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by

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Abstract

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Hyperkähler manifolds are one of the simplest examples of Einstein manifolds. They are Ricci-flat Riemannian manifolds with special holonomy. In dimension 4, hyperkähler 4-manifolds can be purely be described by a triple of symplectic 2-forms that satisfy the pointwise orthonormal condition with respect to the wedge product.

In this dissertation, we proved the compactness of a set of hyperkähler 4-manifolds with boundary under Cheeger-Gromov topology, where we assume only geometric control on the boundary and topological conditions. We showed that our proof can be extended to Einstein 4-manifolds with boundary by assuming only additional topological conditions.

Furthermore, we discuss about the period map for K3 surfaces in a differential geometric setting. We gave a simple proof for the surjectivity of the period map, without invoking Yau's theorem on the Calabi conjecture and any algebraic geometry. The key is to show when a sequence of hyperkähler metrics has bounded period in some sense, then the sequence have a convergent subsequence under Cheeger-Gromov topology.

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Chapter 1

Introduction

A Riemannian metric g on a 4-manifold is called hyperkähler if its holonomy group $Hol(g)$ is contained in $Sp(1) = SU(2)$. A closed hyperkähler 4-manifold is diffeomorphic to either a torus or the K3 manifold, and the moduli space of all hyperkähler metrics are described by Torelli theorems. There have been extensive recent studies on the Gromov-Hausdorff compactification of these moduli spaces, see for example [41, 46].

Hyperkähler metrics in dimension 4 are the simplest models for Riemannian metrics with special holonomy. Little general existence theory is developed for the latter in dimensions greater than 4, except for Calabi-Yau manifolds. Recently Donaldson [19] proposes to study special holonomy metrics on manifolds with boundary and set up suitable elliptic boundary value problems. To make further progress in this direction, it is clear that we need a compactness theory.

In this dissertation, we study the boundary value problem for hyperkähler 4-manifolds, which serves as the first step towards Donaldson's program. We follow the general set-up by Fine-Lotay-Singer [24] in terms of *hyperkähler triples*. A hyperkähler triple on an oriented smooth 4-manifold X is a triple of symplectic forms $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ satisfying the following pointwise condition

$$\omega_i \wedge \omega_j = \frac{1}{3}\delta_{ij}(\omega_1^2 + \omega_2^2 + \omega_3^2).$$

It is well-known that a hyperkähler triple $\boldsymbol{\omega}$ uniquely determines a compatible hyperkähler metric $g_{\boldsymbol{\omega}}$ such that for each i, $\omega_i^2 = 2\mathrm{dvol}_{g_{\boldsymbol{\omega}}}$ and ω_i is parallel with respect to the Levi-Civita connection. Conversely, given a hyperkähler metric g on X, one can choose an orientation and find a compatible hyperkähler triple $\boldsymbol{\omega}$, which is unique up to a constant $O(3)$ rotation.

Now let X be a compact oriented smooth 4-manifold with boundary ∂X. Note ∂X has an induced orientation defined by contracting a volume form of X with an outward vector field. If $\boldsymbol{\omega}$ is a hyperkähler triple on X, then its restriction to ∂X is a closed framing $\boldsymbol{\gamma}$ on ∂X. The following is a natural filling problem, proposed by [24].

Question 1.0.1. *Given a closed framing $\boldsymbol{\gamma}$ on ∂X, does there exist a hyperkähler triple $\boldsymbol{\omega}$ on X extending $\boldsymbol{\gamma}$?*

Notice a framing $\boldsymbol{\gamma}$ defines a Riemannian metric $g_{\boldsymbol{\gamma}}$ on ∂X as follows: first, there exists a unique dual coframe $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$ such that $\gamma_i = \frac{1}{2}\delta^{ijk}\eta_j \wedge \eta_k$ and such that $\eta_1 \wedge \eta_2 \wedge \eta_3$ is compatible with the orientation of ∂X; then the Riemannian metric $g_{\boldsymbol{\gamma}}$ is defined by setting $\boldsymbol{\eta}$ to be orthonormal. When there is no ambiguity, we always use $\boldsymbol{\eta}$ to denote the dual coframe of $\boldsymbol{\gamma}$ defined in this way and denote the Hodge star operator of the Riemannian metric by $*_{\boldsymbol{\gamma}} = *_{\boldsymbol{\eta}}$. It is well-known that if $\boldsymbol{\omega}$ is a hyperkähler triple, then $g_{\boldsymbol{\omega}}|_{\partial X} = g_{\boldsymbol{\gamma}}$; more importantly Bryant [10] observed that the second fundamental form of ∂X is determined intrinsically by $\boldsymbol{\gamma}$ via the matrix $*_{\boldsymbol{\eta}}(\eta_i \wedge d\eta_j)$. In particular, the mean curvature $H_{\boldsymbol{\gamma}}$ is given by one half of the trace of this matrix, i.e., $H_{\boldsymbol{\gamma}} = \frac{1}{2} *_{\boldsymbol{\eta}} (\boldsymbol{\eta} \wedge d\boldsymbol{\eta}^T)$.

There are some previous works on Question 1.0.1. Bryant [10] studied the local "thickening" problem and obtained both positive and negative results. It was shown that any real analytic closed framing on a closed oriented 3-manifold Y can be extended to a hyperkähler triple on $Y \times (-\epsilon, \epsilon)$ for some $\epsilon > 0$, and the extension is essentially unique. On the other hand, there exists a smooth closed framing on an open ball $B^3 \subset \mathbb{R}^3$ that cannot be extended to a hyperkähler triple on $B^3 \times (-\epsilon, \epsilon)$ for any $\epsilon > 0$. Fine-Lotay-Singer[24] studied the local deformation theory for Question 1.0.1 and showed that the boundary framings must deform in certain directions. Roughly speaking, let $X = B^4$ for simplicity, suppose $\boldsymbol{\omega}$ is a hyperkähler triple such that ∂X has positive mean curvature, and $\boldsymbol{\omega}'$ is a nearby hyperkähler triple, then after moduling out diffeomorphisms of ∂X, the dual coframe of $\boldsymbol{\omega}'|_{\partial X}$ must be a small pertubation of that of $\boldsymbol{\omega}|_{\partial X}$ in the direction of negative frenquency of the boundary Dirac operator defined by $g_{\boldsymbol{\omega}}|_{\partial X}$.

Our main result is the following closedness result for Question 1.0.1 :

Theorem 1.0.2. *Let X be a compact oriented smooth 4-manifold with boundary, such that there does not exist $C \in H_2(X, \mathbb{Z})$ with self intersection $C^2 = -2$. Let $\boldsymbol{\omega}_i$ be a sequence of smooth hyperkähler triples on X. Suppose $\boldsymbol{\omega}_i|_{\partial X}$ converges in Cheeger-Gromov sense to a closed framing $\boldsymbol{\gamma}$ on ∂X such that $H_{\boldsymbol{\gamma}} > 0$, then there exists a smooth hyperkähler triple $\boldsymbol{\omega}$ on X with $\boldsymbol{\omega}|_{\partial X} = \boldsymbol{\gamma}$ and $\boldsymbol{\omega}_i$ converges in Cheeger-Gromov sense to $\boldsymbol{\omega}$ on X.*

Here, a sequence of pairs of smooth covariant tensors $(T_i^1, \cdots, T_i^m)$ on a compact manifold M with empty or nonempty boundary is said to converge in *Cheeger-Gromov* sense to $(T^1, \cdots, T^m)$ on M, if there exist diffeomorphisms $f_i : M \to M$ such that $f_i^* T_i^1 \to T^1, \cdots, f_i^* T_i^m \to T^m$ smoothly on M.

The proof of Theorem 1.0.2 includes two parts: the compactness and uniqueness. The former is the main story of this dissertation, and the latter is a consequence of [9] or [5] on unique continuation of Einstein metrics with prescribed boundary metric and second fundamental form. It is worth noting that for the compactness part, no general Riemannian convergence theory can be applied directly. The difficulty here is that we only have data on the boundary, and a priori we do not know anything near the boundary or in the interior. Specifically, we worry about the following three bad geometric behaviours: curvature blow-up, volume collapsing and boundary touching. These things are entangled, making it difficult to rule out any of them. However, we are able to separate these bad behaviours and rule them out. We will also give examples to demonstrate that the assumptions in Theorem 1.0.2 are essential, see Remark 4.2.4 and 4.2.5.

Such C in the assumption of Theorem 1.0.2 is usually called a "-2 *curve*" in X. It appears in Kronheimer's classification of hyperkähler ALE spaces [36, 37]. By analyzing the formation of hyperkähler ALE spaces as bubble limits of volume-noncollapsed hyperkähler 4-manifolds, one can slightly weaken the "no -2 curve" condition. Let us start with the definition of "enhancement". Following [19, 22], an enhancement of a closed framing $\boldsymbol{\gamma}$ on ∂X is an equivalent class in the set of triples of closed 2-forms on X extending $\boldsymbol{\gamma}$, and the equivalence relation is defined by $\boldsymbol{\theta} \sim \boldsymbol{\theta} + d\boldsymbol{a}$ for some triple of smooth 1-forms $\boldsymbol{a}$ on X vanishing on ∂X. From the de Rham cohomology exact sequence of the pair $(X, \partial X)$,

$$H^2(X, \partial X) \to H^2(X) \to H^2(\partial X) \to H^1(X, \partial X),$$

we know $\boldsymbol{\gamma}$ has at least one enhancement if and only if each γ_i lies in the kernel of $H^2(\partial X) \to H^1(X, \partial X)$, and we know the set of all enhancements of $\boldsymbol{\gamma}$ is an affine space over $H^2(X, \partial X) \otimes \mathbb{R}^3$. Let $\hat{\boldsymbol{\gamma}}$ be an enhancement of $\boldsymbol{\gamma}$. Now given a 2-cycle $\Sigma \in H_2(X, \mathbb{Z})$ and $\boldsymbol{\theta} \in \hat{\boldsymbol{\gamma}}$, $\int_\Sigma \boldsymbol{\theta}$ is independent of the choice of $\boldsymbol{\theta}$ and we denote this invariant by $c_{\hat{\boldsymbol{\gamma}},\Sigma} \in \mathbb{R}^3$.

The proof of Theorem 1.0.2 easily adapts to

Theorem 1.0.3. *Let X be a compact oriented smooth 4-manifold with boundary. Let $\boldsymbol{\omega}_i$ be a sequence of smooth hyperkähler triples on X, and $\hat{\boldsymbol{\gamma}}_i$ be the enhancement of $\boldsymbol{\gamma}_i = \boldsymbol{\omega}_i|_{\partial X}$ where $\boldsymbol{\omega}_i$ lie in. Let $a > 0$ be a positive number. Suppose for any $C \in H_2(X, \mathbb{Z})$ with self intersection $C^2 = -2$, $|c_{\hat{\boldsymbol{\gamma}}_i,C}| \geq a$ and $\boldsymbol{\omega}_i|_{\partial X}$ converges in Cheeger-Gromov sense to a closed framing $\boldsymbol{\gamma}$ on ∂X such that $H_{\boldsymbol{\gamma}} > 0$. Then there exists a smooth hyperkähler triple $\boldsymbol{\omega}$ on X with $\boldsymbol{\omega}|_{\partial X} = \boldsymbol{\gamma}$, and $\boldsymbol{\omega}_i$ converges in Cheeger-Gromov sense to $\boldsymbol{\omega}$ on X.*

It is worth noting that Question 1.0.1 is not an elliptic boundary value problem, observed by [24]. This can also be seen from the uniqueness result of [9] or [5]: the restriction of $\boldsymbol{\omega}$ to any open boundary portion determines $g_{\boldsymbol{\omega}}$ in the whole interior up to local isometries. So, it is natural to consider larger class of closed triples of 2-forms on X in Question 1.0.1 to obtain an elliptic boundary value problem. In [19], Donaldson studied the deformation theory of torsion-free G_2 structures on compact oriented 7-manifolds with boundary and set up an elliptic boundary value problem, which can be reduced to dimension 4(See [22]). Hence, the "correct" class of triples in Question 1.0.1 should be those defining torsion free G_2 structures. Specifically, a triple of two forms $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ on X is called torsion-free hypersymplectic if the 3-form ϕ on $X \times T^3$ defined by

$$\phi = dt^1 \wedge dt^2 \wedge dt^3 - \omega_1 \wedge dt^1 - \omega_2 \wedge dt^2 - \omega_3 \wedge dt^3 \tag{1.1}$$

is a torsion-free G_2 structure. Locally, this is a weaker condition than being hyperkähler, and there are examples in [22] or [25].

Similar to the hyperkähler case, a torsion-free hypersymplectic triple $\boldsymbol{\omega}$ defines a Riemannian metric $g_{\boldsymbol{\omega}}$ and a positive definite $SL(3, \mathbb{R})$-valued function $\boldsymbol{Q} = (Q_{ij})$ such that

$$\omega_i \wedge \omega_j = 2Q_{ij}\mathrm{dvol}_{g_{\boldsymbol{\omega}}}$$

We denote $\boldsymbol{Q}'$ the restriction of $\boldsymbol{Q}$ to ∂X. When there is ambiguity, we use notations $\boldsymbol{Q}_{\boldsymbol{\omega}}$, $\boldsymbol{Q}'_{\boldsymbol{\omega}}$ to denote their dependence on $\boldsymbol{\omega}$. One can show that the mean curvature of ∂X has an explicit expression in terms of $\boldsymbol{\gamma}$, $\boldsymbol{Q}'$, and we denote this explicit expression by $H_{\boldsymbol{\gamma},\boldsymbol{Q}'}$. Note that on ∂X, $\boldsymbol{\gamma}, \boldsymbol{Q}'$ are subject to the constraints $d\boldsymbol{\gamma} = 0, d(\boldsymbol{\gamma}(\boldsymbol{Q}')^{-1}) = 0$.

We have the following analogue of Theorem 1.0.3:

Theorem 1.0.4. *Let X be a compact oriented smooth 4-manifold with boundary. Let $\boldsymbol{\omega}_i$ be a sequence of smooth torsion-free hypersymplectic triples on X, and $\hat{\boldsymbol{\gamma}}_i$ be the enhancement of $\boldsymbol{\gamma}_i = \boldsymbol{\omega}_i|_{\partial X}$ where $\boldsymbol{\omega}_i$ lie in. Let $a > 0$ be a positive number. Suppose for any $C \in H_2(X, \mathbb{Z})$ with self intersection $C^2 = -2$, $|c_{\hat{\boldsymbol{\gamma}}_i,C}| \geq a$, and $(\boldsymbol{\gamma}_i, \boldsymbol{Q}'_i)$ converges in Cheeger-Gromov sense to some pair $(\boldsymbol{\gamma}, \boldsymbol{Q}')$ on ∂X, such that $\boldsymbol{\gamma}$ is a framing, $\boldsymbol{Q}'$ is positive definite and $H_{\boldsymbol{\gamma},\boldsymbol{Q}'} > 0$. Then there exists a smooth torsion-free hypersymplectic triple $\boldsymbol{\omega}$ on X with $\boldsymbol{\omega}|_{\partial X} = \boldsymbol{\gamma}$, $\boldsymbol{Q}'_{\boldsymbol{\omega}} = \boldsymbol{Q}'$, and $\boldsymbol{\omega}_i$ converges in Cheeger-Gromov sense to $\boldsymbol{\omega}$.*

Note that Theorem 1.0.4 includes the previous two versions.

We also showed that our techniques to prove compactness results can be generalized to Einstein manifolds, and we get the following:

Theorem 1.0.5. *Let $\mathcal{M}$ be the set of pointed compact oriented Einstein 4-manifolds (M, g, p) with boundary such that $|Ric| \leq 3$, $H_2(M, \mathbb{Z}) = H_1(M, \mathbb{Z}) = 0$, ∂M is diffeomorphic to S^3,*

$$vol_{\partial M}(\partial M) \leq C, |S| \leq C, |\nabla^k_{\partial M} Rm_{\partial M}| \leq C, |\nabla^{k+1}_{\partial M} H| \leq C, inj_{\partial M} \geq i_0, H \geq H_0 > 0, \forall k \geq 0$$

$$d(p, \partial M) \leq K.$$

Then $\mathcal{M}$ is precompact in pointed Cheeger-Gromov topology, and an element in $\partial\mathcal{M}$ is a complete Einstein orbifold with smooth boundary.

Similar results have been obtained for conformally compact Einstein(CCE) 4-manifolds in [11, 12], however, there they need a priori L^2 bound for the Weyl curvature. Our result only imposes geometric control on the boundary, and we face essentially different difficulties from [11, 12].

In the last part of the dissertation, we considered the period map of $K3$ surface in a differential geometric setting. A $K3$ surface is a simply connected complex surface with vanishing first Chern class. Hence, they are simply connected oriented smooth 4-manifolds with signature $(3, 19)$. It is well-known that all complex $K3$ surfaces are diffeomorphic, and their moduli space can be described by period maps.

There are a lot of studies on the moduli space of $K3$ surface, especially the Torelli theorems. However, they invoke a lot of algebraic geometry, complex geometry and their proofs are complicated. The motivation of our study is that our approach could possibly be generalized to studying period maps in higher dimensions, or hypersymplectic 4-manifolds.

Let $\mathcal{M}_{K3}$ be the moduli space of hyperkähler metrics on a $K3$ surface of unit diameter, up to isometries, equipped with the Gromov-Hausdorff topology. One can define a period map

$$\underline{\mathcal{P}} : \mathcal{M}_{K3} \to Gr^+/\Gamma$$

by sending g to $\mathbb{H}_g^+$, the space of self-dual harmonic two forms(of dimension 3). Here Γ is the automorphism group of the K3 lattice $H^2(K3,\mathbb{Z})$, Gr^+ is the space of all positive 3-planes in $H^2(K3,\mathbb{R})$.

Define

$$Gr^{+,\circ} = Gr^+ \backslash \{H \in Gr^+ : \exists \delta . \delta = -2 \ s.t. \int_\delta h = 0, \forall h \in H\}.$$

We proved that

Theorem 1.0.6. *The image of $\underline{\mathcal{P}}$ is contained in $Gr^{+,\circ}$, and*

$$\underline{\mathcal{P}} : \mathcal{M}_{K3} \to Gr^{+,\circ}/\Gamma$$

is a proper, open, surjective map.

Note that in complex geometry setting, the surjectivity part is refered to as Todorov's surjectivity result [49], and later Siu gave a simple proof [45]. Both of the proofs involve Yau's solution to the Calabi conjecture [50].

The main point in our proof is showing the period map $\underline{\mathcal{P}} : \mathcal{M}_{K3} \to Gr_3^{+^\circ}/\Gamma$ is proper, so one needs to show if a sequence of hyperkähler metrics is collapsing, then their periods go to infinity in the locally symmetric space. To show this, we made use of the collapsing theory of hyperkähler 4-manifolds by Sun-Zhang [46], as integrals of the period at specific homology classes can be calculated.

Organization of the dissertation

In Chapter 2, we provide key ingredients in Riemannian geometry needed to prove our main theorems. In Chapter 3, we discuss some known basics about hyperkähler 4-manifolds. In Chapter 4, we prove the compactness results for hyperkähler 4-manifolds and Einstein 4-manifolds with boundary. In Chapter 5, we prove the convergence in triple settings and the uniqueness. In Chapter 6, we prove the statement about the period map.

Disclaimer

This dissertation is based on 2 papers of the author [39, 40], except for Section 4.4.

Notations

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x^n \geq 0\},$$
$$B_r = \{x \in \mathbb{R}^n : |x| < r\},$$
$$B_r^+ = B_r \cap \mathbb{R}^n_+,$$
$$\tilde{B}_r = B_r^+ \cap \partial\mathbb{R}^n_+,$$
$$\partial\tilde{B}_r = \{x \in \mathbb{R}^n : |x| = r, x^n = 0\},$$
$$\partial^+ B_r^+ = \{x \in \mathbb{R}^n : |x| = r, x^n > 0\},$$
$$N_r(\partial M, g) = \{x \in M | d(x, \partial M) \leq r\}.$$

Chapter 2

Riemannian geometry

2.1 Riemannian geometry for manifolds with boundary

2.1.1 Evolution equations of hypersurfaces

We refer to [42] Section 3.2 and [35] for discussions in this section.

Let (M, g) be a Riemannian manifold, let ∇ denotes it's Levi-Civita connection. Given a smooth distance function f on M, i.e., $|\nabla f| = 1$ everywhere, so $\nabla_{\nabla f}\nabla f = 0$, in fact $\nabla_{\nabla f}\nabla f$ is the gradient vector field of $\frac{1}{2}|\nabla f|^2$. Note that the discussion below will apply to Riemannian manifolds with boundary, with f the negative of distance function to the boundary, but the discussions in this subsection is for a general smooth distance function f.

The (symmetric) $(1,1)$ tensor corresponds to $\mathrm{Hess} f = \frac{1}{2}L_{\nabla f}g$ is

$$S(X) = \nabla_X \nabla f, \tag{2.1}$$

and its trace is $H = \Delta f \in C^\infty(M)$.

Let Σ be a level set of f, i.e., $\Sigma = f^{-1}(a)$ for some a. Then $S|_{T\Sigma}$ is a section of $\Gamma(\mathrm{Hom}(T\Sigma, T\Sigma))$. In fact, since ∇ is a metric connection, for any $X \in \Gamma(TM)$, we have $2\langle S(X), \nabla f\rangle = \nabla_X \langle \nabla f, \nabla f\rangle = 0$. $S|_{T\Sigma}$ is usually refered to as the *shape operator* of Σ. The *second fundamental form* of Σ with respect to the unit normal vector ∇f is defined as

$$II(X, Y) := \langle S(X), Y\rangle = \mathrm{Hess} f(X, Y),$$

so H is the *mean curvature* of Σ. The *mean curvature vector* is defined by $\overrightarrow{H} = -H\nabla f$. By tensor calculations, we have evolution equations of second fundamental forms

Proposition 2.1.1.

$$L_{\nabla f} S + S^2 = -R(\cdot, \nabla f)\nabla f, \tag{2.2}$$

$$L_{\nabla f} Hess f - Hess^2 f = -Rm(\cdot, \nabla f, \cdot, \nabla f), \tag{2.3}$$

where

$$Hess^2 f(X,Y) = \langle S^2(X), Y\rangle = \langle S(X), S(Y)\rangle.$$

Proof.

$$\begin{aligned} R(X,\nabla f)\nabla f &= \nabla_X \nabla_{\nabla f}\nabla f - \nabla_{\nabla f}\nabla_X \nabla f - \nabla_{[X,\nabla f]}\nabla f \\ &= -\nabla_{\nabla f}(S(X)) - S([X,\nabla f]) \\ &= -\nabla_{\nabla f}(S(X)) - S(\nabla_X \nabla f - \nabla_{\nabla f} X) \\ &= -\nabla_{\nabla f}(S(X)) - S^2(X) + S(\nabla_{\nabla f} X) \\ &= -\left(\nabla_{\nabla f} S\right)(X) - S^2(X). \end{aligned}$$

Moreover, we have the equality

$$L_{\nabla f} S = \nabla_{\nabla f} S. \tag{2.4}$$

In fact,

$$\begin{aligned} \left(L_{\nabla f}(S)\,(X)\right) &= L_{\nabla f} S(X) - S\left(L_{\nabla f} X\right) \\ &= [\nabla f, S(X)] - S\left(\nabla_{\nabla f} X - \nabla_X \nabla f\right)) \\ &= \nabla_{\nabla f} S(X) - \nabla_{S(X)}\nabla f \\ &\quad - S\left(\nabla_{\nabla f} X - \nabla_X \nabla f\right) \\ &= (\nabla_{\nabla f} S)(\,X) - S^2(X) + S^2(X) \\ &= \left(\nabla_{\nabla f} S\right)(X), \end{aligned}$$

which proves the first equality. For the second equality,

$$\begin{aligned} \left(L_{\nabla f}\text{Hess } f\right)(X,X) &= L_{\nabla f}(\text{Hess } f(X,X)) - 2\text{Hess } f(L_{\nabla f}X, X) \\ &= L_{\nabla f}(g(S(X),X)) - 2\left\langle S\left(L_{\nabla f} X\right), X\right\rangle \\ &= (L_{\nabla f} g)(S(X),X) + \langle L_{\nabla f}(S(X)), X\rangle \\ &\quad + \langle S(X), L_{\nabla f} X\rangle - 2\left\langle S(L_{\nabla f} X), X\right\rangle \\ &= 2\text{Hess } f(S(X),X) + \langle L_{\nabla f}(S(X)), X\rangle \\ &\quad + \left\langle X, S\left(L_{\nabla f} X\right)\right\rangle - 2\left\langle S(L_{\nabla f})X, X\right\rangle \\ &= 2\text{Hess } f(S(X),X) + \langle (L_{\nabla f} S)(X), X\rangle \\ &\quad + \left\langle X, S\left(L_{\nabla f} X\right)\right\rangle + \langle X, S(L_{\nabla f} X)\rangle - 2\left\langle S(L_{\nabla f} X), X\right\rangle \\ &= 2\left\langle S^2(X), X\right\rangle + \left\langle -S^2(X) - R(X,\nabla f)\nabla f, X\right\rangle \\ &= \left\langle S^2(X), X\right\rangle - Rm(X,\nabla f, X, \nabla f). \end{aligned}$$

□

Take the trace of (2.2), and notice that taking the trace commutes with taking the Lie derivative, we have

$$L_{\nabla f} H = -|S|^2 - \text{Ric}(\nabla f, \nabla f), \tag{2.5}$$

where $|S|^2 := \text{Tr}(S^2)$ is the norm square of the shape operator.

Besides the evolution equations, the Gauss equations on Σ is given by

$$\begin{aligned} Rm_M(X,W,Y,Z) =& Rm_\Sigma(X,W,Y,Z) + II(X,Z)II(W,Y) \\ &- II(X,Y)II(W,Z). \end{aligned} \tag{2.6}$$

Take the trace with respect to W, Z, we have

$$\mathrm{Ric}_M = \mathrm{Ric}_\Sigma + \mathrm{Hess}^2 f - H \cdot \mathrm{Hess} f + Rm_M(\cdot, \nabla f, \cdot, \nabla f), \tag{2.7}$$

Take the trace again, we have

$$R_M = R_\Sigma + |S|^2 - H^2 + 2\mathrm{Ric}_M(\nabla f, \nabla f), \tag{2.8}$$

where R denote scalar curvatures. Use equations (2.3) and (2.7) to cancel the curvature term involving the normal vector ∇f, we get

$$L_{\nabla f}\mathrm{Hess} f = \mathrm{Ric}_\Sigma - \mathrm{Ric}_M + 2\mathrm{Hess}^2 f - H \cdot \mathrm{Hess} f. \tag{2.9}$$

One more formula we may need is the Laplace operator on a hypersurface:

Proposition 2.1.2. *Let $\Sigma \subset M$ be a smooth hypersurface, h a smooth function on M,N_0 a unit normal vector field of Σ, then*

$$\Delta_\Sigma h = \Delta h - \mathit{Hess}\ h(N_0, N_0) + \langle \nabla h, \overrightarrow{H}_\Sigma \rangle. \tag{2.10}$$

Proof. Choose an orthonormal frame $\{e_1, \cdots, e_{n+1}\}$ of TM, such that $e_{n+1} = N_0$, then

$$\begin{aligned} \Delta_\Sigma h &= \sum_{i=1}^n (\nabla^\Sigma)^2_{e_i,e_i} h \\ &= \sum_{i=1}^n (\nabla^\Sigma_{e_i}\nabla^\Sigma_{e_i} h - \nabla^\Sigma_{\nabla^\Sigma_{e_i} e_i} h) \\ &= \sum_{i=1}^n (\nabla_{e_i}\nabla_{e_i} h - \nabla_{\nabla_{e_i}e_i - \langle \nabla_{e_i}e_i, e_{n+1}\rangle e_{n+1}} h) \\ &= \sum_{i=1}^n \ \mathrm{Hess} h(e_i,e_i) + \sum_{i=1}^n \langle \nabla_{e_i} e_i, e_{n+1}\rangle \nabla_{e_{n+1}} h \\ &= \sum_{i=1}^n \ \mathrm{Hess} h(e_i,e_i) - \sum_{i=1}^n \langle e_i, \nabla_{e_i} e_{n+1}\rangle \nabla_{e_{n+1}} h \\ &= \Delta h - \mathrm{Hess}\ h(e_{n+1}, e_{n+1}) + \langle \nabla h, \overrightarrow{H}_\Sigma \rangle. \end{aligned}$$

□

2.1.2 The boundary exponential map

Let (M, g) be a complete Riemannian manifold with boundary, which means the induced metric space is complete, or equivalently, the induced distance function to some/any fixed point is proper. Denote $T^\perp\partial M$ the normal line bundle of ∂M, which is a trivialized by the inward unit normal vector field N. We identify $T^\perp\partial M$ with $\partial M \times \mathbb{R}$ via this trivialization. For $p \in \partial M$, denote $\gamma_p(t)$ the geodesic such that $\gamma_p(0) = p, \gamma_p'(0) = N_p$. Denote

$$D(p) = \inf\{t > 0 | \gamma_p(t) \in \partial M\} \in (0, \infty],$$

$$\tau(p) = \sup\{t > 0 | d(\gamma_p(t), \partial M) = t\} \in (0, \infty].$$

We have a subset of $U_{\partial M} \subset T^\perp\partial M$ defined by

$$U_{\partial M} = \{(p, tN_p) \in T^\perp\partial M | 0 \leq t < D(p)\},$$

which is the domain of the boundary exponential map

$$\exp^\perp : U_{\partial M} \to M, (p, s) \mapsto \gamma_p(s), \tag{2.11}$$

and define

$$V_{\partial M} = \{(p, tN_p) \in T^\perp\partial M | 0 \leq t < \tau(p)\} \subset U_{\partial M}.$$

There are some definitions, notations and terminologies related to the boundary exponential map:

- The *boundary injectivity radius* i_b is defined to be the supremum of $s \geq 0$ such that $\exp^\perp |_{\partial M \times [0,s)}$ is a diffeomorphism onto its image. It characterize that to what time the boundary can flow into the interior under geodesic flow without developing singuarities.
- A *focal point* q of ∂M is a critical value of the boundary exponential map (2.11). If q lies in γ_p for some $p \in \partial M$, we say q is a focal point along γ_p.
- A *foot point* of $q \in M$ is a point $p \in \partial M$ such that $d(q, p) = d(q, \partial M)$.
- A *cut point* of ∂M is a point $q \in M$ such that there exists a foot point p of q such that $d(q, p) = \tau(p)$. We also say q is a cut point of p.
- When we say a covariant tensor on M is written in *geodesic gauge*, we mean the pull back of this tensor via $\exp^\perp$.
- For a subset $B \subset \partial M$, we use the notation

  $$C(B, t_1, t_2) = \exp^\perp(B \times [t_1, t_2))$$

  to denote a metric cylinder with base B.
- $N_r(\partial M, g) := \{x \in M | d(x, \partial M) \leq r\}$.

