

AFFINE SPRINGER FIBERS AND GENERALIZED HAIMAN IDEALS

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ABSTRACT. We compute the Borel-Moore homology of unramified affine Springer fibers for GL_n under the assumption that they are equivariantly formal and relate them to certain ideals discussed by Haiman. For $n = 3$, we give an explicit description of these ideals, compute their Hilbert series, generators and relations, and compare them to generalized (q, t) -Catalan numbers. We also compare the homology to the Khovanov-Rozansky homology of the associated link, and prove a version of a conjecture of Oblomkov, Rasmussen, and Shende in this case.

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1. INTRODUCTION

In this paper, we study a broad class of affine Springer fibers and their equivariant Borel-Moore homology, using geometric and algebraic techniques. This homology is related to the Langlands program [22], Hilbert schemes of singular curves [26,

25], and is conjectured to be related to knot homology [27]. Sometimes these relationships are discussed in terms of compactified Jacobians, which are equivalent to affine Springer fibers [23].

Given a matrix $\gamma \in \mathfrak{gl}_n((t))$ (or $\mathfrak{sl}_n((t))$), the affine Springer fiber Sp_γ is a certain ind-subvariety of the affine Grassmannian, see Definition 2.3.

The connection to knot homology is of particular interest. Given γ , its characteristic polynomial $p(\lambda, t) = \det(\gamma - \lambda I)$ defines a singular curve C_γ in \mathbb{C}^2 , called the spectral curve of γ . As long as C_γ is reduced, Sp_γ depends only on its spectral curve C_γ , in particular on its completion at the origin $(C_\gamma, 0)$. In this paper we will work with γ with distinct eigenvalues so that C_γ will be reduced.

Fact 1.1 ([28]). If $(C_\gamma, 0)$ is irreducible, then Sp_γ is a projective variety, but if $(C_\gamma, 0)$ is not irreducible, Sp_γ is an ind-variety with infinitely many irreducible components.

Intersecting the C_γ with a small sphere around the origin (where C_γ is often singular) gives a link L_γ in S^3 . Each irreducible component of $(C_\gamma, 0)$ corresponds to a component of the link, and the intersection numbers of irreducible components are the linking numbers of the corresponding link components. Any smooth components of the curve correspond to unknots. Oblomkov, Rasmussen, and Shende [27] have conjectured that the homology of Sp_γ is closely related to the triply-graded Khovanov-Rozansky homology [20, 19] (also called HHH) of L_γ . This relationship has previously been shown for all torus knots, and for (n, nd) -torus links by Kivinen in [21].

Conjecture 1.2 (ORS [27]). If L_γ is the link associated to γ , we have

$$\mathrm{gr}_P H_*(\mathrm{Sp}_\gamma) \otimes \mathbb{C}[x] \cong \mathrm{HHH}^{a=0}(L_\gamma)$$

where gr_P is a certain perverse filtration on $H_*(\mathrm{Sp}_\gamma)$.

Example 1.3. For the matrix

$$\gamma = \begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix},$$

the characteristic polynomial is given by $p(\lambda, t) = \lambda^2 - t^3$. The associated link to this curve in \mathbb{C}^2 is a trefoil. The affine Springer fiber Sp_γ is isomorphic to \mathbb{P}^1 , and the reduced HHH homology of the trefoil is isomorphic to $H_*(\mathbb{P}^1)$.

In this paper, we will calculate the homology for a large class of affine Springer fibers with γ of the form

$$(1) \quad \gamma = \begin{pmatrix} z_1 t^{d_1} & & 0 \\ & \ddots & \\ 0 & & z_n t^{d_n} \end{pmatrix},$$

with $z_i \in \mathbb{C}^*$ pairwise distinct and $d_i > 0$, under the assumption that Sp_γ is equivariantly formal (see Definition 2.6).

The characteristic polynomial of this γ is $\prod_i (\lambda - z_i t^{d_i})$. The corresponding curve C_γ has n smooth irreducible components with pairwise intersection numbers $d_{ij} = \min(d_i, d_j)$. So the corresponding link L_γ is a link of n unknots with pairwise linking numbers d_{ij} . In Section 6, we calculate HHH of this link L_γ for $n = 3$ and compare the result to $H_*(\mathrm{Sp}_\gamma)$.

1.1. Homology of Affine Springer Fibers. In Section 3, we focus on computing the equivariant Borel-Moore homology $H_*^T(\mathrm{Sp}_\gamma)$ with respect to the natural torus action of $T = (\mathbb{C}^*)^n$ on Sp_γ (explained in Section 2). Given the assumption that Sp_γ is equivariantly formal with respect to T , we can recover the ordinary Borel-Moore homology of Sp_γ by quotienting out by the equivariant parameters, see Fact 2.7.

In order to calculate $H_*^T(\mathrm{Sp}_\gamma)$, we view it as a module over

$$R = H_T^*(pt) \otimes \mathbb{C}[\Lambda] \cong \mathbb{C}[t_1, \dots, t_n, x_1^\pm, \dots, x_n^\pm].$$

Here, the t_i 's are our equivariant parameters, and the x_i 's parametrize the integer lattice Λ which acts on Sp_γ by translations.

In [10], Goresky, Kottwitz, and MacPherson (GKM) conjectured that Sp_γ is pure for all unramified (i.e. diagonal) γ . The following is a more narrow conjecture that is what we will rely on in this paper.

Conjecture 1.4 ([10]). *For γ as above, Sp_γ is equivariantly formal, as defined in Definition 2.6.*

Assuming that Sp_γ is equivariantly formal, we can calculate its equivariant Borel-Moore homology.

Theorem 1.5. *Consider γ as in (1). Define the ideal*

$$\mathcal{J} = \mathcal{J}_\gamma = \bigcap_{i < j} (t_i - t_j, x_i - x_j)^{d_{ij}} \subseteq R$$

with $d_{ij} = \min(d_i, d_j)$. *If Sp_γ is equivariantly formal, then as R -modules,*

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \cong \mathcal{J} \quad \text{where} \quad \Delta = \prod_{i < j} (t_i - t_j)^{d_{ij}}.$$

These \mathcal{J} ideals are slightly more general than the similar ideals considered by Haiman in his work on the Hilbert schemes of points [9], so we refer to them as generalized Haiman ideals. Since Δ is a polynomial, multiplication by Δ is injective, and

$$H_*^T(\mathrm{Sp}_\gamma) \cong \Delta H_*^T(\mathrm{Sp}_\gamma) \cong \mathcal{J}$$

as modules over $R = \mathbb{C}[t_1, \dots, t_n, x_1^\pm, \dots, x_n^\pm]$. It is still useful to keep track of Δ if we want to retain the localization information of $H_*^T(\mathrm{Sp}_\gamma)$, but we can omit Δ when we only care about $H_*^T(\mathrm{Sp}_\gamma)$ as an R -module. Given the assumption that Sp_γ is equivariantly formal, we can recover the ordinary Borel-Moore homology of Sp_γ as well.

Corollary 1.6. *For γ as in (1), if Sp_γ is equivariantly formal, then*

$$H_*(\mathrm{Sp}_\gamma) \cong \mathcal{J}/(\mathfrak{t})\mathcal{J}.$$

Here $(\mathfrak{t}) \subseteq H_T^*(pt) \cong \mathbb{C}[\mathfrak{t}]$ is the maximal ideal generated by t_1, \dots, t_n .

If $n = 3, 4$, it is known that Sp_γ is equivariantly formal, shown in [24] and [7] respectively. It is also known for the equivalued case, when $d_i = d$ for all i , due to GKM [10].

Corollary 1.7. *If $n \leq 4$, or if $d_i = d$ for all i , then*

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \cong \mathcal{J} = \bigcap_{i < j} (t_i - t_j, x_i - x_j)^{d_{ij}}.$$

The equivalued case of 1.7 was previously shown by Kivinen in [21] using GKM theory as defined in [12]. The proof of Theorem 1.5 relies on this result by Kivinen.

In [15] Gorsky, Kivinen and Oblomkov define a graded algebra $\mathcal{A}_G = \bigoplus_{d=0}^{\infty} \mathcal{A}_d$ depending only on a reductive group G with some specific properties, called the graded Coulomb branch algebra. Here we consider the case $G = \mathrm{GL}_n$. One of the key properties is that for any $\gamma \in \mathfrak{g}$, the direct sum of homologies of affine Springer fibers

$$F_\gamma = \bigoplus_{k=0}^{\infty} H_*(\mathrm{Sp}_{t^k \gamma})$$

is a graded module over \mathcal{A}_G or, equivalently, that there is a corresponding quasi-coherent sheaf \mathcal{F}_γ on $\mathrm{Proj} \bigoplus_{d=0}^{\infty} \mathcal{A}_d$. They conjecture the following.

Conjecture 1.8 ([15]). *The module F_γ is finitely generated and the sheaf \mathcal{F}_γ is coherent.*

Theorem 1.9. *Conjecture 1.8 holds for $G = \mathrm{GL}_3$ and γ as in (1).*

This result and many of the results in Section 1.2 rely on the specific combinatorics of the ideal \mathcal{J} when $n = 3$, which is covered in detail in Section 4, with additional proofs in Section 5.

1.2. Generalized Haiman Ideals. For the rest of the introduction we assume that the d_i 's are ordered: $d_1 \leq \dots \leq d_n$. We will consider a similar ideal to \mathcal{J} defined above,

$$J'(d_1, \dots, d_n) = \bigcap_{i < j} (t_i - t_j, x_i - x_j)^{d_{ij}} \subseteq \mathbb{C}[t_1, \dots, t_n, x_1, \dots, x_n].$$

The ideal \mathcal{J} is obtained from $J'(d_1, \dots, d_n)$ by localization in $(x_1 \cdots x_n)$.

In Section 4.1, we define two rational functions, $H(d_1, \dots, d_n)$ and $F(d_1, \dots, d_n)$. The function $F(d_1, \dots, d_n)$ is also known as the generalized (q, t) -Catalan number, see [13].

Conjecture 1.10. *a) The Hilbert series of the ideal $J'(d_1, \dots, d_n)$ equals $H(d_1, \dots, d_n)$.*

b) The Hilbert series of the generating set $J'(d_1, \dots, d_n)/\mathfrak{m}J'(d_1, \dots, d_n)$ equals $F(d_1, \dots, d_n)$, where \mathfrak{m} is the maximal ideal $\mathfrak{m} = (t_1, \dots, t_n, x_1, \dots, x_n)$.

In particular, Conjecture 1.10 implies that $F(d_1, \dots, d_n)$ is a polynomial in q and t with nonnegative coefficients (see [13, Conjecture 1.3]) and provides an explicit algebraic interpretation of these coefficients. Similarly, the conjecture implies that $H(d_1, \dots, d_n)$ is a power series in q and t with nonnegative coefficients. In Theorem 4.10, we show that this Conjecture holds for $n = 3$.

Theorem 1.11. *Conjecture 1.10 holds for $n = 3$.*

If $d_i = d$ for all i , we will say that the ideal J' is equivalued. In [9], Haiman shows the following.

