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COUPLED CHANNEL TREATMENT OF INELASTIC SCATTERING BASED ON MICROSCOPIC NUCLEAR DESCRIPTIONS

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Author

Glendenning, Norman K.

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BASED ON MICROSCOPIC NUCLEAR DESCRIPTIONS

Norman K. Glendenning

February 1968

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COUPLED CHANNEL TREATMENT OF INELASTIC SCATTERING
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Norman K. Glendenning

Lawrence Radiation Laboratory
University of California
Berkeley, California

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Abstract

The coupled channel formalism for treating inelastic scattering based on microscopic nuclear descriptions has been set down. The truncation in both the scattering and structure calculations and how they lead to an effective interaction is discussed. The relevant content of nuclear models can be reduced to a set of amplitudes. These are exhibited explicitly for systems described in terms of two fermion degrees of freedom. The matrix elements appearing in the coupled equations can be developed as a polynomial which is very convenient from the point of view of computer calculation.

*This work was done under the auspices of the U. S. Atomic Energy Commission.

1. Introduction

Until recently the only available description of nuclear states possessing collective properties was a macroscopic one. This description forgoes a derivation of the detailed nucleon correlations resulting from the action of the forces between nucleons, in favor of the assumption of collective vibrational or rotational motion which can be described in terms of several macroscopic parameters^{1,2)}. An analysis of inelastic scattering data in these terms leads to a measurement of the deformation parameters, and a test of the vibrational model when the two phonon triplet is included in the description. Other non-collective transitions however do not fit into this picture, while the ever improving experimental techniques produce a greater accumulation of such data. Recent developments in the theory of nuclear structure have been aimed at describing both single-particle and collective phenomena on the same footing, in terms of the nucleonic motion³⁾. There is considerable evidence, based mainly on energy level systematics, that these more fundamental descriptions contain a certain element of truth. But the energy levels are the easiest nuclear property to calculate (because the Hamiltonian is stationary at its eigenstates). A more rigorous test of the theory must be based on transition rates, both gamma decay and inelastic excitation, and on two-nucleon transfer reactions.

Another motive for beginning to explore microscopic descriptions of inelastic scattering is connected with the expectation that the higher energy region of nuclear spectra will contain various types of collective states. Such states are believed to exist because the nucleon-nucleon interaction is a complicated one, depending as it does on the state of the interacting pair. There is a tendency for a many particle system to form states of special corre-

lations which exploit one component or other of the interaction. The low lying 2^+ level in spherical nuclei is an example of a state whose correlations exploit the $S = 0$, $T = 1$ part of the interaction.

Recent work of a number of people has been concerned with formulating the problem of using microscopic descriptions of nuclear structure to calculate inelastic cross-sections^{4,5)}. So far cross-sections have been calculated in the distorted wave Born approximation, which is valid in the vibrational region for the elastic and enhanced collective transitions, but not generally for the weaker transitions, as we acknowledged in our earlier exploratory work.

The coupled channel method for solving the Schroedinger equation provides the most reliable means of calculating elastic and inelastic cross-sections. The method is by no means new. When something is known about the solution of the equations of motion for an isolated part of a system, the method provides an obvious way of incorporating this partial information into the equations for the whole. Such equations were discussed already in 1933 by Mott and Massey⁶⁾ who provided an exact solution for the case of two degenerate levels, and for the general case, derived the distorted wave Born approximation as a solution to first order in the coupling interaction. However it was not practicable to solve the equations under less restrictive assumptions until very fast electronic computers had been developed. The method has by now been used by several authors in studies based on a macroscopic description of nuclei⁷⁾. We applied it recently to microscopic nuclear descriptions^{8,9)}.

In this work we want to provide the formalism used in these microscopic descriptions of inelastic scattering, as well as the rationale. In particular, we discuss how the truncation, which of necessity must be carried out in the space of scattering states at the energies we have in mind, leads to an

effective interaction. We argue that the diagonal matrix elements of this interaction may be parameterized by a common optical potential. Its non-diagonal elements are probably well represented by the basic interaction, or in the event that the nuclear space was too severely truncated, by its renormalized form.

The second half of the paper is devoted to the reduction of nuclear descriptions based on several models to the form in which it is most conveniently used in the coupled equations, or in the distorted wave approximation which is the first order solution to them.

Some of the background of the present work can be found in ref. ⁵), hereinafter referred to as I.

2. Scattering Formalism and Effective Interactions

2.1 COUPLED EQUATIONS

Typically one wants to use scattering data to test a nuclear model. Assume therefore that we have available a set of wave functions describing the states of a model of the nucleus A, which are eigenfunctions of the model Hamiltonian

$$(H_A - E_{\alpha J}) \Phi_{\alpha J}(A) = 0 \quad (2.1)$$

(Here α denotes quantum numbers additional to J,M.)

For the total system comprising the nucleus plus scattered particle, which for definiteness we take to be a nucleon, adopt the Hamiltonian

$$H = H_A + T + V(\underline{r}, A) \quad (2.2)$$

where T is the kinetic energy of relative motion and V is the interaction between nucleon and nucleus. If the nuclear Hamiltonian were written in terms of collective coordinates, then these would appear in V , whereas if a more fundamental description of the nucleus in terms of the nucleon coordinates $\underline{r}_i, \underline{g}_i, \underline{I}_i$, were available, then V would be a sum of two-body potentials.

