac{(a^2 + b^2 + 2abv) \pm \sqrt{(a+b)^2(a-b)^2 + 4abv(a+b)(b+a)}}{2(1-v)^2} \\
M_{\phi_{3,4}}^2 &= \frac{(a^2 + b^2 + 2abv \cos \theta) \pm \sqrt{(a+b)^2(a-b)^2 + 4abv(a+b \cos \theta)(b+a \cos \theta)}}{2(1-v)^2}
\end{aligned}
\tag{4.2.12}$$

where we've adopted the notation of [4] in defining

$$a = \frac{N_1}{2\pi G_{11}|S_1|}, \quad b = \frac{N_2}{2\pi G_{22}|S_2|},$$

and

$$v = \frac{G_{12}^2}{G_{11}G_{22}},$$

with all quantities evaluated in the vacuum. In addition, we've introduced a new angle, which is defined by the equation

$$\vec{Y}^1 \cdot \vec{Y}^2 = N_1 N_2 \cos \theta,$$

where \vec{Y}^i are related to X^i as in (4.1.22) for each i . In the limit (4.2.8),

$$\theta = \theta_{12} + \mathcal{O}(v),$$

so we can treat this as being the same as the phase which appears in (4.2.7),(4.2.9). Note that in the limit

$$\cos \theta \rightarrow 1,
\tag{4.2.13}$$

we recover the results for a supersymmetric system, with the masses of bosons becoming pairwise degenerate. The other extreme of anti-aligned charges can be shown to correspond to

$$\cos \theta \rightarrow -1,$$

where the results of [5] should be recovered. Indeed by plugging in and rearranging terms we recover the mass formulas from page 25 of [5]. These then provide exact values for the tree-level masses of the component fields in the supersymmetry-breaking vacuum. For any θ , the masses of the bosons are all positive as long as

$$v < 1.$$

This, in turn, is ensured as long as the metric on moduli space is positive definite in the vacuum. So indeed, the system is metastable, as expected.

To get a measure of supersymmetry breaking, let's now compare the masses of the bosons and the fermions in this vacuum. The fermion masses arise from the superspace interaction which appears as

$$\frac{1}{4\pi} \text{Im} \left(\int d^4\theta d^4x \frac{1}{2} \mathcal{F}_{ijk} (\Psi_a^i \theta^a) (\Psi_b^j \theta^b) (X_{cd}^k \theta^c \theta^d) \right),$$

where $\mathcal{F}_{ijk} = \partial_i \partial_j \partial_k \mathcal{F}_0$. For the geometry in question, the prepotential is exact at one-loop order, and the third derivatives vanish except when all derivatives are with respect to the same field. We can then write the fermion mass matrices for a given node (and non-canonical kinetic terms) as

$$M_{ab}^i = \frac{1}{16\pi^2 S_i} \epsilon_{ac} X_{cd}^i \epsilon_{db}$$

Performing a change of basis to give the fermion kinetic terms a canonical form, we can diagonalize the resulting mass matrix and obtain the mass eigenvalues. There are two zero modes,

$$M_{\lambda_{1,2}} = 0,$$

corresponding to two broken supersymmetries. In addition there are two massive fermions, which we label by ψ_i ,

$$M_{\psi_{1,2}} = \frac{(a+b) \pm \sqrt{(a-b)^2 + 2abv(1+\cos\theta)}}{2(1-v)}. \quad (4.2.14)$$

Note that in the supersymmetric limit (4.2.13), the masses of ψ_1 and ψ_2 match those of $\phi_{1,3}$ and $\phi_{2,4}$, which have become pairwise degenerate. For small misalignment, and large separation of the two nodes, the mass splittings of bosons and fermions are easily seen to go like

$$\frac{M_\phi^2 - M_\psi^2}{M_\phi^2 + M_\psi^2} \sim v\theta_{12}^2,$$

where v goes to zero in the limit of large separation, and θ_{12} measures the misalignment of the central charges.

4.2.2. Gauge theory limit

In the gauge theory limit (4.2.4), the vacuum energy (4.2.9) reduces to

$$V_* = \sum_i N_i \frac{b_{NS,i}}{2\pi g_s} \left(1 + \frac{1}{2} \frac{j_i^2}{b_{NS,i}^2}\right) + \frac{1}{8\pi^2} N_1 N_2 \log\left(\frac{a}{\Lambda_0}\right) \left(\frac{j_1}{b_{NS,1}} - \frac{j_2}{b_{NS,2}}\right)^2 + \dots \quad (4.2.15)$$

The first terms are classical contributions, as we saw in section two. The last term comes from a one-loop diagram in string theory, with strings stretched between the two stacks of branes running around the loop.

To begin with, consider the Abelian case,⁴⁸

$$U(1) \times U(1),$$

when the gauge theory has no strong dynamics at low energies. We should be able to reproduce (4.2.15) directly in the field theory by computing the one-loop vacuum amplitude in a theory with FI terms turned on. We can write the classical F- and D-term potential of the gauge theory as

$$V_{tree} = |F_{\Phi_1}|^2 + |F_{\Phi_2}|^2 + |F_Q|^2 + |F_{\tilde{Q}}|^2 + \frac{1}{2} g_{YM,1}^2 (|q|^2 - |\tilde{q}|^2 - \sqrt{2}\xi_1)^2 + \frac{1}{2} g_{YM,2}^2 (|\tilde{q}|^2 - |q|^2 - \sqrt{2}\xi_2)^2,$$

where

$$F_{\Phi_1} = m\phi_1 - q\tilde{q} \quad F_{\Phi_2} = m\phi_2 - q\tilde{q} \quad F_Q = \tilde{q}(a + \phi_2 - \phi_1) \quad F_{\tilde{Q}} = q(a + \phi_2 - \phi_1)$$

and ϕ_i, q, \tilde{q} are the lowest components of the corresponding chiral superfields. The gauge theory quantities are related to those of the string theory construction by

$$\frac{1}{g_{YM,i}^2} = \frac{b_{NS,i}}{4\pi g_s}, \quad \xi_i = \frac{j_i}{4\pi g_s}.$$

The identification between the field theory FI term and the string theory parameter is expected to hold only for small $j_i/b_{NS,i}$. For nonzero $\xi_{1,2}$, supersymmetry appears to be broken since the two D-term contributions cannot be simultaneously set to zero with the F-terms. In fact, we know that if the central charges are aligned, this is just a relic of writing the theory in the wrong superspace.

For large m, a , this potential has a critical point at the origin of field space. At this point, all the F-terms vanish, and there is pure D-term supersymmetry breaking. The spectrum of scalar adjoint and gauge boson masses is still supersymmetric at tree-level, since the only contribution to the masses in the Lagrangian is the FI-dependent piece for the bifundamentals. This means that the only relevant contribution to the one-loop corrected potential is from the bifundamental fields. The scalar components develop a tree-level mass which is simply given by⁴⁹

$$m_q^2 = a^2 + r, \quad m_{\tilde{q}}^2 = a^2 - r \quad (4.2.16)$$

⁴⁸ Although the rank of the gauge group is not large in this case, the geometric transition is still expected to provide a smooth interpolation between the open- and closed-string geometries. For a recent review, see [6], and references therein. It is natural to expect that for small deformations by FI terms that break supersymmetry, the two sides still provide dual descriptions of the same physics.

⁴⁹ One can readily check that for small r , the masses agree with what we expect from string theory. The bifundamental matter is the same as for the 0 – 4 system, with small B -fields turned on along the D4 branes. See, e.g., [103,104].

while the fermion masses retain their supersymmetric value,

$$m_{\psi_q}^2 = m_{\tilde{\psi}_q}^2 = a^2.$$

We have defined the constant

$$r = \sqrt{2}(\xi_2 g_{YM,2}^2 - \xi_1 g_{YM,1}^2) = \sqrt{2}\left(\frac{j_2}{b_{NS,2}} - \frac{j_1}{b_{NS,1}}\right). \quad (4.2.17)$$

The one-loop correction to the vacuum energy density is given by

$$V^{(1-loop)} = \frac{1}{64\pi^2} \left(\sum_b m_b^4 \log \frac{m_b^2}{\Lambda_0^2} - \sum_f m_f^4 \log \frac{m_f^2}{\Lambda_0^2} \right),$$

where $m_{b,f}$ are the boson and the fermion masses, and Λ_0 is the UV cutoff of the theory. The limit in which we expect a good large N dual is when the charged fields are very massive, $r \ll a^2$, and at low energies the theory is a pure gauge theory. Expanding to the leading order in r/a^2 , the one-loop potential is then given by

$$V = V_{tree} + \frac{1}{16\pi^2} r^2 \log \frac{a}{\Lambda_0}.$$

We have omitted the Λ_0 independent terms which correspond to the finite renormalization of the couplings in the Lagrangian and are ambiguous. We see that this exactly agrees with effective potential (4.2.9),(4.2.15) as computed in the dual geometry, after the transition.

In the general, $U(N_1) \times U(N_2)$ case, we have a strongly coupled gauge theory at low energies. Nevertheless, since in the (N_i, \bar{N}_i) sector supersymmetry is preserved, the one-loop contribution of that sector to the vacuum energy density should vanish beyond the classical contribution. Thus, we expect that only the bifundamental fields contribute to the vacuum energy at this level. The one-loop computation then goes through as in the Abelian case, up to the $N_1 N_2$ factor from multiplicity, once again reproducing the answer (4.2.15) from large N dual geometry.

4.2.3. Relation to the work of [6]

We close with a comment on the relation to the work of [6], to put the present work in context. The A_2 model at hand is the same as the geometry used to engineer the Fayet model in [6]. More precisely, the authors there engineered a “retrofitted” Fayet model. The parameter a that sets the mass of the bifundamentals was generated by stringy or fractional gauge theory instantons, and thus was much smaller than the scale set by the FI terms, which were taken to be generic. That resulted in F-term supersymmetry breaking which was dynamical.

In the present context, we still have a Fayet-type model, but we find ourselves in a different regime of parameters of the field theory, where $r/a^2 < 1$, with r defined in terms of the FI parameters as in (4.2.17). Outside of this regime, the vacuum at the origin of

field space, with Q and \tilde{Q} vanishing, becomes tachyonic even in the field theory, as can be seen from (4.2.16). Once this becomes the case, the large N dual presented here is unlikely to be a good description of the physics. For example, for $N_1 = N_2 = N$ and $r/a^2 > 1$, it was found in [6] that the theory has a non-supersymmetric vacuum where all the charged bifundamental fields are massive and the gauge symmetry is broken to $U(N)$. This may still have a description in terms of some dual geometry with fluxes, but not the one at hand. This may be worth investigating.

Thus, unlike the models of [6], those considered here break supersymmetry spontaneously but not dynamically. It would be nice to find a way to retrofit the current models and to generate low scale supersymmetry breaking in this context. This would require finding a natural way of obtaining small FI terms. The mechanism of [6] does not apply here, since the terms in question are D-terms and not F-terms. This may be possible in the context of warped compactifications⁵⁰ and compact Calabi-Yau manifolds, perhaps along the lines of [107].

Appendix 4.A. Fayet-Iliopoulos Terms for ADE Singularities

The large N duality we studied in the previous sections should generalize to other ADE fibered geometries. In this appendix, we demonstrate that the large N dual geometries for these more general spaces have some of the same qualitative features. Consider the ADE type ALE spaces

$$\begin{aligned}
 A_k &: & x^2 + y^2 + z^{k+1} &= 0 \\
 D_r &: & x^2 + y^2 z + z^{r-1} &= 0 \\
 E_6 &: & x^2 + y^3 + z^4 &= 0 \\
 E_7 &: & x^2 + y^3 + yz^3 &= 0 \\
 E_8 &: & x^2 + y^3 + z^5 &= 0
 \end{aligned}$$

which are fibered over the complex t -plane, allowing the coefficients parameterizing the deformations to be t -dependent. The requisite deformations of the singularities are canonical (see [31] and references therein). In fibering this over the t -plane, the z_i become polynomials $z_i(t)$. At a generic point in the t -plane, the ALE space is smooth, with singularities resolved by blowing up r independent two-cycle classes

$$S_i^2, \quad i = 1, \dots, r$$

where r is the rank of the corresponding Lie algebra. This corresponds to turning on Kähler moduli

$$Z_i = \int_{S_i^2} (J + iB_{NS}) = j_i + ib_{NS,i}.$$

⁵⁰ The effects of warping in the context of [4] have been studied in [46]. In [105], the authors constructed supergravity solutions that were subsequently interpreted in [106] to correspond to D5 branes on the conifold with Fayet-Iliopoulos terms turned on, i.e. the theory we studied in section two. We thank Y. Nakayama for directing us to the latter work.

The two-cycles S_i^2 intersect according to the ADE Dynkin diagram of the singularity. Consider now wrapping N_i D5 branes on the i 'th two-cycle class. The theory on the branes is an $\mathcal{N} = 2$ quiver theory with gauge group

$$\prod_i U(N_i),$$

with a bifundamental hypermultiplet Q_{ij}, Q_{ji} for each pair of nodes connected by a link in the Dynkin diagram. The fibration breaks the supersymmetry to $\mathcal{N} = 1$ by turning on superpotentials $W_i(\Phi_i)$ for the adjoint chiral multiplets Φ_i ,

$$W'_i(t) = \int_{S_{i,t}^2} \omega^{2,0},$$

which compute the holomorphic volumes of the two-cycles at fixed t . The superpotentials $W_i(t)$ can be thought of as parameterizing the choice of complex structure of the ALE space at each point in the t -plane. The full tree-level superpotential of the theory is given by

$$W = \sum_i \text{Tr} W_i(\Phi_i) + \sum_{i < j} \text{Tr}(Q_{ij} Q_{ji} \Phi_i - Q_{ij} \Phi_j Q_{ji})$$

where the latter sum runs over nodes that are linked.

For vanishing j_i , the structure of the vacua of the theory was computed in [31]. For each positive root e_I of the lie algebra,

$$e_I = \sum_I n_I^i e_i$$

for positive integers n_I^i , one gets a rigid \mathbb{P}^1 at points in the t -plane

$$t = a_{I,p},$$

where

$$W'_I(a_{I,p}) = \sum_i n_I^i W'_i(a_{I,p}) = 0. \tag{4.A.1}$$

Here I labels the positive root and p runs over all the solutions to (4.A.1) for that root. The choice of vacuum breaks the gauge group down to

$$\prod_{I,p} U(M_{I,p})$$

where

$$N_i = \sum_I M_{I,p} n_I^i.$$

Turning on generic Fayet-Iliopoulos terms for the $U(1)$ centers of the gauge group factors,

$$\Delta\mathcal{L} = \sum_i \frac{j_i}{2\sqrt{2}\pi g_s} \text{Tr} D_i,$$

breaks supersymmetry while retaining (meta)stability of the vacuum as long as j_i is much smaller than the mass of all the bifundamentals in the vacuum.

The ALE fibrations have geometric transitions in which each \mathbb{P}^1 is replaced by a minimal S^3 . The leading order prepotential \mathcal{F}_0 for all these singularities was computed in [52], and is given by

$$2\pi i \mathcal{F}_0(S) = \frac{1}{2} \sum_b S_b^2 \left(\log \left(\frac{S_b}{W_I''(a_b) \Lambda_0^2} \right) - \frac{3}{2} \right) + \frac{1}{2} \sum_{b \neq c} e_{I(b)} \cdot e_{J(c)} S_b S_c \log \left(\frac{a_{bc}}{\Lambda_0} \right) + \dots, \quad (4.A.2)$$

where the sum is over all critical points

$$b = (I, p),$$

and $I(b) = I$ denotes the root I to which the critical point labeled by b corresponds. We are neglecting cubic and higher order terms in the $S_{I,p}$, which are related to higher loop corrections in the open string theory. Above, $e_I \cdot e_J$ is the inner product of two positive, though not necessarily simple, roots. Geometrically, the inner product is the same as minus the intersection number of the corresponding two-cycles classes in the ALE space. In addition, there are fluxes turned on in the dual geometry which are determined by holography:

$$\int_{A_a} H_{RR} = M_a$$

$$\int_{B_a} (H_{RR} + \frac{i}{g_s} H_{NS}) = b_{RR, I(a)} + \frac{i}{g_s} b_{NS, I(a)} \quad , \quad \int_{B_a} dJ = j_{I(a)}.$$

The theory on this geometry without fluxes is an $\mathcal{N} = 2$, $U(1)^k$ gauge theory, where k is the number of S^3 's. The effect of the fluxes on the closed-string theory in this background was determined in [96,97]. The result is a set of electric and magnetic $\mathcal{N} = 2$ Fayet-Iliopoulos terms, which enter the $\mathcal{N} = 2$ superspace Lagrangian,

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left(\int d^4\theta \mathcal{F}_0(\mathcal{A}^a) \right) + \frac{1}{4\pi} \text{Re}(\vec{Y}^a \cdot \vec{E}_a).$$

with

$$\vec{E}_a = \left(\frac{b_a^{NS}}{g_s}, b_{RR,a}, \frac{j_a}{g_s} \right),$$

and where the auxiliary fields \vec{Y}^a are shifted by the magnetic FI term,

$$\vec{M}^a = (0, M^a, 0).$$

The auxiliary field Lagrangian then has the form

$$\mathcal{L}_{aux} = \frac{1}{8\pi} G_{ab} \text{Re}(\vec{Y})^a \cdot \text{Re}(\vec{Y})^b + \frac{1}{4\pi} \text{Re}(\tau_{ab}) \text{Re}(\vec{Y})^a \cdot \vec{M}^b + \frac{1}{4\pi} \text{Re}(\vec{Y})^a \cdot \vec{E}_a$$

and integrating out the auxiliary fields sets them equal to their expectation values,

$$\vec{Y}^a = -G^{ab} \left(\vec{E}_b + \text{Re}(\tau_{bc}) \vec{M}^c \right) + i \vec{M}^a$$

or, to be more precise,

$$-G_{ab} \vec{Y}^b = (b_a^{NS}/g_s, b_a^{RR} + \bar{\tau}_{ab} M^b, j_a/g_s).$$

We can make contact with the more familiar form of this action and its scalar potential by reducing to $\mathcal{N} = 1$ superspace. There, the auxiliary fields \vec{Y}^a get identified with auxiliary fields of the vector and chiral multiplets corresponding to the a 'th S^3 , and the fluxes give rise to the usual flux superpotential

$$\mathcal{W} = \sum_a \alpha_a S_a - M_a \partial_{S_a} \mathcal{F}_0(S).$$

In addition, there are Fayet-Iliopoulos terms for the $U(1)$'s, and the total scalar potential is given by

$$V = \frac{1}{4\pi} G^{ab} \left(\partial_a \mathcal{W} \overline{\partial_b \mathcal{W}} + j_a j_b / g_s^2 \right)$$

where

$$\alpha_a = b_{RR,a} + i b_{NS,a} / g_s.$$

There are vacua at the field values which satisfy

$$\partial_a V \sim \mathcal{F}_{abc} G^{ce} G^{bd} \left((\alpha_e - \bar{\tau}_{em} M^m) (\bar{\alpha}_d - \bar{\tau}_{dn} M^n) + j_e j_d / g_s^2 \right) = 0.$$

At one-loop order, the prepotential has nonvanishing third derivatives only when all derivatives are with respect to the same field. The vacuum condition can be simplified to this order, and upon considering the equation as two real equations for the real and imaginary part, the conditions become

$$\begin{aligned} (b_{RR,a} - \text{Re}(\tau_{ab})) M^b &= 0 \\ G^{ac} G^{ad} (b_{NS,c} b_{NS,d} + j_c j_d) &= (M_a g_s)^2. \end{aligned}$$

The first of these can be solved easily for the phases of the S^a . Moreover, we see that it is equivalent to the condition that the real part of the auxiliary fields Y_2^a vanish for all a in the $\mathcal{N} = 2$ superspace Lagrangian,

$$G_{ab} \text{Re}(Y_2^b) = 0.$$

In light of that result, the second condition can be written as

$$\vec{Y}_a \cdot \vec{Y}_a = 0. \quad (4.A.3)$$

Since the supersymmetry transformations are

$$\delta_\epsilon \Psi_a = X_a \epsilon + \dots$$

where

$$X_a = \frac{i}{\sqrt{2}} \begin{pmatrix} -Y_1^a - i\text{Re}(Y_2)^a + M^a & Y_3^a \\ Y_3^a & Y_1^a - i\text{Re}(Y_2)^a + M^a \end{pmatrix},$$

(4.A.3) is precisely the condition that there exists some supersymmetry transformation on each node which is locally preserved by the vacuum. Of course, for supersymmetry to be conserved globally, these supersymmetry transformations must match for all nodes. The condition for this to be the case is

$$\frac{Y_1^a}{M^a} = \frac{Y_1^b}{M^b} \quad \frac{Y_3^a}{M^a} = \frac{Y_3^b}{M^b},$$

which, along with the requirement that the metric on moduli space be positive definite in the vacuum, requires that

$$Z_a = c_{ab} Z_b$$

for a positive, real constant c_{ab} . This conforms to our intuition from the open-string picture that preserving supersymmetry should require that the complex combination of the FI terms and gauge couplings should have the same phase on each node.

We can also see that the vacuum we just found is metastable, as we expect based on large N duality. Consider the Hessian of the potential,

$$4\pi \partial_a \partial_c V = \frac{1}{8\pi^2 S_a S_c g_s^2} \left(G^{ia} G^{ac} G^{cj} (b_{NS}^i b_{NS}^j + j_i j_j) - G^{ac} M^a M^c g_s^2 \right),$$

$$4\pi \partial_a \partial_{\bar{c}} V = \frac{1}{8\pi^2 S_a \bar{S}_c g_s^2} \left(G^{ia} G^{ac} G^{cj} (b_{NS}^i b_{NS}^j + j_i j_j) + G^{ac} M^a M^c g_s^2 \right),$$

and similarly for complex conjugates. The eigenvalues of the Hessian are manifestly positive in the limit where G_{ab} vanishes for $a \neq b$, which corresponds to widely separated nodes, and where the matrix $\partial\partial V$ is diagonal. Moreover, the determinant of the Hessian is strictly positive for any G_{ab} , so the one-loop Hessian remains positive definite for any G_{ab} .

Finally, we can compute the value of the vacuum energy in the limit where the branes are far separated. The relevant limit in this more general case is

$$S_{a,*} \ll a_{bc} < \Lambda_0.$$

The vacuum energy is then given by

$$V_* = \sum_b M^b \frac{\sqrt{b_{NS,b}^2 + j_b^2}}{2\pi g_s} - \frac{1}{8\pi^2} \sum_{b \neq c} e_{I(b)} \cdot e_{J(c)} M^b M^c \log \frac{a_{bc}}{\Lambda_0} (1 - \cos \theta_{bc})$$

which reduces to the one-loop value in the gauge-theory limit, as in the A_2 case.

Chapter 5

Extended Supersymmetric Moduli Space and a SUSY/Non-SUSY Duality

The last decade has seen great progress in our understanding of the dynamics of $\mathcal{N} = 1$ supersymmetric gauge theories, with string theory playing a large role in these developments thanks to its rich web of dualities. In particular, motivated by string theoretic considerations [18], a perturbative approach was proposed for the computation of glueball superpotentials in certain $\mathcal{N} = 1$ supersymmetric gauge theories using matrix models [19,59], which leads to non-perturbatively exact information for these theories at strong coupling. Further evidence for this proposal was provided through direct computations [57], as well as from consideration of $\mathcal{N} = 1$ chiral rings [108].

