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Author

Gottlieb, Alexander

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Ernest Orlando Lawrence Berkeley National Laboratory

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Alexander Gottlieb

Computing Sciences Directorate
Mathematics Department

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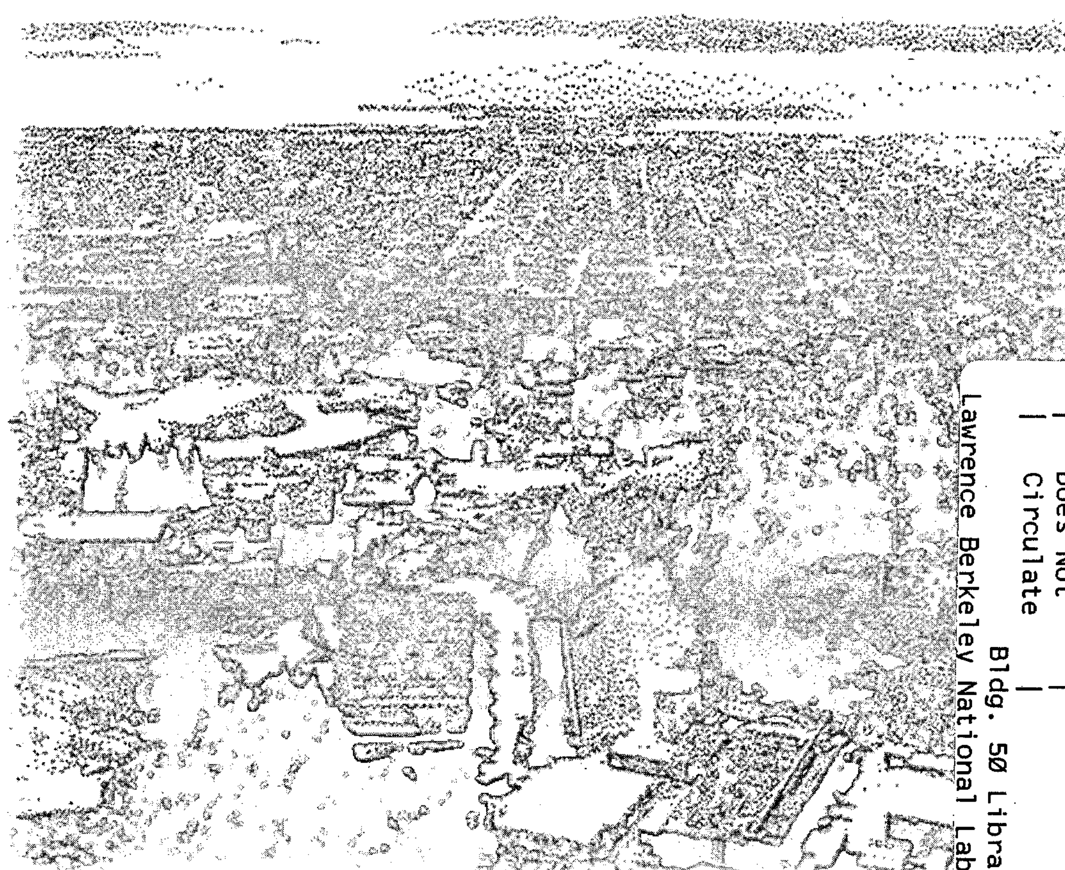

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THE PROPAGATION OF MOLECULAR CHAOS BY MARKOV TRANSITIONS*

Alexander Gottlieb
Mathematics Department
Computing Sciences Directorate
Lawrence Berkeley National Laboratory
Berkeley, CA 94720

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THE PROPAGATION OF MOLECULAR CHAOS BY MARKOV TRANSITIONS

By Alexander Gottlieb

Abstract

We establish a necessary and sufficient condition for the propagation of chaos by a family of many-particle Markov processes, if the particles live in a Polish space S: A sequence of n-particle Markov transition functions $\{K_n\}$ propagates molecular chaos if and only if the sequence $\{K_n(\mathrm{s}_n, \cdot)\}$ is chaotic whenever $\mathrm{s}^n = (s_1^n, s_2^n, \ldots, s_n^n) \in S^n$ is such that $\frac{1}{n}\sum_i \delta(s_i^n)$ converges to a law on S as $n \longrightarrow \infty$. [1] [2]

1 Introduction

The "propagation of chaos" means the persistence in time of molecular chaos (i.e., the stochastic independence of two random particles in a many-particle system) in the limit of infinite particle number. Propogation of chaos is an important concept of kinetic theory that relates the equations of Boltzmann and Vlasov to the dynamics of many-particle systems.

