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HOPF BIFURCATION IN A MODEL OF SOLID FUEL COMBUSTION

Victor Roytburd
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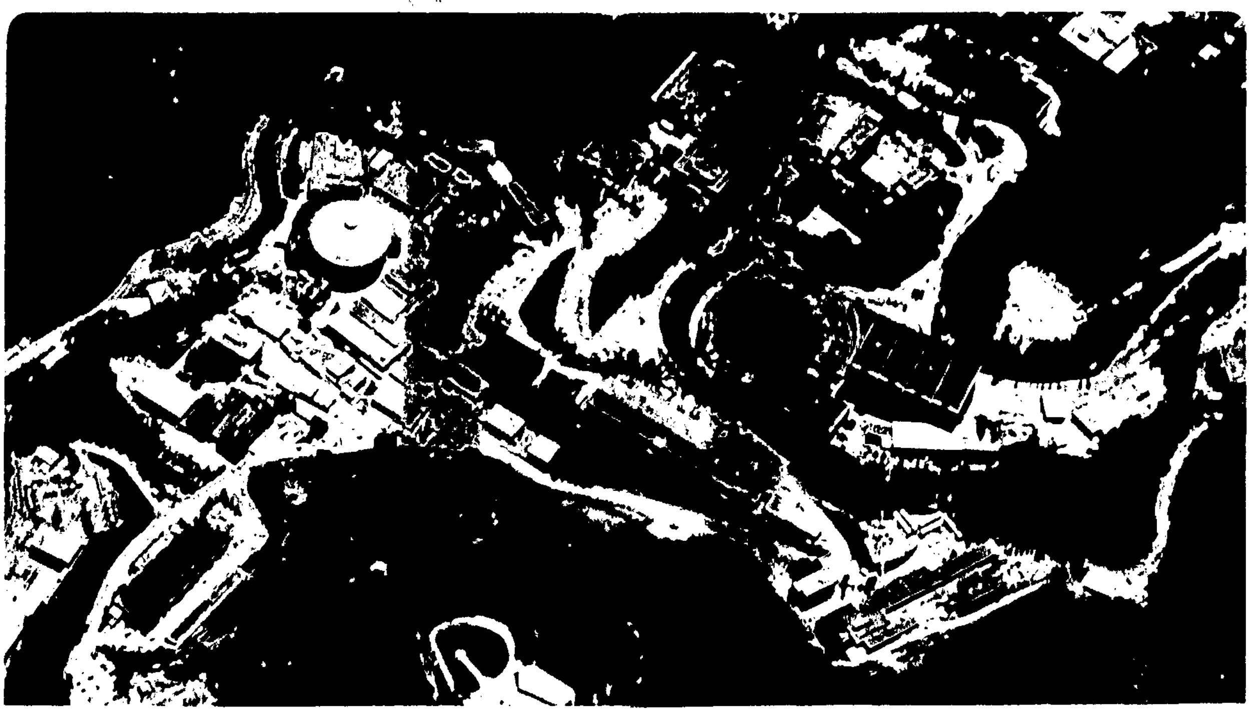

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Ph.D. Thesis

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ABSTRACT

Under some reasonable assumptions, a mathematical model for gasless combustion of one-dimensional solid fuel can be reduced to a free boundary problem for a system of semilinear parabolic equations. The problem is in a sense analogous to the two-phase Stefan problem with nonlinear jump conditions related to heat release. Solutions demonstrate behavior typical for the Hopf bifurcation, i.e., loss of stability for the steadily traveling wave solution as a parameter changes and the emergence of a nonlinearily stable pulsating combustion front.

A formal bifurcation analysis of these phenomena was carried out recently with the help of asymptotic expansions by B. J. Matkowsky and G. I. Sivashinsky. We develop a rigorous treatment of some of these results. We consider the problem as an evolution equation in a Hilbert space. To circumvent difficulties with a possible resonance with the continuous spectrum we introduce appropriate weighted norms. We develop a suitable version of the Hopf bifurcation theorem and prove the existence of time periodic solutions for values of the parameter near some critical value.

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INTRODUCTION

This dissertation is devoted to rigorous mathematical study of a model of the combustion process present in some solid fuels. It should be mentioned that in many problems involving the combustion of condensed systems, the study is complicated by the fact that the combustion occurs in two stages. The gasification of the solid fuel followed by the burning process which in turn yields gaseous combustion products. However there are solid fuels in which combustion is the result of the direct transformation of the solid fuel into solid combustion products.

A realistic model of gasless combustion of one-dimensional solid fuel is given by the system of reaction-diffusion equations

$$T_\tau = \kappa T_{xx} + QCZe^{-E/(RT)} \quad , \quad C_\tau = -CZe^{-E/(RT)} \quad , \qquad (1)$$

$\tau > 0$, $-\infty < x < \infty$. T and C are the temperature and fuel concentration of the limiting component. κ denotes the heat conductivity, Q and E are the constants characterizing the chemical reaction (the heat release and the effective activation energy), Z is a normalizing coefficient and R is the universal gas constant. Note that the fact that the fuel and combustion products are both solid, means that there is no species diffusion term in the second equation.

Shkadinsky, Khaikin and Merzhanov showed [14] by numerical calculations on the model (1), that the range of changing parameters (essentially Q and E) can be divided into two parts, to which there correspond two different regimes of reaction propagation. In addition to uniformly propagating combustion, the authors also describe oscillatory propagation, when the velocity of propagation of the reaction front and the temperature on the front exhibit periodic pulsations. Both types of propagation were observed in experiments with a mixture of 3Nb + 2B by Merzhanov, Filonenko and Borovinskaya [11].

Theoretical explanation of these pulsations was given by Zeldovich, Leypunsky and Librovich [17] (see also [14]). They show that these pulsations which arise at large activation energies, are caused by peculiarities of the heat transfer mechanism in the absence of molecular diffusion.

In a recent paper, Matkowsky and Sivashinsky [10] considered the asymptotic limit of large activation energies for the model in (1). They have shown that the distributed chemical reaction can be approximated by a concentrated heat source on the combustion front. The strength of the source may be modeled by $e^{(T-1)N/2}\,\delta$, where δ is the Dirac function, sitting on the front and N is a nondimensional activation energy. Thus the reaction takes place only on the combustion front which plays the role of a free boundary between two phases, burnt products and the fresh fuel. With this delta function kinetics, one can exclude the equation for the fuel concentration from (1). As the result, Matkowsky and Sivashinsky obtain the equation for the nondimensional temperature, θ,

$$\theta_t = \theta_{yy} - e^{\alpha(\theta(0,t)-1)}\theta_y , \tag{2}$$

subject to the jump conditions at t = 0

$$[\theta] = \theta(0+) - \theta(0-) = 0 \; , \qquad (3)$$
$$[\theta_y] + e^{\alpha(\theta(0,t)-1)} = 0 \; ,$$

with boundary conditions

$$\theta(-\infty,t) = 0 \quad , \quad \theta(\infty,t) < \infty \; . \qquad (4)$$

Here α is some lumped parameter. $\alpha = E(T_b-T_0)/(2RT_b^2)$, where T_b and T_0 are the temperatures of burnt and fresh mixtures. In the case of normal propagation $T_b = T_0 + QC_0$. The equation (2) is written in a coordinate system attached to the front, y is a nondimensional distance to the front and t is nondimensional time. The jump conditions (3) are caused by the substitution of the delta function for the chemical kinetics.

Matkowsky and Sivashinsky noticed that the problem (2) through (4) has a steady state solution, $\theta°(y) = e^y$, $y \leq 0$, $\theta°(y) = 1$, $y > 0$. The problem, linearized around $\theta°$, has two complex conjugate solutions of the form $e^{\lambda(\alpha)t}\phi_\alpha(y)$, where $\lambda(\alpha)$ and $\overline{\lambda}(\alpha)$ cross the imaginary axis if α exceeds some critical value α_0. This fact gives a strong indication that the model in (2) through (4) experiences a Hopf bifurcation and that which can be used to explain the pulsations. In particular, Matkowsky and Sivashinsky in [10] have written a series for the periodic solution of the nonlinear problem, have shown the consistency of the system of equations for terms of the series and have calculated two first non-trivial terms of the series. The paper [10] does not claim to be mathematically rigorous and in fact it has some gaps. For example the influence of the continuous spectrum is not estimated, and the convergence of the series for the periodic solution is not discussed.

Our objective in the present dissertation is to prove the existence of a periodic solution to the problem in (2) through (4). Our main result (Theorem 4.3) is as follows:

Main Theorem (MT). There exist a one-parameter family $u_s(y,t)$ and functions $\alpha(s)$ and $p(s)$ so that the u_s is a solution of the nonlinear problem in (2) through (4), where there is set $\alpha = \alpha(s)$; u_s is periodic with period $p(s)$; u, α and p depend on the parameter infinitely smoothly when s is sufficiently small; $\alpha(0) = \alpha_0$, $p(0) = p_0$ and $u(0) = 0$.

Here p_0 stands for the period corresponding to the critical value, α_0, of the bifurcation parameter. MT claims the existence of a periodic solution if α is sufficiently close to α_0. "Solution" is understood in some weak sense: we consider $u(.,t)$ as a function of t taking values in a Hilbert space

$$H_a = \{f \mid \int_{-\infty}^{\infty} |f|^2 e^{-ay/2} dy < \infty\}$$

for some $0 < a \leq 1$. We consider the problem on hand as a nonlinear evolution equation in H_a. However it could be proved a posteriori that a periodic solution is classical (Theorem 4.7).

As a corollary to MT it could be shown that $u_s \in H_a$ for any $0 < a \leq 1$; this ensures an exponential decay of u_s, at least as $e^{y/2}$, when $y \to -\infty$ and therefore the boundary condition $u(-\infty) = 0$ holds. For $y \to \infty$, we can derive from MT only that u_s may grow not faster than any exponential e^{ay}, $a > 0$. Therefore, in addition to MT we prove that u_s is, in fact, bounded (Theorem 4.9), i.e., both boundary conditions in (4) hold.

In order to prove MT we are compelled to develop some tedious technical tools, however the underlying ideas are rather natural and simple. The goal is to realize the following plan.

Step #1. Write the original nonlinear problem (2) through (4) in the form

$$u_t = A_0 u + R(\alpha,u) \quad , \tag{5}$$

where A_0 is the linearization of the differential operator in (2) provided with the linearized jump conditions, the linearization is done around the steady state solution and at $\alpha = \alpha_0$. Find a solution $T(t)u_0$, to the linear problem

$$u_t = A_0 u \quad , \quad u\big|_{t=0} = u_0 \quad , \tag{6}$$

and show that $T(p_0)$ has two eigenvalues on the unit circle while the remainder of the spectrum lays strictly inside. (The last requirement is standard for the existence of the Hopf bifurcation).

Step #2. Rewrite the Cauchy problem for the equation in (5) using a "variation of constants" formula

$$u(t) = T(t)u_0 + \int_0^t T(t-r)R(\alpha,u(r))\,dr \quad . \tag{7}$$

Step #3. Consider the operator

$$\phi(u)(t) = u(t) - T(pt)u_0 - \int_0^t T(p(t-r))R(\alpha,u(r))\,dr \quad . \tag{8}$$

on the space of p_0 - periodic functions and find its null space. We note that the unknown nondimensional period, p, appears explicitly in the operator in (8) if we change variables $t \to tp$ in (5) and apply the "variation of constants" formula. If u is a zero of the equation $\phi(u) = 0$ in the space of p_0 - periodic functions, then obviously $u_p(t) = u(t/p)$ is a (pp_0) - periodic solution of the nonlinear equation (5).

We realize this outlined program in Chapters 2 through 4 (Chapter 1 is devoted to the derivation of the asymptotic model which is the subject of study).

In Chapter 2 we investigate the linearized operator, A_0, which is an ordinary differential operator of second order with jump conditions at 0 or, equivalently, a system of second order operators on the half-axis. We find out that A_0, considered in L^2, has continuous spectrum, which touches the origin. Thus a straightforward Hopf bifurcation can not take place. It turns out, however, that by introducing the Hilbert spaces H_a, i.e., L^2 with weights $e^{-ay/2}$, $0 < a \leq 1$, one can shift the spectrum from the origin. The spectrum lays on a parabola which passes through the origin if a = 0 and as a increases, deforms and collapses into the half-ray, $y \leq -1/4$, when a = 1. The idea of using weighted spaces for shifting the spectrum belong to Sattinger [12].

In order to construct $T(t)$, a solving operator for the linear problem in (6), we obtain an estimate of the resolvent, $(A_0 - \lambda I)^{-1}$, in some sector of $\mathbb{C}^1$. The estimate shows that A_0 generates an analytic semigroup and $T(t) = e^{A_0 t}$. In the weighted norm, $T(p_0)$ possesses desired spectral properties (see Sec. 2.7).

Also in Chapter 2 we do some preparations for Step #2 of our program. We define fractional norms generated by A_0, $\|f\|_s = \|(-A_0)^s f\|$, $0 \leq s \leq 1$, and show that these norms can be easily estimated, at least for $s = 1$ and $1/2$, since they are equivalent to Sobolev norms (let X^s be a space with the norm $\|.\|_s$ then $X^s \simeq H^{2s}$, where H^{2s} is a Sobolev space).

In Chapter 3 we realize Step #2 of our plan and show that the variation of constants formula is applicable, i.e., that the integral equation in (7) and the differential one in (5) are equivalent.

In order to explain the essence of the problem we have to reveal the explicit forms of the operators that enter into the nonlinear differential equation (5). It is convenient to include jump conditions at the free boundary in the operator itself. This may be done with the help of the delta function, keeping in mind that the equation $u_{xx} + p(u_x,u) = \delta(x)f(x)$, where p is a "nice" function, means that u should satisfy the homogeneous equation for $x \neq 0$ with the jump conditions at $x = 0$, $[u] = 0$ and $[u_x] = f(0)$.

Taking into account the previous remark, for the nonlinear remainder in (5) we obtain

$$R = (1 - e^{\alpha u(0,t)})\ (u_y + 1/2\ u)$$
$$+ (1 + \alpha_0 u(0,t) - e^{\alpha u(0,t)})\ (H(y)e^{y/2} - \delta(y)) \quad , \tag{9}$$

where $H(y)$ is the Heaviside function, equal to 1, for $y < 0$, and to 0 for $y > 0$.

It follows from (9) that the term of the type

$$F(.,t) = \int_0^t f(r)\ T(t-r)\ \delta\ dr$$

should appear in the integral equation (7). We give meaning to this term, showing that the parts of F on positive and negative half-axes satisfy the following parabolic system in the domain $\{x > 0,\ 0 < t < T\}$

$$u_t = u_{xx} - 1/4\ u \quad , \quad v_t = v_{xx} - 1/4\ v - \alpha_0 e^{x/2}\ v(0) \quad ,$$
$$u(0,t) = v(0,t) \quad , \quad u_x(0,t) + v_x(0,t) + \alpha_0 u(0,t) + f(t) = 0 \quad ;$$
$$u\Big|_{t=0} = v\Big|_{t=0} = 0 \quad . \tag{10}$$

The presence of a nonlocal term, $\alpha_0 e^{x/2} v(0)$, prevents us from applying the standard parabolic theory. We construct a solution to (10) with the help of the Laplace transform. As could be expected, the analysis shows that the nonlocal term is of lower order. F can be represented in the form,

$$F(y,t) = \int_0^t G(y,t-r)\ f(r)\ dr \quad .$$

The nonlocal character of the nonlinearity reveals itself once more in the choice of a suitable Hilbert space, where we wish to consider the evolution equation; the functions should have good traces at 0. We choose $H^1 \simeq X^{1/2}$ for the convenience of calculations (although it suffices to have X^s, $s > 1/4$).

Now we can rewrite the integral equation (5) in more explicit form

$$u(t) = T(t)u_0 + \int_0^t T(t-1)\, R_1(\alpha,u(r))\, dr$$
$$+ \int_0^t G(.,t-r)\, (1 + \alpha_0 u)(0,r) - e^{\alpha u(0,r)})\, dr \quad . \tag{11}$$

The second summand on the right-hand side is a "volume potential" while the third one is a "surface potential" generated by the free boundary. It is not hard to show that any solution of the integral equation that is Hölder continuous with respect to t satisfies the differential equation. It is well known that the volume potential is automatically Hölder continuous if R_1 is sufficiently good (which is true in our case).

We carefully study Hölder continuity of the surface potential (Sec. 3.2). We show that the surface potential is Hölder continuous with values in an appropriate Hilbert space if its density is Hölder continuous with the exponent > 1/4. We note that Hölder coefficients can grow like $t^{-\rho}$, $\rho < 1$, as $t \to 0$, therefore we use Hölder spaces with weights t^ρ.

Applying the above results, we prove the equivalence of (5) and (7). We conclude Chapter 3 with an existence and uniqueness theorem (Theorem 3.5), which we include for two reasons: first, it seems to be useful for the stability analysis, second the theorem is obtained by straightforward application of the developed technique. For the bifurcation analysis of Chapter 4 we need only a more restricted result (Proposition 6) which claims the existence and uniqueness for small times. The proposition is rather standard and is proved by application of the contraction mapping method.

In Chapter 4 we realize Step #3 of our plan, we find a periodic solution to the equation in (8). We follow the ideas of Crandall and Rubinowitz [1] and look for a one-parameter family of periodic solutions in the form

$$u^s = s(e_0 + v(s)) \quad , \quad p(s) \ , \quad \alpha(s)$$

where $e_0 \equiv e_0(t)$ is a periodic solution of the linearized problem, $v \in V$, a subspace of periodic functions complementary to $\{e_0, e_1\}$. (Recall that the linearized problem has two linearly independent periodic solutions corresponding to purely imaginary eigenvalues).

To this end following [1] we introduce a function

$$Y(s,p,\alpha,v) = \begin{cases} s^{-1}\phi(p,\alpha,s(e_0+v)) \ , & s \neq 0 \\ \phi_u(p,\alpha,0)(e_0+v) \ , & s = 0 \end{cases}$$

In Sec. 4.1 we carry out the preliminary technical work and show that Y is smooth (in fact infinitely smooth). After that, in Sec. 4.2, we prove that the Frechét derivative of Y with respect to p,α and v at the point $(0,1,\alpha_0,0)$ is an isomorphism and therefore the implicit function theorem is applicable to the equation $Y = 0$. This yields the desired family of solutions to the problem $\phi(u) = 0$. The solution to the original problem in (2) through (4) is $\theta = \theta^0 + u^s$ (recall that θ^0 is a steady state solution).

Having a periodic solution, we study in Section 4.3 its regularity properties. The guiding observation is that the solution θ of the nonlinear equation (2) solves the linear equation

$$u_t = u_{xx} - e^{\alpha(\theta(0,t)-1)} u_x \quad . \tag{12}$$

Considering (12) in the domain $(0,A) \times (0,T)$ with boundary conditions

$$u(0,t) = \theta(0,t) \quad , \quad u(A,t) = \theta(A,t)$$

and the initial data $u(x,0) = \theta(x,0)$ and employing the fact that boundary and initial conditions are, as easily seen, continuous we obtain that u is a classical solution. Therefore, by uniqueness, $\theta = u$ is classical also.

In order to prove boundedness we consider (12) in the domain $(0,\infty) \times (0,2\pi)$ with initial data

$$u(0,t) = \theta(0,t) \quad , \quad u_x(0,t) = \theta_x(0,t)$$

and periodic boundary conditions. Thus (12) is regarded as a Cauchy problem with x playing the role of time.

We consider the Laplace transform of the solution which is a regular function for $\mathrm{Re}\, z > a$, where a is the exponent of the weight determining the growth of the solution. Studying the resolvent of the evolution operator of the Cauchy problem we show that for $\mathrm{Re}\, z \leq a$ all singularities of the Laplace transform, except for a pole at 0, lay in $\{\mathrm{Re}\, z < 0\}$. With this result the solution is seen to be bounded.

In conclusion we should say that although stability analysis is not touched in this dissertation, some helpful tools were developed here. We leave the analysis for the future work.

1. THE MATHEMATICAL MODEL

In the present chapter we derive the model of solid fuel combustion, which was introduced by Matkowsky and Sivashinsky in [10]. To this end we apply the methods of matched asymptotic expansions and multiple scaling. We follow the way outlined in [10] and realized in [15] for a similar problem.

1.1. Nondimensionalization

We start from the system of equations, which describes the combustion of solid fuel. The system describes fuels, in which combustion occurs without preliminary gas phase formation and without gaseous combustion products. Moreover, since the fuel and combustion products are both solid, we suppose that there is no species diffusion

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial y^2} + QW(C,T) \quad , \quad \frac{\partial c}{\partial t} = -W(C,T) \quad , \tag{1}$$

$-\infty < y < \infty$, $t > 0$. T and C represent temperature and fuel concentration of the component which limits the reaction (thus, we regard a one component reaction). The constant κ denotes the heat conductivity of the solid mixture, Q is the heat release in the reaction and W represents the chemical reaction rate. We will assume W is given by Arrhenius law

$$W = ZCe^{-E/(RT)} \quad .$$

Here E is the effective activation energy, R is the universal gas constant and Z is some (big) normalization factor.

We introduce nondimensional variables as follows. Let C_0 and T_0 be the concentration and the temperature of the unburned mixture. For the reaction, propagating uniformly (with some velocity U), temperatures of the unburned mixture, T_0, and of burned one, T_b, are connected by the relation $T_b = T_0 + Q\,C_0$. We set

$$\tilde{C} = C/C_0 \quad , \quad \tilde{T} = T/T_b \quad , \quad \tilde{t} = t\,U^2/\kappa \quad , \quad x = yU/\kappa \quad ,$$

$$\sigma = T_0/T_b < 1 \quad , \quad N = E/(RT_b) \quad , \quad w = W\kappa/(C_0U^2) \quad ,$$

$$A = Z\kappa/(U^2Ne^N) \quad .$$

We will drop tildas at new variables, it will not cause any confusion, since dimensional variables will not appear any more. The new parameters have the following meaning. σ is the ratio of the temperature of the cold solid mixture to that of the hot combustion products, N is a nondimensional activation energy. The normalization of the time and the parameter A are chosen in order to normalize the velocity of normal propagation to one. w represents a nondimensional reaction rate. In new variables, equations (1) take the form

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + (1-\sigma)w(C,T) \quad , \quad \frac{\partial C}{\partial t} = -w(C,T) \quad , \tag{2}$$

where

$$w = ANC\,e^{N(1-1/T)} \quad .$$

1.2. Large Activation Energy Asymptotics

We assume that the combustion front is given by a non-parametric equation of the form

$$x - \phi(t) = 0 .$$

In the coordinate system attached to the front equations (2) become

$$\begin{aligned} \frac{\partial T}{\partial t} - \phi' \frac{\partial T}{\partial y} &= \frac{\partial^2 T}{\partial y^2} + (1-\sigma)\, w(C,T) , \\ \frac{\partial C}{\partial t} - \phi' \frac{\partial C}{\partial y} &= -w(C,T) . \end{aligned} \tag{3}$$

where $\phi' = \partial\phi(t)/\partial t$; y is a distance to the front

$$y = x - \phi(t) .$$

It is convenient to rescale and shift the temperature and introduce θ by

$$T = (1-\sigma)\,\theta + \sigma , \tag{4}$$

so that in new units of measurement the temperature of the cold mixture is 0 and the temperature of the combustion products for normal propagation remains 1. Then (3) takes the form

$$\begin{aligned} \theta_t - \phi'\theta_y &= \theta_{yy} + w_0(C,\theta) \\ C_t = \phi' C_y &= -w_0(C,\theta) , \end{aligned} \tag{5}$$

where the chemical kinetics function is

$$w_0(C,\theta) = ANC\, e^{2\alpha(\theta-1)/(\sigma+(1-\sigma)\theta)} . \tag{6}$$

Here

$$\alpha = (1-\sigma) N/2 , \tag{7}$$

the coefficient $A = O(1)$ could be a function of θ to be specified later on.

We suppose N to be large, $N >> 1$ and $\alpha = O(1)$. For large N the reaction rate depends strongly on temperature. We will show that asymptotically, for $N \to \infty$, the chemical reaction could be approximated by a concentrated heat source on the front.

We assume the chemical reaction taking place only in some vicinity of the flame front, and the thickness of this reaction zone being of order 1/N.

In a region of order unity adjacent to the reaction zone behind the front we have the following regular perturbation expansions

$$\begin{aligned} \theta^b(y,t) &= \theta_0^b(y,t) + \varepsilon\, \theta_1^b(y,t) + \ldots \\ C^b(y,t) &= C_0^b(y,t) + \varepsilon\, C_1^b(y,t) + \ldots \end{aligned} \tag{8}$$

where $\varepsilon = 1/N$ and the superscript "b" stands for burnt. For definitiveness we will assume that the burnt state corresponds to $x > \phi(t)$ (i.e., $y > 0$). Similar expansions are valid for the unburnt region ahead the front with replacement of the superscript "b" by the superscript "u".

We assume that

$$C_0^b(y,t) = 0 \quad , \quad C_0^u(y,t) = 1 . \tag{9}$$

The first assumption means that during the reaction the fuel is burnt totally, and the second one is also rather natural, since there is no diffusion of species and the reaction does not take place in the preheating zone corresponding to the unburnt state.