Theorem 1.12 (Haiman [9]). *For any n ,*

- (1) *The ideal $J'(d, \dots, d)$ is free as a $\mathbb{C}[\mathbf{t}]$ -module.*
- (2) *The ideal $J'(d, \dots, d)$ is equal to a product,*

$$J'(d, \dots, d) = J'(1, \dots, 1)^d.$$

It is easy to see that in general

$$J'(d_1, \dots, d_n) \cdot J'(d'_1, \dots, d'_n) \subseteq J'(d_1 + d'_1, \dots, d_n + d'_n).$$

We conjecture that Theorem 1.12 can be generalized to the non-equivalued case, and that the above inclusion is always an equality.

Conjecture 1.13. *For any n , assume $d_1 \leq d_2 \leq \dots \leq d_n$. Then,*

- (1) *The ideal $J'(d_1, \dots, d_n)$ is free as a $\mathbb{C}[\mathbf{t}]$ -module.*
- (2) *The ideal $J'(d_1, \dots, d_n)$ can be written as the product*

$$J'(d_1, \dots, d_n) = J'(1, \dots, 1)^{d_1} \cdot J'(0, 1, \dots, 1)^{d_2 - d_1} \cdot \dots \cdot J'(0, \dots, 0, 1)^{d_{n-1} - d_{n-2}}.$$

Statement (1) immediately follows in any cases where Sp_γ is known to be equivariantly formal, in particular for $n \leq 4$. In Corollary 4.5 we show that statement (2) holds in the $n = 3$ case.

Theorem 1.14. *The ideal $J'(d_1, d_2)$ can be written as a product*

$$J'(d_1, d_2) = J'(1, 1)^{d_1} \cdot J'(0, 1)^{d_2 - d_1}$$

1.3. Link Homology. In Section 6, we use a recursive process of Hogancamp and Elias [8] to compute $\mathrm{HHH}^{a=0}(L_\gamma)$ for $n = 3$. We are then able to relate $\mathrm{HHH}^{a=0}(L_\gamma)$ to the ideal \mathcal{J} , and thus the homology of Sp_γ , to show that a weaker version of Conjecture 1.2 (without the perverse filtration) holds in this case. In order to show this, we use the y -ified Khovanov-Rozansky homology HY as defined by Gorsky and Hogancamp in [14].

Theorem 1.15. *For $n = 3$ and γ as in 1,*

$$\mathrm{HY}^{a=0}(L_\gamma) \otimes_{\mathbb{C}[\mathbf{x}]} \mathbb{C}[\mathbf{x}, \mathbf{x}^\pm] \cong \Delta H_*^T(\mathrm{Sp}_\gamma)$$

and

$$\mathrm{HHH}^{a=0}(L_\gamma) \otimes_{\mathbb{C}[\mathbf{x}]} \mathbb{C}[\mathbf{x}, \mathbf{x}^\pm] \cong H_*(\mathrm{Sp}_\gamma).$$

We are optimistic that the proof of Theorem 1.15 can be generalized to larger n , and one may actually be able to find an explicit recursion to compute $\mathrm{HHH}^{a=0}(L_\gamma)$ for any n , which would tell us more about the structure of $H_*(\mathrm{Sp}_\gamma)$ beyond $n = 3$.

1.4. The Fundamental Domain. Finally, in the appendix we discuss the cells of the fundamental domain of Sp_γ , as described by Chen in [6], and relate this to the combinatorics of J' for $n = 3$. We show that there is a bijection between half of the cells in the fundamental domain and the generators of the ideal $J'(d_1, d_2)$. We expect that this bijection indicates a stronger relationship between the cells of Sp_γ and the combinatorics of J' and of generalized (q, t) -Catalan numbers.

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2. BACKGROUND

First, some notation. Let $\mathcal{K} = \mathbb{C}((t))$ be the field of Laurent power series in t , and $\mathcal{O} = \mathbb{C}[[t]]$ be the ring of power series in t . For nonzero $f \in \mathcal{K}$, let $\nu(f)$ denote the order of f , which is the degree of the smallest nonzero term. Throughout, $H_*^T(X)$ will refer to the equivariant Borel-Moore homology of X , and $H_*(X)$ refers to regular Borel-Moore homology.

Definition 2.1. A *lattice* $\Lambda \subseteq \mathcal{K}^n$ is a free \mathcal{O} -submodule of \mathcal{K}^n of rank n such that $\Lambda \otimes_{\mathcal{O}} \mathcal{K} = \mathcal{K}^n$. In other words, it is the \mathcal{O} -span of a basis of \mathcal{K}^n over \mathcal{K} .

Definition 2.2. The *affine Grassmanian* $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$ of GL_n over \mathbb{C} is an ind-scheme defined as the space of all lattices $\Lambda \subseteq \mathcal{K}^n$.

We can equivalently define the affine Grassmanian as

$$\mathrm{GL}_n(\mathcal{K})/\mathrm{GL}_n(\mathcal{O}),$$

as $\mathrm{GL}_n(\mathcal{K})$ acts transitively on the space of lattices, and the stabilizer of the standard lattice \mathcal{O}^n is precisely $\mathrm{GL}_n(\mathcal{O})$. We will often conflate a matrix $g \in \mathrm{GL}_n(\mathcal{K})$ with its coset representative in $\mathrm{Gr}(\mathrm{GL}_n)$.

We define $\mathrm{Gr}(\mathrm{SL}_n)$ similarly, either as $\mathrm{SL}_n(\mathcal{K})/\mathrm{SL}_n(\mathcal{O})$, or as the space of lattices of SL_n type. We say that a lattice Λ is of SL_n type if it can be written as $\Lambda = g\mathcal{O}^n$ for some $g \in \mathrm{SL}_n(\mathcal{K})$.

We will always be working over \mathbb{C} , so we will use $\mathrm{Gr}(\mathrm{GL}_n)$ or $\mathrm{Gr}(\mathrm{SL}_n)$ for the affine Grassmanian, or just Gr when the group G is clear.

Definition 2.3 ([18]). The *affine Springer fiber* Sp_γ of an element $\gamma \in \mathfrak{gl}_n(\mathcal{K})$ is a sub ind-scheme of the affine Grassmanian, defined accordingly as the space of lattices $\Lambda \in \mathrm{Gr}_{\mathrm{GL}_n}$ such that $\gamma\Lambda \subseteq \Lambda$, or as the space of $g \in \mathrm{GL}_n(\mathcal{K})/\mathrm{GL}_n(\mathcal{O})$ such that $g^{-1}\gamma g \in \mathfrak{gl}_n(\mathcal{O})$.

Fact 2.4 ([28]). If γ is regular, semi-simple, and topologically nilpotent, then Sp_γ is finite dimensional, although it can still have infinitely many irreducible components. In our cases, these conditions essentially are that γ is diagonalizable, has distinct eigenvalues λ_i (over $\bar{\mathcal{K}}$), and that $\nu(\lambda_i) > 0$ for all i .

Example 2.5 ([28]). Consider the matrix

$$\gamma = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$$

in $\mathfrak{sl}_2(\mathcal{K})$. Then Sp_γ looks like an infinite chain of \mathbb{P}^1 's connected at 0 and ∞ . There is a \mathbb{C}^* action that scales each \mathbb{P}^1 , and a \mathbb{Z} action that translates them.

There is a natural action of the centralizer $C(\gamma)$ on Sp_γ . In the case where γ is diagonal, this gives a torus action of $T = (\mathbb{C}^*)^n$ and lattice action of $\Lambda = \mathbb{Z}^n$ on Sp_γ over GL_n (or $(\mathbb{C}^*)^{n-1}$ and \mathbb{Z}^{n-1} over SL_n) that can be respectively seen as multiplication by the matrices in $C(\gamma)$:

$$\lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

and

$$Z = \begin{pmatrix} t^{m_1} & & \\ & \ddots & \\ & & t^{m_n} \end{pmatrix}.$$

In general for a T -ind-scheme X , $H_*^T(X)$ is naturally a module over $H_T^*(pt)$ via the cap product.

Definition 2.6. We will say that X is *equivariantly formal* if $H_*^T(X)$ is free as a module over $H_T^*(pt)$.

Fact 2.7. (GKM [12]) If X is equivariantly formal, then

$$H_*(X) \cong H_*^T(X)/(\mathfrak{t})H_*^T(X),$$

as modules over $H_T^*(pt)$. Here $(\mathfrak{t}) \subseteq H_T^*(pt) \cong \mathbb{C}[\mathfrak{t}]$ is the maximal ideal generated by t_1, \dots, t_n .

We will need the following localization lemma as stated by Brion.

Lemma 2.8 (Brion [5]). *Let X be a T -ind-scheme and $T' \subseteq T$ a subtorus. If $i : X^{T'} \rightarrow X$ is the inclusion of T' -fixed points, then the induced map*

$$i_* : H_*^T(X^{T'}) \rightarrow H_*^T(X)$$

is an isomorphism after inverting finitely many characters of T that restrict nontrivially to T' . Further, if X is equivariantly formal, then the induced map i_ is injective, and we have that*

$$\prod_{\chi, \chi_{T'} \neq 0} \chi H_*^T(X) \subseteq H_*^T(X^{T'})$$

where we take the product over all characters of T that restrict nontrivially to T' .

We will also make frequent use of the Iwasawa decomposition for $\mathrm{Gr}(\mathrm{GL}_n)$, which tells us that all $g \in \mathrm{Gr}(\mathrm{GL}_n)$ can be represented by a product DU of a diagonal matrix

$$(2) \quad D = \begin{pmatrix} t^{m_1} & & \\ & \ddots & \\ & & t^{m_n} \end{pmatrix}$$

with a unipotent matrix U with 1's on the diagonal and entries χ_{ij} above the diagonal [28]. Further, these χ_{ij} 's are unique up to \mathcal{O} , so we can choose them to have all coefficients of nonnegative powers of t be 0, so that each matrix DU represents a unique element $g \in \mathrm{Gr}$.

Lemma 2.9. *The T -fixed points of $\mathrm{Gr}(\mathrm{GL}_n)$ can be uniquely represented by diagonal matrices D as in the Iwasawa decomposition.*

Proof. Let $\lambda \in T$ and $g = DU$ be as in the Iwasawa decomposition. Since $\lambda^{-1} \in \mathrm{GL}_n(\mathcal{O})$, up to multiplication on the right by $\mathrm{GL}_n(\mathcal{O})$, we get

$$\lambda g = \lambda g \lambda^{-1} = DU',$$

where D is as above, and U' is unipotent with $\frac{\lambda_i}{\lambda_j} \chi_{ij}$ above the diagonal. If g is a fixed point under the action of T , we must have $\frac{\lambda_i}{\lambda_j} \chi_{ij} = \chi_{ij}$ for all i, j and for all $\lambda \in T$. This can only happen if $\chi_{ij} = 0$ for all i, j , so $g = D$ is diagonal as desired. \square

Since the T -action on Sp_γ comes from the action on Gr , the T -fixed points of Sp_γ are simply the T -fixed points of Gr that are contained in Sp_γ .