In any case it is natural to seek a solution of the scattering process defined, with appropriate boundary conditions, by

$$(H - E) \Psi(\underline{r}, A) = 0 \quad (2.3)$$

in the form of an expansion in terms of the nuclear wave functions $\Phi_{\alpha J}$, whose accuracy in describing the nucleus we want to test.

To this end, introduce the spin-orbit functions of the scattering particle

$$\eta_{lsj}^m = \left[Y_\ell(\hat{\underline{r}}) \eta_s(\underline{g}) \right]_j^m \quad (2.4)$$

where $\hat{\underline{r}}$ denotes a unit vector in the direction of the relative coordinate $\underline{r}$, and η is the spin function. Denote by c the whole collection of quantum numbers which define the intrinsic state of the target system, and its relative angular momentum

$$c \equiv \alpha J \quad lsj \quad (2.5)$$

and by c' some other state of intrinsic or relative motion that may be achieved as a result of the collision. Using the spin-orbit function (2.4) and nuclear function (2.1) form the eigenfunction of total angular momentum $I = j+J$ and parity $\pi = (-)^l \pi_\alpha$

$$\phi_{c\pi I}^M(\hat{r}, \alpha, A) = \left[y_{lsj} \phi_{\alpha J} \right]_I^M \quad (2.6)$$

In terms of these an expansion for the solution of (2.3) having given parity and total angular momentums πI , and corresponding to incident particles in the channel c , can be written as,

$$\Psi_{c\pi I}^M(\hat{r}, A) = \frac{1}{r} \sum_c u_c^{c\pi I}(r) \phi_{c, \pi I}^M(\hat{r}, \alpha, A) \quad (2.7)$$

There are two important approximations implied when we write the expansion (2.7). Since it always contains the nucleon as a free particle, any specific effect of reactions, such as pickup, on the inelastic channels that we are interested in are neglected. The expansion is in terms only of the elastic and inelastic channels containing the nucleon. Moreover it neglects exchange of the free nucleon with one in the target, which may take place either by virtue of the indistinguishability of identical particles or by the exchange nature of the forces.

As to the first approximation we may note that the elastic and inelastic processes typically represent the largest part of the direct cross-section. We may be prepared then to treat the other channels implicitly through use of an absorptive part in the diagonal matrix elements of the interaction.

The neglect of exchange is less well founded, but is presumed, on the strength of overlap arguments, to introduce little error particularly for sufficiently high bombarding energies¹⁰⁾

Inserting now the expansion (2.7) for Ψ , into the Schroedinger equation, (2.3), and making use of the orthonormality of the functions ϕ ,

$$\langle \phi_{c'\pi'I'}^{M'} | \phi_{c\pi I}^M \rangle = \delta_{cc'} \delta_{\pi\pi'} \delta_{II'} \delta_{MM'} \quad (2.8)$$

we obtain, for each total angular momentum and parity I, π of the system, a set of coupled equations for the radial functions $u(r)$ of the scattered proton.

For each channel c' they have the form

$$(T_{c'} - E_{c'}) u_{c'}^{c\pi I} = - \sum_{c''} \langle \phi_{c'\pi I}^M | V | \phi_{c''\pi I}^M \rangle u_{c''}^{c\pi I} \quad (2.9)$$

where

$$T_c = \frac{\hbar^2}{2m} \left(- \frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} \right) \quad (2.10)$$

$$E_c = E - E_{\alpha J}$$

The matrix elements remain functions of the distance r of which ϕ is independent.

These are to be solved subject to the boundary conditions that there are incoming spherical waves only in channels containing the nucleus in the ground state, while all channels may have outgoing waves

$$u_{c'}^{c\pi I}(r) \rightarrow \delta_{c',c} I_\ell(k_c r) - (k_c/k_{c'})^{1/2} S_{c',c}^I O_\ell(k_{c'} r) \quad (2.11)$$

This yields the elements of the S-matrix from which cross sections and polarizations can be calculated. In the above I_ℓ and O_ℓ are asymptotic to spherical incoming and outgoing waves

$$I_\ell^* = O_\ell = G_\ell + i F_\ell \quad (2.12)$$

with G and F irregular and regular Coulomb functions and k_c denotes the wave number $(2m E_c/\hbar^2)^{1/2}$ in the channel c . (For more details on the numerical methods see ref. 9).

2.2 TRUNCATION AND EFFECTIVE INTERACTION

As formulated in the preceeding section, the scattering process is described by an infinite set of coupled equations, because of the need to use a complete set in the expansion (2.7). On the other hand information on only a few of the low-lying resolvable levels can be obtained. Thus a description in terms of all the channels, even if possible, would be superfluous. However it is not valid to simply truncate the set in spite of a superficial resemblance to the shell model problem. There, truncation of the space to a few configurations may be justified by rejecting those that lie distant in energy from the region of interest. However at the bombarding energies of interest to us here, the nucleus can be excited into any one of millions of levels, and each of these states of the system corresponds to the same energy. Moreover the reaction cross section, which is about geometric, is several hundred times larger than the cross section of any single level. Therefore the wave function is not dominated by a few "configurations" in the scattering case. The multitude of open channels do have a very strong effect on those of interest, and this must be taken into account. We may employ Feshbach's projection operator

technique¹¹⁾ to focus attention explicitly on the few channels of interest while relegating the others to an implicit role in an effective interaction.*