To deal with the reaction zone we perform rescaling and set

$$Y = \frac{1}{\varepsilon} y .$$

Then the equations in (5) take the form

$$\begin{aligned} \theta_t - \frac{\phi'}{\varepsilon} \phi_Y &= \frac{1}{\varepsilon^2} \theta_{YY} + w_0 (C,\theta) \\ C_t - \frac{\phi'}{\varepsilon} C_Y &= w_0(C,\theta) . \end{aligned} \tag{10}$$

In the reaction zone we employ expansions analogous to (8)

$$\begin{aligned} \theta(Y,t) &= \theta_0(Y,t) + \varepsilon \theta_1(Y,t) + \dots \\ C(Y,t) &= C_0(Y,t) + \varepsilon C_1(Y,t) + \dots \end{aligned} \tag{11}$$

Substituting (11) into (10) and collecting equal orders of ε we obtain a sequence of equations. First for $1/\varepsilon^2$ we have

$$\partial^2\theta_0/\partial Y^2 = 0 .$$

It follows

$$\theta_0(Y,t) = T_0(t) + YT_1(t) .$$

We wish to have an asymptotic expansion for θ valid in some finite region with respect to y, and θ_0 must be bounded there. It immediately follows $T_1 \equiv 0$ and

$$\theta_0(Y,t) = T_0(t) . \tag{12}$$

Comparing expansions in (8) and (11) we see that $\theta_0{}^b(+0,t) = \theta_0(+\infty,t)$ and analogously $\theta_0{}^u(-0,t) = \theta_0(-\infty,t)$, therefore (12) implies

$$\theta_0^u(-0,t) = \theta_0^b(+0,t) \quad . \tag{13}$$

In the reaction zone we will distinguish an 'extinguishing' zone which is a zone where the reaction rate is essentially decreasing due to burning out the fuel. In the extinguishing zone we will assume that the expansion for C in (11) starts from the second term. Using the translation invariance of the problem we can suppose the extinguishing zone to be situated at $Y \geq 0$. Thus our assumption is

$$C_0(Y,t) \equiv 0 \qquad \text{for} \qquad Y \geq 0 \quad . \tag{14}$$

Collecting terms with $1/\varepsilon^2$ in (10) we obtain the system of equations

$$\begin{aligned} \frac{\partial^2\theta_1}{\partial Y^2} + A\, C_0\, e^{2\alpha(\theta_0-1)} &= 0 \\ \phi' \frac{\partial C_0}{\partial Y} - A\, C_0\, e^{2\alpha(\theta_0-1)} &= 0 \quad . \end{aligned} \tag{15}$$

Adding the equations in (15) we get

$$\frac{\partial}{\partial Y} \left(\frac{\partial \theta_1}{\partial y} + \phi' \; C_0\right) = 0 \quad ,$$

which implies that

$$\theta_{1Y}(Y,t) + \phi'(t)C_0(Y,t) = k(t) \quad , \tag{16}$$

where $k(t)$ is a function of t alone. For $Y \geq 0$ (16) gives (recall (14))

$$\theta_{1Y}(Y,t) = k(t) ,$$

and as earlier (see (12)) we get $k(t) = 0$.

If expansions in (8) and (11) have some common domain of validity and if they can be differentiated there, then

$$\frac{\partial\theta_0^b}{\partial y} = \frac{1}{\varepsilon}\frac{\partial\theta_0^b}{\partial Y} = \frac{1}{\varepsilon}\left(\frac{\partial}{\partial Y}(\theta_0 + \varepsilon\theta_1)\right) = \frac{\partial\theta_1}{\partial Y} ,$$

and the boundary condition follows

$$\frac{\partial\theta_0^b}{\partial y}(+0,t) = \frac{\partial\theta_1}{\partial Y}(\infty,t) .$$

A similar condition holds for the unburnt state. Therefore for $Y \to \pm\infty$ we obtain from (16) the following relation

$$\frac{\partial\theta_0^b}{\partial y}(+0,t) - \frac{\partial\theta_0^u}{\partial y}(-0,t) - \phi'(t) = 0 . \tag{18}$$

Remark 1. The relation (18) gives in zeroth approximation a connection between the jump of the temperature gradient across the combustion front and the velocity of the front. To obtain (18) we did not use the explicit form of the kinetic function but only the fact that it is of order $1/\varepsilon$ and that the reaction takes place in a zone of thickness $\sim\varepsilon$.

In order to evaluate $\phi'(t)$ we consider the reaction rate term more thoroughly.

As $\sigma - 1 = 2\alpha/N \sim \varepsilon$, follows from (7), the exponent in (6) can be expanded in an asymptotic series of the form

$$
\begin{aligned}
2\alpha(\theta-1)/[\sigma+(1-\sigma)\theta] &= 2\alpha(\theta-1)/[1+(1-\sigma)(\theta-1)] \\
&= 2\alpha(\theta_0-1) + \varepsilon(2\alpha\ \theta_1-\theta_0+1) + \ldots \\
&= (2\alpha-\varepsilon)(\theta_0-1) + 2\alpha(\theta-\theta_0) + \varepsilon^2(\ldots) + \ldots \qquad (19)
\end{aligned}
$$

where the first term does not depend on Y, in the calculation we used the equality

$$\theta - \theta_0 \doteq \varepsilon\ \theta_1 + \varepsilon^2\ \theta_2 + \ldots \qquad (20)$$

We consider $\theta_0(t)$ as the temperature on the front. We will assume that processes of combustion are similar for all t. Namely we assume that within ε^2

$$A = A\left(\frac{\theta-\theta_0}{\varepsilon}\right) , \qquad (21)$$

where $A \geq 0$ is a sufficiently nice function, say A(s) has a unique maximum at s = 0 and A $L^1(-\infty,0)$.

Now we are in a position to determine the velocity ϕ'. Taking into account k(t) = 0 we can express C_0 from (16) and substitute it into the first equation in (15). We obtain

$$\frac{\partial^2\theta_1}{\partial Y^2} - e^{(2\alpha-\varepsilon)(\theta_0-1)}\ \frac{\theta_1'}{\phi'}\ A\left(\frac{\theta-\theta_0}{\varepsilon}\right) e^{2\alpha(\theta-\theta_0)} = 0 . \qquad (22)$$

Integrating (22) from Y = $-\infty$ to Y = 0 and employing again (20) we have after setting $\varepsilon = 0$

$$\left.\frac{\partial\theta_1}{\partial Y}\right|_0 - \left.\frac{\partial\theta_1}{\partial Y}\right|_{-\infty} - \frac{e^{2\alpha(\theta_0-1)}}{\phi'} \cdot A_0 = 0 , \qquad (23)$$

where

$$A_0 = \lim_{\varepsilon \to 0} \int_{-\infty}^{0} \left(\frac{\theta-\theta_0}{\varepsilon}\right)' A\left(\frac{\theta-\theta_0}{\varepsilon}\right) e^{2\alpha(\theta-\theta_0)} \, dY$$

$$= \lim_{\varepsilon \to 0} \int_{0}^{\infty} A(x) e^{-2\alpha x \varepsilon} \, dx = \int_{0}^{\infty} A(x) \, dx \quad .$$

<u>Remark 2.</u> A_0 is a "universal" constant. Considering the explicit solution, corresponding to the uniform propagation, one may show that $A_0 = 1$.

In the same way we have obtained (18) we can derive from (23) the equality

$$\frac{\partial \theta_0^b}{\partial y} (+0,t) - \frac{\partial \theta_0^u}{\partial y} (-0,t) - \frac{1}{\phi'} e^{2\alpha(\theta_0(t)-1)} = 0 \quad ,$$

which together with (18) give

$$\phi' = e^{\alpha(\theta_0(t)-1)} \quad , \tag{24}$$

where we have chosen the minus sign for ϕ' since in our notation the reaction propagates in negative direction.

Summarizing (9), (13), (18), and (24) we can say than in zeroth approximation the asymptotic model has to satisfy the following conditions

$$[\theta] = 0, \quad [\theta_y] + e^{\alpha(\theta(0,t)-1} = 0 \quad , \quad [C] = -1$$

where $[f] = f(+0)-f(-0)$ denotes the jump of f across the front.

It is easily seen that the same conditions follow if we replace the distributed heat source by a concentrated one with the intensity $e^{\alpha(\theta(t)-1)}$.

The equations in (5) then take the form

$$
\begin{aligned}
\theta_t - \phi'\theta_y &= \theta_{yy} + e^{\alpha(\theta(0,t)-1)}\delta(y) \\
C_t - \phi' C_y &= -e^{\alpha(\theta(0,t)-1)}\delta(y)
\end{aligned}
\tag{25}
$$

where $\delta(y)$ is Dirac's δ-function.

We observe that the assumption in (9) implies that the initial data to (25) must be chosen so that

$$
C(y,t) = \begin{cases} 1 & \text{for } y < 0 \\ 0 & \text{for } y > 0 \end{cases}
$$

Thus the last equation in (25) can be eliminated and we come to the problem governed by the single equation

$$
\theta_t + e^{\alpha(\theta(0,t)-1)}\theta_y = \theta_{yy} + e^{\alpha(\theta(0,t)-1)}\delta(y) \quad . \tag{26}
$$

The equation with δ-function should be understood in the following sense. $\theta(y,t)$ satisfies the equation

$$
\theta_t + e^{\alpha(\theta(0,t)-1)}\ \theta_y = \theta_{yy} \quad , \tag{27}
$$

separately in each of the regions $y > 0$ and $y < 0$ subject to the jump conditions

$$
[\theta] = 0 \quad , \quad [\theta_y] + e^{\alpha(\theta(0,t)-1)} = 0 \quad , \tag{28}
$$

and natural boundary conditions for (27) are

$$\theta(-\infty,t) = 0 \quad , \quad \theta(\infty,t) < \infty \quad . \tag{29}$$

<u>Remark 3</u>. In this paper we use δ-functions only as a convenient shorthand for jump conditions of the type (28).

2. LINEARIZED PROBLEM

In Chapter 1 we derived the model equation

$$\theta_t = \theta_{yy} - e^{\alpha(\theta(0,t)-1)}\,\theta_y \quad , \tag{1}$$

which must be satisfied separately in the regions $y > 0$, and $y < 0$ with jump conditions at 0

$$[\theta] = 0 \quad , \quad [\theta_y] + e^{\alpha(\theta(0,t)-1)} = 0 \quad , \tag{2}$$

subject to boundary conditions

$$\theta(-\infty,t) = 0 \quad , \quad \theta(\infty,t) < \infty \quad . \tag{3}$$

The equation and the jump condition for the derivative are nonlinear. In this chapter we linearize the problem around a time independent solution. We consider the linearized equation as an evolution equation in a Banach space with some weight. We show that the evolution operator has two eigenvectors with eigenvalues crossing the imaginary axis when the parameter α varies. By an appropriate choice of the weight one can achieve that the operator, considered in a subspace complementary to eigenvectors, will generate an analytic semigroup with an estimate for the norm, exponentially decaying in time.

In the last subsections we give a brief account of some properties of semigroups which will be necessary for solving the nonlinear problem in Chapter 3. Also for the future use we introduce fractional norms generated by the evolution operator. We show that they are equivalent to Sobolev norms and therefore can be easily estimated.

All Banach spaces appearing in this chapter are supposed to be real with the customary caveat that the spectral theory is done in the complexification.

2.1. Linearization

As easy to see the nonlinear problem in (1) through (3) has a time independent solution

$$s^0(y) = \begin{cases} e^y , & y < 0 \\ 1 , & y \geq 0 \end{cases} .$$

It follows from (1.24) that the front speed corresponding to this steady state solution is -1. Thus s^0 represents the uniformly propagating plane front and its existence confirms a posteriori correctness of the normalization in Remark 1.2.

s^0 is a steady state solution for all values of the parameter α. Our goal is to study stability of s^0 and its possible time-periodic bifurcations when α changes.

We introduce $u = \theta - s^0$ and rewrite (1) in the form

$$\begin{aligned} u_t \;=\; & u_{yy} - u_y - \alpha u(0,t)\,(s_y^0 - \delta) \\ & + (1 - e^{\alpha u(0,t)})\, u_y + (s_y^0 - \delta)\,(1 + \alpha u(0,t) - e^{\alpha u(0,t)}) \, . \end{aligned} \tag{4}$$

In the derivation of (4) we've used the identity

$$s_{yy}^0 - s_y^0 + \delta = 0 \, .$$

Recall that jump conditions at 0 are served by δ-functions in (4). Boundary conditions for u are the same as for θ

$$u(-\infty,t) = 0 \quad , \quad u(\infty,t) < \infty \ .$$

Separating in (4) the linear part, one can represent the equation (4) as

$$u_t = A(\alpha)u + R(u,u_y) \ .$$

We will study the linear equation $u_t = Au$ which makes the following sense: solve the equations

$$u_t = A(\alpha)u \equiv \begin{cases} u_{yy}-u_y^{-\alpha(0,t)e^y} & , \ y < 0 \\ u_{yy}-u_y & , \ y > 0 \end{cases} \tag{5}$$

subject to the jump conditions

$$[u] = 0 \quad , [u_y] + \alpha u(0,t) = 0 \ , \tag{6}$$

and the boundary conditions

$$u(-\infty,t) = 0 \quad , \ u(\infty,t) < \infty \ . \tag{7}$$

The problem in (5), (6), and (7) is the linearization of the problem (1) through (3).

2.2. Weighted Norms

The operator A is an ordinary differential operator. Its spectral properties could depend on the Banach space on which A acts. As we will see in the next section, A demonstrates in a sense a "bad" behavior if considered in a natural Banach space. But it turns out that introducing an appropriate weighted norm one can "improve" spectral properties of A.

This idea was introduced and successfully employed by Sattinger in [12], [13].

Let B denote the space of complex-valued functions on the rays $x < 0$ and $x > 0$, $B = L^2(-\infty,0) + L^2(0,\infty)$, with norm denoted by $\|.\|$. Let $w(x) > 0$ be a smooth function. We define the norms

$$\|u\|_{w,0} = \|uw\| ,$$

and

$$\|u\|_{w,j} = \|u\|_{w,0} + \|u_x\|_{w,0} + \dots + \|(d/dx)^j u\|_{w,0} .$$

By $B_{w,j}$ we denote the Banach space of functions with finite $\|.\|_{w,j}$ norm

$$B_{w,j} = \{u \,|\, wu \in B,\dots,w(d/dx)^j u \in B\} .$$

There is a simple relation between the norm of an operator K acting on Banach spaces with uniform and weighted norms.

<u>Proposition 1</u>. Let K and K_w be connected by the similarity transformation (8). Then

(i) $\|K_w\| = \|K\|_{w,0}$, where $\|.\|$ and $\| \|_{w,0}$ are the operator norms associated with the vector norms in B and $B_{w,0}$;

(ii) let

$$\sup_x \left|w_x(x)/w(x)\right| = C_w < \infty ,$$

then $\|K\|_{w,0\to1}$,

the norm of K considered as a mapping from $B_{w,0}$ to $B_{w,1}$, can be estimated as follows

$$\|K\|_{w,0\to 1} \leq (1 + C_w) \|K_w\| + \|K_w\|_{0\to 1} .$$

Proof. The proof is straightforward. Let $f \in B_{w,0}$.

(i).

$$\|Kf\|_{w,0} = \|wKf\| = \|wKw^{-1}wf\|$$
$$\leq \|wKw^{-1}\| \|wf\| = \|K_w\| \|f\|_{w,0} ,$$

therefore $\|K\|_{w,0} \leq \|K_w\|$. The inverse inequality follows after the replacement of w by w^{-1}.

(ii). We have by definition

$$\|K_f\|_{w,1} = \|(d/dx)Kf\|_{w,0} + \|Kf\|_{w,0} .$$

By virtue of (i) we get for the second summand on the right-hand side

$$\|(d/dx)Kf\|_{w,0} = \|(d/dx)wKf - (w_x/w) wKf\|$$
$$\leq \|(d/dx)K_w wf\| + C_w\|K_w wf\|$$
$$\leq \|K_w\|_{0\to 1} \|f\|_{w,0} + C_w\|K_w\| \|f\|_{w,0} ,$$

and the proposition follows.

Proposition 1 shows that the estimates in weighted norms can be reduced to estimates in unweighted norms for transformed operators.

2.3. The Resolvent and the Spectrum of the Linearized Operator

In order to study stability properties and bifurcations of the time dependent solution we need some information about the asymptotic behavior of the resolvent transformation $(A-\lambda)^{-1}$ and the spectrum of A.

We consider the family of weight functions $w_b(y) = e^{-by/2}$, $0 \leq b \leq 1$, and denote $B_{b,j}$ and $\| \ \|_{b,j}$ the corresponding weighted spaces and norms. Note that $B_{0,0} = B$.

Proposition 2. Consider A as a transformation on $B_{b,0}$. Then

(i) A has two complex-conjugate eigenvalues $\lambda_\pm(\alpha)$ which cross the imaginary axis when α reaches the value $\alpha_0 = 2 + \sqrt{5}$. Except for $\lambda_\pm(\alpha)$ the exterior of a parabola Π_b is contained in the resolvent set of A, where

$$\Pi_b \quad = \quad \{\lambda \,|\, \mathrm{Re}\ \lambda = b^2/4 - b/2 - (\mathrm{Im}\ \lambda)^2/(1-b)^2\}$$

(ii). There exists an invariant projector, Q, on the space spanned by eigenvectors with eigenvalues λ_+ and λ_- so that QAQ = AQ, PAP = AP, where P = I-Q.

(iii). Let A_p = PAP be the invariant part of A acting on $PB_{b,0}$. Then $(\lambda-A_p)^{-1}$ has the following asymptotic behavior. For any $0 > \varepsilon > b^2/4 - b/2$ and $\delta > 0$ such that the rays $|\arg(\lambda-\varepsilon)| = \pi-\delta$ lay outside Π_b, there exists $C(\delta) > 0$ so that for all

$$\lambda \quad S_{\varepsilon,\delta} \quad = \quad \{\lambda, |\arg(\lambda-\varepsilon)| \leq \pi-\delta \ , \ \lambda \neq \varepsilon\}$$

$$\text{a)} \quad \|(\lambda - A_p)^{-1}u\|_{b,0} \ \leq \ C(\delta) \ |\lambda-\varepsilon|^{-1} \|u\|_{b,0}$$

$$\text{b)} \quad \|(\lambda - A_p)^{-1}u\|_{b,1} \; < \; C(\delta)\,|\lambda-\varepsilon|^{-1/2}\|u\|_{b,0} \quad .$$

Remark 1. The parabolas Π_b cross the real axis at $x = b^2/4-b/2$. We see that for b=0 the spectrum of A, considered as an operator on the space B (without weight), touches the origin. For $b = 1$, Π_b degenerates into the ray $x \leq -1/4$. And for $0 < b < 1$ one has parabolas laying in $\mathrm{Re}\,\lambda < 0$ and separated from the origin.

Remark 2. Passing over to the problem in $B_{b,0}$, $b \neq 0$, we are looking for solutions which presumably violate the boundary condition $|u(+\infty)| < C$. Indeed, the weight $e^{-by/2}$ demands the exponential decay for $y \to -\infty$ but allows an exponential growth for $y \to \infty$. We will show a posteriori that the time periodic solution found in $B_{b,0}$ is in fact bounded (see Sec. 4.3)

Proof of (i). To find the spectrum of A acting on $B_{b,0}$ we must determine λ for which $(A-\lambda)^{-1}$ is unbounded. As follows from Proposition 1, we may consider the operator

$$w_b(A-\lambda)^{-1}\,w_b^{-1} \;=\; (w_bAw_b^{-1}-\lambda)^{-1} \;\equiv\; (A_b-\lambda)^{-1} \quad ,$$

on B. Carrying out multiplications in the definition of A_b we get for A_b the following expression

$$A_bv \;=\; \begin{cases} v_{yy}-(1-b)v_y+(b^2/4-b/2)v \,-\, \alpha v(0)e^{(1-b/2)y} & , \text{ for } y < 0 \\ v_{yy}-(1-b)v_y+(b\ /4-b/2)v \quad , & \text{for } y^2 > 0 \end{cases} \quad . \tag{9}$$

We are looking for bounded solutions of the equation

$$(A_b - \lambda)u = 0 \tag{10}$$

satisfying the same jump conditions as for A (since the weight function is smooth)

$$[u] = 0 \quad , \quad [u_y] + \alpha u(0) = 0 \ . \tag{11}$$

The equation (10) can be easily solved explicitly, and the separate solutions for $y > 0$ and $y < 0$ are

$$\begin{aligned} \phi_\pm(y,\lambda) &= e^{\pm py+y(1-b)/2} \quad , \quad y > 0 \\ \psi_\pm(y,\lambda) &= (1 + \alpha/\lambda)\ e^{\pm py+y(1-b)/2} - \alpha/\lambda\ e^{(1-b/2)y} \quad , \quad y < 0 \end{aligned} \tag{12}$$

with $p = p(\lambda) = (1/4 + \lambda)^{1/2}$, we consider p in the plane, cut from $-\infty$ to $-1/4$ along the negative real axis. For $\lambda = 0$ the solutions ψ_+ and ψ_- take the form

$$\begin{aligned} \psi_+(y,0) &= (1 - \alpha y)\ e^{(1-b/2)y} \\ \psi_-(y,0) &= -\alpha y\ e^{(1-b/2)y} + e^{-by/2} \quad , \end{aligned}$$

we note that $\psi_+(y,\lambda)$ and $\psi-(y,\lambda)$ are regular at $\lambda = 0$.

We obtain a bounded solution of (10) and (11) in two cases:

(a) One can sew together the two decaying solutions $\phi_-(y)$ and $\psi_+(y)$. (b) ψ_+ and ψ_- are both bounded and one can construct a linear combination of them which meets ϕ_- at $y = 0$ with the suitable jump condition. Note, that ϕ_+ is always growing for $y > 0$ except for $b = 1$, $\lambda = -1/4$; $\psi_+(y)$, $y < 0$ is always bounded.

In case (a) the first condition in (11) holds automatically by virtue of the normalization in (12); $\phi_-(0) = \psi_+(0)$. The second jump condition gives the equation

$$4\lambda^2 + (1 + 4\alpha - \alpha^2)\lambda + \alpha = 0 \quad .$$

Since α is real the equation obviously has two complex-conjugate roots which cross the imaginary axis when $\alpha = \alpha_0$, where $\alpha_0 = 2+\sqrt{5}$ is a positive root of the equation $\alpha^2-4\alpha - 1 = 0$. We recall that negative α is physically senseless.

In case (b) one needs that Re $\{(1-b)/2-p(\lambda)y\} \geq 0$ for $|\psi_-| < C$ and Re $\{(1-b)/2-p(\lambda)y\} \geq 0$ for boundedness of ϕ_-. These inequalities, rewritten in terms of Re λ and Im λ, define the parabola Π_b. The jump conditions (11) determine a linear 2x2 system of equations with respect to coefficients C_+ and C_- such that the solution is given by $C_+\psi_+ + C_-\psi_-$ for $y < 0$ and by ϕ_- for $y > 0$. The determinant of the system is the Wronskian of ψ_+ and ψ_- which does not vanish.

Thus we have shown that (10), (11) have bounded solutions when $\lambda \in \Pi_b$ or $\lambda = \lambda_\pm(\alpha)$. It follows that for such λ $(A_b-\lambda)^{-1}$ is unbounded on B.

In order to complete the proof of (i) we need only to show that outside Π_b accept for $\lambda_\pm(\alpha)$ the resolvent is regular. This will be done by proving the estimates in (iii).

<u>Proof of (ii) and (iii)</u>. We turn now to studying the resolvent equation

$$(A_b - \lambda)u = f \quad . \tag{13}$$

In order to simplify some "long" expressions we carry out all calculations for $b = 1$, i.e., for degenerate Π_b. The case $b < 1$ does not cause any additional difficulty. As we underlined before, (13) is being understood as a system of two equations determined separately for $x < 0$ and $x > 0$ and connected with the help of the jump conditions. It is more convenient now to make the reflection of the independent variable, $y \to x = -y$, for the equation given on the negative semiaxis. We arrive at the system (recall that $b = 1$)

$$\begin{aligned} u_{xx} - 1/4\, u - \lambda u &= f \quad , \\ v_{xx} - 1/4\, v - \alpha v(0)\, e^{-x/2} - \lambda v &= g \quad , \end{aligned} \tag{14}$$

determined for $x \geq 0$ subject to boundary conditions

$$u(0) - v(0) = 0 \quad , \tag{15}$$

$$u_x(0) + v_x(0) + \alpha u(0) = 0 \quad . \tag{16}$$

Functions $f,g \in B_+ = L^2[0,\infty]$ are given, we seek solutions of the problem (14) through (16) belonging to B_+.

To solve the problem in (14) through (16) we first consider the boundary value problem for the system

$$u_{xx} - (1/4 + \lambda)u = f \quad , \tag{17}$$

$$v_{xx} - (1/4 + \lambda)v = g + \alpha u_0\, e^{-x/2} \tag{18}$$

$$u(0) = v(0) = u_0 \quad , \quad |u(\infty)| < C \quad , \quad |v(\infty)| < C \quad . \tag{19}$$

and after that we choose u_0 so that the jump condition in (16) would be satisfied.

It is not hard to check that the bounded solution of the BVP

$$u_{xx} - (1/4 + \lambda)u = f$$

$$u(0) = u_0$$

is given by

$$u(x) = u_0 e^{-px} + \int_0^\infty k(x,y)\, f(y)\, dy \quad , \tag{20}$$

where

$$k(x,y) = \frac{1}{2p} \begin{cases} e^{-px}(e^{-py} - e^{py}) & , \quad x > y \\ e^{-py}(e^{-px} - e^{px}) & , \quad x < y \end{cases}$$

is a Green function for zero boundary condition at 0. Plugging into (20) the function $g - \alpha u_0 e^{-x/2}$ in place of f we obtain the solution of equation (18)

$$v(x) = u_0\left(e^{-px} + \frac{\alpha}{\lambda} e^{-px} - \frac{\alpha}{\lambda} e^{-x/2}\right) + \int_0^\infty k(x,y) g(y) dy \quad . \tag{21}$$

Derivatives of u and v at x = 0 are given by the formulas

$$u_x\Big|_{x=0} = -u_0 p - \int_0^\infty e^{-py} f(y) dy \quad ,$$

$$v_x\Big|_{x=0} = u_0(-p - p\alpha/\lambda + \alpha/(2\lambda)) - \int_0^\infty e^{-py}\, g(y)\, dy \quad .$$

Employing the jump condition in (16) we obtain the expression for u_0:

$$u_0 = \int_0^\infty e^{-py}(f+g)dy/(-2p+\alpha-p\alpha/\lambda+\alpha/(2\lambda)) \quad . \tag{22}$$

As easy to check the denominator, $r(\lambda)$, vanishes only at $\lambda = \lambda_\pm(\alpha)$, and u_0 has simple poles at $\lambda = \lambda_\pm(\alpha)$.

For $|p| = (\lambda+1/4)^{1/2}$ sufficiently large, i.e., for $\lambda \geq R$ with some $R \gg 1$

$$r(\lambda) = \left|-2p+\alpha - p\alpha/\lambda + \alpha/(2\lambda)\right| \geq C|p| \quad . \tag{23}$$

Now we need the following result which will be proved after completion of the proof of the Proposition.