The concept of propagation of chaos originated with Kac's Markovian models of gas dynamics [9, 10]. Kac invented a class of interacting particle systems wherein particles collide at random with each other while the *density* of particles evolves deterministically in the limit of infinite particle number. A nonlinear evolution equation analogous to Boltzmann's equation governs the particle density. The processes of Kac were further investigated with regard to their fluctuations about the deterministic infinite particle limit in [13, 22, 23]. McKean introduced propagation of chaos for interacting diffusions and analyzed what are now called McKean-Vlasov equations

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[11, 12]. Independently, Braun and Hepp [4] analyzed the propagation of chaos for Vlasov equations and proved a central limit theorem for the fluctuations. Analysis of the fluctuations and large deviations for McKean-Vlasov processes was carried out in [21, 19, 5]. Other instances of the propagation of chaos have been studied in [15, 17, 8]. Finally, we refer the reader to the long, informative articles by Sznitman [20] and by Méléard [14] in Springer-Verlag's *Lecture Notes in Mathematics.*

This article is organized as follows:

In Section 2, molecular chaos is defined and a class of examples is provided. The weak convergence approach to the study of molecular chaos due to Sznitman and Tanaka is introduced in Theorem 2.1. In Section 3, propagation of chaos is defined, and the main results of this article are stated in Theorem 3.1 and its corollary. Theorem 3.1 is proved in Section 4 using Prohorov's theorem and Theorem 2.1 of Section 2.

2 Molecular Chaos

In what follows, if (S, d_S) is a separable metric space, its Borel algebra will be denoted $\mathcal{B}(S)$, the space of bounded and continuous functions on S will be denoted $C(S)$, and the space of probability laws on S with the weak topology will be denoted $\mathcal{P}(S)$.

Let S^n denote the n-fold product of S with itself;

$$S^n := \{(s_1, s_2, \ldots, s_n) : s_i \in S \quad \text{for} \quad i = 1, 2, \ldots, n\}.$$

S^n is itself metrizable in a variety of equivalent ways that all generate the same topology and the same Borel algebra $\mathcal{B}(S^n)$. If ρ_n is a law on S^n and $k \leq n$, let $\rho_n^{(k)}$ denote the marginal law on the first k coordinates, that is, the law on S^k induced by the map $(s_1, s_2, \ldots, s_n) \mapsto (s_1, \ldots, s_k)$. If $\rho \in \mathcal{P}(S)$, let $\rho^{\otimes n}$ denote the product law on S^n defined by $\int_{S^n} f_1(s_1) \cdots f_n(s_n) \rho^{\otimes n}(ds_1 \cdots ds_n) = \prod_{i=1}^n \int_S f_i(x) \rho(dx)$.

Let Π_n denote the set of permutations of $\{1, 2, \ldots, n\}$. The permutations Π_n act on S^n by permuting coordinates: the map $\pi\cdot : S^n \longrightarrow S^n$ is

$$\pi \cdot (s_1, s_2, \ldots, s_n) := (s_{\pi(1)}, s_{\pi(2)}, \ldots, s_{\pi(n)}).$$

If E is any subset of S^n, define

$$\pi \cdot E = \{\pi \cdot \mathrm{s} : \mathrm{s} \in E\}.$$

A law ρ on S^n is *symmetric* if $\rho(\pi \cdot B) = \rho(B)$ for all $\pi \in \Pi_n$ and all $B \in \mathcal{B}(S^n)$. Products $\rho^{\otimes n}$ are symmetric, for example. The *symmetrization* $\widetilde{\rho}$ of a law $\rho \in \mathcal{P}(S^n)$ is the symmetric law such that

$$\widetilde{\rho}(B) := \frac{1}{n!} \sum_{\pi \in \Pi_n} \rho(\pi \cdot B),$$

for all $B \in \mathcal{B}(S^n)$.

For each n, define the map $\varepsilon_n : S^n \longrightarrow \mathcal{P}(S)$ by

$$\varepsilon_n((s_1, s_2, \dots, s_n)) := \frac{1}{n} \sum_{i=1}^{n} \delta(s_i), \tag{1}$$

where $\delta(x)$ denotes a point-mass at x. These maps are measurable for each n, and $\varepsilon_n(\pi \cdot \mathbf{s}) = \varepsilon_n(\mathbf{s})$ for all $\mathbf{s} \in S^n, \pi \in \Pi_n$.