The simplest class of gauge theories considered in [18] involve an $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theory with an adjoint superfield Φ together with a superpotential

$$\text{Tr}W(\Phi) = \sum_k a_k \text{Tr}\Phi^k.$$

In this chapter, we consider further deforming this theory by the most general set of single-trace chiral operators. This is accomplished by the introduction of superpotential terms

$$\int d^4x d^2\theta \text{Tr}[\alpha(\Phi)\mathcal{W}_\alpha\mathcal{W}^\alpha],$$

where \mathcal{W}_α is the field strength superfield. In string theory, these theories are constructed by wrapping D5 branes on vanishing cycles in local Calabi-Yau threefolds, where the addition of a background B -field which depends holomorphically on one complex coordinate of the threefold leads to the above deformation, with

$$\alpha(\Phi) = B(\Phi) = \sum_k t_k \Phi^k.$$

We show how the strongly coupled IR dynamics of these theories can be understood using both string theoretic techniques (large N duality via a geometric transition) and a direct field theory computation as in [57]. Moreover, following [101], we can consider the limit where $W(\Phi)$ is set to zero, in which case we recover an $\mathcal{N} = 2$ supersymmetric theory with Lagrangian given by

$$\mathcal{L} = \int d^4x d^4\theta \mathcal{F}(\Phi).$$

The prepotential $\mathcal{F}(\Phi)$ is related to $\alpha(\Phi)$ by

$$\mathcal{F}''(\Phi) = \alpha(\Phi), \tag{5.0.1}$$

where Φ is an adjoint-valued $\mathcal{N} = 2$ chiral multiplet. In this limit, our solution reduces to that of the extended Seiberg-Witten theory with general prepotential (5.0.1). Our results are in complete agreement with the beautiful earlier work of [109], which uses Konishi anomaly [108] and instanton techniques [110] to study these same supersymmetric gauge theories.⁵¹

The stringy perspective which we develop, however, sheds light on non-supersymmetric phases of these theories, which will be our main focus. In particular, it turns out that if $\alpha(\Phi)$ is chosen appropriately, there are vacua where supersymmetry is broken. The idea is that a suitable choice of higher-dimensional operators can lead to negative values of g_{YM}^2 for certain factors of the gauge group. Motivated by string theory considerations, we will show that strong coupling effects can make sense of the negative value for g_{YM}^2 , and at the same time lead to supersymmetry breaking. In the string theory construction, this arises from the presence of antibranes in a holomorphic B -field background. When g_{YM}^2 is negative in all the gauge group factors, we propose a complete UV field theory description of these vacua. This is another $U(N)$ gauge theory, already studied in [112,113,96,114], with an adjoint field $\tilde{\Phi}$ and superpotential

$$\int d^2\theta \text{Tr}[\tilde{t}_0 \tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha + \tilde{W}(\tilde{\Phi})], \tag{5.0.2}$$

where

$$\tilde{W}(\tilde{\Phi}) = \sum_k (a_k + 2it_k \theta\theta) \tilde{\Phi}^k.$$

Note that since the spurion auxiliary fields have nonzero vevs t_k , this theory breaks supersymmetry.

A duality between supersymmetric and non-supersymmetric theories may appear contradictory. The way this arises is as follows (see figure 5.1). We have an IR effective $\mathcal{N} = 1$ theory which is valid below a cutoff scale Λ_0 . The IR theory is formulated in terms of chiral fields which we collectively denote by χ (for us, these are glueball fields). The theory depends on some couplings t , and for each value of t we find two sets of vacua – one which is supersymmetric, and one which is not. However, for any given values of t , only one of

⁵¹ A special case of these theories with a particular choice of $W(\Phi)$ was also studied in [111].

these vacua is physical, in that the expectation value of the chiral fields is below the cutoff scale $|\langle\chi\rangle| < \Lambda_0$. The other solution falls outside of this region of validity. In particular, in one regime of parameter space, only the supersymmetric solution is acceptable. As we change t , the supersymmetric solution leaves the allowed region of field space, and at the same time the non-supersymmetric solution enters the allowed region. We obtain in this way a duality between a supersymmetric and a non-supersymmetric theory. Moreover, we are able to identify two dual UV theories. However, unlike the effective IR theory, which is valid for the entire parameter space, each UV theory is valid only for part of the full parameter space. The supersymmetric IR solution matches onto a supersymmetric UV theory, and the non-supersymmetric IR solution matches onto another UV theory where supersymmetry is broken softly by spurions.

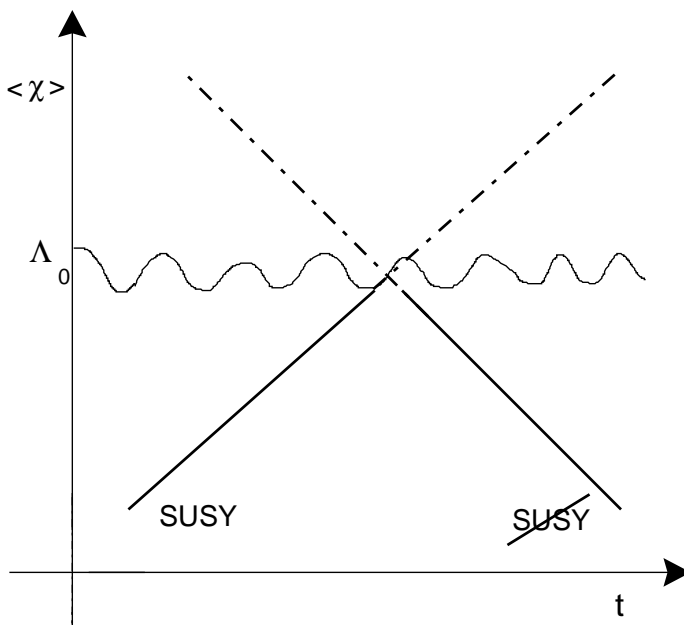


Fig. 5.1. A phase diagram for the supersymmetric/nonsupersymmetric duality. The horizontal axis represents the full parameter space, and the vertical axis represents field vevs. A wavy line at the cutoff Λ_0 is the region where we begin to lose validity of a given solution – we can trust solutions only below this scale.

The organization of this chapter is as follows. In section 2 we establish the basic field theories which will be studied. In section 3 we show how these field theories can be realized in type IIB string theory on local Calabi-Yau threefolds. In section 4 we show how this string theory construction leads to a solution for the IR dynamics of the theory. In section 5 we derive the same result directly from field theory considerations. In section 6 we specialize to the $\mathcal{N} = 2$ case. In section 7 we consider these field theories when some of the gauge couplings g_{YM}^2 become negative. We explain why this leads to supersymmetry-breaking and propose a dual description. Some aspects of the effective superpotential computation are presented in an appendix.

5.1. Field Theory

Consider an $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory with no hypermultiplets. Classically, this theory is described by a holomorphic prepotential $\mathcal{F}(\Phi)$ which appears in the $\mathcal{N} = 2$ Lagrangian,

$$\mathcal{L} = \int d^4x d^4\theta \mathcal{F}(\Phi) \quad (5.1.1)$$

where Φ is an adjoint-valued $\mathcal{N} = 2$ chiral multiplet, and

$$\mathcal{F}(\Phi) = \frac{t_0}{2} \text{Tr} \Phi^2. \quad (5.1.2)$$

Above, t_0 determines the classical gauge coupling and θ angle

$$t_0 = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2}, \quad (5.1.3)$$

and the integral in (5.1.1) is over a chiral half of the $\mathcal{N} = 2$ superspace. The low energy dynamics of this theory were studied in [115], where it was shown that the theory admits a solution in terms of an auxiliary Riemann surface and one-form.

This theory admits a natural extension via the introduction of higher-dimensional single-trace chiral operators,

$$\mathcal{F}(\Phi) = \sum_{k=0} \frac{t_k}{(k+1)(k+2)} \text{Tr} \Phi^{k+2}, \quad (5.1.4)$$

which deform the theory in the ultraviolet. One effect of these new terms is that the effective gauge coupling at a given point in moduli space now depends explicitly on the expectation value of the scalar component ϕ of the superfield Φ ,

$$t_0 \rightarrow \mathcal{F}''(\phi) = \sum_{k=0} t_k \phi^k.$$

We therefore define

$$\alpha(\Phi) \equiv \mathcal{F}''(\Phi).$$

In this chapter, we will solve for the low energy dynamics of this extended Seiberg-Witten theory.

We will also study deformations of the theory (5.1.1) to an $\mathcal{N} = 1$ supersymmetric theory by the addition of a superpotential,

$$\text{Tr} W(\Phi) = \sum_{k=0}^{n+1} a_k \text{Tr} \Phi^k, \quad (5.1.5)$$

for the $\mathcal{N} = 1$ chiral multiplet Φ that sits inside $\mathbf{\Phi}$. In $\mathcal{N} = 1$ language, the full superpotential of the theory then becomes

$$\int d^2\theta \left(\text{Tr} [\alpha(\Phi) \mathcal{W}_\alpha \mathcal{W}^\alpha] - \text{Tr} W(\Phi) \right), \quad (5.1.6)$$

where \mathcal{W}_α is the gaugino superfield.

Classically, the superpotential $W(\Phi)$ freezes the eigenvalues of ϕ at points in the moduli space where

$$W'(\phi) = 0. \quad (5.1.7)$$

For generic superpotential, we can write

$$W'(x) = g \prod_{i=1}^n (x - e_i), \quad (5.1.8)$$

with e_i all distinct, so the critical points are isolated and the choice of a vacuum breaks the gauge symmetry as

$$U(N) \rightarrow \prod_{k=1}^n U(N_k) \quad (5.1.9)$$

for the vacuum with N_i of the eigenvalues of ϕ placed at each critical point $x = e_i$.

As long as the effective gauge couplings of the low-energy theory are positive, i.e.

$$\text{Im}[\alpha(e_i)] = \left(\frac{4\pi}{g_{\text{YM}}^2} \right)_i > 0, \quad i = 1, \dots, n \quad (5.1.10)$$

the general aspects of the low energy dynamics of this theory are readily apparent. In the vacuum (5.1.9), at sufficiently low energies, the theory is pure $\mathcal{N} = 1$ super-Yang-Mills, which is expected to exhibit confinement and gaugino condensation.

When the original $\mathcal{N} = 2$ theory has canonical prepotential (5.1.2), the condition (5.1.10) is satisfied trivially, and in this case the problem of computing the vacuum expectation values of gaugino condensates in the $\mathcal{N} = 1$ theory,

$$S_k = \text{Tr} \mathcal{W}_{\alpha,k} \mathcal{W}_k^\alpha, \quad (5.1.11)$$

has been studied extensively from both string theory [18] and gauge theory [57,108] perspectives. The question can be posed in terms of the computation of an effective glueball superpotential [18],

$$\mathcal{W}_{\text{eff}}(S_i),$$

whose critical points give the supersymmetric vacua of the theory. In this chapter, we will show how to compute \mathcal{W}_{eff} for the $\mathcal{N} = 1$ theory with the more

general prepotential (5.1.4). Note that physically inequivalent choices of $\alpha(\Phi)$ correspond to polynomials in Φ of degree at most $n - 1$. This is because, for the supersymmetric theory, any operator of the form

$$\text{Tr} [\Phi^k W'(\Phi) \mathcal{W}_\alpha \mathcal{W}^\alpha] \sim 0$$

is trivial in the chiral ring [108].

In section 7, we will ask what happens when (5.1.10) is not satisfied and it appears that some of the gauge couplings of (5.1.9) become negative in the vacuum. We will show that in this case, the theory (5.1.6) generically breaks supersymmetry. Moreover, the supersymmetry-breaking vacua still exhibit gaugino condensation and confinement, and we will be able to compute the corresponding expectation values (5.1.11) as critical points of a certain effective *scalar* potential $V_{\text{eff}}(S_i)$.

5.2. The String Theory Construction

In this section we give the string theory realization of the above gauge theory. To begin with, we consider type IIB string theory compactified on an A_1 singularity,

$$uv = y^2, \tag{5.2.1}$$

which is fibered over the complex x -plane. This has a singularity for all x at $u, v, y = 0$, which can be resolved by blowing up a finite \mathbb{P}^1 . Wrapping N D5 branes on the \mathbb{P}^1 gives a $d = 4$ $U(N)$ $\mathcal{N} = 2$ gauge theory at sufficiently low energies. The adjoint scalar ϕ of the gauge theory corresponds to motion of the branes in the x -plane.

In the microscopic $\mathcal{N} = 2$ gauge theory we also have a choice of prepotential $\mathbb{F}(\Phi)$. What does this correspond to geometrically? To answer this, note that the microscopic prepotential determines the bare 4d gauge coupling, which arises in the geometry from the presence of nonzero B -fields,

$$\frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2} = \int_{\mathbb{P}^1} \left(B_{\text{RR}} + \frac{i}{g_s} B_{\text{NS}} \right). \tag{5.2.2}$$

In the undeformed theory with the prepotential (5.1.2), the gauge coupling was a constant t_0 . This translates to the statement that, classically, as the ALE space is fibered over the x -plane, the Kähler modulus of the \mathbb{P}^1 (in particular the B -fields in (5.2.2)) does *not* vary with x . In the extended Seiberg-Witten theory, the complexified gauge coupling becomes ϕ -dependent. Since the adjoint scalar ϕ parameterizes the positions of the D5 branes in the x plane, making the gauge coupling ϕ -dependent should correspond to letting the background B -fields in (5.2.2) be x -dependent,

$$B(x) = \int_{S_x^2} \left(B_{\text{RR}} + \frac{i}{g_s} B_{\text{NS}} \right), \tag{5.2.3}$$

where the integral on the right hand side is over the S^2 at a point in the x plane. In order to reproduce the gauge theory, we require

$$B(x) \rightarrow B_0(x) = \alpha(x) = \sum_{k=0}^{n-1} t_k x^k. \quad (5.2.4)$$

To summarize, the gauge theory in section 2 is realized as the low-energy limit of N D5 branes wrapped on an $A_1 \times \mathbb{C}$ singularity with H -flux turned on,

$$\int_{S_x^2} H_0 = dB_0(x) \neq 0. \quad (5.2.5)$$

It may seem surprising that turning on H -flux does not break supersymmetry down to $\mathcal{N} = 1$.⁵² In the case at hand, the flux we are turning on is due to a B -field that varies *holomorphically* over the complex x -plane. It is known that if the B -field varies holomorphically, the full $\mathcal{N} = 2$ supersymmetry is preserved [116,110].

As was explained in [18], turning on a superpotential $\text{Tr}W(\Phi)$ for the adjoint chiral superfield, as in (5.1.5), corresponds in the geometry to fibering the ALE space over complex x -plane nontrivially,

$$uv = y^2 - W'(x)^2, \quad (5.2.6)$$

where

$$W(x) = \sum_{k=1}^{n+1} a_k x^k.$$

The resulting manifold is a Calabi-Yau threefold and supersymmetry is broken to $\mathcal{N} = 1$. After turning on $W(x)$, the minimal S^2 's (the holomorphic \mathbb{P}^1 's) are isolated at n points in the x -plane, $x = e_i$, which are critical points of the superpotential,

$$W'(x) = g \prod_{i=1}^n (x - e_i).$$

At each of these points, the geometry develops a conifold singularity, which is resolved by a minimal \mathbb{P}^1 . The gauge theory vacuum where the gauge symmetry is broken as in (5.1.9) corresponds to choosing N_i of the D5 branes to wrap the i 'th \mathbb{P}^1 . In particular, the tree-level gauge coupling for the branes wrapping the \mathbb{P}^1 at $x = e_i$ is given by

$$\int_{\mathbb{P}^1_i} B_0 = \left(\frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2} \right)_i = \alpha(e_i), \quad (5.2.7)$$

⁵² The fact that it preserves at least $\mathcal{N} = 1$ supersymmetry is clear for a holomorphic B -field, since the variation of the superpotential $W = \int H \wedge \Omega$ with respect to variations of Ω vanishes if H is holomorphic.

which agrees with the classical values in the gauge theory.

In summary, we can engineer the $\mathcal{N} = 1$ theory obtained from the extended $\mathcal{N} = 2$ theory by the addition of a superpotential $W(\Phi)$ with N D5 branes wrapping the S^2 in the Calabi-Yau (5.2.6), with background flux H_0 . In the next section, we will study the closed-string dual of this theory.

5.3. The Closed String Dual

The open-string theory on the D5 branes has a dual description in terms of pure geometry with fluxes. The gauge theory on the D5 branes which wrap the \mathbb{P}^1 's develops a mass gap as it confines in the IR. The confinement of the open-string degrees of freedom can be thought of as leading to the disappearance of the D5 branes themselves. This has a beautiful geometric realization [18] which we review presently.

In flowing to the IR, the D5 branes deform the geometry around them so that the \mathbb{P}^1 's they wrap get filled in, and the S^3 's surrounding the branes get finite sizes. This is a conifold transition for each minimal S^2 , after which the geometry is deformed from (5.2.6) to

$$uv = y^2 - W'(x)^2 + f_{n-1}(x), \quad (5.3.1)$$

where $f_{n-1}(x)$ is a polynomial in x of degree $n - 1$. This has n coefficients which govern the sizes of the n resulting S^3 's.

In addition, there is H -flux generated in the dual geometry,

$$H = H_{\text{RR}} + \frac{i}{g_s} H_{\text{NS}}.$$

Before the transition, the S^3 's were contractible and had RR fluxes through them due to the enclosed brane charge. After the transition, they are no longer contractible, but the fluxes must remain. In other words we expect the disappearance of the branes to induce (log-)normalizable RR flux, localized near the branes' previous locations, which we denote by H_{RR} . If we denote the S^3 that replaces the k 'th S^2 by A_k -cycles, then

$$\oint_{A_k} H_{\text{RR}} = N_k. \quad (5.3.2)$$

It is also natural to expect that there will be no H_{RR} flux through the B_k -cycles, as there were no branes to generate it. In other words

$$\int_{B_k} H_{\text{RR}} = 0. \quad (5.3.3)$$

In addition to the induced flux H_{RR} , we have a background flux H_0 due to the variation of the background B_0 field, which was present even when there

were no branes, and which we denote by $H_0 = dB_0$. Thus we expect the total flux after the transition to be given by

$$H = H_{\text{RR}} + dB_0.$$

Note that before the transition, there are no compact three-cycles, and so there is no compact flux associated with dB_0 . It is then natural to postulate that after the transition, dB_0 will have no net flux through any of the compact three-cycles. Moreover, far from the branes, we expect B_0 to be given by its value before the transition. For the non-compact three-cycles in the dual geometry, denoted by B_k , we can then explicitly evaluate the periods of H_0 ,

$$\int_{B_k} H_0 = \int_{B_k} dB_0 = \oint_{S^2_{\Lambda_0}} B_0 = B(\Lambda_0). \quad (5.3.4)$$

Because these cycles are non-compact, the integral is regulated by the introduction of a long distance cutoff Λ_0 in the geometry. As usual, we identify this scale with the UV cutoff in the gauge theory.

To summarize, the total flux $H = H_{\text{RR}} + dB_0$ after the transition should be determined by the following facts: H_{RR} is (log-)normalizable, with only nonzero A_k periods (given by N_k), and far from the branes, B_0 is given by its background value (5.2.4), i.e.,

$$dB_0 \sim d\alpha(x) = \sum_{k=1}^{n-1} kt_k x^{k-1}.$$

The fact that the deformed background flux is given by an exact form dB_0 emphasizes the fact that it is cohomologically trivial and has no nonzero periods around compact three-cycles.

The striking aspect of the duality is that in the dual geometry, the gaugino superpotential \mathcal{W}_{eff} becomes purely classical. We will turn to its computation in the next subsection.

5.3.1. The effective superpotential

The effective superpotential is classical in the dual geometry and is generated by fluxes,

$$\mathcal{W}_{\text{eff}} = \int_{CY} (H_{\text{RR}} + H_0) \wedge \Omega,$$

where Ω is a holomorphic three-form on the Calabi-Yau,

$$\Omega = \frac{dx \wedge dy \wedge dz}{z}.$$

This has a simpler description as an integral over the Riemann surface Σ which is obtained from (5.3.1) by setting the $u, v = 0$:

$$0 = y^2 - W'(x)^2 + f_{n-1}(x). \quad (5.3.5)$$

The Riemann surface Σ is a double cover of the complex x -plane, branched over n cuts. The three-cycles A_k and B_k of Calabi-Yau threefold descend to one-cycles on the Riemann surface Σ , with A_k cycles running around the cuts and B_k cycles running from the branch points to the cutoff (see figure 5.2). In addition, H_{RR} descends to a one-form on Σ with periods (5.3.2), (5.3.3). Moreover, Ω descends to a one form on Σ , given by

$$ydx,$$

where y solves (5.3.5). The effective superpotential then reduces to an integral over the Riemann surface,

$$\mathcal{W}_{\text{eff}} = \int_{\text{CY}} (H_{\text{RR}} + H_0) \wedge \Omega = \int_{\Sigma} (H_{\text{RR}} + dB_0) \wedge ydx. \quad (5.3.6)$$

The one-form H_{RR} is defined by its periods

$$\oint_{A_i} H_{\text{RR}} = N_i, \quad \int_{B_i} H_{\text{RR}} = 0,$$

and the asymptotic behavior of B_0 is determined by

$$dB_0(x) \sim \pm d\alpha(x),$$

where \pm correspond to the values of the one-form on the top and bottom sheets of Σ .

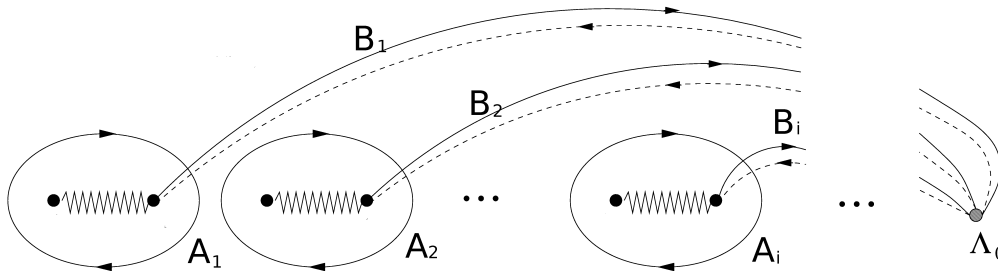


Fig. 5.2. The Calabi-Yau threefold (5.3.1) projects to the x -plane by setting $u = v = 0$. This can be described as a multi-cut Riemann surface Σ , where the nontrivial three-cycles of the Calabi-Yau reduce to one-cycles as drawn.

The evaluation of the superpotential is now straightforward. Using the Riemann bilinear identities, we can evaluate the first term,

$$\int_{\Sigma} H_{\text{RR}} \wedge ydx = \sum_{k=1}^n \oint_{A_k} H_{\text{RR}} \int_{B_k} ydx - \oint_{A_k} ydx \int_{B_k} H_{\text{RR}} = \sum_{k=1}^n N^k \frac{\partial \mathcal{F}_0}{\partial S_k}$$

where

$$\oint_{A_k} y dx = S_k, \quad \int_{B_k} y dx = \frac{\partial \mathcal{F}_0}{\partial S_k},$$

and \mathcal{F}_0 is the genus 0 prepotential of the Calabi-Yau. The background contribution to the superpotential is also straightforward to evaluate, since there are no internal periods for the flux,

$$\int_{\Sigma} dB_0 \wedge y dx = \oint_P B_0(x) y dx \sim \pm \sum_{k=1}^n \oint_{A_k} \alpha(x) y dx,$$

where the last equality follows from the fact that $B_0(x) = \alpha(x)$ for large x by Cauchy's theorem (since the cycle around P is homologous to the sum of all the A_k -cycles).

Thus, the full effective superpotential is

$$\mathcal{W}_{\text{eff}} = \sum_{k=1}^n N_k \frac{\partial}{\partial S_k} \mathcal{F}_0 + \oint_{A_k} \alpha(x) y dx. \quad (5.3.7)$$

This expression is in line with our intuition from the open-string description. Namely, to the leading order we have

$$\oint_{A_k} \alpha(x) y dx \sim \alpha(e_k) S_k + \dots$$

where the omitted terms are higher order in S_i . To this approximation, the superpotential is given by

$$\mathcal{W}_{\text{eff}} \sim \sum_k \alpha(e_k) S_k + N_k \frac{\partial \mathcal{F}_0}{\partial S_k} + \dots$$

Note that the first term above comes from the classical superpotential of the gauge theory, since the A_i -cycle periods S_i in the geometry are identified with glueball superfields in the gauge theory. The coefficient of S_i in the effective superpotential is the microscopic gauge coupling of the $U(N_i)$ gauge group factor in the low energy effective field theory. This is precisely equal to the B -field on the S^2 wrapped by the branes (5.2.7).