<u>Lemma 3</u>. Let K be an integral operator with the kernel

$$k(x,y) = \frac{1}{2p}\begin{cases} e^{-px}(e^{-py}-e^{py}) \ , & x > y \\ e^{-py}(e^{-px}-e^{px}) \ , & x < y \end{cases}$$

where $p = p(\lambda) = (\lambda + 1/4)^{1/2}$. Then for any $0 < \delta < \pi$, $\varepsilon < -1/4$, there exists $C > 0$ so that in the sector

$$S_{\varepsilon,\delta} = \{\lambda \in \mathbb{C}, \ |\arg(\lambda-\varepsilon)| \leq \pi - \delta\}$$

$$\|Kf\|_{L^2[0,\infty]} \leq \frac{C}{|p(\lambda)|^2} \|f\|_{L^2[0,\infty]} \quad . \tag{24}$$

We now estimate the L^2-norm of the solution of the resolvent equation (we denote by $\|.\|$ the norm in $L^2(0,\infty)$). We have from (22) and (23) by Hölder's inequality

$$|u_0| \leq \frac{C}{p} \left|\int e^{-py}(f+g)\right| \leq \frac{C}{p} (\|f\| + \|g\|) \|e^{-py}\| \quad .$$

Noticing that

$$\|e^{-py}\| = (1/(2 \text{ Re } p(\lambda))^{1/2} ,$$

we get for $u_0 \, e^{-px}$ the estimate

$$\|u_0 e^{-px}\| \leq \frac{C}{|p|} (\|f\| + \|g\|) \|e^{-px}\|^2$$

$$\leq \frac{C}{|p| \text{ Re } p} (\|f\| + \|g\|) .$$

When λ is in the sector $S_{\varepsilon,\delta}$ and $|\lambda| \geq R$,

$$|\text{Re } p(\lambda)| \geq C|p(\lambda)| \geq C|\lambda|^{1/2} ,$$

and we obtain

$$\|u_0 e^{-px}\| \leq C/|\lambda| (\|f\| + \|g\|) .$$

For the terms $u_0(e^{-px} \alpha/\lambda - e^{-x/2}\alpha/\lambda)$ entering the expression for $v(x)$ in (21) we have even better estimate with respect to λ.
To estimate the integral terms in (20) and (21) we employ the lemma and get the estimate

$$\|\int k(x,y) \, f(y)dy\| \leq C/|p(\lambda)|^2 \|f\| .$$

for all $\lambda \in S_{\varepsilon,\delta}$.

Thus we have in $S_{\varepsilon,\delta}$ the estimate for the resolvent

$$\|(A_1-\lambda)^{-1}\| \leq C_1/|r(\lambda)p(\lambda)| + C_2/|p(\lambda)|^2 , \qquad (25)$$

where $p(\lambda)$ is regular, and $r(\lambda)$ has poles at $\lambda = \lambda_{\pm}(\alpha)$, and for $|\lambda| \geq R$, $|p(\lambda)|^2 \geq C\lambda$, $|p(\lambda)\ r(\lambda)| \geq C|\lambda|$. The estimate in (25) enable us to define a bounded operator by means of a standard Cauchy integral of operational calculus

$$Q = \frac{1}{2\pi i} \int_C (A_1-\lambda)^{-1}\, d\lambda \quad ,$$

where C is a contour consisting of two circles around the poles. As is easy to verify (see [2]) Q is an invariant projector, $Q^2 = Q$, $QA_1 = A_1Q = QA_1Q$, it follows $P = I-Q$ is an invariant projector too. Thus after subtraction of the singularity the obtained operator function

$$(A_1-\lambda)^{-1} - (A_1-\lambda)^{-1}Q = (PA_1P-\lambda)^{-1} = (A_p-\lambda)^{-1} \quad ,$$

is regular in $S_{\varepsilon,\delta}$ and by virtue of (25) its norm satisfies the desired estimate (iii,a).

To obtain the estimate (iii,b) for the derivative we simply note that the differentiation of expressions (20) and (21) brings down a factor of order $p \sim |\lambda|^{1/2}$ for all terms except for $\alpha u_0 e^{-x/2}/\lambda$ which is itself of order $|\lambda|^{-3/2}$.

<u>Proof of Lemma 3</u>. The expression for Kf can be decomposed into the sum

$$Kf = \int_0^x k(x,y)\ f dy + \int_x^\infty k(x,y)\ f dy \quad . \tag{26}$$

We will estimate only the first summand, the estimate for the second one is similar. Recall that

$$k(x,y) = (e^{-p(x+y)} - e^{-p(x-y)})/(2p) \quad .$$

The contribution from the first summand of k is estimated as follows

$$\|e^{-px}/(2p)\int_0^x e^{-py}f dy\| \leq \|e^{-px}/2p\| \int_0^\infty |e^{-py}f| dy$$

$$\leq C/|p|\ \|e^{-px}\|^2\ \|f\| \leq C/|p|^2\|f\| .$$

We set

$$g(x) = \int_0^x e^{-p(x-y)}f(y)dy \quad , \quad s = \mathrm{Re}\ p > 0 .$$

Then for the contribution from the second summand in k we get

$$|g(x)| \leq e^{-sx}\left(\int_0^{x/2} e^{sy}|f|dy + \int_{x/2}^x e^{sy}|f|dy\right)$$

$$\leq e^{-sx}[(e^{sx}-1)/(2s)]^{1/2}\left(\int_0^{x/2}|f|^2dy\right)^{1/2}$$

$$+ e^{-sx}[(e^{2sx}-e^{sx})/(2s)]^{1/2}\left(\int_{x/2}^x|f|^2dy\right)^{1/2} .$$

We observe that $g(x) \to 0$ as $x \to 0$, since preintegral factors are bounded and both integrals tend to 0. As $x \to \infty$, the first preintegral coefficient and the second integral go to zero, while the other terms remain bounded, therefore $g(x) \to 0$ as $x \to \infty$.

Multiplying the derivative of g, $g_x = -pg + f$, by g and adding the complex-conjugate expression, we get

$$1/2\ (d/dx)|g|^2 = -\mathrm{Re}\ p|g|^2 + 2\mathrm{Re}(f\overline{g}) .$$

We integrate this equality from $x = 0$ to $x = \infty$ taking into account $g(0) = g(\infty) = 0$ and obtain

$$\int |g|^2 \, dx \;=\; 2/\mathrm{Rep} \int \mathrm{Re}(fg)\, dx \;\leq\; 2/\mathrm{Rep}\, \|f\|\, \|g\| \;, \qquad (27)$$

whence $\|g\| < C\|f\|/\mathrm{Rep}$.

2.4. Sobolev spaces

Here we introduce a scale of Sobolev spaces associated with inner boundary conditions (11).

Let $H^0 = L^2(R_+) + L^2(R_-)$, where $R_+ = (0,\infty)$ and $R_- = (-\infty,0)$, $H^2 = H^2(R_+) + H^2(R_-)$. We denote norms in H^0 and H^2 by $|\cdot|_0$ and $|\cdot|_2$, so that if $f = \{f_1,f_2\} \in H^2$ then

$$|f|_2^2 \;=\; |f|_0^2 + |\{f_{1x},f_{2x}\}|_0^2 + |\{f_{1xx},f_{2xx}\}|_0^2 \quad .$$

Let L be the operator

$$L = d^2/dx^2 - 1/4 \ .$$

We consider L acting in the space $\tilde{H}^0 \equiv H^0$ with $D(L) = \tilde{H}^2$, where

$$\tilde{H}^2 \;=\; \{f \in H^2 \,|\, (i)\ \langle f,\phi_1\rangle \;=\; 0 \;\;,\;\; (ii)\ \langle f,\phi_2\rangle \;=\; 0\} \;,$$

the functionals ϕ_1 and ϕ_2 are given by the jump conditions, for $f = \{f_1,f_2\}$

$$\langle f,\phi_1\rangle \;\equiv\; f_1(0) - f_2(0) \quad ,$$

$$\langle f,\phi_2\rangle \;\equiv\; f_{1x}(0) - f_{2x}(0) + \alpha f_1(0) \quad .$$

Conditions (i) and (ii) of the definition of $\widetilde{H}^2$ make sense because, by Sobolev's lemma (see also Lemma 5 below), ϕ_1 and ϕ_2 are continuous on H^2. By the same reason $\widetilde{H}^2$ is a closed subspace of H^0 in the H^2-topology.

L is symmetric on $\widetilde{H}^2$. Indeed integrating by parts we obtain

$$\begin{aligned}(Lf,g) &= \left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) (Lf)g\, dx = -\frac{1}{4}\int_{-\infty}^{\infty} fg\, dx \\ &\quad + f(0^-)\, g(0) - f(0^+)g(0) - \left(\int^{0} + \int^{\infty}\right) f_x\, g_x\, dx \\ &= -\frac{1}{4}(f,g) - g(0)[f_x(0)] \\ &\quad - f(0)g_x(0^-) + f(0)g_x(0^+) + \left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) f\, g_{xx}\, dx \\ &= (f,Lg) + g(0)\alpha f(0) - f(0)\alpha g(0) = (f,Lg) \quad .\end{aligned}$$

Also it is easy to verify that $\widetilde{H}^2$ is dense in H^0.

Analysis, similar to that in Sec. 2.3, shows that the resolvent $(L-z)^{-1}$ is regular on the complex plane with the cut-off along the real axis from $-\infty$ to $-1/4$ except for the point $z_0 = (\alpha^2-1)/4$ which corresponds to the eigenfunction

$$\phi_0 = \begin{cases} \exp(-x\alpha/2) & , \quad x > 0 \\ \exp(x\alpha/2) & , \quad x < 0 \end{cases} \quad .$$

Therefore the operators $(L \pm i)^{-1}$ are bounded, i.e., ranges of $L \pm i$ coincide with H^0. It follows (see e.g., Kato [8, Ch. V, §3]) that L is self-adjoint.

We set $L_0 = a - L$, where $a > z_0$, and define for $2 \geq s \geq 0$

$$|f|_s = |L_0^{s/2} f|_0 \quad .$$

$\widetilde{H}^s$ is defined as a completion of $\widetilde{H}^2$ under $|.|_s$-norm or equivalently as $D(L_0{}^{s/2})$. It is obvious that the norm, just introduced, is equivalent for $s = 2$ to the one introduced earlier.

Fractional powers of L_0 can be defined by means of the spectral resolution of L_0 (or as in Sec. 2.5).

There is another way to introduce a norm for $s = 1$, namely set

$$|||f|||_1^2 \equiv |\{f_{1x}, f_{2x}\}|_0^2 + |f|_0^2 \quad .$$

Proposition 4. The norms $|.|_1$ and $|||.|||_1$ are equivalent, i.e, $|f|_1 \leq C_1 |||f|||_1 \leq C_2 |f|_1$.

Proof. Let $f \in \widetilde{H}^2$, then, integrating by parts we get

$$|f|_1^2 = (L_0^{1/2} f, L_0^{1/2} f) = (L_0 f, f)$$

$$= - (\int_{-\infty}^{0} + \int_{0}^{\infty}) f_{xx} f \, dx + (a + 1/4) (f,f)$$

$$= (f_x(0^+) - f_x(0^-)) f(0) + (\int_{-\infty}^{0} + \int_{0}^{\infty}) f_x f_x dx + (a + 1/4) (f,f)$$

$$= - \alpha |f(0)|^2 + (a + 1/4) |f|_0^2 + (\int_{-\infty}^{0} + \int_{0}^{\infty}) f_x f_x dx$$

$$\leq \max \{a + 1/4, 1\} |||f|||_1^2 \quad .$$

Lemma 5. Let $|||f|||_1 < \infty$. For any $\varepsilon > 0$, there exists $A > 0$ such that

$$|f(0)|^2 \le \varepsilon |\{f_{1x}, f_{2x}\}|_0^2 + A|f|_0^2 \quad ,$$

ε and $A(\varepsilon)$ do not depend on f. Assuming the lemma proved we have

$$|||f|||_1^2 = |f|_0^2 + \left(\int_{-\infty}^{0} + \int_0^{\infty}\right) f_x f_x dx$$

$$= |f|_0^2 - f(0)\,[f_x(0)] - \left(\int_{-\infty}^{0} + \int_0^{\infty}\right) f f_{xx} dx$$

$$= \alpha|f(0)|^2 + |f|_0^2 + (f, (L_0 - 1/4 - a)\, f\,)$$

$$\le \alpha\, \varepsilon\, |f_x|_0^2 + (A\,\alpha + 3/4 - a)\, |f|_0^2 + (f, L_0 f) \quad .$$

Whence

$$(1 - \alpha\varepsilon)\,|||f|||_1^2 \le (A\,\alpha + 3/4 - a - a\varepsilon)\, |f|_0^2 + (f, L_0 f) \,.$$

Choosing ε and a so that

$$\alpha\varepsilon < 1 \quad , \quad A\alpha + 3/4 - \alpha\varepsilon - a \le 0 \quad ,$$

we get

$$|||f|||_1^2 \le |f|_1^2/(1 - \alpha\varepsilon) \quad .$$

It should be mentioned that it is not hard to power that all $|\cdot|_1$ norms obtained for different choices of a are equivalent.

<u>Proof of Lemma 5</u>. Let $e \in C_0^\infty(R_+)$, $e(0) = 1$, $e(x) = 0$, for $x \geq 1$. Set $e_R(x) = e(x/R)$, then, by Schwarz's inequality

$$|f(0)|^2 = \left|\int_0^\infty (f\ e_R)_x\ dx\right|^2 = \int (f_x e_R + (e_R)_x f)dx\Big|^2$$

$$\leq 2\left|\int f_x e_R dx\right|^2 + 2\left|\int f(e_R)_x dx\right|^2$$

$$\leq 2\int |f_x|^2 dx \int |e_R|^2\ dx + 2\int |f|^2\ dx \int |e_{Rx}|^2\ dx \quad ,$$

and the Lemma follows if we notice that

$$\int |e_R|^2\ dx = \int \left|e(x/R)^2\right| dx = R\int |e|^2 dx = C_1 R \quad ,$$

$$\int |(e_R)_x|^2\ dx = \frac{1}{R^2}\int |e_x(x/R)|^2 dx = \frac{1}{R}\int |e_x|^2\ dx = e_2/R \quad ,$$

and choose R so that $2C_1R < \varepsilon$.

2.5. <u>Some Facts From Semigroup Theory</u>

The theory of analytic semigroups is treated in a suitable for us context in a number of books (see e.g., Friedman [5], Yosida [6] and Henry [6]). We adduce here some facts (without proofs) from [6].

We recall the definition of a sectorial operator: a linear operator B is called sectorial in a Banach space X if B is closed densely defined and for some $0 < \delta < \pi/2$ and some $M \geq 1$ and real a, the sector

$$S_{a,\delta} = \{\lambda,\ |\arg(\lambda - a)| < \pi - \delta \quad , \quad \lambda \neq a\} \quad ,$$

is in the resolvent set of B and

$$\|(\lambda - B)^{-1}\| < |M/\ \lambda - a| \qquad \text{for all} \qquad \lambda \in S_{a,\delta} \quad .$$

Fact 1. If B is sectorial it generates an analytic semigroup by formula

$$e^{Bt} = \frac{1}{2\pi i} \int_C (\lambda - B)^{-1} e^{\lambda t} \, d\lambda \quad ,$$

where C is a contour in the resolvent set of B with $\arg\lambda \to \pm\theta$ for some $\theta \in (\pi/2, \pi-\delta)$.

Consider the problem

$$\frac{dx}{dt} - Bx = 0 \quad , \quad t > 0 \; , \quad x(0) = x_0 \quad , \tag{28}$$

where B is a sectorial operator in a Banach space X and $x_0 \in X$. A strong solution of (28) is by definition a function

$$x \in C((0,T),X) \;\cap\; C((0,T),D(B)) \;\cap\; C^1((0,T),X) \quad ,$$

such that x(t) satisfies (28) and $x(t) \to x_0$ in X as $t \to 0^+$.

Fact 2. The problem (28) has a unique solution. The solution is given by $x(t) = e^{Bt}x_0$.

We introduce fractional powers of -B. For any $\alpha > 0$ set

$$(-B)^\alpha = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{tB} \, dt \quad . \tag{29}$$

Fact 3. Let B be sectorial with $\mathrm{Re}\,\sigma(B) < 0$ (i.e., the spectrum of B lays in the left half-plane). Then for any $\alpha > 0$, $(-B)^\alpha$ is a bounded linear operator which is one-one and satisfies $(-B)^\alpha(-B)^\beta = (-B)^{\alpha+\beta}$, $\alpha,\beta > 0$. For $\alpha > 0$, $(-B)^{-\alpha}$ is defined as $((-B)^\alpha)^{-1}$ and $(-B)^\alpha(-B)^\beta = (-B)^{\alpha+\beta}$ on $D((-B)^\gamma)$ where $\gamma = \max(\alpha,\beta,\alpha+\beta)$.

Fact 4. Suppose B is sectorial and $\mathrm{Re}\,\sigma(B) < -d < 0$. Then for $\alpha \geq 0$ there exists $C_\alpha < \infty$ such that

$$\|(-B)^\alpha e^{Bt}\| < C_\alpha t^{-\alpha} e^{-dt} \quad , \quad t > 0 \quad ,$$

and if $0 < \alpha < 1$, $x \in D((-B)^\alpha)$, then

$$\|(e^{Bt}-1)x\| < \alpha^{-1} C_{1-\alpha} t^\alpha \|(-B)^\alpha x\| \quad ,$$

C_α is bounded for α in any compact set of $(0,\infty)$, C_α is also bounded as $\alpha \to 0^+$.

Having a negative sectorial operator one can define an analogue of Sobolev norms. If B is a sectorial operator in a Banach space X, define for each $\alpha \geq 0$

$$X^\alpha = D(B_1^\alpha) \quad \text{with the graph norm}$$

$$\|x\|_\alpha = \|B_1^\alpha x\| \quad , \quad x \in X^\alpha \quad ,$$

where $B_1 = -B - aI$ with a chosen so that $\mathrm{Re}\,\sigma(B_1) > 0$.

Fact 5. Different choices of a given equivalent norms on X^α. X^α is a Banach space in the norm $\|.\|_\alpha$ for $\alpha \geq 0$, $X^0 = X$ and for $\alpha \geq \beta \geq 0$, X^α is a dense subspace of X^β with continuous inclusion.

Fact 6. Let B_1, B_2 be two sectorial operators with the same domain and $\mathrm{Re}\,\sigma(B_j) > 0$, $j=1,2$. Let ${X_j}^\alpha$ be corresponding scales of Banach spaces. If for some $0 \leq \beta \leq 1$, $(B_1 - B_2)(-B_1)^\beta$ is a bounded operator then ${X_1}^\alpha = {X_2}^\alpha$ with equivalent norms for $0 \leq \alpha \leq 1$.

2.6. The Linearized Operator as a Generator of a Semigroup

In Sec. 2.3 we considered the linearized operator in Banach spaces parameterized by b, the exponent of the weight. As before we put for simplicity b = 1, i.e, choose the weight $w = e^{-x/2}$. We eliminate the weight considering instead of A the operator $wA\,w^{-1}$, for which we use the same notation, A (in Sec. 2.3 it was called A_1),

$$Au = \begin{cases} u_{xx} - u/4\ , & x > 0 \\ u_{xx} - u/4 + \alpha u(0)e^{x/2}, & x < 0 \end{cases} .$$

We consider A in the space $X \equiv \tilde{H}^0$, with $D(A) \supseteq \tilde{H}^2$ (for definitions of $\tilde{H}^0$ and $\tilde{H}^2$ see Sec. 2.4), A is well-defined on $\tilde{H}^2$ by virtue of Lemma 5. Consider on D(A) the graph-norm $|f|_0 + |Af|_0$. Due to invertibility of A, D(A) is closed in the graph-norm, hence, by definition, A is closed. We denote by X_1 the space D(A) provided with the norm

$$\|f\|_1 = |Af|_0 \quad ,$$

which is equivalent to the graph-norm since

$$|f|_0 + |Af|_0 \le \|A^{-1}\|\ |Af|_0 + |Af|_0 = C_1\|f\|_1 \le C_1(|f|_0 + |Af|_0) \quad .$$

Recall that $\tilde{H}^2$ is dense in H^0. Thus

$$\begin{array}{l} \text{A is a closed densely defined operator} \\ \text{on X with } D(A) = X_1 \end{array} \tag{30}$$

The estimate of the resolvent of A from Proposition 2 and (30) show that $A_P = PAP$ is sectorial in PX with the sector $S_{\varepsilon,\delta}$, $\varepsilon < 0$. (The invariant projector P and the sector are defined in Proposition 2). Therefore all the results of Sec. 2.5 are valid for A_P. Moreover in the course of the proof of Proposition 2 we have obtained the estimate of the resolvent, (25), which shows that A itself is sectorial in some sector $S_{\varepsilon',\delta'} \subset S_{\varepsilon,\delta}$. $S_{\varepsilon',\delta'}$ should be chosen so that the eigenvalues, $\lambda_\pm(\alpha)$, lay outside of it, thus $\varepsilon' > \mathrm{Re}\lambda_\pm(\alpha)$ and $\delta < \delta' < \pi/2$.

In Sec. 2.4 we have defined a scale of Sobolev spaces $\tilde{H}^s$, $0 \leq s \leq 2$, with norm $|\cdot|_s$. On the other hand, one may define Banach spaces X^α generated by A as in Fact 5, Sec. 2.5 with norms $\|\cdot\|_\alpha$.

<u>Proposition 6.</u> $H^{2s} = X^s$ with equivalent norms for $0 \leq s \leq 1$.

<u>Proof.</u> The proposition almost immediately follows from Fact 6. Set $B_1 = A$, $B_2 = -L_0$, where L_0 is the "Laplacian" generating H^s spaces. Let

$$Du = (B_1 - B_2)u = u(0)e^{x/2} H(x) ,$$

where $H(x) = 1$, $x < 0$, $H(x) = 0$, $x > 0$. Then by Lemma 5 for $u \in \tilde{H}^2$

$$\begin{aligned} |Au|_0 &\leq |B_2 u|_0 + |u(0)|\,|e^{x/2} H(x)|_0 \\ &\leq |u|_2 + C|u|_1 \leq C|u|_2 . \end{aligned}$$

Therefore $\|u\|_1 = |Au|_0 \leq C|u|_2$, similarly $|u|_2 \leq C\|u\|_1$. Thus $\tilde{H}^2 = X^1$ and norms are equivalent.

By the same reason for $s \geq 1/2$

$$|DB_2^{-s}u|_0 \leq C|(B_2^{-s}u)(0)| \leq C|B_2^{-s}u|_1$$
$$= C|B_2^{1/2}B_2^{-s}u|_0 = C|B_2^{-(s-1/2)}u|_0 \leq C|u|_0 .$$

In the calculations we used B_2^{α} has the semigroup property and $B_2^{-(s-1/2)}$ is bounded for $s - 1/2 > 0$ (Fact 3). Thus B_1 and B_2 satisfy the assumptions of Fact 6 and the conclusion of the Proposition follows.

Remark. We shall use Proposition 6 which for $s = 1/2$, by virtue of Proposition 4, means that

$$\|u\|_{1/2}^2 = |(-A)^{1/2}u|_0 \simeq |||u|||_1^2 = |\{u_{1x},u_{2x}\}|_0^2 + |u|_0^2 .$$

2.7 The Semigroup

Taking into account the invariant decomposition, $A = AP + AQ$ (where Q and P are invariant projectors, $Q + P = I$, Q is two-dimensional), one can write the exponential of A as

$$e^{tA}u = (e^{tA_P})u + \sum_{\lambda=\lambda_+,\lambda_-} e^{\lambda t}(u,\phi_{\bar{\lambda}}^*)\,\phi_\lambda , \tag{31}$$

where by $(\cdot,\cdot)$ is denoted the scalar product in X; ϕ_λ, $\lambda = \lambda_+,\lambda_-$, are eigenfunctions of A; ϕ_λ^* are eigenfunctions of the adjoint operator (note that $\bar{\lambda}_+ = \lambda_-$, $\phi_{\lambda-}$, $\bar{\Phi}_{\lambda+} = \phi_{\lambda-}$).

ϕ_λ are given by (12):

$$\phi_\lambda(y) := \begin{cases} (1+\alpha/\lambda)e^{py} - \alpha/\lambda\, e^{y/2} , & y < 0 \\ e^{-py} , & y > 0 \end{cases} , \tag{32}$$

where $p = p(\lambda) = (\lambda+1/4)^{1/2}$ and $\lambda = \lambda_\pm$ are the roots of the equation

$$4\lambda^2 + (1 + 4\alpha - \alpha^2)\lambda + \alpha = 0 \quad . \tag{33}$$

$\phi_\lambda{}^*$ can be easily calculated. Let $u \in D(A)$, $v \in H^2$. Integrating by parts we obtain

$$\begin{aligned}(Au,v) &= \left(\int_0^\infty + \int_{-\infty}^0\right)(u_{xx}-1/4\ u)\,\overline{v}dx + \int_{-\infty}^0 u(0)e^{x/2}\overline{v}(x)dx \\ &= \left(\int_0^\infty + \int_{-\infty}^0\right) u(\overline{v}_{xx}-1/4\overline{v})dx + u_x\overline{v}\Big|_{0_+}^{0^-} \\ &\quad + u(0)\left(\int_{-\infty}^0 e^{x/2}vdx + [v_x]_{x=0}\right) \quad .\end{aligned}$$

If we require that

$$\begin{aligned} v \in D(A^+) = \{v \in \widetilde{H}^2 \,|\, [v]_{x=0} &= 0 \quad ; \\ [v_x] + \alpha v(0) + \int_{-\infty}^\infty e^{x/1}v(x)dx &= 0\} \quad , \end{aligned} \tag{34}$$

and define on $D(A^+)$ the operator

$$A^+ = d^2/dx^2 - 1/4 \quad ,$$

separately for $x > 0$ and $x < 0$, then $(Au,v) = (u,A^+v)$. It is clear that $A^* \supseteq A^+$ (in fact, it is not hard to show, that $A^* = A^+$, i.e, $D(A^*) = D(A^+)$).