To derive the Boltzmann equation for a dilute gas, Boltzmann assumed that a condition of "molecular disorder" obtains, i.e., the velocities of two random molecules in a gas are stochastically independent [2]. This is not an entirely realistic assumption: surely the collisions between the molecules of an n-particle gas must introduce some stochastic dependence, even if the molecules were independent to begin with. Nonetheless, if the number n of molecules is very large, one might well expect that the velocities of two random molecules would be *nearly* independent; the presence of so many other molecules should wash out most of the dependence between two randomly selected molecules. These considerations motivated Kac to define molecular disorder as a kind of asymptotic independence of pairs of particles in the infinite-particle limit. He called the definiendum the "Boltzmann property" instead of "molecular chaos" or simply "chaos," as it is now termed. Kac's definition is

Definition 2.1 (Kac, 1954) *Let (S, d_S) be a separable metric space. Let ρ be a law on S, and for $n = 1, 2, \dots$, let ρ_n be a symmetric law on S^n.*

*The sequence $\{\rho_n\}$ is ρ-**chaotic** if, for each natural number k and each choice $\phi_1(s), \phi_2(s), \dots, \phi_k(s)$ of k bounded and continuous functions on S,*

$$\lim_{n \to \infty} \int_{S^n} \phi_1(s_1)\phi_2(s_2)\cdots\phi_k(s_k)\rho_n(ds_1 ds_2 \dots ds_n) = \prod_{i=1}^{k} \int_S \phi_i(s)\rho(ds).$$

It is now known [18, 21, 7] that a sequence of symmetric laws $\{\rho_n\}$ (with $\rho_n \in \mathcal{P}(S^n)$) is ρ-chaotic if and only if the marginals $\rho_n^{(k)}$ converge weakly to $\rho^{\otimes k}$ as $n \longrightarrow \infty$ for each fixed k. (We suggest that this last condition be taken for the *definition* of chaos in the future, in view of its simplicity and equivalence to Kac's definition.) The following theorem gives conditions that are equivalent to molecular chaos.

Theorem 2.1 (Sznitman, Tanaka) *Let S be a separable metric space, and for each n let ρ_n be a symmetric law on S^n.*

The following are equivalent:

(i) $\{\rho_n\}$ is ρ-chaotic;

(ii) For all $\phi_1, \phi_2 \in C_b(S)$,

$$\lim_{n\to\infty} \int_{S^n} \phi_1(s_1)\phi_2(s_2)\rho_n(ds) = \int_S \phi_1(s)\rho(ds) \int_S \phi_2(s)\rho(ds);$$

(iii) The marginals $\rho_n^{(2)}$ converge weakly to $\rho \otimes \rho$ as n tends to infinity.

(iv) For all k, the marginals $\rho_n^{(k)}$ converge weakly to $\rho^{\otimes k}$ as n tends to infinity.

(v) The laws $\rho_n \circ \varepsilon_n^{-1}$ converge to $\delta(\rho)$ in $\mathcal{P}(\mathcal{P}(S))$ as n tends to infinity;

An important class of chaotic sequences arises in connection with the principle of equivalence of microcanonical and canonical averaging of statistical mechanics. The principle follows from the molecular chaos of sequences of microcanonical distributions. For example, let S be a finite set and let $H : S \longrightarrow \mathbb{R}$ be any function that we call energy: $H(s)$ is the energy of a particle if it is the state $s \in S$. Fix $E \in \mathbb{R}$, to be interpreted as average energy per particle, and $\delta > 0$, and define $\mu_{n,E,\delta} \in \mathcal{P}(S^n)$ to be the law that gives equal probability to each n-particle state $(s_1, s_2, \ldots, s_n)$ that satisfies $\frac{1}{n}\sum_{i=1}^n H(s_i) \in (E - \frac{1}{2}\delta, E + \frac{1}{2}\delta)$ and probability 0 to all other points of S^n. Thus, the "microcanonical distributions" $\mu_{n,E,\delta}$ render equiprobable all n-particle states for which the energy-per-particle is approximately E. The sequence of microcanonical distributions $\{\mu_{n,E,\delta}\}_{n\in\mathbb{N}}$ is γ-chaotic, where $\gamma \in \mathcal{P}(S)$ is the Gibbsian distribution defined by

$$\gamma(t) = e^{-\beta H(t)} / \sum_{s\in S} e^{-\beta H(s)}$$

for all $t \in S$, where the parameter β depends on E and δ. This follows from Sanov's large deviations theorem.

