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Publication Date
2004-07-05

BEAM CONDITIONING AND HARMONIC GENERATION IN FREE ELECTRON LASERS

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Abstract

The next generation of large-scale free-electron lasers (FELs) such as Euro-XFEL and LCLS are to be devices which produce coherent X-rays using Self-Amplified Spontaneous Emission (SASE). The performance of these devices is limited by the spread in longitudinal velocities of the beam. In the case where this spread arises primarily from large transverse oscillation amplitudes, beam conditioning can significantly enhance FEL performance. Future X-ray sources may also exploit harmonic generation starting from laser-seeded modulation. Preliminary analysis of such devices is discussed, based on a novel trial-function/variational-principle approach, which shows good agreement with more lengthy numerical simulations.

Introduction

The next generation of free electron lasers (FELs), such as those proposed at Euro-XFEL or the LCLS, are designed to generate coherent, high-brightness radiation using the Self-Amplified Spontaneous Emission (SASE) process. The SASE mechanism requires high-energy and high brightness electron beams, making the FEL technically challenging. The FEL mechanism is a resonant interaction. It requires that the phase velocity of the beat between the wiggler and radiation fields match the longitudinal particle velocity. In this circumstance, the particle phase $\psi = (k + k_w)z - \omega t$ is slowly-varying, and a strong interaction can occur between the radiation and the particle beam. Thus, one requires that

$$\frac{d\psi}{dz} = (k + k_w) - \frac{\omega}{v_z} \approx 0. \qquad (1)$$

Assuming particles have the appropriate longitudinal velocity v_z, so that resonance is satisfied, SASE can occur and fluctuations in electron beam density grow exponentially, producing coherent radiation. This growth is limited by various effects, including partial overlap of the particle and photon beams as well as Landau damping, which arises from variations, within the beam, of the longitudinal velocity of electrons. The sources of such variation are two-fold: variations in electron energy, and variations in the transverse (betatron) motion of electrons with the same energy.

Therefore, one of the main constraints on SASE FEL performance at short wavelengths, which makes the gun and accelerator design so technically challenging, is the requirement that the geometric beam emittance $\varepsilon = \varepsilon_N / \gamma$ (ε_N is the scaled area in transverse phase space occupied by the beam and γ is the relativistic factor of a typical electron) be of the order the X-ray wavelength or smaller. Particles with large transverse oscillation (i.e., betatron) amplitude will tend to slip backwards axially with respect to a particle with no betatron amplitude and, thus, can fall out of phase with the radiation produced in the wiggler, limiting the gain of the FEL. This requirement drives the design acceleration to high energy, since ε_N is conserved in the acceleration and, therefore, small geometric emittance is easier to obtain at higher energy. If this requirement could be circumvented, the operating energy would be imposed by less severe physical or technological constraints, such as the limited ability to build short wavelength wigglers. Otherwise, for short-wavelength FELs, the emittance requirement is generally the most severe.

Beam Conditioning

A method for circumventing the emittance constraint was proposed many years ago [1]. The idea is to ease the sensitivity on the transverse emittance by introducing a correlation between particle energy and betatron amplitude. Increasing the total energy and, hence, longitudinal velocity, of particles with significant betatron amplitudes compensates for what would otherwise be smaller axial components of velocity, thus allowing FEL operation with larger electron beam emittances than would otherwise be possible. Recently, there has been renewed interest [2],[3] in this concept. The longitudinal velocity of an on-axis electron is

$$\frac{v_z}{c} = 1 - \frac{1}{\gamma^2} - \frac{\bar{v}_\perp^2}{c^2} = 1 - \frac{1 + a_u{}^2}{\gamma^2}, \tag{2}$$

where $a_u = eA_u / mc$ is the dimensionless wiggler vector potential evaluated on axis, $-e$ the electron charge, c the speed of light and m the electron mass. For an electron with transverse actions J_x and J_y, following the notation of Ref. [2], and in the reasonable assumption that the period of oscillation in a wiggler is much shorter the transverse focusing in the wiggler (which arises due to the transverse variation of the wiggler magentic field), we have

$$\left(1 - \frac{v_z}{c}\right) = \frac{1}{2}\left(\frac{1 + a_u{}^2}{\gamma^2} + \frac{2J_x}{\beta_x} + \frac{2J_y}{\beta_y}\right). \tag{3}$$

where, following conventional notion in accelerators, the beta functions $\beta_{x(y)}$ measure the transverse focusing strengths, and the geometric emittances $\varepsilon_{x(y)}$ are the average values of $J_{x(y)}$ over all particles in the beam. The detailed motion of an electron is of course governed by the transverse focusing, but the specifics are unimportant for our discussion.