Solving the equation $A^+\phi = \lambda\phi$ with jump conditions from (34) one finds, the eigenfunctions are

$$\phi_\lambda^*(x) = C_\lambda \begin{cases} e^{px} & , x < 0 \\ e^{-px} & , x > 0 \end{cases} , \quad \lambda = \lambda_\pm , \tag{35}$$

where λ_+ and λ_- satisfy the equation in (33) and normalization constants C_λ are chosen so that $(\phi_\lambda, \phi_\lambda^*) = 1$.

Note that eigenfunctions from (32) or (35) belong to the complexification of the original Banach space X. Since A is real, it is easy to find real two-dimensional invariant spaces. For instance, take

$$x_0 = 2^{-1/2}(\phi_{\lambda_+} + \phi_{\lambda_-}) = 2^{-1/2}(\bar{\phi}_{\lambda_+} + \phi_{\lambda_+}), \quad x_1 = Ax_0/|\lambda| .$$

The space, spanned by x_0 and x_1 is clearly invariant (note that in view of (33), Ax_1 is expressed by means of x_0, x_1).

Analogously,

$$x_0^* = 2^{-1/2}(\phi_{\lambda_+}^* + \phi_{\lambda_-}^*) \quad \text{and} \quad x_1^* = A^+x_0^*/|\lambda|$$

form a basis in a real invariant space. Moreover, as easy to check

$$(x_0, x_0^*) = 1 , \quad (x_1, x_1^*) = 1 ,$$

$$(x_0, x_1^*) = (x_1, x_0^*) = \mathrm{Re}\lambda/|\lambda| . \tag{36}$$

Assume now, that $\alpha = \alpha_0$, then eigenvalues are purely imaginary, $\lambda_\pm = \pm i(\alpha_0)^{1/2}/2 \equiv \pm i\nu$. Denote by $A_0 = A(\alpha_0)$ the operator corresponding to α_0 and by $T(t) = e^{A_0 t}$ the exponential. Let, as before, A_p stand for $A_0 P_0$. Equalities in (36) show that when $\mathrm{Re}\lambda = 0$, i.e., for $\alpha = \alpha_0$, vectors $\{x_0, x_1\}$ and $\{x_0^*, x_1^*\}$

form dual bases. The expression (31) for the exponential can be rewritten in the form

$$T(t)u = e^{tA_p} + (u,x_0^*)\,(x_0 \cos \nu t + x_1 \sin \nu t)$$

$$+ (u,x_1^*)\,(-x_0 \sin \nu t + x_1 \cos \nu t) \quad . \tag{37}$$

The mapping $T(p_0)$, where $p_0 = 2\pi/\nu$, will play an important role in our birfurcation analysis. For an operator A in X, we will denote by N(A) and R(A) its null space and range in X.

Proposition 7. Consider $L \equiv T(p_0)$ on the space X. Then

(i) $\dim N(I-L) = 2$.

(ii) $R(I-L)$ is closed

(iii) $x \in R(I-L)$ if and only if $(x,x_0^*) = (x,x_1^*) = 0$

where x_0^* and x_1^* are the vectors, defined above.

Proof. Let Q be an invariant projector on the space spanned by x_0 and x_1, $Q = (.,x_0^*)x_0 + (.,x_1^*)x_1$, where vectors x_0, x_1, x_0^* and x_1^* were defined before. Obviously, $QA_0 = A_0Q$ and $P = I-Q$ is an invariant projector.

It is clear from (37) that $X_Q = QX \subseteq N(I-L)$. We will show that on a complement of X_Q, the operator I-L is invertable.

Consider A_p on the space $X_p = PX$. By Proposition 2, A_p is sectorial with a sector $S_{\varepsilon,\delta}$, where $\varepsilon < 0$. Then, in view of Fact 4 (Sec. 2.5), we have the estimate

$$\| e^{tA_p} \| \leq C\, e^{-dt} \quad , \quad \varepsilon < d < 0 \quad . \tag{38}$$

Employing (38) we estimate the Neumann series for $((I-L)P)^{-1} = (P-\exp(p_0 A_P))^{-1}$. We get

$$\sum_{n=0}^{\infty} (\exp(p_0 A_p))^n = \sum_{n=0}^{\infty} \exp(np_0 A_p)$$

$$\leq C \sum_{n=0}^{\infty} e^{-dnp_0} < \infty .$$

Therefore I-L is invertable on X_P, hence $N(I-L) \subseteq QX$ and $R((I-L)P) = X_P$ is closed. Automatically $R(I-L)$ is also closed, since $R(I-L) = R((I-1))P) + R((I-L)Q) = R((I-L)P)$.

The last statement of the proposition follows from the definition of Q, since $R(I-L) = X_P = (I-Q)X$.

Corollary 8. Conclusion of Proposition 7 remain true if L is considered on X^s, $s > 0$.

Proof. We observe that

1) Eigenfunctions ϕ_λ and $\phi_{\bar{\lambda}}$ (and hence x_0 and x_1) belong to X^s for any s.

2) $\phi_\lambda{}^*$ and $\phi_{\bar{\lambda}}{}^*$ (hence $x_0{}^*$, $x_1{}^*$) are continuous functionals on X^s. Indeed, by the Schwarz inequality,

$$\left|(x,\phi_\lambda^*)\right| = \bar{\lambda}^{-s}(x,(A_0^*)^s\phi_\lambda^*) = \lambda^{-s}(A_0^s x,\phi_\lambda^*)$$

$$\leq C \|x\|_s \left(\int_{-\infty}^{\infty}\left|\phi_\lambda^*(y)\right|^2 dy\right)^{1/2} = C\|x\|_s .$$

Therefore, Q is an invariant projector also in X^s.

3) Since $P = I-Q$ commutes with A_0, the estimate in (38) is valid in X^s

$$\|\exp(tA_p)u\|_s = \|A_0^s \exp(tA_p)u\| = \|\exp(tA_p)A_0^s u\|$$
$$\leq \|\exp(tA_p)\| \, \|u\|_s \quad .$$

The corollary follows from these observations.

3. LOCAL EXISTENCE AND UNIQUENESS FOR THE NONLINEAR PROBLEM

We return to our nonlinear problem

$$\theta_t = \theta_{yy} - e^{\alpha(\theta(0,t)-1)}\theta_y \quad .$$

More precisely we will be occupied with the equation (2.4) for $u = \theta-\theta^\circ$, where θ° is a steady state solution. Now we drop the boundary conditions $u(-\infty,t) = 0$ and $|u(\infty,t)| < \infty$, and seek a solution belonging to L^2 with the weight $w = e^{-y/2}$. We make the transformation $W: u \to w^{-1}u$ and deal with L^2 without the weight. We should note a fortunate circumstance that nonlinearities of the equation and of jump conditions depend only on $u(0,t)$ and are not affected by W.

After the similarity transformation (2.4) takes the form

$$u_t - u_{yy} + 1/4u + \alpha_0\, u(0,t)\,(S-\delta)$$

$$= (1 - e^{\alpha u(0,t)})(u_y + 1/2u) + (1 + \alpha_0 u(0,t) - e^{\alpha u(0,t)})\,(S-\delta)\ , \tag{1}$$

where $S(y)$ stands for $s_y{}^0(y)$ from (2.4), $S(y) = e^{y/2}$ for $y \leq 0$, $S(y) = 0$ for $y > 0$; α_0 is the critical value of the bifurcation parameter. Recall that (1) incorporates the inner jump conditions via δ-functions.

We wish to solve the Cauchy problem for (1) with the initial data $u(0,y) = u_0(y)$ which will be specified later. Formally we may reason as follows. Consider (1) as a linear non-homogeneous equation

$$u_t - A_0 u = F(\alpha,u)$$

(where $A_0 = A(\alpha_0)$ is the evolution operator on the left-hand side of (1)) and "solve" it using a "variation of constants" formula

$$u(t) = T(t)u_0 + \int_0^t (1-e^{\alpha u(0,r)})\, T(t-r)\, (u_y + \tfrac{1}{2} u)\, dr$$

$$+ \int_0^t (1+\alpha_0 u(0,r) - e^{\alpha u(0,r)})\, T(t-r)\, S dr$$

$$+ \int_0^t (1+\alpha_0 u(0,r) - e^{\alpha u(0,r)})\, T(t-r)\, \delta dr \quad , \tag{2}$$

where $T(t) = e^{A_0 t}$ is the semigroup, solving the linear problem. Immediately the questions arise in connection with the integral equation in (2). How to understand the action of the semigroup on the δ-function, and in what sense the integral equation is equivalent to the Cauchy problem for (1)?

Applying formally the linearized operator A_0 to F, the last term on the right hand side of (2), we observe that it should satisfy the equation

$$(\partial/\partial t - A_0)F = f(t)\delta \quad ; \quad F(y,0) = 0$$

where $f(t) = 1 + \alpha_0 u(0,t) - e^{\alpha u(0,t)}$. In Sec. 3.1 we give meaning to this auxiliary linear problem.

It is well known that in order to prove that a solution u of the integral equation (2) satisfies the differential equation, one needs certain Hölder continuity of u. In Sec. 3.2 we show that the term in (2) generated by the free boundary is Hölder continuous in some Hilbert spaces. In Sec. 3.3 we show that an analogous result is true for first terms in (2). Thus we justify the transfer to the integral equation. In the last section we prove existence and uniqueness results for the integral equations for sufficiently small initial data.

3.1. An Auxiliary Linear Problem

In this section and in the following one we omit the subscript "0" at α_0.

We again represent the problem with jump conditions at 0 as a system on the half-axis $x > 0$ (cf. Sec. 2.3). Consider the problem of finding a solution $u(t,x)$, $v(t,x)$ to the system

$$\begin{aligned} \partial u/\partial t &= u_{xx} - 1/4u \\ \partial v/\partial t &= v_{xx} - 1/4u - \alpha e^{-x/2}\, v(0) \end{aligned} \tag{3}$$

in the domain

$$Q_T = \{0 < t \leq T \;,\; x > 0\}$$

subject to initial and boundary conditions

$$u\Big|_{t=0} = v\Big|_{t=0} = 0 \tag{4}$$

$$\lim_{x \to +0} (u(x,t) - v(x,t)) = 0 \tag{5}$$

$$\lim_{x\to+0} (u_x(x,t) + v_x(x,t) + \alpha u(x,t)) + f(t) = 0 , \tag{6}$$

where f is given.

<u>Theorem 1</u>. The problem in (3) through (6) has a classical solution for any $f \in C((0,T))$ with $\sup|f| < \infty$.

<u>Proof</u>. The parabolic system in (3) has a non-standard feature, the presence of the term $\alpha e^{x/2}v(0) = \alpha e^{x/2}\langle v,\delta\rangle$, which prevents us from applying available theorems of the parabolic theory. However corresponding reasonings (see, e.g., Eidel'man [3, Ch. IV]) go on as well.

We are looking for a solution of the form

$$\begin{aligned} v(x,t) &= \int_0^t G_1(x,t-s)\ f(s)\ ds \\ u(x,t) &= \int_0^t G_2(x,t-s)\ f(s)\ ds \end{aligned} \tag{7}$$

with Green functions G_1 and G_2 given by

$$\begin{aligned} G_1(x,t) &= \int_{a-i\infty}^{a+i\infty} \frac{e^{\lambda t}}{P(\lambda)} \left(e^{-p(\lambda)x}(1 + \tfrac{\alpha}{\lambda}) - \tfrac{\alpha}{\lambda}\, e^{-x/2}\right) d\lambda , \\ G_2(x,t) &= \int_{a-i\infty}^{a+i\infty} \frac{e^{\lambda t}}{P(\lambda)}\ e^{-p(\lambda)x}\ d\lambda , \end{aligned} \tag{8}$$

where $a > 0$,

$$p(\lambda) = (\lambda + 1/4)^{1/2} , \quad P(\lambda) = -2p + \alpha - p\alpha/\lambda + \alpha/(2\lambda) .$$

We will always assume that the factor $1/(2\pi i)$ is included in $d\lambda$. The representation in (8) is certainly motivated by the fact that for any fixed λ the integrands in (8) solve the system of o.d.e. obtained from (3) by means of the Laplace transform. This system was discussed in Sec. 2.3. It is clear, the functions G_1 and G_2 are real.

We have to show that the integrals in (7) converge, define a solution, and satisfy initial and boundary conditions. We will carry out all the estimates only for $v(x,t)$, since the integral for it contains a summand defining $u(x,t)$ also. We divide the subsequent proof into simple steps.

(i) $G_1(x,t)$ is continuous for $x > 0$, $t \geq 0$ and $G_1(x,0) = 0$, $x > 0$.

For any $a > 0$ we have the estimate

$$|G_1(x,0)| \leq \int_{-\infty}^{\infty} (C \exp\{-x(|a| + |y|)^{1/2}\}/(|a| + |y|)^{1/2}$$

$$+ \; Ce^{-x/2}/(|a| + |y|)^{3/2}) \, dy$$

$$\leq C \exp\{-a^{1/2}x\}/x + Ce^{-x/2}/a^{1/2} \quad .$$

Tending $a \to \infty$ we get $G_1(x,0) = 0$. Continuity of $G_1(x,t)$ with respect to t is obvious. Continuity with respect to x follows from the uniform absolute convergence of the integral for G_1 when x takes values from a compact set.

(ii) A new contour of integration.

We recall that $P(\lambda)$ has zeros only at $\lambda = \lambda_+$ and $\lambda = \lambda_- = \overline{\lambda_+}$. It follows from the Cauchy theorem and Jordan's lemma that one can transfer from integration along the imaginary axis $\lambda \in \{a+iR\}$ to integration along a contour composed of two circles ($C^\pm$ around $\lambda_\pm$ and a contour by-passing the cut-off from $-\infty$ to $-1/4$. Then, after introducing $y = \lambda + 1/4$, the expression for G_1 takes the form

$$G_1(x,t) = \Big(\int_{C^+} + \int_{C^-}\Big)$$

$$\pm \frac{1}{2\pi i} \int_{-\infty}^{0} \frac{e^{(y-1/4)t}}{P_\mp(y)} \left(\exp\{\mp iy^{1/2}x\} \left(1 + \frac{\alpha}{y-1/4}\right) - \frac{\alpha}{y-1/4}\, e^{-x/2}\right) dy$$

$$= I_1(x,t) - \frac{1}{2\pi i}\, I_\mp(x,t)\, e^{-t/4} \quad ,$$

where the last term represents the sum of two integrals: one with all upper signs another with all lower signs; by $P_\pm$ are denoted

$$P_\pm(y) = \pm i\, y^{1/2} \left(2 + \frac{\alpha}{y-1/4}\right) + \frac{\alpha}{2y-1/2} + \alpha \ .$$

Integration along $C^\pm$ gives the corresponding residues

$$I_1(x,t) = \sum_{\lambda=\lambda+,\lambda-} C_\lambda e^{\lambda t} \left(e^{-p(\lambda)x}\left(1 + \frac{\alpha}{\lambda}\right) - \frac{\alpha}{\lambda}\, e^{-x/2}\right) \ . \qquad (9)$$

I_1 is a nice (infinitely smooth) function of x and t. We will be concerned with estimates of $I_\pm(x,t)$ and its derivatives. We fix for definiteness the plus sign (and drop subscript at P) and will estimate separately

$$F_1(t) = \int_0^\infty \frac{e^{-yt}dy}{P(y)(y+1/4)} \quad , \tag{10}$$

and

$$F_2(x,t) = \int_0^\infty \frac{e^{-yt}}{P(y)} \left(1 - \frac{\alpha}{y+1/4}\right) \exp\{iy^{1/2}x\}\, dy \quad ,$$

(introducing F_1 and F_2 we have changed variables: $y \to -y$). We note that

$$|F_1(t)| = \left|\int_0^\infty \frac{e^{-yt}dy}{P(y)(y+1/4)}\right| \leq \left|\int_0^a \cdots\right| + \left|\int_a^\infty \cdots\right|$$

$$\leq aC + \int_a^\infty \frac{e^{-yt}}{y^{3/2}}\, dy$$

$$= aC + t^{1/2} \int_{at}^\infty e^{-z} z^{-3/2}\, dz$$

$$\leq aC + t^{1/2} \left(C\, t^{-1/2} + C \int_0^\infty e^{-z} z^{-1/2}\, dz\right)$$

$$\leq C + C\, t^{1/2} \quad .$$

Similarly one can estimate dF_1/dt and obtain

$$|dF_1/dt| \leq C + Ct^{-1/2}$$

(iv) Estimates of F_2 and its derivatives.

As in (iii) we divide the integral defining F_2 into two parts

$$|F_2(x,t)| \leq C + \left|\int_a^\infty \frac{e^{-yt}}{P(y)} \left(1 - \frac{\alpha}{y+1/4}\right) \exp\{iy^{1/2}x\}\, dy\right|$$

$$\leq C + C\int_a^\infty e^{-yt}/y^{1/2}\, dy$$

$$\leq C + C\int_0^\infty e^{-yt}/y^{1/2}\, dy < Ct^{-1/2} .$$

In order to obtain an appropriate estimate of $\partial F_2/\partial x$ we employ the identity

$$\exp\{iy^{1/2}x\} = \frac{2y^{1/2}}{ix} \frac{\partial}{\partial y} \exp\{iy^{1/2}x\} . \tag{11}$$

We write

$$|\partial F_2/\partial x| \leq C + \left|\int_a^\infty \frac{e^{-yt}(1+O(y^{-1}))}{y^{1/2}(1+O(y^{-1/2}))} \exp\{iy^{1/2}x\} y^{1/2} dy\right.$$

and integrate by parts using (11)

$$\leq C + \frac{C}{x} \left|\int_a^\infty e^{-yt}\, y^{1/2} \frac{\partial}{\partial y} \exp\{iy^{1/2}x\}\, dy\right|$$

$$\leq C + \frac{C}{x}\left(C + \int_a^\infty e^{-yt}\,(ty^{1/2} + y^{-1/2})dy\right) \leq C + \frac{C}{xt^{1/2}} .$$

In a similar way one can obtain the estimates

$$|\partial^2F_2/\partial x^2| \leq C + C/(x^2t^{1/2}) \quad , \quad |\partial F_2/\partial t| \leq C + C/(x^2t^{1/2}) \quad .$$

We note that in order to estimate $\partial F_2/\partial t$ and $\partial^2F_2/\partial x^2$ one should integrate by parts twice. It's not hard to show that in general for small x $|\partial^nF_2/\partial x^n| \leq C/(x^nt^{1/2})$.

(v) The main singularity of F_2.

We represent F_2 as

$$F_2 = \frac{1}{2i} \int_0^\infty e^{-yt} \exp\{iy^{1/2}x\}/y^{1/2}\, dy$$

$$+ \int_0^\infty e^{-yt} \exp\{iy^{1/2}x\} \left(\frac{1}{P(y)}\left(1 - \frac{\alpha}{y+1/4}\right) - \frac{1}{2iy^{1/2}}\right) dy \tag{12}$$

We recall that F_2 is obtained by integration along the lower edge of the cut-off in C-plane, and in fact $F_2 \equiv (F_+)_2$. If we add $(F_-)_2$, the integral along the upper edge, to $(F_+)_2$ in (12), then the sum of integrals corresponding to the first term in (12) may be calculated explicitly and gives a fundamental solution of the heat equation. Thus

$$F_2 = (F_+)_2 - (F_-)_2 = \frac{1}{2(\pi t)^{1/2}} \exp\left\{-\frac{x^2}{4t}\right\} + R(x,t) \quad . \tag{13}$$

The remainder can be estimated similarly to estimations in (iv)

$$\left|R(x,t)\right| \leq C + \left|\int_a^\infty e^{-yt} \exp\{iy^{1/2}x\} \frac{C+O(y^{-1/2})}{y+O(y^{1/2})} dy\right|$$

$$\leq C + C\int_a^\infty e^{-yt}/y \, dy \leq C + C\int_0^\infty e^{-z}/z \, dz \leq C \quad .$$

For x-derivative we get

$$\left|\partial R/\partial x\right| \leq C + \int_a^\infty e^{-yt}/y^{1/2} dy$$

$$\leq C + t^{-1/2}\int_0^\infty e^{-z}/z^{1/2} dz \leq Ct^{-1/2} \quad .$$

Summing up the results of the steps (i) through (v) we can make the following conclusions. From (iii) and (iv) it follows that the integral

$$v(x,t) = \int_0^t G_1(x,t-s)f(s)ds \quad ,$$

is well-defined for all $x \geq 0$ if $f \in L^\infty[0,T]$. Moreover for $x > 0$ the integral can be differentiated twice with respect to x, and the differential operator

$$M = \frac{\partial^2}{\partial x^2} - \frac{1}{4} - \alpha e^{-x/2} \langle \cdot, \delta \rangle \quad ,$$

can be applied. For the same reasons $\partial v/\partial t$ can be calculated as follows

$$\frac{\partial v}{\partial t} = G_1(x,0)f(t) + \int_0^t \frac{\partial}{\partial t} G_1(x,t-s)f(s)ds \quad ,$$

and by virtue of (i)

$$\frac{\partial v}{\partial t} = \int_0^t \frac{\partial}{\partial t} G_1(x,t-s)f(s)ds \quad .$$

Therefore, since $\partial G/\partial t = MG$, $v(x,t)$ satisfies (3).

The initial condition (4) follows from (i). It is left only to verify the boundary condition (6), since (5) follows from $G_1(0,t) = G_2(0,t)$.

From (ii) and (v) it follows that $G_1(x,t)$ can be represented in the form

$$G_1(x,t) = e^{-t/4} \frac{1}{2(\pi t)^{1/2}} \exp\{-\frac{x^2}{4t}\} + r(x,t) \quad , \tag{14}$$

where $r(x,t)$ is combined of "remainders" appeared in (9), (10), and (13)

$$r(x,t) = I_1(x,t) - \alpha e^{-x/2-t/4} F_1(t) + e^{-t/4} R(x,t) \quad .$$

From estimates of F_1 and $\partial R/\partial x$ we conclude that

$$\lim_{x\to 0} \frac{\partial}{\partial x} \int_0^t r(x,t-s)f(s)ds = \int_0^t \frac{\partial}{\partial x} r(0,t-s)f(s)ds \quad . \tag{15}$$

As for the first term in (14), there is a well known jump relation for the single-layer potential (see e.g., Friedman [4, Ch.5]). If we denote by $\Gamma(x,t)$ the first term in (14) without the factor $e^{-t/4}$, then the jump relation is as follows

$$\lim_{x\to 0} \frac{\partial}{\partial x} \int_0^t \Gamma(x,t-s)\phi(s)ds = -\frac{1}{2}\phi(t) + \int_0^t \frac{\partial}{\partial x}\Gamma(x,t-s)\phi(s)ds$$

$$= -\phi(t)/2 \quad .$$

Together with (15) the jump relation for $G_1(x,t)$ follows easily.

Recalling the definition of v, it can be written as

$$\lim_{x\to 0} \frac{\partial v}{\partial x}(x,t) = -\frac{1}{2} f(t) + \int_0^t \frac{\partial}{\partial x} G_1(0,t-s)f(s)ds \quad . \tag{16}$$

Taking into account that G_1 and G_2 satisfy the jump condition

$$\frac{\partial}{\partial x} G_1(0,t) + \frac{\partial}{\partial x} G_2(0,t) + \alpha G_1(0,t) = 0 \quad ,$$

we derive the desired jump condition (6) from (16) and from the corresponding relation for u(x,t). Thus Theorem 1 is proved.

3.2. <u>Auxiliary linear problem: Hölder continuity.</u>

In Sec. 3.1 we were concerned with local properties of the solution to the problem, explicitly given in (3). For our purposes we need to study behavior of the solution considered as a function of t with values in appropriate Banach spaces.

We introduce the following:

<u>Definition</u>. A function f: $(0,T) \to B$ (where B is a Banach space) is said to be Hölder continuous of exponent δ $(0 < \delta < 1)$ with singularity $t^{-\rho}(0 \leq \rho < 1)$ or brief $\delta\rho$-continuous if

$$\sup_{0<t<T} \|f(t)\| + H_{\rho,0+\delta}[f] < \infty$$

where

$$H_{\rho,0+\delta}[f] = \sup_{0<t<s<T} \left(t^{\rho} \frac{\|f(t)-f(s)\|}{|t-s|^{\delta}}\right)$$

It is not hard to show that functions satisfying the definition form a Banach space which we denote by $C_{\rho,0+\delta}((0,T),B)$. If $\rho=0$ one obtains uniformly Hölder continuous functions on $[0,T]$. If $\rho\neq 0$ one obtains locally Hölder continuous functions with prescribed growth of Hölder coefficients by the approach to 0.

<u>Theorem 2</u>. Let $w = \{u,v\}$ be a solution to the problem (3) through (6) given in the form (8). Let $f \in C_{\rho,0+\delta}((0,T),\mathbb{R}^1)$ then

(i) $w \in C_{\rho',0+\delta}((0,T),H^0)$, where for any $\varepsilon > 0$

$\rho' = \max(0,\delta-3/4+\varepsilon, \rho-3/4+\varepsilon)$;

(ii) $w \in C_{\rho'',0+\delta}((0,T),H^1)$, where for any $\varepsilon > 0$

$\rho'' = \max(0,\delta-1/4+\varepsilon, \rho-1/4+\varepsilon)$;

if in addition $\delta > 1/4$ then

(iii) $w \in C((0,T),H^2)$

(iv) $w \in C^1((0,T),H^0)$.

<u>Remark</u>. The theorem may be reformulated for the case of $W^{\cdot,p}$ norms. Then in particular the constant 1/4 in (iii) and (iv) should be replaced by $(1-1/p)/2$. Also one might modernize the theorem for the case when f itself has an integrable singularity at 0.

<u>Proof</u>. As in the proof of Theorem 1 we will carry out the proof only for the more complicated component, v, of the solution $w = \{u,v\}$.

We recall that the Green function of the problem can be represented (see (14)) as a multiple of the Gaussian kernel plus a remainder. We will show that the Theorem is true for the function generated by the Gaussian kernel while the remainder generates a term which is in a sense of lower order.