Accordingly let P denote a projection operator onto the space of channels that we choose to treat explicitly. For reasons that will be apparent later, these should include low lying strongly coupled states. We let $Q = 1 - P$ denote the complement of P . Let H_0 denote the non-interacting part of the Hamiltonian (which therefore commutes with P),

$$H_0 = T + H_A \quad (2.13)$$

Substitute the identity $P + Q = 1$ in front of Ψ in (2.3). Multiplying from the left by Q one gets

$$(E - H_0 - V_{QQ}) \Psi_Q = V_{QP} \Psi_P \quad (2.14)$$

where we write

$$V_{QP} \equiv QVP, \quad \Psi_Q \equiv Q\Psi \quad (2.15)$$

The formal solution to (2.14) which satisfies the required boundary conditions that there can be only outgoing waves in Q channels, is

$$\Psi_Q = \frac{1}{E - H_0 - V_{QQ} + i\epsilon} V_{QP} \Psi_P \quad (2.16)$$

On the other hand, Ψ_P satisfies the equation

$$(E - H_0 - V_{PP}) \Psi_P = V_{PQ} \Psi_Q \quad (2.17)$$

* See Ref. 9 for a fuller discussion of the following development.

Using (2.16), this can be rearranged to read

$$(T + H_A + \mathcal{V}_P - E) \Psi_P = 0 \quad (2.18)$$

where

$$\mathcal{V}_P = P \left(V + VQ \frac{1}{E - H_0 - V_{QQ} + i\epsilon} QV \right) P \quad (2.19)$$

The transition matrix corresponding to the problem defined by (2.2, 2.3) is,

$$\mathcal{T}_{\alpha'\alpha} = \langle \phi_{\alpha'} | V | \Psi_{\alpha}^{(+)} \rangle \quad (2.20)$$

where ϕ is the solution in the absence of interaction,

$$(E - H_0) \phi_{\alpha} = 0 \quad (2.21)$$

But we have, using (2.16)

$$\Psi \equiv (P + Q) \Psi = \left(1 + \frac{1}{E - H_0 - V_{QQ} + i\epsilon} V_{QP} \right) \Psi_P \quad (2.22)$$

whence

$$\mathcal{T}_{\alpha'\alpha} = \langle \phi_{\alpha'} | \mathcal{V}_P | \Psi_{P\alpha}^{(+)} \rangle \quad (2.23)$$

provided α and α' belong to P . This last expression, it will be recognized immediately, is the $\mathcal{T}$ matrix for (2.18). As far as the P space is concerned, the two problems are equivalent.

We now have the result that the problem defined by $H = H_0 + V$, which leads to an infinite set of coupled equations, may be replaced by a new one $\mathcal{H} = H_0 + \mathcal{V}_P$, which is to be solved within a chosen subspace of channels and therefore leads to a finite set of equations. But instead of the original interaction V , there now appears an interaction $\mathcal{V}_P$. The usefulness of the transformation lies in the fact that the effect of the enormous number of open channels on those of real interest is now carried in a potential rather than in an infinite set of coupled equations. While we cannot explicitly calculate $\mathcal{V}_P$ since this would require a complete knowledge of the spectrum of H_{QQ} we can infer certain of its properties $\mathcal{V}$ from its defining equation, and seek a parameterization of it. The optical potential for elastic scattering is just a special case of $\mathcal{V}$.

2.3 INTERNAL AND EXTERNAL POLARIZATION

Up till now we have been discussing the truncation of channels describing the scattering process, assuming all the while that the space of configurations in which the nuclear problem (1) was solved, was sufficient that no renormalization effects arise on that account. In such a situation, the interaction V should be the vacuum interaction. The extended shell model treatments of vibrational nuclei intend to be close to exhaustive in this respect. Still the truncation is sufficiently severe as to require use of stronger charges in the electromagnetic decay¹²⁾ and deeper potentials in the inelastic scattering calculations^{5,8)} than the vacuum values.

In other cases, the nuclear problem is treated in such a restricted space that one could, à priori, expect a strong renormalization of the force. This renormalization may be referred to as internal or "core" polarization, and a model for calculating the renormalized force has been proposed recently¹³⁾.

Our treatment of the external polarization is additional to this and it should be understood that V denotes the vacuum interaction (or a phenomenological representation of it) when the nuclear calculation is sufficiently exhaustive, but otherwise it denotes the renormalized force that includes the internal polarization effects. The effective interaction $\mathcal{V}$ includes then the effect of both internal and external polarization.

2.4 MATRIX ELEMENTS OF THE EFFECTIVE INTERACTION

The set of coupled equations to be solved is now understood to be a finite set like (2.9) with V understood to be replaced by the effective interaction appropriate to the chosen subspace.

From its definition (2.19), the effective interaction depends explicitly on the energy, is complex valued and non-local. The second term makes a large contribution to diagonal matrix elements but is probably small for off-diagonal ones. This can be seen by introducing the eigenstates of H_{QQ} in a typical diagonal matrix element,

$$\begin{aligned} \mathcal{V}_{\alpha\alpha}(\underline{r}, \underline{r}') &\equiv \langle \Phi_{\alpha}(\underline{A}) | \mathcal{V} | \Phi_{\alpha}(\underline{A}) \rangle = V_{\alpha\alpha}(\underline{r}) \delta(\underline{r} - \underline{r}') \\ &+ \sum_q \frac{\langle \Phi_{\alpha}(\underline{A}) | V | \Psi_q(\underline{A}, \underline{r}) \rangle \langle \Psi_q(\underline{A}, \underline{r}') | V | \Phi_{\alpha}(\underline{A}) \rangle}{E - E_q + i\epsilon} \end{aligned} \quad (2.24)$$

The sum runs over the Q channels only. (Actually the channels of the total system are labelled according to their parentage based on the states of nucleus A . Therefore q is a very much contracted symbol in the above, for there is a

continuum based on each nuclear state. Accordingly the sum on q implies both a discrete sum over the bound states of the nucleus and an integration over the continuum built on each of these.)