In closing, we note the following pleasant corollary of Theorem 2.1 (see [7] for a proof):

Corollary 2.1 *Let $S = \{s_1, s_2, \dots, s_k\}$ be a finite set, and for each n let $\rho_n \in \mathcal{P}(S^n)$ be a symmetric law.*

If the sequence $\{\rho_n\}$ is p-chaotic, then

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{\mathbf{x}\in S^n}\rho_n(\mathbf{x})\log\rho_n(\mathbf{x})\right) = \sum_{i=1}^{k} p_i \log p_i.$$

3 The Propagation of Chaos

Let (S, d_S) and (T, d_T) be two separable metric spaces. A Markov transition function K is a function on $S \times \mathcal{B}(T)$ that satisfies the following two conditions:

(1) $K(s,\cdot)$ is a probability measure on $(T, \mathcal{B}(T))$ for each $s \in S$, and

(2) $K(\cdot, E)$ is a measurable function on $(S, \mathcal{B}(S))$ for each $E \in \mathcal{B}(T)$.

A Markov transition function on $S \times \mathcal{B}(T)$ is also called a Markov transition from S to T.

A Markov *process* on a state space (S, d_S) determines a family, indexed by time, of Markov transitions from S to itself: $\{K(s, E, t)\}_{t\geq 0}$. The transitions satisfy the Chapman-Kolmogorov equations

$$K(s, E, t+t') = \int_S K(s, dx, t)K(x, E, t'); \quad t, t' \geq 0, s \in S, E \in \mathcal{B}(S)$$

in addition to (1) and (2) above.

For each n, let K_n be a Markov transition from S^n to T^n. We assume that the Markov transition function K_n is such that if π is a permutation in Π_n and A is a Borel subset of T^n,

$$K_n(\pi\cdot\mathbf{s}, \pi\cdot A) = K_n(\mathbf{s}, A). \tag{2}$$

Definition 3.1 *A sequence $\{K_n\}$ satisfying (2)* **propagates chaos** *if, whenever $\{\rho_n\}$ is a ρ-chaotic sequence of symmetric laws on S^n, the sequence*

$$\left\{\int_{S_n} K_n(\mathbf{s},\cdot)\rho_n(d\mathbf{s})\right\}_{n=1}^{\infty}$$

is τ-chaotic for some $\tau \in \mathcal{P}(T)$.

When we say that a family of n-particle Markov *processes* on a state space S **propagates chaos** we mean that, for each fixed time $t > 0$, the family of associated n-particle transition functions $\{K_n(\mathbf{s}, E, t)\}$ propagates chaos.

Propagation of chaos will be seen to imply the existence of a semigroup of (typically nonlinear) operators F_t on $\mathcal{P}(S)$ such that $\{\int K_n(\mathbf{s},\cdot)\rho_n(d\mathbf{s})\}$ is $F_t\rho$-chaotic if $\{\rho_n\}$ is ρ-chaotic. For families of interacting particle systems suited to the study of gases or plasmas, the semigroup $\{F_t\}_{t\geq 0}$ is the semigroup of solution operators for the Boltzmann or the Vlasov equation.

In case S is Polish, and if Definition 3.1 for the propagation of chaos is accepted (see Remark 2 at the end of this section), we have a necessary and sufficient condition for chaos to be propagated, which is expressed in terms of the maps ε_n defined in (1).

Theorem 3.1 *Suppose (S, d_S) is a complete, separable metric space. Let $\{K_n\}$ be a sequence of Markov transitions that satisfy condition (2), and denote the symmetrization of $K_n(\mathbf{s}_n,\cdot)$ by $\widetilde{K}_n(\mathbf{s}_n,\cdot)$.*

$\{K_n\}$ propagates chaos if and only if there exists a continuous function

$$F : \mathcal{P}(S) \longrightarrow \mathcal{P}(T)$$

such that $\left\{\widetilde{K}_n(\mathbf{s}_n,\cdot)\right\}$ is $F(p)$-chaotic whenever the points $\mathbf{s}_n \in S^n$ are such that $\varepsilon_n(\mathbf{s}_n) \longrightarrow p$ in $\mathcal{P}(S)$.