For brevity we disregard the prefactor $e^{-t/4}$ and constants in the Gaussian kernel which will appear in the form

$$h(x,t) = t^{-1/2} \exp\{-x^2/t\} .$$

In what follows we will often use the simple relation valid for $0 \leq a < 1$, $0 \leq b < 1$

$$\int_0^s (s-t)^{-a} t^{-b} dt = C s^{1-a-b} , \tag{17}$$

where $C = \Gamma(1-a)\Gamma(1-b)/\Gamma(2-a-b)$.

The proof is split into several steps.

Step 1. Hölder continuity of the main term in H^0. Set

$$F(x,s) = \int_0^s h(x,s-t)f(t)dt = \int_0^s h(x,t)f(s-t)dt . \tag{18}$$

The following estimate holds for $a \geq 0$

$$|h| = x^{-a} t^{(a-1)/2} (x^2/t)^{a/2} \exp(-x^2/t) \leq C x^{-a} t^{(a-1)/2} . \tag{19}$$

Employing (19) with $a = a_1 > 1/2$ for $x \geq A$ and with $a = a_2 < 1/2$ for $x \leq A$ one obtains

$$\int_0^\infty |F(x,s)|^2\, dx = \left(\int_0^A + \int_A^\infty\right) |F|^2\, dx \leq (\sup|f|)^2 \left(\int_0^A + \int_A^\infty\right) h^2 dt dx$$

$$\leq C|f|_\infty^2 \left(s^{a_2+1} \int_0^A x^{-2a_2}\, dx + s^{a_1+1} \int_A^\infty x^{-2a_1}\, dx\right)$$

$$\leq C|f|_\infty^2 .$$

To get the Hölder continuity we must estimate the difference

$$\Delta(x,s,k) = F(x,s+k)-F(x,s)$$

$$= \int_s^{s+k} hf(s+k-t)dt + \int_0^s (f(s+k-t)-f(s-t)hdt \equiv I_1+I_2 . \tag{20}$$

For large x, (19) with $a \geq 1$ allows us to estimate I_1 and I_2 as follows

$$|I_1| \leq C\, |f|_\infty\, x^{-a}k$$

$$|I_2| \leq Ck^\delta H_{\rho,0+\delta}[f]x^{-a} \int_0^s (s-t)^{-\rho}\, t^{(a-1)/2}\, dt$$

$$\leq Ck^\delta H_{\rho,0+\delta}[f]x^{-a} ,$$

hence

$$\int_A^\infty |\Delta(x,s,k)|^2\, dx \leq C(H_{\rho,0+\delta}[f]k^\delta)^2 . \tag{21}$$

For small x we set $a = 1/2-2\varepsilon$ ($\varepsilon > 0$) in (19), we get

$$|I_1| \leq C|f|_\infty x^{-1/2+2\varepsilon} \int_s^{s+k} t^{-1/4-\varepsilon}\, dt$$

$$= C|f|_\infty x^{-1/2+2\varepsilon} S_1^{-1/4-\varepsilon} k \quad ,$$

where $S < S_1 < s + k$. If $s > k$ then

$$\left(\int_0^A |I_1(x,s,k)|^2 dx\right)^{1/2} \leq C|f|_\infty kS_1^{-1/4-\varepsilon} \leq C|f|_\infty kS^{-1/4-\varepsilon}$$

$$\leq C|f|_\infty k^\delta S^{1-\delta-1/4-\varepsilon} \quad . \tag{22}$$

If $s \leq k$ one may estimate as follows

$$|I_1| \leq C|f|_\infty x^{-1/2+2\varepsilon} \int_0^{s+k} t^{-1/4-\varepsilon}\, dt$$

$$= C|f|_\infty x^{-1/2+2\varepsilon}(s+k)^{3/4-\varepsilon}$$

$$\leq C|f|_\infty x^{-1/2+2\varepsilon} k^\delta s^{-\max(0,\delta-3/4+\varepsilon)}$$

and again (22) holds.

For the second integral employing (19) with $a = 1/2-2\varepsilon$ we obtain the estimate

$$|I_2| \leq Ck^\delta H_{\rho,0+\delta}[f]\, x^{-a} \int_0^s t^{-1/4-\varepsilon}(s-t)^{-\rho}\, dt$$

$$\leq Ck^\delta H_{\rho,0+\delta}[f]\, x^{-a} S^{-\max(0,1/4+\rho+\varepsilon-1)} \quad ,$$

therefore,

$$\left(\int_0^A \left|I_2(x,s,k)\right|^2 dx\right)^{1/2} \leq Ck^{\delta}s^{-\max(0,\rho+\varepsilon-3/4)}H_{\rho,0+\delta}[f] \quad ,$$

this inequality together with (21) and (22) yields the conclusion (i) of the Theorem for the main term.

<u>Step 2</u>. Holder continuity of the main term in the H^1 norm. The proof is almost identical to that in Step 1 and is omitted.

<u>Step 3</u>. Continuity of the main term in H^2 norm.

In analogy with (19) one obtains the inequality, $a \geq 0$,

$$|h_{xx}| = 2\left|t^{-3/2}-2x^2t^{-5/2}\right| \exp(-x^2t) \leq Cx^{-a}t^{(a-3)/2} \quad , \tag{24}$$

which allows to differentiate twice in (18) under the integral sign (setting $a > 1$)

$$F_{xx} = \int_0^s h_{xx}(x,t)f(s-t)dt \quad . \tag{25}$$

First, we will show that $F_{xx}(.,s) \in L^2$. We have from (25) with $a = 2$

$$|F_{xx}| \leq Cx^{-2}|f|_\infty \int_0^s t^{-1/2}dt = C|f|_\infty x^{-2}s^{1/2}$$

and therefore

$$\int_A^\infty |F_{xx}|^2 dx \leq Cs < \infty \quad . \tag{26}$$

In order to get an appropriate estimate of F_{xx} for small x we notice that $h_{xx} = ch_t$ (here c = 4). Therefore one can estimate the integral in (25) as follows (making use of (24) with $a = 1/2-2\varepsilon$)

$$\begin{aligned}|F_{xx}| &= C\left|\int_0^s h_t(x,t)f(s-t)dt\right| \\ &\leq C\left|\int_0^s h_t f(s)dt\right| + \left|\int_0^s h_t(f(s-t)-f(s))dt\right| \\ &\leq C|f(s)|s^{-1/2}\exp(-x^2/t) \\ &+ Cx^{-a}H_{\rho,0+\delta}[f]\int_0^s t^{-5/4-\varepsilon}(s-t)^{-\rho}t^{\delta}dt \quad .\end{aligned}$$

If $\varepsilon < \delta-1/4$ then it follows

$$|F_{xx}| \leq C|f|_\infty s^{-1/2}\exp(-x^2/t) + Cx^{-a}H_{\rho,0+\delta}[f]s^{-\rho} \quad , \tag{27}$$

therefore

$$\int_0^A |F_{xx}|^2\,dx \leq Cs^{-2\max(\rho,1/2)} \quad .$$

Recalling (26) we see that $F_{xx}(.,s) \in L^2$.

In order to prove L^2 contimity of F_{xx} we will again estimate

$$\Delta_{xx} = \int_s^{s+k} h_{xx}f(s+k-t)dt + \int_0^s h_{xx}(f(s+k-t)-f(s-t))dt \equiv I_1 + I_2 ,$$

Equation (24) with $a \geq 3$ yields

$$|\Delta_{xx}|^2 \leq Cx^{-a}H_{\rho,0+\delta}[f]\,(k+Ck^{\delta}) \quad .$$

Hence

$$\int_A^\infty |\Delta_{xx}|^2 dx \to 0 \quad , \text{ as } k \to 0 \ .$$

Now we observe that for any $x > 0$ the integral in (25) for F_{xx} continuously depends on the parameter s when $0 < \varepsilon \leq s \leq T$. Therefore, pointwise

$$|\Delta_{xx}|^2 = |F_{xx}(x,s) - F_{xx}(x,s_0)|^2 \to 0 \quad , \text{ as } s \to s_0 \ ,$$

where $s \geq \varepsilon$. Also Δ_{xx} can be estimated using (27)

$$|\Delta_{xx}(x,s,s_0)|^2 \leq 2(|F_{xx}(x,s)|^2 + |F_{xx}(x,s_0)|^2)$$

$$\leq C\varepsilon^{-1/2} \exp(-x^2/\varepsilon) + Cx^{-2a}\varepsilon^{-\rho} \ .$$

Thus, for any s, $|\Delta_{xx}(x,s,s_0)|^2$ is bounded from above by the integrable function and one can apply Lebesgue's theorem which yields that

$$\int_0^A |\Delta_{xx}|^2 dx \to 0 \ , \quad \text{as } s \to s_0 \ .$$

With results obtained on the previous steps it implies H^2 continuity.

Step 4. Differentiability of the main term in H^0.

The proof is similar to that on Step 3. We'll use (18) in the form

$$F(x,) = \int_0^s h(x,s-t)f(t)dt$$

and estimate the difference

$$F(x,s+k) - F(x,s) = \int_s^{s+k} h(x,s+k-t)f(t)dt$$

$$+ \int_0^s (h(x,s+k-t) - h(x,s-t))f(t)dt \equiv I_1 + I_2$$

We have

$$\frac{1}{k}\left|I_1\right| \leq \left|f\right|_\infty h(x,t_0) = \left|f\right|_\infty t_0^{-1/2}\exp(-x^2/t_0) \to 0 \qquad (28)$$

as $k \to 0$, since $0 < t_0 < k$. Taking into account that $h_t = ch_{xx}$ we see that the estimate in (24) holds for h_t also. Thus we get

$$F_s(x,s) = \int_0^s h_s(x,s-t)f(t)dt = \int_0^s h_{xx}fdt$$

and as it was shown on Step 3, F_s is continuous in L^2. It is left to prove that

$$\|\Phi(.,s,k,k)\|_{L^2} = \|\frac{1}{k}(F(.,s+k)-F(.,s))-F_s\|_{L^2} \to 0 .$$

As before we will show that $|\Phi(.,s,k)|^2$, for $k \leq s-\varepsilon$, is bounded from above by an integrable function. This, in turn, can be reduced to the estimate from above for $|F(x,s+k)-F(x,s)|/k$.

For $x \geq A > 0$ we get from (28) that

$$\left|k^{-1}I_1\right| \leq C\left|f\right|_\infty x^{-1} ,$$

where $C \geq xt_0^{-1/2}\exp(-x/t_0^2)$. For I_2 we have

$$|k^{-1}I_2| \leq \int_0^S |h_s(x,S+\theta(t)-t)|f(t)dt \quad , \tag{29}$$

where $0 < \theta(t) < k$. Employing (24) with $a = 2$ one easily obtains

$$|k^{-1}I_2| \leq C|f|_\infty x^{-2} \quad .$$

Note that x^{-1} and x^{-2} are square integrable on $[A,\infty)$.

For small x, using the same reasonings as on Step 3 one obtains the following analogue of (27)

$$|k^{-1}I_2| \leq C \exp(-x^2/s) + Cx^{-a} \leq Cx^{-a}$$

$a < 1/2$, while the estimate in (28) yields $|k^{-1}I_1| \leq C$.

Thus for $x \leq A$ we get C and Cx^{-a} ($a < 1/2$) as upper bounds. Therefore applying Lebesgue's theorem as on Step 3, we obtain the desired results.

Step 5. The remainder.

Let us take a closer look at the remainder. According to (14) the Green function for the problem is

$$\begin{aligned} G(x,t) &= ce^{-t/4}h(x,4t) - \alpha e^{-x/2-t/4}F_1(t) \\ &\quad + \sum_{\lambda=\lambda_+,\lambda_-} c_\lambda e^{\lambda t}(e^{-p(\lambda)x}(1+\alpha/\lambda) - e^{-x/2}\alpha/\lambda) \\ &\quad + e^{-t/4}R(x,t) \quad . \end{aligned}$$

The first two terms of the remainder are "nice" functions. Indeed $e^{-x/2}$, $e^{-p(\lambda)x}$ (where $\lambda = \lambda_+,\lambda_-$) and their derivatives of any order belong to $L^2(\mathbb{R}_+)$, while $e^{\lambda t}$ and its derivative are bounded and $F_1(t)$ satisfies the estimates (see (iii) of the proof of Theorem 1)

$$|F_1(t)| \leq C \quad , \quad |\partial F_1/\partial t| \leq Ct^{-1/2}$$

Thus conclusions of Theorem 2 are obvious for the first two terms and our concern will be with the last term in the remainder (in what follows we omit a harmless factor $e^{-t/4}$).

We recall (cf. (12)) that R(x,t) can be represented by the sum of two integrals of the form

$$J(x,t) = \int_0^\infty e^{-yt}\exp(iy^{1/2}x)\left(\frac{1}{P(y)}\left(1 - \frac{\alpha}{y+1/4}\right) - \left(2iy^{1/2}\right)\right)dy$$

where

$$P(y) = iy^{1/2}(2 + \alpha/(y-1/4)) + \alpha/(2y-1/2) + \alpha \quad .$$

The second integral differs from J in replacement of i by -i everywhere.

After simple transformations J takes the form

$$J(x,t) = \int_0^\infty e^{-yt}\exp(iy^{1/2}x)\frac{A(y)}{y^{1/2}(y^{1/2}+B(y))}\,dy \quad , \tag{30}$$

where A and B are continuous on $[0,\infty)$ and

$$A(y) = O(y^{-1/2}) + \alpha/4 \ , \quad B(y) = O(y^{-1/2}) + \alpha/2$$

$$A_y(y) = O(y^{-3/2}) \ , \quad B_y(y) = O(y^{-3/2}) \ , \quad \text{as} \quad y \to \infty$$

$$|y^{1/2} + B(y)| \geq C^{-1} > 0 \ .$$

All estimates we need are similar. We estimate for example J_{xx}. We may differentiate (30) under the integral sign since the presence of e^{-yt}, $t > 0$, guarantees the uniform convergence of the integral.

$$J_{xx} = \int_0^\infty e^{-yt} \exp(iy^{1/2}x) \frac{A(y)dy}{1+y^{-1/2}B}$$

$$= \frac{\alpha}{4} \int_0^\infty e^{-yt} \exp(iy^{1/2}x)dy$$

$$+ \int_0^\infty e^{-yt}\exp(iy^{1/2}x) \ C(y)y^{1/2}(y^{1/2}+B)^{-1}dy \ , \qquad (31)$$

where $C(y) = O(y^{-1/2})$ and $C_y(y) = O(y^{-3/2})$ as $y \to \infty$.

The first integral on the right hand side plus the corresponding integral with -i can be easily calculated

$$I_1 = \int_0^\infty e^{-yt}[\exp(iy^{1/2}x) + \exp(-iy^{1/2}x)]dy$$

$$= 2\int_{-\infty}^{\infty} \exp(-z^2t+izx+x^2/(4t))\, zdz\, \exp(-x^2/(4t))$$

$$= 2\, t^{-1} \exp(-x^2/(4t)) \int_{-\infty-ix/(2t)}^{\infty-ix/(2t)} \exp(-u^2)\left(u + \frac{ix}{2t^{1/2}}\right) du$$

$$= Cxt^{-3/2} \exp(-x^2/(4t)). \tag{32}$$

In the course of calculations we have made obvious changes of variables

$$z = y^{1/2} \quad ; \quad u = t^{1/2}(z-ix/(2t)) \quad .$$

The second integral in (31) can be estimated employing the identity

$$\exp(iy^{1/2}x) = -2iy^{1/2}x^{-1}\partial_y \exp(iy^{1/2}x) \quad . \tag{11}$$

We have

$$|I_2| = \frac{1}{x}\left|\int_0^\infty \partial_y \exp(iy^{1/2}x)e^{-yt}yC(y)(y^{1/2}+B)^{-1}dy\right|$$

$$\leq x^{-1}\left(C + t\int_0^\infty e^{-yt}\, O(1)dy + \int_0^\infty e^{-yt}O(y^{-1})dy\right) \leq Cx^{-1} \quad . \tag{33}$$

It should be mentioned that since $|y^{1/2} + B| \geq C^{-1}$ the function $O(y^{-1})$ is integrable at 0 (we omit details).

The estimate of I_2, suitable for small x, is even simpler

$$|I_2| \leq \int_0^\infty e^{-yt} y^{-1/2} dy = Ct^{-1/2} \quad . \tag{34}$$

Thus we observe that R_{xx} can be represented as the sum of two summands: a function satisfying the inequality (32) (which is characteristic for the first derivative of the main term, h_x) plus a remainder satisfying (33) and (34). Therefore the results related to h_x (Step 2) can be applied to R_{xx} and we get even the Hölder continuity in L^2-norm of

$$F_{xx}(\cdot,s) = \int_0^s R_{xx}(\cdot,t)f(s-t)dt \quad .$$

Summing up the reasonings of Step 5 we may say that in a sense smoothness of the remainder is by one order better than smoothness of the main term.

Corollary. Let $f \in C_{\rho,0+\delta}((0,T),\mathbb{R}^1)$, $\delta > 1/4$. Then the problem (3) through (6) has a unique strong solution in H^0.

Proof. Take a solution w of the problem in the form (8). From (iii) and (iv) of the theorem it follows that $w \in C(0,T),H^2) \cap C^1((0,T),H^0)$. Moreover w is continuous at 0. Indeed, it follows from Theorem 1 that $w^2(x,t) \to 0$ as $t \to 0^+$. From the proof of Theorem 2 (Step 1) it is clear that $|w^2(x,t)| \leq \Phi(x)$, where $\Phi(x)$ is integrable. Therefore

$$\|w(.,t)\|_{L^2} \to 0 \quad \text{as} \quad t \to 0_+ .$$

Thus w is a strong solution of the problem (3) through (6) in H^0. If w' is another strong solution, then the difference u = w-w' is a strong solution of the Cauchy problem with homogeneous boundary conditions and u(0) = 0. From the general theory such a solution is unique and given by $u(t) = e^{tA}u(0) = 0$. Hence w' = w.

We conclude this section with a lemma, which will be used only in Chapter 4.

Lemma 3. Let $w(t,.) = \{u(t,.),v(t,.)\}$ be a solution to the problem (3) through (6), let $f \in C_{\rho,0+\delta}((0,T),\mathbb{R}^1)$. Then the mapping

$$(0,\infty) \ni p \to w(p,t,.) = \int_0^t G(.,p(t-r))f(r)dr \in C_{\rho,0+\delta}((0,T),H^1) \quad ,$$

is infinitely differentiable. For any $n \geq 0$ the derivative $\partial_p{}^n\widetilde{w}(p,t)$ is continuous at t = 0 (in the H^1 topology) and $\partial_p{}^n\widetilde{w}(p,0) = 0$.

Proof. Assume that $\rho \geq \delta$, it is not a restriction since obviously $C_{\rho,0+\delta} \in C_{\rho',0+\delta}$ for $\rho \leq \rho'$. By Theorem 2 (ii), $\widetilde{w}(p,t) \in C_{\rho,0+\delta}((0,T),H^1)$. As in the proof of Theorem 2 we represent $\widetilde{w}(p,t)$ in the form

$$\tilde{w}(p,t) = \int_0^t h(x,p(t-s))f(s)ds$$

$$+ \int_0^t r(x,p(t-s))f(s)ds \equiv F(p,t,x)+R(p,t,x) \quad ,$$

where h is Gaussian kernel, the principal part of the Green function, and r is the remainder. Again we restrict ourselves to the consideration of only one component, u, of w.

The main term. First we show that $F(p,t,.) \to 0$ as $t \to 0$. Reasoning as in the proof of Theorem 2 (Step 2), we get for $F_x(p,t,x)$ the following inequalities

$$|F_x|^2 \le |f|_\infty^2 x^{-1-\varepsilon}(pt)^{1/2+\varepsilon/2} \quad , \quad x \ge A$$

$$|F_x|^2 \le |f|_\infty^2 x^{-1+\varepsilon}(pt)^{1/2-\varepsilon/2} \quad , \quad x \le A \quad .$$

$(\varepsilon > 0)$. Whence $\|F(p,t,.)\|_{1/2} \to 0$ as $t \to 0$.

The claim of the lemma concerning derivatives is proved by induction on n. We consider only n = 1.

$$\partial_p\tilde{w}(t,p) = \int_0^t h_t(x,p(t-s))(t-s)f(s)ds$$

$$+ \int_0^t r_t(x,p(t-s))(t-s)f(s)ds \quad ,$$

(where the differentiation under the integral sign will be justified later). We have the estimates

$$\left| th_{tx}(x,pt) \right| \leq \begin{cases} Cx^{-2}p^{-1}, & x \geq A \\ Cx^{-2\varepsilon}p^{-2+\varepsilon}t^{-1+\varepsilon}, & x \leq A \end{cases}$$

where $0 < \varepsilon < 1/2$. Convoluting with f we find that the main term of $\partial_p w$ belongs to H^1 and tends to 0 as $t \to 0$. Any new differentiation with respect to p brings a factor of order $t \cdot x^2/t^2 = x^2/t$ which does not change the integrability properties. The proof of Hölder continuity is a literal repetition of the corresponding piece of the proof of Theorem 2 (Step 2).

The remainder. We omit the proof for the remainder, since it is absolutely analogous to the proof of Theorem 2 (Step 5). As was noted above the main point is that the differentiation with respect to p does not spoil integrability properties.

Remark. Estimating accurately norms of derivatives one might show that in fact w is analytic in p. We don't need this result.

3.3. The Nonlinear Integral Equation

In this section we prove equivalence of the differential equation and the integral one which were discussed in the introduction to the present chapter.

Consider the nonlinear equation

$$u_t - (u_{yy} - 1/4u - \alpha_0 u(0,t)S)$$
$$= (1 - e^{\alpha u(0,t)})(u_y + 1/2u) + (1 + \alpha_0 u(0,t) - e^{\alpha u(0,t)})S \ , \ t>0 \tag{35a}$$

or briefly

$$u_t - A_0 u = F(\alpha,u)$$

which must be satisfied for $y \neq 0$ subject to the jump conditions at 0

$$\begin{aligned} [u(0,t)] &= 0 \\ [u_y(0,t)] + e^{\alpha u(0,t)} - 1 &= 0 \ , \end{aligned} \tag{35b}$$

with the initial data

$$u(y,0) = u_0(y) \ . \tag{35c}$$

The problem (35a), (35b) is just another form of the equation (1), now without δ-functions. Recall that

$$S(y) = \begin{cases} e^{y/2} & , \ y \leq 0 \\ 0 & , \ y > 0 \ . \end{cases}$$

<u>Definition</u>. We say that $u(t) \equiv u(.,t)$ is a solution of the Cauchy problem (35) on $(0,T)$ if

(i) $u \in C((0,T), H^0) \cap C((0,T), H^2) \cap C^1((0,T), H^0)$

(ii) u satisfies (35)

(iii) u is Hölder continuous in H^1-norm,

$u \in C_{\rho,0+\delta}((0,T),H^1)$ with $\delta > 1/4$, $\rho < 1$.

The definition is similar to that for the case of the Cauchy problem in the whole space (without boundary conditions). Usually for the problem

$$u_t + Au = f(x,t) \quad , \quad u(0) = u_0$$

one demands f to be Hölder continuous in t, this requirement corresponds to (iii) of the Definition. The new feature is that the conditions at the free boundary constrain us to require the Hölder exponent to be larger than 1/4.

Let G be the Green function of the auxiliary linear problem studied in Sec. 3.1

$$G(x,t) = \begin{cases} G_1(x,t) & , \quad x > 0 \\ G_2(-x,t) & , \quad x \leq 0 \end{cases} .$$

Consider the integral equation

$$\begin{aligned} u(t) = {} & T(t)u_0 + \int_0^t (1-e^{\alpha u(0,r)})T(t-r)(u_y+1/2u)dr \\ & + \int_0^t (1 + \alpha_0 u(0,r) - e^{\alpha u(0,r)}) \, T(t-r)Sdr \\ & - \int_0^t G(x,t-r)(1+\alpha_0 u(0,r)-e^{\alpha u(0,r)})dr \quad . \end{aligned} \tag{36}$$

or briefly

$$u(t) = \Phi(u) \quad . \tag{36'}$$

We note that the equation in (36) is the same as in (2) with substitution of G for $T\delta$, i.e., one can understand formally G as the result of the semigroup acting on δ-function.

Theorem 4. If u is a solution of (35) then u satisfies (36). Conversely, if u is a continuous function from (0,T) to $X^{1/2}$ such that $u(0,t) \in C_{\rho,0+\delta}((0,T),\mathbb{R}^1)$, $\delta > 1/4$, and the integral equation holds for $0 < t < T$, then u is a solution of the problem (35).

Remark 1. The value of u at x = 0 is unambiguous. Indeed, any $u \in X^{1/2}$ is a limit of functions from X^1 which are continuous at 0 by definition. The limit is being taken in the $\| \|_{1/2}$-norm which by Proposition 2.6 is equivalent to $| |_1$. Since, by Lemma 2.5, convergence in $| |_1$ norm implies convergence at 0, the limit of functions continuous at 0 must be continuous at 0.

Remark 2. The Banach spaces X^s were defined in Sec. 2.6 for any α. From now on we will use only X^s defined for $\alpha = \alpha_0$.

Proof. Part I: If u solves the differential equation then it solves the integral equation.

Let u(t) be a solution of the problem (35). We denote by v(t) the right-hand side of the integral equation (see (36) and (36')), calculated for the solution of the differential equation

$$v = \Phi(u) \quad .$$

We will show that v is a solution of the equation $v_t - A_0 v = F(u)$ with certain jump conditions. It will follow that v = u, i.e., (36) holds.

First we observe that, since u is Hölder continuous in H^1 norm, by virtue of Lemma 2.5 $u(0,.) \in \overset{\circ}{C}_{\rho,0+\delta}((0,T),\mathbb{R}^1)$ with the same ρ and δ. It obviously follows that $e^{\alpha u(0,.)} \in C_{\rho,0+\delta}$. Indeed

$$\left|e^{\alpha u(0,t+h)} - e^{\alpha u(0,t)}\right|$$

$$= \left|u(0,t+h) - u(0,t)\right|\left|\alpha\right|e^{\alpha\theta u(0,t+h)+(1-\theta)\alpha u(0,t)}\Big|$$

$$\leq \alpha h^{\delta} t^{-\rho} H_{\rho,0+\delta}[u(0,.)]e^{\alpha \sup u(0,t)} \quad ,$$

(where $0 \leq \theta \leq 1$).