The idea of conditioning is to correlate transverse action with energy, so as to minimize the variations in resulting longitudinal velocity. That is, in circumstances where FEL performance is limited by transverse beam emittance and not intrinsic energy spread of the beam, one can actually add to the energy variation, but in a correlated manner, so that particles with higher action have higher energy, and thereby reduce the net spread in parallel velocities. Thus, to maximize the number of particles in resonance, it is necessary to minimize the spread in the RHS of Equation (3).

This can be accomplished by introducing the correlation:

$$\Delta\gamma/\gamma = \kappa_x J_x + \kappa_y J_y, \tag{4}$$

where

$$\kappa_{x(y)} = \frac{1}{\beta_{x(y)}}\frac{\gamma^2}{1+a_u{}^2} = \frac{1}{2\beta_{x(y)}}\frac{\lambda_W}{\lambda}. \tag{5}$$

Typical parameters for future X-ray sources imply characteristic values of $\kappa_{x(y)}$ on the order of 1-10 μm^{-1}: that is, for a beam with 1 μm normalized emittance in both transverse planes, a typical electron needs ~1 MeV more energy than a perfectly on-axis electron in order to maintain exact resonance in longitudinal velocity.

The feasibility of conditioning has been the subject of recent interest (see[2],[3]). The main points are, of course, that conditioning must modeled, for lossless systems, so as to obey Hamiltonian (i.e., symplectic) dynamics, and must use realistic electromagnetic fields satisfying Maxwell's equations. This leads to some constraints, both fundamental and technological. Care must also be taken in the design of conditioners, so that the entire beam is properly conditioned (not just one slice of the beam), and that the distribution for all slices remains matched to the lattice.

It is possible to design a beamline in which particle dynamics are governed by the following conditioning Hamiltonian, as in [2]:

$$H = \frac{\mu}{L}J + \frac{\eta}{L}zJ, \tag{6}$$

restricted, for simplicity, to motion in the longitudinal and one transverse direction. Here, L is the length of the conditioner, μ is a phase-advance parameter, η is a coupling parameter, J is the transverse canonical action conjugate to the canonical angle variable ϕ, and z is the position of a particle in the beam with respect to a particular resonant reference orbit, and is canonically conjugate to the energy deviation δ from the design value. The distance s along the beam line is used as the independent evolution variable. Since the angle and energy deviation variables do not appear explicitly in the Hamiltonian, the conjugate action and displacement variables are conserved.

Note that the energy deviation increases by an amount proportional to the action, so under these dynamics, a beam of particles in which the energy deviation and action are initially (*s = 0*) uncorrelated will acquire a correlation at $s = L$ upon passing through the conditioning section:

$$\langle \delta_L J_L \rangle = \eta \langle J_0^2 \rangle. \tag{7}$$

Thus the above Hamiltonian produces the desired conditioning. However, some care is needed in the beamline design [3] to avoid introducing deleterious effects (such as focusing forces which vary with z) that will adversely affect beam quality and render the FEL inoperable [2].

This method of improving FEL performance is most appropriate for long undulators operating in Self-Amplified Spontaneous Emission (SASE) mode, where the laser field is amplified starting from statistical fluctuations in the electron beam current. The radiation power can undergo many e-foldings, or gain lengths, before the FEL saturates. Simulations of sources with and without conditioning using the GENESIS code [4] have been performed, confirming that conditioning can ameliorate reductions in gain lengths by up to a factor of 2, or alternatively increases in the transverse emitance by up to a factor of 4, without performance degradation [2]. Results are shown in Fig. 1 for paramters matching those of the LCLS FEL [5].

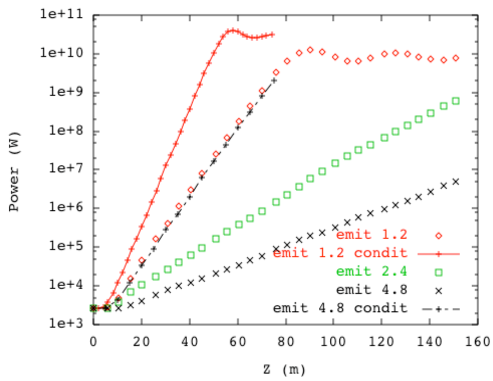

Figure 1

Effect of beam conditioning on SASE performance (output power versus undulator length) for various electron beam emittances, given in μm. LCLS parameters are used.