The numerator in the second term is essentially positive (certainly for $r = r'$) while the denominator changes sign only once in each channel. Therefore there is little chance that it is small, and from the optical model, which has an imaginary part arising from this term, we know that it is not. However the matrix elements connecting the different spaces P and Q will generally be small since states of the latter lie at higher excitation in the nucleus and will have poor overlap with those of the former. Hence the sum, which is over an enormous number of terms, is not dominated by any one term or a few terms, especially if all of the strongly coupled states are explicitly treated. Consequently the sum will be dominated by the high excitation region because of the level density. Therefore we expect the diagonal matrix elements within the space P to be very similar, one to the other, and to vary only slowly from nucleus to nucleus. While we cannot hope to explicitly calculate the second term of $\mathcal{V}$, our experience with the optical potential, which is the phenomenological analogue of the special case of $\mathcal{V}$ when P refers only to the elastic channel, suggests that the diagonal elements can be successfully parameterized.

The non-diagonal matrix elements $\mathcal{V}_{\alpha\beta}$ are another story. In this case the numerator of the sum will fluctuate in sign from term to term so that, in fact, it may very well be small compared to V . This is the assumption we have made in our calculations^{5,8,9}) although it was recognized that V may not be the vacuum interaction, since it must carry any renormalization due to truncation in the nuclear space as discussed in the previous section.

From this discussion we understand that the diagonal matrix elements in the coupled equations should be parameterized as an optical potential. In the non-diagonal elements, we shall assume, unless subsequent developments prove us wrong, that the second part of $\mathcal{V}$ is small compared to V . In this case, these matrix elements will be computed from a real potential, which is supposed to represent the vacuum interaction or, if truncation in the nuclear calculation was severe, an effective interaction which carries the associated internal or core polarization effects. The coupled equations which we solve accordingly are, for each c' including target channels c ,

$$\left[\frac{\hbar^2}{2m} \left(\frac{d^2}{dr^2} - \frac{\ell'(\ell'+1)}{r^2} \right) - U(r) + E_{c'} \right] u_{c'}^{c\pi I}(r) = \sum_{c'' \neq c'} \langle \phi_{c'}^M | V | \phi_{c''}^M \rangle u_{c''}^{c\pi I}(r) \quad (2.25)$$

where $U(r)$ is the optical potential parameterization of $\langle c | \mathcal{V} | c \rangle$, which we assume to be the same in all the included channels, and V is the sum of two-body interactions between the incident and target nucleons,

$$V = \sum_{i=1}^A V(\underline{r}, \underline{r}_i) \quad (2.26)$$

We adopt a very simple form for the direct interaction V .^{*} In the absence of internal polarization, the interaction between scattered particle and the i 'th nucleon in the nucleus is written,

$$V(\underline{r}, \underline{r}_i) = (V_0 + V_1 \underline{\sigma} \cdot \underline{\sigma}_i) g(|\underline{r} - \underline{r}_i|) \quad (2.27)$$

where V_0 and V_1 may in turn depend upon $\underline{\tau}_1 \cdot \underline{\tau}_2$ like $V_0 = V_{00} + V_{01} \underline{\tau}_1 \cdot \underline{\tau}_2$. The multipole expansion of the radial factor we write as

^{*}Generalization to include spin-orbit and tensor components is being carried out in this laboratory by J. Hocquenghem.

$$g(|\underline{r}-\underline{r}'|) = \sum_L \frac{4\pi}{2L+1} v_L(r, r') Y_L(\hat{\underline{r}}) \cdot Y_L(\hat{\underline{r}}') \quad (2.28)$$

so that

$$V(\underline{r}, \underline{r}_1) = \sum_{LSJ} (-)^{L+S+J} v_S \mathfrak{Z}_{LSJ}(r, \underline{r}_1, \underline{g}_1) \cdot \mathfrak{Y}_{LSJ}(\hat{\underline{r}}, \underline{g}) \quad (2.29)$$

where the spherical tensors are

$$\mathfrak{Y}_{LSJ} = \begin{cases} Y_L(\hat{\underline{r}}) \delta_{LJ} & \text{if } S=0 \\ \left[Y_L(\hat{\underline{r}}) \underline{g} \right]_J & \text{if } S=1 \end{cases} \quad (2.30)$$

with Y_L^M a spherical harmonic, and

$$\mathfrak{Z}_{LSJ}(r, \underline{r}_1, \underline{g}_1) = \frac{4\pi}{2L+1} v_L(r, r_1) \mathfrak{Y}_{LSJ}(\hat{\underline{r}}_1, \underline{g}_1) \quad (2.31)$$

$$\mathfrak{Z}_{LSJ}(r, A) = \sum_{i=1}^A \mathfrak{Z}_{LSJ}(r, \underline{r}_i, \underline{g}_i)$$

carries the angular transformation properties of the nuclear part of the interaction. ($\hat{\underline{r}}$ is a unit vector in the direction of $\underline{r}$.)