Propagation of chaos is easily characterized in case the Markov transitions are deterministic, that is, when the n-particle dynamics are given by measurable maps from S^n to T^n that commute with permutations of coordinates. Let $k_n : S^n \longrightarrow T^n$ be a measurable map that commutes with permutations of n-coordinates, i.e., such that

$$k_n(s_{\pi(1)}, s_{\pi(2)}, \ldots, s_{\pi(n)}) = \pi \cdot k_n(s_1, s_2, \ldots, s_n)$$

for each point $\mathbf{s} \in S^n$ and each permutation π of the symbols $1, 2, \ldots, n$. Given k_n, define the Markov transition K_n from S^n to T^n by

$$K_n(\mathbf{s}, E) = \mathbf{1}_E(k_n(\mathbf{s}))$$

when $\mathbf{s} \in S^n$ and $E \in \mathcal{B}_{T^n}$. Say that $\{k_n\}_{n=1}^{\infty}$ *propagates chaos* if the sequence of deterministic transition functions $\{K_n\}$ propagates chaos.

Corollary 3.1 (Deterministic Case) *Let (S, d_S) be a complete and separable metric space, and for each n let k_n be a measurable map from S^n to T^n that commutes with permutations.*

$\{k_n\}$ propagates chaos if and only if there exists a continuous function

$$F : \mathcal{P}(S) \longrightarrow \mathcal{P}(T)$$

such that $\varepsilon_n(k_n(\mathbf{s}_n)) \longrightarrow F(p)$ in $\mathcal{P}(T)$ whenever $\varepsilon_n(\mathbf{s}_n) \longrightarrow p$ in $\mathcal{P}(S)$.

Remark 1: Most Markov processes of interest are characterized by their laws on nice *path spaces*, such as $C([0,\infty),S)$ the space of continuous paths in S, or the space $D([0,\infty),S)$ of right continuous paths in S having left limits. For such processes, the function that maps a state $s \in S$ to the law of the process started at s defines a Markov transition from S to the entire path space. Now, if a sequence of transitions $K_n(s,\cdot)$ from S^n to the path spaces $C([0,\infty),S^n)$ or $D([0,\infty),S^n)$ propagates chaos, then, *a fortiori*, it propagates chaotic sequences of initial laws to chaotic sequences of laws on S^n at any (fixed) later time. We have defined the propagation of chaos for sequences of Markov transitions from S^n to a (possibly) different space T^n, instead of simply from S^n to itself, with the case where T is path space especially in mind. This way, our study will pertain even to those families of processes that propagate the chaos of initial laws to the chaos of laws on the whole path space.
Remark 2: We are adopting here a strong definition of the propagation of chaos. Other authors [14, p. 42][16, p. 98] (define propagation of chaos in a *weaker* sense (before going to prove that chaos propagates in a variety of important cases). According to these authors, a family of Markovian n-particle processes propagates chaos if $\{\int K_n(\mathrm{s},\cdot)\rho^{\otimes n}(d\mathrm{s})\}$ is chaotic for all $\rho \in \mathcal{P}(S)$ and $t > 0$, where $\rho^{\otimes n}$ is product measure on S^n. In other words, only *purely* chaotic sequences of initial measures are required to "propagate" to chaotic sequences. This condition is strictly weaker than the one we adopt for our definition. For example, take $S = \{0,1\}$ and let $\delta(x)$ or δ_x denote a point mass at x. Then, if

$$K_n(\mathrm{s},\cdot,t) = \begin{cases} \delta_{(1,1,\dots,1)} & \text{if } \mathrm{s} \neq (0,0,\dots,0) \\ \delta_{(0,0,\dots,0)} & \text{if } \mathrm{s} = (0,0,\dots,0) \end{cases}.$$

for all $t > 0$, the sequence $\{K_n\}$ propagates chaos in the weak sense, but not in the strong sense of our definition. Under these K_n's, the $\delta(0)$-chaotic sequence $\{\delta_{(0,0,\dots,0)}\}$ is propagated to itself, while other $\delta(0)$-chaotic sequences are propagated to $\delta(1)$-chaotic sequences, and yet other $\delta(0)$-chaotic sequences are not propagated to chaotic sequences at all.

4 Proof of Theorem 3.1

Let (X, d_X) be a metric space, and $D_1 \subset D_2 \subset \cdots$ an increasing chain of Borel subsets of X whose union is dense in X. For each natural number n, let f_n be a measurable real-valued function on D_n.

Consider the following four conditions on the sequence $\{f_n\}$. They are listed in order of decreasing strength.

[A] Whenever $\{\mu_n\}$ is a weakly convergent sequence of probability measures on X with μ_n supported on D_n, then the sequence

$$\left\{\int_X f_n(x)\mu_n(dx)\right\}_{n=1}^{\infty}$$

of real numbers converges as well.