Lemma 2.10. *If γ is diagonal and the orders of its eigenvalues are all nonnegative, then*

$$\mathrm{Sp}_\gamma^T = \mathrm{Gr}^T.$$

Proof. For any γ , $\mathrm{Sp}_\gamma^T \subseteq \mathrm{Gr}^T$ as stated above. If $g \in \mathrm{Gr}^T$, then

$$g = \begin{pmatrix} t^{m_1} & & \\ & \ddots & \\ & & t^{m_n} \end{pmatrix}$$

by Lemma 2.9. As g and γ are diagonal, $g^{-1}\gamma g = \gamma$, and $\gamma \in \mathfrak{gl}_n(\mathcal{O})$, since the eigenvalues of γ are all in \mathcal{O} . So $\mathrm{Gr}^T \subseteq \mathrm{Sp}_\gamma^T$. \square

In particular, this means that the T -fixed points of Sp_γ are discrete and isomorphic to the integer lattice $\Lambda = \mathbb{Z}^n$, so can view $H_*(\mathrm{Sp}_\gamma)$ as a module over

$$H_T^*(\mathrm{Sp}_\gamma^T) \cong H_T^*(pt) \otimes \mathbb{C}[\Lambda] \cong \mathbb{C}[t_1, \dots, t_n, x_1^\pm, \dots, x_n^\pm].$$

Here, the t_i 's are our equivariant parameters, and a monomial $x_1^{a_1} \dots x_n^{a_n}$ corresponds to the fixed point $\mathrm{diag}(t^{a_1}, \dots, t^{a_n})$. The lattice Λ acts on Sp_γ^T and on Sp_γ by translation.

Lemma 2.11. *Fix $i < j$. If $T' \subseteq T$ is a codimension 1 subtorus cut out by $t_i = t_j$, then the T' -fixed points of $\mathrm{Gr}(\mathrm{GL}_n)$ are of the form DU , where D is as in (2) and*

$$U = \begin{pmatrix} 1 & & \chi_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix}$$

with all χ 's zero except for χ_{ij} .

Proof. As before, up to equivalence,

$$\lambda g = \lambda g \lambda^{-1} = DU',$$

where U' has $\frac{\lambda_k}{\lambda_i} \chi_{kl}$ above the diagonal. If g is a fixed point for T , since $\lambda_i = \lambda_j$, χ_{ij} can be arbitrary, but $\chi_{kl} = 0$ for all $(k, l) \neq (i, j)$. So the fixed points are as described. \square

Corollary 2.12. *If T' is a codimension 1 subtorus of T cut out by $t_i = t_j$, then*

$$\mathrm{Gr}(\mathrm{GL}_n)^{T'} \cong \mathrm{Gr}(\mathrm{GL}_2) \times \mathbb{Z}^{n-2}.$$

Proof. Each of the T' fixed points is represented by DU above. Looking at the 2×2 submatrix of DU in rows and columns i, j , we see a copy of $\mathrm{Gr}(\mathrm{GL}_2)$. The rest of the m_i are free integers, and there are $n - 2$ of them. \square

3. BOREL-MOORE HOMOLOGY OF Sp_γ

We want to find the equivariant Borel-Moore homology $H_*^T(\mathrm{Sp}_\gamma)$ of the class of affine Springer fibers with

$$\gamma = \begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_n \end{pmatrix} = \begin{pmatrix} z_1 t^{d_1} & & 0 \\ & \ddots & \\ 0 & & z_n t^{d_n} \end{pmatrix}.$$

Here $z_i \in \mathbb{C}^*$ are pairwise distinct and $d_i \geq 0$. We can assume up to a change of basis that $d_1 \leq \dots \leq d_n$.

It is known that for γ as above, Sp_γ is equivariantly formal (see Definition 2.6) over GL_n for $n \leq 4$ (see [24] for $n = 3$ and [7] for $n = 4$), and it is known to be equivariantly formal for all n if $d_i = d$ for all i [11]. But it is not known over GL_n in general. It would be sufficient to know that the homology of Sp_γ is supported in even degrees, and we conjecture that this is the case for all n . We will need to assume that Sp_γ is equivariantly formal in order to calculate its homology.

We consider $H_*^T(\mathrm{Sp}_\gamma)$ as a module over

$$H_T^*(\mathrm{pt}) \otimes \mathbb{C}[\mathbb{Z}^n] \cong \mathbb{C}[t_1, \dots, t_n, x_1^\pm, \dots, x_n^\pm] = R.$$

Theorem 3.1. *Consider γ as in (1). Define the ideal*

$$\mathcal{J} \subseteq R$$

$$\mathcal{J} = \bigcap_{i < j} (t_i - t_j, x_i - x_j)^{d_{ij}}$$

with $d_{ij} = \min(d_i, d_j)$. If Sp_γ is equivariantly formal, then as R -modules,

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \cong \mathcal{J},$$

where $\Delta = \prod_{i < j} (t_i - t_j)^{d_{ij}}$.

Note that multiplication by Δ is injective, so

$$H_*^T(\mathrm{Sp}_\gamma) \cong \Delta H_*^T(\mathrm{Sp}_\gamma) \cong \mathcal{J}$$

as R -modules. It can be useful to keep track of Δ if we want to retain the localization information of $H_*^T(\mathrm{Sp}_\gamma)$, but we can omit Δ when we only care about $H_*^T(\mathrm{Sp}_\gamma)$ as an R -module.

The rough outline of the proof of Theorem 3.1 is as follows:

- (1) Take a codimension one subtorus $T' \subseteq T$. The T' -fixed points of Sp_γ are essentially isomorphic to an affine Springer fiber $\mathrm{Sp}_{\tilde{\beta}}$ with $\tilde{\beta} \in \mathfrak{gl}_2$ whose homology is known.
- (2) Relate the homology of $\mathrm{Sp}_{\tilde{\beta}}$ to that of Sp_γ using Lemma 2.8.
- (3) Take enough subtori T' and piece together their homologies to find the homology of Sp_γ .

Step 3 will require the assumption that Sp_γ is equivariantly formal.

Lemma 3.2. *If $T' \subseteq T$ is the subtorus cut out by $t_i = t_j$, then up to \mathbb{Z}^{n-2} , the T' -fixed points of Sp_γ are isomorphic to an affine Springer fiber over GL_2 ,*

$$\mathrm{Sp}_\gamma^{T'} \cong \mathrm{Sp}_{\beta_{ij}} \times \mathbb{Z}^{n-2},$$

where

$$\beta_{ij} = \begin{pmatrix} z_i t^{d_i} & 0 \\ 0 & z_j t^{d_j} \end{pmatrix}.$$

Proof. In Lemma 2.11 we've already characterized the T' -fixed points of Gr as DU , where U is unipotent with only a single nonzero χ_{ij} . We just need to check which of those fixed points are in Sp_γ . If $g \in \text{Gr}(\text{GL}_n)^{T'}$, then

$$g^{-1}\gamma g = \begin{pmatrix} z_1 t^{d_1} & & \chi_{ij}(z_i t^{d_i} - z_j t^{d_j}) \\ & \ddots & \\ 0 & & z_n t^{d_n} \end{pmatrix}.$$

Again looking at the 2×2 i, j submatrix, we see a matrix identical to $g^{-1}\beta_{ij}g$. So a T' -fixed point g is in Sp_γ if and only if the 2×2 matrix

$$\begin{pmatrix} t^{m_i} & \chi_{ij} \\ 0 & t^{m_j} \end{pmatrix}$$

is in $\text{Sp}_{\beta_{ij}}$, i.e. $\text{Sp}_\gamma \cong \text{Sp}_{\beta_{ij}} \times \mathbb{Z}^{n-2}$. \square

Lemma 3.3. *Given $\beta_{ij} \in \mathfrak{gl}_2$ as in Theorem 1, we have $\text{Sp}_{\beta_{ij}} \cong \text{Sp}_{\tilde{\beta}_{ij}}$, where*

$$\tilde{\beta}_{ij} = \begin{pmatrix} z_i t^{d_{ij}} & 0 \\ 0 & z_j t^{d_{ij}} \end{pmatrix}$$

and $d_{ij} = \min(d_i, d_j)$.

Proof. Again using the Iwasawa decomposition, write $g = DU$, where

$$U = \begin{pmatrix} 1 & \chi_{ij} \\ 0 & 1 \end{pmatrix}.$$

Then,

$$g^{-1}\gamma g = \begin{pmatrix} \gamma_i & \chi_{ij}(\gamma_i - \gamma_j) \\ 0 & \gamma_j \end{pmatrix}.$$

By definition, $g \in \text{Sp}_{\beta_{ij}}$ if and only if $\chi_{ij}(\gamma_i - \gamma_j) \in \mathcal{O}$. Since we assume that the z_i are distinct, $\nu(\gamma_i - \gamma_j) = \min(\nu(\gamma_i), \nu(\gamma_j)) = \min(d_i, d_j) = d_{ij}$. So $g \in \text{Sp}_{\beta_{ij}}$ if and only if χ_{ij} has order at least $-d_{ij}$. This is the same as the condition for g to be in $\text{Sp}_{\tilde{\beta}_{ij}}$, since

$$g^{-1}\tilde{\beta}_{ij}g = \begin{pmatrix} z_i t^{d_{ij}} & (z_i - z_j)\chi_{ij} t^{d_{ij}} \\ 0 & z_j t^{d_{ij}} \end{pmatrix}.$$

\square

Remark 3.4. The one-dimensional quotient torus T/T' naturally acts on $\text{Sp}_\gamma^{T'}$. On the other hand, T/T' is isomorphic to the one-dimensional torus $(\mathbb{C}^*)^2/\mathbb{C}^*$ which acts on $\text{Sp}_{\beta_{ij}}$ and on $\text{Sp}_{\tilde{\beta}_{ij}}$. The isomorphisms constructed in Lemmas 3.2 and 3.3 are T/T' -equivariant.

The homology for the affine Springer fiber of matrices like $\tilde{\beta}_{ij}$ with all powers the same is known. It is found using GKM theory by Kivinen in [21].

Theorem 3.5 (Kivinen [21]). *If*

$$\gamma = \begin{pmatrix} z_1 t^d & & 0 \\ & \ddots & \\ 0 & & z_n t^d \end{pmatrix},$$

then $\Delta H_*^T(\text{Sp}_\gamma)$ injects into

$$H_T^*(pt) \otimes \mathbb{C}[\mathbb{Z}^n] \cong \mathbb{C}[t_1, \dots, t_n, x_1^\pm, \dots, x_n^\pm],$$

where $\Delta = \prod_{i < j} (t_i - t_j)^d$. As a submodule, there is a canonical isomorphism

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \cong \prod_{i < j} (t_i - t_j, x_i - x_j)^d.$$

Corollary 3.6. *We have the following canonical isomorphism of $\mathbb{C}[t_i - t_j, x_i^\pm, x_j^\pm]$ -modules:*

$$(t_i - t_j)^{d_{ij}} H_*^{T/T'}(\mathrm{Sp}_{\tilde{\beta}_{ij}}) \cong (t_i - t_j, x_i - x_j)^{d_{ij}} \subseteq \mathbb{C}[t_i - t_j, x_i^\pm, x_j^\pm].$$

Here T/T' acts on $\mathrm{Sp}_{\tilde{\beta}_{ij}}$ as in Remark 3.4.