Seeded Harmonic Generation

An alternative mode of operation for an FEL is to use an electron beam with a seeded density perturbation. This allows for greater control of timing and pulse structure. If the density perturbation is sufficiently sharp, it can be used to seed radiation at very short wavelengths. In particular, we consider harmonic generation, where the electron beam first acquires an energy modulation in one undulator while overlapping a seed laser, then is bunched at the modulating wavelength by passage through a dispersive section. A second, radiating undulator can be tuned to a harmonic of the modulating wavelength, as long as the density perturbation has a substantial Fourier component there. The output can be used to seed the next stage of modulation in a fresh section of the electron beam, and the process may be cascaded to achieve still higher frequencies. This design allows for large output power to be produced with a relatively short beamline, and the laser seed can be chosen to be at a convenient wavelength almost regardless of the desired output wavelength. In the LUX study on sources of short X-ray pulses [6], a series of such harmonic generation sections has been modeled to create photons of up to 1 keV using a UV laser to provide the initial modulating seed.

A simple analytic model for predicting and optimizing the FEL output from a prebunched electron beam has been developed [7],[8], with emphasis on applications towards harmonic generation, obviating the need for lengthy nunerical simulations, at least in the early stages of beamline design and optimization. This methodology has the advantage that a simple analytic prescription is used to determine a best fit to the output mode through a trial-function approach, and is particularly well suited to optimization of FEL parameters, because the design parameters can be optimized simultaneously with the trial radiation envelope to maximize the output power. These calculations, however, are applicable only to FELs in the low-gain regime; that is, where the radiator-undulators are shorter than a gain length.

The output from the radiating undulator, or radiator, is here approximated by a simple paraxial Gaussian mode, but is otherwise kept arbitrary, i.e., with adjustable spot size and location of focus:

$$E_y = \mathrm{Re}\, E_0 e^{i\Phi_0} G(x,y,s) \exp(iks - i\omega t), \tag{8}$$

where

$$G(x,y,s) \equiv \frac{Z_R}{Z_R + i(s-s_0)} \exp\left[-\frac{1}{2}\frac{k(x^2+y^2)}{Z_R + i(s-s_0)}\right] \tag{9}$$

characterizes the structure of the mode. The laser wavelength is $\lambda = 2\pi/k$, the frequency is $\omega = ck$, and Z_R is the Rayleigh range, while E_0 represents the real amplitude and Φ_0 the overall phase of the trial mode. The longitudinal coordinate s represents the position along the undulator, such that at $s = s_0$ the laser achieves its waist, with spot size $(Z_R/2k)^{1/2}$. It is possible to generalize this trial envelope to include, for example, admixtures of higher-order transverse modes, elliptically-contoured wavefronts, or other features, as needed. Note that this field only characterizes the output from the radiator, and so is described by vacuum field solutions.

Averaging the acceleration due to the electric field over an undulator period in a planar wiggler yields

$$\frac{d\gamma}{ds} = -\mathrm{Re}\frac{\sqrt{2}k}{2\gamma} a_u a_L G(x,y,s) JJ(\xi) e^{i\Psi}, \tag{10}$$

where $JJ(\xi) = J_0(\xi) - J_1(\xi)$ is a difference of Bessel functions, $\xi = (1/2)a_u^2/(1+a_u^2)$, a_L is the peak normalized vector potential associated with the trial mode, and the ponderomotive phase is $\Psi = ks - \omega t + k_u s$. To leading order in $1/\gamma^2$, this phase evolves according to

$$\frac{d\Psi}{ds} = k_u \left[-\frac{\delta k}{k_r} + 2\frac{\gamma - \gamma_r}{\gamma_r} - \frac{2a_u \delta a_u}{1 + a_u^2} - \sqrt{2}\frac{a_u}{1 + a_u^2} k_u (J_x + J_y) \right], \qquad (11)$$

where we have defined $k = k_r + \delta k$, and the resonant wave vector $k_r \equiv 2\gamma_r^2 k_u / (1 + a_u^2)$ is defined in terms of the scaled resonant energy γ_r. The detuning can be expressed equivalently in terms of δk or as a shift δa_u in undulator strength.