- Our convention is that: the $(1,1)$ tensor S in (2.1) is defined with respect to $f = -d(\cdot, \partial M)$ near ∂M, so $\nabla f = -N$ on ∂M (Recall N is the unit inner normal vector field) , and $II \geq 0$ if ∂M is convex. For example, for the unit ball in $\mathbb{R}^{n+1}$, $f(x) = |x| - 1$, the mean curvature $H = \Delta(|x|-1) = n > 0$ on S^n.

Here are some remarks about some of these definitions:

- It is not hard to show, similar to the case for conjugate point of a point along some geodesic starting at that point, that $\gamma_p(s)$ is a focal point of along γ_p, if and only if there exists a non-zero *∂M-Jacobi field* V along γ_p (a Jacobi field with $V(0) \in T_p\partial M, V'(0) + S(V(0)) \in T_p^{\perp}\partial M$) such that $V(s) = 0$. If there is no focal point along $\gamma_p|_{[0,l)}$, then
$$I_0(W,W) = \int_0^l \langle W', W'\rangle - \langle R(W, \gamma_p')\gamma_p', W\rangle dt - \langle S(W), W\rangle(0) \geq 0 \tag{2.12}$$
for any piecewise smooth vector field W along γ with $W(0) \in T_{\gamma(0)}\partial M$.

- By Lemma 3.2 in [43], τ defines a continuous map from ∂M to $(0, \infty]$. It is well-known that by a second variation argument, the first focal point along γ_p appears no later than $\tau(p)$, and moreover one can argue by contradiction to get (See Lemma 3.6 in [43]), which is similar to the case for cut locus of a point in complete Riemannian manifolds without boundary.

Proposition 2.1.3. *$q \in M$ is a cut point of p if and only if at least one of the following holds:*

- *q is the first focal point of γ_p ;*
- *q has at least two foot points.*

From this, we conclude that $\exp^{\perp}|_{V_{\partial M}}$ is a diffeomorphism and
$$i_b = \inf_{p \in \partial M} \tau(p).$$

It is worth noting that if M is embedded in some complete Riemannian manifold M' without boundary of the same dimension, then ∂M has two sides, hence ∂M can be viewed as an embedded hypersurface of M'. In this case, one can define the focal point of ∂M in M' similarly. It may happen that a focal point of ∂M in M' lies outside M, for example, when ∂M is strictly concave at some point. However, our definition of focal point is intrinsic for the Riemannian manifold with boundary M.

Note that for a complete Riemannian manifold (M, g) with boundary, besides boundary injectivity radius i_b, we have other two types of "injectivity radius": the *interior injectivity radius inj_M*, and the *intrinsic injectivity radius of the boundary $inj_{\partial M}$*. Since ∂M is a complete Riemannian manifold without boundary, $inj_{\partial M}$ is just defined as the injectivity radius of ∂M. For inj_M, it is defined as $\inf_{x \in M \setminus \partial M} inj_x / \min\{1, d(x, \partial M)\}$, where
$$inj_x = \sup\{\rho > 0 \mid \text{ if } d(x, \partial M) > \rho, \text{ then } \exp_x : T_xM \supset B_\rho(0) \to M \text{ is a diffeomorphism}\}.$$

2.1.3 Manifolds with mean convex boundary

In this subsection, we focus on manifolds with mean convex boundary. We will summarize some known results and discuss a new result which is crucial for the proof of our main theorems.

By second variation arguments, one can deduce topological restriction and geometric estimates for manifolds with mean convex boundary, see [38, 20].

Proposition 2.1.4. *[38, 20] Let (M, g) be a compact, connected Riemannian manifold with boundary,* $\mathrm{Ric}_M \geq 0$*. Suppose ∂M has mean curvature $H \geq H_0 > 0$, then*

$$\pi_0(\partial M) = 0,$$

$$\pi_1(M, \partial M) = 0,$$

$$\sup_{q\in M} d(q, \partial M) \leq (n-1)H_0^{-1},$$

$$\mathrm{vol}(M) \leq C(n)H_0^{-1}\mathrm{vol}(\partial M).$$

Proof. If $\pi_0(\partial M) \neq 0$ or $\pi_1(M, \partial M) \neq 0$, we claim that every non-trivial class contains a non-trivial unit speed geodesic $\gamma : [0, l] \to M$ that minimizes the length of all curves in its class. In the case when $\pi_0(M) \neq 0$, this is because the distance between two different boundary components is positive. When $\pi_1(M, \partial M) \neq 0$, we must show that the minimizer exists and is a geodesic that only intersects ∂M at its end points. We can choose a sequence of smooth curves γ_i in the same class whose length converges to the infimum of the length of all smooth curves in the class. Reparameterizing γ_i to be smooth maps $[0, 1] \to M$ with $|\gamma'(t)| \leq C$, we can apply the Arzelà-Ascoli lemma and take a subsequential C^0 limit of γ_i to obtain a continuous curve $\gamma_\infty : [0, 1] \to M$. The end points of γ_∞ are on ∂M and γ_∞ is also in the same class. Then γ_∞ only intersects ∂M at end points, otherwise it can be homotoped to have less length. Now we divide γ_∞ to sufficiently many pieces such that every interval is sufficiently small. We conclude each piece must be a geodesic, otherwise, one can replace that piece with a length minimizing geodesic to get a curve with less length. Similarly, one conclude γ_∞ must be C^1 at the break points. Hence γ_∞ is a smooth geodesic.

From the first variation formula, γ intersects boundary perpendicularly at both end points. Pick an orthonormal basis $V_i, 1 \leq i \leq n-1$ of $T_{\gamma(0)}\partial M$ and parallel them transport along γ to get $V_i(t)$. Let $\gamma_{i,s}(t)$ be a family of curves centered at γ with variation field $V_i(t)$. By second variational formula,

$$0 \leq \sum_{i=1}^{n-1} \frac{d^2}{ds^2}\Big|_{s=0} E(\gamma_{i,s}) = \int_0^l -\mathrm{Ric}(V_i(t), V_i(t))dt - H(\gamma(0)) - H(\gamma(l)) < 0,$$

which is a contradiction.

Now we prove the estimate for $\sup_{q\in M} d(q, \partial M)$. If for some $q \in M$, $\tilde{l} := d(q, \partial M) > (n-1)H_0^{-1}$. Let p be a foot point of q. Let $\bar{V}_i, 1 \leq i \leq n-1$ be an orthonormal basis of $T_p\partial M$ and parallel them transport along γ_p to get $\bar{V}_i(t)$, and denote $\tilde{V}_i(t) = (\tilde{l} - t)\bar{V}_i(t)$, $\tilde{\gamma}_{i,s}(t)$ a family of curves centered at γ_p with variation field $\tilde{V}_i(t)$, then

$$0 \leq \sum_{i=1}^{n-1} \frac{d^2}{ds^2}\Big|_{s=0} E(\tilde{\gamma}_{i,s}) = (n-1)\tilde{l} - \int_0^{\tilde{l}} \mathrm{Ric}(t\bar{V}_i(t), t\bar{V}_i(t))dt - \tilde{l}^2 H(\gamma(0)) < 0,$$

which is a contradiction. The volume upper bound is by volume comparison, see [31]. □

Remark 2.1.5. Note that when M is connected, $\pi_1(M, \partial M) = 0$ is equivalent to that ∂M is connected and the natural map $\pi_1(\partial M) \to \pi_1(M)$ is surjective, the latter of which means that fix a point $p_0 \in \partial M$, then for any closed path $x(t) : 0 \leq t \leq 1$ in M with $x(0) = x(1) = p_0$, there exists a homotopy $F_s(t) : 0 \leq s, t \leq 1$ with $F_s(0) = F_s(1) = p_0, F_0(t) = x(t)$ such that $F_1(t) \in \partial M$.

The following result is well-known, see Lemma 6.3 of [34].

Proposition 2.1.6. *[34] Let M be a complete Riemannian manifold with nonempty compact boundary. If there are no focal points whose distance to ∂M equals i_b, then there exists a smooth geodesic of length $2i_b$ that is perpendicular to ∂M at both endpoints.*

Proof. $i_b = \inf_{p \in \partial M} \tau(p) > 0$. Suppose the infimum is achieved at $p_1 \in \partial M$. Then by assumption and Proposition 2.1.3 $\gamma_{p_1}(i_b)$ has another foot point p_2. We claim $\gamma'_{p_1}(i_b) = -\gamma'_{p_2}(i_b)$, so $\gamma_{p_1} : [0, 2i_b] \to M$ is the smooth geodesic we want. By assumption, we can find smooth distance functions h_1, h_2 extending $d(\cdot, \partial M)$ near $\gamma_{p_1}|_{[0,i_b]}$, $\gamma_{p_2}|_{[0,i_b]}$, respectively. Consider the smooth hypersurface $\Sigma = (h_1 - h_2)^{-1}(0)$ near $q := \gamma_{p_1}(i_b) = \gamma_{p_2}(i_b)$. Then $v = \nabla h_1(q) + \nabla h_2(q) \in T_q\Sigma$. If it is non-zero, then $\langle \nabla h_1(q) + \nabla h_2(q), v\rangle > 0$. Without loss of generality, assume $\langle \nabla h_1, v\rangle > 0$. Then in the direction of $-v$ in Σ, we have some point $q' \in \Sigma$ with $h_1(q') < h_1(q)$. Hence q' has two foot points and $d(q', \partial M) = h_1(q') < i_b$, which is a contradiction. □

It is worth noting that in this proof, Kodani used the first order variation of h_1 on Σ to lead a contradiction. We can also investigate the second order variation of h_1 on Σ and prove the following:

Proposition 2.1.7. *Let M be a compact Riemannian manifold with mean convex boundary, $\mathrm{Ric}_M \geq 0$, then there exists a focal point of ∂M whose distance to ∂M is equal to i_b.*

Proof. Suppose not, by Proposition 2.1.6, we have a smooth geodesic of length $2i_b$ which is perpendicular to ∂M at both end points. We use the notations p_1, p_2, q, h_1, h_2, and Σ as given there. We claim $\Delta_\Sigma h_1(q) < 0$, so we get another point $q'' \in \Sigma$ near p with $h_1(q'') < h_1(q) = i_b$ and get a contradiction. Denote $N_0 = \nabla h_1(q) = -\nabla h_2(q), \Sigma_1 = h_1^{-1}(i_b), \Sigma_2 = h_2^{-1}(i_b)$, then N_0 is a common unit normal vector for $\Sigma, \Sigma_1, \Sigma_2$ at q. A graph visualization is depicted in

Figure 2.1. Let $II_\Sigma, II_{\Sigma_1}, II_{\Sigma_2}$ be second fundamental forms with respect to N_0 at q. Then at q,

$$II_\Sigma = \frac{1}{|\nabla(h_1-h_2)|}\mathrm{Hess}(h_1-h_2) = \frac{1}{2}\mathrm{Hess}(h_1-h_2) = \frac{1}{2}(II_{\Sigma_1}+II_{\Sigma_2}),$$

hence

$$H_\Sigma = \frac{1}{2}(H_{\Sigma_1}+H_{\Sigma_2}), \overrightarrow{H}_\Sigma = \frac{1}{2}(\overrightarrow{H}_{\Sigma_1}+\overrightarrow{H}_{\Sigma_2}).$$

From the formula of Laplace operator on a hypersurface (2.10), we know that at q,

$$\Delta_\Sigma h_1 = \Delta h_1 - \mathrm{Hess}h_1(N_0,N_0) + \langle \nabla h_1, \overrightarrow{H}_\Sigma\rangle,$$

$$\Delta_{\Sigma_1} h_1 = \Delta h_1 - \mathrm{Hess}h_1(N_0,N_0) + \langle \nabla h_1, \overrightarrow{H}_{\Sigma_1}\rangle.$$

Since h_1 is a constant on Σ_1, $\Delta_{\Sigma_1}h_1 = 0$, hence

$$\begin{aligned}\Delta_\Sigma h_1 &= \Delta_\Sigma h_1 - \Delta_{\Sigma_1}h_1 = \langle \nabla h_1, \overrightarrow{H}_\Sigma - \overrightarrow{H}_{\Sigma_1}\rangle = \langle \nabla h_1, \frac{1}{2}(\overrightarrow{H}_{\Sigma_2} - \overrightarrow{H}_{\Sigma_1})\rangle \\ &= -\frac{1}{2}(H_{\Sigma_2} - H_{\Sigma_1}).\end{aligned} \tag{2.13}$$

Since ∂M is mean convex, $\mathrm{Ric}_M \geq 0$, by the evolution equation of mean curvature (2.5), we have $-H_{\Sigma_1}(q) > H_{\partial M}(p_1) > 0$, $H_{\Sigma_2}(q) > H_{\partial M}(p_2) > 0$. Hence $\Delta_\Sigma h_1(p) < 0$, which completes the proof.

□

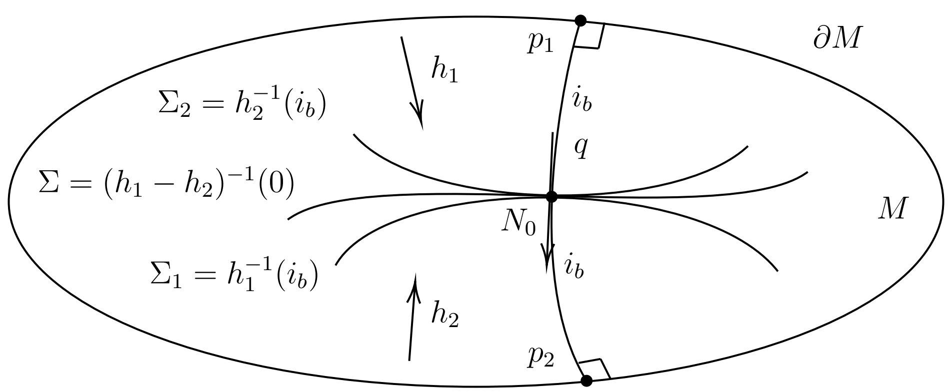

Figure 2.1: Existence of a focal point

A moment thought about the arguments in the end of the previous proof yields that $\mathrm{Ric}_M \geq 0$ is not so necessary, since we can make use of the evolution equation (2.5) to get an ordinary differential inequality for the mean curvature.

Proposition 2.1.8. *If in Proposition 2.1.7 we assume instead* $\mathrm{Ric}_M \geq -(n-1)c$ *for some* $c > 0$*, and* $H \geq H_0 > 0$*. If* $i_b < -\frac{1}{2\sqrt{c}} \ln \left|\frac{H_0-(n-1)\sqrt{c}}{H_0+(n-1)\sqrt{c}}\right|$*, then there exists a focal point of* ∂M *whose distance to* ∂M *is equal to* i_b*.*

Proof. Suppose the conclusion is not true, follow the arguments of Proposition 2.1.7 except for the last part. Let $S_i(X) = -\nabla_X \nabla h_i$, $H_i = TrS_i = -\Delta h_i$, and identify a neighborhood of $\gamma_{p_i}|_{[0,i_b]}$ with a subset of $\partial M \times \mathbb{R}$ via $\exp^\perp$. Then by (2.5),

$$\partial_t H_i = |S_i|^2 + \mathrm{Ric}(\nabla h_i, \nabla h_i) \geq \frac{1}{n-1} H_i^2 - (n-1)c,$$

and $H(p_i, 0) \geq H_0$. Let f solves the ODE on $[0, i_b]$

$$f' = \frac{1}{n-1} f^2 - (n-1)c, \tag{2.14}$$

and $f(0) = H_0$, then by ODE comparison we have $H_i(p_i, t) \geq f(t)$. In particular, $H_i(p_i, i_b) \geq f(i_b) > 0$, which leads to a contradiction as before.

We now show $f(i_b) > 0$ in detail. The autonomous ordinary differential equation (2.14) has two equilibrium solutions $f = \pm(n-1)\sqrt{c}$. If $H_0 \geq (n-1)\sqrt{c}$, then f is increasing, hence $f(i_b) > 0$. If $H_0 < (n-1)\sqrt{c}$, then f is strictly decreasing with $\lim_{t\to\infty} f(t) = -(n-1)\sqrt{c}$. The solution of (2.14) is given by

$$\frac{1}{2\sqrt{c}} \ln \left|\frac{f(t)-(n-1)\sqrt{c}}{f(t)+(n-1)\sqrt{c}}\right| - \frac{1}{2\sqrt{c}} \ln \left|\frac{H_0-(n-1)\sqrt{c}}{H_0+(n-1)\sqrt{c}}\right| = t,$$

hence the unique zero point of f is given by $-\frac{1}{2\sqrt{c}} \ln \left|\frac{H_0-(n-1)\sqrt{c}}{H_0+(n-1)\sqrt{c}}\right|$, so $f(i_b) > 0$. □

Proposition 2.1.7 implies

Corollary 2.1.9. *Let* (M, g) *be a compact Riemannian manifold with boundary,* $K > 0, \lambda > 0$ *are constants. Suppose* $\sec \leq K$*,* $S \leq \lambda$*,* $H > 0$*,* $\mathrm{Ric}_M \geq 0$*, then* $i_b \geq \frac{1}{\sqrt{K}} \mathrm{arccot} \frac{\lambda}{\sqrt{K}}$*.*

Proof. By Proposition 2.1.7, there exists $p \in \partial M$ such that $\gamma_p(i_b)$ is a focal point along γ_p. If $i_b < \frac{1}{\sqrt{K}} \mathrm{arccot} \frac{\lambda}{\sqrt{K}}$, from comparison theorem for Jacobi fields, we know $\gamma_p(i_b)$ cannot be a focal point along γ_p, which is a contradiction. □

Similarly, Proposition 2.1.8 implies

Corollary 2.1.10. *Let* (M, g) *be a compact Riemannian manifold with boundary. Suppose* $|Rm| \leq C$*,* $|S| \leq C$*,* $H \geq H_0 > 0$*, then we can find* i_0 *depending explicitly on* C, H_0 *such that* $i_b \geq i_0$*.*

Remark 2.1.11. In the previous two corollaries, if the sectional curvature and Ricci curvature bounds only holds for $N_1(\partial M, g)$, then we also have a i_b lower bound. In the case of Corollary 2.1.9, we have $i_b \geq \min\{\frac{1}{\sqrt{K}} \mathrm{arccot} \frac{\lambda}{\sqrt{K}}, 1\}$.

Remark 2.1.12. In the same setting as the previous two corollaries, [32] Lemma 2.2 claimed to prove a lower bound for i_b, using a similar method as [7] Lemma 2.4. In both papers, there is a logic problem that they get a contradiction with an unjustified statement: Let M be a Riemannian manifold with boundary, $\gamma : [0, l] \to M$ be a geodesic that is perpendicular to the boundary at both end points, and suppose there is no focal point along γ for both boundary portions, then $I_1(V, V) \geq 0$ for any smooth vector field along γ with $V(0), V(l) \in T\partial M$. Here

$$I_1(V, V) = \int_0^l \langle V', V' \rangle - \langle R(V, \gamma')\gamma', V \rangle dt - \langle S(V(0)), V(0) \rangle - \langle S(V(l)), V(l) \rangle.$$

In fact, this unjustified statement is not true, and one can easily think of an example: let

$$\Sigma_1 = \{(x', x^n) \in \mathbb{R}^n \big| |x'|^2 + (1 - x^n)^2 = R_1^2\},$$
$$\Sigma_2 = \{(x', x^n) \in \mathbb{R}^n \big| |x'|^2 + (1 + x^n)^2 = R_2^2\},$$

$R_1, R_2 > 2$ and $\gamma(t) = (0, 1 - t), 0 \leq t \leq 2$, then $I_1(V, V) = -\frac{1}{R_1} - \frac{1}{R_2} < 0$ for any unit-norm parallel vector field along γ with $V(0) \in T_{\gamma(0)}\Sigma_1$. In this case, there exist no focal points on γ for both Σ_1, Σ_2.

In fact, focal points give crucial information for index form defined by one submanifold and one point. However, as seen from the example, they do not fit well with the index form defined for two submanifolds. Indeed, there is some notion of "conjugate point" defined for two submanifolds, see [1].

It is easy to see focal points can "pass to the limit", since they arise from kernels of the differential of exponential maps. Though one can use Corollary 2.1.10 directly in many situations, we point out this fact here, which may help in contradiction arguments.

Proposition 2.1.13. *Let M be a manifold with boundary, g_i be a sequence of Riemannian metrics on M, and $p_i \in \partial M$. Suppose g_i converges to a Riemannian metric g_∞ in C^2 and p_i converges to $p_\infty \in \partial M$, γ_{p_i} is defined on $[0, b]$ and $\gamma_{p_i}(t_i)$ is a focal point along γ_{p_i} with $0 < a \leq t_i \leq b$. Then for a subsequence, γ_{p_i} converges in C^2 to γ_{p_∞}, $t_i \to t_\infty$ and $\gamma_{p_\infty}(t_\infty)$ is a focal point along γ_{p_∞}.*

Proof. $\gamma'_{p_i}(0) \in TM$ is a bounded sequence, hence subconverges to some $v \in TM$, which must equal to $\gamma'_{p_\infty}(0)$. Hence by ODE theories, γ_{p_i} converges in C^2 to γ_{p_∞}. Suppose for a subsequence $t_i \to t_\infty$. Let $J_i : [0, t_i] \to TM$ be ∂M-Jacobi fields along γ_i with $J_i(t_i) = 0, |J'_i(t_i)| = \frac{t_\infty}{t_i}$. Normalize these geodesics and Jacobi-fields by $\bar{\gamma}_i(t) = \gamma_i(\frac{t_i}{t_\infty}t)$, $\bar{J}_i(t) = J_i(\frac{t_i}{t_\infty}t)$, so $\bar{\gamma}_i, \bar{J}_i$ are defined on the same interval $[0, t_\infty]$, and

$$\begin{aligned} \bar{J}''_i(t) + R_{g_i}(\bar{J}_i(t), \bar{\gamma}'_i(t))\bar{\gamma}'_i(t) = 0, \\ \bar{J}_i(t_\infty) = 0, |\bar{J}'_i(t_\infty)| = 1. \end{aligned} \tag{2.15}$$

For a subsequence $\bar{J}_i{}'(t_\infty) \to w$ with $|w| = 1$. Let J_∞ be the non-trivial Jacobi-field along γ_{p_∞} with $J_\infty(t_\infty) = 0, J'_\infty(t_\infty) = w$, then J_i converges in C^2 as maps $[0, t_\infty] \to TM$ to J_∞ by

ODE theories. Hence J_∞ is a ∂M-Jacobi field, which implies $\gamma_{p_\infty}(t_\infty)$ is a focal point along γ_{p_∞}. □

2.1.4 Volume estimates near boundary

In this subsection, we show some volume lower bounds near boundary under some geometric control. These estimates can be found in [2] and [32].

Proposition 2.1.14. *Let (M, g) is a Riemannian manifold with boundary, suppose* $\mathrm{Ric} \geq -(n-1)c$ *for some* $c \geq 0$, *and* $\exp^\perp$ *is an diffeomorphism in* $B \times [0, T)$ *for some open subset* B *of* ∂M, *then*

$$\sup_{B\times\{t\}} H \leq \max\{(n-1)\sqrt{2c}, 4(n-1)^{-1}T^{-1}\},$$

$$\mathrm{vol}(C(B, t_1, t_2)) \geq C(n, T, c)\mathrm{vol}_{\partial M}(B)|t_2 - t_1|,$$

where $0 \leq t, t_1, t_2 \leq \frac{1}{2}T$.

Proof. By the evolution equation (2.5),

$$\partial_t H = |S|^2 + \mathrm{Ric}(N_t, N_t), \tag{2.16}$$

hence

$$\partial_t H \geq \frac{1}{n-1}H^2 - (n-1)c. \tag{2.17}$$

If for some $t_0 \in [0, \frac{1}{2}T]$, $z_0 \in B$, we have $H(z_0, t_0) \geq \delta$ and $\delta \geq (n-1)\sqrt{2c}$, then $\partial_t H(z_0, t) \geq 0$ and $H(z_0, t) \geq (n-1)\sqrt{2c}$ for $t \in [t_0, T)$. Hence

$$\partial_t H(z_0, t) \geq \frac{1}{2}(n-1)^{-1}H(z_0, t)^2,$$

$$H^{-1}(z_0, t) \leq H^{-1}(z_0, t_0) - \frac{1}{2}(n-1)(t - t_0) \leq \delta^{-1} - \frac{1}{2}(n-1)(t - \frac{1}{2}T).$$

Then the continuity of $H(z_0, \cdot)$ in $[0, T)$ forces $\delta \leq 4(n-1)^{-1}T^{-1}$, which proves the mean curvature estimate.

For $t \in [0, \frac{1}{2}T]$, let $B_t = \exp^\perp(B \times \{t\})$, then

$$\frac{d}{dt}\mathcal{H}^{n-1}(B_t) = -\int_{B_t} H d\mathcal{H}^{n-1}(B_t) \geq -C_1(n, T, c)\mathcal{H}^{n-1}(B_t),$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure of M. Hence

$$\mathcal{H}^{n-1}(B_t) \geq e^{-C_1(n,T,c)t}\mathrm{vol}_{\partial M}(B) \geq e^{-\frac{1}{2}C_1(n,T,c)T}\mathrm{vol}_{\partial M}(B)$$

and when $0 \leq t_1 < t_2 \leq \frac{1}{2}T$,

$$\mathrm{vol}(C(B, t_1, t_2)) = \int_{t_1}^{t_2} \mathcal{H}^{n-1}(B_t)dt \geq C(n, T, c)\mathrm{vol}_{\partial M}(B)(t_2 - t_1).$$

□

Proposition 2.1.15. *Let M be a Riemannian manifold with boundary, $p \in \partial M$. Suppose $B(p, 2r_0)$ has compact closure,*

$$\sup_{B(p,2r_0)} |Rm| \leq C,$$

$$\mathrm{vol}_{\partial M}(B_{\partial M}(p, r_0)) \geq v_0,$$

$\exp^{\perp}$ *is an diffeomorphism in $B_{\partial M}(p, r_0) \times [0, r_0)$, and on $B_{\partial M}(p, r_0)$*

$$\mathrm{Ric}_{\partial M} \geq -(n-2)c_0, \;\; |S| \leq C,$$

then there exists $r_1 > 0, v_1 > 0$ depending on n, C, c_0, r_0, v_0, such that for $q = \exp^{\perp}(p, 2r_1)$, we have

$$\mathrm{vol}(B(q, r_1)) \geq v_1.$$

Proof. By definition $C(B_{\partial M}(p, r_0), 0, r_0) \subset B(p, 2r_0)$. We claim that there exists $r_2 > 0, C_1 > 0$ such that for any $p_1, p_2 \in B_{\partial M}(p, r_0)$,

$$d_{\Sigma_t}(\exp^{\perp}(p_1, t), \exp^{\perp}(p_2, t)) \leq C_1 d_{\partial M}(p_1, p_2)$$

when $0 \leq t \leq r_2$, where Σ_t is the image of $B_{\partial M}(p, 2r_0)$ under $\exp^{\perp}(\cdot, t)$. In fact, by (2.2)(2.4),

$$\nabla_{\nabla t} S = S^2 + R(\cdot, \nabla t)\nabla t,$$

hence

$$\frac{d}{dt}|S| \leq |S|^2 + C.$$

Integrate the inequality, we have

$$\arctan(\frac{|S|}{\sqrt{C}})(x, t) - \arctan(\frac{|S|}{\sqrt{C}})(x, 0) \leq \sqrt{C}t.$$

Hence there exist $r_2 > 0, C_2 > 1$ such that $|S| \leq \log C_2$ when $0 \leq t \leq r_2$. Now let $\gamma_0(s)$ be a smooth curve in $B_{\partial M}(p, 2r_0)$ that connects p_1, p_2, and let $\gamma_t(s) = \exp^{\perp}(\gamma_0(s), t) \in \Sigma_t$, then we have

$$\frac{d}{dt}\log|\gamma_t'(s)| = -\frac{\langle S_{\Sigma_t}(\gamma_t'(s)), \gamma_t'(s)\rangle}{\langle \gamma_t'(s), \gamma_t'(s)\rangle} \leq |S_{\Sigma_t}(\gamma_t(s))| \leq \log C_2.$$

It follows that $|\gamma_t'(s)| \leq C_1|\gamma_0'(s)|$ with $C_1 = C_2^{r_2}$ and the claim follows from integration. Now take

$$r_1 = \min\{2C_1 r_0, \frac{1}{4}r_2\},$$

we have

$$C(B_{\partial M}(p, \frac{r_1}{2C_1}), \frac{3r_1}{2}, \frac{5r_1}{2}) \subset B(q, r_1),$$

then we apply Proposition 2.1.14 and Bishop-Gromov volume comparison on ∂M to get the desired conclusion. □

Remark 2.1.16. It is easy to give a quantitative version of the lemma from the proof. However, to the author's knowledge, we cannot prove the last inclusion in the proof without a control of curvature. It may be possible that a metric ball of the boundary becomes "long and thin" under the flow of $\nabla d(\cdot, \partial M)$, while maintains an area lower bound.

2.2 Convergence theory of Riemannian manifolds

2.2.1 Gromov-Hausdorff convergence

This section is intended to summerize some results of Gromov-Hausdorff convergence for Riemannian manifold without boundary.