[B] Whenever $\{\mu_n\}$ is a sequence of probability measures on X that converges weakly to $\delta(x)$ for some $x \in X$, and μ_n is supported on D_n, then the sequence $\left\{\int_X f_n(x)\mu_n(dx)\right\}$ also converges.

[C] Whenever $\{d_n\}$ is a convergent sequence of points in X, with $d_n \in D_n$, then $\{f_n(d_n)\}$ also converges.

[D] There exists a continuous function f on X towards which the functions f_n converge uniformly on compact sets: for any compact $K \subset X$, and for any $\epsilon > 0$, there exists a natural number N such that, whenever $n \geq N$ and $d \in D_n \cap K$, then $|f(d) - f_n(d)| < \epsilon$.

Lemma 4.1 *If X is a Polish space, i.e., if X is homeomorphic to a complete and separable metric space, and $\sup_{d \in D_n}\{|f_n(d)|\} \leq B$ for all n, then conditions [A], [B], [C], and [D] are all equivalent.*

Sketch of Proof:

[A] $\Longrightarrow$ [B] $\Longrightarrow$ [C] $\Longrightarrow$ [D] even when X is not Polish.

To get [D] $\Longrightarrow$ [A], we are assuming that X is Polish, for then there must exist, by Prohorov's theorem, compact sets which support all of the measures of the convergent sequence $\{\mu_n\}$ to within any $\epsilon > 0$. (Prohorov's theorem states that if $(X, \mathcal{T})$ is Polish, then $\Sigma \subset \mathcal{P}(X)$ is tight if and only if its closure is compact in $\mathcal{P}(X)$.) ■

Let S and T be Polish spaces. For each natural number n, let K_n be a Markov transition from S^n to T^n that satisfies the permutation condition (2). Henceforth, the notation ε_n is used both for the map from S^n to n-point empirical measures on S and for the same kind of map on T^n. $\varepsilon_n(S^n)$ denotes the subset of $\mathcal{P}(S)$ that consists of discrete laws of the form (1).

Now, Markov transitions K_n from S^n to T^n induce Markov transition functions H_n from $\varepsilon_n(S^n)$ to $\varepsilon_n(T^n)$ defined by

$$H_n(\zeta, G) := \int_{\mathbf{s} \in S^n} J_n(\zeta, d\mathbf{s}) K_n\left(\mathbf{s}, \varepsilon_n^{-1}(G)\right) \tag{3}$$

for $\zeta \in \varepsilon_n(S^n)$ and G a measurable subset of $\varepsilon_n(T^n)$, where J_n is the Markov transition from $\varepsilon_n(S^n)$ to S^n such that $J_n(\zeta, \cdot)$ that allots equal probability to each of the points in $\varepsilon^{-1}(\{\zeta\})$. Note that $J_n(\zeta, \cdot) \circ \varepsilon_n^{-1} = \delta(\zeta)$.

Theorem 2.1 shows that propagation of chaos by a sequence $\{K_n\}$ is equivalent to the following condition on the sequence of induced transitions $\{H_n\}$:

Lemma 4.2 *The sequence of Markov transitions $\{K_n\}$ propagates chaos if and only if, whenever $\{\mu_n\}$ converges to $\delta(p) \in \mathcal{P}(\mathcal{P}(S))$ for some $p \in \mathcal{P}(S)$, with μ_n supported on $\varepsilon_n(S_n)$ for each n, the sequence*

$$\left\{ \int_{\varepsilon_n(S^n)} H_n(\zeta, \cdot)\mu_n(d\zeta) \right\}_{n=1}^{\infty} \tag{4}$$

converges in $\mathcal{P}(\mathcal{P}(T))$ to $\delta(q)$, for some $q \in \mathcal{P}(T)$.

Proof:

By condition (v) of Theorem 2.1, $\{\mu_n \in \mathcal{P}(\varepsilon_n(S^n))\}$ converges to a point mass $\delta(p) \in \mathcal{P}(\mathcal{P}(S))$ if and only if $\left\{ \int_{\varepsilon_n(S^n)} J_n(\zeta, \cdot)\mu_n(d\zeta) \right\}$ is chaotic. Therefore, the sequence of transitions $\{K_n\}$ propagates chaos if and only if

$$\left\{ \int_{S^n} K_n(\mathbf{s}, \cdot) \int_{\varepsilon_n(S^n)} J_n(\zeta, d\mathbf{s})\mu_n(d\zeta) \right\}_{n=1}^{\infty} \tag{5}$$

is chaotic whenever $\mu_n \longrightarrow \delta(p)$. By part (v) of Theorem 2.1 again, the sequence (5) is chaotic if and only if