However, this cannot be the whole story. After the deformation, the location of the \mathbb{P}^1 is no longer well defined, as the \mathbb{P}^1 at the point $x = e_k$ has disappeared and been replaced by an S^3 which is a branch cut on the x -plane. The geometry has been deformed around the branes and the two sheets of the Riemann surface connect through a smooth throat. We need to specify *where* the gauge coupling is to be evaluated, and since the point in the x -plane has been replaced by a throat, the most natural guess is that we smear the B -field

over the cuts. This is precisely what (5.3.7) does! In the appendix, we provide more details for the derivation of (5.3.7) based on the use of the Riemann bilinear identities.

In the next section, we will show that the same effective superpotential follows from a direct gauge theory computation. Moreover, we will relate the gauge theory computation to an effective matrix model. We will also give a more explicit expression for \mathcal{W}_{eff} ,

$$\mathcal{W}_{\text{eff}} = \sum_{k=0}^{n-1} t_k \frac{\partial}{\partial a_k} \mathcal{F}_0 + N_k \frac{\partial}{\partial S_k} \mathcal{F}_0, \quad (5.3.8)$$

which arises from the following nontrivial identity that we prove in section 5 using the formulation of the topological string in terms of matrix models [58]:

$$\oint_P \alpha(x) y dx = \sum_{k=0}^{n-1} t_k \frac{\partial}{\partial a_k} \mathcal{F}_0.$$

Equations (5.3.7) and (5.3.8) agree with the results of [109,111].

The form of the superpotential (5.3.8) suggests a dual role played by (a_k, S_k) and (t_k, N_k) – indeed it suggests a formulation in terms of fluxes [17] (see also [113,96]). We can think of the fluxes N_k as turning on auxiliary fields for the S_k superfields in the $\mathcal{N} = 2$ effective theory, where S_k is the lowest component of the superfield,

$$S_k \rightarrow S_k + \dots + 2iN_k \theta_2 \theta_2 + \dots$$

The $\mathcal{N} = 1$ superpotential arises by the integration over half of the chiral $\mathcal{N} = 2$ superspace

$$\int d^4\theta \mathcal{F}_0(S_k) = \int d^2\theta N_k \frac{\partial \mathcal{F}_0}{\partial S_k} + \dots$$

Similarly, we can view the background parameters a_k as scalar components of non-normalizable superfields, and the t_k as the corresponding fluxes leading to vevs for their associated auxiliary fields,

$$a_k \rightarrow a_k + \dots + 2it_k \theta_2 \theta_2 + \dots$$

Thus the full superpotential can be obtained from the $\mathcal{N} = 2$ formulation simply by giving vevs (t_k, N_k) to the auxiliary fields of (a_k, S_k) .

5.3.2. Extrema of the superpotential

With the closed-string dual of our gauge theory identified, we turn to the extremization of the flux superpotential. We wish to solve

$$\frac{\partial \mathcal{W}_{\text{eff}}}{\partial S_k} = \int_{\Sigma} (H_{\text{RR}} + H_0) \wedge \frac{\partial}{\partial S_k} y dx = 0. \quad (5.3.9)$$

From (5.3.8) this can be written as

$$\sum_{i=0}^{n-1} t_i \eta_{ik} = \sum_{i=1}^n N_i \tau_{ik} \quad (5.3.10)$$

where η is an $n \times n$ matrix,

$$\eta_{ik} = \frac{\partial^2 \mathcal{F}_0}{\partial a_i \partial S_k} \quad (5.3.11)$$

and τ_{ik} is the usual period matrix,

$$\tau_{ik} = \frac{\partial^2 \mathcal{F}_0}{\partial S_i \partial S_k}. \quad (5.3.12)$$

Note that for a fixed choice of Higgs branch, specified by N_i , the number of parameters specifying the choice of $B_0(x)$ and the number of parameters determining the normalizable deformations of the geometry, given by $f_{n-1}(x)$, are both equal to n . Therefore we would expect to generically have a one-to-one map. This allows us to invert the problem. Instead of asking how B_0 determines f_{n-1} , i.e.,

$$B_0 \rightarrow f_{n-1},$$

we can instead ask for which choice of $B_0(x)$ we obtain a given deformed geometry, $f_{n-1}(x)$, i.e.,

$$B_0 \leftarrow f_{n-1}.$$

In this formulation, the extremization problem has a simple solution. We choose a set of complex structure moduli for the Riemann surface,

$$y^2 = (W'(x; a))^2 - f_{n-1}(x; a, S),$$

by picking values for the S_i (or equivalently for the coefficients of f_{n-1}). This completely determines the matrices τ_{ij} and η_{ij} through (5.3.11) and (5.3.12). The equations (5.3.9), (5.3.10) can then be thought of as n linear equations for the n coupling constants $\{t_i\}_{i=0}^{n-1}$, thus determining $B_0(x)$.

The equations (5.3.9) determine the explicit form of the flux $H_{\text{RR}} + H_0$ on the solution. Recall that, off-shell, $H_{\text{RR}} + H_0$ was defined by its compact periods,

$$\oint_{A_i} H_{\text{RR}} + H_0 = N_i \quad \int_{B_i} H_{\text{RR}} + H_0 = \alpha(\Lambda_0), \quad (5.3.13)$$

and asymptotic behavior for large x ,

$$H_{\text{RR}} + H_0(x) \sim \pm dB(x).$$

The equations of motions (5.3.9) then imply that the one-form $H_{\text{RR}} + H_0$ is holomorphic on the punctured Riemann surface $\Sigma - \{P, Q\}$, and given by

$$H_{\text{RR}} + H_0 = \sum_{k=1}^n N_k \frac{\partial}{\partial S_k} y dx - \sum_{k=0}^{n-1} t_k \frac{\partial}{\partial a_k} y dx. \quad (5.3.14)$$

Above, P and Q correspond to points at infinity of the top and the bottom sheet of the Riemann surface, and

$$\frac{\partial}{\partial S_k} y dx$$

are linear combinations of the $n - 1$ holomorphic differentials on Σ ,

$$\frac{x^k dx}{y}, \quad k = 0, \dots, n - 2,$$

together with $x^{n-1} dx/y$, which has a pole at infinity.

To derive this, we note that (5.3.9) implies that $H_{\text{RR}} + H_0$ is orthogonal to the complete set of holomorphic differentials in the interior. This implies that $H_{\text{RR}} + H_0$ is holomorphic away from the punctures. We can also show that (5.3.14) has the correct periods and asymptotic behavior. Consider the periods of $\omega_i = \frac{\partial}{\partial S_i} y dx$,

$$\oint_{A_k} \omega_i = \delta_i^k, \quad \int_{B_k} \omega_i = \tau_{ik} \quad (5.3.15)$$

and the periods of $\rho_i = \frac{\partial}{\partial a_i} y dx$,

$$\oint_{A_k} \rho_i = 0, \quad \int_{B_k} \rho_i = \eta_{ik} + \Lambda_0^i. \quad (5.3.16)$$

The reason for the Λ_0^i term in (5.3.16) is that $\frac{\partial \mathcal{F}_0}{\partial S_i}$ is the B_i -period with boundary term subtracted. The A_k periods also match – this is because the $\frac{\partial}{\partial a_k}$ derivative is taken at fixed S_k , per definition. Using these periods and (5.3.10), we immediately see that (5.3.14) has the correct periods (5.3.13). It is also clear that the large x behavior is dominated by ρ_i and this yields $d\alpha(x)$ for the large x behavior of $H_{\text{RR}} + H_0$ as required.

5.4. Gauge Theory Derivation

In this section we will sketch the derivation of the effective glueball superpotential directly in the gauge theory language, and show that this exactly reproduces the results of the string theoretic derivation. In [57] the effective superpotential for the glueball superfields was computed by explicitly integrating out the chiral superfield Φ . This is possible as long as we are only interested in the chiral $\int d^2\theta$ terms in the effective action. In the absence of the deformation (5.1.6), computation of the relevant gauge theory Feynman graphs with Φ running around loops directly translates into the computation of planar diagrams in a certain auxiliary matrix integral. We will see that this is the case even after the deformation, albeit with a novel deformation of the relevant matrix integral.

Let us review the results of [57]. For simplicity, consider the vacuum where the $U(N)$ gauge symmetry is unbroken. The propagators for Φ can be written in the Schwinger parameterization as

$$\int ds_i \exp[-s_i(p_i^2 + \mathcal{W}^\alpha \pi_\alpha + m)],$$

where s_i are the Schwinger times, p_i are the bosonic momenta, and π_α the fermionic momenta. The mass parameter m is given by $m = W''(\phi_0)$. These propagators have the property that each Φ loop brings down two insertions of the glueball superfield \mathcal{W}_α . Using the chiral ring relation

$$\{\mathcal{W}_\alpha, \mathcal{W}_\beta\} \sim 0, \tag{5.4.1}$$

only those operators of the form

$$S^k = (\text{Tr} \mathcal{W}_\alpha \mathcal{W}^\alpha)^k$$

are nontrivial as F-terms. In particular, there must be *at most* two insertions of \mathcal{W}_α per index loop. This implies that only planar Φ -diagrams contribute to the superpotential – nonplanar graphs have fewer index loops than momentum loops.

The integration over bosonic and fermionic loop momenta in a planar diagram with h holes gives a constant factor,

$$NhS^{h-1}, \tag{5.4.2}$$

independent of the details of the diagram. The planar graphs have one more index loop (hole) than momentum loop, and there is one insertion of S per momentum loop, with h choices of which index loop goes unoccupied. At the same time, the index summation for the unoccupied loop leads to the factor of N .

The rest of the computation, namely combinatorial factors, contributions of vertices, and an additional factor of $1/m^{h-1}$ from the propagators, is captured by a zero-dimensional, auxiliary holomorphic matrix theory with path integral

$$Z_M = \frac{1}{\text{Vol}(U(M))} \int d\Phi \exp(-\text{Tr} W(\Phi)/g_{\text{top}}), \tag{5.4.3}$$

where Φ is an $M \times M$ matrix, and $W(\Phi)$ is the same superpotential as in (5.1.5). The coefficient

$$\mathcal{F}_{0,h}$$

of (5.4.2) is computed by summing over the planar graphs of Z_M with h holes and extracting the coefficient of $M^h g_{\text{top}}^{h-2}$. In other words, by rewriting the sum

$$\mathcal{F}_0(S) = \sum_h \mathcal{F}_{0,h} S^h$$

where

$$Z_M \sim \exp(-\mathcal{F}_0/g_{\text{top}}^2).$$

In the semiclassical approximation, the effective superpotential of the undeformed theory is simply

$$\mathcal{W}_{\text{eff}} = t_0 S + N \partial_S \mathcal{F}_0(S).$$

In the full answer, \mathcal{F}_0 contains a $\frac{1}{2} S^2 \log S$ piece which, in the matrix model, comes from the volume of the gauge group in (5.4.3).

5.4.1. The deformed matrix model

Now consider the gauge theory with the more general tree-level superpotential (5.1.6) (for a special form of the superpotential, this theory was studied in [111]). In this case, the propagators of the theory are unchanged, but there are now additional vertices coming from the first term in (5.1.6). What is the effect of this? Clearly, it is still only the planar graphs that can contribute to the amplitude, since nonplanar graphs still have too few index loops to absorb the \mathbb{W}_α insertions. This, together with (5.4.1), implies that the extra vertices from $\text{Tr}[\alpha(\Phi)\mathcal{W}_\alpha\mathcal{W}^\alpha]$ can only be brought down once for each planar graph, where they are inserted on the sole index loop that would have otherwise been unoccupied. The prescription for extracting the contributions of these new graphs from the matrix model is now clear. Consider the deformed matrix model

$$Z_M = \frac{1}{\text{Vol}(U(M))} \int d\Phi \exp(-\text{Tr}W(\Phi)/g_{\text{top}} + \text{Tr} \Lambda \alpha(\Phi)/g_{\text{top}}), \quad (5.4.4)$$

where the matrix Λ stands for $\mathcal{W}_\alpha\mathcal{W}^\alpha$ insertions that do not come from the propagators. Summing over planar graphs, the matrix integral now has the form

$$Z_M \sim \exp(-\mathcal{F}_0/g_{\text{top}}^2 - \text{Tr} \Lambda \mathcal{G}_0/g_{\text{top}} + \dots)$$

where the omitted terms contain higher powers of traces of Λ that will not play any role. The effective superpotential, including the contribution of the new vertices from $\text{Tr}[\alpha(\Phi)\mathcal{W}_\alpha\mathcal{W}^\alpha]$, is now

$$\mathcal{W}_{\text{eff}} = S \mathcal{G}_0(S) + N \partial_S \mathcal{F}_0(S).$$

Note that it is manifest in the matrix model that the effective superpotential is invariant under the addition to $\alpha(\Phi)$ of terms the form $\Phi^k W'(\Phi)$, as mentioned in section 2. These terms can be removed by a shift in Φ

$$\Phi \rightarrow \Phi + \Lambda \Phi^k,$$

and as such they do not affect the matrix integral.

It is easy to generalize this to vacua of the gauge theory where the gauge group is broken as in (5.1.9). The superpotential in these vacua is computed by the same matrix model, but where one now considers the perturbative expansion about the more general vacuum, where the gauge symmetry of the matrix model is broken to $\prod_{k=1}^n U(M_k)$ [60]. The contributions of insertions of

$$\text{Tr}[\alpha(\Phi_k)\mathcal{W}_{\alpha,k}\mathcal{W}_k^\alpha]$$

are now captured by deforming the matrix model to

$$Z_M = \frac{1}{\prod_k \text{Vol}(U(M_k))} \int \prod_k d\Phi_k \dots \exp\left(-\frac{1}{g_{\text{top}}} \sum_k (\text{Tr}W(\Phi_k) + \text{Tr}\Lambda_k \alpha(\Phi_k))\right)$$

where the omitted terms \dots are gauge fixing terms [60] corresponding to the choice for Φ to be block diagonal, and breaking the gauge symmetry to $\prod_k U(M_k)$. Summing over the planar graphs returns

$$Z_M \sim \exp\left(-\mathcal{F}_0/g_{\text{top}}^2 - \sum_k \text{Tr}\Lambda_k \mathcal{G}_{0,k}/g_{\text{top}} - \dots\right)$$

where \mathcal{F}_0 and $\mathcal{G}_{0,k}$ are functions of the matrix model 't Hooft couplings $g_{\text{top}}M_k$. These are identified with the glueballs S_i in the physical theory. The effective superpotential is now given by

$$\mathcal{W}_{\text{eff}} = \sum_k S_k \mathcal{G}_{0,k} + N_k \partial_{S_k} \mathcal{F}_0,$$

and all that remains is to compute the new terms in $\mathcal{G}_{0,k}$.

5.4.2. Matrix model computation

Now let us compute the relevant correction from the matrix model. Since we are only interested in the planar graphs linear in $\text{Tr}\Lambda_k$, the contribution of interest can be extracted from the special case where we choose

$$\Lambda_k = \lambda_k \mathbf{1}_{M_k \times M_k}$$

The matrix model partition function then becomes

$$Z_M = \int \dots \exp\left(-\sum_k \lambda_k \text{Tr}\alpha(\Phi_k)/g_{\text{top}}\right) \sim \exp\left(-\mathcal{F}_0/g_{\text{top}}^2 - \sum_k M_k \lambda_k \mathcal{G}_{0,k}/g_{\text{top}}\right)$$

which implies

$$\mathcal{G}_{0,k} = \langle \text{Tr}[\alpha(\Phi_k)] \rangle / M_k,$$

where the expectation value is evaluated in the planar limit of the $\prod_k U(M_k)$ vacuum of the undeformed matrix model. These can be computed using well known large M matrix model saddle point techniques [19,59]. The answer can be formulated in terms of a Riemann surface,

$$y^2 - (W'(x))^2 + f_{n-1}(x) = 0,$$

with a one-form ydx , where the coefficients of f_{n-1} are chosen so that

$$M_k g_{\text{top}} = \oint_{A_k} ydx.$$

Namely, the result is that

$$\langle \text{Tr } \alpha(\Phi_k) \rangle = \frac{1}{g_{\text{top}}} \oint_{A_k} \alpha(x) ydx.$$

Since the glueballs S_k are identified with $M_k g_{\text{top}}$ in the matrix model, we can write the corresponding contribution to the effective superpotential

$$\delta \mathcal{W}_{\text{eff}} = \sum_k S_k \mathcal{G}_{0,k}$$

simply as

$$\delta \mathcal{W}_{\text{eff}} = \sum_k \oint_{A_k} \alpha(x) ydx.$$

A look back at (5.3.7) shows that this agrees with the result of our string theoretic analysis. Moreover, this is consistent with the results of [108] for the expectation values of the corresponding chiral ring elements.

In the next subsection, we will use matrix model technology to derive the identity (5.3.8) for expressing $\delta \mathcal{W}_{\text{eff}}$, as a function of S_k .

5.4.3. Evaluation of $\delta \mathcal{W}_{\text{eff}}$

To begin with, note that $\delta \mathcal{W}_{\text{eff}}$ can be rewritten as

$$\delta \mathcal{W}_{\text{eff}} = \sum_k \frac{S_k \langle \text{Tr } \alpha(\Phi_k) \rangle}{M_k} = g_{\text{top}} \sum_k \langle \text{Tr } \alpha(\Phi_k) \rangle = g_{\text{top}} \langle \text{Tr } \alpha(\Phi) \rangle$$

where the trace is over the $M \times M$ matrix Φ .⁵³ The expectation value is now straightforward to compute. The problem amounts to the computation of

$$\langle \text{Tr } \Phi^k \rangle, \quad k = 0, \dots, n-1$$

⁵³ This leads to the same expression (5.3.7) for the large M average using $y(x) = W'(x) + g_{\text{top}} \langle \frac{1}{x-\Phi} \rangle$, and the fact that the sum over the A_k -cycles is homologous to the cycle around infinity in x -plane.

in the matrix model. Recall that

$$W(\Phi) = \sum_{k=0}^{n+1} a_k \Phi^k,$$

which implies that, for $k = 0, \dots, n-1$

$$\langle \text{Tr} \Phi^k \rangle = -\frac{g_{\text{top}}}{Z_M} \frac{\partial Z_M}{\partial a_k}$$

with Z_M as defined in (5.4.3). In particular, since

$$Z_M \sim \exp\left(-\frac{1}{g_{\text{top}}^2} \mathcal{F}_0(S, a)\right),$$

it follows that

$$\langle \text{Tr} \Phi^k \rangle = \frac{1}{g_{\text{top}}} \frac{\partial \mathcal{F}_0}{\partial a_k}.$$

Thus we have derived (5.3.8),

$$\delta \mathcal{W}_{\text{eff}} = \sum_{k=0}^{n-1} t_k \frac{\partial \mathcal{F}_0}{\partial a_k}.$$

5.5. The $\mathcal{N} = 2$ Gauge Theory

5.5.1. Extended Seiberg-Witten theory

With the results of the previous section in hand, we are now in position to recover the solution to the extended $\mathcal{N} = 2$ theory with classical prepotential

$$\mathcal{F}(\Phi) = \sum_{k=0} \frac{t_k}{(k+1)(k+2)} \text{Tr} \Phi^{k+2}. \quad (5.5.1)$$

The analysis of this section closely mirrors the approach taken in [101], and the results also follow from [109].

To begin with, consider a special case of the $\mathcal{N} = 1$ theories studied in the previous section. We deform the extended $U(N)$ $\mathcal{N} = 2$ theory (5.5.1) to $\mathcal{N} = 1$ by the addition of a degree $N+1$ superpotential,

$$W(\Phi) = \sum_{k=0}^{N+1} a_k x^k \quad (5.5.2)$$

with

$$W'(\Phi) = g \prod_{k=1}^N (x - e_k). \quad (5.5.3)$$

In particular, we now study a generic vacuum on the Coulomb branch of the theory, where the gauge symmetry is broken as

$$U(N) \rightarrow U(1)^N.$$

This is important, because if we now take the limit of vanishing superpotential (5.5.2) while keeping the expectation value of the adjoint fixed,

$$g \rightarrow 0, \quad e_k = \text{const},$$

we expect to recover the $\mathcal{N} = 2$ vacuum at the same point in moduli space. As discussed in section 3, this corresponds in string theory language to reverting to studying N D5 branes on the \mathbb{P}^1 in the A_1 ALE space, but with a holomorphically varying B -field turned on. The nontrivial B -field background corresponds in the low energy theory on the branes to turning on the higher dimensional terms in the classical prepotential (5.5.1).

We found in section 4 that the critical point of this theory corresponds to a Riemann surface

$$y^2 = (W'(x; a))^2 - f_{N-1}(x; S, a) \quad (5.5.4)$$

where the N parameters t_k in (5.5.1) are determined in terms of the complex structure moduli S_i of (5.5.4) by extremizing the superpotential (5.3.9). Moreover, at the critical point, the net flux $H_{\text{RR}} + H_0$ is given by a holomorphic one-form on the Riemann surface (5.5.4),

$$H_{\text{RR}} + H_0 = \sum_{k=1}^N \frac{\partial}{\partial S_k} y dx - \sum_{k=0}^{N-2} t_k \frac{\partial}{\partial a_k} y dx, \quad (5.5.5)$$

with periods

$$\begin{aligned} \oint_{A_i} H_{\text{RR}} + H_0 &= 1 \\ \int_{B_i} H_{\text{RR}} + H_0 &= \alpha(\Lambda_0) \\ \oint_P x^{-k} (H_{\text{RR}} + H_0) &= k t_k, \quad k = 1, \dots, N-2. \end{aligned}$$

It turns out that all of the holomorphic information about the $\mathcal{N} = 2$ theory in the infrared can be recovered from calculations in the $\mathcal{N} = 1$ theory, just as in [101]. To observe this, we note that if we extract an overall factor of g from y in (5.5.4) and use new g -independent functions $\widetilde{W} \equiv \frac{1}{g} W$ and $\widetilde{f}_{N-1} \equiv \frac{1}{g^2} f_{N-1}$, then

$$y = g \sqrt{\widetilde{W}(x)^2 + \widetilde{f}_{N-1}(x)},$$

and the periods of y have a trivial g -dependence. In particular,

$$\frac{1}{g}S_i, \quad \frac{1}{g}\frac{\partial\mathcal{F}_0}{\partial S_i},$$

are independent of g . Consequently, the period matrix

$$\tau_{ij} = \frac{\partial^2\mathcal{F}_0}{\partial S_i\partial S_j} = \frac{\partial}{\partial(S_i/g)}\left(\frac{1}{g}\frac{\partial\mathcal{F}_0}{\partial S_j}\right)$$

is independent of g . This fact can be made more manifest by considering the geometry in question,

$$\frac{y^2}{g^2} = \widetilde{W}(x)^2 + \widetilde{f}_{N-1}(x).$$

It is clear that the variation of g can just be absorbed into a rescaling of the coordinate y .

It is also crucial that in the process of sending $g \rightarrow 0$, the values of t_k for which the Riemann surface in question satisfies the equations of motion remain fixed. The superpotential

$$W_{\text{eff}} = \int_{\Sigma} (H_{\text{RR}} + H_0) \wedge y dx$$

is simply proportional to g , and hence its critical points are g -independent.

Lastly, we note that the Seiberg-Witten one-form on the Riemann surface can be recovered from the $\mathcal{N} = 1$ analysis as well. First note that the H -flux $H_{\text{RR}} + H_0$ at the critical point of the superpotential is given by a g -independent holomorphic one-form (5.5.5). Just as in [101], it follows that the Seiberg-Witten one-form on the Riemann surface is given by

$$\lambda_{\text{SW}} = x(H_{\text{RR}} + H_0) \tag{5.5.6}$$

which we can read off from the $\mathcal{N} = 1$ theory. This can be seen as follows. Periods of λ_{SW} compute the masses of dyons in the $\mathcal{N} = 2$ theory. However, these dyons can be identified with D3 branes wrapping Lagrangian three-cycles in the Calabi-Yau, or one-cycles on the Riemann surface, and their mass can be derived from string theory to be given by periods of the one-form (5.5.6).