Definition 2.2.1 (Gromov-Hausdorff distance)**.** *Let X, Y be two compact metric spaces, the Gromov-Hausdorff distance between X, Y is defined to be*

$$\begin{aligned} d_{GH}(X,Y) = \inf\{d_H^Z(i(X), j(Y)) \mid \exists \textit{ a metric space } Z \textit{ and isometric embeddings} \\ i : X \hookrightarrow Z, j : Y \hookrightarrow Z\}, \end{aligned}$$

where d_H^Z denotes the Hausdorff distance in Z, i.e.,

$$d_H^Z(i(X), j(Y)) = \max\{\sup_{x\in i(X)} d(x, j(Y)), \sup_{y\in j(Y)} d(y, i(X))\}.$$

The Gromov-Hausdorff distance between X, Y measures how far X, Y are from being isometric. It is easy to check that d_{GH} is a metric on the set of all compact metric spaces, say $\mathcal{M}_{cms}$. The topology induced by d_{GH} is refered to as Gromov-Hausdorff topology. It is proved that

Theorem 2.2.2. *$(\mathcal{M}_{cms}, d_{GH})$ is a complete metric space.*

Theorem 2.2.3 (Gromov compactness theorem)**.** *If (M_i, g_i) are compact Riemannian manifolds, $Ric_{M_i} \geq -(n-1)c$, $diam(M_i, g_i) \leq C$, then a subsequence converges in Gromov-Hausdoff sense to a compact metric space (Z_∞, d_∞).*

The point is that we have Bishop-Gromov volume comparison to conclude that for any fixed $\epsilon > 0$, we can cover each M_i by a uniform number of metric balls of radius ϵ. The set of all limits (Z_∞, d_∞) are refered to as *Ricci limit spaces*.

Similarly, one can define the pointed Gromov-Hausdorff distance for pointed compact metric spaces

$$\begin{aligned} d_{GH}((X,p),(Y,q)) = \inf\{d_H^Z(i(X), j(Y)) + d^Z(i(p), j(q)) \mid \exists \text{ a metric space } Z \\ \text{and isometric embeddings } i : X \hookrightarrow Z, j : Y \hookrightarrow Z\}. \end{aligned}$$

In the case of point Riemannian manifolds(without boundary, not necessary complete)

Definition 2.2.4. *A sequence of pointed Riemannian manifolds (M_i, g_i, p_i) is said to converge in (pointed) Gromov-Hausdorff sense to a pointed complete metric space $(Z_\infty, d_\infty, p_\infty)$, if $\exists R_i \to \infty$, such that $\bar{B}(p_i, R_i)$ is compact and*

$$d_{GH}((\bar{B}(p_i, R_i), p_i), (\bar{B}(p_\infty, R_i), p_\infty)) \to 0.$$

In general, Gromov-Hausdorff topology is a very weak topology: the limit space may be bad, and convergence in Gromov-Hausdorff topology is not satisfactory in many contradiction arguments. However, a general philosophy is that, if we impose geometry control in the converging sequence, then we have regularity results on the limit space. If we assume the regularity on the limit space, then the convergence can be improved to a stronger topology. This is one of the key ideas in Cheeger-Colding theory.

Definition 2.2.5. *A sequence of closed Riemannian manifold (M_i, g_i) is said to converge in $C^{k,\alpha}$ topology to a $C^{k,\alpha}$ Riemannian manifold (M_∞, g_∞), if there exists $C^{k+1,\alpha}$ diffeomorphisms $\varphi_i : M_\infty \to M_i$ such that $\varphi_i^* g_i \xrightarrow{C^{k,\alpha}} g_\infty$ as tensors on M_∞.*

It is proved by Anderson and Colding that

Theorem 2.2.6 ([3, 17])**.** *If (M_i, g_i) are closed Einstein manifolds, $Ric_{M_i} = (n-1)c$ and $(M_i, g_i) \xrightarrow{\text{GH}} (M_\infty, g_\infty)$ for some closed smooth Riemannian manifold (M_∞, g_∞), then $(M_i, g_i) \xrightarrow{C^{k,\alpha}} (M_\infty, g_\infty)$, $\forall k, \alpha$.*

Definition 2.2.7. *Suppose $Ric_{M_i} \geq -(n-1)c$, and $(M_i, g_i, p_i) \xrightarrow{\text{GH}} (X_\infty, d_\infty, p_\infty)$. If for some $\lambda_i \to \infty$, $(M_i, \lambda_i g_i, p_i) \xrightarrow{\text{GH}} (Z_\infty, q_\infty)$, then Z_∞ is called a bubble limit assosciated to the sequence. If for some $\mu_i \to \infty$, $(X_\infty, \mu_i d_\infty, p_\infty) \xrightarrow{\text{GH}} (W_\infty, q'_\infty)$, then W_∞ is a called a tangent cone at p_∞.*

In application, we usually take λ_i to be the maximum of the curvature norm. If the rescaled metric is volume non-collapsing, then our bubble limit will be a complete Riemannian manifold of the same dimension. In fact, in this dissertation, we will encounter bubble limits with maximum volume growth.

2.2.2 Harmonic radius and convergence theory

Convergence theory of Riemannian manifolds is a powerful tool to prove conclusions in Riemannian geometry through contradiction arguments when explicit bounds is not required. In this section, we will restate some results of [2], follow the proof there, and discuss some direct corollaries. We note that while we discuss about manifold with boundary here, it applies to manifolds without boundary (which is easier and was proved much earlier).

Let (M, g) be a Riemannian manifold with boundary, $m \in \mathbb{N}$, $0 < \alpha < 1$, $Q > 1$. For $p \in M$, define $r_h^{m,\alpha}(p, g, Q)$ to be the supremum of $\rho > 0$ such that if $d(p, \partial M) > \rho$, then there exists a neighborhood U of p in M and a interior coordinate chart $\varphi : B_{\frac{\rho}{2}} \to U$, $\varphi(0) = p$, and if $d(p, \partial M) \leq \rho$, then there exists a neighborhood U of p in M and a boundary coordinate chart $\varphi : B^+_{4\rho} \to U$, $\varphi((0, d(p, \partial M))) = p, \varphi(\tilde{B}_{4\rho}) = U \cap \partial M$, and in either $B_{\frac{\rho}{2}}$ or $B^+_{4\rho}$, we have

$$\Delta_M \varphi^{-1} = 0,$$

$$Q^{-2}(\delta_{ij}) \leq (g_{ij}) \leq Q^2(\delta_{ij}),$$

$$\rho^{m+\alpha}\sum_{|\beta|=m}|\partial_\beta g_{ij}(x)-\partial_\beta g_{ij}(y)|\leq(Q-1)|x-y|^\alpha.$$

We call such a coordinate chart a (ρ, Q, m, α)-harmonic coordinate chart centered at p. Note that the second condition implies there exists r_1, r_2, depending on ρ, Q, such that $B(p, r_1) \subset U \subset B(p, r_2)$.

Definition 2.2.8. *Fix an integer $m \geq 0$, and $0 < \alpha < 1$. We say a sequence of Riemannian manifold with boundary (M_i, g_i, p_i) converges in pointed $C^{m,\alpha}$ to $(M_\infty, g_\infty, p_\infty)$ if there exists precompact open subsets Ω_i of M_i and $\Omega_{\infty,i}$ of M_∞, and $\sigma_i > \rho_i \to \infty$ such that $B(p_i, \rho_i) \subset \bar{\Omega}_i \subset B(p_i, \sigma_i)$, $B(p_\infty, \rho_i) \subset \bar{\Omega}_i \subset B(p_\infty, \sigma_i)$ and there exists diffeomorphisms $F_i : \Omega_{\infty,i} \to \Omega_i$, $F_i : \Omega_{\infty,i} \cap \partial M_i \to \Omega_i \cap \partial M_\infty$ such that $F_i^* g_i \to g$ in $C^{m,\alpha}$ topology, and $F_i^{-1}(p_i) \to p_\infty$. If we replace $C^{m,\alpha}$ by C^∞, we say the convergence is in pointed Cheeger-Gromov sense.*

Remark 2.2.9. (M_∞, g_∞) is automatically a complete $C^{m,\alpha}$ or C^∞ Riemannian manifold with boundary from the definitions. Sometimes we only need that one metric ball converges, so one can modify the definitions above: suppose $\bar{B}(p_i, r) \subset \Omega_i$ for some precompact open set $\Omega_i \subset M_i$ and there exists a Riemannian manifold with boundary $(\Omega_\infty, g_\infty)$, a point $p_\infty \in \Omega_\infty$, and diffeomorphisms $F_i : \Omega_\infty \to \Omega_i$ mapping $\partial\Omega_\infty$ onto $\Omega_i \cap \partial M_i$ such that $F_i^* g_i \to g_\infty$ in $C^{m,\alpha}$ or C^∞ topology and $F_i^{-1}(p_\infty) \to p_i$, we say $B(p_i, r)$ converges in $C^{m,\alpha}$ or Cheeger-Gromov sense to $B(p_\infty, r)$.

The following theorem mentioned in [2] is well-known and is a fundamental theorem of Riemannian convergence theory.

Proposition 2.2.10. *[2] Let (M_i, g_i) be a sequence of complete Riemannian manifold with boundary, $p_i \in M_i$. Suppose there exists some $Q > 1$, and a positive function $r : (0, \infty) \to (0, \infty)$, such that $r_h^{m,\alpha}(p, g_i, Q) \geq r(R)$ for any $p \in B(p_i, R)$, then for a subsequence, (M_i, g_i, p_i) converges in pointed $C^{m,\beta}$ sense to $(M_\infty, g_\infty, p_\infty)$ for any $0 < \beta < \alpha$. If the above assumption holds for only one R, then $B(p_i, R)$ converges in $C^{m,\beta}$ sense to $B(p_\infty, R)$.*

Next, we discuss under what geometric control we can get a harmonic radius lower bound. We state and prove the following local version of Theorem 3.2.1 in [2], with simplified arguments in some parts, when derivative bounds is assumed.

Theorem 2.2.11. *Fix $m \geq 1$. Let (M, g) be a Riemannian manifold with boundary, and $\Sigma \subset \partial M$ be a boundary metric ball that has nonempty boundary and compact closure. Suppose $\exp^\perp$ maps $\Sigma \times [0, i_0)$ diffeomorphically onto its image Ω,*

$$inj_\Omega \geq i_0,\ inj_\Sigma \geq i_0, \tag{2.18}$$

and in Ω,

$$|\nabla^l Ric_M| \leq \Lambda, 0 \leq l \leq m, \tag{2.19}$$

on Σ

$$|\nabla^l_{\partial M} Ric_{\partial M}| \leq \Lambda, |\nabla^{l+1}_{\partial M} H| \leq \Lambda, 0 \leq l \leq m. \tag{2.20}$$

Then for any $Q > 1$, $\alpha \in (0,1)$, $p \in \Omega$,

$$r_h^{m+1,\alpha}(p,g,Q) \geq r_0(i_0,\Lambda,m,\alpha,Q)d(p,\partial^+\Omega), \tag{2.21}$$

where $\partial^+\Omega = \bar{\Omega}\backslash(\Omega \cup \partial M)$.

Remark 2.2.12. Recall that $inj_\Omega \geq i_0$ means that $inj_x \geq i_0 \min\{1, d(x,\Omega^c)\}, \forall x \in \Omega$ and $inj_\Sigma \geq i_0$ means that $inj_{\partial M,y} \geq i_0 \min\{1, d_{\partial M}(y,\Sigma^c)\}, \forall y \in \Sigma$.

Proof. If not, we have a sequence $(M_k, \tilde{g}_k)$ and Σ_k, Ω_k that satisfies the conditions, but there exists $p_k \in \Omega_k$ with

$$\frac{r_h^{m+1,\alpha}(p_k,\tilde{g}_k,Q)}{d_{\tilde{g}_k}(p_k,\partial^+\Omega_k)} = \inf_{p\in\Omega_k} \frac{r_h^{m+1,\alpha}(p_k,\tilde{g}_k,Q)}{d_{\tilde{g}_k}(p,\partial^+\Omega_k)} \to 0.$$

Rescale the metric $g_k = (r_h^{m+1,\alpha}(p_k,\tilde{g}_k,Q))^{-2}\tilde{g}_k$, so $r_h^{m+1,\alpha}(p_k,g_k,Q) = 1$, then $d_{g_k}(p_k,\partial^+\Omega_k) \to \infty$, and $r_h^{m+1,\alpha}(p,g_k,Q) \geq \frac{1}{2}$ if $d_{g_k}(p,p_k) \leq R$, $k \geq k(R)$. Fix any $\beta \in (0,\alpha)$. Then there are two cases:

Case 1 $d_{g_k}(p_k,\Sigma_k) \to \infty$ for some subsequence.

Then for any $\beta \in (0,\alpha)$, a subsequence (M_k, g_k, p_k) converges in pointed $C^{m+1,\beta}$ sense to a complete Riemannian manifold $(M_\infty, g_\infty, p_\infty)$. So $\mathrm{Ric}_{M_\infty} = 0$, $inj_{M_\infty} = \infty$. By Cheeger-Gromoll splitting theorem, (M_∞, g_∞) is isometric to flat $\mathbb{R}^n$.

Hence for any $L > 0$, there exist a coordinate $\varphi_{0,k} : B_{L+5} \to U_k \subset M_k$, $\varphi_{0,k}(0) = p_k$, such that

$$\|g_{k,ij} - \delta_{ij}\|_{C^{m+1,\beta}(B_{L+5})} \to 0, 1 \leq i,j \leq n.$$

We solve the Dirichlet problem for functions u_k^ν, $1 \leq \nu \leq n$:

$$\Delta_{M_k} u_k^\nu = 0 \text{ in } B_{L+5}, u_k^\nu|_{\partial B_{L+5}} = x^\nu \ . \tag{2.22}$$

Recall the formula

$$\Delta_g = g^{ij}\partial_i\partial_j + \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij})\partial_j, |g| = \det(g_{ij}).$$

Then we have

$$\|u_k^\nu - x^\nu\|_{C^{m+2,\beta}(B_{L+5})} \leq C\|\Delta_{M_k}(u_k^\nu - x^\nu)\|_{C^{m,\beta}(B_{L+5})} \to 0.$$

Hence, we get a new coordinate system $(u_k^1, \cdots, u_k^n)$ and we discard the original coordinate system, and we use the same notation for tensors written in the new coordinate system, so in the new coordinate system we have

$$\|g_{k,ij} - \delta_{ij}\|_{C^{m+1,\beta}(B_{L+3})} \to 0.$$

Now we want to improve the convergence of $g_{k,ij}$ from elliptic equations. We have a system of equations

$$\Delta_{M_k} g_{k,ij} + B_{ij}(g_k, \partial g_k) = -2\mathrm{Ric}_{M_k,ij}.$$

where $B_{ij}(g, \partial g)$ are polynormials of $g, \partial g$ and are quadratic in ∂g. From $W^{m+2,p}$ estimates, Morrey embeddings, and

$$|\nabla^l \mathrm{Ric}_{M_k}| \to 0, 0 \le l \le m,$$

we have for $1 \le i, j \le n$

$$\begin{aligned} \|g_{k,ij} - \delta_{ij}\|_{C^{m+1,\alpha}(B_{L+2})} \le & C(\|\Delta_{M_k}(g_{k,ij} - \delta_{ij})\|_{C^m(B_{L+3})} \\ & + \|g_{k,ij} - \delta_{ij}\|_{L^\infty(B_{L+3})}) \to 0. \end{aligned} \tag{2.23}$$

Hence we get a $(2(L+2), Q, m+1, \alpha)$ harmonic coordinate chart centered at p'_k, with $d_{g_k}(p'_k, p_k) \to 0$, then $r_h^{m,\alpha}(p_k, g_k, Q) \ge 2(L+1)$ for large k, which is a contradiction.

Case 2 $d_{g_k}(p_k, \Sigma_k) \le K$.

A subsequence (M_k, g_k, p_k) converges in pointed $C^{m+1,\beta}$ sense to a complete Riemannian manifold with boundary $(M_\infty, g_\infty, p_\infty)$ and $(\partial M_k, g_k, q_k)$ converges in $C^{m+1,\beta}$ sense to $(\partial M_\infty, g_\infty|_{\partial M_\infty}, q_\infty)$, where $q_k \in \Sigma_k$ is the unique foot point of p_k in Σ_k. Then $\mathrm{Ric}_{M_\infty} = 0$, $\mathrm{Ric}_{\partial M_\infty} = 0$, $H_\infty = 0$, $inj_{\partial M_\infty} = \infty$, $i_{b,M_\infty} = \infty$. Hence $(\partial M_\infty, g_\infty|_{\partial M_\infty})$ is isometric to flat $\mathbb{R}^{n-1}$. By (2.8), we have $S_\infty = 0$. Then by Lemma 4.1.6, (M_∞, g_∞) is a smooth Riemannian manifold with boundary and $Rm_{M_\infty} = 0$. Since also $i_{b,M_\infty} = \infty$, (M_∞, g_∞) is a isometric to flat $\mathbb{R}^n_+$.

Hence for any $L > 2K+10$, there exist a coordinate $\varphi_{0,k} : B^+_{L+5} \to U_k \subset M_k$, $\varphi_{0,k}(0) = q_k$, $\varphi_{0,k}(\tilde{B}_{L+5}) = U_k \cap \partial M_k$ such that

$$\|g_{k,ij} - \delta_{ij}\|_{C^{m+1,\beta}(B^+_{L+5})} \to 0, 1 \le i, j \le n.$$

First, we solve for functions v^ν_k, $1 \le \nu \le n-1$,

$$\Delta_{\partial M_k} v^\nu_k = 0 \text{ in } \tilde{B}_{L+5}, v^\nu_k|_{\partial \tilde{B}_{L+5}} = x^\nu, \tag{2.24}$$

Then we have

$$\|v^\nu_k - x^\nu\|_{C^{m+2,\beta}(\tilde{B}_{L+5})} \le C\|\Delta_{\partial M_k}(v^\nu_k - x^\nu)\|_{C^{m,\beta}(\tilde{B}_{L+5})} \to 0.$$

Next, we solve for $1 \le \nu \le n-1$,

$$\Delta_{M_k} u^\nu_k = 0 \text{ in } B^+_{L+5}, u^\nu_k|_{\tilde{B}_{L+5}} = v^\nu_k, u^\nu_k|_{\partial^+ B^+_{L+5}} = x^\nu. \tag{2.25}$$

Note that ∂B^+_{L+5} is not a C^1-boundary, but it satisfies exterior sphere condition, so we can solve the equations by Perron's method to get a unique solution $u^\nu_k \in C^\infty((B^+_{L+5})^\circ) \cap C^0(\overline{B^+_{L+5}})$. From definitions and the estimates above, we have

$$\|\Delta_{M_k}(u^\nu_k - x^\nu)\|_{C^{m,\beta}(B^+_{L+5})} \to 0,$$

$$\|u_k^\nu - x^\nu\|_{C^{m+2,\beta}(\tilde{B}_{L+5})} \to 0,$$
$$\|u_k^\nu - x^\nu\|_{L^\infty(\partial B_{L+5}^+)} \to 0,$$

then by maximum principle, we have

$$\|u_k^\nu - x^\nu\|_{L^\infty(B_{L+5}^+)} \to 0,$$

and by Schauder estimates

$$\begin{aligned}\|u_k^\nu - x^\nu\|_{C^{m+2,\beta}(B_{L+4}^+)} \leq C(\|\Delta_{M_k}(u_k^\nu - x^\nu)\|_{C^{m,\beta}(B_{L+5}^+)} + \|u_k^\nu - x^\nu\|_{L^\infty(B_{L+5}^+)} \\ + \|u_k^\nu - x^\nu\|_{C^{m+2,\beta}(\tilde{B}_{L+5})}) \to 0.\end{aligned} \tag{2.26}$$

Next, we construct u_k^n by solving

$$\Delta_{M_k} u_k^n = 0 \ in \ B_{L+5}^+, u_k^n|_{\partial B_{L+5}^+} = x^n.$$

We have

$$\begin{aligned}\|u_k^n - x^n\|_{C^{m+2,\beta}(B_{L+4}^+)} \leq & C(\|\Delta_{M_k}(u_k^n - x^n)\|_{C^{m,\beta}(B_{L+5}^+)} \\ & + \|u_k^n - x^n\|_{L^\infty(B_{L+5}^+)}) \to 0.\end{aligned} \tag{2.27}$$

Hence we get a new coordinate system $(u_k^1, \cdots, u_k^n)$ and we discard the original coordinate system, and we use the same notation for tensors written in both coordinate systems, so in the new coordinate system we have

$$\|g_{k,ij} - \delta_{ij}\|_{C^{m+1,\beta}(B_{L+3}^+)} \to 0. \tag{2.28}$$

Now we want to improve the convergence of $g_{k,ij}$ from elliptic equations with Neumann boundary conditions. We have equations

$$\Delta_{\partial M_k} g_{k,ij} + \tilde{B}_{ij}(g_k, \partial g_k) = -2\mathrm{Ric}_{\partial M_k, ij} \tag{2.29}$$

$$\Delta_{M_k} g_{k,ij} + B_{ij}(g_k, \partial g_k) = -2\mathrm{Ric}_{M_k, ij} \tag{2.30}$$

Fix $\theta \in (\beta, 1), p = \frac{n}{1-\theta}$, from $W^{m+2,p}$ estimates, Morrey embeddings, and

$$|\nabla_{\partial M_k}^l \mathrm{Ric}_{\partial M_k}| \to 0, 0 \leq l \leq m,$$

we have for $1 \leq i, j \leq n-1$,

$$\begin{aligned}\|g_{k,ij} - \delta_{ij}\|_{C^{m+1,\theta}(\tilde{B}_{L+2.5}^+)} \leq & C(\|\Delta_{\partial M_k}(g_{k,ij} - \delta_{ij})\|_{C^m(\tilde{B}_{L+3}^+)} \\ & + \|g_{k,ij} - \delta_{ij}\|_{L^\infty(\tilde{B}_{L+3}^+)}) \to 0.\end{aligned} \tag{2.31}$$

By Theorem 8.33 in [29],

$$\begin{aligned}\|g_{k,ij}-\delta_{ij}\|_{C^{m+1,\theta}(B^+_{L+2})} \leq & C(\|\Delta_{M_k}(g_{k,ij}-\delta_{ij})\|_{C^m(\tilde{B}^+_{L+2.5})} \\ & + \|g_{k,ij}-\delta_{ij}\|_{L^\infty(B^+_{L+2.5})} \\ & + \|g_{k,ij}-\delta_{ij}\|_{C^{m+1,\theta}(\tilde{B}^+_{L+2.5})}) \to 0.\end{aligned} \tag{2.32}$$

Note that

$$N_k g_k^{nn} = -2(n-1)H_k g_k^{nn}, \tag{2.33}$$

$$N_k g_k^{in} = -(n-1)H_k g_k^{in} + \frac{1}{2\sqrt{g_k^{nn}}} g_k^{ij}\partial_j g_k^{nn}, \tag{2.34}$$

where $N_k = \frac{g_k^{jn}\partial_j}{\sqrt{g_k^{nn}}}$ is the unit normal vector of ∂M_k, $1 \leq i \leq n-1$ and j sums from 1 to n, then we have Neumann boundary conditions for (2.30). For simplicity, assume for a while $m = 0$. Since

$$\|g_{k,ij}-\delta_{ij}\|_{C^{1,\beta}(B^+_{L+2})} \to 0,$$

$$|\mathrm{Ric}_{M_k,ij}|_{C^0(B^+_{L+2})} \to 0, 1 \leq i,j \leq n,$$

$$|H_k|_{C^1(\tilde{B}_{L+2})} \to 0,$$

we have

$$\|\Delta_{M_k}(g_k^{nn}-\delta^{nn})\|_{C^0(B^+_{L+2})} \to 0, \|N_k g_k^{nn}\|_{C^1(\tilde{B}_{L+2})} \to 0,$$

then by Morrey embeddings (together with extensions), and $W^{2,p}$ estimates for Neumann boundary problems (for example, see a priori estimate 2.3.1.1 in [30]),

$$\begin{aligned}\|g_k^{nn}-\delta^{nn}\|_{C^{1,\theta}(B^+_{L+1.7})} \leq & C\|g_k^{nn}-\delta^{nn}\|_{W^{2,p}(B^+_{L+1.8})} \\ \leq & C(\|g_k^{nn}-\delta^{nn}\|_{L^p(B^+_{L+2})} + \|\Delta_{M_k}(g_k^{nn}-\delta^{nn})\|_{L^p(B^+_{L+2})} \\ & + \|N_k g_k^{nn}\|_{W^{1-\frac{1}{p},p}(\tilde{B}_{L+2})} \to 0.\end{aligned} \tag{2.35}$$

Now for $1 \leq l \leq n-1$, since

$$\|\Delta_{M_k}(g_k^{ln}-\delta^{ln})\|_{C^0(B^+_{L+2})} \to 0,$$

and

$$\begin{aligned}\|N_k g_k^{ln}\|_{W^{1-\frac{1}{p},p}(\tilde{B}_{L+1.5})} & \leq C(\|g_k^{nn}-\delta^{nn}\|_{W^{2-\frac{1}{p},p}(\tilde{B}_{L+1.5})} + \|H_k\|_{W^{1-\frac{1}{p},p}(\tilde{B}_{L+1.5})}) \\ & \leq C(\|g_k^{nn}-\delta^{nn}\|_{W^{2,p}(B^+_{L+1.7})} + \|H_k\|_{C^1(\tilde{B}_{L+1.7})}) \to 0.\end{aligned} \tag{2.36}$$

Then

$$\begin{aligned}\|g_k^{ln}-\delta^{ln}\|_{C^{1,\theta}(B^+_{L+1.1})} \leq & C\|g_k^{ln}-\delta^{ln}\|_{W^{2,p}(B^+_{L+1.2})} \\ \leq & C(\|g_k^{ln}-\delta^{ln}\|_{L^p(B^+_{L+1.5})}+\|\Delta_{M_k}(g_k^{ln}-\delta^{ln})\|_{L^p(B^+_{L+1.5})} \\ & +\|N_k g_k^{ln}\|_{W^{1-\frac{1}{p},p}(\tilde{B}_{L+1.5})}) \to 0.\end{aligned} \tag{2.37}$$

Hence

$$\|g_{k,ij}-\delta_{ij}\|_{C^{1,\theta}(B^+_{L+1.1})} \to 0, 1 \leq i,j \leq n.$$

For general $m \geq 1$, take m-th derivatives of (2.30) and the Neumann boundary conditions (2.33)(2.34), and note that

$$[\partial_i, N_k] = (\frac{\partial_i g_k^{jn}}{\sqrt{g_k^{nn}}} - \frac{g_k^{jn}\partial_i g_k^{nn}}{2\sqrt{g_k^{nn}}^3})\partial_j,$$

so we get a system of second order elliptic equations with Neumann boundary conditions in $\partial_\gamma g_k^{nn}$ and $\partial_\gamma g_k^{ln}$, $|\gamma| = m, 1 \leq l \leq m-1$, with other terms freezed. Apply the previous estimates in the case $m = 0$ and use (2.28)(2.31), we get

$$\|g_k^{nn}-\delta^{nn}\|_{C^{m+1,\theta}(B^+_{L+1})} \to 0,$$

and then for $1 \leq l \leq n-1$,

$$\|g_k^{ln}-\delta^{ln}\|_{C^{m+1,\theta}(B^+_{L+1})} \to 0,$$

hence

$$\|g_{k,ij}-\delta_{ij}\|_{C^{m+1,\theta}(B^+_{L+1})} \to 0, 1 \leq i,j \leq n$$

In particular, take $\theta = \alpha$, one can we get a $(\frac{L+1}{4}, Q, m+1, \alpha)$ harmonic coordinate chart centered at p'_k , with $d_{g_k}(p'_k, p_k) \to 0$. Then $r_h^{m+1,\alpha}(p_k, g_k, Q) \geq \frac{L}{4}$ for large k, which is a contradiction.

□

Remark 2.2.13. Note that the case $m = 0$ is also true, and one should be a little careful with the geometric arguments in the proof. Actually, the arguments in [2] prove a C_*^{m+2} harmonic radius lower bound.

Remark 2.2.14. The proof also shows that if M is complete, $i_b \geq i_0$, $inj_M \geq i_0$, $inj_{\partial M} \geq i_0$ and (2.19)(2.20) hold, then for any $p \in M$

$$r_h^{m+1,\alpha}(p, g, Q) \geq r_0(i_0, \Lambda, m, \alpha, Q). \tag{2.38}$$

The following corollary is a version we will use often.

Corollary 2.2.15. *Let* (M_i, g_i) *be a sequence of complete Einstein manifolds with boundary. Suppose* $i_b \geq i_0$, $inj_{\partial M} \geq i_0$, $|Rm| \leq C$, $|S| \leq C$, $|\nabla^k_{\partial M} Ric_{\partial M}| \leq C_k$, $|\nabla^{k+1}_{\partial M} H| \leq C_k$, $\forall k \geq 0$. *Then for any* $p_i \in M_i$*, there exists some subsequence such that* (M_i, g_i, p_i) *converges in pointed Cheeger-Gromov sense to a complete Einstein manifold with boundary* $(M_\infty, g_\infty, p_\infty)$*. If the bounds only hold for* $k = 0, 1$*, then the convergence is in* $C^{2,\alpha}$*.*

Proof. By Proposition 2.1.15, $|Rm| \leq C$, $|S| \leq C$, $i_b \geq i_0$, together imply volume lower bounds of interiors balls of some fixed radius near boundary, hence also gives an interior injectivity radius lower bound by the following lemma. Then use Remark 2.2.14 and Proposition 2.2.10. □

The following lemma is well-known, which is a qualitative version of Theorem 4.3 in [14] and can also be easily proved by contradiction arguments. Remark that the proof of our main theorems should be in this flavor.

Lemma 2.2.16. *Let* (M, g) *be a Riemannian manifold, and* $B(p, r)$ *be a metric ball that has compact closure. Suppose*

$$\sup_{B(p,r)} |Rm| \leq C, \mathrm{vol}(B(p, r)) \geq v,$$

then there exists $r_0 > 0$ *depending on* n, C, v, r *such that* $\exp_q : B_{r_0}(0) \subset T_q M \to B(q, r_0) \subset M$ *is a diffeomorphism for any* $q \in B(p, \frac{r}{4})$.