Now in order to piece together these homologies of $\mathrm{Sp}_{\tilde{\beta}_{ij}}$, we use the following fact:

Lemma 3.7 (Algebraic Hartogs' lemma). *If A is an integrally closed Noetherian integral domain, then*

$$A = \bigcap_{p \text{ codimension } 1} A_p,$$

where we take the intersection over all codimension 1 prime ideals of A of A_p inside the fraction field $\mathrm{Frac}(A)$. If M is a free A -module, then we also have

$$M = \bigcap_{p \text{ codimension } 1} M_p,$$

where the intersection is taken inside $M \otimes_A \mathrm{Frac}(A)$.

Since we assumed that Sp_γ is equivariantly formal, $H_*^T(\mathrm{Sp}_\gamma)$ is free over $H_T^*(pt)$. Now we can prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 2.8, we have that, up to localization away from $(t_i - t_j)$,

$$H_*^T(\mathrm{Sp}_\gamma) \cong_{loc} H_*^T(\mathrm{Sp}_\gamma^{T'}) \cong H_*^{T/T'}(\mathrm{Sp}_\gamma^{T'}) \otimes H_{T'}^*(pt) = H_*^{T/T'}(\mathrm{Sp}_{\tilde{\beta}_{ij}} \times \mathbb{Z}^{n-2}) \otimes H_{T'}^*(pt).$$

Here \cong_{loc} indicates an isomorphism after localization. Note that after localization, $\Delta = (t_i - t_j)^{d_{ij}}$ up to an invertible factor, so by Lemma 3.3 we get

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \cong_{loc} (t_i - t_j)^{d_{ij}} H_*^{T/T'}(\mathrm{Sp}_{\tilde{\beta}_{ij}}) \otimes H_{T'}^*(pt) \otimes \mathbb{C}[\mathbb{Z}^{n-2}] = (t_i - t_j, x_i - x_j)^{d_{ij}} \subseteq \mathbb{C}[\mathbf{t}, \mathbf{x}^\pm].$$

We have inclusion map

$$\mathrm{Sp}_\gamma^{T'} \xrightarrow{i} \mathrm{Sp}_\gamma$$

and by Lemma 2.8

$$\prod_{(k,\ell) \neq (i,j)} (t_k - t_\ell) H_*^T(\mathrm{Sp}_\gamma) \subseteq i_* H_*^T(\mathrm{Sp}_\gamma^{T'}).$$

We also have that

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \subseteq \prod_{(k,\ell) \neq (i,j)} (t_k - t_\ell) (t_i - t_j)^{d_{ij}} H_*^T(\mathrm{Sp}_\gamma).$$

By the above, this is contained in

$$i_* (t_i - t_j)^{d_{ij}} H_*^T(\mathrm{Sp}_\gamma^{T'}) = (t_i - t_j, x_i - x_j)^{d_{ij}}.$$

We conclude that

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \subseteq (t_i - t_j, x_i - x_j)^{d_{ij}}.$$

This holds for all codimension-1 subtori $T' = (t_i - t_j) \subseteq T$ with $i < j$, so we have

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \subseteq \bigcap_{i < j} (t_i - t_j, x_i - x_j)^{d_{ij}}.$$

In fact we have already seen that $(t_i - t_j, x_i - x_j)^{d_{ij}}$ is exactly the localization $H_*^T(\mathrm{Sp}_\gamma)_p$ where $p = (t_i - t_j)$.

So by Lemma 3.7 we conclude that

$$\Delta H_*^T(\mathrm{Sp}_\gamma) \cong \bigcap_{i < j} (t_i - t_j, x_i - x_j)^{d_{ij}}.$$

□

This proof required the assumption that Sp_γ is equivariantly formal. If $d_i = d$ for all i , then it is known to be equivariantly formal [11] and we recover the homology result of Theorem 3.5 from Kivinen [21].

Conjecture 3.8. Sp_γ is equivariantly formal for all d_1, \dots, d_n and all n .

4. GENERALIZED HAIMAN IDEAL J FOR $n = 3$

When $n = 3$, it is known that Sp_γ is equivariantly formal, so by Theorem 3.1, up to denominators (Δ) , its equivariant Borel-Moore homology is isomorphic to the ideal $\mathcal{J} \subseteq \mathbb{C}[t_1, t_2, t_3, x_1^\pm, x_2^\pm, x_3^\pm]$ defined as

$$\mathcal{J} = \mathcal{J}(d_1, d_2) = (t_1 - t_2, x_1 - x_2)^{d_1} \cap (t_1 - t_3, x_1 - x_3)^{d_1} \cap (t_2 - t_3, x_2 - x_3)^{d_2}$$

seen as a module over $\mathbb{C}[t_1, t_2, t_3, x_1^\pm, x_2^\pm, x_3^\pm]$. Here we are assuming that $d_1 \leq d_2 \leq d_3$, so that d_1, d_1, d_2 are equal to the pairwise minima d_{ij} .

We consider a similar ideal $J' \subset \mathbb{C}[t_1, t_2, t_3, x_1, x_2, x_3]$

$$J' = J'(d_1, d_2) = (t_1 - t_2, x_1 - x_2)^{d_1} \cap (t_1 - t_3, x_1 - x_3)^{d_1} \cap (t_2 - t_3, x_2 - x_3)^{d_2}$$

It is easy to see that $\mathcal{J} = J' \otimes_{\mathbb{C}[\mathbf{x}]} \mathbb{C}[\mathbf{x}^\pm]$, so the generators for \mathcal{J} over $\mathbb{C}[\mathbf{t}, \mathbf{x}^\pm]$ will be the same as generators of J' over the polynomial ring $\mathbb{C}[\mathbf{t}, \mathbf{x}]$. Next we will do a change of variables:

$$a = t_1 - t_2, \quad b = x_1 - x_2, \quad c = t_3 - t_2, \quad d = x_3 - x_2$$

and consider the ideal

$$J = J(d_1, d_2) = (a, b)^{d_1} \cap (c, d)^{d_2} \cap (a - c, b - d)^{d_1}$$

over $R = \mathbb{C}[a, b, c, d]$. Clearly, we get

$$J'(d_1, d_2) = J(d_1, d_2) \otimes_{\mathbb{C}} \mathbb{C}[x_1 + x_2 + x_3, t_1 + t_2 + t_3],$$

so again all three ideals have the same generators up to this change of variables.

We can also consider these as bigraded ideals, where the t_i 's (or a and c) have bidegree q , and the x_i 's (b and d) have bidegree t .

We will frequently use the polynomial

$$ad - bc = d(a - c) - c(b - d) \in (a, b) \cap (c, d) \cap (a - c, b - d).$$

Theorem 4.1. *The ideal $J = (a, b)^{d_1} \cap (c, d)^{d_2} \cap (a - c, b - d)^{d_1}$ over $R = \mathbb{C}[a, b, c, d]$ has the following families of generators ($0 \leq j \leq d_1$):*

- (1) $A_{i,j} = a^{d_1-j} c^{d_2-j} (a-c)^i (b-d)^{d_1-j-i} (ad-bc)^j$, $1 \leq i \leq d_1 - j$. These generators have bidegree $q^{d_1+d_2-j+i} t^{d_1-i}$, and there are $d_1 - j$ of these for a fixed j . They are characterized by $\deg_t < d_1$.

- (2) $B_{i,j} = a^{d_1-j-i} b^i d^{d_2-j} (b-d)^{d_1-j} (ad-bc)^j$, $1 \leq i \leq d_1 - j$. Such generators have bidegree $q^{d_1-i} t^{d_1+d_2-j+i}$, and there are $d_1 - j$ of these for a fixed j . They are characterized by $\deg_q < d_1$.
- (3) $C_{i,j} = a^{d_1-j} c^i d^{d_2-j-i} (b-d)^{d_1-j} (ad-bc)^j$, $1 \leq i \leq d_2 - j$. Such generators have bidegree $q^{d_1+i} t^{d_1+d_2-j-i}$, and there are $d_2 - j$ of these for a fixed j . They are characterized by $\deg_q > d_1$, $\deg_t \geq d_1$.
- (4) $D_j = a^{d_1-j} d^{d_2-j} (b-d)^{d_1-j} (ad-bc)^j$ has bidegree $q^{d_1} t^{d_1+d_2-j}$, there is one such generator for each j . They are characterized by $\deg_q = d_1$.

Remark 4.2. For $j = d_1$, the generators $A_{i,j}$ and $B_{i,j}$ are not defined, while $C_{i,d_1} = c^i d^{d_2-d_1-i} (ad-bc)^{d_1}$ for $1 \leq i \leq d_2 - d_1$ and $D_{d_1} = d^{d_2-d_1} (ad-bc)^{d_1}$.

In particular, it is easy to see that there is at most one generator in each (q, t) -bidegree, see also Proposition 4.12. Also notice that we chose the generators in Theorem 4.1 such that the monomial factor in $A_{i,j}$ does not contain b or d , and the monomial factors in $B_{i,j}, C_{i,j}, D_j$ do not contain bc (unless $j = d_1$).

Theorem 4.1 follows from Proposition 4.3, which we prove in Section 5.1.

Proposition 4.3. *The ideal $J(d_1, d_2)$ has the following basis (over \mathbb{C}):*

$$m(a, c)A_{i,j}, m(a, b, d)B_{i,j}, m(a, c, d)C_{i,j}, m(a, d)D_k \quad (j \leq d_1 - 1)$$

where m are arbitrary monomials in the corresponding variables. For $j = d_1$ we have to add all polynomials of the form

$$a^\alpha b^\beta c^\gamma d^\delta (ad-bc)^{d_1}, \quad \gamma + \delta \geq d_2 - d_1.$$

Example 4.4. For $d_1 = d_2 = 1$ we get the following 5 generators of $J(1, 1)$:

$$A_{1,0} = ac(a-c), \quad B_{1,0} = bd(b-d), \quad C_{1,0} = ac(b-d), \quad D_0 = ad(b-d), \quad D_1 = (ad-bc).$$

We can change the variables back to see that the generators of \mathcal{J} over $\mathbb{C}[t_1, t_2, t_3, x_1^\pm, x_2^\pm, x_3^\pm]$ are

$$\begin{aligned} A_{1,0} &= (t_1 - t_2)(t_3 - t_2)(t_1 - t_3), & B_{1,0} &= (x_1 - x_2)(x_3 - x_2)(x_1 - x_3), \\ C_{1,0} &= (t_1 - t_2)(t_3 - t_2)(x_1 - x_3), & D_0 &= (t_1 - t_2)(x_3 - x_2)(x_1 - x_3), \end{aligned}$$

and

$$D_1 = \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ t_1 & t_2 & t_3 \end{pmatrix}$$

Corollary 4.5. *We have that*

$$J(d_1, d_2) = J(1, 1)^{d_1} \cdot J(0, 1)^{d_2-d_1},$$

$$J'(d_1, d_2) = J'(1, 1)^{d_1} \cdot J'(0, 1)^{d_2-d_1}$$

and

$$\mathcal{J}(d_1, d_2) = \mathcal{J}(1, 1)^{d_1} \cdot \mathcal{J}(0, 1)^{d_2-d_1}.$$

Proof. We prove the first equation, and the other two equations follow immediately.