In summary, we can obtain the full $\mathcal{N} = 2$ curve and the Seiberg-Witten one-form λ_{SW} that capture the low energy dynamics of the extended $\mathcal{N} = 2$ theory (5.5.1). These results are consistent with those obtained recently in [117] using very different techniques. There, the authors formulate the solution of the $\mathcal{N} = 2$ theory in terms of a hyperelliptic curve of genus $N - 1$

$$y^2 = \prod_{i=1}^N (x - a_{i,+})(x - a_{i,-}), \tag{5.5.7}$$

and a holomorphic one-form $d\Phi$ with the properties that

$$\begin{aligned} \oint_{A_i} d\Phi &= 1 \\ \int_{B_i} d\Phi &= 0 \\ \oint_P x^{-k} d\Phi &= kt_k, \quad k = 1, \dots, N-2 \end{aligned}$$

and which is related to the Seiberg-Witten one-form by

$$\lambda_{\text{SW}} = xd\Phi.$$

Comparing with our results, it is clear that $d\Phi$ should be identified with $H_{\text{RR}} + H_0$.

The agreement is almost complete, apart from two points. First, our Seiberg-Witten curve (5.5.4) is not a generic genus N hyperelliptic curve like (5.5.7), but rather is one where all the dependence on the parameters t_k is in the polynomial $f_{N-1}(x)$ of degree $N-1$. More precisely, note that the defining equation of the hyperelliptic curve has $2N$ parameters and generally all such parameters appear. However, half the parameters correspond to the choice of the point on the Coulomb branch e_i , while the other half define the quantum deformation which depends on the choice of the $\alpha(x)$. In our formulation, there is a natural way to separate how these parameters appear in the defining equation of the Seiberg-Witten curve. Secondly, there is an apparent discrepancy in that in the current solution, the B_i periods of $H_{\text{RR}} + H_0$ do not all vanish, but are instead equal to $\alpha(\Lambda_0)$. It is possible that in the definition of the B_k integrals (5.5.1) of [117], there is a hidden subtraction of the value of the integral at infinity, which would account for the vanishing B_k periods and resolve this discrepancy.

5.6. Duality and Supersymmetry Breaking

In this section we study the phase structure of the $\mathcal{N} = 1$ models under consideration. We find that there is a region in the parameter space where supersymmetry is broken. This leads to a novel and calculable mechanism for breaking supersymmetry. Even though this method for supersymmetry breaking is motivated by string theoretic considerations, we will see that it can also be phrased entirely in terms of the underlying $\mathcal{N} = 1$ supersymmetric gauge theory.

The organization of this section is as follows. We first discuss some general features of the phase structure for these theories, and point out a region where classical considerations are not sufficient to provide a reasonable picture. We next turn to focus on the meaning of this new phase and show how string

dualities can shed light on its meaning. Furthermore, we show that, generically, supersymmetry is spontaneously broken in the new phase. We propose UV dual field theory descriptions for some of these phases which turn out to be $\mathcal{N} = 1$ supersymmetric gauge theories with supersymmetry broken softly by nonzero expectation values for the auxiliary components of spurion superfields.

5.6.1. Parameter space with $g_{\text{YM}}^2 < 0$

Consider the $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theory studied in the previous sections, with adjoint field Φ together with superpotential $W(\Phi)$, and gauge kinetic term in Lagrangian is captured by $\alpha(\Phi)$ as below

$$\int d^4x d^2\theta \text{Tr} [\alpha(\Phi) \mathcal{W}_\alpha \mathcal{W}^\alpha].$$

As already discussed, the classical vacua correspond to all the ways of distributing the eigenvalues of ϕ among the critical points of $W'(\phi) = 0$. For concreteness, let

$$W'(\Phi) = g \prod_{i=1}^n (\Phi - e_i),$$

and consider the classical vacuum with N_i eigenvalues of Φ equal to e_i . For generic superpotential, Φ will be massive, and at sufficiently low energies the light degrees of freedom describe pure $\mathcal{N} = 1$ supersymmetric Yang-Mills theory with gauge symmetry $\prod_i U(N_i)$. The coupling constant of each of the $U(N_i)$ in the UV is given by

$$\alpha_i = \alpha(e_i).$$

As long as the gauge coupling for each factor of the gauge group is positive, i.e.,

$$\text{Im}[\alpha_i] = \frac{4\pi}{(g_{\text{YM}})_i^2} > 0 \tag{5.6.1}$$

for all i with $N_i \neq 0$, we expect a supersymmetric theory in the IR to which the analysis of the previous section applies. This suggests the question: *What is the meaning of the phase where (5.6.1) is not satisfied for some i ?* It is to this question which we now turn our attention.

One may be inclined to consider such cases as pathological, as one is not able to give a meaning to such a theory in the UV. However, we also know from various examples that the appearance of a negative g_{YM}^2 is often the smoking gun for the existence of a dual description. Thus all we can conclude is that when $\text{Im}[\alpha(e_i)]$ do not have the correct sign, the original UV picture is not appropriate, and we should look for an alternative description.

Generically⁵⁴, for an arbitrary choice of $W(\Phi)$ and $\alpha(\Phi)$, $\text{Im} \alpha(e_i)$ will not have the same sign for all the critical points, and thus some vacua will

⁵⁴ Generic in the sense of generic functions $\alpha(x)$ and $W(x)$. From a field theory perspective, it is natural for the nonrenormalizable operators in $\alpha(\Phi)$ to be suppressed by large mass scales, in which case the phenomenon discussed in this section will be unusual.

have gauge group factors with $g_{\text{YM}}^2 < 0$. We have a practical way to analyze the IR theory in these vacua directly from the field theory approach. We can start with parameters such that the UV theory makes sense, and then compute the effective IR action in terms of the glueball superfields, as discussed in the previous sections. We then change the parameters so that the UV theory would formally develop a negative value of g_{YM}^2 for some of the gauge group factors. However, the effective IR theory *still* makes sense when we do this, so we can simply study the IR action, without worrying about the dual UV description. As we will show, in the IR theory this change of parameters leads to supersymmetry-breaking.

We are thus naturally led to ask: *What is the corresponding UV theory in such cases?* When only some of the gauge couplings are negative, we will argue that supersymmetry is broken, but we will not have a full field theory description in the UV. However, if they are *all* negative, we can formulate a complete UV field theory description for which supersymmetry is manifestly broken. In all cases, the UV description provided by string theory exists, and we will argue that it involves both branes and antibranes.

In the general, these theories have two sources of supersymmetry breaking. One, which comes from any of the gauge factors with negative $\text{Im} \alpha(e_i)$, corresponds to giving a nonzero vev to spurion auxiliary fields. The other effect comes from the fact that when both signs of $\text{Im} \alpha(e_i)$ are present, the interaction between the gauge group factors are not supersymmetric, as each factor tries to preserve a different supersymmetry.

We first study the situation of the first kind – all $\text{Im} \alpha(e_i)$ negative – where the internal dynamics of the gauge theory softly break supersymmetry. For this case, we quantify the supersymmetry-breaking effect in terms of a dimensionless parameter which measures fractional mass splittings in the supermultiplets. Moreover, we motivate and provide strong evidence for the existence of a dual non-supersymmetric UV theory. We motivate this from field theory as well as describing its natural explanation in the context of string theory.

We then move to the multi-sign case and show that when some $\text{Im} \alpha(e_i)$ have different signs, there is an additional effect which breaks supersymmetry. Essentially, this arises from each factor of the gauge group trying to preserve a different half of a background $\mathcal{N} = 2$ supersymmetry, and charged bifundamental matter communicates supersymmetry breaking. For this case, we only have a stringy dual description in the UV.

5.6.2. Negative gauge couplings and duality

We now discuss, from both string theory and field theory perspectives, how a gauge coupling squared becoming negative can be sensibly understood in terms of the dual description. The simple example which we review, where both the original and the dual theories are supersymmetric, has already been studied in [4].

Consider N D5 branes on the resolved conifold geometry with a single \mathbb{P}^1 .

As in section 3, we can view this geometry as obtained by fibering an A_1 ALE singularity over the x -plane as

$$uv = y^2 - W'(x)^2 \tag{5.6.2}$$

where

$$W(x) = \frac{1}{2}mx^2. \tag{5.6.3}$$

We turn on a *constant* B -field through the S^2 at the tip of the ALE space,

$$\alpha = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2} = \int_{S_x^2} (B_{\text{RR}} + \frac{i}{g_s} B_{\text{NS}}).$$

In the language of section 2, this means that the gauge coupling is independent of ϕ . More generally, the effective gauge coupling of the $4d$ $U(N)$ theory is given by $4\pi/g_{\text{YM}}^2 = \sqrt{r^2 + B_{\text{NS}}^2}/g_s$, where r is the physical volume of the \mathbb{P}^1 . This is usually written in terms of single complex variable t , the complexified Kähler class, given by $t = B_{\text{NS}} + ir$, as $4\pi/g_{\text{YM}}^2 = |t|/g_s$. In the present chapter we have permanently set $r = 0$, so $t = B_{\text{NS}}$.

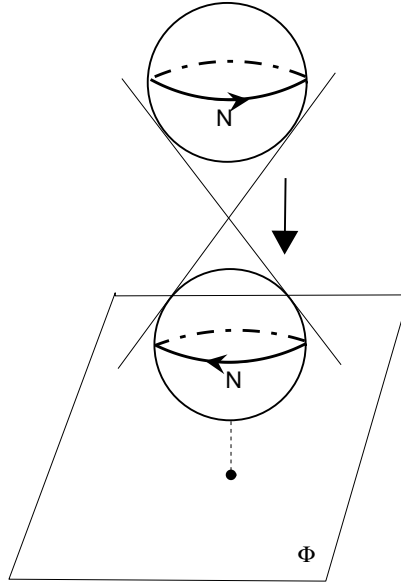


Fig. 5.3. By changing the B -field, an S^2 undergoes a flop, and N branes on the S^2 become N antibranes on the flopped S^2 . If the B -field is constant on x -plane, then the antibrane system preserves an $\mathcal{N} = 1$ supersymmetry *opposite* to that of the brane system. If the B -field varies holomorphically, then the B -field and antibranes preserve orthogonal $\mathcal{N} = 1$ supersymmetries, leading to a stable $\mathcal{N} = 0$ vacuum.

Now consider the same geometry, but with the complexified Kähler class varied so that it undergoes a flop (see figure 5.3), corresponding to $t \rightarrow -t$. We now get a *new* \mathbb{P}^1 . Moreover, the charge of the wrapped D5 branes on this flopped \mathbb{P}^1 is opposite to what it was before the flop. Therefore, in order to

conserve D5 brane charge across the flop, we will end up with anti-D5 branes on the new \mathbb{P}^1 . In the case of constant B -field, we again obtain a $U(N)$ gauge theory with $\mathcal{N} = 1$ supersymmetry at low energies. However, the $\mathcal{N} = 1$ supersymmetry that the theory preserves after the flop has to be *orthogonal* to the original one, since branes and antibranes preserve different supersymmetries.

This stringy duality is directly manifested in field theory. It turns out, as we now review, that this situation has a simple and elegant realization in terms of the glueball superfields which emerge as the IR degrees of freedom. Consider first the situation before the flop. In the IR, we have a deformed conifold geometry where S , the modulus of the deformation, is identified with the glueball superfield, $S = \text{Tr} \mathcal{W}_\alpha \mathcal{W}^\alpha$. The Veneziano-Yankielowicz superpotential, which can be derived in either the field theory or the dual string theory, is given by

$$\mathcal{W}(S) = -\alpha S + N \partial_S \mathcal{F}_0 = -\alpha S + \frac{1}{2\pi i} N S \left(\log \left(\frac{S}{m \Lambda_0^2} \right) - 1 \right).$$

As was already reviewed in previous sections, in the gravitational dual picture, the two terms above correspond to flux contributions to the superpotential. One should note that this effective description is only valid for field values where $|S/m| \ll |\Lambda_0^2|$.

Extremizing \mathcal{W} with respect to S gives

$$\partial_S \mathcal{W} = 0 \rightarrow S^N = (m \Lambda_0^2)^N \exp(2\pi i \alpha). \quad (5.6.4)$$

As long as the bare UV gauge coupling satisfies

$$\text{Im}[\alpha] = \frac{4\pi}{g_{\text{YM}}^2} (4.1.14) < 0,$$

this is an acceptable solution in the sense that S is within the allowed region of field space. Note that in addition to the chiral superfield, the theory in the IR still has a $U(1)$ vector multiplet, because only the $SU(N) \subset U(N)$ is confined. In the string theory construct, the extra $U(1)$ is identified with the reduction of the four-form IIB gauge potential on the deformed S^3 . In other words, this theory describes a massive chiral multiplet consisting of S and its fermionic partner ψ , as well as a massless photon A and its partner λ ,

$$(S, \psi), \quad (A, \lambda). \quad (5.6.5)$$

Together these would form an $\mathcal{N} = 2$ chiral multiplet before the supersymmetry is broken to $\mathcal{N} = 1$ by fluxes.

Now consider the same theory, but in the limit where

$$\text{Im}(\alpha) \ll 0,$$

which *would have corresponded* to $1/g_{\text{YM}}^2 \ll 0$. Then the above solution (5.6.4) is not valid anymore, since $|S/m|(4.1.14)|\Lambda_0^2|$ lies outside the regime of validity of

the effective theory. Thus the original supersymmetry is broken, since we cannot set $\partial_S \mathcal{W}$ to zero. Even so, as was shown in [4], there are still physical vacua which minimize an effective scalar potential V_{eff} . Moreover, the theory in these minima is exactly the same as one would expect for the IR limit of an $\mathcal{N} = 1$ supersymmetric $U(N)$ theory, with a positive squared gauge coupling. In fact, a new supersymmetry does re-emerge! It turns out that ψ becomes the massless goldstino of the original supersymmetry which is broken, whereas λ picks up a mass and becomes the superpartner of S under the new supersymmetry, giving realigned supermultiplets

$$(S, \lambda), \quad (A, \psi). \quad (5.6.6)$$

This beautifully reflects the physics of the string theory construction. After the flop, the D5 branes are replaced by anti-D5 branes, which still give rise to a $U(N)$ gauge theory with $\mathcal{N} = 1$ supersymmetry, albeit a *different* supersymmetry than the original one, explaining the above realignment.

Let us review in more detail how the flop is manifested in the IR field theory of [4]. When $\text{Im}(\alpha) \ll 0$, we must look for critical points of the physical potential

$$V_{\text{eff}} = g^{S\bar{S}} |\partial_S \mathcal{W}|^2. \quad (5.6.7)$$

At leading order, the theory spontaneously breaks an underlying $\mathcal{N} = 2$ supersymmetry, so the tree-level Kähler metric should be determined by special geometry. While we do not expect this to be an exact statement, we nevertheless make the assumption for the remainder of this section that the Kähler metric is that of the $\mathcal{N} = 2$ theory⁵⁵. Thus the action for the IR dual is given by

$$\int d^4x d^2\theta d^2\bar{\theta} \Lambda^{-4} [\bar{S}_i \partial_i \mathcal{F}_0 - c.c.] + \left[\int d^4x d^2\theta W(S_i) + c.c. \right] \quad (5.6.8)$$

where Λ^4 gets identified with M_{string}^4 in the string context. This leads to the Kähler metric

$$\mathcal{G}_{S\bar{S}} = \text{Im}(\tau) \cdot \Lambda^{-4},$$

where

$$\tau(S) = \partial_S^2 \mathcal{F}_0 = \frac{1}{2\pi i} \log \left(\frac{S}{m\Lambda_0^2} \right).$$

The effective potential can then be made explicit,

$$V_{\text{eff}} = \frac{2i}{(\tau - \bar{\tau})} |\alpha - N\tau|^2,$$

and the critical points, $\partial_S V_{\text{eff}} = 0$, are located at the solutions to

$$\frac{2i}{(\tau - \bar{\tau})^2} \partial_S^3 \mathcal{F}_0 (\bar{\alpha} - N\bar{\tau}) (\alpha - N\bar{\tau}) = 0.$$

⁵⁵ See [46] for a discussion of stringy corrections to the Kähler metric.

This can be satisfied through either

$$\alpha - N\tau = 0 \quad \text{or} \quad \alpha - N\bar{\tau} = 0. \quad (5.6.9)$$

The first solution preserves the manifest $\mathcal{N} = 1$ supersymmetry, and corresponds to the solution of $\partial_S \mathcal{W} = 0$. The second solution does not preserve the original supersymmetry as $\partial_S \mathcal{W} \neq 0$. Only *one* of these two solutions is valid at a given point in parameter space if S is to be within the field theory cut-off of $|S| \ll |m\Lambda_0^2|$. For $\text{Im}(\alpha) > 0$ the first solution is physical, and this is the supersymmetric solution we discussed above. However, for $\text{Im}(\alpha) = 1/g_{\text{YM}}^2 < 0$, it is the second solution which is physical, and we obtain

$$S^N = (m\Lambda_0^2)^N \exp(2\pi i \bar{\alpha}). \quad (5.6.10)$$

This solution looks very much like the solution (5.6.4) for the original $U(N)$ confining theory, except that $\alpha \rightarrow \bar{\alpha}$. This is what one would expect if we were discussing the theory of N antibranes on the flopped geometry. In fact, as discussed in detail in [4] one can show that this theory is indeed supersymmetric, with supermultiplets aligned as in (5.6.6).

5.6.3. Supersymmetry breaking by background fluxes

Now consider the same geometry as in the previous subsection, but with a holomorphically varying B -field introduced. If we wrap branes on the conifold, this gives rise to the supersymmetric theories considered in sections 3-4. However, in the case of antibranes, we will see that supersymmetry is in fact broken. This is due to the fact that, while branes preserve the same half of the background $\mathcal{N} = 2$ supersymmetry as the B -field, antibranes preserve an opposite half.

As in the previous section, we will consider branes and antibranes on the conifold geometry (5.6.2) with superpotential given by (5.6.3), but now with the holomorphically varying B -field given by⁵⁶

$$B(\Phi) = t_0 + t_2 \Phi^2. \quad (5.6.11)$$

We will study this from the perspective of the IR effective field theory of the glueball superfield S . Because of the underlying $\mathcal{N} = 2$ structure of this theory, we will have a good description regardless of whether it is branes or antibranes which are present. In the next subsection, we will provide UV *field* theories describing both situations.

The superpotential in the dual geometry is given by (5.3.7), which we repeat here for convenience

$$\mathcal{W}(S) = - \oint_A B(x) y dx + N \frac{\partial \mathcal{F}_0}{\partial S}. \quad (5.6.12)$$

⁵⁶ We could have also added a term linear in Φ , but this has no effect due to the symmetry of the problem.

An explicit computation in the geometry yields an exact expression for the first term,

$$\oint_A B(x)ydx = t_0 S + t_2 \frac{S^2}{m}.$$

The scalar potential is again given by (5.6.7) with the same metric and prepotential \mathcal{F}_0 , but now with superpotential (5.6.12). There are two vacua which extremize the potential, $\partial_S V_{\text{eff}} = 0$,

$$\begin{aligned} - \left(t_0 + 2t_2 \frac{S}{m} \right) + N\tau &= 0, \\ - \left(t_0 + 2t_2 \frac{S}{m} \right) + N\bar{\tau} + 4\pi i(\tau - \bar{\tau})t_2 \frac{S}{m} &= 0. \end{aligned} \tag{5.6.13}$$

The first solution satisfies $\partial\mathcal{W} = 0$. This has solutions in the case where branes are present, with

$$\text{Im}[\alpha](4.1.14)0.$$

Here α is defined as $\alpha = t_0 + 2t_2 \frac{S}{m}$, and large positive values of $\text{Im}[\alpha]$ give $|S/m| \ll |\Lambda_0^2|$ within the allowed region. This vacuum is manifestly supersymmetric, and we have studied it in sections 3-4.

We can instead study antibranes by allowing the geometry to undergo a flop, so

$$\text{Im}[\alpha] \ll 0.$$

Then the supersymmetric solution is unphysical, and we instead study solutions to the second equation in (5.6.13). We already know that the manifest supersymmetry is entirely broken in this vacuum, because $\partial\mathcal{W} \neq 0$. Moreover the fact that the second equation in (5.6.13) is not holomorphic in S suggests that no accidental supersymmetry emerges here, unlike the cases in previous subsection and [4]. We can directly observe the fact that supersymmetry is broken in this vacuum by computing the tree-level masses of the bosons and fermions in the theory, and showing that there is a nonzero mass splitting.

From the $\mathcal{N} = 1$ Lagrangian, we can read off the fermion masses,

$$\begin{aligned} \Lambda^{-4} m_\psi &= \frac{1}{2i(\text{Im}\tau)^2} \frac{1}{2\pi i S} \left(t_0 + N\bar{\tau} + 2t_2 \frac{S}{m} \right) + \frac{1}{\text{Im}\tau} \frac{2t_2}{m} \\ \Lambda^{-4} m_\lambda &= \frac{1}{2i(\text{Im}\tau)^2} \frac{1}{2\pi i S} \overline{\left(t_0 + N\tau + 2t_2 \frac{S}{m} \right)}, \end{aligned}$$

while the bosonic masses are given by

$$\Lambda^{-4} m_{b,\pm}^2 = \frac{1}{\text{Im}\tau} (\partial\bar{\partial}V_{\text{eff}} \pm |\partial\partial V_{\text{eff}}|).$$

Evaluating the masses in the brane vacuum, we see that λ is massless and acts as a partner of the gauge field A , while ψ is a superpartner to S . In other words, supersymmetry pairs up the bosons and fermions as in (5.6.5).

Evaluating the masses in the antibrane vacuum, ψ becomes the massless goldstino. However, there is no longer a bose/fermi degeneracy like where the background B -field was constant. Instead,

$$m_{b,\pm}^2 = |m_\lambda|^2 \pm 4\pi\Lambda^4 |m_\lambda \partial\alpha|. \quad (5.6.14)$$

This mass splitting shows quite explicitly that all supersymmetries are broken in this vacuum. We can capture the strength of this breaking with a dimensionless quantity,

$$\epsilon = \frac{\Delta m_b^2}{m_b^2} \sim 2\pi\Lambda^4 \left| \frac{2t_2/m}{m_\lambda} \right| \sim \frac{t_2 S \log |S|}{m N}.$$

We can get a heuristic understanding of this measure of supersymmetry breaking as follows. The reason supersymmetry is broken in this phase is that B -field varies in a way incompatible with the normalizable fluxes/branes. Thus its variation over the cut in the IR geometry is a natural way to quantify supersymmetry breaking. More precisely, we expect that measuring

$$\epsilon = \Delta B$$

across the cut should give a quantification of the supersymmetry breaking by a dimensionless number. Evaluating this explicitly yields $\epsilon = t_2 S/m$, which is in rough agreement (up to a factor of order $\log |S|/N$) with the dimensionless quantity coming from the mass splittings.

5.6.4. A susy/non-susy duality

Motivated by the considerations of the previous example, we now formulate a duality between two field theories – one which is manifestly supersymmetric, and the other in which supersymmetry is broken softly by spurions. Consider an $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theory with an adjoint field Φ and superpotential terms

$$\int d^2\theta_1 \text{Tr} [B(\Phi)\mathcal{W}_\alpha\mathcal{W}^\alpha + W(\Phi)] \quad (5.6.15)$$

where, as before,

$$B(\Phi) = \sum_{k=0}^{n-1} t_k \Phi^k, \quad W(\Phi) = \sum_{k=0}^{n+1} a_k \Phi^k.$$

Consider a choice of parameters (a_k, t_k) such that

$$\text{Im}B(e_k) < 0 \quad (5.6.16)$$

for all e_k with $W'(e_k) = 0$. Then this theory is not sensible in this regime as it has no unitary vacuum. However, we propose that this theory is dual to

another $U(N)$ gauge theory already studied in [114], with an adjoint field $\tilde{\Phi}$ and superpotential term

$$\int d^2\theta_2 \text{Tr}[\bar{t}_0 \tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha + \tilde{W}(\tilde{\Phi})] \quad (5.6.17)$$

where

$$\tilde{W}(\tilde{\Phi}) = \sum_{k=1}^{n+1} (a_k + 2it_k \theta_2 \theta_2) \tilde{\Phi}^k.$$

Note that since the auxiliary field in the spurion supermultiplets have vevs t_k , this theory breaks supersymmetry. Also, the fermionic parts of the superspaces for these two actions are not related in any way. Indeed, they are orthogonal subspaces of an underlying $\mathcal{N} = 2$ superspace. This is indicated by the first theory being formulated in terms of coordinates θ_1^α , and the second in terms of θ_2^α – two *different* $\mathcal{N} = 1$ superspaces.