Consider separately all terms on the right-hand side of the integral equation (36). Set

$$v_0(t) = T(t)u_0 \quad ,$$

then v_0 solves the problem

$$\partial v_0/\partial t - A_0 v = 0 \quad ,$$

$$[v_0]_{x=0} = 0 \quad , \quad [v_{0x}]_{x=0} = -\alpha_0 v_0 0,t) \quad ,$$

$$v_0(x,0) = u_0(x) \quad . \tag{37}$$

as follows from Fact 2, Sec. 2.5.

Let

$$v_1(t) = \int_0^t (1-e^{\alpha u(0,r)})T(t-r)(u_y+1/2u)dr \quad , \tag{38}$$

the integral converges, since T is bounded and $u \in C_{\rho,0+\delta}((0,T),H^1)$ i.e., in particular $\sup_{0<t<T} |u|_1 < \infty$. Consider the integral

$$\int_0^t (1 - e^{\alpha u(0,r)})A_0T(t-r)(u_y(r)+1/2u(r))dr$$

$$= \int_0^t (1-e^{\alpha u(0,t)})A_0T(t-r)(u_y(t)+1/2u(t))dr$$

$$+ \int_0^t A_0T(t-r)\ (1-e^{\alpha u(0,s)})(u_y(s) + 1/2u(s))\Big|_{s=t}^{s=r} dr \quad . \qquad (39)$$

The first integral on the right-hand side is obviously equal to

$$I_1(t) = (1-e^{\alpha u(0,t)})\ (I-T(t))\ (u_y(t)+1/2u(t)) \quad .$$

The second integral can be estimated using Hölder continuity of u. Denoting by $\phi(s) = 1-e^{\alpha u(0,s)}$ and $w(s) = u_y(s)+1/2u(s)$, we have

$$\|I_2(t)\|_0 \le \|\int_0^t \phi(t)A_0T(t-r)(w(r)-w(t))dr\|_0$$

$$+ \|\int_0^t (\phi(r)-\phi(t))A_0T(t-r)w(r)dr\|_0 \quad ,$$

and taking into account that $\|A_0T(s)\| \le Cs^{-1}$ (Fact 4, Sec. 5.2)

$$\|I_2(t)\|_0 < |\phi(t)| \int_0^t C(t-r)^{-1}|w(r)-w(t)|_0 dr$$

$$+ \int^t |\phi(2)-\phi(t)| \; C(t-r)^{-1} \; w(r) \; dr$$

$$\leq C|\phi(t)| \int_0^t (t-r)^{-1} \; |u(r)-u(t)|_1 \; dr$$

$$+ \alpha \exp \{2\alpha \sup u(0,t)\} \; H_{\rho,0+\delta}[u(0,.)]$$

$$\int_0^t (t-r)^{-1+\delta} r^{-\rho} |u(r)|_1 dr$$

$$\leq Ct^{-\max(0,\rho-\delta)} \quad ,$$

where C depends on the Hölder constant of u, $\sup|u(r)|_1$, $\sup|u(r)|$ and $|1-e^{\alpha u(0,t)}|$. Thus we see that the integral (39) converges. The Riemann sums for it satisfy the relation

$$\sum \phi(r_i)A_0T(t-r_i)w(r_i)\Delta_i = A_0 \sum \phi(r_i)T(t-r_i)w(r_i)\Delta_i \quad . \tag{40}$$

Therefore, by closedness of A_0, the limit of the right-hand side of (40) is equal to $A_0v_1(t)$ and equals to the left-hand side integral in (39).

In a similar way it can be shown that $\partial v_1/\partial t$ is well defined and

$$\partial v_1/\partial t = A_0v_1 + (1 - e^{\alpha u(0,t)})(u_y + 1/2u)$$

Obviously the traces of v_1 and $\partial v_1/\partial x$ at 0 are well defined (since $v_1 \in D(A_0)$) and satisfy linear jump condition in (37). Also it is clear from the definition of v_1, (38) that $v_1(x,0) = 0$.

The treatment of

$$v_2 = \int_0^t (1 + \alpha_0 u(0,r) - e^{\alpha u(0,r)}) T(t-r) S \, dr ,$$

is similar to that of v_1 and even simpler because S does not depend on time. One can obtain that v_2 satisfies the equation

$$\partial v_2/\partial t = A_0 v_2 + (1 + \alpha_0 u(0,t) - e^{\alpha u(0,t)}) S ,$$

with the linear jump conditions as in (37) and zero initial data.

The last term in (36)

$$v_3 = - \int_0^t G(x,t-r) (1+\alpha_0 u(0,r) - e^{\alpha u(0,r)}) dr ,$$

by Theorem 1, satisfies the equation

$$\partial v_3/\partial t = A_0 v_3 ,$$

with zero initial data and jump conditions $[v_3]_{x=0} = 0$

$$[(v_3)_x]_{x=0} = -\alpha_0 v_3(0,t) + 1 + \alpha_0 u(0,t) - e^{\alpha u(0,t)} .$$

Summing up we see that $v = v_0 + v_1 + v_2 + v_3$ is a strong solution of the problem

$$\partial v/\partial t - A_0 v = F(u) ,$$

$$[v]_{x=0} = 0 , \quad [v_x]_{x=0} = \alpha_0 u(0,t) - \alpha_0 v(0,t) + 1 - e^{\alpha u(0,t)} ,$$

$$v(x,0) = u_0(x) .$$

Hence, the difference $w = v-u$ solves the homogeneous linear problem $\partial w/\partial t - A_0 w = 0$

$$[w]_{x=0} = 0 \quad , \quad [w_x]_{x=0} = \alpha_0 w(0,t) \quad , \quad w(x,0) = 0 \quad ,$$

therefore $w \equiv 0$ (uniqueness), and $u = v = \Phi(u)$, i.e., u solves the integral equation in (36).

Part II: A solution of the integral equation is a solution of the differential equation.

In Part I we have shown that if $u \in C_{\rho,0+\delta}((0,T),H^1)$ then $v = \Phi(u)$ is a solution of the problem (41).

Remark. In fact we could require less than $u \in C_{\rho,0+\delta}$. Estimating the integral in (39) we have used only the fact that the function is integrable at 0 (not necessarily bounded!) and that the Hölder coefficient grows at $t^{-\rho}$.

Thus if one shows that the solution of the integral equation $u = \Phi(u)$ is Hölder continuous in the sense of the Remark, then it will follow that u is a solution of (41) with v replaced by u. That is, u is a solution of (36). As in Part I we consider separately all terms on the right-hand side of the integral equation.

For $T(t)u_0$ we have

$$\begin{aligned} \|T(t+h)u_0 - T(t)u_0\|_{1/2} &= \|(T(h)-1)T(t)u_0\|_{1/2} \\ &\le Ch^{\delta}\|(-A_0)^{\delta}T(t)u_0\|_{1/2} \\ &\le Ch^{\delta}t^{-1/2-\delta}\|u_0\|_0 \quad , \end{aligned} \tag{42}$$

(Fact 4, Sec. 2.5). And also

$$\|T(t)u_0\|_{1/2} \leq Ct^{-1/2}\|u_0\|_{1/2} \quad .$$

For

$$u_1(t) = \int_0^t (1-e^{\alpha u(0,r)})T(t-r)(u_y+1/2u)dr \equiv \int_0^t \phi(r)T(t-r)w(r)dr \tag{43}$$

we have

$$u_1(t+h)-u_1(t) = \int_t^{t+h} \phi(r)T(t+h-r)w(r)dr$$

$$+ \int_0^t \phi(r)(T(h)-1)T(t-r)w(r)dr \equiv I_1 + I_2 \quad .$$

Estimating analogously to (42) one obtains

$$\|I_2\|_{1/2} \leq Ch^\delta \int_0^t (t-r)^{-1/2-\delta} \|w(r)\|_0 |\phi(r)| dr$$

$$\leq Ch^\delta \sup_{0<\tau<t} (|1-e^{\alpha u(r)}|) \int_0^t (t-r)^{-1/2-\delta} \|u(r)\|_{1/2} dr$$

For the first integral one gets

$$\|I_1\|_{1/2} \leq C \int_t^{t+h} \left|\phi(r)\right| \|(-A)^{1/2}T(t+h-r)w(r)\|_0 dr$$

$$\leq C \int_t^{t+h} \left|\phi(r)\right| (t+h-r)^{-1/2} \|w(r)\|_0 dr$$

$$\leq Ch^{1/2} \sup_{t<r<t+h} \left(\|u(r)\|_{1/2} \left|1-e^{\alpha u(0,r)}\right|\right)$$

The estimates for the term u_2 generated by S are similar to those for u_1. At last for

$$u_3 = -\int_0^t G(x,t-r)\ (1+\alpha_0 u(0,r) - e^{\alpha u(0,r)})dr \quad ,$$

we get Hölder continuity from Theorem 2 since $u(0,r)$ is Hölder continuous by assumption and $e^{\alpha u(0,r)}$ is also Hölder continuous (see Remark following the statement of Theorem 4). This completes the proof of Theorem 4.

3.4. Existence and Uniqueness for the Nonlinear Problem

Now we are in a position to formulate our existence and uniqueness theorem.

Theorem 5. Consider the Cauchy problem in (35). For any $T > 0$ there exists $\alpha(T) > 0$ such that for any α, $|\alpha-\alpha_0| < \alpha(T)$, the problem has a unique solution if the initial data is sufficiently small

$$\|u_0\|_{1/2} \leq L(\alpha,T) \quad ,$$

where $L(\alpha,t) > 0$ is some constant. For fixed T,

$$L(\alpha,T) = (C_0(T) - C_1(T)|\alpha-\alpha_0|)^2 \quad ; \quad C_0,C_1 > 0 \ .$$

Proof. The idea of the proof is rather standard (cf. Friedman [4, Chapter 7]). First, employing Theorem 4 we replace the differential equation by the integral one. In the latter we freeze the coefficients dependent on $u(0,t)$, namely we introduce a function $\phi(t)$ and consider the linear integral equation

$$\begin{aligned} u(t) = & \int_0^t (1-e^{\alpha\phi(r)})T(t-r)(u_y+1/2u)dr \\ & + [T(t)u_0 + \int_0^t (1+\alpha_0\phi(r)-e^{\alpha\phi(r)})T(t-r)Sdr \\ & - \int_0^t G(x,t-r)\ (1+\alpha_0\phi(r)-e^{\alpha\phi(r)})dr] \ . \end{aligned} \tag{44}$$

We show that if ϕ is Hölder continuous, $\delta > 1/4$, then (44) has a unique solution, u, Hölder continuous in $X^{1/2}$. Thus, one can define a mapping $\phi \to F(\phi) = u(0,.)$. We apply to F the Schauder fixed point theorem and obtain a solution of the original integral equation. In order to prove uniqueness we show that the mapping defined by the right-hand side of (44) is a contraction in $C_{\rho,0+\delta}((0,T_0),X^{1/2})$ for some T_0 small enough.

Now we turn to the detailed proof.

Step 1. Solving the linear equation (44).

Let $\phi \in C_{\rho,0+\delta}((0,T),\mathbb{R}^1)$, $1/4 < \delta < 1/2$. We always assume $\rho \geq \delta$, it isn't a restriction, since $C_{\rho,0+\delta} \subseteq C_{\rho',0+\delta}$ if $\rho \leq \rho'$. Let $f(t)$ be the inhomogeneous term in (44)

$$f(t) = T(t)u_0 + \int_0^t (1-e^{\alpha\phi}+\alpha_0\phi)T(t-r)S\,dr$$

$$- \int_0^t G(x,t-r)(1+\alpha_0\phi(r)-e^{\alpha\phi(r)})dr \quad .$$

In order to estimate $f(t)$ we first estimate $T(t)u_0$. If $u_0 \in X^{1/2}$, then

$$\|T(t)u_0\|_{1/2} = \|A_0^{1/2}T(t)u_0\|_0 \leq C\|u_0\|_{1/2} \quad ,$$

$$\|(T(t+h)-T(t))u_0\|_{1/2} \leq Ch^{\delta}t^{-\delta}\|u_0\|_{1/2} \quad .$$

One may estimate f as follows

$$\|f(t)\|_{1/2} \leq C\|u_0\|_{1/2} + \sup_{0<r<t} \left|1+\alpha\phi(r)-e^{\alpha\phi(r)}\right| \int_0^t (t-r)^{-1/2}dr\,\|S\|_0$$

$$+ C \sup \left|1+\alpha_0\phi(r)-e^{\alpha\phi(r)}\right|$$

$$\leq C\|u_0\|_{1/2} + C \sup \left|1+\alpha_0\phi(r)-e^{\alpha\phi(r)}\right| \quad . \tag{45}$$

Analogously

$$\|f(t)-f(t+h)\|_{1/2} \le Ch^{\delta}t^{-\delta}\|u_0\|_{1/2}$$

$$+ h^{\delta}\sup\left|1+\alpha_0\phi(r)-e^{\alpha}\phi(r)\right| \int_0^t (t-r)^{-1/2-\delta}dr\|S\|_0$$

$$+ h^{\delta}H_{\rho,0+\delta}[1+\alpha_0\phi(r)-e^{\alpha\phi(r)}]\ t^{-\rho+1/4-\varepsilon}$$

$$\le Ch^{\delta}t^{-\delta}\|u_0\|_{1/2} + Ch^{\delta}\sup\left|1+\alpha_0\phi(r)-e^{\alpha\phi(r)}\right|$$

$$+Ch^{\delta}t^{-\rho+1/4-\varepsilon}H_{\rho,0+\delta}[\phi]\sup|\phi| \quad . \tag{46}$$

Thus $f \in C_{\rho',0+\delta}(0,/T,X^{1/2})$ where $\rho' = \rho-1/4+\varepsilon$ for any small $\varepsilon > 0$. (We've made use of Fact 4, Sec. 2.5, and Theorem 2).

We have estimated the inhomogeneous term in the equation (44), now we observe that the norm of the integral operator was in fact estimated in the proof of Theorem 4 (see (43) and further). Summarizing those calculations we have

$$\|K_{\phi}u\|_{1/2} \equiv \|\int_0^t (1-e^{\alpha\phi(r)})T(t-r)(u_y+1/2u)dr\|_{1/2}$$

$$\le C\sup\left(\left|1-e^{\alpha\phi(r)}\right|\|u(r)\|_{1/2}\right)\int_0^t (t-r)^{-1/2}dr$$

$$= Ct^{1/2}\sup_{0<r<t}\left(\left|1-e^{\alpha\phi(r)}\right|\|u(r)\|_{1/2}\right) \quad , \tag{47}$$

and in addition

$$\|(K_\phi u)(t+h)-(K_\phi u)(t)\|_{1/2} \le \sup\left|1-e^{\alpha\phi(r)}\right| \sup\|u(r)\|_{1/2}(Ct^{1/2-\delta}h^{\delta}+Ch^{1/2}) \tag{48}$$

Now we consider the equation

$$u = K_\phi u + f , \tag{44'}$$

in the Banach space $L^\infty([0,T], X^{1/2})$. The inequality (47) shows that K_ϕ is a contraction if T is small enough (T depends only on the magnitude of ϕ). Therefore (44') has a unique solution. For arbitrary T one can divide [0,T] into small intervals and again obtain a solution. In addition, it follows from (47) that for any $L > 0$ and any $\phi \in C_{\rho,0+\delta}$ such that $\sup_{0<t<T}|\phi(t)| < L$

$$\sup_{0<t<T} \|u(t)\|_{1/2} \le C(L) \sup_{0<t<T} \|f(t)\|_{1/2} . \tag{49}$$

It is clear from estimates in (46) and (48) that if $w \in L^\infty([0,T], X^{1/2})$ satisfies (44') then $u \in C_{\rho',0+\delta}((0,T), X^{1/2})$ with $\rho' = \max(\rho-1/4+\varepsilon, 1/2-\delta) < \rho$. Hence $u \in C_{\rho,0+\delta}$ because $H_{\rho,0+\delta}[g] \le T^{\rho-\rho'}H_{\rho',0+\delta}[g]$ for any g.

<u>Step 2</u>. The operator on the boundary.

In view of Step 1, there is a well defined mapping $\phi \to u$, where u solves (44). Taking the trace of u at $x = 0$ (which is defined since $u(.,t) \in X^{1/2}$) one obtains the operator

$$F : \phi \to \psi \equiv u(0,t) .$$

Firstly we show that F is continuous in $C_{\rho,0+\delta}$. Let $\phi_1, \phi_2 \in C_{\rho,0+\delta}((0,T),\mathbb{R}^1)$. We denote by u_1, u_2 the solutions of the corresponding equations (44') and let $\psi_i = F(\phi_i)$, i=1,2. As follows from Lemma 2.5.

$$\left|\psi_1(t)-\psi_2(t)\right| \leq C\|u_1(t)-u_2(t)\|_{1/2} \;.$$

In order to estimate 1/2-norm we employ (44') and get

$$\|u_1(t)-u_2(t)\|_{1/2} \leq \|f_1-f_2\|_{1/2} + \|K_{\phi 1}u_1-K_{\phi 2}u_2\|_{1/2} \;. \qquad (50)$$

For the first summand on the right-hand side one obtains (similarly to (45))

$$\|f_1-f_2\|_{1/2} \leq \left\|\int_0^t (\alpha(\phi_1-\phi_2)-e^{\alpha\phi_1(r)}+e^{\alpha\phi_2(r)})dr\right\|_{1/2}$$

$$+ \left\|\int_0^t G(x,t-r)(\alpha_0(\phi_1(r)-\phi_2(r))+e^{\alpha\phi_2(r)}-e^{\alpha\phi_1(r)})dr\right\|_{1/2}$$

$$\leq C \sup_{0<r<t} \left|\phi_1-\phi_2\right| \left|\alpha-\alpha_0\right| \;, \qquad (51)$$

where C is uniform for all ϕ with $|\phi| < L$.

The estimate for the second summand in (50) is similar to (47). Put for brevity

$$v_i(r) = 1 - e^{\alpha\phi_i(\tau)} \;, \quad w_i = u_{iy} + 1/2u_i \;;$$

then

$$\|\int_0^t v_1(r)T(t-r)w_1dr - \int_0^t v_2(r)T(t-r)w_2dr\|_{1/2}$$

$$\leq \|\int_0^t (v_1(r)-v_2(r))T(t-r)w_1dr\|_{1/2}$$

$$+ \|\int_0^t (v_2(r)T(t-r)(w_1-w_2)dr\|_{1/2}$$

$$\leq C_1 \sup |\phi_1-\phi_2| \sup \|u_1(r)\|_{1/2}t^{1/2} + C_2 \sup \|u_1-u_2\|_{1/2}\cdot t^{1/2} \quad . \tag{52}$$

If t is chosen so that $C_2t^{1/2} \leq 1-a$, $a < 0$ then using (51) and (52) one can derive from (5) the following estimate

$$a\|u_1(t)-u_2(t)\|_{1/2} \leq C \sup_{0<r<t} |\phi_1-\phi_2| \, |\alpha-\alpha_0|$$

$$+ C \sup |\phi_1-\phi_2| \sup \|u_1(r)\|_{1/2}t^{1/2} \quad . \tag{53}$$

We have obtained this estimate for small t but since C_2 in (52) depends only on L (which is the upper bound for $\phi_i(t)$) one can divide [0,T] into small intervals and get from (53)

$$\sup_{0<t<T} |\psi_1(t)-\psi_2(t)| \leq C_1 \sup |\phi_1-\phi_2| \, |\alpha-\alpha_0| + C_2 \sup |\phi_1-\phi_2| \quad , \tag{54}$$

where we have used (49) and (45), C_2 depends on L and $\|u_0\|_{1/2}$.

To complete the proof of continuity of F one should compare Hölder coefficients of $\phi_1-\phi_2$ and $\psi_1-\psi_2$. Again

$$H_{\rho,0+\delta}[\psi_1-\psi_2] \leq CH_{\rho,0+\delta}[u_1-u_2, X^{1/2}]$$

$$\leq CH_{\rho,0+\delta}[f_1-f_2] + CH_{\rho,0+\delta}[K_1u_1 - K_2u_2]$$

Repeating the calculations from (45) one easily obtains that

$$\|(f_1-f_2)\Big|_t^{t+h}\|_{1/2} \leq Ch^\delta \sup|\phi_1-\phi_2|\;|\alpha-\alpha_0|$$

$$+ Ch^\delta t^{-\rho+1/4-\varepsilon} H_{\rho,0+\delta}[\phi_1-\phi_2] \quad . \tag{55}$$

And analogously to (48) and (52)

$$\|(K_1u_1 - K_2u_2)\Big|_t^{t+h}\|_{1/2}$$

$$\leq \sup\left|e^{\alpha\phi_1}-e^{\alpha\phi_2}\right| \sup\|u_1(r)\|_{1/2}\;(Ct^{1/2-\delta}h^\delta+Ch^{1/2})$$

$$+ \sup\left|1-e^{\alpha\phi_2}\right| \sup\|u_1-u_2\|_{1/2}\;(Ct^{1/2-\delta}h^\delta+Ch^{1/2}) \quad .$$

Thus, recalling (53), we have

$$\sup_{0<t<T,h>0} t^\rho h^{-\delta}\,\|(K_1u_1-K_2u_2)\Big|_t^{t+h}\|_{1/2} \leq C\sup|\phi_1-\phi_2|\,\sup\|u_1(r)\|_{1/2}$$

$$+ C\sup\left|1-e^{\alpha\phi_2}\right|\,\sup|\phi_1-\phi_2| \quad , \tag{56}$$

(we've made use of the inequality $\rho + 1/2 - \delta > 0$).

Transforming in the same way the inequality in (55) we obtain finally

$$H_{\rho,0+\delta}[\psi_1-\psi_2] \leq CH_{\rho,0+\delta}[\phi_1-\phi_2] + C \sup |\phi_1-\phi_2| \ . \tag{57}$$

The estimates in (54) and (57) show that F is continuous in $C_{\rho,0+\delta}$.

Step 3. A ball, invariant under F.

Here we show that there exists such L that the ball of the radius L in $C_{\rho,0+\delta}((0,T),R^1)$ is invariant under F. All the information we need was, in fact, obtained on Step 1.

From (47) and (48) we have

$$\sup \|K_\phi u\|_{1/2} + H_{\rho,0+\delta}[K_\phi u] \leq C \sup |1-e^{\alpha\phi}| \sup \|u\|_{1/2} \ ,$$

and recalling (49) we can continue

$$\leq C \sup |1-e^{\alpha\phi}| \sup \|f\|_{1/2} \ . \tag{58}$$

Further, from (44'), (58) we obtain

$$|\psi|_{\rho,\delta} \equiv \sup |\psi| + H_{\rho,0+\delta}[\psi] = \sup |\psi| + \sup t^\rho \frac{\psi(x+h)-\psi(x)}{h^\delta}$$

$$\leq (1 + C \sup |1-e^{\alpha\phi}|) \sup \|f\|_{1/2} + H_{\rho,0+\delta}[f] \ , \tag{59}$$

and by (45) and (46)

$$\leq (1 + C \sup |1-e^{\alpha\phi}|) (C\|u_0\|_{1/2} + C \sup |1 + \alpha_0\phi - e^{\alpha\phi}|)$$

$$+ C\|u_0\|_{1/2} + C \sup |1 + \alpha_0\phi - e^{\alpha\phi}| + CH_{\rho,0+\delta}[\phi] \sup |\phi| \ . \tag{59a}$$

Let $L = |\phi|_{\rho,\delta}$. For small L, (59a) yields the following inequality

$$|\psi|_{\rho,\delta} \leq C_0 \|u_0\|_{1/2} + C_1 L^2 + C_2 L|\alpha-\alpha_0| \quad .$$

Tracing accurately all constants one can verify that C_0, C_1 and C_2 depend only on T. Thus choosing $\alpha(T)$ so that $C_2\alpha(T) < 1$ and taking such a L_0 that

$$L(\alpha,T) = (L_0 - C_1 L_0^2 - C_2 L_0|\alpha-\alpha_0|)/C_0 \quad ,$$

is maximal, we achieve our goal, $|\psi|_{\rho,\delta} \leq L$, if $\|u_0\|_{1/2} \leq L(\alpha,T)$ (we remind that $|\alpha-\alpha_0| < \alpha(T)$). It's easy to see that

$$L(\alpha,T) = (1-C_2|\alpha-\alpha_0|)^2/(4C_0C_1) \quad , \tag{60}$$

Now set $\rho' = \delta' < \delta$. Consider in the Banach space $C_{\delta',0+\delta'}((0,T),R^1)$ a subset $P_L = \{\phi: |\phi|_{\rho,\delta} < L\}$, where L is the constant just introduced. It is not hard to prove that P_L is relatively compact in $C_{\delta',0+\delta'}$ (for the proof see Friedman [4, Chapter 7]. Thus, a continuous operator F acts on a relatively compact set, therefore, by Schauder's theorem it has a fixed point ϕ_0. Solving the integral equation (44) with coefficients defined by ϕ_0 we obtain a solution, u, whose trace, by definition, equals ϕ_0, i.e., $u(0,t) = \phi_0(t)$. Replacing ϕ_0 by $u(0,.)$ in (44) we get the original nonlinear equation.

Uniqueness is a corollary of the following proposition, which itself is of some interest.

<u>Proposition 6</u>. Consider the Cauchy problem (35). For any a, U > 0 there exists T > 0 so that the problem has a unique solution on the interval (0,T) if $|\alpha-\alpha_0| < a$, $\|u_0\|_{1/2} < U$.

<u>Proof</u>. The proposition is a standard local existence result. For the proof we'll show that the integral operator is a contraction for small t.