A renormalization of the direct interaction due to internal polarization can be accounted for through a simple modification of v_L as was shown recently by Love and Satchler¹³).

The non-diagonal matrix elements needed in the coupled equations (2.25) in the representation (6) are

$$\langle \phi_{c_1 \pi_1}^M | V | \phi_{c_2 \pi_2}^M \rangle = \sum_{LSJ} v_S C_{LSJ}^{\pi_1}(c_1, c_2) \langle \alpha_1 J_1 || \mathfrak{Z}_{LSJ}(r, A) || \alpha_2 J_2 \rangle \quad (2.32)$$

where

$$c_{LSJ}^{\pi I}(c_1, c_2) = (-)^{L+S+J+j_2+J_1+I} \begin{Bmatrix} I & J_1 & j_1 \\ J & j_2 & J_2 \end{Bmatrix} \langle j_1 \| \mathfrak{Y}_{LSJ} \| j_2 \rangle \quad (2.33)$$

is purely geometrical. For nucleons, the reduced matrix element*

$\langle j_1 \| \mathfrak{Y}_{LSJ} \| j_2 \rangle$ is given by eq. (23) in paper I. The last factor in (2.32) is called the nuclear form factor.

3. Nuclear Form Factors

3.1 EXPANSION IN TERMS OF SINGLE-PARTICLE FORM FACTORS

Information about the nuclear structure enters the scattering calculation only through the reduced matrix elements appearing in (2.32), and they are therefore of great interest. They are referred to as form factors for the transition between a pair of states, and for brevity we denote them by

$$\mathcal{F}_{LSJ}^{\alpha_1 \alpha_2}(r) = \langle \Phi_{\alpha_1 J_1}(\underline{A}) \| \mathfrak{S}_{LSJ}(r, \underline{A}) \| \Phi_{\alpha_2 J_2}(\underline{A}) \rangle \quad (3.1)$$

The structure of the nuclear form factors (3.1) is actually rather simple as follows from the fact that the scattered particles acts as a one-body operator on the nuclear coordinates[†] because the interaction between nucleons is two-body. (That is to say that $\mathfrak{S}$ (2.31) is a sum of one-body operators on the nuclear coordinates.) Thus however complicated the structure of the nuclear states may be, it is always possible to write

$$\mathcal{F}_{LSJ}^{\alpha_1 \alpha_2}(r) = \sum_{a,b} S_J^{ab}(\alpha_1 \alpha_2) F_{LSJ}^{ab}(r) \quad (3.2)$$

where the amplitudes S depend on the structure of the two states α_1 and α_2 , while

* There are several definitions of reduced matrix elements. We use Racah's original definition also used by Edmonds¹⁴).

[†] This is with neglect of exchange.

$$F_{LSJ}^{ab}(r) = \langle \psi_a(\underline{r}') \| \mathfrak{F}_{LSJ}(r, \underline{r}') \| \psi_b(\underline{r}') \rangle \quad (3.3)$$

denotes the form factor corresponding to the promotion of a single nucleon between the state a and b . We use a to denote the collection of single-particle quantum numbers $a \equiv n_a \ell_a j_a$, and ψ_a denotes the corresponding wave function.

The problem of employing a microscopic calculation of the structure of nuclear states in a scattering calculation amounts, then, to calculating the amplitudes S for the single-particle transitions which connect the various states. This is done explicitly in section 3.3 for several currently popular nuclear models. In general however, when the structure calculation is carried out in a space involving many particles or quasiparticles it would be pointless to try to anticipate the various conventions that may be adopted. In such a situation it would be most helpful if these amplitudes were calculated by the structure theorist. They are also needed for calculating electromagnetic transitions.

From the definition of $\mathfrak{F}$, the single-particle form factor (3.3) is

$$F_{LSJ}^{ab}(r) = 4\pi R_L^{ab}(r) \langle j_a \| \mathfrak{Y}_{LSJ} \| j_b \rangle \quad (3.4)$$

The radial shape is determined by the factor

$$R_L^{ab}(r) = \frac{1}{2L+1} \int u_a(r') v_L(r, r') u_b(r') r'^2 dr' \quad (3.5)$$

where u_a is the radial part of ψ_a . We can gain an immediate impression of the shape by considering a zero-range interaction. In that case

$$v_L(r, r') = \frac{2L+1}{4\pi} \frac{\delta(r-r')}{r r'} \quad (3.6)$$

and

$$R_L^{ab}(r) = u_a(r) u_b(r)/4\pi \quad (\text{zero range}) \quad (3.7)$$

However for a finite range interaction this becomes more smoothly behaved⁵).

The reduced matrix element of $\mathcal{Y}$ appearing in (3.4) is given by eq. (23) of paper I.

3.2 SELECTION RULES

The transition from one nuclear state to another can be accomplished by the transfer between nucleus and scattered particle, of various combinations of orbital, spin and total angular momentum L, S, J , consistent with the conservation laws.

$$\begin{aligned} J_1 + J_2 + J &= 0, & J &= L + S \\ S &= 0 \text{ or } 1, & (-)^L &= \pi_1 \pi_2 \end{aligned} \quad (3.8)$$

where π_1 denotes the parity of the nuclear state $\alpha_1 J_1$. To each of these there corresponds a form factor. The form factors associated with the spin dependent and independent parts of the interaction (2.27) are referred to as vector ($S = 1$) and scalar ($S = 0$). The former are associated with the spin-flip process.