Proof. For $q \in B(p, r/3)$, let $inj_{g,q}$ denote the injectivity radius at q. We claim that for some $\delta > 0$,

$$inj_{g,q}/d_g(q, \partial B_g(p, \frac{r}{3})) \geq \delta.$$

Otherwise, we would have a sequence (M_i, g_i, p_i, q_i) with q_i achieving the maximum of $inj_{g_i,q}/d_{g_i}(q, \partial B_{g_i}(p, \frac{r}{3}))$, which converges to 0. By Bishop-Gromov volume comparison, $\mathrm{vol}_{g_i}(B_{g_i}(q_i, \frac{2}{3}r)) \geq \mathrm{vol}_{g_i}(B_{g_i}(p_i, \frac{1}{3}r)) \geq v_1$. Let $\tilde{g}_i = inj^{-2}_{g_i,q_i} g_i$. Then $d_{\tilde{g}_i}(q_i, \partial B_{g_i}(p, \frac{r}{3})) \to \infty$, and $inj_{\tilde{g}_i,q_i} = 1, inj_{\tilde{g}_i,q} \geq \frac{1}{2}, \forall q \in B_{\tilde{g}_i}(q_i, R)$, $i \geq i(R)$. By passing to a subsequence, we may assume that $(M_i, \tilde{g}_i, q_i)$ converges in pointed $C^{1,\alpha}$ topology to $(M_\infty, \tilde{g}_\infty, q_\infty)$. $(M_\infty, \tilde{g}_\infty, q_\infty)$ is a complete flat manifold of maximum volume growth; hence, it must be isometric to Euclidean $\mathbb{R}^n$, contradicting $inj_{\tilde{g}_i,q_i} = 1$. □

Chapter 3

Hyperkähler 4-manifolds and closed framings

3.1 Hyperkähler triples

The discussions in this section are well-known facts about hyperkähler 4-manifolds.

3.1.1 Pointwise theory

Let V be an oriented 4-dimensional vector space, and let $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ be a triple of 2-forms on V, so that $\boldsymbol{\omega} \in \Lambda^2(V) \otimes \mathbb{R}^3$. Suppose that $\boldsymbol{\omega}$ is a *definite* triple, which means that $\omega_1, \omega_2, \omega_3$ span a maximal positive subspace of $\Lambda^2(V)$ with respect to the wedge product. Then the triple $\boldsymbol{\omega}$ defines a unique conformal structure on V by making each 2-form ω_i self-dual. We fix a volume form μ_0 on V that defines the orientation of V, and we write $\omega_i \wedge \omega_j = 2q_{ij}\mu_0$. We define a matrix $\boldsymbol{Q}$ associated with the definite triple $\boldsymbol{\omega}$ by $Q_{ij} = \frac{q_{ij}}{\det(q_{ij})^{\frac{1}{3}}}$, which does not depend on the choice of μ_0. We denote the inverse matrix of $\boldsymbol{Q}$ by $\boldsymbol{Q}^{-1} = (Q^{ij})$. If we write

$$\omega_i \wedge \omega_j = 2Q_{ij}\mu,$$

then μ is a volume form that is intrinsically defined by $\boldsymbol{\omega}$. We define a unique metric $\langle , \rangle_{\boldsymbol{\omega}}$ on V in the conformal structure by choosing μ to be the volume form. More explicitly, this metric is given by

$$\langle u, v \rangle_{\boldsymbol{\omega}} = \frac{1}{6} \sum_{i,j,k=1}^{3} \delta^{ijk} \frac{\iota_u \omega_i \wedge \iota_v \omega_j \wedge \omega_k}{\mu}.$$

Therefore, we have

$$\langle u, u \rangle_{\boldsymbol{\omega}} = \frac{\iota_u \omega_1 \wedge \iota_v \omega_2 \wedge \omega_3}{\mu}.$$

We denote the Hodge star operator defined by this metric as $*_{\boldsymbol{\omega}}$.

Let W be an orientated 3-dimensional vector space, and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ be triple of 2-forms on W, so $\boldsymbol{\gamma} \in \Lambda^2(W^*) \otimes \mathbb{R}^3$. Suppose $\boldsymbol{\gamma}$ is a *framing* on W, i.e., $\{\gamma_1, \gamma_2, \gamma_3\}$ forms a basis for $\Lambda^2 W^*$. Then by elementary linear algebra, there exists a coframe $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$ such that

$$\gamma_i = \frac{1}{2}\delta^{ijk}\eta_j \wedge \eta_k. \tag{3.1}$$

Such $\boldsymbol{\eta}$ is uniquely determined up to a sign and we choose $\boldsymbol{\eta}$ such that $\eta_1 \wedge \eta_2 \wedge \eta_3$ defines the orientation of W and we denote this volume form by $\mathrm{vol}_{\boldsymbol{\gamma}}$. There is a unique metric on W, denoted by $\langle , \rangle_{\boldsymbol{\gamma}}$, that makes $\boldsymbol{\eta}$ an orthonormal coframe. Denote $*_{\boldsymbol{\gamma}} = *_{\boldsymbol{\eta}}$ by the Hodge star operator of $\langle , \rangle_{\boldsymbol{\gamma}}$. We denote $e_i \in W$ the dual vector of η_i, so $\eta_i(e_j) = \delta_{ij}$. Then $\boldsymbol{e} = (e_1, e_2, e_3)$ is a frame of W. Conversely, given a coframe $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$ on W compatible with the orientation, one can define a framing $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ via (3.1), and a volume form $\mathrm{vol}_{\boldsymbol{\eta}} = \mathrm{vol}_{\boldsymbol{\gamma}}$, a metric $\langle , \rangle_{\boldsymbol{\eta}} = \langle , \rangle_{\boldsymbol{\gamma}}$, a Hodge star opertaor $*_{\boldsymbol{\eta}} = *_{\boldsymbol{\gamma}}$.

Now in the case $W \subset V$ is a 3-dimensional subspace, the restriction of a definite triple $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \in \Lambda^2(V^*)$ to W defines a triple of 2-forms on W, say $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$. Let $\langle , \rangle_W$ be the restriction of the metric $\langle , \rangle_{\boldsymbol{\omega}}$ on W, which defines a Riemannian volume form vol_W and a Hodge star operator $*_W$ compatible with the orientation of W. Since ω_i are self-dual, we can write

$$\omega_i = \nu^* \wedge *_W \gamma_i + \gamma_i,$$

where $\nu^* = *_{\boldsymbol{\omega}} \mathrm{vol}_W$, then we have

$$\omega_i \wedge \omega_j = 2\nu^* \wedge *_W \gamma_i \wedge \gamma_j = 2\langle \gamma_i, \gamma_j \rangle_W \nu^* \wedge \mathrm{vol}_W = 2\langle \gamma_i, \gamma_j \rangle_W \mu.$$

Hence on W,

$$Q_{ij} = \langle \gamma_i, \gamma_j \rangle_W,$$

Since $\det(\boldsymbol{Q}) = 1$, we conclude $\boldsymbol{\gamma}$ is a framing, and $\mathrm{vol}_{\boldsymbol{\gamma}} = \mathrm{vol}_W$. If furthermore we assume $Q_{ij} = \delta_{ij}$, i.e.,

$$\omega_i \wedge \omega_j = \frac{1}{3}\delta_{ij}(\omega_1^2 + \omega_2^2 + \omega_3^2) \tag{3.2}$$

then $\langle \gamma_i, \gamma_j \rangle_W = \delta_{ij} = \langle \gamma_i, \gamma_j \rangle_{\boldsymbol{\gamma}}$. In this case, $\langle , \rangle_{\boldsymbol{\gamma}} = \langle , \rangle_W$ as an inner product on $W = \Lambda^1 W$, so in particular $*_W = *_{\boldsymbol{\gamma}}$.

3.1.2 Hyperkähler triples

Now we move our pointwise discussions to manifolds. Let X be an oriented 4-manifold, let $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ be a smooth section in $\Gamma(X, \Lambda^2 T^* X \otimes \mathbb{R}^3)$ such that it is a definite triple pointwise. By the discussions above, $\boldsymbol{\omega}$ defines a matrix valued function $\boldsymbol{Q} = (Q_{ij})$ and a volume form μ such that

$$\det \boldsymbol{Q} = 1,$$

$$\omega_i \wedge \omega_j = 2Q_{ij}\mu,$$

and $\boldsymbol{\omega}$ defines a Riemannian metric $g_{\boldsymbol{\omega}}$ which equals to $\langle , \rangle_{\boldsymbol{\omega}}$ on the tangent spaces at each point. We denote by $\nabla^{g_{\boldsymbol{\omega}}}$, $\nabla^{\boldsymbol{\omega}}$, or simply ∇, the Levi-Civita connection of $g_{\boldsymbol{\omega}}$.

Definition 3.1.1. *Let $\boldsymbol{\omega}$ be a definite triple of 2-forms.*

- *$\boldsymbol{\omega}$ is a hypersymplectic triple if $d\omega_i = 0$ for $i = 1, 2, 3$.*
- *$\boldsymbol{\omega}$ is a torsion-free hypersymplectic triple if $d\omega_i = 0$ for $i = 1, 2, 3$ and $\sum_{j=1}^{3} d(Q^{ij}\omega_j) = 0$ for $i = 1, 2, 3$.*
- *$\boldsymbol{\omega}$ is a hyperkähler triple if $d\omega_i = 0$ for $i = 1, 2, 3$ and $Q_{ij} = \delta_{ij}$ for $i, j = 1, 2, 3$.*

We will discuss torsion-free hypersymplectic triples in detail later, but for now we focus on hyperkähler triples.

It is well-known that

Proposition 3.1.2. *If $\boldsymbol{\omega}$ is a hyperkähler triple, then $Hol(g_{\boldsymbol{\omega}}) \subset SU(2)$, in particular $Ric_{g_{\boldsymbol{\omega}}} = 0$. Conversely, if (M, g) is a Riemannian manifold with $Hol(g) \subset SU(2)$, then there exists a hyperkähler triple $\boldsymbol{\omega}$, which is unique up to a constant $O(3)$ rotation, such that $g = g_{\boldsymbol{\omega}}$.*

Proof. For the first part, by elementary linear algebra, the almost complex structures I_1, I_2, and I_3 defined by ω_1, ω_2, ω_3, and $g_{\boldsymbol{\omega}}$ satisfy the quaternion relation, i.e., $I_1^2 = I_2^2 = I_3^2 = -1$ and $I_1 I_2 = I_3$. By direct computation, one knows that $\nabla\omega_i = 0$ for $i = 1, 2, 3$, where ∇ is the Levi-Civita connection defined by $g_{\boldsymbol{\omega}}$. Hence, I_1, I_2, and I_3 are also parallel, which implies $Hol(g_{\boldsymbol{\omega}}) \subset SU(2)$. With respect to I_1, ω_1 is Kähler, and $\Omega := \omega_2 + i\omega_3$ is a holomorphic volume form satisfying the Calabi-Yau equation $\omega_1^2 = \frac{1}{2}\Omega \wedge \bar{\Omega}$, so $Ric_{g_{\omega}} = 0$.

For the second part, one can define a triple $\boldsymbol{\omega}$ from 3 parallel almost complex structures I_1, I_2, and I_3 satisfying the quaternion relation. Elementary linear algebra implies $\boldsymbol{\omega}$ satisfies the orthonormal condition (3.2). Since $\nabla I_1 = 0$, we know ω_1 is Kähler with respect to I_1, so $d\omega_1 = 0$, and similarly $d\omega_2 = d\omega_3 = 0$, so $\boldsymbol{\omega}$ is hyperkähler. For the uniqueness part, suppose we have two hyperkähler triples $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ which define the same Riemannian metric g. Then, $\exists$ an $O(3)$-valued function $\boldsymbol{A}$ with $\boldsymbol{\omega}' = \boldsymbol{A}.\boldsymbol{\omega}$, and $\nabla^g\boldsymbol{\omega}' = \nabla^g\boldsymbol{\omega} = 0$ imply A is a constant matrix. □

Proposition 3.1.3. *Let M be an oriented 4-manifold admitting a hyperkähler triple, then either*

- *M is diffeomorphic to a torus T^4.*
- *M is diffeomorphic to a K3 surface.*

Proof. As in the previous proof, one can find a complex structure I_1 such that $(\omega_1, \omega_2 + i\omega_3)$ is Calabi-Yau, in particular, the canonical bundle K_M is trivial and ω_1 is Kähler. By classifiation of complex surfaces we know (M, I_1) is either a complex torus or a $K3$ surface. □

Remark 3.1.4. A $K3$ *surface* is a compact complex surface X such that $H^1(X, \mathcal{O}_X) = 0$ and K_X is trivial. It is well-known that all $K3$ surfaces are diffeomorphic, and we call the underlying oriented smooth manifold the $K3$ *manifold*. It is simply connected, with $b_2^+ = 3, b_2^- = 19$. A Riemannian metric on the $K3$ manifold means a Riemannian metric whose Riemannian volume form coincides with the orientation.

Proposition 3.1.5.

- *If g is a Ricci-flat metric on T^4, then g is flat and hyperkähler;*
- *If g is a Ricci-flat metric on the $K3$ manifold, then g is hyperkähler.*

Proof. For a closed oriented Riemannian 4-manifold X, we have the Chern-Gauss-Bonnet formula and signature formula, since g is Ricci-flat, we have

$$\frac{1}{8\pi^2}\int_X |W^+|^2 + |W^-|^2 = \chi(X),$$

$$\frac{1}{12\pi^2}\int_X |W^+|^2 - |W^-|^2 = \tau(X),$$

where W^+, W^- denote the self-dual, anti-self-dual Weyl curvature, respectively, χ, τ denote the Euler characteristic and signature, respectively.

When X is homeomorphic to T^4, $\chi(X) = \tau(X) = 0$, thus $\int_X |W^+|^2 = \int_X |W^-|^2 = 0$, hence $W^+ = W^- = 0$, so g is flat and any orthonormal frame in Λ^2_+ gives a hypekähler triple.

When X is homeomorphic to the $K3$ manifold, $\chi(X) = 24$ and $\tau(X) = -16$, thus we have $\int_X |W^+|^2 = 0$, hence $W^+ = 0$. Since $\pi_1(X) = 0$, we conclude that Λ^2_+ is trivial and any orthonormal frame in Λ^2_+ gives a hyperkähler triple.

□

The moduli space of hyperkähler metrics on the $K3$ manifold is well understood. It is characterized by the period map. See Chapter 6 some detailed discussions.

For complete non-compact hyperkähler manifolds, there have been intensive study on this by assuming decaying conditions at infinity, for a survey, refer to [28]. In this dissertation, we will only encounter complete Ricci-flat manifolds with maximum volume growth.

Definition 3.1.6. *A complete Riemannian manifold (M, g) is called ALE(asymptotical locally Euclidean) of order $\tau > 0$, if there exists a finite subgroup Γ of $SO(n)$, acting freely on S^{n-1} and there exists $R > 0$, a compact set $K \subset M$ and a diffeomorphism $\pi : (\mathbb{R}^n \backslash B_R(0))/\Gamma \to M\backslash K$ such that*

$$|\nabla^k(\pi^* g - g_{Euc})|_{g_{Euc}} = O(|x|^{-k-\tau}), \forall k \in \mathbb{N}$$

It was proved by Bando-Kasue-Nakajima that

Theorem 3.1.7. *[8] If (M, g) is a complete, Ricci-flat 4-manifold such that*

$$\int_M |Rm|^2 < \infty, \tag{3.3}$$

and

$$\exists p \in M, c > 0 \text{ such that } vol(B(p,r)) \geq cr^4, \forall r > 0,$$

then (M, g) is an ALE of order 4.

Thus, in the 4d Ricci-flat case, we can simply call (M, g) is ALE if it satisfies the conditions of the above theorem. We do not specify the order as it depends on the coordinate at infinity.

Remark 3.1.8. Actually, Theorem 1.13 in [15] implies that one can remove the condition (3.3) in Theorem 3.1.7 and reach the same conclusion, as other conditions in the Theorem 3.1.7 imply (3.3). However, we will not make use of this theorem of Cheeger-Naber, as condition (3.3) arises natural in our settings.

In the case of 4-dimensional hyperkähler geometry, 4-dimensional hyperkähler ALE spaces have been completely classified by Kronheimer [36, 37], which plays an important role in this dissertation. In fact, we only need the topological classification.

Theorem 3.1.9 ([36, 37] Classification of hyperkähler ALE spaces)**.**

- *Let (X, g) be a 4 dimensional hyperkähler ALE space, then X is diffeomorphic to the minimal resolution of $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $SU(2)$.*
- *Let X be an oriented smooth manifold underlying the minimal resolution of $\mathbb{C}^2/\Gamma$, then the image of the period map $\mathcal{P} : \mathcal{M}_{ALE} \to H^2(X, \mathbb{R}) \otimes \mathbb{R}^3$ is equal to*

$$\{\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in H^2(X, \mathbb{R}) \otimes \mathbb{R}^3 \mid \forall C \in H_2(X, \mathbb{Z}) \text{ such that } C^2 = -2, \int_C \boldsymbol{\alpha} \neq 0\},$$

where $\mathcal{M}_{ALE}$ is the space of all ALE hyperkähler triples on X, and $\mathcal{P}$ sends the triple to its cohomology class. Moreover, if $\mathcal{P}(\boldsymbol{\omega}) = \mathcal{P}(\boldsymbol{\omega}')$, then $\exists$ diffeomorphism $f : X \to X$ such that $f^*\boldsymbol{\omega} = \boldsymbol{\omega}'$.

Remark 3.1.10. There is a one-to-one correspondene between non-trivial finite subgroups of $SU(2)$ and Dynkin diagrams of ADE type. In the minimal resolution of $\mathbb{C}^2/\Gamma$, the intersection matrix of exceptional divisors is the negative of the corresponding Cartan matrix of Dynkin diagrams. In particular, any non-flat hyperkähler ALE space has a homology class $C \in H_2(X, \mathbb{Z})$ with $C^2 = -2$.

3.1.3 Examples of hyperkähler 4-manifolds

There are various ways to construct hyperkähler 4-manifolds. One of the most basic idea is using the Gibbons-Hawking ansatz (e.g., see [27]):

Gibbons-Hawking Ansatz

Let $U \subset \mathbb{R}^3$ be an open set in $\mathbb{R}^3$, and P is a principle S^1 bundle over U, with a connection 1-form θ. Suppose V is a positive harmonic function on U which solves the monopole equation on U

$$*dV = d\theta, \tag{3.4}$$

where $*$ is the standard Hodge star operator on $\mathbb{R}^3$, then one can construct 2-forms $\omega_1, \omega_2, \omega_3$ on P by

$$\begin{aligned} \omega_1 &= V dx_2 \wedge dx_3 + dx_1 \wedge \theta, \\ \omega_2 &= V dx_3 \wedge dx_1 + dx_2 \wedge \theta, \\ \omega_3 &= V dx_1 \wedge dx_2 + dx_3 \wedge \theta. \end{aligned} \tag{3.5}$$

It is direct to check $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is a hyperkähler triple. The corresponding Riemannian metric on P is given by

$$g = V(dx_1^2 + dx_2^2 + dx_3^2) + V^{-1}\theta^2.$$

Hence, $V^{-1/2}$ is the length of S^1 fibers.

A typical choice is $U = \mathbb{R}^3 \backslash \{p_1, \cdots, p_n\}$ and

$$V = \lambda + \sum_{i=1}^{n} \frac{m_i}{2|x - p_i|},$$

for $n \geq 0, m_i \in \mathbb{N}^*, \lambda \geq 0$. In this case, one has $\frac{1}{2\pi} * dV \in H^2(U, \mathbb{Z})$, since its integral on a sphere around each p_i is an integer. Thus, one can find a principal S^1 bundle P and a connection 1-form θ that solves (3.4). This allows one to define the corresponding hyperkähler triple via (3.5).

One can further compactify P (via metric completion) around p_i by adding a point $\tilde{p}_i$; as points in U approach p_i, the length of S^1 fibers goes to 0. This yields a complete hyperkähler orbifold $\bar{P}$. The unique tangent cone at each $\tilde{p}_i$ is flat $\mathbb{R}^4/\mathbb{Z}_{m_i}$.

In particular, when each $m_i = 1$, $\bar{P}$ is a complete hyperkähler 4-manifold. When $\lambda = 0$, $\bar{P}$ has the maximum volume growth rate and is therefore ALE. When $\lambda > 0$, $\bar{P}$ has cubic volume growth rate and is ALF.

Example 3.1.11. *Take* $U = \mathbb{R}^3 \backslash \{0\}$, $V = \frac{1}{2|x|}$, *then the principle* S^1 *bundle is given by the Hopf fibration* $\pi : \mathbb{R}^4 \backslash \{0\} \to \mathbb{R}^3 \backslash \{0\}$, *which in complex coordinates is* $(z_1, z_2) \to (z_1 \bar{z}_2, \frac{1}{2}(|z_1|^2 - |z_2|^2))$ *and clearly extend to a smooth map* $\mathbb{R}^4 \to \mathbb{R}^3$. *Here* $\mathbb{R}^4 = \mathbb{C}^2$ *and* $\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}$. *After metric completion, we get the Euclidean metric on* $\mathbb{R}^4$. *If we take* $V = 1 + \frac{1}{2|x|}$ *and take the metric completion, then we get the Taub-NUT metric on* $\mathbb{R}^4$.

Example 3.1.12. *Take* $U = \mathbb{R}^3 \backslash \{0, p\}$*,* $V = \frac{1}{2|x|} + \frac{1}{2|x-p|}$*. After metric completion, we get the Eguchi-Hanson spaces* $\bar{X}_p$*, which is diffeomorphic to* T^*S^2 *(also diffeomorphic to the minimal resolution of* $\mathbb{C}^2/\mathbb{Z}_2$*). Let* C_p *denotes the* S^1 *bundle over any path joining* $0, p$*, which is a sphere of self intersection* -2*. By direct computation,* $\int_{C_p} \boldsymbol{\omega} = |p|$*. As* p *approach* 0*,* $\int_{C_p} \boldsymbol{\omega} \to 0$*, and the* $(\bar{X}_p, \tilde{0})$ *converges in pointed Gromov-Hausdorff sense to the flat orbifold* $\mathbb{R}^4/\mathbb{Z}_2$*.*

Conversely, given a free tri-Hamiltonian S^1 action on a hyperkähler 4-manifold $(X, \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3))$, the moment map μ defines a principle S^1 bundle and reverses the Gibbons-Hawking ansatz.

Recall that an S^1 action on $(X, \boldsymbol{\omega})$ is tri-Hamiltonian if there exists a smooth map $\mu = (\mu_1, \mu_2, \mu_3) : X \to Lie(S^1)^* \otimes \mathbb{R}^3 \cong \mathbb{R}^3$ such that

$$\iota_\xi \omega_i = d\mu_i,$$

where $\xi(x) = \frac{d}{dt} \exp(it) \cdot x$ is the vector field on X generated by the S^1 action.

Now 3-components of μ give the coordinates x_1, x_2, x_3. Define $V := |\xi|^{-2}$, and define a 1-form θ on X by the metric dual of ξ, then $\omega_1, \omega_2, \omega_3$ is given by (3.5).

Next we discuss some known constructions for hyperkähler metrics on the $K3$ manifolds.

Kummer construction

Let T^4 be the flat torus defined by $\mathbb{R}^4/\mathbb{Z}^4$. Let (x_1, x_2, x_3, x_4) be the coordinates of $\mathbb{R}^4$. The standard hyperkähler triple on T^4 is given by

$$\begin{aligned} \omega_1 &= dx_2 \wedge dx_3 + dx_1 \wedge dx_4, \\ \omega_2 &= dx_3 \wedge dx_1 + dx_2 \wedge dx_4, \\ \omega_3 &= dx_1 \wedge dx_2 + dx_3 \wedge dx_4, \end{aligned} \tag{3.6}$$

which gives rise to the flat metric

$$g_{flat} = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

The antipodal map $(x_1, x_2, x_3, x_4) \mapsto (-x_1, -x_2, -x_3, -x_4)$ gives a $\mathbb{Z}_2$ action on T^4 with 16 fixed points defined by $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$, say $p_1, p_2, \cdots, p_{16}$. The hyperkähler triple is preserved by the $\mathbb{Z}_2$ action, so it descends to an orbifold hyperkähler triple on $T^4/\mathbb{Z}_2$. By abuse of notation, we also denote p_i by the orbifold points in the quotient. The tangent cones at each p_i is the flat cone $\mathbb{R}^4/\mathbb{Z}_2$. The idea of the Kummer construction is that, around each point p_i, one cuts out a small ball $B_{k,i}$ of radius $\frac{1}{k}$ and glues in a smooth, non-compact, complete hyperkähler 4-manifold whose asymptotic geometry matches the geometry near p_i. So, in particular, the asymptotic cone of the non-compact manifold must be $\mathbb{R}^4/\mathbb{Z}_2$, thus the correct choice is the Eguchi-Hanson space $Y_{k,i}$. Denote the resulting smooth closed 4-manifold by X_k.

By Mayer-Vietoris sequence, we know the Euler characteristic of X_k is given by

$$\begin{aligned}\chi(X_k) &= \chi(T^4/\mathbb{Z}_2\backslash\cup_{i=1}^{16} B_{k,i}) + \sum_{i=1}^{16}\chi(Y_{k,i}) - \sum_{i=1}^{16}\chi(\mathbb{RP}^3)\\ &= \frac{1}{2}\chi(T^4\backslash\cup_{i=1}^{16}\tilde{B}_{k,i}) + \sum_{i=1}^{16}\chi(Y_{k,i}) - \sum_{i=1}^{16}\chi(\mathbb{RP}^3)\\ &= \frac{1}{2}(\chi(T^4) - \sum_{i=1}^{16}\chi(\tilde{B}_{k,i}) + \sum_{i=1}^{16}\chi(S^3)) + \sum_{i=1}^{16}\chi(Y_{k,i}) - \sum_{i=1}^{16}\chi(\mathbb{RP}^3)\\ &= \frac{1}{2}(0-16*1+16*0)+16*2-16*0\\ &= 24\end{aligned} \tag{3.7}$$

where $\tilde{B}_{k,i} \subset T^4$ is the preimage of $B_{k,i}$ under the $\mathbb{Z}_2$ quotient map.

The first analytic approach to making the Kummer construction precise is due to Donaldson in [21]. He developed an analytic method for gluing two Riemannian manifolds with cylindrical ends (in the Kummer construction, the end of an Eguchi-Hanson space is diffeomorphic to $\mathbb{RP}^3 \times \mathbb{R}$). Actually, he carried out the Kummer construction in the Kähler geometry setting. He viewed T^4 as $\mathbb{C}^2/\mathbb{Z}^4$, and the Eguchi-Hanson metric as the Ricci-flat Kähler metric on the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_2$. He constructed Kähler forms and holomorphic volume forms on X_k that are approximated solutions to the Calabi-Yau equation. He then perturbed the Kähler form in the Kähler class to satisfy the genuine Calabi-Yau equation. In particular, this argument involves linear analysis using the implicit function theorem and does not require Yau's solution to the Calabi conjecture.

In conclusion,

Theorem 3.1.13. *[21] There exists hyperkähler triples $\boldsymbol{\omega}_k$ on X_k such that (X_k, g_k) converges in Gromov-Hausdorff sense to $(T^4/\mathbb{Z}_2, g_{flat})$, and there exists $C_{k,i} \in H_2(X_k,\mathbb{Z})$, $C_{k,i}.C_{k,j} = -2\delta_{ij}$, $i,j = 1,2,\cdots,16$ such that $\int_{C_{k,i}}\boldsymbol{\omega}_k \to 0$, and the given 16 Eguchi-Hanson spaces arise as bubble limits associated with the converging sequence.*

In particular, since $\chi(X_k) = 24$ and X_k admits a hyperkähler triple, Proposition 3.1.3 implies that X_k is diffeomorphic to the $K3$ manifold. In the Kummer construction, the hyperkähler manifolds X_k are volume non-collapsing, i.e., for some $q_k \in X_k$, $\mathrm{vol}_{g_k}(B(q_k, g_k)) \geq v_0$.

There are also other gluing constructions to get hyperkähler metrics on the $K3$ manifold. One is

Foscolo's construction

In [27], Foscolo used a different construction for collapsing limits of hyperkähler metrics on the $K3$ manifold. Consider T^3. There is a $\mathbb{Z}_2$ action on T^3 induced by the antipodal map, which has 8 fixed points $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$, denoted by $p_1, p_2, \cdots, p_8$. Pick any $n \geq 0$ distinct points $q_1, \cdots, q_n \in T^3 \backslash \{p_1, p_2, \cdots, p_8\}$ and take integers $k_i > 0$ and $m_j \geq 0$ $(1 \leq i \leq n, 1 \leq j \leq 8)$ such that

$$\sum_{i=1}^{n} k_i + \sum_{j=1}^{8} m_j = 16.$$

Denote by ρ_j and $\rho_i^{\pm}$ the distance functions to p_j and $\pm q_i$, respectively. The above conditions guarantee that one can solve for a positive harmonic function V with prescribed singularities at p_j and q_i:

$$V \sim \frac{2m_j - 4}{2\rho_j} \text{ around } p_j, \quad V \sim \frac{k_i}{2\rho_i^{\pm}} \text{ around } \pm q_i,$$

and guarantee that $*dV$ gives the curvature of some connection on some principal S^1 bundle P over $T^3 \backslash \{p_1, \cdots, p_8, \pm q_1, \cdots, \pm q_n\}$. The $\mathbb{Z}_2$ action on T^3 induces a $\mathbb{Z}_2$ action on P by simultaneously reversing the base and fibers. From $P/\mathbb{Z}_2$, one can rescale such that the size of the base $(T^3 \backslash \{p_1, \cdots, p_8, \pm q_1, \cdots, \pm q_n\})/\mathbb{Z}_2$ is almost fixed, and the length of the S^1 fiber shrinks to 0. And then one cuts off small neighborhoods of $[p_j], [\pm q_i]$, and glues in ALF-D_{m_j}, ALF-$A_{k_i - 1}$ gravitational instantons, respectively, to get a closed 4-manifold X_k. The gluing construction gives rise to hypersymplectic triples that are close to being hyperkähler. By performing linear analysis on the hyperkähler triple equation, one can perturb the approximate hyperkähler triple to a genuine hyperkähler triple.

As before, one can check that $\chi(X_k) = 24$ using the Mayer-Vietoris sequence. Then X_k admitting a hyperkähler triple implies X_k is diffeomorphic to the $K3$ manifold.

In sum, Foscolo proved

Theorem 3.1.14. *[27] There exists hyperkähler metrics (X_k, g_k) which converges in Gromov-Hausdorff sense to $T^3/\mathbb{Z}_2$, such that away from $[p_j], [\pm q_i]$, the collapsing happens with bounded curvature, and the given ALF-D_{m_j}, ALF-A_{k_i-1} type gravitational instantons arise as bubble limits assosciated with the collapsing sequence.*

Remark 3.1.15. It is worth noting that in Kummer construction, one can take $T^4_\epsilon = T^3 \times S^1_\epsilon$, where S^1_ϵ denotes a circle with radius ϵ and $\epsilon \to 0$. One can do the Kummer construction to each $T^4_\epsilon/\mathbb{Z}_2$. By a diagonal argument, one can also get a sequence of hyperkähler manifolds (X_k, g_k) collapsing to $T^3/\mathbb{Z}_2$, and away from 8 orbifold points in $T^3/\mathbb{Z}_2$, the collapsing happens with bounded curvature. However, one can only see Eguchi-Hanson space as bubble limits. This is a essentially different construction from Foscolo's.