Note that this is natural from the string theory perspective. In the regime of parameters where (5.6.16) holds, one should describe the physics in terms of the flopped geometry, and ask how the antibrane theory perceives the geometry. Since the background B -field is holomorphic, it breaks supersymmetry. Indeed the tension of the antibranes will vary as they change position in the x -plane (and we do not expect a canceling term as would be the case for branes). We thus expect the potential to depend on x through a term proportional to the B -field,

$$V_{\text{eff}} \sim \text{Im}B(x). \quad (5.6.18)$$

Indeed, the soft supersymmetry-breaking term in (5.6.17) gives precisely this contribution when we identify the eigenvalues of Φ with positions in the x -plane. Moreover, note that in going from (5.6.15) to (5.6.17) we have flipped the sign of $\text{Im}(t_0) \sim 1/g_{\text{YM}}^2$, which is consistent with the fact that (5.6.17) describes the same physics from the antibrane perspective. As an aside, note that in this section (unlike in much of the rest of the chapter), t_0 and $t_{k>0}$ enter on different footings.

We now provide evidence for this duality. We will show that both theories (5.6.15) and (5.6.17) have the *same* IR description in terms of glueball fields. The effective superpotential for the supersymmetric theory we have already discussed, and is given by

$$\int d^2\theta_1 \mathcal{W}_{\text{eff}}(S_i, a_k) \quad (5.6.19)$$

where

$$\mathcal{W}_{\text{eff}} = \sum_i t_0 S_i + \sum_{k>0} t_k \frac{\partial \mathcal{F}_0}{\partial a_k} + \sum_i N_i \frac{\partial \mathcal{F}_0}{\partial S_i}.$$

The effective glueball theory for the non-supersymmetric theory, in which auxiliary spurion fields have nonzero vevs, has been studied in [113,96,114].

As shown in [114], turning on soft supersymmetry-breaking terms that give spurionic F-terms to the a_i in the UV theory has the expected effect in the IR of simply giving spurionic F-terms to $a_{k>0}$ in that theory,

$$\int d^2\theta_2 \widetilde{\mathcal{W}}_{\text{eff}}(S_i, a_k + 2it_k\theta_2\theta_2) \quad (5.6.20)$$

where

$$\widetilde{\mathcal{W}}_{\text{eff}} = \bar{t}_0 S_i + \sum_i N_i \frac{\partial \mathcal{F}_0}{\partial S_i}.$$

We will see that the two effective glueball theories are in fact identical!

As we reviewed in section 4, one way to arrive at the effective IR theory is via a dual gravity theory. Both theories (5.6.19) and (5.6.20) originate from the same Calabi-Yau after the transition, and so have the *same* underlying $\mathcal{N} = 2$ theory with prepotential $\mathcal{F}_0(S, a)$ at low energies,⁵⁷

$$\text{Im}\left(\int d^2\theta_1 d^2\theta_2 \mathcal{F}_0(S_i, a_k)\right),$$

with appropriate fluxes or auxiliary spurion fields turned on. In fact, it was shown in [25,22] that turning on fluxes is also equivalent to giving vevs to auxiliary fields, so both (5.6.19) and (5.6.20) can be thought of as originating from the $\mathcal{N} = 2$ theory with prepotential $\mathcal{F}_0(S, a)$, where auxiliary fields for the glueball fields S_i and the background fields a_k are subsequently given vevs. This breaks supersymmetry explicitly to $\mathcal{N} = 1$ in the case of (5.6.19), and to $\mathcal{N} = 0$ in case of (5.6.20).

To be more precise, (5.6.19) can be obtained by shifting the auxiliary fields of the $\mathcal{N} = 2$ multiplets containing S and a according to

$$S_i \rightarrow S_i + 2iN_i\theta_2\theta_2, \quad a_k \rightarrow a_k + 2it_k\theta_2\theta_2, \quad k > 0,$$

and integrating over θ_2 . Meanwhile, (5.6.20) arises from instead shifting

$$S_i \rightarrow S_i + 2iN_i\theta_1\theta_1 \quad a_k \rightarrow a_k + 2it_k\theta_2\theta_2, \quad k > 0,$$

and integrating over θ_1 .

These two situations differ in how they shift the auxiliary fields F_{11}^i and $F_{22}^i = \overline{F}_{11}^i$ which lie in the $\mathcal{N} = 2$ chiral multiplet containing S_i ,

$$\mathcal{S}_i = S_i + \dots + \theta_1\theta_1 F_{11}^i + \theta_2\theta_2 F_{22}^i.$$

Shifts of fields alone cannot affect any aspect of the physics if the shift can be undone by an allowed field redefinition. Indeed, the difference between the

⁵⁷ More precisely, the Lagrangian also contains the $\mathcal{N} = 2$ FI terms $t_0 F_{11}^i + \bar{t}_0 F_{22}^i$ where F^i 's are the auxiliary fields discussed in the text.

shifts of (5.6.19) and (5.6.20) is an allowed auxiliary field redefinition, so these theories are equivalent! Put another way, in integrating out the auxiliary fields, we end up summing over all of their values, so any difference between the two theories will disappear. Note that, if F_{11}^i and F_{22}^i were independently fluctuating degrees of freedom, we could use this argument to say that both theories were equivalent to the original $\mathcal{N} = 2$ theory. They are not, however, since the auxiliary field shifts we made cannot be undone by a field redefinition obeying $F_{22}^i = \overline{F}_{11}^i$, which the fluctuating part of the auxiliary fields must satisfy.

To make this duality more explicit, we will show that both theories give rise to the same IR effective potential, $V_{\text{eff}}(S_i)$. For the supersymmetric theory (5.6.19), the superpotential (5.6.15) is

$$\mathcal{W}_{\text{eff}} = t_k \frac{\partial \mathcal{F}_0}{\partial a_k} + N_i \frac{\partial \mathcal{F}_0}{\partial S_i},$$

which leads to an effective potential

$$V_{\text{eff}} = \mathcal{G}^{i\bar{j}} (N^k \tau_{ki} + t_0 + t^k \eta_{ki}) \overline{(N^r \tau_{rj} + t_0 + t^r \eta_{rj})},$$

where in the summation $t^k \eta_{ki}$, we have removed the $m = 0$ term and written it explicitly. This will be convenient for the manipulations below, where we will continue to use this summation convention. We can rewrite V_{eff} grouped by order in t^k ,

$$\begin{aligned} V_{\text{eff}} &= \mathcal{G}^{i\bar{j}} N^k \tau_{kj} \overline{N^r \tau_{rj}} + \mathcal{G}^{i\bar{j}} (t_0 + t_k \eta_{ki}) \overline{(t_0 + t_r \eta_{rj})} \\ &+ \mathcal{G}^{i\bar{j}} N^k \tau_{ki} \overline{(t_0 + t_r \eta_{rj})} + \mathcal{G}^{i\bar{j}} (t_0 + t_k \eta_{ki}) \overline{N^r \tau_{rj}}. \end{aligned} \quad (5.6.21)$$

Now we will show that the effective potential of the non-supersymmetric theory (5.6.17) agrees with (5.6.19). The Lagrangian can be written in $\mathcal{N} = 1$ superspace,

$$\begin{aligned} \mathcal{L} &= \text{Im} \left(\int d^2\theta d^2\overline{\theta} S_i \frac{\partial \mathcal{F}_0}{\partial S_i} \right) + \text{Im} \left(\int d^2\theta \frac{1}{2} \frac{\partial^2 \mathcal{F}_0}{\partial S_i \partial S_j} \mathcal{W}_{\alpha,i} \mathcal{W}_j^\alpha \right) \\ &+ \left(\int d^2\theta \widetilde{\mathcal{W}}_{\text{eff}}(S) + c.c. \right), \end{aligned} \quad (5.6.22)$$

and the superpotential of this non-supersymmetric theory is simply

$$\widetilde{\mathcal{W}}_{\text{eff}} = \bar{t}_0 S_i + N_i \frac{\partial \mathcal{F}_0}{\partial S_i}.$$

Let F_i be the auxiliary field in the S_i superfield. Performing the $d^2\theta$ integral for the superpotential term (the last terms of (5.6.22)) and picking out the spurion contribution (note that $\frac{\partial^2 \mathcal{F}_0}{\partial S_i \partial a_k} = \eta_{ik}$), gives

$$\int d^2\theta \widetilde{\mathcal{W}}_{\text{eff}}(S) = (\bar{t}_0 + N_i \tau_{ij}) F_j + 2i N_i \eta_{ik} t_k.$$

The remaining terms come from the Kähler potential term (the first term of (5.6.22)). This gives $\mathcal{G}_{i\bar{j}}F_i\bar{F}_j$ before spurion deformation, while the spurions produce additional terms, giving a total contribution

$$\text{Im} \left(\int d^2\theta d^2\bar{\theta} S_i \frac{\partial \mathcal{F}_0}{\partial S_i} \right) = \mathcal{G}_{i\bar{j}}F_i\bar{F}_j + F_i\bar{\eta}_{ik}\bar{t}_k + \bar{F}_i\eta_{ik}t_k + \dots$$

With the full F-term Lagrangian, it is now easy to check that integrating out the auxiliary fields F_i , produces precisely the effective potential (5.6.21), which arose from the supersymmetric theory (5.6.19).

We have seen that the tree-level effective potentials for the supersymmetric theory (5.6.15) and the non-supersymmetric theory (5.6.17) agree exactly, corroborating the proposed the duality between the two theories.

5.6.5. Multi-cut geometries and supersymmetry breaking

In the previous subsections we have focused on the case where all gauge couplings have the same sign, positive or negative. We now shift to consider the more general case in which both signs are present. For simplicity, we will focus on the case where the superpotential has two critical points, with a brief discussion of the generalization to an arbitrary number of critical points reserved for the end of this subsection.

In particular, we now consider the UV theory where the superpotential appearing in the geometry (5.6.2) is given by

$$W(\Phi) = g\text{Tr} \left(\frac{1}{3}\Phi^3 - m^2\Phi \right)$$

and the holomorphic variation of the B -field gives rise to an effective field-dependent gauge coupling

$$\alpha(\Phi) = t_0 + t_1\Phi.$$

The two critical points of the superpotential are given by $\Phi = \pm m$, at which points the gauge coupling takes values

$$\alpha_{\pm} \equiv \alpha(\pm m) = t_0 \pm mt_1.$$

We wish to study the case where the imaginary parts of gauge couplings have opposite signs (see figure 5.4). Without loss of generality, we then consider

$$\text{Im}(\alpha_-) \ll 0 \ll \text{Im}(\alpha_+). \tag{5.6.23}$$

We will consider the vacuum where the $U(N)$ gauge group is broken to $U(N_1) \times U(N_2)$ with N_i both nonzero. It is clear from the discussion in section 7.2 that this theory is that of N_1 branes wrapping the S^2 at e_1 and N_2 antibranes wrapping the flopped S^2 at e_2 .

There are now two sources of supersymmetry breaking present. First, for the N_2 antibranes (even if $N_1 = 0$), supersymmetry is broken due to the holomorphic variation of the B -field, as discussed in section 7.3. However, this effect is secondary to that which arises from the fact that branes and antibranes are both present and preserve disparate halves of the background supersymmetry. This more dominant source of supersymmetry breaking was studied in a slightly simpler context in [4,37,5].

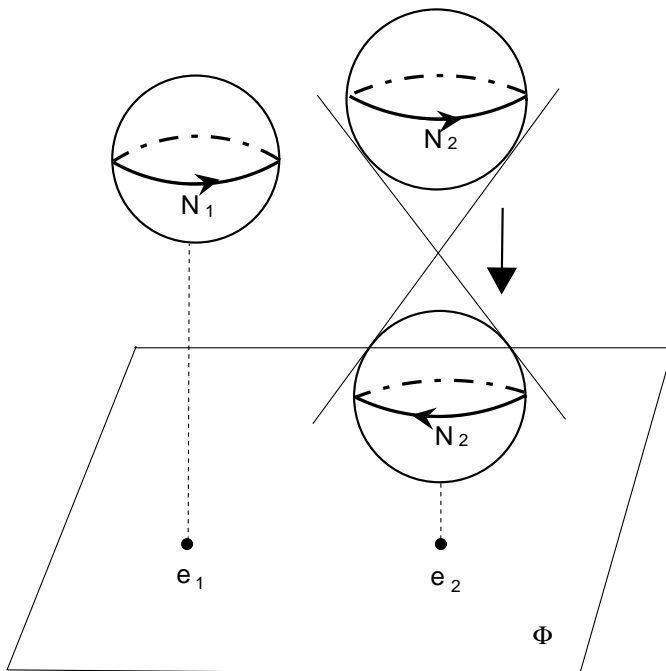


Fig. 5.4. By changing the parameters of the B -field continuously, it can be arranged for *only* the second S^2 to undergo a flop, with the N_2 branes replaced by N_2 antibranes on the flopped S^2 at e_2 . This configuration clearly breaks supersymmetry, as branes and antibranes preserve orthogonal supersymmetries.

We now show that this stringy UV picture is borne out in the dual IR theory. The superpotential for the closed-string dual geometry is given by (5.3.8)

$$\mathcal{W}(S_1, S_2) = t_0(S_1 + S_2) + t_1 \frac{\partial \mathcal{F}_0}{\partial a_1} + N_k \frac{\partial \mathcal{F}_0}{\partial S_k}.$$

In the large N_i limit, it is a sufficient approximation to work to 1-loop order in the associated matrix model. For the geometry in question, the superpotential then takes the form

$$\mathcal{W}(S_1, S_2) = \alpha_+ S_1 + \alpha_- S_2 + N_1 \frac{\partial \mathcal{F}_0}{\partial S_1} + N_2 \frac{\partial \mathcal{F}_0}{\partial S_2},$$

where $a_1 = -m^2g$ and \mathcal{F}_0 was computed in [18],

$$\begin{aligned}\partial_{S_1}\mathcal{F}_0 &\approx \frac{1}{2\pi i} \left(-W(e_1) + S_1 \left(\log \frac{S_1}{8gm^3} - 1 \right) - 2(S_1 + S_2) \log \left(\frac{\Lambda_0}{2m} \right) \right), \\ \partial_{S_2}\mathcal{F}_0 &\approx \frac{1}{2\pi i} \left(-W(e_2) + S_2 \left(\log \frac{S_2}{8gm^3} - 1 \right) - 2(S_1 + S_2) \log \left(\frac{\Lambda_0}{2m} \right) \right).\end{aligned}\tag{5.6.24}$$

Note that at this order, the effect of the varying B -field is just to change the effective coupling constants in the superpotential from α_0 to α_{\pm} . As a result, the only supersymmetry-breaking effects which appear are due to the presence of antibranes.

This theory has no physical supersymmetric vacua, so in order to study its low energy dynamics, we minimize the physical scalar potential,

$$\Lambda^{-4} V_{\text{eff}} = \mathcal{G}^{i\bar{j}} \partial_i \mathcal{W} \overline{\partial_{\bar{j}} \mathcal{W}},$$

where again the Kähler metric is determined by $\mathcal{N} = 2$ supersymmetry,

$$\mathcal{G}_{i\bar{j}} = \text{Im}(\tau_{ij}) = \text{Im} \left(\frac{\partial^2 \mathcal{F}_0}{\partial S_i \partial S_j} \right).$$

The critical points are given by solutions to

$$\mathcal{G}^{i\bar{a}} \mathcal{G}^{b\bar{j}} \mathcal{F}_{abk} (\alpha_i - N^l \bar{\tau}_{li}) \overline{(\alpha_j - N^r \tau_{rj})} = 0.$$

At one-loop order in the matrix model, \mathcal{F}_{ijk} only has nonzero diagonal elements, in which case the vacuum equations simplify. In particular, for the case at hand they simplify to

$$\begin{aligned}N_1 \tau_{11} &= \alpha_+ - N_2 \bar{\tau}_{12}, \\ N_2 \bar{\tau}_{22} &= \alpha_- - N_1 \tau_{12},\end{aligned}$$

and using the expression for the Kähler metric arising from (5.6.24), we obtain following explicit solutions

$$\begin{aligned}(S_1)^{N_1} &= (2gm\Lambda_0^2)^{N_1} \exp(2\pi i \alpha_+) \left(\frac{\Lambda_0^2}{4m^2} \right)^{-N_2} \\ (-S_2)^{N_2} &= (2gm\Lambda_0^2)^{N_2} \exp(2\pi i \bar{\alpha}_-) \left(\frac{\Lambda_0^2}{4m^2} \right)^{-N_1}.\end{aligned}$$

In addition, we can compute the vacuum energy, and find it to be

$$\frac{V_{\text{eff}*}}{\Lambda^4} = 4N_2 |\text{Im} \alpha_-| + \frac{4}{\pi} N_1 N_2 \log \left| \frac{\Lambda_0}{2m} \right|.$$

The first term we identify as the brane tension due to antibranes on the flopped \mathbf{P}^1 , which agrees with (5.6.18), while the second term suggests a Coulomb *repulsion* between brane stacks preserving opposite supersymmetries. A similar expression for the potential energy between branes and antibranes can be found in [4,37,5].

We can further study the masses of the bosonic and fermionic excitations about the non-supersymmetric vacua. At the current order of approximation, most of the expressions from [4] still hold. We obtain four distinct bosonic masses, given by [4]

$$(m_{\pm,c})^2 = \frac{(a^2 + b^2 + 2abcv) \pm \sqrt{(a^2 + b^2 + 2abcv)^2 - 4a^2b^2(1-v)^2}}{2(1-v)^2} \quad (5.6.25)$$

where $c = \pm 1$,

$$a \equiv \Lambda^4 \left| \frac{N_1}{2\pi S_1 \text{Im}\tau_{11}} \right|, \quad b \equiv \Lambda^4 \left| \frac{N_2}{2\pi S_2 \text{Im}\tau_{22}} \right|$$

$$v \equiv \frac{(\text{Im}\tau_{12})^2}{\text{Im}\tau_{11} \text{Im}\tau_{22}},$$

and Λ is a mass scale in the action (5.6.8). The tree-level fermionic masses can also be computed from the off-shell $\mathcal{N} = 1$ Lagrangian. As in [4], they are given by⁵⁸

$$m_{\psi_i} = \left(\frac{a}{1-v}, 0 \right), \quad m_{\lambda_i} = \left(0, \frac{b}{1-v} \right). \quad (5.6.26)$$

The presence of two massless fermions can be thought of as representing two goldstinos due to the breaking of off-shell $\mathcal{N} = 2$ supersymmetry. Alternatively, this fermion spectrum can be viewed as the natural result of breaking supersymmetry collectively with branes and antibranes. There is a light gaugino localized on both the branes and the antibranes. However, since these preserve different supersymmetries, we see the gauginos as arising one from the gaugino sector and one from the sfermion sector with respect to a given $\mathcal{N} = 1$ superspace.

For a generic choice of parameters, supersymmetry breaking is not small, and there is no natural way to pair up bosons and fermions in order to write a mass splitting as a measure of how badly supersymmetry is broken. However from the mass formula we have given, it is clear that in the limit $v \rightarrow 0$, the

⁵⁸ Note that the relation [118]

$$\sum_{\text{boson}} m^2 - \sum_{\text{fermion}} m^2 = \text{Tr}(-)^F m^2 = 0$$

holds for our system, as well as for (5.6.14).

spectrum becomes supersymmetric, and there does emerge a natural pairing of bosonic and fermionic excitations. In this limit, v becomes a good dimensionless measure of the mass splitting, and we can write it in terms of parameters $(\Lambda_0, \alpha_{\pm}, m, N_i)$ as

$$v = \frac{N_1 N_2 \left(\log \left| \frac{\Lambda_0}{2m} \right| \right)^2}{\left(\pi |\operatorname{Im}(\alpha_+)| + \Delta N \log \left| \frac{\Lambda_0}{2m} \right| \right) \left(\pi |\operatorname{Im}(\alpha_-)| - \Delta N \log \left| \frac{\Lambda_0}{2m} \right| \right)}.$$

where $\Delta N = N_1 - N_2$. For $\Delta N = 0$, this further simplifies to

$$v = \frac{N^2 \left(\log \left| \frac{\Lambda_0}{2m} \right| \right)^2}{\pi^2 (|\operatorname{Im}(\alpha_+)|) (|\operatorname{Im}(\alpha_-)|)}.$$

This vanishes and supersymmetry is restored for large separation mt_1 (4.1.14)1, corresponding to the extreme weak-coupling limit. One can also consider another extreme where N_1 (4.1.14) N_2 . In this limit we again expect supersymmetry to be restored. Indeed, in this limit $v \propto N_2/N_1$, and so vanishes.

It should be noted that, unlike the case where all gauge couplings are negative and the background flux is small, in this case the dimensionless parameter v does depend explicitly on the cutoff Λ_0 . This may be related to the fact that, in this case, there is no field theory description in the UV. Namely, even though we know that this system should be described by branes and antibranes, these brane configurations do not admit a good field theory limit. Nevertheless the arguments of the previous section can be used to show that *below* the scale of gauge symmetry breaking, there is an effective field theory description in terms of a $\prod_i U(N_i)$ gauge theory which breaks supersymmetry and captures the same IR physics. In this theory, the gauge group factors with positive gauge couplings have an effective field dependent gauge coupling, while those with negative gauge couplings have supersymmetry softly broken by spurions. However, this is not a satisfactory description for the full dual UV theory.

Before concluding this section, let us briefly consider the generalization of the previous discussion to the n -cut geometry. Here, the superpotential in (5.6.2) is given by

$$W'(\Phi) = g \prod_{i=1}^n (\Phi - e_i).$$

Starting with D5 branes wrapped on n shrinking \mathbb{P}^1 's at $x = e_i$, we perform a geometric transition and study the dual closed-string geometry with n finite S^3 's. The distance between critical points are

$$\Delta_{ij} \equiv e_i - e_j.$$

From the period expansion of [18] we have following expressions in a semiclassical

sical regime

$$2\pi i\tau_{ii} = 2\pi i \frac{\partial^2 \mathcal{F}_0}{\partial S_i^2} \approx \log \left(\frac{S_1}{W''(e_i)\Lambda_0^2} \right) + \mathcal{O}(S)$$

$$2\pi i\tau_{ij} = 2\pi i \frac{\partial^2 \mathcal{F}_0}{\partial S_i \partial S_j} \approx -\log \left(\frac{\Lambda_0^2}{\Delta_{ij}^2} \right) + \mathcal{O}(S)$$

Generalizing the vacuum condition from the two cut geometry, the physical minima of effective potential are then determined by

$$0 = -\text{Re}(\alpha_i) + \sum_j \text{Re}(\tau)_{ij} N_j,$$

$$0 = -\text{Im}(\alpha_i) + \sum_j \text{Im}(\tau)_{ij} N_j \delta_j$$

where $\delta_i \equiv \text{sign}[\text{Im} \alpha_i]$. The expectation values of S_i are expressed explicitly below,

$$\langle S_i \rangle^{N_i} = (W''(e_i)\Lambda_0^2)^{N_i} \prod_{j \neq i}^{\delta_i \delta_j > 0} \left(\frac{\Lambda_0}{\Delta_{ij}} \right)^{2N_j} \prod_{k \neq i}^{\delta_i \delta_k < 0} \left(\frac{\Lambda_0}{\Delta_{ik}} \right)^{-2N_k} \exp(2\pi i \alpha_i), \quad \delta_i > 0$$

$$\langle S_i \rangle^{N_i} = (W''(e_i)\Lambda_0^2)^{N_i} \prod_{j \neq i}^{\delta_i \delta_j > 0} \left(\frac{\Lambda_0}{\Delta_{ij}} \right)^{2N_j} \prod_{k \neq i}^{\delta_i \delta_k < 0} \left(\frac{\Lambda_0}{\Delta_{ik}} \right)^{-2N_k} \exp(2\pi i \bar{\alpha}_i), \quad \delta_i < 0.$$

The vacuum energy density formula is now given by

$$\frac{V_{\text{eff}*}}{\Lambda^4} = 2 \sum_i N_i (|\text{Im} \alpha_i| - \text{Im} \alpha_i) + \left(\sum_{i,j}^{\delta_i > 0, \delta_j < 0} \frac{2}{\pi} N_i N_j \log \left| \frac{\Lambda_0}{\Delta_{ij}} \right| \right), \quad (5.6.27)$$

where the first term is the brane tension contribution from each flopped \mathbb{P}^1 with negative g_{YM}^2 (matching with (5.6.18)), and the second term suggests that opposite brane types interact to contribute a *repulsive* Coulomb potential energy (as in the cases considered in [5])

5.6.6. Decay mechanism for non-supersymmetric systems

It is straightforward to see how the non-supersymmetric systems studied in this section can decay. This is particularly clear in the UV picture. If the gauge coupling constants are all negative, the branes want to sit at the critical point which minimizes $|\text{Im} B(e_i)|$, as this will give the smallest vacuum energy according to (5.6.27). Thus we expect that in this case the system will decay to one which is the $U(N)$ theory of antibranes in a holomorphic B -field background. This still breaks supersymmetry, but it is completely stable. Considering that RR charge has to be conserved, no further decay is possible.