If one of the nuclear states has zero spin, the above rules yield the condition that J must equal the spin of the other and further, that

$$\left. \begin{array}{l} L = J \\ S = 0 \text{ and } 1 \text{ (if } J \neq 0) \end{array} \right\} \text{ if } \pi_2 = \pi_1(-)^J$$

or

$$\left. \begin{array}{l} L = J \pm 1 \\ S = 1 \end{array} \right\} \text{ if } \pi_2 = \pi_1(-)^{J+1}$$

4. Nuclear Structure Amplitudes

It was seen earlier that the calculation of nuclear form factors for inelastic scattering can be reduced to the problem of finding the amplitudes S_J^{ab} defined by (3.2) for the nuclear model in question. These are the amplitudes corresponding to the promotion of a nucleon between the single-particle states a and b accompanied by a transfer of total angular momentum J .

4.1 A SIMPLE CASE

To illustrate its calculation in a simple example consider the two states

$$|(j^2)_{J_1}\rangle, J_1 = \text{even} \quad (4.1)$$

$$|(j j')_A J_2\rangle \equiv \frac{1}{\sqrt{2}} \left\{ |j(1)j'(2)J_2\rangle - |j(2)j'(1)J_2\rangle \right\}$$

Then

$$\mathcal{F}_{LSJ}^{1,2}(r) = \langle (j)^2_{J_1} \| \mathfrak{F}_{LSJ}(r, A) \| (j j')_A J_2 \rangle \quad (4.2)$$

$$\begin{aligned}
 &= \sqrt{2} \langle j(1) j(2) J_1 \| \mathfrak{Y}_{LSJ}(r, r_2) \| j(1) j'(2) J_2 \rangle \\
 &= (-)^{j+j'+J+J_1} \left[2 \hat{J}_1 \hat{J}_2 \right]^{1/2} \left\{ \begin{matrix} j & j' & J \\ J_2 & J_1 & j \end{matrix} \right\} F_{LSJ}^{jj'}(r)
 \end{aligned} \tag{4.3}$$

In this case only one amplitude for each J exists corresponding to $j \rightarrow j'$ and it can be read from this equation. ($\hat{J}$ means $2J+1$)

4.2 TWO-QUASIPARTICLE OR TWO-PARTICLE STATES

One method of approximately solving the many nucleon system consists in isolating the pairing effects of the residual interaction by solving the BCS equations^{3,15}). This corresponds to transforming from the real nucleons of the system to quasiparticles, which, as far as pairing effects are concerned, are almost non-interacting. The transformation between particles, for which we denote a creation operator by β_{jm}^+ , and quasiparticles, has the form

$$\beta_{jm}^+ = U_j \alpha_{jm}^+ + V_j \tilde{\alpha}_{jm} \tag{4.4}$$

where

$$\tilde{\alpha}_{jm} = (-)^{j-m} \alpha_{j, -m} \tag{4.5}$$

Here U and V are the coefficients of the Bogoliubov-Valatin transformation.

We assume Condon-Shortly phases with the consequence that for a state of minimum energy

$$UV \geq 0 \tag{4.6}$$

In addition they satisfy the normalization condition

$$U^2 + V^2 = 1 \tag{4.7}$$

The remaining part of the interaction is treated approximately by a diagonalization in a subspace of two-quasiparticle configurations, so that the nuclear wave functions in this approximation are of the form

$$|\alpha_{JM}\rangle = \frac{1}{2} \sum_{ab} \eta_{ab}^{\alpha J} A_{JM}^{+}(ab) |\omega\rangle \quad (4.8)$$

where $|\omega\rangle$ denotes the BCS ground state, and

$$\begin{aligned} A_{JM}^{+}(ab) &= (-)^{j_a - j_b + J} A_{JM}^{+}(b, a) \\ &= - \left[\alpha_a^{+} \alpha_b^{+} \right]_J^M \end{aligned} \quad (4.9)$$

is a pair creation operator. Note that the sums on both a and b are each over all configurations.

Matrix elements of a one-body operator on the nucleons, such as

$$\mathcal{F}_{JM} \equiv \sum_{ab m_a m_b} \langle a m_a | \mathcal{F}_{JM} | b m_b \rangle \beta_{a m_a}^{+} \beta_{b m_b} \quad (4.10)$$

can be calculated by expressing the nucleon operators β in terms of the quasiparticle operators. An expression for the quasiparticle representation of $\mathcal{F}_J$ is given by eq. (35) in paper I. The calculation is tedious and we reproduce only the results. For transitions involving the ground state (BCS vacuum) we have

$$S_J^{ab}(0, \alpha J) = - \eta_{ab}^{\alpha J} v_a u_b \quad (4.11)$$

while for transitions between a pair of two-quasiparticle states

$$S_J^{ab}(\alpha_1, \alpha_2) = (-)^{J_1+J_2+J+1} \left[\hat{J}_1 \hat{J}_2 \right]^{1/2} \times \left\{ u_a u_b x_{ab}(\alpha_1 \alpha_2) - (-)^{j_a-j_b+J} v_a v_b x_{ba}(\alpha_1 \alpha_2) \right\} \quad (4.12)$$

where

$$x_{ab}(\alpha_1 \alpha_2) = \sum_d \eta_{da}^{\alpha_1 J_1} \eta_{bd}^{\alpha_2 J_2} \begin{Bmatrix} J_2 & J & J_1 \\ j_a & j_d & j_b \end{Bmatrix} \quad (4.13)$$