Assuming that the energy extracted from the beam is converted into the single radiation mode defined above, the evolution of this mode can be described by

$$\frac{da_L}{ds} = i\frac{I}{I_A}\frac{2\sqrt{2}a_u}{\gamma Z_R} JJ(\xi) \left\langle G^*(x,y,s)e^{-i\Psi} \right\rangle, \qquad (12)$$

where I is the beam current and I_A is the Alfvén current. This expression assumes that the sum of the energy in the beam and the expected laser mode is conserved. The above average is a correction to the usual bunching parameter, $b \equiv \left\langle e^{-i\Psi} \right\rangle$. In fact, a generalized bunching parameter may be defined as

$$B(s) \equiv \left\langle G^*(x,y,s)e^{-i\Psi} \right\rangle. \qquad (13)$$

Again, harmonic generation, for example in the LUX design concept, uses a seed laser to generate an energy modulation in one undulator, which is then converted into microbunching by means of a chicane. The additional slippage which results from the chicane is characterized by the slippage parameter R_{56}, defined by $c\Delta t = R_{56}(\gamma - \gamma_0)/\gamma_0$, where γ_0 is the average (scaled) beam energy. Following this chicane, the bunched beam produces radiation while passing through a second undulator. Because the bunching includes Fourier components at harmonics of the initial laser seed, this second, radiating undulator can be tuned to a higher harmonic of the laser seed. Here, we consider a simplified case where the modulator applies an energy modulation γ_M which depends solely on the phase Ψ of the electrons. The energy distribution after modulation then takes the form

$$f = f\left[(\gamma - \gamma_0 - \kappa_x J_x - \kappa_y J_y + \gamma_M \sin\Psi_M)/\sigma_\gamma \right], \qquad (14)$$

where σ_γ is the scaled RMS energy spread of the beam. The phase Ψ_M is typically a sub-harmonic of the ponderomotive phase of the outgoing radiation, $\Psi = n\Psi_M$. The parameters κ_x and κ_y indicate the possibility for a correlation between energy and transverse amplitude, including fully-conditioned beams, for which $\kappa_x = \kappa_y = k/2k_u\beta$. The free-streaming evolution of this distribution function allows for the generalized bunching parameter, $B(s)$, to be calculated throughout the undulator in the low-gain regime. This is all that is necessary to determine the amount of power output by the FEL.

The result is still not fully defined because Z_R and s_0 remain free parameters. In general, after fixing Z_R and s_0, any paraxial radiation field can be described using a sum of normal modes (fundamental and higher-order modes), but here we are restricting attention to a single, Gaussian mode. Because the exact result will include the power contained within all these modes, the analytic result is expected to always fall below the correct value. This suggests varying the free parameters to maximize the output power, yielding a greatest lower bound to the correct result.

This method is essentially a trial function approach, and any trial function which is a valid vacuum laser field can be used. The closer the trial function is to the exact result, the more accurate this estimate for the power will be. Furthermore, the prediction for the laser power is expected to be second-order accurate compared to errors in the relative shape of the optimized trial function; in other words, even a relatively poor approximation to the laser field can result in a reasonably good estimate for the total output power. In the configurations being considered, a pure Gaussian mode is expected to be a reasonable approximation to the FEL output except in the emittance-dominated regime, where $\varepsilon_N / \gamma_0 \geq \lambda / 4\pi$. Here, only a simplified FEL configuration is considered, but the trial function method applies to more general cases as well. The analytic predictions are compared with GENESIS simulations in Fig. 2. The electron beam energy is taken to be 3.1 GeV, and the normalized emittance is 1.2 μm.

Using the Guassian mode, the resulting integrals are simple enough to implement as a *Mathematica* script, which allows for rapid optimization. Because the optimization procedure amounts to maximizing the output power, any additional constraints (undulator field strength, chicane paramter(s), or energy modulation) can be simultaneously optimized to obtain the largest possible output power. Thus any optimizations performed on the beamline can occur in parallel with the trial-function optimization for the radiation mode paramters Z_R and s_0, greatly reducing the computational time required. In the regime where the induced energy modulation is larger than the instrinsic energy spread in the electron beam, simple numerical fits for the optimization of output power can be obtained. In particular, the optimal value for the correlation between energy and transverse amplitude is $\kappa = (k / 2k_u\beta_u) k_u L / (k_u L + kR_{56})$. For typical undulator lengths, roughly speaking, the output power can be doubled through the use of appropriate beam conditioning.