If there are some critical points where $\text{Im}B(e_i)$ is positive, there is no unique stable vacuum. Instead, there are as many as there are ways of distributing N branes amongst the critical points $x = e_i$ where $\text{Im}B(e_i) > 0$. Thus, we find numerous supersymmetric vacua which could be the end point of the decay process, each one minimizing the potential energy to zero. As in [4], these decays can be reformulated in the closed-string dual in terms of Euclidean D5 brane instantons, which effectively transfer branes/flux from one cut to another.

Appendix 5.A. Computation of \mathcal{W}_{eff}

Here we provide more detail on the derivation of (5.3.7) using the Riemann bilinear identity and its extension to a non-compact Riemann surface Σ . In particular, we wish to compute the integral

$$\int_{\Sigma} \chi \wedge \lambda \tag{5.A.1}$$

for closed one-forms χ and λ which are now allowed to have arbitrarily bad divergences at infinity. We need to be extra careful due to this worse-than-usual behavior at infinity. In particular, the contribution of the interior of the Riemann surface will be exactly the same as the usual case, with the only difference coming from a careful treatment of contributions coming from the boundary at infinity.

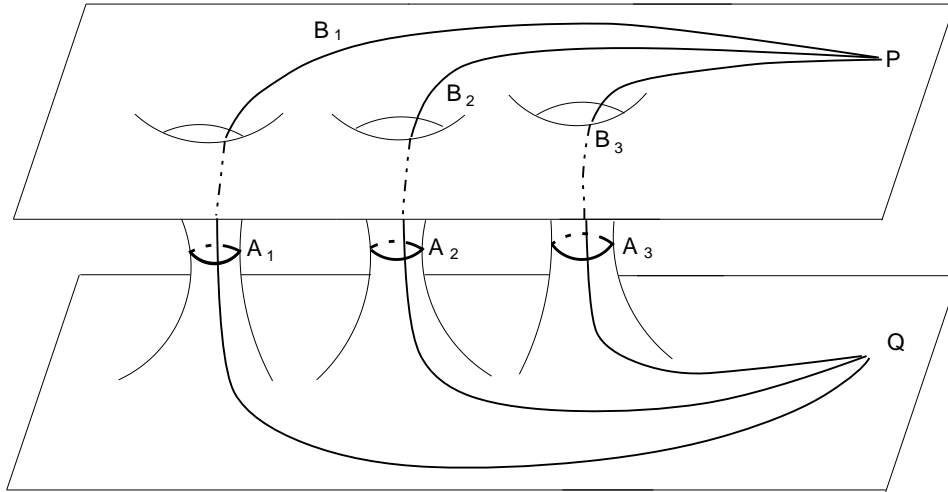


Fig. 5.5. A non-compact Riemann surface represented as a compact Riemann surface Σ with two points P and Q at infinity removed.

We can represent the non-compact Riemann surface Σ as a compact Riemann surface of genus n with two points representing the points at infinity on the top and bottom sheet (labeled by P and Q , respectively) removed. The derivation of the Riemann bilinear identity on the surface then goes through

as usual, by cutting the Riemann surface open into a disk, except that we get an additional contribution from the boundary piece connecting the points P and Q (see figure 5.5). In particular, the contributions of the $n - 1$ compact B -cycles $B_i - B_{i+1}$ and the dual $n - 1$ compact A -cycles are the usual ones. The contribution from the boundary at infinity is given by

$$\oint_P fl + \oint_Q fl - \oint_P \chi \int_Q^P l \quad (5.A.2)$$

where $\chi = df$ and f is a function defined on the simply connected domain which represents the cut-open surface Σ .⁵⁹ Evaluating this for our case of interest, with

$$\begin{aligned} \lambda &= ydx \\ \chi &= H_{\text{RR}} + H_0, \end{aligned}$$

(5.A.2) gives a contribution

$$\oint_P B(x)ydx - \oint_P (H_{\text{RR}} + H_0) \int_Q^P ydx$$

where we have used the fact that $\oint_P = -\oint_Q$ and that $H_{\text{RR}} + H_0 \sim dB(x)$ for large x (and so at the contour around P). Combining all contributions, the superpotential can indeed be rewritten as

$$\mathcal{W}_{\text{eff}} = \sum_{i=1}^n \oint_{A_i} B(x)ydx - \sum_{i=1}^n N_i \partial_{S_i} \mathcal{F}_0.$$

⁵⁹ Note that when f has at worst a logarithmic divergence at P and Q , and λ has at worst a simple pole, then we can write

$$\oint_P fl + \oint_Q fl = (f(P) - f(Q)) \oint_P l = \int_Q^P \chi \oint_P l$$

which returns the standard form for the integral (5.A.2). However, in the case where f has poles at P and Q , the resulting equations are modified.

Chapter 6

The Geometry of D-Brane Superpotentials

Much of the progress in our understanding of topological string theory on Calabi-Yau threefolds has been driven by its numerous intersections with physical superstring theory. For a non-compact Calabi-Yau, input from string dualities led to a computation of both open and closed topological string amplitudes to all orders in perturbation theory by means of the topological vertex [119]. Recently, these results have been verified by mathematicians [120]. In a lesser measure, progress has also been made with regard to topological string amplitudes on *compact* Calabi-Yau manifolds. For example, closed topological string amplitudes have been computed in perturbation theory up a very high genus in [121].

For the open topological string on the quintic, the disk amplitude has been computed for an involution brane in a ground-breaking paper by Walcher [122]. In this case there are no massless open-string deformations, and the disk amplitude depends on closed-string moduli alone. The results of [122] have subsequently been verified by mathematicians in [123]. More such examples were studied in [124,125,126], the results of which were formalized in [127] within the framework of Griffiths' normal functions [128].

Despite the successes of [122] and subsequent papers, a more general framework is desirable. In particular, the topological string disk amplitude can depend on massless open-string moduli – a situation which lies outside the scope of [122,127]. Moreover the topological string on a disk computes the D-brane superpotential. The superpotential, corresponding to the classical brane action in topological string field theory, is naturally an off-shell quantity. For example, the superpotential for a B-brane wrapping a curve is given by

$$W(\mathcal{C}) = \int_{B(\mathcal{C})} \Omega^{(3,0)} \tag{6.0.1}$$

where \mathcal{C} is *any* curve, not necessarily holomorphic, and $B(\mathcal{C})$ is a three-chain with \mathcal{C} as its boundary. The critical points of (6.0.1) with respect to variations of the brane embedding are holomorphic curves. Restricting to the critical locus, one recovers the normal functions of [122,127].

A method for computing the off-shell superpotential (6.0.1) for “toric branes”, first defined in [30], has been proposed in [129,130], following [131,132,133], and extended to compact Calabi-Yaus in [134,135,136]. For these branes, the superpotentials (as well as the open-string flat coordinates) are the solutions to a system of “open/closed” Picard-Fuchs equations, which arise as a consequence of “ $\mathcal{N} = 1$ special geometry”. For closed string periods, the Picard-Fuchs equations can be read off from the associated gauged linear sigma model (GLSM). The authors of [135] extend this formalism and associate an auxiliary GLSM to the open/closed Picard-Fuchs system, thus treating open and closed-string moduli at the same footing, and allowing systematic computation of superpotentials for a large class of branes.

While these results are remarkable, their physical underpinnings remain somewhat mysterious. In formulating the Picard-Fuchs system, one must specify a divisor in the B-model geometry. This divisor is only a part of the combinatoric data which enters into the definition of the curve \mathcal{C} as a toric brane. The role of this divisor then requires some explanation. In addition, the appearance of the auxiliary GLSM begs for a physical interpretation. Finally, the methods of [122] extend beyond the class of toric D-branes, and so there should be some appropriate generalization of the techniques of [129,130] which allows for the treatment of these other cases.

In this chapter, we show that duality of the physical superstring explains these remarkable results. Consider the theory obtained by wrapping a D3 brane on a holomorphic two-cycle \mathcal{C} in the compact threefold X_3 . The theory on the brane has $\mathcal{N} = (2, 2)$ supersymmetry in two dimensions, with the superpotential (6.0.1) computed by the disk partition function of the topological B-model with boundary on \mathcal{C} . We will argue that the same superpotential is generated by a modified brane configuration, with an additional D5 brane wrapping a divisor D , and the D3 brane dissolved as world-volume flux. Note that this is *not* a duality; the modified configuration only produces the same answers for a certain subset of physical quantities. In particular, the moduli space for D is not equivalent to the configuration space of the curve \mathcal{C} . The superpotentials of the two theories must agree only for those variations of \mathcal{C} which are encoded in variations of the moduli of the divisor. In fact, given X_3 and \mathcal{C} we argue that the superpotential is the same for any configuration of D3, D5 and D7 branes where the D3 charge brane ends up localized on \mathcal{C} .

The superpotential is also the same for any other D-brane configuration related by dimensional reduction/oxidation in $\mathbf{R}^{3,1}$. For example, the superpotentials for a D3 brane and a D5 brane wrapping the curve \mathcal{C} are the same. Our choice is such that the D-branes have codimension two in $\mathbf{R}^{3,1}$. For D-branes of lower codimension, only a subset of these models are consistent due

to RR tadpoles. Tadpole cancellation in these cases requires the introduction of ingredients, such as orientifold planes, that are extraneous to the problem at hand.⁶⁰ For higher codimension, the branes break more spacetime symmetries, complicating the problem. For this reason, the superstring embedding we have chosen is the most natural.

S -duality of type IIB string theory relates the D5 branes on D to NS5 branes, with the flux remaining invariant since it is generated by a dissolved D3 brane. By further compactifying and T -dualizing one of the directions transverse to both X_3 and the NS5 branes, the branes are geometrized. The resulting configuration is then type IIA on a *non-compact* Calabi-Yau fourfold X_4 . The flux on the NS5 brane is T -dualized to RR four-form flux G_4 on X_4 , which generates a flux superpotential of the form,

$$W = \int_{X_4} G_4 \wedge \Omega^{(4,0)}, \quad (6.0.2)$$

where $\Omega^{(4,0)}$ is the holomorphic four-form. Duality and the BPS nature of the superpotential guarantee that the superpotential (6.0.2) is the same as (6.0.1) as a function on appropriate moduli space. Moreover, since (6.0.2) can be expressed in terms of closed-string periods of a Calabi-Yau fourfold, it is guaranteed that the superpotential will satisfy a system of Picard-Fuchs equations which also encode the appropriate flat coordinates. This explains the appearance of the auxiliary toric data of Calabi-Yau fourfolds in [131,135]. Our approach is more general, however, in principle allowing one to go beyond the category of toric branes.

The formalism of [129,130,134,135,136], then, does *not* strictly reflect the physics of B-branes wrapping curves, but rather that of B-branes wrapping divisors D with non-trivial world-volume flux.⁶¹ In particular, to extract the disk amplitude for a B-brane on \mathcal{C} , one must specify not only the divisor, but the first Chern class of the gauge bundle as well; on the Calabi-Yau fourfold, this corresponds to the choice of the RR flux. However, the distinction is for most purposes immaterial, as long as one is only interested in the superpotential.

The chapter is organized as follows. In section two, we discuss the relation between D3 branes on \mathcal{C} and D5 branes on divisors containing \mathcal{C} . In particular, we show that upon the introduction of the appropriate fluxes on the D5 branes, the superpotentials for the two configurations agree. We then turn the chain of dualities that relates the D-brane geometry in IIB to a IIA flux compactification on a non-compact Calabi-Yau fourfold, which we give explicitly. We explain the

⁶⁰ For example, D5 branes wrapping curves on a compact Calabi-Yau can be dissolved in the D7 branes, after introducing an appropriate number of orientifold planes so that the net D-brane charge vanishes, or by working in F-theory on a compact Calabi-Yau fourfold.

⁶¹ A different proposal for how to geometrize the D-brane superpotential by blowing up the divisor D , resulting in a threefold which is not Ricci-flat, has been proposed recently in [137].

role of mirror symmetry for Calabi-Yau threefolds and fourfolds in this context. In section three, we present detailed computations for a number of examples, which illustrate a variety of circumstances in which our prescription is of use. Among other things, we show that we can reproduce earlier results of [138] obtained from matrix factorizations. In an appendix, we discuss the relation of the methods developed here to the toric geometry approach of [135].

6.1. B-Brane Superpotentials on Calabi-Yau Threefolds

Consider a D3 brane which wraps a curve \mathcal{C} inside a Calabi-Yau threefold X_3 . This gives, in the non-compact directions, a $1+1$ dimensional theory with $\mathcal{N} = (2, 2)$ supersymmetry. When \mathcal{C} is genus zero, there are no bundle moduli associated with the gauge fields on the D3 brane, and so any light degrees of freedom will arise from variations of the D-brane embedding. The number of massless chiral fields is equal to $H^0(\mathcal{C}, \mathcal{N}_{\mathcal{C}})$, the number of holomorphic sections of the normal bundle to the curve in X_3 , which encode infinitesimal, holomorphic deformations of the curve. On general grounds, the normal bundle splits as $\mathcal{N}_{\mathcal{C}} = \mathcal{O}(-1+n) \oplus \mathcal{O}(-1-n)$. Since an $\mathcal{O}(k)$ bundle has $k+1$ holomorphic sections, the number of massless deformations for the curve is n . For $n=1$, there is one massless adjoint chiral field. This does not imply that the curve has finite holomorphic deformations – there may be obstructions at higher order. Such an obstruction is encoded by a superpotential for the moduli [139]. This superpotential is computed at string tree-level by the topological B-model, with boundaries on \mathcal{C} . Alternatively, this amplitude can be computed in terms of the classical geometry of the brane configuration [30],

$$W(\mathcal{C}) = \int_{B(\mathcal{C}, \mathcal{C}_*)} \Omega^{(3,0)}, \quad (6.1.1)$$

where $\Omega^{(3,0)}$ is the holomorphic three-form on X_3 , and $B(\mathcal{C}, \mathcal{C}_*)$ is a three-chain with one boundary on \mathcal{C} , and the other on a homologous, reference two-cycle \mathcal{C}_* .

In the generic case, the normal bundle to a curve is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. This corresponds to $n=0$, and there are no massless fields on the brane. The B-model disk amplitude then depends on the closed-string moduli alone, i.e., it measures the superpotential (6.1.1) evaluated at its critical point with respect to open-string variations. This scenario has been studied in [122]. The dependence of the physical D-brane superpotential on massive brane deformations can nevertheless be interesting and relevant to the low-energy effective theory above a certain scale [139].

In practice, evaluating these chain integrals directly from first principles is difficult. For the mirrors of non-compact, toric Calabi-Yau threefolds, the computations are rendered tractable by the relative simplicity of the geometry. This is not the case when the Calabi-Yau is compact, and so we will instead explore an alternate approach.

There are other brane configurations in string theory that give rise to the same superpotential (6.1.1). For example, we may consider a D5 brane which wraps a divisor $D \subset X_3$. The divisor has $h^{2,0}(D)$ complex moduli. Each such modulus corresponds to a massless chiral field on the D5 brane. This moduli space is lifted when there is non-trivial flux on the D5 brane world-volume. In particular, take the flux F to be Poincaré dual to a curve $\mathcal{C} \subset D$,

$$F = \text{PD}_D[\mathcal{C}]. \quad (6.1.2)$$

For a generic D5 brane embedding, supersymmetry will be broken by the flux. A condition for unbroken supersymmetry is that the gauge bundle on the brane be holomorphic,⁶²

$$F^{(0,2)} = 0. \quad (6.1.3)$$

This condition is equivalent to the requirement that the curve \mathcal{C} be holomorphic – the same requirement that is enforced at the critical points of (6.1.1). In fact, it can be shown that the flux (6.1.2) generates *precisely* the superpotential (6.1.1) for an appropriate subset of open-string deformations. The superpotential due to the flux (6.1.2) can be written [141-145]

$$W(D) = \int_{\Gamma(D, D_\star)} F \wedge \Omega^{(3,0)}, \quad (6.1.4)$$

where Γ is a five-chain which interpolates between D and a homologous reference divisor D_\star and F is the appropriate extension of the flux as a closed form onto Γ (obtained by taking the Poincaré dual to $B(\mathcal{C}, \mathcal{C}_\star)$). By Poincaré duality, this superpotential can be rewritten as

$$W(D) = \int_{B(\mathcal{C}, \mathcal{C}_\star)} \Omega^{(3,0)}, \quad (6.1.5)$$

which matches (6.1.1) for deformations which are common to the two brane systems.

Alternatively, one could consider a D7 brane which wraps all of X_3 . The superpotential for the brane is the holomorphic Chern-Simons functional,

$$W(X_3) = \int_{X_3} A \wedge \bar{\partial}A \wedge \Omega^{(3,0)}.$$

Consider now turning on world-volume flux, $F \wedge F$, which is Poincaré dual to the curve \mathcal{C} ,

$$F \wedge F = \text{PD}_{X_3}[\mathcal{C}]. \quad (6.1.6)$$

⁶² There is an additional supersymmetry constraint on $J \wedge F$, where J is the Kähler form on the four-cycle [140]. This is interpreted in four dimensions as a D-term constraint, and so does not affect the superpotential.

It is easy to see [30] that the D7 brane superpotential is the same as (6.1.1). Namely, locally, near \mathcal{C} we can write

$$\Omega^{(3,0)} = d\omega$$

for a two form ω . Integrating by parts and using (6.1.6) we can write

$$W(X_3) = \int_{\mathcal{C}} \omega,$$

which is the same, up to a constant as (6.1.1).

From the superstring perspective, the superpotentials (6.1.1), (6.1.4), (6.1.6) are all identical since they all have the same origin – the D3 brane charge that is supported on \mathcal{C} . For example, world-volume flux on a D5 brane of the type described above carries D3 brane charge due to a Wess-Zumino coupling on the brane world-volume of the form

$$S_{WZ} \sim \int_{D5} F \wedge C^{(4)}, \quad (6.1.7)$$

where $C^{(4)}$ is the four-form RR potential. When F is as in (6.1.2), this reduces to

$$S_{WZ} \sim \int_{\mathcal{C} \times \mathbf{R}^{1,1}} C^{(4)}, \quad (6.1.8)$$

so the D5 brane carries the charge of a D3 brane wrapping \mathcal{C} . Similarly, turning on (6.1.6) on the D7 brane, gives it a charge of a D3 brane supported on \mathcal{C} .

6.1.1. String duality and Calabi-Yau fourfolds

The reformulation of the superpotential computation in terms of a D5 brane wrapping a divisor is of great use due to a duality which relates the problem to the classical geometry of Calabi-Yau fourfolds.

First, type IIB S -duality exchanges D5 branes and NS5 branes, leaving D3 branes invariant. So, we could have equivalently obtained (6.1.4) as the superpotential for an NS5 brane on the divisor D . Even though S -duality exchanges strong and weak coupling, the superpotential remains invariant when we compare the two theories at weak coupling. One way to see this is to note that the supersymmetry constraints (6.1.3), which are reproduced by the superpotential (6.1.4), are the same for both D5 and NS5 branes. Next, we compactify and T -dualize on one direction of $\mathbf{R}^{3,1}$ transverse to the NS5 brane. T -duality on this circle relates IIB to IIA and geometrizes the NS5 branes. The resulting geometry preserves $1 + 1$ dimensional Lorentz invariance and $\mathcal{N} = (2, 2)$ supersymmetry, so it is a Calabi-Yau fourfold, which we denote by X_4 . Since one of the two directions transverse to the NS5 brane remained non-compact, the fourfold is non-compact as well.

The fourfold X_4 can be described explicitly as follows [94,146,147]. In X_3 , to the divisor D is associated a line bundle \mathcal{L}_D over X_3 and a section f_D , such that D is the zero-locus of the section,

$$f_D(x) = 0. \tag{6.1.9}$$

The Calabi-Yau X_4 is a \mathbb{C}^* fibration over X_3 which degenerates over D , and can be described globally as

$$uv = f_D. \tag{6.1.10}$$

Here u, v are sections of line bundles \mathcal{L}_1 and \mathcal{L}_2 over X_3 , where $\mathcal{L}_1 \otimes \mathcal{L}_2 = \mathcal{L}_D$. The locus where u or v go to infinity is deleted, so the manifold is non-compact. The fiber over a given point of X_3 is a copy of \mathbb{C}^* described by

$$uv = \text{const.}$$

This fiber has the topology of a cylinder, and is mirror to the $\mathbf{R} \times S^1$ formed by the two directions transverse to the NS5 brane and to X_3 . It degenerates to $uv = 0$ over the divisor D that was wrapped by the NS5 brane.⁶³

There are several immediate and important consequences of this correspondence. First, the moduli of the divisor entering into the choice of section f_D become complex structure moduli of X_4 . The holomorphic three-form on X_3 lifts to the holomorphic four-form on X_4 ,

$$\Omega^{(4,0)} = \Omega^{3,0} \wedge du/u.$$

The compact, Lagrangian four-cycles of X_4 come in two different flavors. First, every closed Lagrangian three-cycle lifts to a Lagrangian four-cycle when combined with the S^1 in the fiber. In addition, there are four-cycles which project to three-chains in X_3 with boundaries on D . The generic fiber over the chain is still an S^1 , but the circle now degenerates over D , capping off a closed cycle in X_4 . Note that this means that the Lagrangian T^4 fibration of X_4 is related to the Lagrangian T^3 fibration of X_3 by simple inclusion of the S^1 fiber.

These observations imply that the superpotential (6.1.1) can be represented, on the Calabi-Yau fourfold X_4 , as the period of $\Omega^{(4,0)}$ over an appropriately chosen four-cycle $L_4(B)$ which is the S^1 fibration over the three-chain $B(\mathcal{C}, \mathcal{C}_\star)$,

$$W = \int_{L_4(B)} \Omega^{(4,0)}. \tag{6.1.11}$$

⁶³ The correspondence between the open/closed-string geometry of the D3/NS5 brane system on X_3 and the geometry of X_4 is related to the duality of IIB on Calabi-Yau orientifolds to F-theory on Calabi-Yau fourfolds [144]. In the present case, X_4 is a fibration over X_3 with *non-compact* fiber. The main simplification here is that, due to the low codimension of branes, there are no tadpoles on X_3 to begin with. Correspondingly, the dual fourfold is non-compact.

Such a superpotential for the complex structure moduli can only be generated by the presence of RR four-form flux G_4 on X_4 . The flux generated superpotential is [21]

$$W_{flux} = \int_{X_4} G_4 \wedge \Omega^{(4,0)}. \quad (6.1.12)$$

We now show that T -duality indeed implies that flux on the NS5 branes maps to the four-form flux of just the right value that W_{flux} is equal to W in (6.1.11), thus providing a check for the duality.

One way to study the superpotential generated by fluxes is to study the corresponding BPS domain walls [21]. To begin with, the superpotential (6.1.1) in IIB on X_3 is generated by world-volume flux F on the D5 brane which is supported on a curve \mathcal{C} . Different vacua correspond to different curves $\mathcal{C}_i \subset D$ which are homologous in X_3 , but distinct in D . If $B(\mathcal{C}_1, \mathcal{C}_2)$ is the three-chain that interpolates between two curves, the relevant domain wall is a D3 brane which wraps $B(\mathcal{C}_1, \mathcal{C}_2)$, with boundaries on the D5 brane. Under S/T -duality, the D3 brane domain wall becomes a D4 brane wrapping a special Lagrangian four-cycle $L_4(B)$ in X_4 , obtained as the S^1 fibration over $B(\mathcal{C}_1, \mathcal{C}_2)$. This domain wall interpolates between vacua where RR four-form flux shifts by an amount Poincaré dual to $L_4(B)$,

$$G_4 = \text{PD}_{X_4}[L_4(B)]. \quad (6.1.13)$$

Inserting this in W_{flux} (6.1.12) we precisely recover (6.1.11).