From these the nuclear form factors can now be calculated according to (3.2).*

Note that the sums on a and b are unrestricted! The connection between the configuration amplitudes η in (4.8), where the sum is unrestricted, and those over ordered configurations

$$|\alpha JM\rangle = \sum_{a \leq b} \frac{c_{ab}^{aJ}}{\sqrt{1+\delta_{ab}}} A_{JM}^+(a, b) |\omega\rangle \quad (4.14)$$

is

$$\eta_{ab}^{\alpha J} = (-)^{j_a-j_b+J} \eta_{ba}^{\alpha J} = c_{ab}^{\alpha J} \sqrt{1+\delta_{ab}} \quad (4.15)$$

Usually configuration amplitudes are quoted in terms of an ordered group of configurations (in which only (ab) or (ba) but not both occur), but it is much simpler to state the results for S in terms of the unordered convention.

* In refs. 5 and 9 the amplitudes S_J depended upon L and S as well as J . This can be avoided by summing in a different way. I am indebted to J. Hocquenghem for pointing this out.

The results (4.11) and (4.12) are general when two fermions are involved, whether they be particles, quasiparticles or hole states of either of these. For example (4.3) can be obtained from (4.12). In this case

$$\begin{aligned} c_{jj}^{(1)} &= 1 \quad \text{so} \quad \eta_{jj}^{(1)} = \sqrt{2} \\ c_{jj'}^{(2)} &= 1 \quad \text{so} \quad \eta_{jj'}^{(2)} = 1, \quad \eta_{j'j}^{(2)} = (-)^{j-j'+J_2} \end{aligned} \quad (4.16)$$

$$U_j = U_{j'} = 1$$

We find

$$x_{jj'}(1,2) = \sqrt{2} (-)^{j-j'+J_2} \begin{Bmatrix} J_2 & J & J_1 \\ j & j & j' \end{Bmatrix} \quad (4.17)$$

$$x_{j,j}(1,2) = 0 \quad (4.18)$$

which lead directly to (4.3). More generally, (4.8) becomes a two-particle state upon choosing $U \equiv 1$. The structure amplitudes connecting these are given by (4.12).

4.3 PARTICLE-HOLE STATES

In lowest approximation the vibrations in closed shell nuclei may be described as a superposition of particle hole states. The formalism of the preceeding section can be specialized to this case. Let the occupied levels of the ground state be referred to as $h(\equiv n_h \ell_h j_h)$ and the levels above by p . Then choose $V_h = 1$. $U_p = 1$ in which case the BCS vacuum becomes

$$|\omega\rangle = \beta_1^+ \dots \beta_A^+ |0\rangle \quad (4.19)$$

where $|0\rangle$ is the particle vacuum and $1...A$ enumerates the closed shell levels. Also

$$\alpha_h^+ \rightarrow -\tilde{\beta}_h, \quad \alpha_p^+ \rightarrow \beta_p^+ \quad (4.20)$$

so that (4.8) becomes

$$|\alpha JM\rangle = - \sum_{p,h} c_{ph} \left[\beta_p^+ \tilde{\beta}_h \right]_J^M |\omega\rangle \quad (4.21)$$

where h is summed over the levels below, and p over the levels above the Fermi surface. Structure amplitudes connecting the ground to particle-hole states is obtained from (4.11), as

$$S_J^{hp}(0, \alpha J) = (-)^{j_p + j_h + J} c_{ph}^{\alpha J} \quad (4.22)$$

$$S_J^{ph}(0, \alpha J) = 0$$

where we convert to the C 's according to (4.15) when the sums are restricted.

For the amplitudes connecting a pair of particle-hole states we must use (4.12). For scattering of the particle from one state to another we have

$$S_J^{pp'}(\alpha_1, \alpha_2) = (-)^{J_1 + J_2 + J + 1} (\hat{J}_1 \hat{J}_2)^{1/2} x_{pp'}(\alpha_1 \alpha_2) \quad (4.23)$$

while the amplitude for scattering of the hole is

$$S_J^{hh'}(\alpha_1, \alpha_2) = (-)^{J_1 + J_2 + j_h - j_{h'}} (\hat{J}_1 \hat{J}_2)^{1/2} x_{h'h}(\alpha_1 \alpha_2) \quad (4.24)$$

where

$$X_{pp'} = \sum_h (-)^{j_h - j_p + J_1} c_{ph}^{J_1} c_{p'h}^{J_2} \begin{Bmatrix} J_2 & J & J_1 \\ p & h & p' \end{Bmatrix} \quad (4.25)$$

$$X_{h'h} = \sum_p (-)^{j_h - j_p + J_2} c_{ph'}^{J_1} c_{ph}^{J_2} \begin{Bmatrix} J_2 & J & J_1 \\ h' & p & h \end{Bmatrix} \quad (4.26)$$