We introduce the notation

$$v = u_y + 1/2u \quad , \quad w(r) = 1 - e^{\alpha u(0,r)} \quad , \quad z(r) = w(r) + \alpha_0 u(0,r)$$

$$\phi(u) \; = \; \int_0^t w(r)T(t-r)v(r)dr + \int_0^t z(r)(T(t-r)S-G(.,t-r))dr \quad .$$

Then the integral equation, equivalent to the problem (35), takes the form

$$u \; = \; T(t)u_0 + \Phi(u) \quad .$$

Consider the operator

$$u \; \to \; T(t)u_0 + \Phi(u) \; = \; B(u) \quad .$$

We will show that this is a contraction in $C_{\rho,0+\delta}((0,T),X^{1/2})$ for some T. We provide all the functions with subscripts 1 and 2, and calculate the difference,

$$B(u_1)-B(u_2) \; = \; \int_0^t w_1(r)T(t-r)(v_1-v_2)dr$$

$$+ \int_0^t (w_1-w_2)T(t-r)v_2dr + \int_0^t (z_1-z_2)(T(t-r)S-G(.,t-r)dr \quad . \tag{61}$$

As many times before, we have

$$\left\| \int_0^t w_1(r)T(t-r)(v_1-v_2)dr \right\|_{1/2} \leq \int_0^t (t-r)^{-1/2} |w_1(r)| \cdot \|v_1-v_2\|_0 dr$$

$$\leq t^{1/2} C \sup |w_1(r)| \sup \|u_1-u_2\|_{1/2} ,$$

and

$$\int_0^t w_1(r)T(t-r)(v_1-v_2)dr \Big|_{t=s}^{t=s+h} \quad 1/2$$

$$\leq \sup |w_1(r)| \sup \|u_1-u_2\|_{1/2} \; C(S^{1/2-\delta}h^{\delta}+Ch^{1/2}) \quad .$$

Thus

$$\left| \int_0^t w_1 T(t-r)(v_1-v_2) \right|_{\rho,\delta} \leq t^{\min(1/2,\rho)} C \sup |w_1(r)| \sup \|u_1-u_2\|_{1/2} , \tag{62}$$

where $|.|_{\rho,\delta}$ is defined in (59).

The analogous estimates hold for other terms in (61)

$$\left| \int_0^t (w_1-w_2)T(t-r)v_2 dr \right|_{\rho,\delta} \leq t^q C \sup |w_1-w_2| \sup \|u_2\|_{1/2} , \tag{63}$$

$$\left| \int_0^t (z_1-z_2)T(t-r)S dr \right|_{\rho,\delta} \leq C \sup |z_1-z_2| \; t^q , \quad q = \min(\rho,1/2) . \tag{64}$$

At last,

$$\|\int_0^t (z_1-z_2)G(.,t-r)dr\|_{1/2} \leq \sup |z_2-z_2|Ct^{1/4-\varepsilon} \quad ,$$

(see Step 2 of the proof of Theorem 2) and

$$\left|\int_0^t (z_1-z_2)G(.,t-r)dr\right|_{\rho,\delta} \leq C|z_1-z_2|_{\rho,\delta}t^p \quad , \tag{65}$$

where $p = \min(\rho-\delta+1/4-\varepsilon,\ 1/4-\varepsilon)$, as follows from Theorem 2.

Collecting together estimates in (62) through (65) we obtain from (61)

$$\begin{aligned}|B(u_1)-B(u_2)|_{\rho,\delta} \leq\ & t^r(C_1 \sup |w_1| \sup \|u_1-u_2\|_{1/2} \\ & + C_2 \sup |w_1-w_2| \sup \|u_2\|_{1/2} + C_3|z_1-z_2|_{\rho,\delta}) \quad .\end{aligned} \tag{66}$$

where $r = \min(p,q) \geq 1/4-\varepsilon$.

We fix $t' > 0$ (the length of the existence interval, T, will be chosen later smaller than t'). Let $\ell = |T(.)u_0|_{\rho,\delta}$, calculated on the interval $(0,t')$. We have seen before that $\ell \leq C\|u_0\|_{1/2} \leq CU$.

Suppose that u_1 and u_2 are such that, on $(0,t')$, $|u_i|_{\rho,\delta} \leq 2\ell$ We start to estimate the right-hand side of (66)

$$\begin{aligned}\sup|w_1-w_2| &= \sup|e^{\alpha u_1}-e^{\alpha u_2}| = \sup(|u_1-u_2|e^{\alpha(\theta u_1+(1-\theta)u_2)}\alpha) \\ &\leq C\|u_1-u_2\|_{1/2}\ \alpha e^{\alpha 2\ell} \quad .\end{aligned}$$

Further

$$\left|z_1-z_2\right| = \left|\alpha_0(u_1-u_2) + e^{\alpha u_2}-e^{\alpha u_1}\right| = \left|u_1-u_2\right| \left|\alpha_0-\alpha e^{\alpha(\theta u_1+(1-\theta)u_2)}\right| ,$$

it follows

$$\left|z_1-z_2\right|_{\rho,\delta} \leq C\|u_1-u_2\|_{1/2} (\alpha_0+\alpha e^{2\ell\alpha}) + H_{\rho,\delta}(X^{1/2},[u_1-u_2])(\alpha_0+\alpha e^{2\ell\alpha})$$

$$+ C\|u_1-u_2\|_{1/2} H_{\rho,\delta}[\theta u_1+(1-\theta)u_2]\alpha e^{2\alpha\ell}$$

$$\leq C\left|u_1-u_2\right|_{\rho,\delta} (\alpha_0+\alpha e^{2\ell\alpha}+2\ell\alpha e^{2\alpha\ell}) = C_0(\alpha,\ell) \left|u_1-u_2\right|_{\rho,\delta} .$$

And finally we obtain from (66)

$$\left|B(u_1)-B(u_2)\right|_{\rho,\delta} \leq t^r(C_1(1+e^{2\alpha\ell})+C_2 2\ell\alpha e^{2\alpha\ell}+C_3C_0(\alpha,\ell)) \left|u_1-u_2\right|_{\rho,\delta} . \tag{67}$$

Now we take $T < t'$ so small that, for $t = T$, (67) yields

$$\left|B(u_1)-B(u_2)\right|_{\rho,\delta} \leq 1/2 \left|u_1-u_2\right|_{\rho,\delta} .$$

If we put in a standard fashion

$$u^{-1} = 0 , \quad u^0 = B(u^{-1}) = T(t)u_0, \ldots, u^n = Bu^{n-1}$$

then

$$\left|u^0\right|_{\rho,\delta} \leq \ell , \quad \left|u^1\right|_{\rho,\delta} = \left|u^{-1}-u^0+u^0\right|_{\rho,\delta}$$

$$= \left|B(u^0-u^{-1}) + u^0\right|_{\rho,\delta} \leq 3/2\ell < 2\ell ,$$

and by induction

$$|u^n|_{\rho,\delta} \leq (2-2^{-n})\ell < 2\ell \quad .$$

Thus, the following series converges

$$u = \sum_{n=0}^{\infty} (u^n - u^{n-1}) \quad ,$$

and gives a solution to the problem $u = Bu$. If u' is another solution then by taking T even smaller we can make $|u'|_{\rho,\delta} < \ell$ and on this interval u' should coincide with u.

4. PERIODIC SOLUTIONS

We consider the non-linear problem from (3.35) ($t \geq 0$)

$$u_t - A_0 u - F(\alpha,u) = 0$$

$$[u(0,t)] = 0$$

$$[u_y(0,t)] + e^{\alpha u(0,t)} - 1 = 0 \quad , \tag{1}$$

not imposing initial conditions. In this chapter we show that, for α sufficiently close to α_0, the problem in (1) has a time periodic solution. Recall that α_0 corresponds to crossing the imaginary axis by eigenvalues of $A(\alpha)$, $A_0 = A(\alpha_0)$.

In order to prove the existence we develop an analogue of the Hopf bifurcation theorem. Our approach is very close to the treatment of the theorem by Crandall and Rabinowitz in [1] where they consider the case of the operator with compact resolvent (cf. also Ize [7]). The most essential distinction of our problem is due to conditions at the free inner boundary (the non-linear function in the differential equation contains δ-functions). As we have seen in Chapter 3, this circumstance forces us to be careful about the choice of Banach spaces.

Without entering into details, we can describe the procedure of finding a periodic solution as follows. Instead of (1) we consider the corresponding integral equation

$$\Phi(\tau,\alpha,u)\ (t) = 0 \quad , \tag{2}$$

where p is a non-dimensionalized unknown period which appears explicitly after some change of the time variable. Regarding Φ as an operator on a space of periodic functions with the fixed period p_0, we seek zeros of Φ.

We use the information that if $\alpha = \alpha_0$ then the linearized operator has a p_0-periodic solution, $e_0(t)$, and construct the function

$$Y(s,p,\alpha_1,v) = \begin{cases} s^{-1}\Phi(p,\alpha,s(e_0+v)), & s\neq 0 \\ \Phi_u(p,\alpha,0)(e_0+v), & s=0 \end{cases}, \tag{3}$$

where v belongs to V, a complement of e_0 (and e_1) in the space of p_0-periodic functions.

We show that Y is a mapping of class C^1 from a neighborhood of $(0,1,0,0)$ in $\mathbb{R}^3 \times V$, $Y(0,1,0,0) = 0$ and that the implicit function theorem is applicable. Then the solutions of $Y = 0$ are given by continuously differentiable functions $p(s)$, $\alpha(s)$ and $v(s)$. Setting $u(s)(t) = s(e_0(t) + v(s)(t))$ we find a desired solution to (2). $u(s)$ is a solution on the interval $[0,2\pi]$ and can be continued periodically on the whole line. A priori the continuation may be nondifferentiable at points $2\pi k$. In order to show that it is not so, we apply the existence theory of Chapter 3 and construct a solution, v, on the interval $(2\pi-\varepsilon, 2\pi+\varepsilon)$ with initial data $v(2\pi-\varepsilon) = u(2\pi-\varepsilon)$. By uniqueness $u = v$ which is smooth at 2π.

In the last section, as easy consequence of parabolic theory we obtain that a periodic solution is classical. We prove also that the solution is bounded. Recall that the solution, u, belongs to some weighted space, and thus $u \sim e^{ax/2}$ as $x \to \infty$. We consider the equation (1) as a Cauchy problem in the half-strip $(0,2\pi) \times (0,\infty)$ with x playing the role of time and with periodic boundary conditions. Studying the spectrum of this ill-posed problem we prove that the condition $|u| < Ce^{ax/2}$ yields boundedness of u.

4.1. Regularity Properties of the Integral Operator

We are seeking a solution of (1), periodic with some unknown period p'. p' should be close to $p_0 = 2\pi i/\lambda_+(\alpha_0)$. We introduce $p = p'/p_0$ explicitly in the equation, making a change of variables, $t \to tp$ and seek a p_0-periodic solution of the equation

$$u_t - p\,(A_0 u + F(\alpha,u)) = 0 \quad , \tag{4}$$

with p close to 1 and (α,u) near $(\alpha_0,0)$.

Employing results of Sec. 3.3, we replace the differential equation (4) by the following integral equation

$$\Phi(p,\alpha,u)(t) \equiv u(t)-T(pt)u(0)$$

$$- p\,[\int_0^t (1-e^{\alpha u(0,r)})T(p(t-r))(u_y+1/2u)dr$$

$$+ \int_0^t (1+\alpha_0 u(0,r)-e^{\alpha u(0,r)})T(p(t-r))Sdr$$

$$- \int_0^t G(x,p(t-r))(1+\alpha_0 u(0,r)-e^{\alpha u(0,r)})dr] = 0 \quad . \tag{5}$$

Denote by C_p the Banach space

$$C_p = \{h \in C_{\rho,0+\delta}((0,p_0),X^{1/2}) \cap C([0,p_0],X^{1/2}) \mid h(0) = h(p_0)\} ,$$

and by C_0 the Banach space

$$C_0 = \{h \in C_{\rho,0+\delta}((0,p_0),X^{1/2}) \cap C([0,p_0],X^{1/2}) \mid h(0) = 0\} ,$$

where $\rho = \delta$, $1/4 < \delta < 1/2$. If we find a zero of Φ, considered as an operator on C_p, then, by Theorem 3.4, it will solve the differential equation (4) on the interval $(0,p_0)$. We will show later that its periodic continuation solves (4) for all $t \in \mathbb{R}^1$.

Denote by $\Omega \subseteq \mathbb{R} \times \mathbb{R} \times C_p$ that subset for which the expression (5), defining $\Phi(p,\alpha,u)$, makes sense.

Proposition 1. Φ is a mapping from Ω into C_0, analytic with respect to u and α, infinitely differentiable with respect to p. $\Phi(p,\alpha,0) = 0$ and for $v(.,t) \in C_p$

(i) $$(\Phi_u(1,\alpha_0,0)v)(t) = v(t)-T(t)v(0) ,$$

(ii) $$(\Phi_{pu}(1,\alpha_0,0)v)(t) = -tA_0T(t)v(0) ,$$

(iii) $$(\Phi_{\alpha u}(1,\alpha_0,0)v)(t) = \int_0^t v(0,r)T(t-r)S\,dr - \int_0^t G(.,t-r)v(0,r)\,dr .$$

Proof. At first we show that Φ maps Ω into C_0. We have seen in the proof of Theorem 3.5 that the operator similar to Φ (there was p=1) maps the set

$$\{h \in C_{\rho,0+\delta}((0,T),X^{1/2}) \mid h(0) \in X^{1/2}\}$$

into $C_{\rho,0+\delta}((0,T),X^{1/2})$ continuously.

Considering each summand on the right-hand side of (5) separately we see that

$$\begin{aligned} \|v_1(t)\|_{1/2} &\equiv \|u(t)-T(pt)u(0)\|_{1/2} \\ &\leq \|u(t)-u(0)\|_{1/2} + \|(I-T(pt))u(0)\|_{1/2} \\ &= \|u(t)-u(0)\|_{1/2} + \|(I-T(pt))(-A_0)^{1/2}u(0)\|_0 \to 0 \quad , \end{aligned}$$

as $t \to 0$, since $u(t) \in C([0,p_0]),X^{1/2})$ and for analytic semigroup, $T(t)v \to v$ as $t \to 0$ for any $v \in X$ (put $v = (-A_0)^{1/2}u(0)$). For

$$v_2(t) = \int_0^t (1-e^{\alpha u(0,r)})T(p(t-r))(u_y+1/2u)dr \quad .$$

We have

$$\begin{aligned} \|v_2(t)\|_{1/2} &\leq \int_0^t (p(t-r))^{-1/2} \left|1-e^{\alpha u(0,r)}\right| \|u_y+1/2u\|_0 \, dr \\ &\leq Ct^{1/2} \to 0 \quad , \text{ as } t \to 0 \ . \end{aligned}$$

For the next term on the right-hand side of (5), $v_3(t)$, a similar estimate holds. And at last for

$$v_4(t) = \int_0^t G(x,p(t-r))(1+\alpha_0 u(0,r)-e^{\alpha u(0,r)})dr \quad ,$$

we obtain $v_4(0) = 0$ by Lemma 3.3 (Sec. 3.2). Thus, we have shown that $\Phi(p,\alpha,u)(t) \to 0$ in $X^{1/2}$ as $t \to 0$.

We observe that $\Phi(p,\alpha,u)$ is linear with respect to u, $u(0,r)$ and $e^{\alpha u(0,r)}$. It is almost obvious that the mapping $(\alpha,f) \to e^{\alpha f}$ is analytic from $I \times C_{\rho,0+\delta}((0,p_0), \mathbb{R}^1)$ to $C_{\rho,0+\delta}((0,p_0),\mathbb{R}^1)$. Indeed

$$p_n(\alpha f) = \sum_{i=0,n}(\alpha f)^i/i! - e^{\alpha f} \to 0$$

in sup-norm and

$$k^{-\delta}\left|p_n(\alpha f)(t)-p_n(\alpha f)(t+k)\right| = \frac{|p_n(\alpha f)(t+h)-p_n(\alpha f)(t)|}{\alpha\ f(t+k)-f(t)}$$

$$\times \quad \alpha k^{-\delta}\left|f(t+k)-f(t)\right| = (p_n)_x\Big|_{x=\theta} \quad \alpha t^{-\rho}H_{\rho,0+\delta}[f] \to 0 \quad ,$$

if $|f|_{\rho,\delta} \leq L < \infty$ for any $L > 0$ since $p_n \to 0$ with derivatives on any finite interval.

We can deduce the proposition from the chain rule if enough regularity can be established for the maps

$$(p,x) \to T(pt)x \quad , \quad (0,\infty) \times X^{1/2} \to C_{\rho,0+\delta}([0,p_0]),X^{1/2})$$

$$(p,u) \to \int_0^t T(p(t-r))u(r)dr \quad ; \quad (0,\infty) \times C([0,p_0],X) \to C_0$$

$$(p,f) \to \int_0^t G(.,p(t-r))f(r)dr \quad ; \quad (0,\infty) \times C_{\rho,0+\delta}((0,p_0), X^{1/2}) \to C_0 \ . \tag{6}$$

The last map in (6) is infinitely differentiable by Lemma 3.3, while it turns out that the first two maps are analytic. We reproduce here the proof of analyticity from Crandall and Rabinowitz [1] (where the observation is attributed to Henry [6]).

Since $T(t)$ is a holomorphic semigroup, the following estimate holds

$$\|A_0^m T(t)x\|_{1/2} = \|(A_0 T(t/m))^m x\|_{1/2} \le \left(\frac{Cm}{t}\right)^m \|x\|_{1/2} \quad , \tag{7}$$

for $x \in X^{1/2}$, $0 < t \le t_0$, C depends only on t_0. It follows the series,

$$\sum_{m=0}^{\infty} (t\tilde{p})^m A_0^m T(tp)x/m! = \sum \tilde{p}^m/m! f_m(tp) \quad , \tag{8}$$

as a function of t converges in $L^\infty([0,p_0],X^{1/2})$ if $\tilde{p}/p$ is small enough (say, smaller than $1/Ce$). The series in (8) is obviously equal to $T((p+\tilde{p})t)x$.

The inequality in (7) shows that $t^m A_0{}^m T(t)$ is a bounded operator in $X^{1/2}$ for all $0 < t \le t_0$. If $x \in D(A_0{}^{m+1})$ then $t^m A_0{}^m T(t)x \to 0$ as $t \to 0$, for $m > 0$. Since $D(A_0{}^{m+1})$ is dense in $X^{1/2}$, it follows that for any $x \in X^{1/2}$ the functions $f_m(t,p)$ from (8) may be extended by continuity to have the value 0 at $t = 0$, for $m > 0$.

Let us prove convergence of the series in (8) in Hölder's norm.

$$\|f_m(t+h,p)-f_m(t,p)\|_{1/2} \leq \|(t+h)^m-t^m)A_0^m T((t+h)p)x\|_{1/2}$$

$$+ \quad \|t^m A_0^m(T((t+h)p) - T(tp))x\|_{1/2}$$

$$\leq \quad m(t+h)^{m-1}\cdot h\cdot\left(\frac{Cm}{(t+h)p}\right)^m \|x\|_{1/2}$$

$$+ \quad t^m(tp)^{-\delta}(hp)^\delta \left(\frac{Cm}{tp}\right)^m \|x\|_{1/2}$$

$$= \quad (Cm/p)^m \|x\|_{1/2} \left((m-1)h/(t+h)+(h/t)^\delta\right) \quad .$$

Noticing that for $0 \leq \delta \leq 1$

$$h/(t+h) \quad < \quad \begin{cases} 1 \leq (h/t)^\delta & \text{for } h \geq t \\ h/t \leq (h/t)^\delta & \text{for } h \leq t \end{cases} \quad ,$$

we arrive at the estimate

$$\|f_m(t+h,p)-f_m(t,p)\|_{1/2} \leq (Cm/p)^m m\, h^\delta t^{-\delta} \|x\|_{1/2} \quad . \tag{9}$$

Therefore

$$f_m \in C_{\delta,0+\delta}((0,p_0), X^{1/2})$$

and as before (8) converges if $|\tilde{p}/p| \leq C_0$ where C_0 is sufficiently small.

Thus, we have proved that the first map in (6) is analytic. The proof for the second one is almost the same. Instead of (7), one should start from the inequality

$$\|A_0^m T(t)x\|_{1/2} \leq (Cm/t)^m t^{-1/2} \|x\|_0 \quad . \tag{10}$$

Employing (10) one can show that the series

$$\sum \tilde{p}^m/m! \int_0^t f_m(t-r,p)u(r)dr$$

converges in C_0.

Computation of the derivatives appearing in the statement of the Proposition, is routine. It reduces to formal differentiation with respect to the corresponding variables and this is omitted.

Denote by L the operator $\Phi_u(1,\alpha_0,0)$. By Proposition 1,

$$(Lv)(t) \equiv (\Phi_u(1,\alpha_0,0)v)(t) = v(t)-T(t)v(0) \quad ,$$

where $v \in C_p$. Let A be a linear operator, we will denote by $N(A)$ and $R(A)$ its null space and range.

For the bifurcation theorem we need some properties of the Jacobian L, which we summarize in the following lemma.

Lemma 2. Consider L as an operator from C_p to C_0. Then

(i) $v \in N(L)$ if and only if $v(t) = T(t)x$ for some $x \in N(I-T(p_0))$.

(ii) $h \in R(L)$ if and only if $h(p_0) \in R(I-T(p_0))$.

(iii) $\dim N(L) = 2$.

(iv) $R(L)$ is a closed subspace in C_0, $\operatorname{codim} R(L) = 2$.

Remark. Note that $I-T(p_0)$ is regarded here as an operator on $X^{1/2}$.

Proof. Statement (i) as obvious. For (ii), if

$$h(t) = u(t)-T(t)u(0) \quad ,$$

and $u \in C_p$, then

$$h(p_0) = u(p_0)-T(p_0)u(0) = (I-T(p_0))u(0) \quad .$$

Conversely let $h(p_0) = (I-T(p_0))x$, set $u(t) = T(t)x+h(t)$. Since $x \in X^{1/2}$, $T(t)x$ is Hölder continuous and $u(0) = u(p_0) = x$ therefore $u \in C_p$. Obviously,

$$h(t) = u(t)-T(t)u(0) \quad .$$

Thus (ii) holds.

We now recall some results of Sec. 2.7. It was proved (Corollary 2.8) that there exist x_0^* and x_1^*, continuous linear functionals on $X^{1/2}$ so that

$$R(I-T(p_0)) = \{x \in X^{1/2} \mid (x,x_0^*) = (x,x_1^*) = 0\} \quad .$$

In view of (ii) this means, that

$$R(I-L) = \{h \in C_0 \mid (h(p_0),\overset{*}{x}_0) = (h(p_0),\overset{*}{x}_1) = 0\} \quad .$$

The functionals $h \to (h(p_0),x_i^*)$, $i = 1,2$ are continuous on C_0, and statement (iv) follows. (iii) follows from (i) and Proposition 2.7, where it was proved that $\dim N(I-T(p_0)) = 2$.

4.2. The Bifurcation Theorem

With the information of the previous section we can now formulate a version of the Hopf bifurcation theorem. Except for calculations in two-dimensional space, the proof is almost identical to one in Crandell and Rabinowitz [1]. We include it here for completeness.

Theorem 3. Let Φ be the integral operator defined by (5). Then there exists a number $\eta > 0$ and smooth functions $(p,\alpha,u):(-\eta,\eta) \to \mathbb{R} \times \mathbb{R} \times C_p$ such that

$$\Phi(p(s),\alpha(s),u(s)) = 0 \quad \text{for} \quad |s| < \eta \ ,$$

$$\alpha(0) = \alpha_0 \ , \quad p(0) = 1 \ , \quad u(0) = 0$$

$$\text{and} \qquad u(s) \neq 0 \quad \text{if} \quad 0 < |s| < \eta \ .$$

Proof. By Lemma 2, $N(\Phi_u(1,\alpha_0,0))$ is two-dimensional. Let V be a complement of $N(\Phi_u(1,\alpha_0,0))$ in C_p, then clearly V is a Banach subspace of codimension 2.

Let $x_0 \in N(I-T(p_0))$ and set $e_0(t) = T(t)x_0$, then $e_0 \in N(\Phi_u(1,\alpha_0,0))$. Define

$$Y(s,p,\alpha,v) = \begin{cases} s^{-1}\Phi(p,\alpha,s(e_0+v)) & , \quad s \neq 0 \\ \Phi_u(p,\alpha,0)(e_0+v) & , \quad s \neq 0 \end{cases} \quad .$$

By Proposition 1, Y is an infinitely differentiable mapping from a neighborhood of $(0,1,\alpha_0,0)$ in $\mathbb{R}^3 \times V$ to C_0. Note that

$$s^{-1}\Phi(p,\alpha,s(e_0+v)) = s^{-1}(\Phi(p,\alpha,s(e_0+v))-\Phi(p,\alpha,0)) \quad ,$$

is smooth with respect to s, since $\Phi(p,\alpha,u)$ is smooth in u.

Obviously $Y(0,1,\alpha_0,0) = 0$ and, as follows from Proposition 1, the Frechet derivative of the map $(p,\alpha,v) \to Y(s,p,\alpha,v)$ at $(0,1,\alpha_0,0)$ is the linear operator

$$Z(p,\alpha,v)(t) = v(t)-T(t)v(0)-ptA\ T(t_0)x_0$$

$$+ \quad \alpha(\int_0^t e_0(0,r)T(t-r)Sdr - \int_0^t G(.,t-r)e_0(0,r)dr) \quad .$$

We will show that Z is an isomorphism. Once this is shown, the implicit function theorem implies that the solutions (s,p,α,v) of $Y = 0$ near $(0,1,\alpha_0,0)$ are given by smooth functions $(p(s),\alpha(s),u(s))$.

Setting

$$u(s)t = s(e_0(t) + v(s)(t)) \quad ,$$

we obtain the desired curve of solutions of $\Phi = 0$.

$v \to v-T(.)v(0)$ maps V onto $R(\Phi_u(1,\alpha_0,0))$ as follows from Lemma 2 and from the closed graph theorem. Moreover both spaces have codimension 2. Thus Z is an isomorphism if the relation

$$-ptA_0T(t)x_0 + \alpha \int_0^t e_0(0,r)(T(t-r)S - G(.,t-r))dr \quad R(\Phi_u(1,\alpha_0,0)) \quad ,$$

implies $p = \alpha = 0$. By Lemma 2, the inclusion is equivalent to the condition

$$(-pp_0A_0T(p_0)x_0 + \alpha \int_0^{p_0} e_0(0,r)(T(p_0-r)S - G(.,p_0-r))dr, x_i^*) = 0 ,$$

$$i = 0, 1 . \tag{11}$$

Noticing that $T(p_0)x_0 = x_0$ and employing the relations from Sec. 2.7, $(x_i,x_j^*) = \delta_{ij}$, $A_0x_0 = Cx_1$, we observe that if

$$d \equiv (\int_0^{p_0} e_0(0,r)(T(p_0-r)S - G(.,p_0-r))dr, x_0^*) \neq 0 , \tag{12}$$

then (11) implies $\alpha = 0$, hence $p = 0$.