The containment $J(1, 1)^{d_1} \cdot J(0, 1)^{d_2-d_1} \subseteq J(d_1, d_2)$ is clear, so it is sufficient to show that any generator of $J(d_1, d_2)$ can be written as a product of d_1 generators of $J(1, 1)$ (listed in Example 4.4) and $d_2 - d_1$ generators of $J(0, 1) = (c, d)$. Indeed:

$$A_{i,j} = a^{d_1-j} c^{d_2-j} (a-c)^i (b-d)^{d_1-j-i} (ad-bc)^j = (ac(a-c))^i \cdot (ac(b-d))^{d_1-j-i} \cdot (ad-bc)^j \cdot c^{d_2-d_1},$$

$$B_{i,j} = a^{d_1-j-i} b^i d^{d_2-j} (b-d)^{d_1-j} (ad-bc)^j = (ad(b-d))^{d_1-j-i} (bd(b-d))^i (ad-bc)^j d^{d_2-d_1},$$

$$C_{i,j} = a^{d_1-j} c^j d^{d_2-j-i} (b-d)^{d_1-j} (ad-bc)^j = \\ (ac(b-d))^x (ad(b-d))^{d_1-j-x} (ad-bc)^j c^{i-x} d^{d_2-d_1-i+x},$$

where $x = \max(0, i + d_1 - d_2)$. Note that $i \leq d_2 - j$, so $i + d_1 - d_2 \leq d_1 - j$ and hence $x \leq d_1 - j$. Also, $d_1 \leq d_2$, so $i + d_1 - d_2 \leq i$ and $x \leq i$. Therefore all exponents are indeed nonnegative. Finally,

$$D_j = a^{d_1-j} d^{d_2-j} (b-d)^{d_1-j} (ad-bc)^j = (ad(b-d))^{d_1-j} (ad-bc)^j d^{d_2-d_1}.$$

□

Remark 4.6. Note that by Remark 4.2 the polynomials $a^\alpha b^\beta c^\gamma d^\delta (ad-bc)^{d_1}$, $\gamma + \delta \geq d_2 - d_1$ are either multiples of C_{i,d_1} or of D_{d_1} .

In [15], Gorsky, Kivinen, and Oblomkov define a graded algebra $\mathcal{A}_G = \bigoplus_{d=0}^{\infty} \mathcal{A}_d$, depending only on a reductive group G , with some specific properties. One of the key properties is that for any $\gamma \in \mathfrak{g}$, the direct sum of homologies of affine Springer fibers

$$F_\gamma = \bigoplus_{k=0}^{\infty} H_*(\mathrm{Sp}_{t^k \gamma})$$

is a graded module over \mathcal{A}_G , or equivalently, that there is a corresponding quasi-coherent sheaf \mathcal{F}_γ on $\mathrm{Proj} \bigoplus_{d=0}^{\infty} \mathcal{A}_d$. They conjecture that F_γ is finitely generated and that this sheaf is coherent [15, Conjecture 8.1]. In the case where $G = \mathrm{GL}_n$, they show that this graded algebra is generated in degrees 0 and 1, and that $\mathrm{Proj} \bigoplus_{d=0}^{\infty} \mathcal{A}_d = \mathrm{Hilb}^n(\mathbb{C}^* \times \mathbb{C})$. A special case of this conjecture follows from Theorem 4.1 and its corollaries.

Indeed, it is proved in [15] that \mathcal{A}_0 is the space of symmetric polynomials in $\mathbb{C}[t_1, \dots, t_n, x_1^\pm, \dots, x_n^\pm]$, and \mathcal{A}_1 is the space of antisymmetric polynomials.

Theorem 4.7. *In the case of $G = \mathrm{GL}_3$ and γ as in (1), the graded module F_γ is finitely generated over \mathcal{A}_G , and defines a coherent sheaf on $\mathrm{Hilb}^3(\mathbb{C}^* \times \mathbb{C})$, i.e. Conjecture 8.1 holds in this case.*

Proof. By Corollary 3.8.3 in [9], the ideal generated by \mathcal{A}_1 is exactly $\mathcal{J}(1, 1)$. There is a natural inclusion of ideals

$$\mathcal{J}(d_1, d_2) \cdot \mathcal{J}(1, 1) \rightarrow \mathcal{J}(d_1 + 1, d_2 + 1).$$

It follows from Corollary 4.5 that this map is actually surjective as well. Since $H_*(\mathrm{Sp}_{t^k \gamma})$ corresponds to the ideal $\mathcal{J}(d_1 + k, d_2 + k)$, this shows that the module F_γ is generated in degree 0, and therefore finitely generated by the generators of $\mathcal{J}(d_1, d_2)$. □

4.1. Hilbert Series. Let us introduce two rational functions

$$H(d_1, \dots, d_n) = \sum_T \frac{z_1^{d_n} \dots z_n^{d_1}}{(1-q)^n (1-t)^n} \prod_{i=2}^n \frac{1}{(1-z_i^{-1})} \prod_{i < j} \omega(z_i/z_j)$$

and

$$F(d_1, \dots, d_n) = \sum_T z_1^{d_n} \dots z_n^{d_1} \prod_{i=2}^n \frac{1}{(1-z_i^{-1})(1-qtz_{i-1}/z_i)} \prod_{i < j} \omega(z_i/z_j).$$

Here the sums are over standard tableaux T with n boxes, z_i is the (q, t) -content $q^{c-1} t^{r-1}$ of the box labeled by i in row r and column c in T , and $\omega(x) = \frac{(1-x)(1-qt)}{(1-qx)(1-tx)}$.

By convention, all the factors of the form $(1 - 1)$ in the above products (either in the numerator or in denominator) should be ignored.

The function $F(d_1, \dots, d_n)$ is also known as generalized (q, t) -Catalan number, see [13] for more details and context. Note that the order of the d_i is reversed here compared to [13].

Conjecture 4.8. *We have that:*

- a) *The Hilbert series of the ideal $J'(d_1, \dots, d_n)$ equals $H(d_1, \dots, d_n)$.*
- b) *The Hilbert series of the generating set $J'(d_1, \dots, d_n)/\mathfrak{m}J'(d_1, \dots, d_n)$ equals $F(d_1, \dots, d_n)$. Here \mathfrak{m} is the maximal ideal $\mathfrak{m} = (t_1, \dots, t_n, x_1, \dots, x_n)$.*

In particular, this conjecture implies that $F(d_1, \dots, d_n)$ is a polynomial in q and t with nonnegative coefficients (see [13, Conjecture 1.3]) and provides an explicit algebraic interpretation of these coefficients. Similarly, the conjecture implies that $H(d_1, \dots, d_n)$ is a power series in q and t with nonnegative coefficients.

Example 4.9. For $n = 2$ we get $J(d_1, d_2) = (x_1 - x_2, y_1 - y_2)^{d_1}$. We change coordinates to $x_1 - x_2 = a, y_1 - y_2 = b$ and $\bar{x} = x_1 + x_2, \bar{y} = y_1 + y_2$, then $J(d_1, d_2)$ has generating set $a^{d_1}, a^{d_1-1}b, \dots, b^{d_1}$, so the Hilbert series for the generating set equals

$$q^{d_1} + q^{d_1-1}t + \dots + t^{d_1} = \frac{q^{d_1}}{1 - t/q} + \frac{t^{d_1}}{1 - q/t} = F(d_1, d_2).$$

Similarly, $J(d_1, d_2)$ is free over $\mathbb{C}[\bar{x}, \bar{y}]$ with basis $a^\alpha b^\beta, \alpha + \beta \geq d_1$, so by Lemma 5.6 below we get the Hilbert series

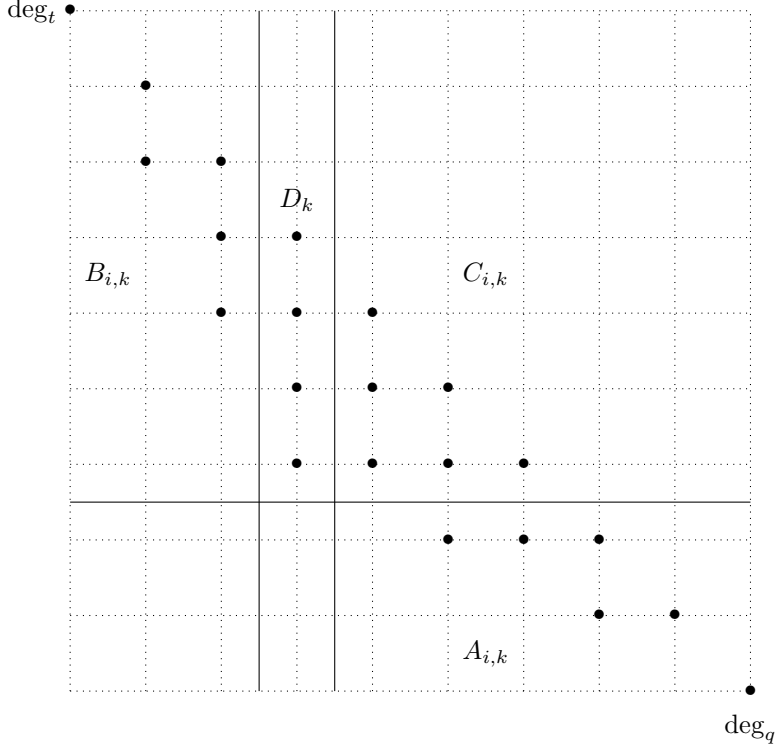
$$\frac{1}{(1-q)(1-t)} \sum_{\alpha+\beta \geq d_1} q^\alpha t^\beta = \frac{q^{d_1}}{(1-q)^2(1-t)(1-t/q)} + \frac{t^{d_1}}{(1-q)(1-t)^2(1-q/t)}.$$

Theorem 4.10. *Conjectures 4.8(a) and 4.8(b) hold for $n = 3$ for all d_1, d_2, d_3 .*

The statement of (a) follows from the Hilbert series calculation in Section 5.2, and the proof of (b) will be in Section 4.2.

4.2. Combinatorics of J . We've already seen in Example 4.4 that $J(1, 1)$ has 5 generators, and that in general the generators of J each have a unique (q, t) -bidegree. We can plot the bidegrees of these generators as below.