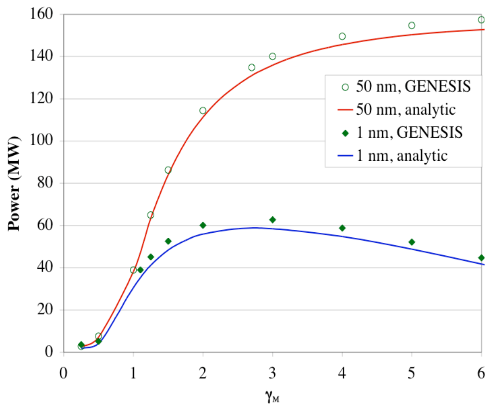

Figure 2

Comparison of analytical predictions of FEL performance with GENESIS simulations, showing output power as a function of induced energy modulation. Two cases are considered: for harmonic generation from a 200 nm seed to 50 nm output radiation, and from a 3 nm seed to 1 nm output radiation. Note that the analytic trial function always provides a lower bound.

Variational Principle for Spontaneous Wiggler Radiation

To summarize our general strategy for approximating the radiation in each given stage of the harmonic cascade: we first model the structure of the fields in terms of one or more free-space paraxial modes described by certain adjustable parameters. Some of these parameters may be subsequently determined directly by dynamical considerations, but some remain free, and are determined at the end of the calculation so as to maximize the resulting radiated power assuming the particular mode shape. This procedure seems physically reasonable and intuitively plausible. In fact, we have justified it rigorously using a Maximum-Power Variational Principle (MPVP) of quite general validity [9]. This methodology is now briefly described.

The variational principle emerges within the framework of a Hilbert space theory, applicable to the classical spontaneous radiation from prescribed harmonic current sources. Results can be derived easily in the paraxial limit (which is all that is actually needed in the present application), informed by the well-known parallels between the Schrodinger equation in non-relativistic quantum mechanics and the paraxial wave equation of classical physical optics. Guided by this important special case, one can employ a Green function treatment and spherical wave expansion of the general three-dimensional radiation fields to generalize these results to the case of non-paraxial fields.

Although developed within the present context of undulator radiation from relativistic electron beams, these tools are generally suitable to the numerical or analytic approximation of features of most forms of synchrotron or magnetic

Bremsstrahlung radiation, and after some suitable generalization, may be more broadly applicable to the cases of Cerenkov, transition, wave-guide, Smith-Purcell, or other types of radiation as well.

By spontaneous emission, we mean that the trajectories of the charged particles constituting the source for the radiation can, in principle, be considered prescribed functions of time, independent of the actual radiation fields emitted. That is, the electron trajectories are assumed to be determined by initial conditions, externally applied wiggler or other guiding fields, and possibly even (Vlasov) space-charge effects, while any back-action of the radiation itself on the particles, via recoil, absorption, or multiple scattering, may be neglected. Thus the MPVP provides an approximate alternative to calculation of the radiation-zone fields via the usual Lienard-Weichart potentials or related expressions.

For the specific case of a wiggler, this implies that any gain due to ponderomotive feedback and dynamic bunching over the radiative formation length remains small – hence our earlier restriction to the low-gain FEL regime. However, we stress that the effects of arbitrary amounts of pre-bunching established before the beam enters any particular undulator can be included. In addition, we assume that the sources remain localized in space during the emission (so that the far-field may be defined) and remain at least weakly localized in time (so that Fourier transforms exist.)

Our starting point is the Coulomb-gauge, frequency-domain wave equation in dimensionless variables:

$$\left(\nabla^2 + \omega^2\right)\mathbf{a}(\mathbf{x};\omega) = -\mathbf{j}_\perp(\mathbf{x};\omega), \tag{15}$$

together with the gauge condition $\nabla \cdot \mathbf{a}(\mathbf{x};\omega) = 0$, for the scaled vector potential $\mathbf{a}(\mathbf{x};\omega)$, which in principle includes certain near fields as well as all the radiation fields, and where the source $\mathbf{j}_\perp(\mathbf{x};\omega)$ is the solenoidal component of the scaled, frequency-domain current density $\mathbf{j}(\mathbf{x};\omega)$, assumed known.