For other types of gluing constructions, refer to Foscolo's survey paper [28] for details and references.

3.2 Closed framings

In this section, we discuss the the geometry of the boundary framing of a hyperkähler triple. This was originally due to Bryant [10].

Suppose $(X, \boldsymbol{\omega})$ is a hyperkähler 4-manifold with boundary. Denote $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ the restriction $\boldsymbol{\omega}$ to ∂X. As in subsection 3.1.1, $\boldsymbol{\gamma}$ defines a dual coframe $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3) \in \Omega^1(\partial X) \otimes \mathbb{R}^3$, and the dual frame is denoted by $\boldsymbol{e} = (e_1, e_2, e_3) \in \Gamma(\partial X, T\partial X) \otimes \mathbb{R}^3$, such that $\eta_i(e_j) = \delta_{ij}$. This coincide with metric dual, by definition. Recall that the induced boundary metric of $g_{\boldsymbol{\omega}}|_{\partial X}$ coincide with the metric defined purely by $\boldsymbol{\gamma}$ in hyperkähler case.

Let ν be the outward unit normal vector field of ∂X. We are going to calculate the second fundamental form of ∂X with respect to ν, say $II(v, w) = \langle \nabla_v \nu, w \rangle_{\partial X}$. Let $S \in \Gamma(\partial X, \mathrm{End}\ T\partial X)$ be the shape operator, i.e., $S(v) = \nabla_\nu v$, so $\langle S(v), w \rangle_{\partial X} = II(v, w)$. Denote $\Gamma = (\Gamma_{ij})$ the symmetric matrix

$$\Gamma_{ij} = \frac{1}{2}\langle \gamma_i, d(*_{\boldsymbol{\gamma}}\gamma_j)\rangle_{\boldsymbol{\gamma}} + \frac{1}{2}\langle \gamma_j, d(*_{\boldsymbol{\gamma}}\gamma_i)\rangle_{\boldsymbol{\gamma}} = \frac{1}{2} *_{\boldsymbol{\eta}} (\eta_i \wedge d\eta_j + \eta_j \wedge d\eta_i),$$

which is completely determined by $\boldsymbol{\gamma}$. Denote the matrix $II(e_i, e_j)$ by A and the mean curvature is given by $H = TrS = TrA$.

Lemma 3.2.1. *If $\boldsymbol{\omega}$ is a hyperkähler triple, then*

$$A = \frac{1}{2}(Tr\Gamma)I - \Gamma,$$

In particular,

$$2H = *_{\boldsymbol{\eta}}(\boldsymbol{\eta} \wedge d\boldsymbol{\eta}^T) = \langle \gamma_1, d(*_{\boldsymbol{\gamma}}\gamma_1)\rangle_{\boldsymbol{\gamma}} + \langle \gamma_2, d(*_{\boldsymbol{\gamma}}\gamma_2)\rangle_{\boldsymbol{\gamma}} + \langle \gamma_3, d(*_{\boldsymbol{\gamma}}\gamma_3)\rangle_{\boldsymbol{\gamma}},$$

$$|S|^2 = Tr\Gamma^2 - H^2.$$

Proof. Fix a point $p \in \partial X$, choose a semi-geodesic coordinate system centered at p, say (x^1, x^2, x^3, t), such that a neighborhood of p is identified with $\{t \geq 0\}$ and its intersection with ∂X is identified with $\{t = 0\}$. So (x_1, x_2, x_3) is a coordinate system on ∂X and t-curves are unit speed geodesics perpendicular to the boundary.

The hyperkähler triple can be written as

$$\boldsymbol{\omega} = -dt \wedge *_{\boldsymbol{\gamma}_t}\boldsymbol{\gamma}_t + \boldsymbol{\gamma}_t,$$

where $\boldsymbol{\gamma}_t$ is a smooth family of closed framings on ∂X such that $\boldsymbol{\gamma}_0 = \boldsymbol{\gamma}$ and because $d\boldsymbol{\omega} = 0$, $\boldsymbol{\gamma}_t$ satisfies the evolution eqution

$$\frac{\partial \boldsymbol{\gamma}_t}{\partial t} = -d(*_{\boldsymbol{\gamma}_t}\boldsymbol{\gamma}_t).$$

We can further choose (x^1, x^2, x^3) to be a normal coordinate system for ∂X at p of the metric $g_{\boldsymbol{\omega}}|_{\partial X}$ such that $e_i = \partial_{x^i}$ at p. Write $\partial_{x^i} = a_i^k(x,t)e_k(x,t)$, where $\boldsymbol{e}_k(x,t) = (e_1(x,t), e_2(x,t), e_3(x,t))$ is the dual frame of $\boldsymbol{\eta}_t = *_{\boldsymbol{\gamma}_t}\boldsymbol{\gamma}_t$. Then at p,

$$II(e_i, e_j) = II(\partial_{x^i}, \partial_{x^j}) = -\frac{1}{2}\partial_t g_{ij} = -\frac{1}{2}\partial_t(a_i^k a_j^k) = -\frac{1}{2}(\partial_t a_i^k + \partial_t a_j^k). \tag{3.8}$$

Note that at p,

$$\partial_t \eta_p = \partial_t a_m^p dx^m = \partial_t a_m^p \eta_m,$$

then we have

$$\begin{aligned} \langle \gamma_i, d(*_{\boldsymbol{\gamma}}\gamma_j)\rangle_{\boldsymbol{\gamma}} &= -\langle \gamma_i, \partial_t \gamma_j\rangle_{\boldsymbol{\gamma}} \\ &= -\langle \frac{1}{2}\delta^{ikl}\eta_k \wedge \eta_l, \frac{1}{2}\partial_t(\delta^{jpq}\eta_p \wedge \eta_q)\rangle_{\boldsymbol{\gamma}} \\ &= -\frac{1}{4}\delta^{ikl}\delta^{jpq}(\partial_t a_m^p \langle \eta_k \wedge \eta_l, \eta_m \wedge \eta_q\rangle_{\boldsymbol{\gamma}} \\ &\quad + \partial_t a_n^q \langle \eta_k \wedge \eta_l, \eta_p \wedge \eta_n\rangle_{\boldsymbol{\gamma}}) \\ &= -\frac{1}{4}\delta^{ikl}\delta^{jpq}(\partial_t a_m^p(\delta^{km}\delta^{lq} - \delta^{kq}\delta^{lm}) + \partial_t a_n^q(\delta^{kp}\delta^{ln} - \delta^{kn}\delta^{lp})) \\ &= -\delta^{ij}\partial_t a_k^k + \partial_t a_j^i. \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we have

$$\Gamma = (TrA)I - A,$$

so $Tr\Gamma = 2TrA$, $A = \frac{1}{2}(Tr\Gamma)I - \Gamma$.

□

This lemma says that the second fundamental form of ∂X is just in algebraic terms of $\boldsymbol{\eta}$ and $d\boldsymbol{\eta}$.

Chapter 4

Convergence of hyperkähler 4-manifolds with boundary

4.1 Curvature estimates near the boundary

This section serves as a first step for the proof of our main theorem. For an Einstein manifold with boundary, if the boundary intrinsic and extrinsic geometry are controlled well and i_b is bounded from below, we hope to control the interior geometry within i_b. To the author's knowledge, we do not know any general statement. We will first state and prove a version we need, and then discuss some lemmas needed in the proof.

Theorem 4.1.1. *Let (M, g) be a complete hyperkähler 4-manifold with compact boundary. Suppose $|S| \leq C$, $|\nabla^j_{\partial M} Rm_{\partial M}| \leq C_j, |\nabla^{j+1}_{\partial M} H| \leq C_j, j = 0, 1, \cdots$, $inj_{\partial M} \geq i_0$, $i_b \geq i_0$, $\int_M |Rm|^2 \leq C$. Then for any $r_1 < i_0$, there exists $C' > 0$, depending on C, C_j, i_0, r_1, such that $\sup\limits_{N_{r_1}(\partial M, g)} |Rm| \leq C'$.*

Proof. Without loss of generality, assume $i_0 = 1$. Denote $\alpha = r_1$, $\beta = \frac{1}{4}(1-\alpha)$. Suppose the conclusion is not true, we have a sequence (M_i, g_i) satisfying the conditions, but

$$\sup_{N_\alpha(\partial M_i, g_i)} |Rm_{g_i}| \to \infty.$$

Let $p_i \in N_\alpha(\partial M_i, g_i)$ achieves this supremum. Note that the equations (2.6)(2.7) imply that $|Rm_{g_i}|$ is uniformly bounded on ∂M_i, so $p_i \notin \partial M_i$ for large i.

Claim 1 There exists a subsequence such that

$$d_{g_i}(p_i, \partial M_i)^2 |Rm_{g_i}(p_i)| \to \infty. \tag{4.1}$$

If this is not true, we have $\sup_i d_{g_i}(p_i, \partial M_i)^2 |Rm_{g_i}|(p_i) < \infty$. Rescale $\tilde{g}_i = |Rm_{g_i}(p_i)| g_i$, then $|Rm_{\tilde{g}_i}(p_i)| = 1$, and $|Rm_{\tilde{g}_i}| \leq 1$ in $N_{\alpha|Rm(p_i)|^{\frac{1}{2}}}(\partial M_i, \tilde{g}_i)$, and $\sup_i d_{\tilde{g}_i}(p_i, \partial M_i) < \infty$, $i_{b,\tilde{g}_i} \geq$

$|Rm_{g_i}(p_i)|^{\frac{1}{2}}$ for all i. Hence by Corollary 2.2.15, (M_i, g_i, p_i) subconverges in pointed Cheeger-Gromov sense to $(M_\infty, \tilde{g}_\infty, p_\infty)$, which is a complete Ricci-flat 4-manifold with flat, totally geodesic boundary, hence must be flat by Lemma 4.1.6. This contradicts that $|Rm_{\tilde{g}_\infty}(p_\infty)| = 1$ and proves Claim 1.

Now rescale g_i in another way, let $g'_i = d_{g_i}(p_i, \partial M_i)^{-2} g_i$, so $d_{g'_i}(p_i, \partial M_i) = 1$. Since $d_{g_i}(p_i, \partial M_i) \le \alpha$, the rescaled metric g'_i satisfies $i_{b,g'_i} \ge \alpha^{-1}$ as well as all other conditions of the assumptions of the theorem, but with different bounds, regardness of whether $d_{g_i}(p_i, \partial M_i)$ is uniformly bounded from below or not. Moreover, (4.1) is equivalent to $|Rm_{g'_i}(p_i)| \to \infty$.

By the ϵ-regularity Theorem 4.1.4, there exists a universal constant ϵ_0 such that for sufficiently large i,

$$\int_{B_{g'_i}(p_i,\beta)} |Rm_{g'_i}|^2 \ge \epsilon_0.$$

Claim 2 There exists a subsequence such that

$$\sup_{N_\alpha(\partial M_i, g'_i)} |Rm_{g'_i}| \to \infty,$$

If not, we have $\sup_{N_\alpha(\partial M_i, g'_i)} |Rm_{g'_i}| \le C$. By Proposition 2.1.15 and Bishop-Gromov volume comparison, $\mathrm{vol}(B_{g'_i}(p_i, \beta)) \ge v$. Since $B_{g'_i}(p_i, \beta) \subset N_{\alpha^{-1}(\alpha+\beta)}(\partial M_i, g'_i) \subset N_{\alpha+\beta}(\partial M_i, g_i)$, and the last one is diffeomorphic to $\partial M_i \times [0, \alpha+\beta]$, we conclude that there is no integral homology class of self-intersection -2 in $B_{g'_i}(p_i, \beta)$. By Proposition 4.1.5, $|Rm_{g'_i}(p_i)|$ is bounded, which is a contradiction to Claim 1 and finishes the proof of Claim 2. One can denote p'_i by the point in $N_\alpha(\partial M_i, g'_i)$ that achieves the supremum of $|Rm_{g'_i}|$. A graph visualization is presented in Figure 4.1.

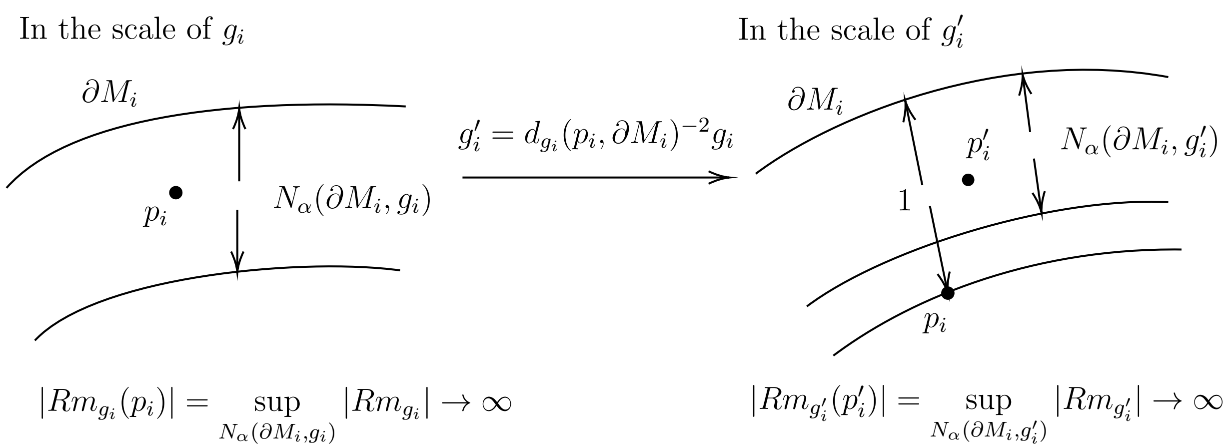

Figure 4.1: Change of scale

Now Claim 2 enables us to obtain, by induction, for each fixed positive integer N, N sequences of metrics $g_i^{(0)} = g_i$, $g_i^{(1)} = g'_i, \dots, g_i^{(N)}$, and points $p_i^{(j)} \in N_\alpha(\partial M_i, g_i^{(j)})$ for $0 \le$

$j \leq N-1$, $p_i^{(0)} = p_i, p_i^{(1)} = p_i'$, such that for $0 \leq j \leq N-1$, $p_i^{(j)}$ achieves the supremum of $|Rm_{g_i^{(j)}}|$ in $N_\alpha(\partial M_i, g_i^{(j)})$, and

$$|Rm_{g_i^{(j)}}(p_i^{(j)})| \to \infty,$$

$$g_i^{(j+1)} = d_{g_i^{(j)}}(p_i^{(j)}, \partial M_i)^{-2} g_i^{(j)},$$

$$d_{g_i^{(j+1)}}(p_i^{(j)}, \partial M_i) = 1,$$

$$\int_{B_{g_i^{(j+1)}}(p_i^{(j)}, \beta)} |Rm_{g_i^{(j+1)}}|^2 \geq \epsilon_0,$$

$$B_{g_i^{(j+1)}}(p_i^{(j)}, \beta) \subset N_{\alpha^{-1}(\alpha+\beta)}(\partial M_i, g_i^{(j+1)}) \subset N_{\alpha+\beta}(\partial M_i, g_i^{(j)}),$$

$$B_{g_i^{(j+1)}}(p_i^{(j)}, \beta) \cap N_{\alpha+\beta}(\partial M_i, g_i^{(j+1)}) = \emptyset.$$

It follows that for each fixed i, $\bigcup_{j=0}^{N-1} B_{g_i^{(j+1)}}(p_i^{(j)}, \beta)$ is a disjoint union. Since $\int_{M_i} |Rm_{g_i}|^2 \leq C$, we have $N\epsilon_0 \leq C$. This is a contradiction, since N can be any positive integer. □

Remark 4.1.2. This theorem is purely local. In fact, by slightly modifying the proof, we can show that if $\exp^\perp$ maps $B_{\partial M}(p, r_0) \times [0, r_1)$ diffeomorphically onto the metric cyclinder $C(B_{\partial M}(p, r_0), 0, r_1)$, and such that $B_{\partial M}(p, r_0)$ has compact closure. Assume all the bounds hold on $B_{\partial M}(p, r_0)$ and assume a L^2-curvature bound, then we have uniformly bounded curvature in any interior metric cylinder $C(B_{\partial M}(p, r_0'), 0, r_1')$ with fixed $r_0' < r_0$ and $r_1' < r_1$.

In other words,

Theorem 4.1.3. *Let (M, g) be a hyperkähler 4-manifolds with boundary (X, g_0) . Assume $p \in X$ such that $B_{g_0}(p, 1)$ has compact closure, on which*

$$|S| \leq C, |\nabla_{g_0}^j Rm_{g_0}| \leq C_j, |\nabla_{g_0}^{j+1} H| \leq C_j, j = 0, 1, inj_{g_0} \geq i_0$$

Suppose $\exp^\perp$ maps $B_g(p, 1) \times [0, 1)$ diffeomorphically onto its image in M which is also denoted by $B_g(p, 1) \times [0, 1)$. If

$$\int_{B_{g_0}(p.1) \times [0,1)} |Rm|^2 \leq C,$$

then

$$\sup_{B_{g_0}(p, \frac{1}{2}) \times [0, \frac{1}{2})} |Rm| \leq C'.$$

Proof. (Sketch) Note that we only need $j = 0, 1$ in the assumption to get $C^{2,\alpha}$ convergence. We need to show that the quantity

$$d(x, \partial^+(B_{g_0}(p,1) \times [0,1)))^2 |Rm|(x)$$

is uniformly bounded in $B_{g_0}(p,1) \times [0,1)$, where ∂^+ is the "parabolic boundary", i.e., the topological boundary \ manifold boundary. We argue by contradiction. Suppose this is not bounded, then we can find a sequence of manifolds and a point p_i that achieves the maximum of this quantity, and rescale the metric by $|Rm(p_i)|$ and $d(p_i, X_i)^{-2}$ to get a contradiction, as before. □

Now we provide the lemmas that we used in the proof.

The following collapsing ϵ-regularity theorem is originally due to Cheeger-Tian in [16]. Recently, in the hyperkähler case, [46] gives a simple proof by a blow-up argument and studying complete collapsing limits of hyperkähler 4-manifolds with bounded curvature.

Theorem 4.1.4. *[16] There exists ϵ, c such that the following holds: Let (M^4, g) be an Einstein 4-manifold, $|\mathrm{Ric}| \leq 3$, $r \leq 1$, and $B(p,r)$ is a metric ball that has compact closure. If*

$$\int_{B(p,r)} |Rm|^2 \leq \epsilon,$$

then

$$\sup_{B(p,\frac{r}{2})} |Rm| \leq cr^{-2}.$$

The following proposition plays an important role. To avoid redundancy, we state it here without proving, but we will prove a more general version later, see Proposition 5.2.1.

Proposition 4.1.5. *Let (M, g) be an hyperkähler 4-manifold. Suppose $B(p,5)$ has compact closure, $B(p,3)$ contains no integral homology class of self intersection -2, if*

$$\mathrm{vol}(B(p,1)) \geq v,$$

$$\int_{B(p,3)} |Rm|^2 \leq C,$$

then there exists $C' > 0$, depending on v, C such that

$$\sup_{B(p,1)} |Rm| \leq C'.$$

The following lemma originally dates back to Koiso in [35], and is an incredibly special case of the result in [9, 5]. Since it plays an important role throughout the dissertation, we provide a detailed proof here following [35]. Note that this is a one-sided version of Koiso's.

Lemma 4.1.6. *Let (M, g) be a connected C^2 Riemannian manifold with boundary. Suppose* $\mathrm{Ric}_M = 0$ *and for some open boundary portion T, $S|_T = 0$ and $Rm_{\partial M}|_T = 0$, then g is smooth and $Rm_M = 0$.*

Proof. Fix any point p in T. First, we show g can be extended across the boundary near p. Choose a semi-geodesic coordinate system $(x^1, \cdots, x^n)$ near p, with ∂M identified with $\{x^n = 0\}$, and interior identified with $\{x^n > 0\}$, $\nabla x^n = \partial_{x^n}$, and $g_{ij}(x', 0) = \delta_{ij}, 1 \leq i, j \leq n-1$, where $x' = (x^1, \cdots, x^{n-1})$. We extend the metric tensor g by reflection across $\{x^n = 0\}$, i.e., set $g_{ij}(x', x^n) = g_{ij}(x', -x^n), 1 \leq i, j \leq n$. Then $g \in C^0(B) \cap C^2(B^+)$, where B^+ is the boundary coordinate ball and B is its extension after reflection. We need to show $g \in C^2(B)$. In fact, $S = 0$ is equivalent to $\frac{\partial g_{ij}}{\partial x^n_+}(x', 0) = 0, 1 \leq i, j \leq n-1$, then we also have $\frac{\partial g_{ij}}{\partial x^n_-}(x', 0) = -\frac{\partial g_{ij}}{\partial x^n_+}(x', 0) = 0$, hence $\frac{\partial g_{ij}}{\partial x^n}(x', 0) = 0$ and $g_{ij} \in C^1(B)$. Since $g \in C^2(B^+)$, we have for $1 \leq l \leq n-1$,

$$\frac{\partial}{\partial x^n_+}\frac{\partial g_{ij}}{\partial x^l}(x', 0) = \frac{\partial}{\partial x^l}\frac{\partial g_{ij}}{\partial x^n_+}(x', 0) = \frac{\partial}{\partial x^l}0 = 0,$$

$$\frac{\partial}{\partial x^n_-}\frac{\partial g_{ij}}{\partial x^l}(x', 0) = -\frac{\partial}{\partial x^n_+}\frac{\partial g_{ij}}{\partial x^l}(x', 0) = 0.$$

$$\begin{aligned}\frac{\partial}{\partial x^n_-}\frac{\partial g_{ij}}{\partial x^n}(x', 0) &= \lim_{x^n \to 0^-}\frac{1}{x^n}\frac{\partial g_{ij}}{\partial x^n}(x', -x^n) \\ &= \lim_{x^n \to 0^+}\frac{1}{x^n}\frac{\partial g_{ij}}{\partial x^n}(x', x^n) = \frac{\partial}{\partial x^n_+}\frac{\partial g_{ij}}{\partial x_n}(x', 0).\end{aligned} \tag{4.2}$$

Hence $g_{ij} \in C^2(B)$. Finally, we have $g_{nn} = 1, g_{ln} = g_{nl} = 0, 1 \leq l \leq n-1$, hence $g \in C^2(B)$.

By elliptic regularity, all harmonic coordinate charts in B give rise to a real analytic structure in B such that g is real analytic. Hence if t is a distance function such that ∂M is defined by $t^{-1}(0)$ near p, then t is real analytic near p. Choose a real analytic coordinate (z, t) near p. Since $t^{-1}(0)$ is totally geodesic, $\frac{\partial g}{\partial t}(z, 0) = 0$. The evolution equation (2.9) is equivalent to the second order PDE

$$\frac{\partial^2 g}{\partial t^2} = 2ric\ g - \frac{1}{2}tr_g(\frac{\partial g}{\partial t})\frac{\partial g}{\partial t} + (\frac{\partial g}{\partial t})^2, \tag{4.3}$$

where $ric\ g$ is the Ricci tensor of level sets of t. By the uniqueness part of Cauchy-Kovalevskaya theorem, we know $g(z, t) = g(z, 0)$. Hence $Rm_M = 0$ near p. Since Rm_M is real analytic in the interior of M, $Rm_M = 0$ in M.

□

4.2 Convergence of hyperkähler metrics

Now we state and prove our main theorem in the sense of Riemannian geometry.

Theorem 4.2.1. *Let (X_i, g_i) be a sequence of compact, connected hyperkähler 4-manifold with boundary, suppose on ∂X_i, we have*

$$H_i \geq H_0 > 0, |S_i| \leq C, |\nabla^{j+1}_{\partial X_i} H| \leq C_j,$$

$$inj_{\partial X_i} \geq i_0, \mathrm{diam}_{g_i|_{\partial X_i}}(\partial X_i) \leq C, |\nabla^j_{\partial X_i} Rm_{\partial X_i}| \leq C_j, \forall j \geq 0,$$

and $\chi(X_i) \leq C$. Assuming there exists no $C \in H_2(X_i, \mathbb{Z})$ with $C^2 = -2$, then there exists a subsequence such that (X_i, g_i) converges in Cheeger-Gromov sense to a compact, connected hyperkähler 4-manifold with boundary (X_∞, g_∞).

Remark 4.2.2. By Chern-Gauss-Bonnet formula, our assumptions imply

$$\int_{X_i} |Rm|^2 \leq C.$$

Note that for a compact connected Einstein 4-manifold (M, g) with boundary, the Chern-Gauss-Bonnet formula says

$$\frac{1}{8\pi^2}\int_M |Rm|^2 = \chi(M) - \frac{1}{2\pi^2}\int_{\partial M}\prod_{i=1}^{3}\lambda_i - \frac{1}{8\pi^2}\int_{\partial M}\sum_{\sigma\in S_3} K_{\sigma_1\sigma_2}\lambda_{\sigma_3}.$$

Here $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of the shape operator S of ∂M, Let e_i be eigenvectors of eigenvalue λ_i such that $\{e_1, e_2, e_3\}$ is an orthonormal basis, then $K_{ij} = \sec(e_i, e_j)$. See for example (1.16) in [6].

Remark 4.2.3. If we drop the condition $\mathrm{diam}_{g_i|_{\partial X_i}}(\partial X_i) \leq C$, and replace $H_i \geq H_0 > 0$ by $H_i > 0$, $\chi(X_i) \leq C$ by $\int_{X_i} |Rm_{g_i}|^2 \leq C$, then for any point $p_i \in X_i$ with $d(p_i, \partial X_i) \leq K$, a subsequence of (X_i, g_i, p_i) converges in pointed Cheeger-Gromov sense to a complete hyperkähler 4-manifolds with boundary $(X_\infty, g_\infty, p_\infty)$.

Remark 4.2.4. The positive mean curvature condition is necessary. The following counterexample is natural and was observed by Donaldson in [20]. Consider the standard unit ball B^4 inside Euclidean $\mathbb{R}^4$, "squeeze" the ball such that the north pole and the south pole of the boundary S^3 comes together, so we get a sequence of embedded B^4 in $\mathbb{R}^4$ converging in Hausdorff sense to a limit homeomorphic to the wedge sum of two B^4, whose boundary is an immersed S^3 intersecting itself at one point. For this sequence, all other assumptions are satisfied except for the positive mean curvature. Slightly modifying the process, one can also have a sequence of B^4 of dumbbell shape such that the middle cyclinder $B^3 \times [0, 1]$ collapses to $[0, 1]$, then they have a Hausdorff limit which is homeomorphic to two B^4 joint by a line segment.

In these types of examples, the curvatures are uniformly bounded, and the global volume are non-collapsing before taking the limit.

Remark 4.2.5. If we allow integral homology classes with self-intersection -2 and do not assume the positive mean curvature condition, something worse will happen: consider the Kummer construction (See also subsection 3.1.3 Theorem 3.1.13, Remark 3.1.15). Let $T^4/\mathbb{Z}_2$ be the flat 4-orbifold with 16 singularities, remove small neighborhoods of the 16 singularities, and glue in 16 copies of T^*S^2. By varing the sizes of these glue-in regions and perturbing the metrics, we get a sequence of hyperkähler 4-manifolds (M_i, h_i), each of which is diffeomorphic to the $K3$ manifold, such that (M_i, h_i) converges in Gromov-Hausdorff sense to $T^4/\mathbb{Z}_2$, and converges in Cheeger-Gromov sense away from these 16 singularities. Now let $T^3/\mathbb{Z}_2 \subset T^4/\mathbb{Z}_2$ be the flat 3-orbifold, such that the last coordinate equal to 0. Let $\tilde{X}_i \subset T^4/\mathbb{Z}_2$ be the closure of the tubular neighborhood of $T^3/\mathbb{Z}_2$ of width i^{-1}, then $\partial\tilde{X}_i$ is connected, totally geodesic, isometric to the same flat T^3. Now for each i, choose $n(i)$ large enough such that one can find $X_i \subset M_{n(i)}$ which are compact domains with smooth boundary, $d_{GH}(X_i, \tilde{X}_i) \to 0$, $|\nabla^j_{\partial X_i} S_{\partial X_i}| \to 0, \forall j \geq 0$, ∂X_i converges in Cheeger-Gromov sense to flat T^3. Let $g_i = h_{n(i)}|_{X_i}$, then (X_i, g_i) converges in Gromov-Hausdorff sense to flat $T^3/\mathbb{Z}_2$. In this case, $\mathrm{vol}(X_i) \to 0$, $\sup_{X_i} |Rm_{g_i}| \to \infty$, and there is no interior ball in X_i of a uniform size.

Note that in this example $\partial\tilde{X}_i$ cannot be perturbed in flat $T^4/\mathbb{Z}_2$ to have positive mean curvature. In fact, take a small tubular neighborhood of $\partial\tilde{X}_i$ of width less than i^{-1}, whose boundary has two totally geodesic connected components T_1, T_2. Suppose $\partial\tilde{X}_i$ can be perturbed to T' such that its mean curvature has a strict sign, say that its mean curvature vector points towards T_1. Then T', T_1 bound a region W. By [20] Proposition 7, T' and T_1 are isometric, so T' is totally geodesic, which is a contradiction. However, it is unknown whether ∂X_i could have positive mean curvature or not.

The major step of the proof in Theorem 4.2.1 is to show that these Riemannian manifolds have a nice neighborhood of definite size. We show that the boundary injectivity radius has a lower bound, so that the interior geometry within the boundary injectivity radius is nicely controlled by Theorem 4.1.1.