Example 4.11. Here is an example where $d_1 = d_2 = 3$:



Proposition 4.12. *The generators of J are in bijection with the integer lattice points inside the trapezoid bounded by the following inequalities:*

$$2d_1 + d_2 \geq x + y \geq d_1 + d_2, \quad x + 2y \geq 2d_1 + d_2, \quad 2x + y \geq 2d_1 + d_2.$$

Proof. It is easy to check that all of the generators of J satisfy these inequalities on their bidegrees $(x, y) = (\deg_q, \deg_t)$. Since the generators all have unique (q, t) -bidegree, it is sufficient to count the number of points in the integer lattice and check that it is the same as the number of generators.

We will count the lattice points going by diagonals, starting with the top diagonal $x + y = 2d_1 + d_2$. On this diagonal, $x + 2y \geq 2d_1 + d_2$ and $2x + y \geq 2d_1 + d_2$ are trivially satisfied, since $x, y \geq 0$. So x and y can both range from 0 to $2d_1 + d_2$, and there are $2d_1 + d_2 + 1$ points on this diagonal.

On the next diagonal, $x + y = 2d_1 + d_2 - 1$, we have that $x + 2y = 2d_1 + d_2 - 1 + y$ and $2x + y = 2d_1 + d_2 - 1 + x$ are both at least $2d_1 + d_2$. This implies that $x, y \geq 1$ so we have points

$$(1, 2d_1 + d_2 - 2), (2, 2d_1 + d_2 - 1), \dots, (2d_1 + d_2 - 2, 1),$$

which amounts to 3 less points than the first diagonal. If we keep going, each diagonal will have 3 less points than the last, until the final diagonal $x + y = d_1 + d_2$, which will have $d_2 - d_1 + 1$ points. If we index the diagonal $x + y = 2d_1 + d_2 - j$ by j , j will range from 0 to d_1 . So in total the number of points in the lattice is

$$\sum_{j=0}^{d_1} [2d_1 + d_2 + 1 - 3j].$$

If we count the generators of J as laid out in Theorem 4.1, we get

$$A_{i,j} : \sum_{j=0}^{d_1} (d_1 - j), \quad B_{i,j} : \sum_{j=0}^{d_1} (d_1 - j), \quad C_{i,j} : \sum_{j=0}^{d_1} (d_2 - j), \quad D_j : \sum_{j=0}^{d_1} 1,$$

and combining the sums, we get the same count. So we have shown the desired bijection. \square

Proof of Theorem 4.10(b). In Example 1.2 in [13], the authors show that

$$F(d_1, d_2, d_3) = [2d_1 + d_2 + 1]_{q,t} + qt[2d_1 + d_2 - 2] + \cdots + q^{d_1} t^{d_1} [d_2 - d_1 + 1]_{q,t},$$

where $[n + 1]_{q,t} := q^n + q^{n-1}t + \cdots + qt^{n-1} + t^n$.

We can see from the proof of Proposition 4.12 that this exactly matches up with the coordinates of the lattice points, grouped by diagonals, and therefore by Proposition 4.12, this is in bijection with the generators of J and their (q, t) degrees.

Any choice of basis of $J(d_1, \dots, d_n)/\mathfrak{m}J(d_1, \dots, d_n)$ lifts to a set of generators of J by Nakayama's lemma. So the Hilbert series of $J(d_1, \dots, d_n)/\mathfrak{m}J(d_1, \dots, d_n)$ is precisely the degree count of the generators of J , so indeed it is $F(d_1, d_2, d_3)$. \square

5. PROOFS

5.1. Proof of Theorem 4.1. After doing the change of variables to $\mathbb{C}[a, b, c, d]$, the upshot is that we've reduced the number of variables, and we can use the fact that $J = M \cap (a - c, b - d)^{d_1}$, where $M = (a, b)^{d_1} \cap (c, d)^{d_2}$ is a monomial ideal. In this section we will find a basis for J over \mathbb{C} by characterizing when elements of $(a - c, b - d)^{d_1}$ are in the monomial ideal M , proving Proposition 4.3 and by extension Theorem 4.1. First, we will need a few key lemmas.

Since M is a monomial ideal, a polynomial f is in M if and only if all monomials m of f that have nonzero coefficients are in M . If m is a monomial, then let $\deg_1(m)$ be the combined (a, b) degree of m , i.e. the sum of its a and b degrees, and similarly let $\deg_2(m)$ be its combined (c, d) degree. Then the monomial m is in M if and only if $\deg_1(m) \geq d_1$, and $\deg_2(m) \geq d_2$. Note that these degrees should not be confused with \deg_q and \deg_t defined above.

Consider some $f \in R$ of the form

$$f = \varphi(b - d) + \psi(ad - bc) \in M$$

for some $\varphi, \psi \in R$. Notice that for any $\gamma \in R$, we can modify the coefficients φ, ψ by simultaneously substituting:

$$(3) \quad \varphi \rightarrow \varphi + \gamma(ad - bc), \quad \psi \rightarrow \psi - \gamma(b - d)$$

without changing f .

Lemma 5.1. *If*

$$f = \varphi(b - d) + \psi(ad - bc) \in M$$

for some $\varphi, \psi \in R$, then up to the relation (3), we can assume that $\varphi \in M$.

Proof. Since M is a monomial ideal, $\varphi(b - d) + \psi(ad - bc) \in M$ if and only if every monomial term of $\varphi(b - d) + \psi(ad - bc)$ is in M . If m is a monomial in the expansion of this expression, either m is in M , or m cancels with some other monomial.

When we expand, all monomials come in pairs from distributing $b - d$ or $ad - bc$. These pairs look like $m \left(\frac{b}{d}\right) - m$ or $m \left(\frac{ad}{bc}\right) - m$, where each term is appropriately

divisible so that there are no denominators. If one of these monomials cancels with another monomial, that other monomial also must be part of a pair like the above. For example,

$$\frac{m}{cd}(ad - bc) + \frac{m}{d}(b - d) = m\left(\frac{b}{d}\right)\left(\frac{ad}{bc}\right) - m\left(\frac{b}{d}\right) + m\left(\frac{b}{d}\right) - m = m\left(\frac{b}{d}\right)\left(\frac{ad}{bc}\right) - m.$$

We can continue to follow a chain of cancellation until either we get two terminal monomials that do not cancel with anything, or we eventually reach a monomial that cancels with the starting monomial m , creating a cycle. We can visualize these chains of cancellations by oriented paths in a 2 dimensional lattice. Vertices represent monomials, vertical edges represent a difference of two monomials of the form $m\left(\frac{b}{d}\right) - m$, and horizontal edges represent a difference of the form $m\left(\frac{ad}{bc}\right) - m$. The full path represents the sum of all the pairs of monomials represented by each edge, and the end vertices of the path are the terminal monomials. For example, the above cancellation can be represented by the path:

$$\begin{array}{ccc} m\left(\frac{b}{d}\right) & \longrightarrow & m\left(\frac{ad}{bc}\right)\left(\frac{b}{d}\right) \\ \uparrow & & \\ m & & \end{array}$$

Cycles in these paths correspond exactly to the equivalence

$$\varphi(b - d) + \psi(ad - bc) = (\varphi + \gamma(ad - bc))(b - d) + (\psi - \gamma(b - d))(ad - bc).$$

To see this, let m be a monomial. A cycle looks like this:

$$\begin{array}{ccc} m\left(\frac{b}{d}\right) & \longrightarrow & m\left(\frac{ad}{bc}\right)\left(\frac{b}{d}\right) \\ \uparrow & & \downarrow \\ m & \longleftarrow & m\left(\frac{ad}{bc}\right) \end{array}$$

This corresponds to the identity

$$\frac{m}{d}(b - d) + \frac{m}{cd}(ad - bc) - \frac{ma}{bc}(b - d) - \frac{m}{bc}(ad - bc) = 0.$$

For all of these terms to be monomials, b, c, d must all divide m . Multiplying through by bcd and grouping, we get

$$-m(ad - bc)(b - d) + m(b - d)(ad - bc) = 0.$$

Adding this cycle corresponds to using the above equivalence with $\gamma = m$. Adding any number of these cycles along our path corresponds to modifying the coefficients φ and ψ with relation (3) without changing the overall sum f .

We can add and subtract this square loop to any path in order to both eliminate any loops, and to reorder cancellation. For example:

$$\begin{array}{c}
 m\left(\frac{b}{d}\right) \longrightarrow m\left(\frac{ad}{bc}\right)\left(\frac{b}{d}\right) \\
 \uparrow \\
 m
 \end{array}
 =
 \begin{array}{c}
 m\left(\frac{b}{d}\right) \longrightarrow m\left(\frac{ad}{bc}\right)\left(\frac{b}{d}\right) \\
 \uparrow \qquad \qquad \downarrow \\
 m \longleftarrow m\left(\frac{ad}{bc}\right)
 \end{array}
 +
 \begin{array}{c}
 m\left(\frac{ad}{bc}\right)\left(\frac{b}{d}\right) \\
 \uparrow \\
 m \longrightarrow m\left(\frac{ad}{bc}\right)
 \end{array}$$

So up to relation (3), we can make any path into one with vertical edges first and horizontal edges after, going from m to $m\left(\frac{b}{d}\right)^k$ to $m\left(\frac{b}{d}\right)^k\left(\frac{ad}{bc}\right)^j$, with k and j possibly negative or 0.

$$\begin{array}{c}
 m\left(\frac{b}{d}\right)^k \cdots \cdots \cdots m\left(\frac{ad}{bc}\right)^j\left(\frac{b}{d}\right)^k \\
 \uparrow \\
 \vdots \\
 m
 \end{array}$$

This reduced path corresponds to $\varphi(b-d) + \psi(ad-bc)$, where

$$\varphi = \frac{m\left(\frac{b}{d}\right)^k - m}{b-d} \quad \text{and} \quad \psi = \frac{m\left(\frac{b}{d}\right)^k\left(\frac{ad}{bc}\right)^j - m\left(\frac{b}{d}\right)^k}{ad-bc}.$$

Notice for any monomial m , $m\left(\frac{ad}{bc}\right)^j \in M$ if and only if $m \in M$, as multiplying by $\frac{a}{b}$ does not change the combined (a, b) degree of a monomial, and the same for $\frac{c}{d}$.

Given a reduced path as above, we know that the terminal monomials m and $m\left(\frac{b}{d}\right)^k\left(\frac{ad}{bc}\right)^j$ are in M . But since $m\left(\frac{b}{d}\right)^k\left(\frac{ad}{bc}\right)^j \in M$, by the above, $m\left(\frac{b}{d}\right)^k$ is in M . Now we want to show that $\varphi \in M$.