The variational principle involves consideration of a family of trial (solenoidal) radiation envelopes $\boldsymbol{\chi}(\mathbf{x};\omega;\boldsymbol{\alpha})$ which are chosen to satisfy the gauge constraint as well as the homogeneous, i.e., source-free, or free-space, Helmholtz equation. These variational solutions may be formally decomposed into ingoing- and outgoing-wave components

$$\boldsymbol{\chi}(\mathbf{x};\omega;\boldsymbol{\alpha}) = \boldsymbol{\chi}^{in}(\mathbf{x};\omega;\boldsymbol{\alpha}) + \boldsymbol{\chi}^{out}(\mathbf{x};\omega;\boldsymbol{\alpha}), \tag{16}$$

each satisfying the corresponding Sommerfeld boundary conditions, and each depending on a set $\boldsymbol{\alpha}$ of parameters determining the overall amplitude, spatial shape, and polarization of the trial mode. In the paraxial case, any solenoidal, free-space radiation fields are uniquely specified by the carrier frequency and the (complex) profile for two independent polarization components in any one transverse plane, and may be expanded in a discrete set of orthogonal eigenfunctions such as the Gauss-Hermite or Gauss-Laguerre modes. In the non-paraxial case, the solutions will be more complicated, but from an expansion into vector spherical harmonics, we know at least that they also form a separable Hilbert space, parameterized by the multipole expansion coefficients.

The variational parameters may then be adjusted (either analytically for simple cases, or else numerically) so as to optimize a radiated-power condition derived from a Cauchy-Schwarz inequality in the Hilbert space picture:

$$\tilde{\boldsymbol{\alpha}} = \arg\min\Big[\tfrac{\partial}{\partial\omega} P_{EM}[\boldsymbol{\chi}^{out}](\varsigma;\omega;\boldsymbol{\alpha})\Big] , \qquad (17)$$

subject to a constraint determined by energy conservation:

$$-\tfrac{1}{2}\tfrac{\partial}{\partial\omega} P_{mech}[\boldsymbol{\chi};\mathbf{j}](\omega;\boldsymbol{\alpha}) = \tfrac{\partial}{\partial\omega} P_{EM}[\boldsymbol{\chi}^{out}](\varsigma;\omega;\boldsymbol{\alpha}) \ . \qquad (18)$$

Here $\frac{\partial}{\partial\omega} P_{EM}[\boldsymbol{\chi}^{out}](\varsigma;\omega;\boldsymbol{\alpha})$ is the power spectral density of outgoing radiation passing through some closed surface $S(\varsigma)$ of characteristic radius ς large enough to enclose all the sources, associated with the outgoing component of the variational approximation $\boldsymbol{\chi}(\mathbf{x};\omega;\boldsymbol{\alpha})$ to the actual vector potential $\mathbf{a}(\mathbf{x};\omega)$, and may be defined in terms of the scaled Poynting flux:

$$\tfrac{\partial}{\partial\omega} P_{EM}[\boldsymbol{\chi}^{out}](\varsigma;\omega;\boldsymbol{\alpha}) \equiv \mathrm{Re}\int_{S(\varsigma)} d^2\sigma\hat{\mathbf{n}}\cdot\Big[i\omega\boldsymbol{\chi}^{out}(\mathbf{x};\omega;\boldsymbol{\alpha})\times\Big(\nabla\times\boldsymbol{\chi}^{out}(\mathbf{x};\omega;\boldsymbol{\alpha})\Big)\Big] \qquad (19)$$

By choosing ς to be sufficiently large, the trial solution need only be decomposed into ingoing and outgoing components in the far-field region, which is straightforward.

The quantity $\frac{\partial}{\partial\omega} P_{mech}[\boldsymbol{\chi};\mathbf{j}](\omega;\boldsymbol{\alpha})$ represents the power spectral density of mechanical work which would be performed on the charges contributing to $\mathbf{j}(\mathbf{x};\omega)$, by the solenoidal electric field associated with the full trial solution $\boldsymbol{\chi}(\mathbf{x};\omega;\boldsymbol{\alpha})$ if it were actually present, and is given by the usual Joule-like expression:

$$\tfrac{\partial}{\partial\omega} P_{mech}[\boldsymbol{\chi};\mathbf{j}](\omega;\boldsymbol{\alpha}) \equiv \mathrm{Im}\int d^3\mathbf{x}\Big[\omega\boldsymbol{\chi}^*(\mathbf{x};\omega;\boldsymbol{\alpha})\cdot\mathbf{j}(\mathbf{x};\omega)\Big]. \qquad (20)$$

The factor of one-half appears in the conservation constraint above in order to avoid over-counting in the energetics; the radiated power in the outgoing component of the variational approximation is being related to the mechanical power which would be delivered by the sources to the full source-free fields, if they were actually present in the region of the sources, rather than the actual, inhomogeneous fields.