Proposition 4.2.6. *There exists $i_1 > 0$, depending on the constants in Theorem 4.2.1 such that $i_{b,g_i} \geq i_1$.*

Proof. Suppose not, we have a subsequence of hyperkähler metrics g_i, with $i_{b,g_i} \to 0$. Rescale the metric $\tilde{g}_i = i^{-2}_{b,g_i} g_i$, then $i_{b,\tilde{g}_i} = 1$. For any point $p_i \in \partial X_i$, the restriction metric $(\partial X_i, \tilde{g}_i|_{\partial X_i}, p_i)$ converges in pointed Cheeger-Gromov sense to flat $\mathbb{R}^3$, $|S_{\tilde{g}_i}| \to 0$ and $|\nabla^{j+1}_{\partial X_i} H_i| \to 0$ uniformly on ∂X_i. Consider $\sup_{B_{\tilde{g}_i}(p_i,4)} |Rm_{\tilde{g}_i}|$. We have two cases

Case 1 $\sup_{B_{\tilde{g}_i}(p_i,4)} |Rm_{\tilde{g}_i}| \leq C$.

We have a subsequence $(B_{\tilde{g}_i}(p_i, 3), \tilde{g}_i)$ converges in Cheeger-Gromov sense to a Riemannian manifold with boundary (B_∞, g_∞), so (B_∞, g_∞) is Ricci flat, and all boundary components are flat, totally geodesic. By Lemma 4.1.6, (B_∞, g_∞) is flat. We need to choose good points p_i to lead to a contradiction. In fact, by Proposition 2.1.7, there exists $p_i \in \partial X_i$ such that

$\gamma_{p_i}(1)$ is a focal point along γ_{p_i}. Let p_∞ be the limit of p_i. By Proposition 2.1.13, we get a limit geodesic $\gamma_{p_\infty} : [0,1] \to B_\infty$, such that $\gamma_{p_\infty}(1)$ is a focal point along γ_{p_∞}, contradiction.

Case 2 For some subsequence we have $\sup\limits_{B_{\tilde{g}_i}(p_i,4)} |Rm_{\tilde{g}_i}| \to \infty$.

Then we can find points $q_i \in B_{\tilde{g}_i}(p_i, 4)$ such that $|Rm_{\tilde{g}_i}(q_i)| \to \infty$. By Theorem 4.1.1, we have $d_{\tilde{g}_i}(q_i, \partial X_i) \geq \frac{1}{2}$ for large i. By Theorem 4.1.1, Proposition 2.1.15, and the fact $d_{\tilde{g}_i}(q_i, \partial X_i) \leq 4$, we have $\mathrm{vol}_{\tilde{g}_i}(B_{\tilde{g}_i}(q_i, \frac{r_0}{2})) \geq v_0$ for some r_0, v_0. Since there is no integral homology class in X_i with self-intersection -2, then by Proposition 4.1.5, $\sup_{B_{\tilde{g}_i}(q_i, \frac{r_0}{10})} |Rm_{\tilde{g}_i}| \leq C$, which is a contradiction.

□

Then we can finish the proof of Theorem 4.2.1 as follows: by Theorem 4.1.1, $|Rm_{g_i}|$ is uniformly bounded in $N_{\frac{i_1}{2}}(\partial X_i, g_i)$. By Proposition 2.1.15, there exist $r_1 < \frac{i_1}{10}, v_1$, such that $\mathrm{vol}_{g_i}(B_{g_i}(p, r_1)) \geq v_1$ for any p with $d_{g_i}(p, \partial X_i) = 2r_1$. By Proposition 2.1.4, $\sup\limits_{q \in X_i} d_{g_i}(q, \partial X_i) \leq 3H_0^{-1}$, hence $\mathrm{diam}(X_i, g_i) \leq C$. Then from Bishop-Gromov volume comparison, we know $\mathrm{vol}_{g_i}(B_{g_i}(p, r_1)) \geq v_2$ for any p with $d_{g_i}(p, \partial X_i) \geq 5r_1$. By Proposition 4.1.5, for these p, $|Rm_{g_i}(p)|$ is uniformly bounded, with the bound independent of i and p. Hence $\sup_X |Rm_{g_i}|$ is uniformly bounded. Then by Corollary 2.2.15, a subsequence (X_i, g_i) converges in Cheeger-Gromov sense to a smooth Riemannian manifold with boundary (X, g), so g is a hyperkähler metric.

Remark 4.2.7. It is also possible to prove Theorem 4.2.1 directly by rescaling the maximum curvature norm to be 1. Suppose that the maximum curvature norm is achieved at point p_i. From Corollary 2.1.9, we know that for the rescaled metrics, $i_b \geq i_0$. If the distance between p_i and the boundary ∂X_i is bounded, then the pointed Riemannian manifolds converge in the pointed Cheeger-Gromov sense to a Ricci-flat manifold with flat and totally geodesic boundary. Hence, the limit is flat by Lemma 4.1.6, which leads to a contradiction. If, on the other hand, $d(p_i, \partial X_i) \to \infty$ for a subsequence, we rescale the metrics again such that this distance is 1. However, it is not clear whether the curvature is bounded in a fixed-size neighborhood of the boundary at this scale. Using the idea of Theorem 4.1.1, we can keep rescaling until this happens at some point, possibly at different points p_i'. This leads to a contradiction, which is the same as the one in Case 2 of Proposition 4.2.6. Since volume lower bounds can be passed within a finite distance, so by Proposition 4.1.5, the curvature bounds can also be passed within a finite distance.

Alternatively, one can prove Theorem 4.2.1 by rescaling the harmonic radius. All of these methods eventually turn out to use essentially the same ingredient.

4.3 Local limits

Suppose X is a connected complete metric space, and there exists a finite set $\Sigma = \{p_1, \cdots, p_m\}, m \geq 0$ such that $X \backslash \Sigma$ is a smooth flat hyperkähler 4-manifold with nonempty boundary, and each boundary component $Y_1, \cdots, Y_n, n \geq 1$ is isometric to flat $\mathbb{R}^3$, $X \backslash \cup_{i=1}^n Y_i$ is a flat hyperkähler 4-orbifold such that Σ is the set of all orbifold points. Our goal is to classify all such X.

The motivation of this problem is the following:

Proposition 4.3.1. *Let (X_i, g_i) be a sequence of compact hyperkähler 4-manifolds with boundary, such that*

$$i_{b,g_i} \geq i_0, \quad \text{and} \quad inj_{\partial X_i} \to \infty,$$

$$|S_i| \to 0, \quad |\nabla^{j+1}_{\partial X_i} H_i| \to 0, \quad |\nabla^j_{\partial X_i} Rm_{\partial X_i}| \to 0$$

uniformly on ∂X_i for all $j \geq 0$, and

$$\int_{X_i} |Rm_{g_i}|^2 \leq C.$$

Then for any $p_i \in X_i$ such that $d_{g_i}(p_i, \partial X_i) \leq K$, a subsequence of (X_i, g_i, p_i) converges in the pointed Gromov Hausdorff sense to a complete metric space $(X_\infty, d_\infty, p_\infty)$. The limit (X_∞, d_∞) satisfies all the properties of X stated above.

By Theorem 4.1.1, the geometry near boundary is nicely controlled, so this theorem is a consequence of [4, 8, 48] etc.

Remark 4.3.2. In Theorem 4.2.1, under the mean positive condition, if we allow homology class of self-intersection -2 in X_i, then we are unable to show that i_{b,g_i} has a lower bound. In fact, the bad case is that focal points are exactly those curvature blow-up points, and they approach the boundary in a moderate rate. However, we can say something within the scale of the boundary injectivity radius. Suppose $i_{b,g_i} \to 0$, then we rescale the metric such that the boundary injectivity radius becomes 1, then the rescaled metric satisfies the assumption of Proposition 4.3.1.

Theorem 4.3.3. *We must have $(m, n) \in \{(0,1), (0,2), (1,1)\}$ and X is isometric to one of the following:*

- *(m=0, n=1) $\mathbb{R}^4_+$;*
- *(m=0, n=2) the region in $\mathbb{R}^4$ bounded by two parallel hyperplanes;*
- *(m=1, n=1) the connected component of 0 in $(\mathbb{R}^4 \backslash H)/\mathbb{Z}_2$, where H is a hyperplane such that $0 \notin H$.*

Proof. Consider another copy of X, glue them together along $Y_1, \cdots, Y_n$, we get a complete flat hyperkähler orbifold $\hat{X} = X \sqcup_{id} X, id : \cup_{i=1}^n Y_i \to \cup_{i=1}^n Y_i \subset X$, which contains $Y_1, \cdots, Y_n$ as smooth hypersurfaces. Then $\hat{X}$ is a $(SU(2), \mathbb{R}^4)$ orbifold in the sense of [47]. Let $\tilde{X}$ be the universal covering orbifold of $\hat{X}$, then we have a developing map $D : \tilde{X} \to \mathbb{R}^4$, and since $\tilde{X}$ is a complete orbifold, D is a covering map. Hence $\tilde{X}$ is homeomorphic to $\mathbb{R}^4$, and $\hat{X}$ is isometric to $\mathbb{R}^4/\Gamma$, where Γ is a discrete subgroup of $\mathbb{R}^4 \rtimes SU(2)$. Let r be the projection map to the second factor, then $r(\Gamma)$ is a finite subgroup of $SU(2)$, and we have a short exact sequence

$$0 \to \Gamma \cap \mathbb{R}^4 \to \Gamma \to r(\Gamma) \to 0.$$

Then $\mathrm{rank}(\Gamma \cap \mathbb{R}^4) \leq 1$, since otherwise $\hat{X}$ is a quotient of $\mathbb{R}^2 \times T^2$ and cannot contain a flat $\mathbb{R}^3$ as a Riemannian submanifold.

If $\Gamma \cap \mathbb{R}^4 = \{0\}$, then $\hat{X} \cong \mathbb{R}^4/r(\Gamma)$ and hence $r(\Gamma) = \{1\}$, since otherwise $\hat{X}$ has exactly one orbifold point, contradiction. Hence, $m = 0, n = 1$ and X is isometric to $\mathbb{R}^4_+$.

If $\Gamma \cap \mathbb{R}^4 = \mathbb{Z}a$ for some $0 \neq a \in \mathbb{R}^4$, let $\pi : \mathbb{R}^4 \to \mathbb{R}^4/\Gamma$ be the covering map, then $\pi^{-1}(Y_1)$ is a complete totally geodesic submanifold of $\mathbb{R}^4$, hence a countable disjoint union of parallel hyperplanes. Pick one of them, denoted by Z, then $\pi^{-1}(Y_1)$ is a disjoint union of $\gamma.Z$ for $\gamma \in \Gamma$. Suppose Z is defined by $b^T x + c = 0$, then $\gamma^{-1}.Z$ is defined by $b^T r(\gamma)x + c' = 0$. Since they are parallel to each other, $b^T = \pm b^T r(\gamma)$, and hence ± 1 is an eigenvalue of $r(\gamma)$, which forces $r(\gamma) = \pm 1$, as $r(\gamma) \in SU(2)$. Hence $r(\Gamma) = \{1\}$ or $r(\Gamma) = \mathbb{Z}_2$. If $r(\Gamma) = \{1\}$, then Γ is generated by $x \mapsto x + a$, so $m = 0, n = 2$ and X is isometric to the region in $\mathbb{R}^4$ bounded by two parallel hyperplanes; If $r(\Gamma) = \mathbb{Z}_2$, then Γ is generated by $x \mapsto -x + d$ and $x \mapsto x + a$ for some $d \in \mathbb{R}$. Hence $m = 1$ and $\mathbb{R}^4/\Gamma \backslash \{Y_1, Y_2, \cdots, Y_n\}$ has $n + 1$ connected components. But from the gluing construction and that X is connected, we know $\hat{X} \backslash \{Y_1, Y_2, \cdots, Y_n\}$ has exactly two connected components. Hence $n + 1 = 2, n = 1$, and X is isometric to the last case in the conclusion.

□

4.4 Einstein 4-manifolds with boundary

Our arguments for hyperkähler 4-manifolds can actually be generalized to Einstein 4-manifolds.

Theorem 4.4.1. *Let (M, g) be a compact, connected Einstein 4-manifold with boundary, $|Ric| \leq 3$. Suppose $\mathbb{RP}^3$ cannot be smoothly embedded in $M \backslash \partial M$, and for $k = 0, 1$,*

$$|S| \leq C, |\nabla^k_{\partial M} Rm_{\partial M}| \leq C, |\nabla^{k+1}_{\partial M} H| \leq C, H \geq H_0 > 0, inj_{\partial M} \geq i_0, vol_{\partial M}(\partial M) \leq C,$$

$\chi(M) \leq C$. Then there exists i_0' such that $i_b \geq i_0'$. If in addition $Ric \geq 0$, then one can replace $H \geq H_0 > 0$ by $H > 0$.

Proof. The Chern-Gauss-Bonnet formula together with equations (2.6) and (2.7) imply $\int_M |Rm|^2 \leq C$. Suppose the conclusion is not true, then we can find a sequence $(M_i, \tilde{g}_i)$ such that $i_{b,\tilde{g}_i} \to 0$. In particular, $i_{b,\tilde{g}_i} < -\frac{1}{2} \ln \left|\frac{H_0 - 3}{H_0 + 3}\right|$, hence by Proposition 2.1.8, there exists focal points $p_i \in M_i$ whose distance to ∂M_i is equal to $i_{b,\tilde{g}_i}$. Rescale the metric $g_i = i^{-2}_{b,\tilde{g}_i} \tilde{g}_i$, so that $i_{b,g_i} = 1$. From now on we are in the scale of g_i.

Let q_i'' be any point on ∂M_i. By Lemma 4.4.3 below and Corollary 2.2.15, we know $C(B_{\partial M_i}(q_i'', 1), 0, 0.9999)$ converges in $C^{2,\alpha}$ sense to the flat product metric on $B_1 \times [0, 0.9999)$.

It follows that $\mathrm{vol}(B(q_i', 0.01)) \geq 0.4\mathrm{vol}(B_{0.01})$ for any q_i' with $d(q_i', \partial M_i) \in [0.999, 1.001]$, where $B_{0.01}$ is the Euclidean ball of radius 0.01. Hence, by Proposition 4.4.4 we have $\sup_{B(p_i,0.002)} |Rm| \leq C$. Let q_i be a foot point of p_i. Then $C(B_{\partial M_i}(q_i, 0.0001), 0, 0.9999) \cup B(p_i, 0.001)$ converges in $C^{2,\alpha}$ sense to the a Ricci-flat metric with flat totally geodesic boundary, hence the metric must be flat by Lemma 4.1.6, which contradicts the fact that p_i is a focal point of ∂M_i.

□

Remark 4.4.2. The point here is in the scale of g_i, the volume of a metric ball near the focal point p_i have volume greater than Euclidean half ball of the same radius. Then nearly the focal point, the worst case is that we get a $\mathbb{R}^4/\mathbb{Z}_2$ singularity in the limit, which is avoided by the topological assumption.

The lemma presented below is analogous to Theorem 4.1.1, and the proof is very similar except for the step of ruling out the bubbles, as integral homology class of self-intersection -2 does not arise naturally in Einstein case.

Lemma 4.4.3. *Let (M_i, g_i) be a sequence of complete Einstein 4-manifold with compact boundary, such that $\mathbb{RP}^3$ cannot be smoothly embedded in $N_{i_0}(\partial M_i, g_i)$. Suppose $|Ric_{g_i}| \to 0$,*

$$|S_i| \to 0, inj_{\partial M_i} \to \infty, i_{b,g_i} \geq i_0, |\nabla^k_{\partial M_i} Rm_{\partial M_i}| \to 0, |\nabla^{k+1}_{\partial M_i} H_i|_{Lip(\partial M_i)} \to 0, k = 0, 1,$$

and

$$\int_{M_i} |Rm_{g_i}|^2 \leq C.$$

Then for any $r_1 < i_0$,

$$\sup_{N_{r_1}(\partial M, g_i)} |Rm_{g_i}| \to 0.$$

Proof. Without loss of generality, assume $i_0 = 1$ and $r_1 > 0.9999$. Denote $\alpha = r_1$, $\beta = \frac{1}{4}(1-\alpha)$. It suffices to show

$$\sup_{N_\alpha(\partial M, g_i)} |Rm_{g_i}| \leq C'. \tag{4.4}$$

This implies $\sup_{N_{\alpha-\beta}(\partial M_i, g_i)} |Rm_{g_i}| \to 0$ and one can take a larger r_1 to get the desired conclusion. In fact, suppose $\sup_{N_{\alpha-\beta}(\partial M_i, g_i)} |Rm_{g_i}|$ is achieved at $s_i \in M_i$ with $d_{g_i}(s_i, \partial M_i) \leq \alpha - \beta$, let $s_i' \in \partial M_i$ be the foot point of s_i, then by Corollary 2.2.15 and Lemma 4.1.6 we know $(B_{g_i}(s_i', \alpha - \frac{1}{2}\beta), g_i)$ converges in $C^{2,\alpha}$ sense to Euclidean $B^+_{\alpha-\frac{1}{2}\beta}$, so $\sup_{N_{\alpha-\beta}(\partial M_i, g_i)} |Rm_{g_i}| = |Rm_{g_i}(s_i)| \to 0$.

Now we argue by contradiction to prove (4.4). Suppose this is not true, then by passing to a subsequence we may assume

$$\sup_{N_\alpha(\partial M_i, g_i)} |Rm_{g_i}| \to \infty.$$

Let $p_i \in N_\alpha(\partial M_i, g_i)$ achieves this supremum, then $p_i \notin \partial M_i$.

Claim 1 There exists a subsequence such that

$$d_{g_i}(p_i, \partial M_i)^2 |Rm_{g_i}(p_i)| \to \infty. \tag{4.5}$$

If this is not true, we have $\sup_i d_{g_i}(p_i, \partial M_i)^2 |Rm_{g_i}|(p_i) < \infty$. Rescale $\tilde{g}_i = |Rm_{g_i}(p_i)| g_i$, then $|Rm_{\tilde{g}_i}(p_i)| = 1$, and $|Rm_{\tilde{g}_i}| \le 1$ in $N_{\alpha|Rm_{g_i}(p_i)|^{\frac{1}{2}}}(\partial M_i, \tilde{g}_i)$, and $\sup_i d_{\tilde{g}_i}(p_i, \partial M_i) < \infty$, $i_{b,\tilde{g}_i} \ge |Rm_{g_i}(p_i)|^{\frac{1}{2}}$ for all i. Hence (M_i, g_i, p_i) subconverges in pointed $C^{2,\alpha}$ sense to $(M_\infty, \tilde{g}_\infty, p_\infty)$, which is a complete Ricci-flat 4-manifold with flat, totally geodesic boundary, hence must be flat by Lemma 4.1.6. This contradicts that $|Rm_{g_\infty}(p_\infty)| = 1$ and proves Claim 1.

Now rescale g_i in another way, let $g_i' = d_{g_i}(p_i, \partial M_i)^{-2} g_i$, so $d_{g_i'}(p_i, \partial M_i) = 1$. Since $d_{g_i}(p_i, \partial M_i) \le \alpha$, the rescaled metric g_i' satisfies $i_{b,g_i'} \ge \alpha^{-1}$ as well as other conditions in this lemma. Moreover, (4.5) is equivalent to $|Rm_{g_i'}(p_i)| \to \infty$.

By the ϵ-regularity theorem of Cheeger-Tian [16], there exists a universal constant ϵ_0 such that for sufficiently large i,

$$\int_{B_{g_i'}(p_i, \beta)} |Rm_{g_i'}|^2 \ge \epsilon_0.$$

Claim 2 There exists a subsequence such that

$$\sup_{N_\alpha(\partial M_i, g_i')} |Rm_{g_i'}| \to \infty,$$

If not, we have $\sup_{N_\alpha(\partial M_i, g_i')} |Rm_{g_i'}| \le C$. Hence, we have $\sup_{N_{\alpha-\beta}(\partial M_i, g_i')} |Rm_{g_i'}| \to 0$, which follows from the argument presented at the beginning of the lemma.

By Bishop-Gromov volume comparison and note that $|Ric_{g_i'}| \to 0$, we have

$$\frac{\mathrm{vol}(B_{g_i'}(q_i, \beta))}{\mathrm{vol}(B_\beta)} \ge 0.9 \frac{\mathrm{vol}(B_{g_i'}(q_i, 0.1))}{\mathrm{vol}(B_{0.1})} \ge 0.36$$

for any q_i with $d_{g_i'}(q_i, \partial M_i) \in [1 - 5\beta, 1 + 5\beta]$. Note that $B_{g_i'}(p_i, \beta) \subset N_{\alpha^{-1}(\alpha+\beta)}(\partial M_i, g_i') \subset N_{\alpha+\beta}(\partial M_i, g_i)$. By Proposition 4.4.4, $|Rm_{g_i'}(p_i)|$ is bounded, which is a contradiction to Claim 1 and finishes the proof of Claim 2.

Now Claim 2 enables us to obtain, by induction, for each fixed positive integer N, N sequences of metrics $g_i^{(0)} = g_i$, $g_i^{(1)} = g_i', \ldots, g_i^{(N)}$, and points $p_i^{(j)} \in N_\alpha(\partial M_i, g_i^{(j)})$ for $0 \le j \le N-1$, $p_i^{(0)} = p_i, p_i^{(1)} = p_i'$, such that for $0 \le j \le N-1$, $p_i^{(j)}$ achieves the supremum of $|Rm_{g_i^{(j)}}|$ in $N_\alpha(\partial M_i, g_i^{(j)})$, and

$$|Rm_{g_i^{(j)}}(p_i^{(j)})| \to \infty,$$

$$g_i^{(j+1)} = d_{g_i^{(j)}}(p_i^{(j)}, \partial M_i)^{-2} g_i^{(j)},$$

$$d_{g_i^{(j+1)}}(p_i^{(j)}, \partial M_i) = 1,$$

$$\int_{B_{g_i^{(j+1)}}(p_i^{(j)},\beta)} |Rm_{g_i^{(j+1)}}|^2 \geq \epsilon_0,$$

$$B_{g_i^{(j+1)}}(p_i^{(j)}, \beta) \subset N_{\alpha^{-1}(\alpha+\beta)}(\partial M_i, g_i^{(j+1)}) \subset N_{\alpha+\beta}(\partial M_i, g_i^{(j)}),$$

$$B_{g_i^{(j+1)}}(p_i^{(j)}, \beta) \cap N_{\alpha+\beta}(\partial M_i, g_i^{(j+1)}) = \emptyset.$$

Consequently, for any fixed i, the union of the balls $\bigcup_{j=0}^{N-1} B_{g_i^{(j+1)}}(p_i^{(j)}, \beta)$ consists of non-overlapping balls. Additionally, since $\int_{M_i} |Rm_{g_i}|^2 \leq C$, we have $N\epsilon_0 \leq C$. However, this contradicts the fact that N can be any positive integer. □

The Proposition below is used to rule out certain bubbles in Einstein manifolds.

Proposition 4.4.4. *Let (M, g) be a Einstein 4-manifold with $|Ric| \leq 3$. Suppose $B(p, 5)$ has compact closure, $\mathbb{RP}^3$ cannot be smoothly embedded in $B(p, 5)$, and for any $q \in B(p, 2)$,*

$$vol(B(q, 1)) \geq (\frac{1}{3} + \delta) vol(B_1) \tag{4.6}$$

Then there exists a constant $C = C(\delta)$ such that

$$\sup_{B(p,1)} |Rm| \leq C.$$

Proof. Suppose the conclusion is not true, then we have a sequence (M_i, g_i, p_i) satisfies the conditions, but there exists $q_i' \in B(p_i, 1)$ with $|Rm_{g_i}(q_i')| \to \infty$. By the following point selection Lemma 4.4.5, we can find points $q_i \in B(p_i, 2)$ such that $|Rm_{g_i}(q_i)| \geq |Rm_{g_i}(q_i')|$, and

$$\sup_{B_{g_i}(q_i, |Rm_{g_i}(q_i')|^{\frac{1}{2}} |Rm_{g_i}(q_i)|^{-\frac{1}{2}})} |Rm_{g_i}| \leq 4|Rm_{g_i}(q_i)|.$$

Rescale the metric $\tilde{g}_i = |Rm_{g_i}(q_i)| g_i$. Then we have for large i,

$$\sup_{B_{\tilde{g}_i}(q_i, |Rm_{g_i}(q_i')|^{\frac{1}{2}})} |Rm_{\tilde{g}_i}| \leq 4,$$

$$|Rm_{\tilde{g}_i}(q_i)| = 1,$$

$$\mathrm{vol}_{\tilde{g}_i}(B_{\tilde{g}_i}(q_i, r)) \geq (\frac{1}{3} + \frac{\delta}{2}) \mathrm{vol}(B_r), \forall r \leq |Rm_{g_i}(q_i)|^{\frac{1}{2}},$$

Hence, for a subsequence, $(M_i, \tilde{g}_i, q_i)$ converges in pointed $C^{2,\alpha}$ topology to a complete non-flat Ricci-flat 4-manifold $(M_\infty, g_\infty, q_\infty)$ with maximum volume growth. By Cheeger-Naber [15] and Bando-Kasue-Nakajima [8], M_∞ is a Ricci-flat ALE space whose tangent cone at infinity is $\mathbb{R}^4/\mathbb{Z}_2$, hence $\mathbb{RP}^3$ can be smoothly embedded into $B_{g_i}(p_i, 5)$ for large i, which contradicts our assumption.

□

The following point selection lemma is well-known and elementary. The argument holds for any function in place of $|Rm|$.

Lemma 4.4.5. *Let (M, g) be a Riemannian manifold. Suppose $\sup_{B(p,2)} |Rm| < \infty$, $|Rm(p)| \neq 0$. Then there exists a point $q \in B(p, 2)$ such that $|Rm(q)| \geq |Rm(p)|$, and*

$$\sup_{B(q,|Rm(p)|^{\frac{1}{2}}|Rm(q)|^{-\frac{1}{2}})} |Rm| \leq 4|Rm(q)|.$$

Proof. If this is not true, let $A = |Rm(p)|^{\frac{1}{2}}$, then there exist $q_1 \in B(p, 2)$ such that $d(q_1, p) < 1$ and $|Rm(q_1)| > 4|Rm(p)| = 4A^2$. By induction, we can find a sequence of points $q_0 = p, q_1, q_2, \cdots$ with $d(q_{j+1}, q_j) < |Rm(q_j)|^{-\frac{1}{2}} A$, $d(q_{j+1}, p) < 2 - 2^{-j}$ and $|Rm(q_{j+1})| > 4|Rm(q_j)| \geq 4^{j+1}A^2$. This is an obvious contradiction since $\sup_{B(p,2)} |Rm| < \infty$. □

Condition (4.6) does not arise as natural for interior balls, as we cannot in general get a precise volume estimate. To rule out interior bubbles for Einstein manifolds, we need additional topological conditions, as indicated in [11].

Proposition 4.4.6. *Let (M, g) be an oriented Einstein 4-manifold with $|Ric| \leq 3$. Suppose $B(p, 5)$ has compact closure and embeds smoothly as an open subset of a homology 4-sphere, $H_2(M, \mathbb{R}) = 0$ and for any $q \in B(p, 2)$,*

$$vol(B(q, 1)) \geq v_0.$$

Then there exists a constant C such that

$$\sup_{B(p,1)} |Rm| \leq C.$$

Proof. This follows from the arguments in [11] and we refer there for details. If the curvature blows up, the same arguments as in Proposition 4.4.4 imply that there exists a oriented non-flat 4d Ricci-flat ALE space E that can be smoothly embedded in an open subset of $B(p, 5) \subset$ a homology 4-sphere, then a topological result of Crisp-Hillman [18] implies the boundary at infinity of E is diffeomorphic to S^3/Γ, where Γ is the group Q_8 or the perfect group. As the universal cover of E is also a Ricci-flat ALE space, we know E must have finite fundamental group, hence $H_1(E, \mathbb{R}) = 0$. By our assumption $H_2(M, \mathbb{R}) = 0$ and the Mayer-Vietoris sequence, we know $H_2(E, \mathbb{R}) = 0$. By a result of Shen-Sormani [44], we know $H_3(E, \mathbb{Z}) = 0$. Since E is an open 4-manifold, $H_4(E, \mathbb{R}) = 0$, so $\chi(E) = 1$. This contradicts the Chern-Gauss-Bonnet formula and the signature formula applied to E by computing the η-invariant of S^3/Γ when $\Gamma = Q_8$ or the perfect group. □

Now follow the proof of Theorem 4.1.1 we obtain the following:

Theorem 4.4.7. *Let $\mathcal{M}$ be the set of pointed compact Einstein 4-manifolds (M, g, p) with boundary such that $|Ric| \leq 3$, $\mathbb{RP}^3$ cannot be smoothly embedded in $M\backslash\partial M$, ∂M is diffeomorphic to S^3, for $k = 0, 1$,*

$$vol_{\partial M}(\partial M) \leq C, |S| \leq C, |\nabla^k_{\partial M} Rm_{\partial M}| \leq C, |\nabla^{k+1}_{\partial M} H| \leq C, inj_{\partial M} \geq i_0, H \geq H_0 > 0,$$

$$d(p, \partial M) \leq K, \chi(M) \leq C.$$

Then $\mathcal{M}$ is precompact in pointed Gromov-Hausdorff topology, and an element in $\partial\mathcal{M}$ is a complete $C^{2,\alpha}$ Einstein orbifold with smooth boundary, $\forall \alpha \in (0, 1)$, and the convergence to the limit is $C^{2,\alpha}$ away from finitely many orbifold points.