Since $m\left(\frac{b}{d}\right)^k \in M$, it follows that $\frac{m}{d^k} \in (c, d)^{d_2}$, since multiplying by b^k does not change the (c, d) degree. Since m is in $(a, b)^{d_1}$, it follows that $\frac{m}{d^k} \in (a, b)^{d_1}$, since it has the same (a, b) degree as m . So $\frac{m}{d^k} \in M$, and therefore indeed

$$\varphi = \frac{m}{d^k} \frac{b^k - d^k}{b-d} \in M.$$

For a general $\varphi(b-d) + \psi(ad-bc)$, we can break the terms into a sum of discrete cancellation chains,

$$\sum_i [\varphi_{1,i}(b-d) + \varphi_{2,i}(ad-bc)].$$

For each cancellation chain, we have shown that up to equivalence, $\varphi_{1,i} \in M$, and therefore

$$\varphi = \sum_i \varphi_{1,i} \in M.$$

□

Lemma 5.2. *We have that $\varphi(b-d) \in M$ if and only if $\varphi \in M$, and $\varphi(a-c) \in M$ if and only if $\varphi \in M$.*

Proof. This is essentially what the final argument of the above proof shows. If we expand the expression $\varphi(b-d)$ into monomials, then as in the proof of Lemma 5.1,

we will get chains of cancellations with two terminal monomials that do not cancel, which looks like the path:

$$\begin{array}{c} m \left(\frac{b}{d}\right)^k \\ \vdots \\ m \end{array}$$

This chain corresponds to

$$\varphi = \frac{m \left(\frac{b}{d}\right)^k - m}{b - d},$$

and as before, $m, m \left(\frac{b}{d}\right)^k \in M$ together imply that $\varphi \in M$. Any general $\varphi(b-d)$ can be split into the sum of distinct cancellation chains, and thus $\varphi \in M$.

The same argument applies to $\varphi(a-c)$ since M is symmetric in $(b, d), (a, c)$. \square

Next, we characterize how we can best express $f \in (a-c, b-d)^{d_1}$ in order to see when $f \in M$.

Lemma 5.3. *Any $f \in (a-c, b-d)$ can be written as*

$$f = \alpha_1(a-c) + \alpha_2(b-d) + \alpha_3(ad-bc),$$

where $\alpha_i \in R$ and α_1 is a polynomial in a and c only.

Proof. If $f \in (a-c, b-d)$, then

$$f = \gamma_1(a-c) + \gamma_2(b-d)$$

for some $\gamma_1, \gamma_2 \in R$. Observe that

$$b(a-c) = a(b-d) + (ad-bc),$$

and

$$d(a-c) = c(b-d) + (ad-bc).$$

So by applying this to any term in γ_1 with a factor of b or d , we can ensure that γ_1 only depends on a and c . \square

Lemma 5.4. *If $f \in (a-c, b-d)^{d_1} \cap M$, we can write $f = \sum_i f_i$, where*

$$f_i = \sum_{j=0}^{d_1-i} \alpha_{i,j} (a-c)^i (b-d)^{d_1-i-j} (ad-bc)^j$$

and each $f_i \in M$.

Proof. Consider $f \in (a-c, b-d)^{d_1}$. Following Lemma 5.3, we can express f as a linear combination of products of $(a-c)$, $(b-d)$, and $(ad-bc)$,

$$f = \sum_{i,j} \alpha_{i,j} (a-c)^i (b-d)^{d_1-i-j} (ad-bc)^j.$$

Further, we can assume that every coefficient of a term with a factor of $(a-c)$ depends only on a and c , because for any term with coefficient α with $i > 0$, we can write

$$\alpha(a-c)^i (b-d)^{d_1-i-j} (ad-bc)^j$$

$$\begin{aligned}
 &= [\alpha(a-c)](a-c)^{i-1}(b-d)^{d_1-i-j}(ad-bc)^j \\
 &= [\alpha_1(a-c) + \alpha_2(b-d) + \alpha_3(ad-bc)](a-c)^{i-1}(b-d)^{d_1-i-j}(ad-bc)^j
 \end{aligned}$$

where α_1 only depends on a and c by Lemma 5.3. We can continue this process by induction until all the modified coefficients $\alpha_{i,j}$ only depend on a and c .

Now we will group terms of f by their combined (b, d) degree. For all $i > 0$, since $\alpha_{i,j}$ depends only on a and c , we know that the combined b, d degree of every monomial of

$$\alpha_{i,j}(a-c)^i(b-d)^{d_1-i-j}(ad-bc)^j$$

is $k = \deg_t(m) = d_1 - i$. If any monomial from this term cancels, it must cancel with another term with the same (b, d) degree. So in fact, every monomial with $\deg_t(m) = k$ must come from the sum

$$\begin{aligned}
 f_i &= \sum_{j=0}^k \alpha_{i,j}(a-c)^i(b-d)^{k-j}(ad-bc)^j \\
 &= (a-c)^i \sum_{j=0}^k \alpha_{i,j}(b-d)^{k-j}(ad-bc)^j
 \end{aligned}$$

with fixed i . In other words, monomials can cancel within each f_i , but not between them. This implies that for all $i > 0$, each $f_i \in M$, since after internal cancellation, each f_i is a sum of monomials in M .

Since $f = \sum_i f_i$ is in M and $f_i \in M$ for all $i > 0$,

$$f_0 = \sum_{j=0}^{d_1} \alpha_{i,j}(b-d)^{k-j}(ad-bc)^j$$

must also be in M . □

By Corollary 5.2, we know that

$$f_i = (a-c)^i \sum_{j=0}^k \alpha_{i,j}(b-d)^{k-j}(ad-bc)^j \in M$$

implies that

$$(4) \quad \sum_{j=0}^k \alpha_{i,j}(b-d)^{k-j}(ad-bc)^j \in M.$$

Now we fix i and look at a single f_i , which we will call f to avoid unnecessary indices. Also let $k = d_1 - i$ as before.

Proposition 5.5. *If*

$$f = \sum_{j=0}^k \alpha_j(b-d)^{k-j}(ad-bc)^j \in M,$$

for some $\alpha_j \in R$, then there exists $\alpha'_j \in R$ such that

$$f = \sum_j \alpha'_j(b-d)^{k-j}(ad-bc)^j$$

and each term $\alpha'_j(b-d)^{k-j}(ad-bc)^j$ of the sum is in M .

Proof. We can rewrite f as

$$(5) \quad f = \sum_{j=0}^{k-1} [\varphi_j(b-d) + \psi_j(ad-bc)] (b-d)^{k-1-j} (ad-bc)^j,$$

where initially $\varphi_j = \alpha_j$ for all $0 \leq j \leq k-1$, $\psi_{k-1} = \alpha_k$, and the rest of the ψ_j 's are 0. Essentially we have taken the previous sum for f and added some redundant terms; in particular, ψ_j and φ_{j+1} are coefficients for like terms for $0 \leq j \leq k-1$. So we have two relations we can use to modify the coefficients of (5) without changing the sum:

$$(6) \quad \psi_j \rightarrow \psi_j + \gamma, \quad \varphi_{j+1} \rightarrow \varphi_{j+1} - \gamma$$

$$(7) \quad \varphi_j \rightarrow \varphi_j + \gamma(ad-bc), \quad \psi_j \rightarrow \psi_j - \gamma(b-d).$$

The first comes from the redundant coefficients, and the second is the same relation (3) used in Lemma 5.1.

We will induct on $k = d_1 - i$. If $k = 0$, then $i = d_1$, and our sum (4) is just the single term $\alpha_0(a-c)^{d_1}$, which is in M by assumption. Now assume by induction that if

$$\sum_{j=0}^{k-1} \alpha_j (b-d)^{k-1-j} (ad-bc)^j$$

is in M , then we can modify the coefficients using only (7) to get all terms $\alpha_j (b-d)^{k-1-j} (ad-bc)^j$ in M . We can apply this to (5) with $\alpha_j = \varphi_j(b-d) + \psi_j(ad-bc)$. Using (7) on the α_j 's actually just amounts to using (6) on the φ_j 's and ψ_j 's, as adding $\gamma(ad-bc)$ to α_j is the same as adding γ to ψ_j , and subtracting $\gamma(b-d)$ from α_j is the same as subtracting γ from φ_j . So the inductive hypothesis implies that up to (6), we can get

$$[\varphi_j(b-d) + \psi_j(ad-bc)] (b-d)^{k-1-j} (ad-bc)^j$$

in M for all j . By Corollary 5.2, this implies that

$$(8) \quad [\varphi_j(b-d) + \psi_j(ad-bc)] (ad-bc)^j \in M.$$

Notice that multiplying a monomial (polynomial) by $(ad-bc)$ raises its combined (a, b) degree and (c, d) degree each by one. So (8) is in $M = (a, b)^{d_1} \cap (c, d)^{d_2}$ if and only if $[\varphi_j(b-d) + \psi_j(ad-bc)] \in N = (a, b)^{d_1-j} \cap (c, d)^{d_2-j}$. Now we apply Lemma 5.1 on N to get both $\varphi_j(b-d)$ and $\psi_j(ad-bc)$ in N , and then when multiply by $(ad-bc)^j$, we get that both terms of (8) are in M .

So we have shown that if

$$f = \sum_{j=0}^{k-1} [\varphi_j(b-d) + \psi_j(ad-bc)] (b-d)^{k-1-j} (ad-bc)^j,$$

up to relations (6) and (7), we can get all terms of this sum to be in M . Now simply recombine like terms to get

$$f = \sum_j \alpha_j (b-d)^{d_1-i-j} (ad-bc)^j$$

with all terms in M as desired. □

Now we can lay out a basis for the ideal

$$J(d_1, d_2) = (a, b)^{d_1} \cap (c, d)^{d_2} \cap (a - c, b - d)^{d_1}$$

over \mathbb{C} .

Proof of Proposition 4.3. We know by Proposition 5.5 that $J(d_1, d_2)$ is generated by polynomials of the form

$$\alpha_{i,j}(a - c)^i (b - d)^{d_1 - i - j} (ad - bc)^j,$$

where $\alpha_{i,j}$ only depends on a, c for $i > 0$. Here $0 \leq j \leq d_1$ and $0 \leq i \leq d_1 - j$. So as a vector space, $J(d_1, d_2)$ is generated by

$$a^\alpha b^\beta c^\gamma d^\delta (a - c)^i (b - d)^{d_1 - i - j} (ad - bc)^j$$

with the conditions that $\alpha + \beta + j \geq d_1$, $\gamma + \delta + j \geq d_2$, and $\beta = \delta = 0$ if $i > 0$. Among these generators, the only kind of relations remaining are those that come from the fact that $bc = ad - (ad - bc)$.

If $i > 0$, then this relation is irrelevant, and we get linearly independent generators of the form

$$m(a, c)A_{i,j}.$$

If $i = 0$ and $\beta, \gamma > 0$, then we can write

$$a^\alpha b^\beta c^\gamma d^\delta = a^{\alpha+1} b^{\beta-1} c^{\gamma-1} d^{\delta+1} + a^\alpha b^{\beta-1} c^{\gamma-1} d^\delta (ad - bc).$$

Continue reducing bc this way until we end up in one of the following situations:

- (1) $\gamma = 0$, $\beta \neq 0$, in which case we are left with

$$m(a, b, d)B_{\beta,j}$$

with $0 \leq j \leq d_1$ and $1 \leq \beta \leq d_1 - j$.

- (2) $\beta = 0$, $\gamma \neq 0$, in which case we are left with

$$m(a, c, d)C_{\gamma,j}$$

with $0 \leq j \leq d_1$ and $1 \leq \gamma \leq d_2 - j$.