The constant, d, from (17) is closely connected with $\phi_{u\alpha}(1,\alpha_0,0)$. More precisely, the following lemma holds.

Lemma 4. The constant, d, defined in (12) equals to

$$d = \mathrm{Re}(d\lambda(\alpha)/d\alpha\big|_{\alpha=\alpha_0})p_0 , \tag{13}$$

where $\lambda(\alpha)$ is one of two eigenvalues $\lambda_\pm$.

The lemma follows that $d \neq 0$, since for α near α_0, $\mathrm{Re}\lambda = (1+4\alpha-\alpha^2)/8$ whence $d\mathrm{Re}\lambda/d\lambda \neq 0$ at $\alpha = \alpha_0 = 2 + \sqrt{5}$.

The theorem is proved.

Proof of Lemma 4. We recall that in Sec. 2.3 we have found two eigenfunctions $\phi(\lambda_i,x)$, $i = 1,0$, $\lambda_1 = \lambda_0$, so that

$$A(\alpha)\phi(\lambda_i,.) = \lambda_i\phi(\lambda_i,.) \quad ,$$

and the functions $\psi_i(\lambda,t) = e^{\lambda_i t}\phi_i(\lambda,.)$ solve the Cauchy problem

$$\frac{du}{dt} - A(\alpha)u = 0 \quad , \quad u\big|_{t=0} = \phi_i(\lambda) \quad . \tag{14}$$

On the other hand, representing (14) in the form

$$\frac{du}{dt} - A_0 u = (A(\alpha)-A_0)u \quad , \quad u_{\ t=0} = \phi_i(\lambda) \quad , \tag{15}$$

one sees that the solution of (15) should satisfy the integral equation

$$u(t) = T(t)\phi_i(\lambda) + \int_0^t T(t-r)(A(\alpha)-A_0)u(r)dr \quad , \tag{16}$$

where, as always, $T(t)$ is a semigroup, generated by A_0.

Now, we observe that

$$(A(\alpha)-A_0) = (\alpha-\alpha_0)(S-\delta) \quad ,$$

and the integral in (16) should be understood as

$$(\alpha-\alpha_0)\int_0^t u(0,r)(T(t-r)S - G(.,t-r))dr \quad .$$

We plug $u = \psi_0(\lambda,t)$ into (16), take $t = p_0$ and differentiate with respect to α. At $\alpha = \alpha_0$ we obtain

$$d\lambda/d\alpha\,|_{\alpha=0}\ \phi_0 p_0 + d\phi_0/d\alpha \;=\; T(p_0)d\phi_0/d\alpha\,|_{\alpha=\alpha_0}$$

$$+ \int_0^t \phi_0(0,r)(T(p_0-r)S - G(.,t-r))dr \quad , \tag{17}$$

where we've used $\exp(\lambda_0(\alpha_0)p_0) = 1$.

Multiplying (17) by x_0^* and noticing that, since

$$((-I+T(p_0))d\phi_0/d\alpha,x_0^*) \;=\; (d\phi_0/d\alpha,(-I+T^*(p_0))x_0^*) \;=\; 0 \quad ,$$

the terms with $d\phi_0/d\alpha$ disappear, we obtain

$$p_0\mathrm{Re}(d\lambda/d\alpha(\phi_0,x_0^*)) \;=\; (\int_0^{p_0} \mathrm{Re}\phi_0(0,r)(T(p_0-r)S-G(.,p_0-r)dr,x_0^*) \quad . \tag{18}$$

Recall that for $\alpha = \alpha_0$ a real basis x_0, x_1 from Sec. 2.7 has the form

$$x_0 \;=\; 2^{1/2}(\phi_0+\overline{\phi}_0)$$

$$x_1 \;=\; A_0x_0/\nu \;=\; 2^{1/2}i(\phi_0-\overline{\phi}_0) \quad ,$$

whence $\phi_0 = (x_0-ix_1)2^{-1/2}$. Taking into account that $(x_i,x_j^*) = \delta_{ij}$ we get from (18) the desired result, (13).

With the help of Theorem 3 we obtain a solution, u, of the differential equation (4) on the interval $(0,p_0)$ such that $u(0) = u(p_0)$. As a simple consequence of results of Chapter 3 we get the following.

<u>Corollary 4</u>. Let u be a solution of problem (4) obtained in Theorem 3. Set

$$\tilde{u}(t+kp_0) = u(t) \ , \quad 0 \leq t \leq p_0 \quad , \quad k = 0,\pm 1,\ldots$$

then $\tilde{u}$ is a solution of (4) on $\mathbb{R}^1$.

<u>Proof</u>. Let $U = \sup_{0\leq t\leq p_0} \|u(t)\|_{1/2}$. By Proposition 3.6, there exists a $T > 0$, such that the problem

$$v_t - p(A_0 v + F(\alpha,v)) = 0$$

$$v\big|_{t=p_0-T/2} = u(p_0-T/2) \quad ,$$

has a unique solution v on the interval $(p_0-T/2,\ p_0+T/2)$. By uniqueness, $v(t)$ should coincide with $u(t)$ for $p_0-T/2 < t < p_0$ and with $u(t-p_0)$ for $p_0 < t < p_0 + T/2$. Thus $v(t) \equiv u(t)$ on $(p_0-T/2,\ p_0+T/2)$, hence $u(t)$ satisfies the equation also at $t = p_0$.

Note, that for the proof we could use Theorem 3.5 as well.

We conclude this section with the following uniqueness result.

Theorem 5. (Crandall and Rabinowitz). Under the conditions of Theorem 3 there exists $\varepsilon > 0$ such that if (α_1, u_1) is a solution of (1) of period p_1, where $|p_1 - p_0| < \varepsilon$, $|\alpha_1 - \alpha_0| < \varepsilon$ and $\|u_1\|_{1/2} < \varepsilon$, then there are numbers $s \in [0,\eta)$ and $\theta \in [0,p_0)$ such that

$$u_1(p_1 t/p_0) \;=\; u(s)(t+\theta) \quad ,$$

here $u(s)(.)$ is a solution of (4) found in Theorem 3.

The proof is a literal repetition of the proof of Theorem 1.11(c) from [1] and is omitted.

4.3. Regularity and Boundedness of the Solution

It should be noted that in previous sections we dealt with the nonlinear equation which differs from the original model equation (2.4) by a similarity transformation. In order to obtain a periodic solution to the original model problem one should multiply the solution $u(t)$ from Sec. 4.2 by the weight $w = e^{x/2}$.

Al alternative approach is to deal directly with the original problem not introducing explicitly the similarity transformation and estimating weighted norms instead of unweighted norms.

In this way one can show that the reasonings of Sec. 4.1 and 4.2 are applicable to the original problem considered in Banach spaces with any weight $e^{ax/2}$, $0 < a < 1$, and the following holds.

Proposition 6. Let u be a periodic solution to the problem

$$u_t \;=\; p(u_{yy} - e^{\alpha u(0,t)} u_y + (1 - e^{\alpha u(0,t)}) e^y) \quad , \quad y < 0 \tag{19}$$

$$u_t = p(u_{yy} - e^{\alpha u(0,t)} u_y) \quad , \quad y > 0 \tag{20}$$

$$[u(0)] = 0 \quad , \quad [u_y]_{y=0} + e^{\alpha u(0,t)} - 1 = 0 \quad . \tag{21}$$

(u exists under the conditions of Theorem 3, $u = e^{y/2}v$, where v is a solution to (4)). Then for any $0 < a \leq 1$ u is a solution (in the sense of Definition Sec. 3.3) in a Banach space with the weight $e^{ax/2}$. This means, in particular, that

$$u \in C^1([0,p_0],H_a^0) \cap C_{\rho,0+\delta}((0,p_0),H_a^1) \quad ,$$

where the norms in $H_a{}^0$ and $H_a{}^1$ are given by

$$\|f\|^2_{H_a^0} = (\int_0^\infty + \int_{-\infty}^0) e^{ax} |f|^2 \, dx \quad ,$$

$$\|f\|_{H_a^1} = \|f\|_{H_a^0} + \|f_x\|_{H_a^0} \quad .$$

We won't go into details of the proof of Proposition 6, because in the present section we will obtain a stronger result. Note only that, by virtue of Proposition 6, the periodic solution decays as $e^{x/2}$ when $x \to -\infty$ and grows slower than any exponential e^{ax}, $a > 0$, as $x \to \infty$. From Theorem 3 alone it follows that $u(x,.)$ and u_x grow not faster than $e^{x/2}$ as $x \to \infty$.

From now on we will consider all equations only for $x \geq 0$, reasonings for $x < 0$ are similar.

Our main observation is that a solution of the nonlinear equation (20) solves the linear equation

$$v_t = v_{xx} - f(t)v_x \quad , \quad x > 0 \quad , \tag{22}$$

where $f(t) = e^{\alpha u(0,t)}$ (note that we have omitted the immaterial factor p).

First, we will establish the following regularity result.

Theorem 7. Let u be a periodic solution to the problem (19) through (21) in the weak sense which was treated in Proposition 6. Then u is a classical solution.

Proof. We consider the equation in (22) in the domain $0 \leq x \leq A$, $0 \leq t < T$ with initial data

$$v(x,0) = u_0(x) \equiv u(x,0) \quad , \tag{23}$$

and boundary conditions

$$v(0,t) = \phi_1(t) \equiv u(0,t) \quad , \quad v(A,t) = \phi_2(t) \equiv u(A,t) \quad . \tag{24}$$

As we know, $u \in C_{\rho,0+\delta}((0,T),H^1)$. It follows that the functions ϕ_1, ϕ_2 and the coefficient $f(t) = e^{\alpha u(0,t)}$ are Hölder continuous on $(0,T)$. Moreover, they are Hölder continuous on some larger interval $(-a,T+a)$ because u is a solution on $\mathbb{R}^1$. Since $u(x,0) \in H^1$, it is continuous also.

Thus we have an initial-boundary value problem with continuous initial and boundary values and with Hölder continuous coefficients. It is a well known fact of the parabolic theory (see Friedman [4, Chap. 3, Th. 8]) that such a problem has a classical solution. Therefore a solution, v, of the problem (22) through (24) is classical. By uniqueness, $u \equiv v$.

In order to prove boundedness of the periodic solution we will consider the same linear equation (21) in another domain and with different initial and boundary conditions.

Consider the following problem in the domain $[0,2\pi] \times [0,\infty)$

$$u_{xx}(x,t) - f(t)u_x(x,t)-u_t(x,t) = 0 \quad , \tag{25}$$

with initial data

$$u(0,t) = u_0(t) \quad , \quad u_x(0,t) = v_0(t) \quad , \quad 0 \le t \le 2\pi \quad , \tag{26}$$

subject to boundary conditions

$$u(x,0) = u(x,2\pi) \quad , \quad x \ge 0 \quad . \tag{27}$$

Here f, u_0 and v_0 are given 2π-periodic functions, and f is continuous.

We will regard the problem in (25) through (27) as a Cauchy problem in some Hilbert space of periodic functions, with x playing the role of time. In a standard fashion, we transfer from the equation of second order (25) to the system

$$\begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ \partial_t \end{pmatrix} \begin{pmatrix} 1 \\ f \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad .$$

We will define the Hilbert space with the help of the operator $i\partial_t$, which we consider now in some detail. $i\partial_t$ with the domain of definition $D = \{\phi \in H^1(0,2\pi) \mid \phi(0) = \phi(2\pi)\}$ (where H^1 stands for the Sobolev space) is, clearly, self-adjoint. The exponentials e^{int} form a complete system of eigenfunctions and the eigenvalues are $0,\pm 1,\ldots$ If c is real, $c \notin \mathbb{Z}$, then the operator $(c+i\partial_t)$ is self-adjoint and invertible. We define

$$T = (c + i\partial_t)^{1/2} \quad , \tag{28}$$

as the arithmetical square root of $(C+i\partial_t)$.

Let $(.,.)$ be the standard scalar product in $H^0 = L^2(0,2\pi)$. We denote by L a subspace of the space of two-vectors $H^0 \dot{+} H^0$. Set $L = D(T) \dot{+} H^0$ where the domain of definition is generated by periodic boundary conditions. If $w = \{u,v\} \in L$, then the scalar product is defined as

$$\langle w,w\rangle_1 = 1/2\,[(Tu,Tu) + (v,v)] \quad .$$

It is easy to show that T is closed and therefore that L is a Hilbert space.

Another description of L may be obtained via Fourier coefficients. Let $\{a_n\}$ and $\{b_n\}$, $n = 0,\pm 1,\ldots$, be Fourier coefficients of u and v, then for $w = \{u,v\}$ we have

$$\langle w,w\rangle_1 = 1/2\,\Big(\sum_{n=-\infty}^{\infty} (C^2+|n|)\,|a_n|^2 + \sum_{n=-\infty}^{\infty} |b_n|^2\Big) \quad .$$

By obvious reasons the norm, just defined, is equivalent to

$$\langle w,w\rangle = 1/2(|a_0|^2 + \sum |n| \, |a_n|^2 + \sum |b_n|^2) \quad , \tag{29}$$

this norm will be denoted by $\|.\|$ (i.e., $\|w\|^2 = \langle w,w\rangle$). Thus L is a subspace of $\ell^2 + \ell^2$ defined by the condition $\|w\| < \infty$.

It is easily seen that the vectors

$$e_0 = \{1,1\} \; , \; f_0 = \{1,-1\} \; ,$$
$$e_n = \{(in)^{-1/2}e^{inx} \; , \; e^{inx}\} \; , \; f_n = \{(in)^{-1/2}e^{inx}, -e^{inx}\} \; . \tag{30}$$

where $n = \pm 1, \pm 2, \ldots$, form an orthonormal basis in L.

The following assertion can be made about the Cauchy problem

$$\begin{pmatrix} u \\ v \end{pmatrix}_x = M \begin{pmatrix} u \\ v \end{pmatrix} \quad , \quad \begin{pmatrix} u \\ v \end{pmatrix}_{x=0} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \text{ where } M = \begin{pmatrix} 0 & 1 \\ \partial_t & f \end{pmatrix} \; , \tag{31}$$

in the space L.

<u>Theorem 8</u>. Let $w = \{u,v\}$ be a solution of the problem in (31). Assume that

$$\int_0^\infty \|w(x,.)\|^2 \, e^{-2ax} \, dx < 0 \; , \tag{32}$$

for some $a < \mu$, where

$$\mu = (2\pi)^{-1} \int_0^{2\pi} f(t)dt \quad , \quad \mu > 0 \; . \tag{33}$$

Then $\|w(x,.)\| \leq C < \infty$.

The meaning of the condition $a < \mu$ will be cleared up later on. It turns out that μ is the minimal positive eigenvalue of M.

Proof. The plan of the proof. First, we will find the eigenvalues of M. In the degenerate case $f \equiv 0$, we will estimate the resolvent. We will then obtain the estimate for the general case by applying perturbation theory to the former one. Using spectral properties of M we will show that the Laplace transform of the solution can be continued analytically from the half plane, $\{\mathrm{Re} z > a\}$ into some sector containing $\{\mathrm{Re} z \geq 0\}$. The desired result then follows.

Solving the equation $Mw = \lambda w$, where $w = \{u,v\}$ one sees that

$$u_t + \lambda f u - \lambda^2 u = 0 \quad , \quad v = \lambda u \quad . \tag{34}$$

Solutions of (34) are given explicitly by

$$u(t) = C \exp \left(\lambda^2 t - \lambda \int_0^t f(s)ds\right) \quad , \tag{35}$$

therefore u is 2π-periodic if and only if, for some $n \in \mathbb{Z}, \lambda^2 - \lambda\mu = in$. The last condition yields two series of characteristic numbers

$$\xi_n = \mu/2 + (\mu^2/4 + in)^{1/2} \quad , \quad \eta_n = \mu/2 - (\mu^2/4 + in)^{1/2} \quad . \tag{36}$$

Note that ξ_n and η_n lay on two branches of the hyperbola $(\mathrm{Re} z - \mu/2)^2 - (\mathrm{Im} z)^2 = \mu^2/4$. If

$$F(t) = \int_0^t (f(s) - \mu)ds \quad ,$$

then, in virtue of (35), the eigenfunctions, corresponding to $\lambda = \xi_n$ and $\lambda = \eta_n$, can be written as follows

$$\begin{aligned} \tilde{e}_n &= C\ \{\exp(int-\xi_n F(t)) \quad , \quad \xi_n \exp(int-\xi_n F(t))\} \quad , \\ \tilde{f}_n &= C\ \{\exp(int-\eta_n F(t)) \quad , \quad \eta_n \exp(int-\eta_n F(t))\} \quad . \end{aligned} \tag{37}$$

Now, we will find generalized eigenvectors. Let w be an eigenvector with the eigenvalue λ and w' be a generalized eigenvector such that $(M-\lambda)^2 w' = 0$. Then from the equation $(M-\lambda)w' = w$ we obtain the following nonhomogeneous analogue to (34)

$$u'_t + \lambda f u' - \lambda^2 u' = 2\lambda u - fu, \quad v' = u + \lambda u' \quad .$$

Solving the equation with respect to u', one may calculate that

$$u' = u\cdot(2\lambda t - \mu t - F(t)) \quad .$$

It follows that if u' is 2π-periodic then necessarily $\lambda = \mu/2$. Comparing λ with eigenvalues from (36) one sees that the only possibility is $\mu = 0$ and $\lambda = \xi_0 = \eta_0 = 0$. It is easy to show that for $\mu = 0$, the multiplicity of zero-eigenvalue is no bigger than 2.

Now, we consider the degenerate case, when the coefficient, f, is identically 0. Then, of course, $\mu = 0$ and from (37) we obtain eigenvectors,

$$\tilde{e}_n = C\{\exp(int) \quad , \quad (in)^{1/2}\exp(int)\} \ ,$$

$$\tilde{f}_n = C\{\exp(int) \quad , \quad -(in)^{1/2}\exp(int)\} \ ,$$

which after the normalization coincide for $n \neq 0$ with basis vectors, e_n and f_n from (30). For $n = 0$, we add to $\tilde{e}_0 = \tilde{f}_0 = (1,0)$ the generalized eigenvector $\mathring{e}_0' = (1,1)$.

Thus we see that the operator,

$$M_0 = \begin{pmatrix} 0 & 1 \\ \partial_t & 0 \end{pmatrix},$$

has the complete system of eigenvectors, $\{e_n, f_n\}$, $n \neq 0$, $\mathring{e}_0' = e_0$ and $\tilde{f}_0$. Solving the resolvent equation

$$(M_0 - \lambda)\ (\textstyle\sum a_n e_n + \sum_{n\neq 0} b_n f_n + b_0 \tilde{f}_0) = \sum a_n' e_n + \sum_{n\neq 0} b_n' f_n + b_0' \tilde{f}_0 \quad ,$$

one observes that

$$\| R(\lambda, M\ ) \| \leq C|\lambda|^{-2} + \rho(\lambda, \text{ spec } M_0)^{-1} \quad , \tag{38}$$

where $\| . \|$ means the operator norm in L and $\rho(\lambda, \text{spec } M_0)$ is the distance from λ to the spectrum of M_0. Taking into account that the spectrum of M_0 lays on lines $\text{Re} z = \pm \text{Im} z$, we obtain the estimate

$$\| R(\lambda, M_0) \| \leq C/\ \lambda \quad , \tag{39}$$

which holds in the domain

$$\Omega_{\varepsilon,r} = \mathbf{C}^1 - \{ |\arg(\lambda - \pi/4 - k\pi/2)| < \varepsilon \ ; \ k = 0,1,2,3\}$$

$$- \{ |\lambda| < r\} \ . \tag{40}$$

for any $\varepsilon > 0$ and $r > 0$.

We now turn to the general case, $f \equiv 0$. We observe that

$$M = \begin{pmatrix} 0 & 1 \\ \partial_t & f \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \partial_t & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix} \equiv M_0 + F \quad .$$

If f is continuous, then F is a bounded operator with the L-norm,

$$\| F \| = \max_{0 \leq t \leq 2\pi} \left| f(t) \right| \quad .$$

For the resolvent of M one obtains

$$(M_0 + F - \lambda)^{-1} = (M_0 - \lambda)^{-1} \, (I + (M_0 - \lambda)^{-1} F)^{-1} \quad .$$

Thus, if the constants ε and r defining $\Omega_{\varepsilon,r}$ are chosen so that $\| (M_0 - \lambda)^{-1} \| \leq C < \| F \|^{-1}$ in $\Omega_{\varepsilon,r}$ then

$$\| R(\lambda, M) \| \leq C / \lambda \quad , \tag{41}$$

holds in $\Omega_{\varepsilon,r}$.

In order to control the spectrum of M inside the circle $|\lambda| < r$, we do the following. We consider some circle, $\Gamma_R = \{ |\lambda| = R \}$, which does not touch the spectrum of M_0. Applying standard results of the perturbation theory (see Kato [8, Chapter [IV, §3]), we observe that there exists a homotopy $\Gamma(s)$ of the contour Γ_R such that the number of eigenvalues of $M(s) = M_0 + sF$ in $\Gamma(s)$ is equal to that of M_0 in Γ_R. Taking R sufficiently large we can show that the only singularities of the resolvent $R(\lambda, M)$ inside $\{ |\lambda| < r \}$ are the eigenvalues of M, i.e., ξ_n and η_n from (36) such that $|\xi_n|$, $|\eta_n| < r$.

Remark. It could be proved that the spectrum of M is discrete. We need here only the above result about a bounded part of the spectrum.

Now we pass to the analytical part of the proof. Let $W(z)$ be the Laplace transform of the solution. By the Schwartz inequality we have for $\mathrm{Re}z > a$

$$\|W(z)\|^2 = \|\int_0^\infty w(x)e^{-zx}dx\|^2 \le \int_0^\infty \|w(x)\|^2 e^{-2ax}dx \int_0^\infty e^{-2(\mathrm{Re}z-a)x}dx \quad .$$

Thus, in virtue of the assumption (33),

$$\|W(z)\| < C(\mathrm{Re}z-a)^{-1/2} \quad , \tag{42}$$

therefore $W(z)$ is regular for $\mathrm{Re}z \ge \mu-\varepsilon$.

It is well known that $W(z) = R(z,M)w(0)$ (see Yosida [16]). We will employ the information about the resolvent, obtained above, in order to continue $W(z)$ analytically into the left half-plane. We choose two rays

$$\Gamma_\pm = \{z \mid \mathrm{Im}z = \pm\delta\gamma(\mathrm{Re}z-\delta) \quad , \quad \mathrm{Re}z \le \delta < 0\} \quad , \quad \gamma > 0 \quad ,$$

so that the larger sector between Γ_+ and Γ_- doesn't contain any eigenvalue with non-positive real part except for 0. $W(z)$ can be continued in the sector and has there only a simple pole at 0. (We recall that for $f \equiv 0$ all eigenvalues are simple).

$w(x)$ can be restored using the inverse Laplace transform

$$w(x) = \int_{a-\varepsilon-i\infty}^{a-\varepsilon+i\infty} e^{zx} W(z)dz \quad . \tag{43}$$

Employing the fact that there are no singularities of the resolvent between $\{\mathrm{Re}z = a-\varepsilon\}$ and the contour $\Gamma = \Gamma_+ \cup \Gamma_-$ except for 0 and recalling the estimate in (39), one can change the contour of integration in (43)

$$w(x) = \int_\Gamma e^{zx} W(z)dz + \mathrm{Res}_{z=0} W(z) .$$

And the estimate $\|w(x)\| \leq C + e^{-\delta x} \leq C$ follows. The proof is completed.

The following theorem is a simple corollary of Theorem 8.

Theorem 9. For α sufficiently close to α_0, the periodic solution of the nonlinear problem is bounded.

Proof. Let u_1 be the periodic solution of the problem in (1), obtained in Sec. 4.3. By definition, this means that

$$u_1(.,t) \in C^1((-T,T),H^0) \cap C_{\delta,0+\delta}((-T,T)H^1)$$

for any T, therefore

$$u_1 \in C^1([0,2\pi], H^0) \cap C([0,2\pi],H^1) .$$

Since $u = e^{x/2}u_1$ is a solution of the nonlinear problem in (19) through (21) it follows that u satisfies the inequalities

$$\int_0^{2\pi} dt \int_0^\infty |u_x(x,t)|^2 e^{-x} dx < \infty ,$$
$$\int_0^{2\pi} dt \int_0^\infty |u_t(x,t)|^2 e^{-x} dx < \infty . \tag{44}$$

By Theorem 7, u_x and u_t are continuous functions and one may apply Fubini's theorem to the integrals in (44). Changing the order of integrations in (44) and noticing that for any $\phi = \Sigma a_n e^{int}$

$$\int_0^{2\pi} |\phi_t|^2 dt = \Sigma n^2 |a_n|^2 \geq |a_0|^2 + \Sigma |n| |a_n|^2 ,$$

one obtains

$$\int_0^\infty \|\{u, u_x\}\|^2 e^{-x} dx$$

$$\leq \int_0^\infty dx\, e^{-x} \int_0^{2\pi} |u_t(t,x)|^2 dt + \int_0^\infty dx e^{-x} \int_0^{2\pi} |u_x(t,x)|^2 dt < \infty .$$

Thus $\{u, u_x\}$ satisfies the assumption (32) of Theorem 8 with $a = 1/2$.

It is left to notice that for some $\varepsilon > 0$ if $|\alpha - \alpha_0| < \varepsilon$ then $u(0,t)$ is small. Thus for $f = e^{\alpha u(0,t)}$ we have

$$\mu = (2\pi)^{-1} \int_0^{2\pi} f(t) dt > 1/2 = a .$$

Therefore in virtue of Theorem 8 u and u_x are bounded.

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