Proof. The Chern-Gauss-Bonnet formula imply $\int_M |Rm|^2 \leq C$. By Theorem 4.4.1, we know $i_b \geq i_0$. Hence $N_{\frac{2i_0}{3}}(\partial M, g)$ is diffeomorphic to $S^3 \times [0, 1] \subset S^4$. Following the proof of Theorem 4.1.1 we know that $\sup_{N_{\frac{i_0}{2}}(\partial M, g)} |Rm| \leq C$, where we use Proposition 4.4.6 to rule out Ricci-flat ALE bubbles. So by Proposition 2.1.15 and Bishop-Gromov volume comparisons, we know that for any fixed $R > 0$, $\mathrm{vol}(B(q, d(q, \partial M))) \geq v_R d(q, \partial M)^4$ for any $q \in M$ with $d(q, \partial M) \leq R$. Then the conclusion follows from Corollary 2.2.15 and [4, 8, 48]. □

Also, we are ready to prove the Theorem 1.0.5 listed in the introduction:

Proof. Topologically glue two identical copies of M along ∂M, we get a homology 4-sphere by Mayer-Vietoris sequence. Then follow the proof of Theorem 4.4.7 and use Proposition 4.4.6. □

Chapter 5

Convergence of triples

5.1 Moduli space formulation

Now we turn to the setting of Theorem 1.0.2. X will denote a compact oriented 4-manifold with boundary $\partial X = Y$. Let $\mathcal{N}^+$ be the set of closed framings $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ on Y that satisfies the "positive mean curvature" condition

$$\sum_{i=1}^{3} \langle \gamma_i, d(*_{\boldsymbol{\gamma}} \gamma_i) \rangle_{\boldsymbol{\gamma}} > 0. \tag{5.1}$$

Let $\mathcal{M}^+$ be the set of smooth hyperkähler triples $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ whose restriction to the boundary lies in $\mathcal{N}^+$, so we have a restriction map $p_0 : \mathcal{M}^+ \to \mathcal{N}^+$, which induces a map

$$p : \mathcal{M}^+/\mathcal{G}_X \to \mathcal{N}^+/\mathcal{G}_Y. \tag{5.2}$$

Here $\mathcal{M}^+$, $\mathcal{N}^+$ are equipped with Fréchet topology defined by smooth convergence, $\mathcal{G}_X, \mathcal{G}_Y$ are orientation preserving diffeomorphism groups of X, Y, respectively. It is obvious that p_0, p are continuous. Then the compactness part of Theorem 1.0.2 is equivalent to:

Theorem 5.1.1. *When there is no $C \in H_2(X, \mathbb{Z})$ with $C^2 = -2$, the map $p : \mathcal{M}^+/\mathcal{G}_X \to \mathcal{N}^+/\mathcal{G}_Y$ is proper.*

Proof. Suppose that for some $\phi_i \in \mathcal{G}_Y$, and $\boldsymbol{\gamma}_i \in \mathcal{N}^+$, we have $\phi_i^* \boldsymbol{\gamma}_i \to \boldsymbol{\gamma} \in \mathcal{N}^+$ and there exist $\boldsymbol{\omega}_i \in \mathcal{M}^+$ with $\boldsymbol{\omega}_i|_Y = \boldsymbol{\gamma}_i$. We want to show that there exists $\psi_i \in \mathcal{G}_X$ and $\boldsymbol{\omega} \in \mathcal{M}^+$ such that $\psi_i^* \boldsymbol{\omega}_i \to \boldsymbol{\omega}$. Let g_i be the Riemannian metric defined by $\boldsymbol{\omega}_i$. By the assumptions, we have a uniform positive lower bound for the mean curvature H_i of Y for the metric $g_i = g_{\boldsymbol{\omega}_i}$, and bounds for $|\nabla_Y^l S_i|$ for all $l \geq 0$. Moreover, $(Y, g_i|_Y)$ converges in the Cheeger-Gromov sense. Hence, (X, g_i) satisfies all conditions in Theorem 4.2.1. Then, for a subsequence, there exists a diffeomorphism $\psi_i : X \to X$ such that $\psi_i^* g_i \to g$ smoothly as tensors. One can assume that ψ_i is orientation-preserving; otherwise, for a subsequence, compose them with a fixed orientation-reversing diffeomorphism of X. Since $|\psi_i^* \boldsymbol{\omega}_i|^2 = 3$, $\psi_i^* \boldsymbol{\omega}_i$ is parallel. We conclude that for some subsequence, $\psi_i^* \boldsymbol{\omega}_i \to \boldsymbol{\omega}$ smoothly, and $\boldsymbol{\omega} \in \mathcal{M}^+$.

□

5.2 Enhancements

For the proof of the compactness part of Theorem 1.0.3, one only needs the following proposition in place of Proposition 4.1.5. Then we argue in the same way as the proof of Theorem 1.0.2.

Proposition 5.2.1. *Let $(M, g, \boldsymbol{\omega})$ be a hyperkähler 4-manifold. Suppose $B(p, 5)$ has compact closure, and for any homology class Σ of self intersection -2 in $B(p, 3)$,*

$$\Big|\int_\Sigma \boldsymbol{\omega}\Big| \geq a > 0,$$

$$\mathrm{vol}(B(p,1)) \geq v,$$

$$\int_{B(p,3)} |Rm|^2 \leq C.$$

Then there exists $C' > 0$, depending on a, v, C such that

$$\sup_{B(p,1)} |Rm| \leq C'.$$

Proof. For all $q \in B(p, 2)$, by volume comparison, we have

$$\mathrm{vol}(B(q,1)) \geq 3^{-4}\mathrm{vol}(B(q,3)) \geq 3^{-4}\mathrm{vol}(B(p,1)) \geq 3^{-4}v.$$

Suppose the conclusion is not true. Then we have a sequence (M_i, g_i, p_i) that satisfies the conditions, but there exists $q_i' \in B(p_i, 1)$ with $|Rm_{g_i}(q_i')| \to \infty$. By the point selection Lemma 4.4.5, we can find points $q_i \in B(p_i, 2)$ such that $|Rm_{g_i}(q_i)| \geq |Rm_{g_i}(q_i')|$, and

$$\sup_{B_{g_i}(q_i, |Rm_{g_i}(q_i')|^{\frac{1}{2}}|Rm_{g_i}(q_i)|^{-\frac{1}{2}})} |Rm_{g_i}| \leq 4|Rm_{g_i}(q_i)|.$$

We rescale the metric $\tilde{g}_i = |Rm_{g_i}(q_i)|g_i$ and $\tilde{\boldsymbol{\omega}}_i = |Rm_{g_i}(q_i)|\boldsymbol{\omega}_i$, so $\tilde{\boldsymbol{\omega}}_i$ defines $\tilde{g}_i$. Then we have

$$\sup_{B_{\tilde{g}_i}(q_i, |Rm_{g_i}(q_i')|^{\frac{1}{2}})} |Rm_{\tilde{g}_i}| \leq 4,$$

$$|Rm_{\tilde{g}_i}(q_i)| = 1,$$

$$\mathrm{vol}_{\tilde{g}_i}(B_{\tilde{g}_i}(q_i, r)) \geq 3^{-4}vr^n, \forall r \leq |Rm_{g_i}(q_i)|^{\frac{1}{2}},$$

and

$$\int_{B_{\tilde{g}_i}(q_i, |Rm_{g_i}(q_i)|^{\frac{1}{2}})} |Rm_{\tilde{g}_i}|^2 \leq \int_{B_{\tilde{g}_i}(p_i, 3|Rm_{g_i}(q_i)|^{\frac{1}{2}})} |Rm_{\tilde{g}_i}|^2 \leq C.$$

Hence, for a subsequence, $(M_i, \tilde{g}_i, q_i)$ converges in Cheeger-Gromov topology to a complete non-flat hyperkähler 4-manifold $(M_\infty, g_\infty, q_\infty)$ with maximum volume growth and

$$\int_{M_\infty} |Rm_{g_\infty}|^2 \leq C.$$

Since $\tilde{\boldsymbol{\omega}}_i$ are parallel, and $|\tilde{\boldsymbol{\omega}}_i|^2 = 6$, $\tilde{\boldsymbol{\omega}}_i$ also subconverges to a hyperkähler triple $\boldsymbol{\omega}_\infty$ that defines g_∞. By [8], (M_∞, g_∞) is a hyperkähler ALE space of order 4. Hence, from Kronheimer's classification [37, 36], there exists a smooth 2-sphere C_∞ of self intersection -2 in $B_{g_\infty}(q_\infty, R)$ for some $R > 0$. Let $\phi_i : B_{g_\infty}(q_\infty, R) \to V_i \subset M_i$ be diffeomorphisms, such that $\phi_i^* \tilde{\boldsymbol{\omega}}_i \to \boldsymbol{\omega}_\infty$ smoothly. Let $C_i = (\phi_i)_* C_\infty$, then C_i is a homology class of self-intersection -2 in $B_{g_i}(p_i, 3)$ for large i and

$$\int_{C_i} \boldsymbol{\omega}_i = |Rm_{g_i}(q_i)|^{-1} \int_{C_i} \tilde{\boldsymbol{\omega}}_i = |Rm_{g_i}(q_i)|^{-1} \int_{C_\infty} \phi_i^* \tilde{\boldsymbol{\omega}}_i \to 0,$$

which contradicts our assumption.

□

5.3 Uniqueness

It is natural to ask whether a subsequential limit $\boldsymbol{\omega}$ is unique(up to a diffeomorphism) in Theorem 1.0.2, Theorem 1.0.3. The answer is yes. In fact, both [9] and [5] proved a unique continuation theorem for Einstein metrics with prescribed boundary metric and second fundamental form, which implied uniqueness in our case.

Biquard proved

Theorem 5.3.1. *[9] Let M be a compact smooth manifold with boundary. Let g, h be two smooth Riemannian metrics on M such that $g|_{\partial M} = h|_{\partial M}$ and $II_g = II_h$ on ∂M, then $g_{\exp^\perp_{g_0}(x,t)} = h_{\exp^\perp(x,t)}$ as pointwise inner products for $(x, t) \in \partial M \times [0, \min\{i_{b,g}, i_{b,h}\})$.*

Proposition 5.3.2. *Let X be a connected oriented 4-manifold with boundary. Suppose $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are two smooth hyperkähler triples on X. If $\boldsymbol{\omega}_1|_{\partial X} = \boldsymbol{\omega}_2|_{\partial X}$, then in the geodesic gauges of $g_{\boldsymbol{\omega}_1}$ and $g_{\boldsymbol{\omega}_2}$, we have $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2$ near ∂X.*

Proof. We have $g_{\boldsymbol{\omega}_1}|_{\partial X} = g_{\boldsymbol{\omega}_2}|_{\partial X}$ and $II_{g_{\boldsymbol{\omega}_1}} = II_{g_{\boldsymbol{\omega}_2}}$ on ∂X. By Theorem 5.3.1, in the geodesic gauges provided by $\exp^\perp$, we have $g_{\boldsymbol{\omega}_1} = g_{\boldsymbol{\omega}_2} := g$. Since $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are parallel, we have $\nabla^g |\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2|_g^2 = 0$, since $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2$ on ∂X. Therefore, $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2$ everywhere near ∂X. □

Now the following global uniqueness result follows from an analytic continuation argument (See [33] Chapter VI, Section 6):

Theorem 5.3.3. *Let X be a connected 4-manifold with boundary such that $\pi_1(X, \partial X) = 0$. Suppose $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$ are two smooth hyperkähler triples on X, and $\phi_0 : \partial X \to \partial X$ is a diffeomorphism, such that $\boldsymbol{\omega}_1|_{\partial X} = \phi_0^*(\boldsymbol{\omega}_2|_{\partial X})$, then there exists a diffeomorphism $\phi : X \to X$, $\phi|_{\partial X} = \phi_0$ such that $\boldsymbol{\omega}_1 = \phi^* \boldsymbol{\omega}_2$ on X.*

Remark 5.3.4. This theorem implies that the map $p : \mathcal{M}^+/\mathcal{G}_X \to \mathcal{N}^+/\mathcal{G}_Y$ defined in section 5 is injective, provided that $\mathcal{M}^+$ is nonempty, so one must have $\pi_1(X, \partial X) = 0$ by Proposition 2.1.4.

Proof. By the Proposition 5.3.2, there exists a collar neighborhood U of ∂X and a diffeomorphism $\phi_1 : U \to V \subset X$ such that $g_{\boldsymbol{\omega}_1} = \phi_1^* g_{\boldsymbol{\omega}_2}$, $\phi_1|_{\partial X} = \phi_0$. Since $g_{\boldsymbol{\omega}_1}, g_{\boldsymbol{\omega}_2}$ are real analytic, ϕ_1 is real analytic. Fix $p_0 \in U$ and a small neighborhood U_0 of p_0 in X. For any $p \in X \backslash \partial X$, choose a path $x(t), 0 \le t \le 1$ such that $x(0) = p_0, x(1) = p, x(t) \in X\backslash\partial X$, then an analytic continuation of the isometry $\phi_1|_{U_0}$ along $x(t)$ gives rise to an isometry defined near p. We claim that if we have two paths and two analytic continuations, then they define the same germ at p. In fact, one only needs to show that for the closed path $y_0(t)$ formed by concatenating these two paths, the isometry near p_0 given by an analytic continuation of $\phi_1|_{U_0}$ along $y_0(t)$ has the same germ as $\phi_1|_{U_0}$ at p_0. Since $\pi_1(X, \partial X) = 0$, $y_0(t)$ can be homotoped to a path $y_1(t)$ contained in U via paths $y_s(t)$ in $X\backslash\partial X$, such that $y_s(0) = y_s(1) = p_0$. Since $\phi_1 : U \to V$ is a globally defined isometry, by uniqueness of analytic continuation, we know that any analytic continuation of $\phi_1|_{U_0}$ along $y_1(t)$ must coincide with ϕ_1, which finishes the proof of the claim by invariance of analytic continuation via homotopy. This shows that $\phi_1 : U \to V$ can be extended to a global isometry $\phi : X \to X$ by analytic continuation. Hence $\boldsymbol{\omega}_1 = \phi^* \boldsymbol{\omega}_2$. □

Given the compactness result Theorem 5.1.1, Theorem 5.3.3, together with the theorem of Ebin-Palais on properness of diffeomorphism group action on the space of Riemannian metrics on a closed manifold (See [23]), one can complete the proof of Theorem 1.0.2, Theorem 1.0.3 as follows:

Suppose we have two sequences of diffeomorphism ϕ_i, ψ_i of X such that $\phi_i^* \boldsymbol{\omega}_i \to \boldsymbol{\omega}$, $\psi_i^* \boldsymbol{\omega}_i \to \boldsymbol{\omega}'$, then $(\phi_i|_{\partial X})^* \boldsymbol{\gamma}_i \to \boldsymbol{\omega}|_{\partial X}$, $(\psi_i|_{\partial X})^* \boldsymbol{\gamma}_i \to \boldsymbol{\omega}'|_{\partial X}$, where $\boldsymbol{\gamma}_i = \boldsymbol{\omega}_i|_{\partial X}$. Since $\boldsymbol{\gamma}_i$ converges to $\boldsymbol{\gamma}$ in Cheeger-Gromov sense, there exists diffeomorphisms $u_i : \partial X \to \partial X$ such that $u_i^* \boldsymbol{\gamma}_i \to \boldsymbol{\gamma}$. By the theorem of Ebin-Palais, we have for a subsequence $(\phi_i|_{\partial X})^{-1} \circ u_i, (\psi_i|_{\partial X})^{-1} \circ u_i$ converge to some diffeomorphisms u, u' on ∂X, respectively(because their inverses converge). Hence we also have $u_i^* \boldsymbol{\gamma}_i \to u^* \boldsymbol{\omega}|_{\partial X}$, $u_i^* \boldsymbol{\gamma}_i \to (u')^* \boldsymbol{\omega}'|_{\partial X}$, so $\boldsymbol{\omega}|_{\partial X} = (u' \circ u^{-1})^* \boldsymbol{\omega}'|_{\partial X}$. Note that the positive mean curvature condition implies that $\pi_1(X, \partial X) = 0$ (See Proposition 2.1.4), then by Theorem 5.3.3, there exists a diffeomorphism φ on X with $\boldsymbol{\omega}' = \varphi^* \boldsymbol{\omega}$.

5.4 Torsion-free hypersymplectic manifolds with boundary

5.4.1 Preliminaries

Recall that a hypersymplectic triple $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ on an oriented 4-manifold X with boundary is a definite triple of symplectic forms. Write

$$\omega_i \wedge \omega_j = 2Q_{ij}\mu.$$

Denote $\boldsymbol{Q} = (Q_{ij})$, $\boldsymbol{Q}^{-1} = (Q^{ij})$, and $g = g_{\boldsymbol{\omega}}$ the Riemannian metric. Recall that $\boldsymbol{\omega}$ is called torsion-free if for each i,

$$d(Q^{ij}\omega_j) = 0, \tag{5.3}$$

which is equivalent to

$$dQ^{ij} \wedge \omega_j = 0. \tag{5.4}$$

Let us begin with some arguments and results in [25]. Let $Sym_{>0}$ denotes the set of symmetric positive-definite 3 by 3 matrices. There are two Riemannian metrics on $Sym_{>0}$: the first one is the Euclidean metric

$$\langle A, B\rangle = Tr(AB), \tag{5.5}$$

and the second one is the symmetric space metric, which has non-positive sectional curvature

$$\langle A, B\rangle_Q = Tr(Q^{-1}AQ^{-1}B) \tag{5.6}$$

at each point $Q \in Sym_{>0}$.

Then $\boldsymbol{Q}$ can be regarded as a map $\boldsymbol{Q} : X \to Sym_{>0}$. Let $\Delta\boldsymbol{Q}, \hat{\Delta}\boldsymbol{Q}$ denote the harmonic Laplacian of this map with respect to (5.5),(5.6), respectively. Explicitly, their components are related by

$$(\hat{\Delta}\boldsymbol{Q})_{ij} = \Delta Q_{ij} - Q^{km}\langle dQ_{ik}, dQ_{mj}\rangle, \tag{5.7}$$

For a hypersymplectic triple $\boldsymbol{\omega}$, the calculations in [26] showed that the torsion-free condition is equivalent to

$$\hat{\Delta}\boldsymbol{Q} = 0, \ \text{Ric} = \frac{1}{4}\langle d\boldsymbol{Q} \otimes d\boldsymbol{Q}\rangle_{\boldsymbol{Q}}, \tag{5.8}$$

where $\langle d\boldsymbol{Q} \otimes d\boldsymbol{Q}\rangle_{\boldsymbol{Q}}(u, v) = \langle \nabla_u\boldsymbol{Q}, \nabla_v\boldsymbol{Q}\rangle_{\boldsymbol{Q}}$. Hence if $\boldsymbol{\omega}$ is torsion-free, then $\boldsymbol{Q}$ is a harmonic map with respect to (5.6) and $\text{Ric} \geq 0$. Then the scalar curvature R of g is

$$R = \frac{1}{4}|d\boldsymbol{Q}|^2_{\boldsymbol{Q}} \geq 0,$$

which is a multiple of the energy density of the harmonic map $\boldsymbol{Q}$. Take the trace of (5.7), we get

$$\Delta Tr\boldsymbol{Q} = Q^{pq}\langle dQ_{kp}, dQ_{qk}\rangle \geq 0. \tag{5.9}$$

Moreover, [25] showed that the function R satisfies the inequality

$$R\Delta R \geq \frac{1}{2}|\nabla R|^2 + \frac{1}{2}R^3, \tag{5.10}$$

hence a contradiction argument implies that everywhere

$$\Delta R \geq 0. \tag{5.11}$$

Then they used standard geometric analysis arguments for inequality (5.10) and the fact $\mathrm{Ric} \geq 0$ to conclude

Theorem 5.4.1. *[25] Suppose $(X, g, \boldsymbol{\omega})$ is a torsion-free hypersymplectic manifold, $B(p, r) \subset X$ has compact closure, and $\partial B(p, r) \neq \emptyset$, then $R(p) \leq \frac{32}{r^2}$. In particular, if g is complete, then there exists a constant matrix $B \in SL(3, \mathbb{R})$ such that $\boldsymbol{\omega} B$ is a hyperkähler triple.*

5.4.2 Compactness for the boundary value problem

Now suppose X is an oriented 4-manifold with compact boundary $Y = \partial X$, then through the boundary exponential map, a neighborhood U of Y is diffeomorphic to $Y \times [0, a)$. Let t denote the distance function $d(\cdot, Y)$, i.e., the projection $Y \times [0, a) \to [0, a)$, then in U, $\boldsymbol{\omega}$ can be written as

$$\boldsymbol{\omega} = -dt \wedge *_{Y_t} \boldsymbol{\gamma}_t + \boldsymbol{\gamma}_t.$$

where $Y_t = Y \times \{t\}$ and $*_{Y_t}$ is the Hodge star operator of $g|_{Y_t}$. $\boldsymbol{\omega}$ being closed is equivalent to

$$d_{Y_t} \boldsymbol{\gamma}_t = 0, \tag{5.12}$$

$$\frac{\partial \boldsymbol{\gamma}_t}{\partial t} = -d_{Y_t}(*_{Y_t} \boldsymbol{\gamma}_t). \tag{5.13}$$

(5.4) is equivalent to

$$d_{Y_t} Q^{ij} \wedge *_{Y_t} \gamma_{t,j} + \frac{\partial Q^{ij}}{\partial t} \gamma_{t,j} = 0, \tag{5.14}$$

$$d_{Y_t} Q^{ij} \wedge \gamma_{t,j} = 0. \tag{5.15}$$

Note that (5.14) is equivalent to

$$\frac{\partial Q^{ij}}{\partial t} = \frac{d_{Y_t} Q^{ik} \wedge \eta_{t,j} \wedge *_{Y_t} \gamma_k}{\mathrm{vol}_t}, \tag{5.16}$$

where $\mathrm{vol}_t = \eta_{t,1} \wedge \eta_{t,2} \wedge \eta_{t,3}$ and equals to the Riemannian volume form of $g|_{Y_t}$. From calculations in Lemma 3.2.1 and (5.13), (5.16) , it is easy to see that the second fundamental form $II(e_i, e_j)$ is in algebraic terms of $\boldsymbol{\eta}, d_{\partial X} \boldsymbol{\eta}, \boldsymbol{Q}, d_{\partial X} \boldsymbol{Q}$.

Now let us try to prove Theorem 1.0.4, starting with basic observations. In Theorem 1.0.4, suppose $\boldsymbol{\omega}_i|_{\partial X}, \boldsymbol{Q}_i$ converge in Cheeger-Gromov sense to the limit, then we have

$\mathrm{diam}(\partial X, g_i|_{\partial X}) \leq C, \mathrm{vol}_{g_i}(\partial X) \leq C, inj_{\partial X, g_i|_{\partial X}} \geq i_0$, $|\nabla^j_{\partial X} Rm_{\partial X, g_i}| \leq C_j$, $|\nabla^j_{\partial X} S_i| \leq C_j$. By (5.11) and maximum principle, R_i is uniformly bounded on X, so $|\mathrm{Ric}_{g_i}|$ is uniformly bounded on X since Ric_{g_i} is non-negative and its trace is uniformly bounded. Due to the uniform mean positive curvature condition, and $\mathrm{Ric}_{g_i} \geq 0$, we have an upper bound of $\sup_{p\in X} d_{g_i}(p, \partial X)$, $\mathrm{vol}_{g_i}(X)$ by Proposition 2.1.4, and in particular an upper bound of $\mathrm{diam}(X, g_i)$. The Chern-Gauss-Bonnet formula thus gives an upper bound of $\int_X |Rm_{g_i}|^2$. Also, by (5.9) and maximum principle, $Tr\boldsymbol{Q}_i$ is uniformly bounded on X, so $\boldsymbol{Q}_i$ is uniformly bounded on X and then $|d\boldsymbol{Q}_i|$ is uniformly bounded on X.

Given these conclusions, we need to verify that all propositions that were used to prove Theorem 1.0.3 adapt to the torsion-free hypersymplectic setting. Firstly, we digress to discuss elliptic regularity for torsion-free equations (5.8). In harmonic coordinates in B_2 or B_2^+, view g as a 4 by 4 matrix of functions, then we have a system of PDEs in g, Q:

$$\Delta_g Q_{kl} - Q^{pq}\langle dQ_{kp}, dQ_{ql}\rangle_g = 0, \tag{5.17}$$

$$\Delta_g g_{ij} + B_{ij}(g, \partial g) = -\frac{1}{2} Q^{ab}Q^{cd}\partial_i Q_{bc}\partial_j Q_{da}. \tag{5.18}$$

From this, by a boostrapping argument, one sees interior regularity: fix $\beta \in (0,1)$. If $\boldsymbol{Q}$ is C^1 bounded, g is $C^{1,\beta}$ bounded, and they are uniform positive on B_2, then all derivatives of $\boldsymbol{Q}$ and g are bounded. For boundary regularity, there is no techinical difficulty to get the following version from Neumann boundary conditions (2.33),(2.34): if all tangential derivatives of $\boldsymbol{Q}$, g, H are bounded on $\tilde{B}_2$, and $\boldsymbol{Q}$ is C^1 bounded, g is $C^{1,\beta}$ bounded, and they are uniformly positive on B_2^+, then all derivatives of $\boldsymbol{Q}$ and g are bounded in B_1^+. If in both cases, we also assume g is $C^{k,\beta}$ close to identity in B_2 or B_2^+, then similarly, g is $C^{k,\alpha}$ close to identity for any $\alpha \in (\beta, 1)$.

Following the proof of Theorem 2.2.11, we get the following two propositions.

Proposition 5.4.2. *Let $(X, g, \boldsymbol{\omega})$ be a torsion-free hypersymplectic manifold and $B(p, r)$ is a metric ball that has compact closure, $\partial B(p,r) \neq \emptyset$. Suppose for any $q \in B(p, r)$,*

$$inj_q \geq cd(q, \partial B(p, r)),$$

$$Tr\boldsymbol{Q}, R \leq C.$$

Fix $\Lambda > 1, 0 < \alpha < 1$, then for any $k \geq 0$, $q \in B(p, r)$,

$$r_h^{k,\alpha}(q, g, \Lambda) \geq C'_k d(q, \partial B(p, r)).$$

In particular, $|\nabla^k \boldsymbol{Q}| \leq C''_k$ in $B(p, \frac{r}{2})$.

Proposition 5.4.3. *Let $(X, g, \boldsymbol{\omega})$ be a compact torsion-free hypersymplectic manifold with boundary. Suppose*

$$i_b \geq i_0, inj_X \geq i_0, inj_{\partial X} \geq i_0,$$

On ∂X we have

$$Tr\boldsymbol{Q}, R \leq C, |\nabla^j_{\partial X} Rm_{\partial X}| \leq C_j, |\nabla^k_{\partial X} S| \leq C_j, |\nabla^j_{\partial X}\boldsymbol{Q}| \leq C_j, \forall j \geq 0.$$

Fix $\Lambda > 1, 0 < \alpha < 1$, then for any $k \geq 0$, $q \in X$,

$$r_h^{k,\alpha}(q, g, \Lambda) \geq C_k'''.$$

From this we conclude Proposition 5.2.1 holds for torsion-free hypersymplectic manifolds $(X, g, \boldsymbol{\omega})$, provided an upper bound of $Tr\boldsymbol{Q}$ in $B(p, 5)$.

Proposition 5.4.4. *Let $(M, g, \boldsymbol{\omega})$ be a torsion-free hypersymplectic manifold. Suppose $B(p, 5)$ has compact closure, and for any homology class Σ of self intersection -2 in $B(p, 3)$,*

$$\Big| \int_\Sigma \boldsymbol{\omega} \Big| \geq a > 0,$$

$$\mathrm{vol}(B(p,1)) \geq v,$$

$$\int_{B(p,3)} |Rm|^2 \leq C_1,$$

$$\sup_{B(p,5)} Tr\boldsymbol{Q} \leq C_2.$$

Then there exists $C' > 0$, depending on a, v, C_1, C_2 such that

$$\sup_{B(p,1)} |Rm| \leq C'.$$

Proof. The proof is almost the same as there. Let us list the ingredients here:

- We have Bishop-Gromov volume comparison, since $\mathrm{Ric}_{g_i} \geq 0$,
- Before rescaling, $R_i, |\mathrm{Ric}_{g_i}|$ are automatically bounded by Theorem 5.4.1.
- For the rescaled metric $\tilde{g}_i$, the curvature bound and the volume non-collapsing condition imply injectivity radius lower bound on compact sets, hence by Proposition 5.4.2, we have harmonic radius lower bounds as well as bounds for derivatives of $\boldsymbol{Q}_i$ on compact sets, so we have pointed Cheeger-Gromov convergence of a subsequence $(M_i, g_i, \boldsymbol{\omega}_i, q_i)$.
- The limit $\boldsymbol{\omega}_\infty$ is a hyperkähler triple up to a constant $SL(3, \mathbb{R})$ rotation, because g_∞ is scalar flat, or because of Theorem 5.4.1.

So, we get the contradiction in the same way as in Proposition 5.2.1. □

With the above three Propositions as tools and ϵ-regularity, argue the same way as in Theorem 4.1.1, one gets an analogous version of Theorem 4.1.1, i.e., curvature control within i_b, assuming $i_b \geq i_0$.

Now one can finish the proof of compactness part of Theorem 1.0.4 by the same arguments as the proof of 1.0.3 . Note that $\mathrm{Ric} \geq 0, H > 0$ is enough for the focal point argument.

Remark 5.4.5. We make a remark about the proof of ϵ-regularity for torsion-free hypersymplectic manifolds here. Firstly, the proof in [46] Theorem 3.21 directly applies to this case by using Theorem 5.4.1. Alternatively, we can apply Remark 8.22 in [16] to conclude that g has $C^{1,\alpha}$ bounded covering geometry when curvature L^2 norm is small. By (5.8), we have $|\nabla \mathrm{Ric}| \leq C$, hence g has $C^{2,\alpha}$ bounded covering geometry and $|Rm|$ is bounded.

For the convenience of the reader, we provide some details based on Sun-Zhang's proof of ϵ-regularity [46] on hyperkähler 4-manifolds.

Lemma 5.4.6. *(ϵ-regularity) Given $K > 0$, there exists constants ϵ_0, C depending on K such that if $(M, g, \boldsymbol{\omega})$ is a torsion-free hypersymplectic manifold and $B(p, 5)$ has compact closure,*

$$\sup_{B(p,5)} R \leq K,$$

$$\int_{B(p,5)} |Rm|^2 < \epsilon_0,$$

then

$$\sup_{B(p,1)} |Rm| \leq C.$$

Proof. Suppose this is not true, then we can find a sequence $(M_i, g_i, \boldsymbol{\omega}_i, p_i)$ with

$$\sup_{B_{g_i}(p_i,5)} R_{g_i} \leq K,$$

$$\int_{B_{g_i}(p_i,5)} |Rm_{g_i}|^2 \to 0,$$

but

$$\sup_{B_{g_i}(p_i,1)} |Rm_{g_i}| \to \infty.$$

Denote $A_i = \sup_{B_{g_i}(p_i,1)} |Rm_{g_i}|^{\frac{1}{2}}$ and suppose this supremum is achieved at p_i'. By the point selection Lemma 4.4.5 we can find points $q_i \in B_{g_i}(p_i', 2)$ such that $|Rm_{g_i}(q_i)| \geq A_i^2 \to \infty$ and

$$\sup_{B_{g_i}(q_i, A_i|Rm_{g_i}(q_i)|^{-\frac{1}{2}})} |Rm_{g_i}| \leq 4|Rm_{g_i}(q_i)|.$$

Then rescale the metric $\tilde{g}_i = |Rm_{g_i}(q_i)| g_i$ we have

$$|Rm_{\tilde{g}_i}(q_i)| = 1,$$

$$\sup_{B_{\tilde{g}_i}(q_i, A_i)} |Rm_{\tilde{g}_i}| \leq 4,$$

$$\sup_{B_{\tilde{g}_i}(q_i, A_i)} R_{\tilde{g}_i} \to 0,$$

$$\int_{B_{\tilde{g}_i}(q_i, A_i)} |Rm_{\tilde{g}_i}|^2 \to 0.$$

For a subsequence one can take pointed Gromov-Hausdorff limit of (M_i, g_i, q_i) to a complete metric space $(X_\infty, d_\infty, q_\infty)$.