Thus the problem of computing the open-string superpotential (6.1.1) is rephrased as determining the periods of the holomorphic four-form on X_4 which control the flux superpotential (6.1.12). Before proceeding to the calculation of such periods in explicit examples, however, we discuss the role played by mirror symmetry for the Calabi-Yau fourfolds in these geometries.

6.1.2. T -duality and mirror symmetry for fourfolds

Mirror symmetry provides another piece of evidence for the proposed correspondence. We recall in advance that mirror symmetry for Calabi-Yau n -folds can be interpreted as T^n duality on Lagrangian T^n fibers [148].

To begin with, consider IIB on X_3 with a D3 brane wrapping \mathcal{C} before adding D5 or NS5 branes. Mirror symmetry for Calabi-Yau threefolds is a T^3 -duality on the special Lagrangian T^3 fibers of X^3 . This maps X_3 to its mirror Y_3 , IIB to IIA, and D3 branes on \mathcal{C} to D4 branes wrapping a Lagrangian three-cycle L . Mirror symmetry for threefolds also exchanges the topological A- and B-models, so the superpotential for the low-energy effective theory on the D4 branes is computed by the disk amplitude of the topological A-model. This amplitude receives contributions from holomorphic maps of a worldsheet with the topology of a disk into Y_3 , with boundaries on L . For $b_1(L) = n$, there are n non-contractible one-cycles in L , which are contractible in Y_3 since $b_1(Y_3) = 0$. The one-cycles can then be filled in to disks in Y_3 . Let u denote complexified

Kähler volume of a minimal area disk, and t the closed-string Kähler modulus. The at large radius in both closed and open-string moduli, the disk amplitude has the form

$$W = P_2(u, t) + \sum_{n=1} \sum_{q, Q} \frac{N_{q, Q}}{n^2} e^{-n(qu+Qt)} \quad (6.1.14)$$

where (q, Q) denotes the relative homology class of the disk, the sum over n is a sum over multi-covers and $N_{q, Q}$ are *integers*. We have added a polynomial quadratic in u to cover the case when the moduli are actually massive, i.e., the Lagrangian has the topology of an S^3 and $b_1(L) = 0$. There can still be holomorphic disks ending on the brane, whose Kähler moduli u are expressible in terms of the Kähler moduli t of the Calabi-Yau, corresponding to the extrema of the quadratic part of the superpotential. An example of this in the non-compact setting was given in [149].

Now consider adding NS5 branes on the divisor D to the IIB setup on X_3 . If we perform a T^3 -duality on the Lagrangian T^3 fibers now, the NS5 branes will again be geometrized. This is because a T -duality on an odd number of circles transverse to the NS5 branes geometrizes them. Here, the NS5 branes wrap a divisor, so only one of the T -dualized circles is transverse to them. The theory preserves the $1 + 1$ dimensional Lorentz invariance and $\mathcal{N} = (2, 2)$ supersymmetry, and so the dual geometry is again a Calabi-Yau fourfold. In fact, this fourfold Y_4 is nothing but the mirror of X_4 ! To see this, note that the T^4 fiber of X_4 is an S^1 fibration over the T^3 fiber of X_3 . The statement then follows upon refining mirror symmetry on the fourfold as T^4 -duality in two steps: a T -duality relating X_4 to NS5 branes on X_3 , followed by a T^3 -duality relating this to IIA on Y_4 .

So, after introducing an NS5 brane on the divisor D , the mirror of X_3 is no longer the Calabi-Yau threefold with Lagrangian D4 brane, but rather is a Calabi-Yau fourfold with flux. However, we have argued in this note that the physics of the superpotential must remain the same. We will now show how the superpotential (6.1.14) is reproduced in the mirror. Mirror symmetry relates complex structure moduli of X_4 to Kähler moduli of Y_4 . A superpotential for these Kähler moduli can be generated by 0-, 2-, 4-, 6- and 8-form fluxes [21,150,151]. To determine the superpotential, we again follow the BPS domain walls through the chain of dualities. In the context of IIB on X_3 with NS5 branes on D , the domain walls were D3 branes wrapping special Lagrangian three-chains $B(\mathcal{C}, \mathcal{C}')$ and ending on the NS5 branes. The T^3 -duality that maps to IIA on Y_4 sends the D3 branes to D4 branes on a four-chain $D_4(B)$ that interpolates between the Lagrangians L_1 and L_2 which are mirror to \mathcal{C}_1 and \mathcal{C}_2 . The RR flux that shifts across these is a four-form flux, Poincaré dual to $D_4(B)$,

$$G_4 = \text{PD}_{Y_4}[D_4(B)]. \quad (6.1.15)$$

This implies that the superpotential on Y_4 is

$$W_{flux} = \int_{Y_4} G_4 \wedge k \wedge k, \quad (6.1.16)$$

where k is the complexified Kähler form. In particular, inserting the jump in the flux over the D4 brane domain wall (6.1.15), we precisely recover the BPS tension of D4 brane on $L_4(B)$.

It is a remarkable fact, and a check of the duality chain proposed here, that the flux superpotential (6.1.16) in the fourfold has, at large radius, the integral expansion (6.1.14) [152,151]. For this, it is crucial that there are only four-form fluxes turned on, which is exactly what is needed for the theory to be dual to IIA on Calabi-Yau threefold Y_3 with D4 branes wrapping a Lagrangian three-cycle L .

Before we turn to examples, note that we have made no a-priori restriction on D . In particular, the divisor D does not have to be a “toric” divisor. When D is toric (and X_3 a hypersurface in a toric variety), it is easy to see that the Calabi-Yau fourfold (6.1.10) has the same complex structure as those in [135], so our results are guaranteed to agree.

6.2. Examples

In this section, the observations of the previous sections are applied to a number of brane configurations on families of Calabi-Yau threefolds. The first examples will focus on D3 branes wrapping degree-one rational curves on the quintic in the vicinity of the Fermat point. These brane configurations have been studied in [138] using matrix factorization/worldsheet CFT methods, and the results reported here will agree with those obtained previously. The second set of examples draws from the class of “toric branes” near the large complex structure point of the mirror quintic. These branes have previously been discussed in [135]. The present treatment will agree with those results, but will provide a new interpretation for some of the methods involved in the calculations. Finally, we consider several examples of branes on complete intersection Calabi-Yaus which were studied recently in [126].

In all of these examples, our approach will be results-oriented and will not include an exhaustive analysis of the fourfold geometries involved. In particular, we will not attempt to derive the complete set of Picard-Fuchs operators for the fourfolds, and will instead settle for a set of differential operators which uniquely determines the periods of interest given the desired leading-order behavior. We will borrow the overall normalization of the superpotentials from results elsewhere in the literature, leaving a careful intersection/monodromy analysis for future study.

6.2.1. D-branes on the Fermat quintic

The starting point for this first set of examples is the Fermat quintic, given by the hypersurface in \mathbb{P}^4 ,

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0.$$

There are continuous families of rational curves on this quintic, which we specify by their parameterizations in terms of homogeneous coordinates (u, v) on a \mathbb{P}^1 as

$$\begin{aligned} (x_1, x_2, x_3, x_4, x_5) &= (au, bu, cu, v, -\eta v) \\ a^5 + b^5 + c^5 &= 0, \end{aligned} \tag{6.2.1}$$

where η is a fifth root of unity. There are fifty such families, corresponding to ten partitions of the x_i into groups of two and three, and five values of η . For later convenience, we denote these families thusly,

$$\Sigma_m^{ijk}(a, b, c) : (x_i, x_j, x_k) = (au, bu, cu), \quad \eta = e^{2\pi im/5}. \tag{6.2.2}$$

The coefficients (a, b, c) are only defined up to an overall scaling, so each family is parameterized by a complex one-dimensional curve which is a hypersurface in \mathbb{P}^2 . Each family of curves intersects another family at points where one of the homogeneous coordinates vanishes.

The presence of continuous families of rational curves is non-generic, and perturbing the bulk complex structure will lift (some of) the families, leaving only isolated curves. Such a perturbation should therefore generate a superpotential for D3 branes which wrap these cycles. Such a scenario was explored in [138], where worldsheet CFT techniques were used to compute the superpotential at leading order in such a bulk perturbation as an analytic function on the open-string moduli space.

In order to compute the superpotential for one of these families of curves using the methods of section two, a family of divisors must be chosen such that each divisor is transverse to the family of curves, and each member of the family of curves is subsumed by a single member of the family of divisors. In particular, choosing divisors

$$D(\phi) : \quad x_4 + \phi x_5 = 0, \tag{6.2.3}$$

the Picard-Fuchs equations for the associated fourfold will encode the superpotential for any of the fifteen families Σ_m^{ijk} . For a given value of ϕ , this divisor encompasses the curve with $\phi = -b/c$, and so allows for a good parameterization of the family away from the points at $c = 0$.⁶⁴

Following [138], we further consider perturbations to the complex structure of the threefold which are of the form

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + x_1^3 g(x_3, x_4, x_5) \tag{6.2.4}$$

where

$$g(x_3, x_4, x_5) = \sum_{p+q+r=2} g_{pqr} x_3^p x_4^q x_5^r. \tag{6.2.5}$$

⁶⁴ By making the change of variables $\phi \rightarrow \phi^{-1}$, one can equally well recover the physics in the neighborhood of $c = 0$.

For simplicity, we restrict to monomial perturbations – the case for more general perturbations can be treated by the same methods. In particular, the following two bulk perturbations will lead to qualitatively different physics on the D3 branes,

$$g_1(x_3, x_4, x_5) = \psi_1 x_4 x_5 \quad g_2(x_3, x_4, x_5) = \psi_2 x_4^2. \quad (6.2.6)$$

The superpotential for the family of D3 branes is encoded in the periods of a non-compact Calabi-Yau fourfold X_4 . The line bundles introduced in section two to facilitate construction of the fourfold construction can be identified simply as the bundles $\mathcal{O}(n) \rightarrow \mathbb{P}^4$ restricted to X_3 ,

$$\begin{aligned} \mathcal{L}_D &= \mathcal{O}(1) \rightarrow \mathbb{P}^5|_{X_3}, \\ \mathcal{L}_1 &= \mathcal{O}(1) \rightarrow \mathbb{P}^5|_{X_3}, \\ \mathcal{L}_2 &= \mathcal{O}(0) \rightarrow \mathbb{P}^5|_{X_3}. \end{aligned} \quad (6.2.7)$$

It follows that X_4 is a complete intersection in $\mathbb{P}^6 \times \mathbb{C}$ given by

$$Q(\psi_i) = 0 \quad P(\phi) = x_6 x_7 + D(\phi) = 0, \quad (6.2.8)$$

with the points $(0 : 0 : 0 : 0 : 0 : x_6 ; x_7)$ deleted. The periods of $\Omega^{(4,0)}$ for these geometries can be computed using standard methods, as we now summarize.

For the first bulk perturbation $g_1(x_3, x_4, x_5)$, the D3 brane moduli spaces for the families Σ_m^{245} , Σ_m^{345} are lifted, with the only remaining holomorphic curves in these families being located at the points in (6.2.1) where

$$b \cdot c = 0. \quad (6.2.9)$$

This leads to five distinct curves at $b = 0$ and another five at $c = 0$. The divisors (6.2.3) provide a description of the configurations with $b = 0$. The normal bundle to these curves is $\mathcal{N}_\Sigma = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, rendering the theory on the D3 branes massive.

Picard-Fuchs operators⁶⁵ for the fourfold (6.2.8) can be derived using the residue representation for periods of the holomorphic four-form [128],

$$\Pi_\alpha(\psi, \phi) = \int_{\gamma_1 \times \gamma_2 \times \Gamma_\alpha} \frac{\Delta}{Q(\psi_1)P(\phi)}, \quad (6.2.10)$$

where $\gamma_1 \times \gamma_2 \times \Gamma_\alpha$ is a tubular neighborhood constructed about the desired four-cycle,

$$\Delta = \sum_{i=1}^6 (-1)^i w_i x_i dx_1 \wedge \dots \wedge \widehat{dx}_i \wedge \dots \wedge dx_6 \wedge dx_7,$$

⁶⁵ In all examples, we use the scaling symmetries of the ambient projective space as an efficient way to produce GKZ-type operators which are guaranteed to annihilate the compact periods of the fourfold.

and the w_i are the scaling dimensions of the homogeneous coordinates. We initially represent the hypersurface in terms of redundant parameters on the fourfold moduli space,

$$Q(a_i) = \sum_{i=1}^5 a_i x_i^5 + a_0 x_1^3 x_4 x_5 \quad P(b_i) = x_6 x_7 + b_4 x_4 + b_5 x_5. \quad (6.2.11)$$

where the algebraic coordinates on the moduli space can be given in terms of these parameters by a rescaling of the x_i ,

$$\psi_1 = \frac{a_0}{a_1^{3/5} a_4^{1/5} a_5^{1/5}}, \quad \phi = \frac{b_5 a_4^{1/5}}{b_4 a_5^{1/5}}.$$

The periods expressed in terms of the redundant parameters,

$$\widehat{\Pi}_\alpha(a_i, b_i) = \int_{\gamma_1 \times \gamma_2 \times \Gamma_\alpha} \frac{\omega}{Q(a_i)P(b_i)}, \quad (6.2.12)$$

can be related to the physical periods (6.2.10) according to

$$\widehat{\Pi}_\alpha(a_i, b_i) = \frac{1}{(a_1 a_2 a_3 a_4 a_5)^{1/5}} \Pi_\alpha(\psi_1, \phi). \quad (6.2.13)$$

It is easy to produce differential operators which annihilate the periods $\widehat{\Pi}_\alpha(a_i, b_i)$ because differentiation with respect to the a_i, b_i can be performed under the integral. As such, we obtain the following operators,

$$\begin{aligned} L_1 &= \partial_{a_1}^3 \partial_{a_3} \partial_{a_4} - \partial_{a_0}^5, \\ L_2 &= \partial_{b_3}^5 \partial_{a_5} - \partial_{b_5}^5 \partial_{a_3}. \end{aligned} \quad (6.2.14)$$

These operators in turn are equivalent to relations on the periods written in terms of the algebraic moduli,

$$\mathcal{L}_i \Pi_\alpha(\psi, \phi_i) = 0,$$

where, after factorizing, the operators can be written as

$$\begin{aligned} \mathcal{L}_1 &= \prod_{k=0, \widehat{3}}^4 (\theta_{\psi_1} - k) + 3 \left(\frac{\psi_1}{5} \right)^5 (3\theta_\psi + 11)(3\theta_\psi + 1) \times \\ &\quad (\theta_\psi + \theta_\phi + 1)(\theta_\psi - \theta_\phi + 1) \\ \mathcal{L}_2 &= \theta_\phi(\theta_\phi - \theta_{\psi_1} - 1) - \phi^5 \theta_\phi(\theta_\phi + \theta_\psi + 1). \end{aligned} \quad (6.2.15)$$

with $\theta_z \equiv z \partial_z$.

Among the solutions to these Picard-Fuchs equations are four that depend only on ψ , and which correspond to solutions which would have arisen in a similar analysis of the periods of X_3 . In addition, there are four ϕ -dependent solutions, from amongst which we identify two of interest,

$$\begin{aligned} t &= \phi \left(1 + \frac{1}{15} \phi^5 + \frac{7}{275} \phi^{10} + \frac{77}{187500} \phi^5 \psi_1^5 + \dots \right), \\ \Pi_1 &= \phi^2 \psi_1 \left(1 + \frac{8}{35} \phi^5 + \frac{3}{25} \phi^{10} + \frac{36}{15625} \phi^5 \psi_1^5 + \dots \right). \end{aligned} \tag{6.2.16}$$

The first of these defines the flat open-string coordinate. The second, by its leading behavior, can be identified as the superpotential induced by D3 brane flux in the class of Σ_0^{345} .

These results are in agreement with those of [138]. In particular, the term in Π_1 which is linear in ψ_1 can be written in terms of a hypergeometric function as

$$\Pi_1 = (\psi_1 \phi^2) {}_2F_1 \left(\frac{2}{5}, \frac{4}{5}; \frac{7}{5}; \phi^5 \right) + \dots \tag{6.2.17}$$

Up to the overall normalization, this exactly matches equation (3.14) of [138] for this choice of bulk deformation. It is clear from this derivation, however, that the physical basis for writing the superpotential is *not* in terms of ϕ , but rather in terms of the flat coordinate t which represents an appropriate period of X_4 . Moreover, the overall normalization of the superpotential should be adjusted by the fundamental period of the fourfold as in conventional mirror symmetry calculations. We note that the fundamental period, $\Pi_0(z)$ for these fourfolds is *identical* to that for the related threefold – in particular, it is independent of ϕ – and so the change in normalization doesn't effect the superpotential at leading order in the ψ_1 . However, for higher order corrections, it should be the normalized result which would match any CFT computation such as those performed in [138]. In light of these considerations, we display the *physical* superpotential,

$$\begin{aligned} \mathcal{W}(z, t) = \frac{\Pi_1(z, t)}{\Pi_0(z)} &= t^2 z \left(1 + \frac{2}{21} t^5 + \frac{17}{31250} z^5 + \frac{2}{99} t^{10} + \right. \\ &\quad \left. \frac{68}{46875} t^5 z^5 + \frac{38299}{128906250000} z^{10} + \dots \right) \end{aligned} \tag{6.2.18}$$

The second bulk perturbation also lifts the D3 brane moduli space for the families Σ_m^{245} and Σ_m^{345} leaving only the holomorphic curves given by (6.2.1) along with

$$b^2 = 0. \tag{6.2.19}$$

For each family, this leads to five solutions, each of degeneracy two, at $b = 0$. Thus, D3 branes wrapping these curves find themselves at the critical point of a higher-order superpotential. The normal bundle to each of these curves is $\mathcal{N}_\Sigma = \mathcal{O}(1) \oplus \mathcal{O}(-3)$. However, only *one* of the holomorphic sections of the normal bundle is encoded by the variation of ϕ for the divisor.⁶⁶ As a result,

⁶⁶ An analysis of the full moduli space of the divisor D would encode all holomorphic deformations of the curves.

the following derivation produces the superpotential with respect to only one of the massless deformations.

As before, the differential relations for $\widehat{\Pi}_\alpha(a_i, b_i)$ can be determined,

$$\begin{aligned} L_1 &= \partial_{a_1}^3 \partial_{a_4}^2 - \partial_{a_0}^5 \\ L_2 &= \partial_{a_3} \partial_{b_5}^5 - \partial_{a_5} \partial_{b_4}^3, \end{aligned} \tag{6.2.20}$$

where by rescaling the homogeneous coordinates the algebraic moduli for the fourfold can be found in terms of the redundant parameters,

$$\psi_2 = \frac{a_0}{a_1^{3/5} a_4^{2/5}}, \quad \phi = \frac{b_5 a_4^{1/5}}{b_4 a_5^{1/5}}.$$

The resulting Picard-Fuchs operators which annihilate the periods $\Pi_\alpha(\psi_2, \phi_2)$ are (after factorizing) given by

$$\begin{aligned} \mathcal{L}_1 &= \prod_{k=0, \widehat{3}}^4 (\theta_{\psi_2} - k) + 3 \left(\frac{\psi_2}{5} \right)^5 (3\theta_{\psi_2} + 11)(3\theta_{\psi_2} + 1) \times \\ &\quad (2\theta_{\psi_2} - \theta_\phi + 6)(2\theta_{\psi_2} - \theta_\phi + 1), \\ \mathcal{L}_2 &= \theta_\phi(\theta_\phi - 2\theta_{\psi_2} - 1) - \phi^5 \theta_\phi(\theta_\phi + 1). \end{aligned} \tag{6.2.21}$$

There are four ϕ -independent solutions which are determined purely by the geometry of the threefold, and four additional ϕ -dependent periods. Of these, the relevant flat coordinate and superpotential are given by

$$\begin{aligned} t &= \phi \left(1 + \frac{1}{15} \phi^5 + \frac{7}{275} \phi^{10} + \dots \right), \\ \Pi_2 &= \phi^3 \psi_2 \left(1 + \frac{3}{10} \phi^5 + \frac{54}{325} \phi^{10} + \dots \right). \end{aligned} \tag{6.2.22}$$

It is worth noting that there is no solution which could correspond to a *massive* superpotential for ϕ . The superpotential Π_2 can be written at leading order in ψ_2 in hypergeometric form,

$$\Pi_2 = (\psi_2 \phi^3) {}_2F_1 \left(\frac{3}{5}, \frac{4}{5}; \frac{8}{5}; \phi^5 \right) + \dots \tag{6.2.23}$$

which agrees with equation (3.14) of [138]. The superpotential should again be expressed in the physical normalization in terms of flat open and closed-string coordinates, giving

$$\begin{aligned} \mathcal{W}(z, t) = \frac{\Pi_2(t, z)}{\Pi_0(z)} &= t^3 z \left(1 + \frac{1}{10} t^5 + \frac{69}{31250} z^5 + \frac{248}{10725} t^{10} - \right. \\ &\quad \left. \frac{24}{78125} t^5 z^5 + \frac{98999}{5371093750} z^{10} + \dots \right) \end{aligned} \tag{6.2.24}$$

6.2.2. *Toric branes on the mirror quintic*

A class of D-branes which has been well studied in the context of open-string mirror symmetry are “toric branes,” first described in [30], to which we defer for their description in terms of toric geometry. They were originally introduced as non-compact brane geometries in local Calabi-Yau threefolds, but more recently, progress has been made in understanding the extension to the compact case [134,135]. We consider one of these examples from the perspective of the dual fourfold, and reproduce the same Picard-Fuchs equations and superpotentials which were derived in [135] based on a somewhat formal application of toric geometry/GLSM techniques.

The bulk geometry is the one-parameter family of quintic hypersurfaces in \mathbb{P}^4 given by⁶⁷

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \psi x_1 x_2 x_3 x_4 x_5 = 0.$$

The D3 brane geometries of interest are degree-two rational curves, with the following parameterization,

$$(x_1, x_2, x_3, x_4, x_5) = (u^2, \alpha u^2, v^2, \beta v^2, (-\alpha\beta\psi)^{1/4} uv),$$

where $\alpha^5 = \beta^5 = -1$. These are rigid curves, so there are no massless open-string moduli.

In [135], these branes were described in the framework of toric geometry by identifying them as components of the intersection in the mirror quintic of two of the following three toric divisors,

$$x_1^5 + x_2^5 = 0, \quad x_3^5 + x_4^5 = 0, \quad x_5^5 + \psi x_1 x_2 x_3 x_4 x_5 = 0.$$

By choosing one of these as the physical divisor for our prescription, we expect to reproduce the D3 brane superpotential with respect to certain massive open-string deformations as well as the bulk modulus ψ . As opposed to the previous examples, there are no privileged massive deformations that are singled out by the brane geometry. Consequently, there is no preferred choice for the physical divisor, and the off-shell superpotentials for the different branes will not match. However, the *on-shell* value of the superpotential with respect to the open-string degrees of freedom should be independent of the choice of divisor. Since the first and second divisors are related by a permutation symmetry, we will consider

$$D_1(\phi_1) = x_1^5 + \phi_1 x_2^5, \quad D_2(\phi_2) = x_5^5 + \phi_2 x_1 x_2 x_3 x_4 x_5, \quad (6.2.25)$$

⁶⁷ For our purposes, this can be either a one-dimensional slice of the complex structure moduli space of the quintic, or the \mathbf{Z}_5^3 orbifold of these geometries which constitutes the mirror quintic.

which contain information about the supersymmetric D3 branes at $\phi_1 = 1$ and $\phi_2 = \psi$, respectively. The relevant line bundles for this configuration are⁶⁸

$$\begin{aligned}\mathcal{L} &= \mathcal{O}(5) \rightarrow \mathbb{P}^4|_{X_3}, \\ \mathcal{L}_1 &= \mathcal{O}(5) \rightarrow \mathbb{P}^4|_{X_3}, \\ \mathcal{L}_2 &= \mathcal{O}(0) \rightarrow \mathbb{P}^4|_{X_3}.\end{aligned}\tag{6.2.26}$$

The dual fourfold is a complete intersection in the weighted projective space $\mathbb{P}_{111151}^5 \times \mathbb{C}$, with the points $(0 : 0 : 0 : 0 : 0 : x_6, ; x_7)$ removed. The defining equations are

$$Q(\psi) = \sum_{i=1}^5 x_i^5 + \psi x_1 x_2 x_3 x_4 x_5 = 0, \quad P(\phi_i) = x_6 x_7 + D_i(\phi_i) = 0.\tag{6.2.27}$$

We now turn to the derivation and solution of the Picard-Fuchs equations for these fourfolds.