$$\left(\int_{S^n} K_n(\mathbf{s}, \cdot) \int_{\varepsilon_n(S^n)} J_n(\zeta, d\mathbf{s})\mu_n(d\zeta) \right) \circ \varepsilon_n^{-1} \longrightarrow \delta(q)$$

for some $q \in \mathcal{P}(T)$. But, by definition (3) of the transitions H_n,

$$\left(\int_{S^n} K_n(\mathbf{s}, \cdot) \int_{\varepsilon_n(S^n)} J_n(\zeta, d\mathbf{s})\mu_n(d\zeta) \right) \circ \varepsilon_n^{-1} = \int_{\varepsilon_n(S^n)} H_n(\zeta, \cdot)\mu_n(d\zeta),$$

so $\{K_n\}$ propagates chaos if and only if the sequence (4) converges to $\delta(q)$ for some $q \in \mathcal{P}(T)$ whenever $\{\mu_n\}$ converges to $\delta(p) \in \mathcal{P}(\mathcal{P}(S))$. ■

Having Lemmas 4.1 and 4.2 in hand, we proceed to the

Proof of Theorem 3.1:

For each bounded and continuous function $\phi \in C_b(\mathcal{P}(T))$ define the functions $\widehat{\phi}_n : \varepsilon_n(S^n) \longrightarrow \mathbb{R}$ by

$$\widehat{\phi}_n(\zeta) := \int_{\mathcal{P}(T)} \phi(\eta) H_n(\zeta, d\eta), \tag{6}$$

where H_n is as defined in (3). Note that the functions $\widehat{\phi}_n$ are bounded uniformly in n since ϕ is bounded.

Assume first that $\{K_n\}$ propagates chaos. We will show that there exists a continuous function $F : \mathcal{P}(S) \longrightarrow \mathcal{P}(T)$ such that $\left\{\widetilde{K}_n(\mathbf{s}_n, \cdot)\right\}$ is $F(p)$-chaotic whenever the points $\mathbf{s}_n \in S^n$ are such that $\varepsilon_n(\mathbf{s}_n) \longrightarrow p$ in $\mathcal{P}(S)$.

Since $\{K_n\}$ propagates chaos, Lemma 4.2 implies that whenever $\{\mu_n \in \mathcal{P}(\varepsilon_n(S^n))\}$ converges in $\mathcal{P}(\mathcal{P}(S))$ to $\delta(p)$, then

$$\int_{\varepsilon_n(S^n)} \widehat{\phi}_n(\zeta)\mu_n(d\zeta) \longrightarrow \phi(q) \tag{7}$$

for some $q \in \mathcal{P}(T)$. In fact, Lemma 4.2 implies that q does not depend on our choice of ϕ: the same q works for all ϕ in (7).

Condition (7) resembles condition [B] above, so Lemma 4.1 shows that there exists a continuous function $G_\phi(p)$, depending on ϕ, such that if $\{\mathbf{s}_n \in S^n\}$ is a sequence satisfying $\varepsilon_n(\mathbf{s}_n) \longrightarrow p$ in $\mathcal{P}(S)$, then

$$\widehat{\phi}_n(\varepsilon_n(\mathbf{s}_n)) \longrightarrow G_\phi(p).$$

By (7), $G_\phi(p) = \phi(q)$ for some $q \in \mathcal{P}(T)$ *that does not depend on* ϕ. The only way that all the G_ϕ's can have this form and yet all be continuous is for the dependence of q on p to be continuous: there must be a continuous F from $\mathcal{P}(S)$ to $\mathcal{P}(T)$ such that $G_\phi(p) = \phi(F(p))$ for all $\phi \in C_b(\mathcal{P}(T))$.

Thus, there exists a continuous function F from $\mathcal{P}(S)$ to $\mathcal{P}(T)$ such that

$$[\varepsilon_n(\mathbf{s}_n) \longrightarrow p] \implies \left[\widehat{\phi}_n(\varepsilon_n(\mathbf{s}_n)) \longrightarrow \phi(F(p))\right]$$

for all $\phi \in C_b(\mathcal{P}(T))$. This fact, and the definitions (3) and (6) of H_n and $\widehat{\phi}$, imply that

$$[\varepsilon_n(\mathbf{s}_n) \longrightarrow p] \implies \left[\widetilde{K}_n(\mathbf{s}_n, \cdot) \circ \varepsilon_n^{-1} \longrightarrow \delta(F(p))\right].$$