4.4 TWO-PHONON STATES

While the two quasiparticle method mentioned above gives rise to a collective 2^+ state, it cannot describe anything like a two-phonon state. However in the spirit of the vibrational model we can use the coherent operator that creates the 2_1^+ state to generate a triplet of states. Presumably these are not eigenfunctions of the problem and would mix with neighboring two-quasiparticle states, but they can be considered as an extreme limit. Such states have the structure

$$|2 \text{ Phon}, J\rangle = \frac{1}{\sqrt{2}} \sum_{abcd} \eta_{ab} \eta_{cd} \left[A_2^+(ab) A_2^+(cd) \right]_J |\omega\rangle \quad (4.27)$$

where the η 's are the amplitudes of the coherent 2^+ state. Using the quasiboson commutation relations for the pair creation operators A^+ we easily obtain for the form factor connecting each member of the triplet to the 2^+ state

$$\begin{aligned} \mathcal{F}_{LSJ}(r) &\equiv \langle 2_1 \| \mathcal{F}_{LSJ} \| 2 \text{ Phon}, J_2 \rangle \\ &= \delta_{J,2} \left(\frac{2 J_2}{5} \right)^{1/2} \mathcal{F}_{LSJ}^{0,2_1}(r) \end{aligned} \quad (4.28)$$

where the form factor on the right is that connecting ground and 2_1^+ state.

In other words

$$s_J^{ab}(2_1, 2 \text{ Phon } J_2) = -\delta_{J,2} \left(\frac{2 \hat{J}_2}{5} \right)^{1/2} \eta_{ab} v_a u_b \quad (4.29)$$

4.5 SYMMETRIES OF THE STRUCTURE AMPLITUDES

Since the matrix (2.32) is symmetric, the form factors must have the same phase under interchange of the states as does the geometrical factor C of (2.33). This is

$$C_{LSJ}^{\pi I}(c_1, c_2) = (-)^{J_1+J_2+L+S+J} C_{LSJ}^{\pi I}(c_2, c_1) \quad (4.30)$$

Also note that

$$F_{LSJ}^{ab} = (-)^{j_a-j_b+L+S+J} F_{LSJ}^{ba} \quad (4.31)$$

whence from its definition (3.2)

$$s_J^{ab}(\alpha_1 \alpha_2) = (-)^{j_a-j_b+J_1+J_2} s_J^{ba}(\alpha_2 \alpha_1) \quad (4.32)$$

5. Polynomial Expansion of the Form Factor

When the shape of the potential (2.27) is Gaussian

$$g(|\underline{r}-\underline{r}'|) = \exp(-\beta |\underline{r}-\underline{r}'|^2) \quad (5.1)$$

and the single-particle states of the nucleus are represented by harmonic oscillator functions, then the single-particle form factors, and hence the nuclear form factors (3.2) can be written in a simple closed form involving a polynomial and a Gaussian factor¹⁰). Indeed in paper I the formulae were given which express the single-particle form factor, F , as

$$F_{LSJ}^{ab}(r) = e^{-\gamma r^2} \sum_{n=L}^{N_a+N_b} g_n (\beta r/\sqrt{\nu})^n \quad (5.2)$$

where

$$g_n \equiv g_n(LSJ;ab) = 4\pi \langle j_a || \mathfrak{Y}_{LSJ} || j_b \rangle G_{(n-L)/2}^L(a,b) \quad (5.3)$$

is defined for $n-L \geq 0$ and even, and the coefficients G were defined in the appendix of I. In the upper limit on the sum, $N_a = 2(n_a - 1) + \ell_a$, is the oscillator quantum number. Also $\nu = m\omega/\hbar$ is the oscillator constant, and $\gamma = \nu\beta/(\nu+\beta)$, where $\beta^{-1/2}$ is the force range of the direct interaction (5.1).

Using the above results we can now express the coupling matrix elements in the same simple form:

$$\langle \phi_{c_1} \pi I | V | \phi_{c_2} \pi I \rangle = e^{-\gamma r^2} \sum_{n=L}^{\bar{n}} \gamma_n^I(c_1, c_2) (\beta r/\sqrt{\nu})^n \quad (5.4)$$

where $L_<$ is the smallest value of L for which g_n is non-vanishing and the upper limit, $\bar{n}$, is the maximum value of $N_a + N_b$ for the configurations entering the nuclear description. The coefficients γ_n are given in terms of the previously defined quantities as

$$\gamma_n^I(c_1, c_2) = \sum_{LSJ} V_S^I C_{LSJ}^I(c_1, c_2) \sum_{a,b} S_J^{ab}(\alpha_1, \alpha_2) g_n(LSJ; ab) \quad (5.5)$$

which is defined in the range $L_< \leq n \leq \bar{n}$.

The usefulness of these formulae in a coupled channel calculation can be gauged by the fact that all of the nuclear structure and geometrical information is concentrated in a few coefficients, γ , for each matrix element (5.4) and these can be computed in a few seconds on a fast machine for all the states one would want to include in a calculation.

6. Summary

The formalism and rationale for calculating inelastic scattering from nuclei described in microscopic terms has been set down. The relevant content of a nuclear description for this purpose can be summarized in a set of amplitudes. These have been exhibited explicitly for several nuclear models in which two fermion degrees of freedom are treated explicitly. In the general case, this is not possible but it is an easy task for the structure theorist to generate them for his particular calculation, and thus make it accessible to the test which scattering and polarization data provides. Finally explicit formulae for the polynomial development of the form factors, or matrix elements appearing in the coupled equations, are given.

The formalism has been applied to the nickel and tin isotopes treated in the two-quasiparticle representation, as reported earlier^{8,9}).

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