For the first divisor, $D_1(\phi_1)$, we proceed in a manner analogous to the previous examples and find, after factorization, the following Picard-Fuchs operators,⁶⁹

$$\begin{aligned}\mathcal{L}_1 &= \theta_z^4 - \theta_z^2 \theta_{\phi_1}^2 + z \prod_{k=1}^4 (5\theta_z + k), \\ \mathcal{L}_2 &= \theta_{\phi_1} (\theta_{\phi_1} + \theta_z) - \phi_1 \theta_{\phi_1} (\theta_{\phi_1} - \theta_z).\end{aligned}\tag{6.2.28}$$

where we expand about the large complex structure point of the mirror quintic, so have introduced $z = \psi^{-5}$. The holomorphic curves in question are located at $\phi_1 = 1$, and so we look for solutions to these equations expanded about that point, as a function of $\widehat{\phi} = \phi_1 - 1$. There are precisely two solutions which are functions of $\widehat{\phi}_1$ and finite at $z = 0$,

$$\begin{aligned}t &= \widehat{\phi} - \frac{1}{2}\widehat{\phi}^2 + \frac{1}{3}\widehat{\phi}^3(1 - 60z) - \frac{1}{4}\widehat{\phi}^4(1 - 120z) + \dots \\ \Pi_1 &= \sqrt{z} \left(1 + \frac{5005}{9}z + \frac{52055003}{75}z^2 + \frac{283649836041}{245}z^3 + \frac{8908737478232449}{3969}z^4 \right. \\ &\quad \left. + \frac{1}{8}\widehat{\phi}^2 - \frac{1}{8}\widehat{\phi}^3 + \frac{15}{128}\widehat{\phi}^4 + \dots \right).\end{aligned}\tag{6.2.29}$$

In addition, there are four solutions which comprise the usual set of closed-string periods on the mirror quintic. After appropriately normalizing, these

⁶⁸ Alternatively, one could choose $\mathcal{L}_u = \mathcal{O}(a)$ and $\mathcal{L}_v = \mathcal{O}(b)$ for any $a + b = 5$. Such a choice does not affect the present discussion.

⁶⁹ These can be obtained most easily as the GKZ-operators associated with the charge vectors which define the toric bulk/brane geometry, as in [135]. However, it is easy to see that the fourfold introduced above gives rise to the same operators.

are the open-string flat coordinate and superpotential, respectively. This leads to the following expressions for the physically normalized superpotential as a function of the flat open/closed-string coordinates,

$$\mathcal{W}_{phys}(q, t) = \mathcal{W}_{closed}(q) + \frac{15}{8}q^{1/2}t^2 (1 - 265q + \dots) \quad (6.2.30)$$

Where \mathcal{W}_{closed} is the *on-shell* superpotential as a function of the closed-string moduli when the open-string coordinate is fixed at its critical point,

$$\mathcal{W}_{closed}(q) = 15q^{1/2} + \frac{2300}{3}q^{3/2} + \frac{2720628}{5}q^{5/2} + \frac{23911921125}{49}q^{7/2} + \dots \quad (6.2.31)$$

This result precisely matches those of [122,135], which were obtained through a variety of different methods.

We now turn to the second divisor, $D_2(\phi_2)$. This is precisely the divisor which was studied in [135], and we reproduce the results here for completeness. Following [135], we choose to work in terms of algebraic variables

$$z_1 = -\phi_2^{-1}\psi^{-4}, \quad z_2 = -\phi_2\psi^{-1},$$

with respect to which the Picard-Fuchs operators are given by

$$\begin{aligned} \mathcal{L}_1 &= \theta_1^5 - \theta_1^4\theta_2 - z_1 \prod_{k=1}^4 (4\theta_1 + \theta_2 + k)(\theta_1 - \theta_2), \\ \mathcal{L}_2 &= \theta_1^2 - \theta_1\theta_2 - z_2(4\theta_1 + \theta_2 + 1)(\theta_1 - \theta_2), \\ \mathcal{L}'_1 &= \theta_2\theta_1^4 + z_1z_2 \prod_{k=1}^5 (4\theta_1 + \theta_2 + k). \end{aligned} \quad (6.2.32)$$

The expected critical point is at $z_2 = -1$, $z_1 = \psi^{-5}$. To find a good expansion for the solutions to the Picard-Fuchs equations, we introduce coordinates

$$u = z_1^{-1/4}(1 + z_2), \quad v = z_1^{-1/4},$$

as a function of which the superpotential can be found in a power-series expansion,

$$\begin{aligned} \Pi_2 &= \frac{1}{8}u^2 + 15v^2 + \frac{5}{48}u^3v - \frac{15}{2}uv^3 + \frac{1}{46080}u^6 + \frac{35}{384}u^4v^2 - \\ &\quad \frac{15}{8}u^2v^4 + \frac{25025}{3}v^6 + \dots \end{aligned} \quad (6.2.33)$$

It can be verified that this superpotential has a critical point with respect to the open-string variation at $u = 0$, as predicted by the geometry. Moreover, by setting the open-string deformation to zero, normalizing, and expressing the result in terms of the flat closed-string coordinate, the derived *on-shell* superpotential matches the results of [122],

$$\mathcal{W} = 15q^{1/2} + \frac{2300}{3}q^{3/2} + \frac{2720628}{5}q^{5/2} + \frac{23911921125}{49}q^{7/2} \dots \quad (6.2.34)$$

This matches the results obtained above using the first divisor (6.2.31), although the physical theories on the different NS5 branes are inequivalent.

6.2.3. $\mathbb{P}^5[3, 3]$

Our next example is a one-parameter complete intersection Calabi-Yau, studied recently in [126]. The geometry is mirror to the intersection of two cubics in \mathbb{P}^5 , and can be described as the quotient⁷⁰

$$X_3 = \{W_1 = 0, W_2 = 0\}/G$$

Where the W_1 and W_2 are the most general cubic polynomials invariant under the appropriate discrete symmetry group $G = \mathbf{Z}_3^2 \times \mathbf{Z}_9$,

$$\begin{aligned} W_1 &= \frac{x_1^3}{3} + \frac{x_2^3}{3} + \frac{x_3^3}{3} - \psi x_4 x_5 x_6 \\ W_2 &= \frac{x_4^3}{3} + \frac{x_5^3}{3} + \frac{x_6^3}{3} - \psi x_1 x_2 x_3 \end{aligned}$$

The curves studied in [126] are determined by the intersection of two hyperplane divisors in X_3 ,

$$D_1 = \{x_1 + x_2 = 0\}, \quad D_2 = \{x_4 + x_5 = 0\}.$$

This intersection is reducible, being comprised of one line and two degree four curves, the rational parameterizations of which can be found in [126].

We introduce a fivebrane on the divisor D_1 , which we embed in the one-parameter family of divisors,

$$D_1(\phi) = x_1 + \phi x_2 \tag{6.2.35}$$

The line bundles for this configuration are then

$$\begin{aligned} \mathcal{L}_D &= \mathcal{O}(1) \rightarrow \mathbb{P}^5|_{X_3}, \\ \mathcal{L}_1 &= \mathcal{O}(1) \rightarrow \mathbb{P}^5|_{X_3}, \\ \mathcal{L}_2 &= \mathcal{O}(0) \rightarrow \mathbb{P}^5|_{X_3}, \end{aligned} \tag{6.2.36}$$

and the dual fourfold is a complete intersection

$$W_1(\psi) = 0, \quad W_2(\psi) = 0, \quad P(\phi) = x_6 x_7 + D_1(\phi) = 0, \tag{6.2.37}$$

in $\mathbb{P}^6 \times \mathbb{C}$ with the points $(0 : 0 : 0 : 0 : 0 : 0 : x_7 ; x_8)$ removed. Picard-Fuchs operators can be derived in the usual way, leading to

$$\begin{aligned} \mathcal{L}_1 &= (\theta_z^4 - \theta_w^2) \theta_z^2 - 9z(3\theta_z + 2)^2(3\theta_z + 1)^2, \\ \mathcal{L}_2 &= \theta_w(\theta_z + \theta_w) + w\theta_w(\theta_z - \theta_w), \end{aligned} \tag{6.2.38}$$

⁷⁰ As in the examples on the mirror quintic, one can equally well consider this to be a special case of the larger A-mode geometry, $\mathbb{P}^5[3, 3]$.

where we've introduced local variables $z = (3\psi)^{-6}$, $w = \phi^3$. The holomorphic curves in question are located at $\phi = 1$, so we find solutions expanded about the point $\widehat{w} = w - 1 = 0$. There are two \widehat{w} -dependent solutions which are finite at $z = 0$,

$$\begin{aligned} t &= \widehat{w} - \frac{1}{2}\widehat{w}^2 + \frac{1}{3}\widehat{w}^3 - \frac{1}{4}\widehat{w}^4 - 6z\widehat{w}^3 + \frac{1}{5}\widehat{w}^5 + 9z\widehat{w}^4 + \dots \\ \Pi &= \Pi_{closed}(z) + \sqrt{z}\widehat{w}^2\left(\frac{1}{8} - \frac{1}{8}\widehat{w} + \frac{15}{128}\widehat{w}^2 - \frac{1225}{384}z\widehat{w}^2 - \frac{7}{64}\widehat{w}^3 + \dots\right) \end{aligned} \quad (6.2.39)$$

Where Π_{closed} is the part of the superpotential which depends only on the closed-string moduli, i.e., the *on-shell* part that is accessible to the methods of [126].

$$\Pi_{closed} = \sqrt{z}\left(1 + \frac{1225}{9}z + \frac{1002001}{25}z^2 + \frac{19200813489}{1225}z^3 + \frac{28214528710225}{3969}z^4\right). \quad (6.2.40)$$

These periods are the flat open-string coordinate and superpotential, respectively. Again, the periods should be normalized by the fundamental period of X_3 and expressed in terms of the flat coordinates. The resulting superpotential is

$$\mathcal{W}(q, t) = \mathcal{W}_{cl} + q^{1/2}t^2 \left(\frac{9}{4} - \frac{243}{2}q - \frac{10935}{4}q^2 + \frac{3}{64}t^2 + \dots \right) \quad (6.2.41)$$

where, \mathcal{W}_{cl} is the *on-shell* superpotential obtained by setting $t \rightarrow 0$,

$$\begin{aligned} \mathcal{W}_{cl}(q) &= 18q^{1/2} + 182q^{3/2} + \frac{787968}{25}q^{5/2} + \\ &\quad \frac{323202744}{49}q^{7/2} + \frac{15141625184}{9}q^{9/2} + \dots \end{aligned} \quad (6.2.42)$$

This matches equation (2.37) of [126].

6.2.4. $\mathbb{P}_{112112}^5[4, 4]$

As our final example, we consider another one-parameter complete intersection Calabi-Yau from [126]. This time, the A-model geometry is the intersection of two degree-four hypersurfaces in the weighted projective space \mathbb{P}_{112112}^5 . The mirror geometry is given by the quotient

$$X_3 = \{W_1 = 0, W_2 = 0\}/G$$

Where the W_1 and W_2 are the most general degree-four polynomials invariant under the appropriate discrete symmetry group $G = \mathbf{Z}_2^2 \times \mathbf{Z}_{16}$,

$$\begin{aligned} W_1 &= \frac{x_1^4}{4} + \frac{x_2^4}{4} + \frac{x_3^2}{2} - \psi x_4 x_5 x_5, \\ W_2 &= \frac{x_4^4}{4} + \frac{x_5^4}{4} + \frac{x_6^2}{2} - x_1 x_2 x_3. \end{aligned}$$

The curves in question are contained in the intersection in X_3 of the two divisors

$$D_1 = \{x_1^2 + \alpha_1 \sqrt{2}x_3 = 0\}, \quad D_2 = \{x_2^2 + \alpha_2 \sqrt{2}x_6 = 0\}.$$

where $\alpha_i = \pm i$. The intersection consists of one line and three degree-five curves.

We introduce a fivebrane on the divisor D_1 , which is embedded into the one-parameter family of divisors given by

$$D(\phi) = x_1^2 + \phi x_3. \quad (6.2.43)$$

The line bundles for this configuration can be chosen as

$$\begin{aligned} \mathcal{L}_D &= \mathcal{O}(2) \rightarrow \mathbb{P}^5|_{X_3}, \\ \mathcal{L}_1 &= \mathcal{O}(1) \rightarrow \mathbb{P}^5|_{X_3}, \\ \mathcal{L}_2 &= \mathcal{O}(1) \rightarrow \mathbb{P}^5|_{X_3}, \end{aligned} \quad (6.2.44)$$

and the dual fourfold is a complete intersection

$$W_1(\psi) = 0, \quad W_2(\psi) = 0, \quad P(\phi) = x_6 x_7 + D(\phi) = 0, \quad (6.2.45)$$

in $\mathbb{P}_{11211211}^8$ with the points $(0 : 0 : 0 : 0 : 0 : 0 : x_7 : x_8)$ removed. Picard-Fuchs operators can be derived in the usual way, leading to

$$\begin{aligned} \mathcal{L}_1 &= (\theta_z + \theta_w)(2\theta_z - \theta_w - 1)(2\theta_z - \theta_w)\theta_z^2 - \\ &\quad 16z(4\theta_z + 3)^2(4\theta_z + 2)(4\theta_z + 1)^2, \\ \mathcal{L}_2 &= (\theta_z + \theta_w)\theta_w - \frac{1}{2}w(2\theta_z - \theta_w)\theta_w, \end{aligned} \quad (6.2.46)$$

where we've introduced local variables $z = (8\psi)^{-4}$, $w = \phi^2$. The holomorphic curves in question are located at $\phi = \pm i\sqrt{2}$, so we look for solutions expanded about the point $\hat{w} = w+1 = 0$. Amongst the \hat{w} -dependent solutions, we identify the periods which correspond to open-string superpotentials,

$$\begin{aligned} \Pi_1 &= z^{1/3} \left(1 + \frac{8281}{16}z + \frac{38130625}{49}z^2 + \frac{80263989481}{49}z^3 + \frac{1}{36}\hat{w}^2 + \frac{1}{81}\hat{w}^3 + \dots \right), \\ \Pi_2 &= z^{2/3} \left(1 + \frac{559504}{625}z + \frac{15557323441}{10000}z^2 + \frac{10590422929849}{3025}z^3 + \frac{1}{162}\hat{w}^3 + \dots \right). \end{aligned} \quad (6.2.47)$$

Along with the closed-string periods, we can compute the on-shell superpotential in terms of closed-string flat coordinates, finding

$$\begin{aligned} \mathcal{W}(q) &= 24q^{1/3} + 150q^{2/3} + \frac{2571}{2}q^{4/3} + \frac{417024}{25}q^{5/3} + \\ &\quad \frac{45420672}{49}q^{7/3} + \frac{131074059}{8}q^{8/3} + \dots \end{aligned}$$

Again, we've chosen the linear combination of solutions to match the results of [126].

Appendix 6.A. Toric Methods

In this appendix, we demonstrate the equivalence of the fourfolds derived by T-duality with those obtained using toric geometry/GLSM techniques in the case of toric branes [135]. We will take one of the branes of section 3.2 as our example. From the toric data which defines the brane geometry in question, the methods of [153] allow for a derivation of the appropriate B-model geometry.

Recall that in the context of toric geometry, the quintic is described by the charge vector of the associated GLSM,

$$Q_1 = (-5, 1, 1, 1, 1, 1).$$

Moreover, a certain class of “toric branes” can also be encoded in a similar charge vector [30]. In particular, for the charge vector

$$Q_2 = (1, -1, 0, 0, 0, 0),$$

the mirror B-brane on the mirror quintic wraps the divisor

$$x_1^5 + \phi x_1 x_2 x_3 x_4 x_5 = 0.$$

The approach of [135,130,132] was to “enhance” these charge vectors to define an auxiliary GLSM for the open/closed-string geometry as follows

$$\begin{aligned} Q'_1 &= (-5, 1, 1, 1, 1, 1; 0, 0), \\ Q'_2 &= (1, -1, 0, 0, 0, 0; 1, -1). \end{aligned} \tag{6.A.1}$$

This toric data then defines a system of GKZ differential operators. Defining coordinates on the complex structure moduli space for the B-model geometry according to

$$z_a = (-)^{Q'_{a,0}} \prod_{i=0}^7 a_i^{Q'_{a,i}}, \tag{6.A.2}$$

the differential operators which annihilate the periods of the holomorphic four-form are

$$\prod_{k=1}^{\ell_0^a} (\theta_{a_0} - k) \prod_{\ell_i^a > 0} \prod_{k=0}^{\ell_i^a - 1} (\theta_{a_i} - k) - (-1)^{\ell_0^a} z_a \prod_{k=1}^{-\ell_0^a} (\theta_{a_0} - k) \prod_{\ell_i^a < 0} \prod_{k=0}^{-\ell_i^a - 1} (\theta_{a_i} - k), \tag{6.A.3}$$

where $\theta_{a_i} = \sum_a Q'_{a,i} \theta_{z_a}$. It is straightforward to check that the operators defined in this way match those obtained by our methods in section three.

Moreover, by applying the methods of [153], we can further derive the B-model geometry which is defined by this toric data. Starting with the non-compact toric variety given by (6.A.1), the mirror Landau-Ginzburg theory has twisted superpotential

$$\widetilde{W} = \sum_{a=1}^2 \Sigma_a \left(\sum_{i=0}^7 Q_{a,i} Y_i - t_a \right) + \sum_{i=0}^7 e^{-Y_i}. \tag{6.A.4}$$

The period integral which computes BPS masses in the related compact theory is then given by⁷¹

$$\Pi = \int d\Sigma_1 d\Sigma_2 \left(\prod_{i=0}^7 dY_i \right) (5\Sigma_1 - \Sigma_2) \exp \left(-\widetilde{W} \right), \quad (6.A.5)$$

where the term $(5\Sigma_1 - \Sigma_2)$ has been inserted to render the theory (partially) compact. Manipulation of this period integral leads to a representation of the related B-model Calabi-Yau geometry as a hypersurface as follows:

$$\begin{aligned} \Pi &= \\ &= \int d\Sigma_1 d\Sigma_2 \left(\prod_{i=0}^7 dY_i \right) \frac{\partial}{\partial Y_0} \exp \left(-\sum_{a=1}^2 \Sigma_a \left(\sum_{i=0}^7 Q_{a,i} Y_i - t_a \right) \right) \exp \left(-\sum_{i=0}^7 e^{-Y_i} \right), \\ &= \int d\Sigma_1 d\Sigma_2 \left(\prod_{i=0}^7 dY_i \right) e^{-Y_0} \exp \left(-\sum_{a=1}^2 \Sigma_a \left(\sum_{i=0}^7 Q_{a,i} Y_i - t_a \right) \right) \exp \left(-\sum_{i=0}^7 e^{-Y_i} \right), \\ &= \int \left(\prod_{i=0}^7 dY_i \right) e^{-Y_0} \delta \left(\sum_{i=0}^7 Q_{1,i} Y_i - t_1 \right) \delta \left(\sum_{i=0}^7 Q_{2,i} Y_i - t_2 \right) \exp \left(-\sum_{i=0}^7 e^{-Y_i} \right). \end{aligned} \quad (6.A.6)$$

At this point, there are two sets of manipulations which lead to different, but equivalent, representations of the B-model geometry. In order to make contact with the results of section three, one may make the following change of variables,

$$\begin{aligned} e^{-Y_0} &= P \\ e^{-Y_i} &= e^{-t_1/5} P \frac{z_1^5}{x_1 x_2 x_3 x_4 x_5} \quad i = 1, \dots, 5 \\ e^{-Y_6} &= Z, \end{aligned} \quad (6.A.7)$$

the first delta function is satisfied automatically. This change of variables is only one-to-one after dividing out by a $\mathbf{C}^* \times \mathbf{Z}_5^3$ action under which (6.A.7) is invariant. Evaluating the second delta function to fix Y_7 as well, the periods become,

$$\Pi = \int \left(\prod_{i=1}^5 dx_i \right) \frac{dZ}{Z} \delta \left(\mathcal{G}(x_i) \right) \exp \left(Z \left(1 + e^{-t_2 - t_1/5} \frac{x_1^5}{x_1 x_2 x_3 x_4 x_5} \right) \right) \quad (6.A.8)$$

where

$$\mathcal{G}(x_i) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \psi x_1 x_2 x_3 x_4 x_5, \quad (6.A.9)$$

⁷¹ In all manipulations, we suppress the explicit contours of integration, which are period-dependent.

and $\psi = \exp(t_1/5)$. Introducing new fields \tilde{u} and \tilde{v} and inserting the identity in the form,

$$1 = Z \int d\tilde{u} d\tilde{v} e^{\tilde{u}Z\tilde{v}},$$

(6.A.8) becomes

$$\Pi = \int \left(\prod_{i=1}^5 dx_i \right) du dv \delta(\mathcal{G}(x_i)) \delta \left(\tilde{u}\tilde{v} + 1 + \phi \frac{x_1^5}{x_1 x_2 x_3 x_4 x_5} \right). \quad (6.A.10)$$

A final coordinate redefinition,

$$\begin{aligned} u &= x_1 x_2 x_3 x_4 x_5 \tilde{u} \\ v &= \tilde{v}, \end{aligned} \quad (6.A.11)$$

leads to an expression which describes periods of the holomorphic four-form on a non-compact Calabi-Yau fourfold,

$$\Pi = \int \left(\prod_{i=1}^5 dx_i \right) du dv \delta(\mathcal{G}(x_i; \psi)) \delta(\mathcal{P}(x_i; u, v; \phi)), \quad (6.A.12)$$

where

$$\mathcal{P}(\phi) = uv + x_1^5 + \phi x_1 x_2 x_3 x_4 x_5, \quad (6.A.13)$$

and $\phi = \exp(t_2 + t_1/5)$. This precisely matches the fourfold which was obtained by T-duality considerations in section three.

Alternatively, starting with (6.A.6), we can introduce variables as follows,

$$\begin{aligned} e^{-Y_0} &= P \\ e^{-Y_i} &= e^{-t_1/5} P \frac{x_1^5}{x_1 x_2 x_3 x_4 x_5} \quad i = 1, \dots, 5 \\ e^{-Y_6} &= P \frac{e^z}{x_1 x_2 x_3 x_4 x_5}. \end{aligned} \quad (6.A.14)$$

Again, the first delta function is satisfied automatically, while the second delta function can be enforced to fix Y_7 in terms of the other variables. The resulting periods are of the form

$$\Pi = \int \left(\prod_{i=1}^5 dx_i \right) dz dP \exp \left(P \mathcal{H}(\psi, \phi) \right), \quad (6.A.15)$$

where

$$\mathcal{H} = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \psi x_1 x_2 x_3 x_4 x_5 + e^w (x_1^5 + \phi x_1 x_2 x_3 x_4 x_5) \quad (6.A.16)$$

with ψ and ϕ defined above. These periods are then given by integrals of the holomorphic four-form on the Calabi-Yau fourfold defined by $\mathcal{H} = 0$. This formulation makes manifest the structure of the fourfold as a fibration of a Calabi-Yau threefold over a cylinder, as in [131,154,155].

A similar analysis to these can be carried out for the other toric examples. However, the derivation in terms of T-duality is more general, since it also applies to cases which cannot be described within the ‘‘toric brane’’ framework.

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