Finally, by Theorem 2.1, we have that

$$[\varepsilon_n(\mathbf{s}_n) \longrightarrow p] \implies \left\{\widetilde{K}_n(\mathbf{s}_n, \cdot)\right\}_{n=1}^{\infty} \text{ is } F(p)\text{–chaotic.}$$

This demonstrates the necessity of the condition of Theorem 3.1. Next we demonstrate its sufficiency, i.e., that $\{K_n\}$ propagates chaos *if* there exists a continuous function $F : \mathcal{P}(S) \longrightarrow \mathcal{P}(T)$ such that $\{\widetilde{K}_n(\mathbf{s}_n, \cdot)\}$ is $F(p)$-chaotic whenever $\varepsilon_n(\mathbf{s}_n)$ converges weakly to $p \in \mathcal{P}(S)$.

Suppose $P_n \in \mathcal{P}(S^n)$ is p-chaotic. Let $\mu_n = P_n \circ \varepsilon_n^{-1}$. Then $\mu_n \longrightarrow \delta(p)$ in $\mathcal{P}(\mathcal{P}(S))$ by Theorem 2.1. Our goal is to prove that

$$\int_{\mathcal{P}(S)} H_n(\zeta, d\eta)\mu_n(d\zeta) \longrightarrow \delta(F(p)),$$

where H_n is as defined in (3). This is enough, by Lemma 4.2, to demonstrate that chaos propagates.

By hypothesis, if $\{\mathbf{s}_n \in S^n\}$ is such that $\varepsilon_n(\mathbf{s}_n)$ converges to p then $\{\widetilde{K}_n(\mathbf{s}_n, \cdot)\}$ is $F(p)$-chaotic. By Theorem 2.1 and the fact that

$$\widetilde{K}_n(\mathbf{s}_n, \cdot) \circ \varepsilon_n^{-1} = H_n(\varepsilon_n(\mathbf{s}_n), \cdot),$$

the hypothesis is equivalent to the statement that, if $p_n \in \varepsilon_n(S^n)$ for each n, then

$$[p_n \longrightarrow p] \implies [H_n(p_n, \cdot) \longrightarrow \delta(F(p))]\,. \qquad (8)$$

The hypothesis (8) and equation (6) imply that for each $\phi \in C_b(\mathcal{P}(T))$,

$$\lim_{n\to\infty} \widehat{\phi}_n(p_n) = \phi(F(p))$$

when $p_n \longrightarrow p$ with $p_n \in \varepsilon_n(S^n)$. We are assuming S is complete and separable, therefore so is $\mathcal{P}(S)$ [6].

We may now apply Lemma 4.1 with

$$X = \mathcal{P}(S), \quad D_n = \varepsilon_n(S^n), \quad f_n = \widehat{\phi}_n,$$

and conclude that

$$\lim_{n\to\infty} \int_{\mathcal{P}(S)} \widehat{\phi}_n(\zeta)\mu_n(d\zeta) = \phi(F(p)) \qquad (9)$$

for any sequence $\{\mu_n\}$ that converges to $\delta(p)$ in $\mathcal{P}(\mathcal{P}(S))$.

By equations (9) and (6),

$$\begin{aligned} \phi(F(p)) &= \lim_{n\to\infty} \int_{\mathcal{P}(S)} \widehat{\phi}_n(\zeta)\mu_n(d\zeta) \\ &= \lim_{n\to\infty} \int_{\mathcal{P}(S)} \int_{\mathcal{P}(T)} \phi(\eta) H_n(\zeta, d\eta)\mu_n(d\zeta) \\ &= \lim_{n\to\infty} \int_{\mathcal{P}(T)} \phi(\eta) \int_{\mathcal{P}(S)} H_n(\zeta, d\eta)\mu_n(d\zeta), \end{aligned}$$

for all $\phi \in C_b(\mathcal{P}(T))$. This implies that $\int_{\mathcal{P}(S)} H_n(\zeta, d\eta)\mu_n(d\zeta) \longrightarrow \delta(F(p))$ in $\mathcal{P}(\mathcal{P}(S))$, completing the proof. ■

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MATHEMATICS DEPARTMENT
COMPUTING SCIENCES DIRECTORATE
LAWRENCE BERKELEY NATIONAL LAB
1 CYCLOTRON ROAD, BERKELEY, CA 94720

ERNEST ORLANDO LAWRENCE BERKELEY NATIONAL LABORATORY
ONE CYCLOTRON ROAD | BERKELEY, CALIFORNIA 94720

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