The optimized trial mode shape (or more accurately, the outgoing component thereof) is then the best guess, within the manifold of possibilities allowed by the shapes parameterized by $\boldsymbol{\alpha}$, of the actual field profile in the region beyond the sources, and its Poynting flux yields a lower bound on the actual power spectral density of the radiation at the frequency under consideration. The approximation will improve monotonically as additional independent parameters are included to allow for more general envelope shapes.

This variational principle can be variously interpreted according to one's tastes or application. As we have seen, the best variational approximation maximizes the radiated power consistent with the constraint that this energy could have arisen from work extracted from the actual sources. It also minimizes a Hilbert-space distance between the actual fields and the parameterized family of solenoidal, free-space fields, and in many cases may be regarded as an orthogonal projection into this manifold of trial solutions. It also maximizes, for each frequency component, the spatial overlap, or physical resemblance, between the actual current density and the trial fields, extrapolated back into the region of the sources assuming source-free propagation.

Equivalently, one can say the optimal field profile is that which, if it actually were incident on the sources, would maximally couple to the given sources and would experience maximal small-signal gain; and, furthermore, the "virtual" gain delivered would be equal to the estimated power spontaneously radiated. In amplifier or stimulated emission situations, we naturally expect to observe in the presence of gain that mode which grows the fastest, but this intuition is also applicable in the spontaneous regime, because arguments along the lines of Einstein's derivation of the A and B coefficients or its generalization to FEL physics in the form of Madey's theorem lead to definite connections between spontaneous emission, stimulated emission, and stimulated absorption, even when the radiation is completely classical.

In fact, the only essential difference between the present case, and say, Madey's theorem is that by taking completely prescribed sources, we implicitly assume that any radiation, once emitted by one part of the source, cannot induce appreciable recoil in that part of the source or subsequently be re-scattered or absorbed by any other part of the source. So in fact we find a relationship between the spontaneous emission spectrum and that of the "bare" stimulated emission, not the "net" response given by the difference between stimulated emission and absorption as in Madey's theorem.

Note that this variational principle is reminiscent of, but actually distinct from, the Rayleigh-Ritz variational principle familiar from textbook quantum mechanics, as well as various Rumsey reaction-based principles commonly used in waveguide, antenna, and cavity analyses, and also to action-principles in Lagrangian formulations of electrodynamics, and thus adds to the large family of variational techniques available for electromagnetic problems in general, and undulator/FEL radiation in particular. Mathematical details aside, at its most essential, the MPVP is really just a straightforward consequence of two simple and rather obvious constraints: the power radiated in any one source-free mode of the electromagnetic far-field may not exceed the total power in all the modes (i.e., Bessel inequality); and the power radiated must be attributable to power delivered by the sources, even in the regime where we ignore back-action on the sources (i.e., conservation of energy.) However simple, even trivial, these observations are not without practical content or application to undulator systems, and possibly other radiation problems.

Discussion

Accelerator designs for SASE X-ray FELs may be able to benefit from the implementation of new ideas (and some old ones) for beam conditioning and harmonic geneneration. Theory, based on a novel trial-function/variational-principle apporach, is seen to be in good agreement with simulation for harmonic generation FELs. This use of a laser to initiate bunching for harmonic generation is but one of a series of possible applications of lasers in the generation of X-rays. It has been proposed [10] to use an ultra-short laser pulse to shift the energy of a very short section of the electron beam which would subsequently radiate an ultra-short X-ray pulse. Further applications [11] of lasers involve optical pre-bunching to enhancing the SASE FEL interaction. The result of these ideas may be X-ray FELs that are less costly, and better suited for many applications benefiting from high-brightness X-ray sources.

We thank W. Fawley, A. Sessler, and A. Zholents for useful discussions. This work was supported by the Director, Office of Science, of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098 and by the Division of High-Energy Physics.

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