If $\dim X_\infty = 4$, the sequence is volume non-collapsing and the limit $(X_\infty, d_\infty, q_\infty)$ is a complete hyperkähler 4-manifold with $|Rm_{\tilde{g}_i}(q_i)| = 1$ and $\int_{X_\infty} |Rm_{\tilde{g}_i}|^2 = 0$, which is a contradiction.

If $\dim X_\infty = 1, 2,$ or 3, then the sequence is collapsing with bounded curvature and Cheeger-Fukaya-Gromov theory [13] together with its improvement for hyperkähler 4-manifolds developed in Section 3 [46] applies. It is easy to see that Proposition 3.1 in [46] is still true, since we only need the limit geometry of the local universal cover to be hyperkähler. Therefore, X_∞ is a smooth, complete Riemannian manifold endowed with some additional global structure. Certain types of Liouville theorems imply that X_∞ is flat. Moreover, the explicit expression for hyperkähler triples with nilpotent symmetry implies that the limit geometry of the local universal cover is flat, which contradicts $|Rm_{\tilde{g}_i}(q_i)| = 1$.

□

5.4.3 Uniqueness

Finally, we prove the uniqueness part of Theorem 1.0.4.

Proposition 5.4.7. *Let $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$ be two torsion-free hypersymplectic triples on an oriented 4-manifold X with compact boundary. Suppose $\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_2, \boldsymbol{Q}_1 = \boldsymbol{Q}_2$ on ∂X, then $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2$ in geodesic gauges of $g_{\boldsymbol{\omega}_i}$ near ∂X.*

Proof. $\boldsymbol{\omega}_i$ defines a torsion-free G_2 structure ϕ_i on $X \times T^3$ via (1.1), which defines a warpped product metric

$$g_{\phi_i} = g_{\boldsymbol{\omega}_i} + Q_{ij} dt^i dt^j. \tag{5.19}$$

In the geodesic gauge of $g_{\boldsymbol{\omega}_i}$, write

$$\boldsymbol{\omega}_i = -dt \wedge *_{Y_t} \boldsymbol{\gamma}_t + \boldsymbol{\gamma}_t,$$

where t is the distance function $d_{g_{\boldsymbol{\omega}_i}}(\cdot, \partial X)$, Y_t is the level set of t. Hence

$$\phi_i = -dt \wedge \theta_{t,i} + \rho_{t,i}, \tag{5.20}$$

where

$$\rho_{t,i} = dt^1 \wedge dt^2 \wedge dt^3 - \gamma_i^1 \wedge dt^1 - \gamma_i^2 \wedge dt^2 - \gamma_i^3 \wedge dt^3,$$

$$\theta_{t,i} = - *_{Y_t} \gamma_{t,i}^1 \wedge dt^1 - *_{Y_t}\gamma_{t,i}^2 \wedge dt^2 - *_{Y_t}\gamma_{t,i}^3 \wedge dt^3.$$

By (5.19), t can also be viewed as $d_{g_{\phi_i}}(\cdot, \partial X \times T^3)$, so (5.20) is written in the geodesic gauge of g_{ϕ_i}. By the calculations in [20] Section 2.2, for $i = 1, 2$, both $g_{\phi_i}|_{\partial X \times T^3}$ and the second fundamental forms of $\partial X \times T^3$ are equal to each other, since they are explicitly in terms of $\theta_{0.i}, \rho_{0,i}$, which are in terms of $\boldsymbol{\gamma}_i, \boldsymbol{Q}_i$. Since g_{ϕ_i} are Ricci-flat, by [9] Theorem 4, $g_{\phi_1} = g_{\phi_2} := g$. Since $\nabla^g|\phi_1 - \phi_2|^2 = 0$ and $\phi_1 - \phi_2 = 0$ at one point, we have $\phi_1 = \phi_2$, $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2$.

□

Note that for a torsion-free hypersymplectic triple $\boldsymbol{\omega}$, the metric $g_{\boldsymbol{\omega}}$ is real analytic with respect to the analytic structure defined by harmonic coordinates, due to elliptic regularity of (5.17)(5.18), so the arguments in Section 5.3 shows global uniqueness:

Theorem 5.4.8. *Let X be a connected 4-manifold with boundary such that $\pi_1(X, \partial X) = 0$. Suppose $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$ are two smooth torsion-free hypersymplectic triples on X, and $\varphi_0 : \partial X \to \partial X$ is a diffeomorphism, such that $\boldsymbol{\omega}_1|_{\partial X} = \varphi_0^*(\boldsymbol{\omega}_2|_{\partial X})$, $\boldsymbol{Q}_1|_{\partial X} = \varphi_0^*\boldsymbol{Q}_2|_{\partial X}$, then there exists a diffeomorphism $\varphi : X \to X$, $\varphi|_{\partial X} = \varphi_0$, such that $\boldsymbol{\omega}_1 = \varphi^*\boldsymbol{\omega}_2$ on X.*

Chapter 6

Period map of the K3 manifold

6.1 Surjectivity

In this chapter, we give a simple proof of the Todorov's surjectivity result [49] on the period map of K3 surfaces in a differential geometric setting. Our proof makes use of collasping geometry of hyperkähler 4-manifolds developped by Sun-Zhang in [46], and does not rely on the solution to the Calabi conjecture.

On an oriented smooth 4-manifold, a hyperkähler metric g is a Riemannian metric with holonomy contained in $SU(2)$. This is equivalent to saying that the bundle of self-dual forms is flat and trivial, which implies that there is a triple $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ of closed 2-forms satisfying $\omega_\alpha \wedge \omega_\beta = 2\delta_{\alpha\beta}\mathrm{dVol}_g$. Such a triple is often referred to as a hyperkähler triple.

Let $\mathcal{K}$ be the K3 manifold, which is by definition the unique oriented smooth 4-manifold underlying a complex K3 surface. It is simply connected and the intersection form on $\Lambda \equiv H^2(\mathcal{K};\mathbb{Z})$ has signature $(3, 19)$. Denote by $\mathcal{N}$ the set of all hyperkähler metrics on $\mathcal{K}$ with diameter 1. Given $g \in \mathcal{N}$, the space $\mathbb{H}^+_g$ of self-dual harmonic 2-forms with respect to g is a 3-dimensional subspace in $\Lambda_{\mathbb{R}} \equiv H^2(\mathcal{K};\mathbb{R})$ which is positive definite with respect to the intersection form. Indeed, any choice of a hyperkähler triple gives rise to a basis of $\mathbb{H}^+_g$, and they are up to a constant $O(3)$ rotation.

Define the positive Grassmannian Gr^+ to be the space of all 3-dimensional positive definite subspaces of $\Lambda_{\mathbb{R}}$. It is an open subset in the standard Grassmannian $Gr(3, \Lambda_{\mathbb{R}})$. We define the period map

$$\mathcal{P} : \mathcal{N} \to Gr^+; g \mapsto \mathbb{H}^+_g.$$

The diffeomorphism group $\mathbf{Diff}(\mathcal{K})$ acts on $\mathcal{N}$ by $\varphi.g = \varphi^* g$, which induces a homomorphism $\Phi : \mathbf{Diff}(\mathcal{K}) \to \mathrm{Aut}(\Lambda)$, where $\Gamma := \mathrm{Aut}(\Lambda)$ is the automorphism group of the lattice Λ preserving the intersection form. There is a natural action of Γ on Gr^+, hence $\mathcal{P}$ induces a map

$$\underline{\mathcal{P}} : \mathcal{M} = \mathcal{N}/\mathbf{Diff}(\mathcal{K}) \to \mathcal{D} \equiv Gr^+/\Gamma. \tag{6.1}$$

The left-hand side is the set of isometry classes of hyperkähler metrics on $\mathcal{K}$. It is endowed with a natural Cheeger-Gromov topology. A sequence $[g_j]$ converges to $[g_\infty]$ if there are

$\varphi_j \in \mathbf{Diff}(\mathcal{K})$ such that $\varphi_j^* g_j$ converges smoothly to g_∞. Since hyperkähler metrics are Ricci-flat, it follows from the Cheeger-Colding theory that this topology coincides with the Gromov-Hausdorff topology. We also endow the period domain $\mathcal{D}$ with the quotient topology. One can check that $\underline{\mathcal{P}}$ is continuous, and the Γ action on Gr^+ is properly discontinous.

For any homology class $\delta \in H_2(\mathcal{K}; \mathbb{Z}) \cong H^2(\mathcal{K}; \mathbb{Z}) = \Lambda$ with $\delta.\delta = -2$, we define $\xi^\perp$ to be subspace in Gr^+ consisting of hyperkähler metrics g such that $\int_\delta \xi = 0$ for all $\xi \in \mathbb{H}_g^+$. We denote

$$Gr^{+,\circ} = Gr^+ \setminus \bigcup_{\delta \in \Lambda, \delta.\delta = -2} \delta^\perp$$

and $\mathcal{D}^\circ = Gr^{+,\circ}/\Gamma$.

Theorem 6.1.1. *The image of* $\underline{\mathcal{P}}$ *is* $\mathcal{D}^\circ$.

We prove Theorem 6.1.1 in a few steps.

Step 1. We show that the image of $\mathcal{P}$ is contained in $Gr^{+,\circ}$. In particular, the image of $\underline{\mathcal{P}}$ is contained in $\mathcal{D}^\circ$.

The proof that we know uses some complex geometry. Suppose there is a $g \in \mathcal{N}$ and $0 \neq \delta \in H_2(\mathcal{K}; \mathbb{Z})$ such that $\delta.\delta \geq -2$ and $\int_\delta \xi = 0$ for all $\xi \in \mathbb{H}_g^+$. Choose a hyperkähler triple $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ for g. Then with respect to a compatible complex structure J we see ω_1 is a Kähler form, and $\Omega = \omega_2 + \sqrt{-1}\omega_3$ is a holomorphic volume form. By the Hodge decomposition it follows that δ is a $(1,1)$ class with respect to J. So $\delta = c_1(L)$ for some non-trivial holomorphic line bundle L. By the Hirzebruch-Riemann-Roch theorem it follows that $h^0(\mathcal{K}, L) + h^0(\mathcal{K}, L^{-1}) = h^1(\mathcal{K}, L) + 2 + \frac{1}{2}\delta^2 > 0$. Without loss of generality we assume L has a non-zero holomorphic section S. Its zero set is a complex curve dual to δ. It follows that $\int_\delta \omega > 0$. Contradiction.

Step 2: We show that $\mathcal{P}$ is an open map. In particular, the image of $\underline{\mathcal{P}}$ is open.

This follows from the standard deformation theory. The arguments below follows from [46]. Suppose $g \in \mathcal{N}$. We fix a hyperkähler triple $\boldsymbol{\omega}$ associated to g. In the following, we will often identify an element in $\Omega_+^2 \otimes \mathbb{R}^3$ (i.e., a triple of self-dual 2-forms) with a 3×3 matrix-valued function in $\Omega^0 \otimes \mathbb{R}^{3\times 3}$: $\boldsymbol{\eta} \in \Omega_+^2 \otimes \mathbb{R}^3$ corresponds to $\boldsymbol{A} = (A_{\alpha\beta}) \in \Omega^0 \otimes \mathbb{R}^{3\times 3}$ if $\eta_\alpha = \sum_{\beta=1}^3 A_{\alpha\beta}\omega_\beta$, or concisely $\boldsymbol{\eta} = \boldsymbol{A}.\boldsymbol{\omega}$. We claim that for any fixed small triple of anti-self-dual harmonic 2-forms $\boldsymbol{h}^- \in \mathbb{H}_g^- \otimes \mathbb{R}^3$, $\boldsymbol{\omega}' := \boldsymbol{\omega} + \boldsymbol{h}^- + \boldsymbol{h}^+ + dd^*(\boldsymbol{f}.\boldsymbol{\omega})$ defines a hyperkähler triple for some small $(\boldsymbol{h}^+, \boldsymbol{f}) \in \mathbb{H}_g^+ \otimes \mathbb{R}^3 \oplus C^{2,\gamma}(\Omega_+^2 \otimes \mathbb{R}^3)$.

Denote $\mathscr{S}_0(\mathbb{R}^3)$ the set of trace-free 3×3 symmetric matrices, and $\mathfrak{F}$ the inverse of the map $\mathscr{S}_0(\mathbb{R}^3) \to \mathscr{S}_0(\mathbb{R}^3)$, $A \mapsto \mathrm{tf}(A + A^T + AA^T)$ near 0, where $\mathrm{tf}(B) = B - \frac{1}{3}\mathrm{Tr}(B)$. Let $\boldsymbol{A}$ be the self-dual part of $\boldsymbol{\omega}' - \boldsymbol{\omega}$ and $\boldsymbol{\theta}^-$ be the anti-self-dual part of $\boldsymbol{\omega}' - \boldsymbol{\omega}$, the hyperkähler condition for $\boldsymbol{\omega}'$ is equivalent to

$$\mathrm{tf}(\boldsymbol{A} + \boldsymbol{A}^T + \boldsymbol{A}\boldsymbol{A}^T) = \mathrm{tf}(-I_3 - S_{\boldsymbol{\theta}^-}),$$

where $S_{\boldsymbol{\theta}^-} = (\theta^-_\alpha \wedge \theta^-_\beta / 2\mathrm{dVol}_g)$. If we require $\boldsymbol{A}$ to be symmetric trace-free, then this is equivalent to

$$\boldsymbol{A} = (\mathrm{tf}(-S_{\boldsymbol{\theta}^-})).$$

Denote $\mathfrak{A} \subset C^{2,\gamma}(\Omega^2_+ \otimes \mathbb{R}^3)$, $\mathfrak{B} \subset C^{\gamma}(\Omega^2_+ \otimes \mathbb{R}^3)$, $\mathfrak{C} \subset \mathbb{H}^+_g \otimes \mathbb{R}^3$ denote the subspace consisting of trace-free symmetric matrices, respectively. For $\boldsymbol{u} = (\boldsymbol{h}^+, \boldsymbol{f}) \in \mathfrak{C} \oplus \mathfrak{A}$, define $\mathscr{F} : \mathfrak{C} \oplus \mathfrak{A} \to \mathfrak{B}$ by

$$\mathscr{F}(\boldsymbol{u}) := \boldsymbol{h}^+ + d^+d^*(\boldsymbol{f}.\boldsymbol{\omega}) - \mathfrak{F}(-\mathrm{tf}(S_{\boldsymbol{h}^- + d^- d^*(\boldsymbol{f}.\boldsymbol{\omega})})).$$

Then the condition $\boldsymbol{\omega}'$ being hyperkähler is equivalent to the equation

$$\mathscr{F}(\boldsymbol{u}) = 0.$$

To solve the equation, we write $\mathscr{F}(\boldsymbol{u}) = \mathscr{L}(\boldsymbol{u}) + \mathscr{N}(\boldsymbol{u})$, where $\mathscr{L}(\boldsymbol{u}) = \boldsymbol{h}^+ + d^+d^*(\boldsymbol{f}.\boldsymbol{\omega}) = \boldsymbol{h}^+ + (\Delta_g \boldsymbol{f}).\boldsymbol{\omega}$, $\mathscr{N}(\boldsymbol{u}) = -\mathfrak{F}(-\mathrm{tf}(S_{\boldsymbol{h}^- + d^- d^*(\boldsymbol{f}.\boldsymbol{\omega})}))$. Then by standard elliptic theory that $\mathscr{L}$ is a bounded linear map which is surjective with a bounded right inverse, and $\|\mathscr{N}(\boldsymbol{u}) - \mathscr{N}(\boldsymbol{v})\| \leq C(\|\boldsymbol{h}^-\| + \|\boldsymbol{u}\| + \|\boldsymbol{v}\|)(\|\boldsymbol{u} - \boldsymbol{v}\|)$. Then the following implicit function theorem Lemma 6.1.2 implies that there exists a $\delta > 0$ such that for any $\|\boldsymbol{h}^-\| < \delta$, $\mathscr{F}(\boldsymbol{u}) = 0$ has a solution $\boldsymbol{u}$ with $\|\boldsymbol{u}\| < C(\delta)$, which finishes the proof of the claim. Now the map $\Psi : \mathbb{H}^-_g \otimes \mathbb{R}^3 \to Gr^+$, $\boldsymbol{h}^- \mapsto \mathrm{span}\{\boldsymbol{\omega} + \boldsymbol{h}^-\}$ defines a homeomorphism from a neighborhood of 0 to a neighborhood of $\mathcal{P}(g)$, and $\mathrm{span}\{\boldsymbol{\omega} + \boldsymbol{h}^-\} = \mathrm{span}\{\boldsymbol{\omega} + \boldsymbol{h}^- + \boldsymbol{h}^+ + dd^*(\boldsymbol{f}.\boldsymbol{\omega})\}$ as elements in Gr^+, it follows that the image of a neighborhood of g under $\mathcal{P}$ contains a neighborhood of $\mathcal{P}(g)$, hence $\mathcal{P}$ is an open map.

The following version of implicit function theorem is well-known and is used in [46]. For completeness, we provide the proof here.

Lemma 6.1.2. *Let $(X, \|\cdot\|))$, $(Y, \|\cdot\|)$ be two Banach spaces, $B_\delta(0) \subset X$, $F : B_\delta(0) \to Y$ is a map with $F(0) = 0$. Suppose one can write $F = L + N$ such that $L : X \to Y$ a bounded linear map with a bounded right-inverse R, i.e., $L \circ R = id_Y$, $\|R\| \leq M$. If furthermore $\|N(x) - N(y)\| \leq (2M)^{-1}\|x - y\|$. Then for any $c \in Y$, $\|c\| < (2M)^{-1}\delta$, the equation $F(u) = c$ has a solution with $\|u\| \leq 2M\|c\|$.*

Proof. For fixed $c \in Y$, define $T_c(w) = w - (F(R(w)) - c) = -N(R(w)) + c$ for $w \in Y$, $\|w\| < M^{-1}\delta$. Then

$$\|T_c(w) - T_c(w')\| \leq \frac{1}{2}\|w - w'\|.$$

In particular

$$\|T_c(w) - T_c(0)| \leq \frac{1}{2}\|w\|,$$

then

$$\|T_c(w)\| \leq \|c\| + \frac{1}{2}\|w\|.$$

Hence T_c is a contraction map from $\bar{B}_{\|2c\|}(0)$ to $\bar{B}_{\|2c\|}(0)$. By Banach fixed point theorem we know $T_c(w) = w$ has a solution with $\|w\| \leq 2\|c\|$. Let $u = R(w)$, then $F(u) = c$ and $\|u\| \leq 2M\|c\|$. □

Step 3. We show that $\underline{\mathcal{P}} : \mathcal{M} \to \mathcal{D}^\circ$ is a proper map.

Suppose otherwise, then we can find a sequence of hyperkähler metrics $g_j \in \mathcal{N}$ which do not converge smoothly modulo **Diff**$(\mathcal{K})$, but there exist $\gamma_j \in \Gamma$ such that $\gamma_j.\mathcal{P}(g_j)$ converges to a positive 3-dimensional subspace P_∞ in $Gr^{+,\circ}$. Choose a hyperkähler triple $\boldsymbol{\omega}_j = (\omega_{j,1}, \omega_{j,2}, \omega_{j,3})$ for g_j. Denote $v_j^4 = 2\text{Vol}(g_j)$. We define the renormalized triple $\widetilde{\boldsymbol{\omega}}_j = v_j^{-2}\boldsymbol{\omega}_j$, then $\int_{\mathcal{K}} \widetilde{\omega}_{j,\alpha} \wedge \widetilde{\omega}_{j,\beta} = \delta_{\alpha\beta}$.

Now we fix an norm $\|\cdot\|$ on $\Lambda_{\mathbb{R}}$. By abusing notation we also denote by $\|\cdot\|$ the standard norm on $\mathbb{R}^3$, or the induced norm on $\Lambda_{\mathbb{R}} \otimes \mathbb{R}^3$.

Lemma 6.1.3. *$\|\gamma_j.[\widetilde{\boldsymbol{\omega}}_j]\|$ is uniformly bounded.*

Proof. Otherwise, by passing to a subsequence and $O(3)$ rotations we may assume only the first component of $\gamma_j.[\widetilde{\boldsymbol{\omega}}_j]$ is non-zero and $\|\gamma_j.[\widetilde{\boldsymbol{\omega}}_j]\| = \lambda_j \to \infty$. Denote $\zeta_j = \lambda_j^{-1}\gamma_j.[\widetilde{\boldsymbol{\omega}}_j]$, then passing to a subsequence we may assume ζ_j converges to an element ζ_∞ in $\Lambda_{\mathbb{R}}$ with $\|\zeta_\infty\| = 1$ and with $\zeta_\infty \cup \zeta_\infty = 0$. But the line spanned by ζ_∞ is contained in P_∞ which is positive definite with respect to the intersection form. Contradiction. □

Given the Lemma, by passing to a subsequence we may assume $\gamma_j.[\widetilde{\boldsymbol{\omega}}_j]$ converges to a limit $\boldsymbol{\eta}_\infty$ in $\Lambda_{\mathbb{R}} \otimes \mathbb{R}^3$. Notice $[\eta_{\infty,\alpha}] \cup [\eta_{\infty,\beta}] = \delta_{\alpha\beta}$, so $\boldsymbol{\eta}_\infty$ forms a basis for P_∞.

Proposition 6.1.4. *Passing to a subsequence, for j large, there exists a non-zero homology class $C_j \in H_2(\mathcal{K}, \mathbb{Z})$ satisfying $C_j.C_j \in \{0, -2\}$, and $\| \int_{C_j} \widetilde{\boldsymbol{\omega}}_j \| \to 0$.*

Proof. Passing to a subsequence we may assume $(\mathcal{K}, g_j)$ converges to a Gromov-Hausdorff limit X_∞, which is a compact metric space.

If $v_j \geq \epsilon > 0$ for all j, then it follows from the classical results ([4, 8, 48]) that X_∞ is a hyperkähler orbifold. Let $p_j \in \mathcal{K}$ be such that $\lambda_j := \max_{\mathcal{K}} |Rm(g_j)|$ is achieved at p_j. By assumption $\lambda_j \to \infty$. Then passing to a subsequence we can take a pointed Gromov-Hausdorff limit of $(\mathcal{K}, p_j, \lambda_j^{1/2} g_j)$ to get a complete ALE hyperkähler 4-manifold Z. By Kronheimer's classification [36] we know Z must contain a homology class C_∞ with $C_\infty.C_\infty = -2$. This gives rise to a sequence of -2 class C_j in $\mathcal{K}$ such that $\| \int_{C_j} \boldsymbol{\omega}_j \| \to 0$, so $\| \int_{C_j} \widetilde{\boldsymbol{\omega}}_j \| \to 0$ as well. Notice here we only need the topological classification in Kronheimer's result.

If $v_j \to 0$, then the conclusion follows from the results of Sun-Zhang [46]. The point is that away from finitely many points, the hyperkähler triple $\boldsymbol{\omega}_j$(up to $O(3)$ rotations) has almost local nilpotent symmetry, and it can be perturbed to a new hyperkähler triple $\boldsymbol{\omega}_j'$ such that $\boldsymbol{\omega}_j'$ has local nilpotent symmetry and $\boldsymbol{\omega}_j - \boldsymbol{\omega}_j'$ is exact. In particular, $\boldsymbol{\omega}_j'$ has explicit expression and the integration of $\boldsymbol{\omega}_j'$ over a cycle is the same as the integration of $\boldsymbol{\omega}_j$ over that cycle. We divide into 3 cases. The first two cases only use the analysis over the regular region in [46].

- $\dim X_\infty = 2$. In this case $\boldsymbol{\omega}_j'$ is locally T^2-invariant. Take C_j to be the class of a T^2 fiber. Then we get $\int_{C_j} \omega_{j,1} = \int_{C_j} \omega_{j,2} = 0$ and $\int_{C_j} \omega_{j,3} \sim v_j^4$. We get $\| \int_{C_j} \widetilde{\boldsymbol{\omega}}_j \| \sim v_j^2 \to 0$.

- $\dim X_\infty = 1$. Locally there are two cases, either there is a T^3 symmetry or a Heisenberg symmetry. In the former case the metric corresponding to $\boldsymbol{\omega}_j'$ is locally a flat product $S^1(r_{j,1}) \times S^1(r_{j,2}) \times S^1(r_{j,3}) \times I$, where I is an open interval and we assume $r_{j,1} \leq r_{j,2} \leq r_{j,3}$, then we take C_j to be the homology class of $S^1(r_{j,1}) \times S^1(r_{j,2})$, then $\int_{C_j} \omega_{j,1} = \int_{C_j} \omega_{j,2} = 0$ and $0 < \int_{C_j} \omega_{j,3} = r_{j,1} r_{j,2}$ whereas $v_j^4 \sim r_{j,1} r_{j,2} r_{j,3}$. So $\| \int_{C_j} \widetilde{\boldsymbol{\omega}}_j \| \sim r_{j,1}^{1/2} r_{j_2}^{1/2} r_{j,3}^{-1/2} \to 0$. In the latter case the metric corresponding to $\boldsymbol{\omega}_j'$ is locally given by the Gibbons-Hawking ansatz applied to a nonconstant linear function on $T^2 \times I$. Take C_j to be the homology class of the 2-torus given by the total space of the corresponding circle bundle over a circle S^1 in $T^2 \times I$, then one can arrange that $\int_{C_j} \omega_{j,1} = \int_{C_j} \omega_{j,2} = 0$ and $0 < \int_{C_j} \omega_{j,3} \sim r_{j,1} r_{j,2}$, where $r_{j,1}$ is the size of the S^1 fiber in the Gibbons-Hawking construction, and $r_{j,2}$ is the size of the flat T^2 base. Notice the volume $v_j^4 \sim r_{j,1} r_{j,2}^2$. So $\| \int_{C_j} \widetilde{\boldsymbol{\omega}}_j \| \sim r_{j,1}^{1/2} \to 0$.

- $\dim X_\infty = 3$. Here we need some global result from [46]. It is proved there that X_∞ must be a flat orbifold $T^3/\mathbb{Z}_2$ and $\boldsymbol{\omega}_j'$ is given by Gibbons-Hawking construction on the complement of a small neighborhood of the 8 orbifold points. After hyperkähler rotations, one can write
$$\omega_{j,1}' = V dx_2 \wedge dx_3 + dx_1 \wedge \theta,$$
$$\omega_{j,2}' = V dx_3 \wedge dx_1 + dx_2 \wedge \theta,$$
$$\omega_{j,3}' = V dx_1 \wedge dx_2 + dx_3 \wedge \theta,$$
where V is given by a positive constant. Let C_j be given by the circle bundle over the the closed geodesic $x_3 = \pm\epsilon$, then we have $\int_{C_j} \omega_{j,1}' = \int_{C_j} \omega_{j,2}' = 0$ and $\int_{C_j} \omega_{j,3}' = \int_{C_j} dx_3 \wedge \theta = r_j \sim v_j^4$ So $\| \int_{C_j} \widetilde{\boldsymbol{\omega}}_j \| \sim r_j^{1/2} \to 0$.

□

Now we derive a contradiction. Denote $C_j' = \gamma_j^{-1}.C_j$. Since C_j' is integral and non-zero, we know $\|C_j'\|$ has a uniform positive lower bound. By passing to a subsequence we may assume $\|C_j'\|^{-1}C_j'$ converges to a limit $C_\infty' \in \Lambda_{\mathbb{R}}$ with $\|C_\infty'\| = 1$. First suppose $C_j.C_j = 0$, then $C_\infty'.C_\infty' = 0$ and

$$\int_{C_\infty'} \boldsymbol{\eta}_\infty = \lim_{j\to\infty} \|C_j'\|^{-1} \int_{C_j'} \boldsymbol{\eta}_\infty = \lim_{j\to\infty} \|C_j'\|^{-1} \int_{C_j'} \gamma_j.[\widetilde{\boldsymbol{\omega}}_j] = \lim_{j\to\infty} \|C_j'\|^{-1} \int_{C_j} \widetilde{\boldsymbol{\omega}}_j = 0. \qquad (6.2)$$

This contradicts the fact that the intersection form on $\Lambda_{\mathbb{R}}$ has signature $(3, 19)$. Now suppose $C_j.C_j = -2$. If $\|C_j'\|$ is unbounded, then $C_\infty'.C_\infty' = 0$, and we have $\int_{C_\infty'} \boldsymbol{\eta}_\infty = 0$ as in (6.2), hence we obtain a contradiction in the same way. If $\|C_j'\|$ is bounded, by passing to a further subsequence we may assume C_j' converges to a limit $C_\infty'' \in \Lambda$ with $C_\infty''.C_\infty'' = -2$ and $\int_{C_\infty''} \boldsymbol{\eta}_\infty = 0$. This contradicts to $P_\infty \in Gr^{+,\circ}$ and finishes the proof of Step 3.

Finally, it is worth mentioning that $\mathcal{M}$ is non-empty. Hyperkähler metrics in dimension 4 can be constructed by various ways(twistor methods, moduli space of monopoles, Yau's theorem, gluing constructions etc). For a survey, refer to [28]. See Section 3.1.3 for some discussion gluing constructions. In particular, one does not need to invoke Yau's theorem to show $\mathcal{M}$ is nonempty.

Now, $\underline{\mathcal{P}} : \mathcal{M} \to \mathcal{D}^\circ$ being proper and $\mathcal{D}^\circ$ being locally compact, Hausdorff imply $\underline{\mathcal{P}} : \mathcal{M} \to \mathcal{D}^\circ$ is a closed map. Together with the image of $\underline{\mathcal{P}}$ being open, non-empty, $\mathcal{D}^\circ$ being connected, we conclude $\underline{\mathcal{P}} : \mathcal{M} \to \mathcal{D}^\circ$ is surjective.

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