- (3) $\beta = 0$, $\gamma = 0$, in which case we are left with

$$m(a, d)D_j$$

with $0 \leq j \leq d_1$.

- (4) The exponent of $(ad - bc)$ is greater than or equal to d_1 , in which case we are left with a linear combination of terms of the form

$$m(a, b, c, d)(ad - bc)^{d_1}.$$

□

Theorem 4.1 immediately follows from Proposition 4.3.

5.2. Hilbert Series Calculation. Let us compute the Hilbert series using the basis in Proposition 4.3.

Lemma 5.6. *We have*

$$\sum_{\alpha+\beta \geq s} q^\alpha t^\beta = \frac{q^s}{(1-q)(1-t/q)} + \frac{t^s}{(1-t)(1-q/t)}.$$

Proof. We have

$$\begin{aligned} \sum_{\alpha+\beta \geq s} q^\alpha t^\beta &= \sum_{\beta=0}^{s-1} \frac{q^{s-\beta} t^\beta}{(1-q)} + \sum_{\beta=s}^{\infty} \frac{t^\beta}{1-q} = \\ &= \frac{q^s - t^s}{(1-q)(1-t/q)} + \frac{t^s}{(1-q)(1-t)}. \end{aligned}$$

Now we can use the identity

$$\frac{1}{(1-q)(1-t)} - \frac{1}{(1-q)(1-t/q)} = \frac{1}{(1-t)(1-q/t)}.$$

□

Theorem 5.7. *The bigraded Hilbert series of $J(d_1, d_2)$ is equal to*

$$\begin{aligned} &\frac{q^{2d_1+d_2}}{(1-q)^2(1-t/q)(1-t/q^2)} + \frac{t^{2d_1+d_2}}{(1-t)^2(1-q/t)(1-q/t^2)} + \\ &\frac{q^{d_1} t^{d_2} (1+t)}{(1-q)(1-t)(1-q/t)(1-t^2/q)} + \frac{q^{d_2} t^{d_1} (1+q)}{(1-t)(1-q)(1-t/q)(1-q^2/t)}. \end{aligned}$$

Proof. We compute the contribution of various basis elements.

1. The contribution of $m(a, c)A_{i,j}$, $j \leq d_1 - 1$ equals

$$\begin{aligned} &\frac{1}{(1-q)^2} \sum_{j=0}^{d_1-1} \sum_{i=1}^{d_1-j} q^{d_1+d_2-j+i} t^{d_1-i} = \\ &\frac{1}{(1-q)^2} \sum_{j=0}^{d_1-1} \frac{q^{d_1+d_2-j+1} t^{d_1-1} - q^{2d_1+d_2-2j+1} t^{j-1}}{(1-q/t)} = \\ &\frac{q^{d_1+d_2+1} t^{d_1-1} - q^{d_2+1} t^{d_1-1}}{(1-q)^2(1-q^{-1})(1-q/t)} - \frac{q^{2d_1+d_2+1} t^{-1} - q^{d_2+1} t^{d_1-1}}{(1-q)^2(1-q/t)(1-t/q^2)}. \end{aligned}$$

2. The set $m(a, b, d)B_{i,j} \cup m(a, d)D_j$, $j \leq d_1 - 1$ consists of elements

$$a^\alpha b^\beta d^\gamma (b-d)^{d_1-j} (ad-bc)^j, \quad \alpha + \beta \geq d_1 - j, \quad \gamma \geq d_2 - j,$$

so by Lemma 5.6 we get the Hilbert series

$$\begin{aligned} &\sum_{j=0}^{d_1-1} \left[\frac{q^{d_1-j}}{(1-q)(1-t/q)} + \frac{t^{d_1-j}}{(1-t)(1-q/t)} \right] \frac{q^j t^{d_1+d_2-j}}{(1-t)} = \\ &\sum_{j=0}^{d_1-1} \left[\frac{q^{d_1} t^{d_1+d_2-j}}{(1-t)(1-q)(1-t/q)} + \frac{q^j t^{2d_1+d_2-2j}}{(1-t)^2(1-q/t)} \right] = \\ &\frac{q^{d_1} t^{d_1+d_2} - q^{d_1} t^{d_2}}{(1-t^{-1})(1-t)(1-q)(1-t/q)} + \frac{t^{2d_1+d_2} - q^{d_1} t^{d_2}}{(1-q/t^2)(1-t)^2(1-q/t)} \end{aligned}$$

3. Similarly, for $m(a, c, d)C_{i,j} \cup m(a, d)D_j$, $j \leq d_1 - 1$ we get

$$a^\alpha c^\beta d^\gamma (b-d)^{d_1-j} (ad-bc)^j, \quad \alpha \geq d_1 - j, \quad \beta + \gamma \geq d_2 - j,$$

so the Hilbert series equals

$$\sum_{j=0}^{d_1-1} \left[\frac{q^{d_2-j}}{(1-q)(1-t/q)} + \frac{t^{d_2-j}}{(1-t)(1-q/t)} \right] \frac{q^{d_1} t^{d_1}}{(1-q)} =$$

$$\frac{q^{d_1+d_2} t^{d_1} - q^{d_2} t^{d_1}}{(1-q^{-1})(1-q)^2(1-t/q)} + \frac{q^{d_1} t^{d_1+d_2} - q^{d_1} t^{d_2}}{(1-t^{-1})(1-q)(1-t)(1-q/t)}.$$

4. We overcount by $m(a, d)D_j$, $j \leq d_1 - 1$ which contribute

$$\frac{1}{(1-q)(1-t)} \sum_{j=0}^{d_1-1} q^{d_1} t^{d_1+d_2-j} = \frac{q^{d_1} t^{d_1+d_2} - q^{d_1} t^{d_2}}{(1-t^{-1})(1-q)(1-t)}$$

5. For $j = d_1$ we have special terms

$$a^\alpha b^\beta c^\gamma d^\delta (ad - bc)^{d_1}, \quad \gamma + \delta \geq d_2 - d_1,$$

which contribute

$$\frac{1}{(1-q)(1-t)} \left[\frac{q^{d_2-d_1}}{(1-q)(1-t/q)} + \frac{t^{d_2-d_1}}{(1-t)(1-q/t)} \right] q^{d_1} t^{d_1}.$$

6. Finally, we collect the coefficients at similar terms:

$$q^{d_1+d_2} t^{d_2} \left[\frac{qt^{-1}}{(1-q)^2(1-q^{-1})(1-q/t)} + \frac{1}{(1-q^{-1})(1-q)^2(1-t/q)} \right] = 0;$$

$$q^{d_2} t^{d_1} \left[-\frac{1}{(1-q)^2(1-q^{-1})(1-q/t)} + \frac{1}{(1-q)^2(1-q/t)(1-t/q^2)} - \right.$$

$$\left. \frac{1}{(1-q^{-1})(1-q)^2(1-t/q)} + \frac{1}{(1-q)(1-t)(1-q)(1-t/q)} \right] = \frac{q^{d_2} t^{d_1} (1+q)}{(1-t)(1-q)(1-t/q)(1-q^2/t)}.$$

$$q^{2d_1+d_2} \left[-\frac{qt^{-1}}{(1-q)^2(1-q/t)(1-t/q^2)} \right] + \frac{t^{2d_1+d_2}}{(1-q/t^2)(1-t)^2(1-q/t)} =$$

$$\frac{q^{2d_1+d_2}}{(1-q)^2(1-t/q)(1-t/q^2)} + \frac{t^{2d_1+d_2}}{(1-t)^2(1-q/t)(1-q/t^2)};$$

$$q^{d_1} t^{d_1+d_2} \left[\frac{1}{(1-t^{-1})(1-t)(1-q)(1-t/q)} + \frac{1}{(1-t^{-1})(1-q)(1-t)(1-q/t)} - \right.$$

$$\left. \frac{1}{(1-t^{-1})(1-q)(1-t)} \right] = 0;$$

$$q^{d_1} t^{d_2} \left[-\frac{t^{-1}}{(1-t^{-1})(1-t)(1-q)(1-t/q)} - \frac{qt^{-2}}{(1-q/t^2)(1-t)^2(1-q/t)} - \right.$$

$$\left. \frac{t^{-1}}{(1-t^{-1})(1-q)(1-t)(1-q/t)} + \frac{t^{-1}}{(1-t^{-1})(1-q)(1-t)} + \frac{1}{(1-q)(1-t)(1-t)(1-q/t)} \right] =$$

$$\frac{q^{d_1} t^{d_2} (1+t)}{(1-q)(1-t)(1-q/t)(1-t^2/q)}.$$

□

6. BRAID RECURSION

In this section we compute Khovanov-Rozansky homology for the link L_γ associated to γ as in (1) for $n = 3$. It is the closure of the braid $\beta_{d_1, d_2} = (\text{FT}_2)^{d_2 - d_1} \cdot (\text{FT}_3)^{d_1}$ where $\text{FT}_2 = \sigma_1^2$ and $\text{FT}_3 = (\sigma_1^2)\sigma_2\sigma_1^2\sigma_2$ are the full twist braids on two and three strands respectively. We will also use the Jucys-Murphy braids $\text{JM}_2 = \text{FT}_2$ and $\text{JM}_3 = \text{FT}_2^{-1}\text{FT}_3 = \sigma_2\sigma_1^2\sigma_2$, so that $\beta_{d_1, d_2} = \text{JM}_2^{d_2} \text{JM}_3^{d_1}$. It is well-known that JM_2 and JM_3 commute in the braid group.

We use the recursion for triply graded Khovanov-Rozansky homology defined by Elias and Hogancamp in [8], as described in [16]. Each braid-like diagram represents a complex of Soergel bimodules, where crossings correspond to Rouquier complexes, and stacking a complex above (or below) represents taking a tensor product with that complex on the right (or left respectively). The K_n are certain complexes of Soergel bimodules defined in [8] that satisfy the relations below.

(a) K_1 is represented by a single vertical strand with a box labeled K_1 on it.

(b) K_n is represented by a vertical strand with a box labeled K_n and a crossing above it. This is equal to K_n with a crossing below it, which is equal to K_n with a crossing above it.

(c) K_{n+1} is represented by a vertical strand with a box labeled K_{n+1} and a loop on it. This is equal to $(t^n + a)$ times K_n .

(d) K_n is represented by a vertical strand with a box labeled K_n and a crossing below it. This is equal to t^{-n} times K_{n+1} , which is equal to q times K_n .

For this section, we think of Khovanov-Rozansky homology HHH as a certain functor from complexes of Soergel bimodules to triply graded vector spaces over \mathbb{C} . For a braid, $\text{HHH}(\beta)$ is HHH of the complex associated to its braid diagram. For some braids, we can use the above recursions to compute HHH by starting with its braid diagram and applying these rules until all strands are closed up. We refer to [16] and references therein for all details.

The three gradings of $\text{HHH}(\beta)$ are typically denoted Q, T , and A . These are related to the lowercase q, t , and a by the change of variables

$$q = Q^2, \quad t = T^2 Q^{-2}, \quad a = A Q^{-2}.$$

Definition 6.1. For any braid or braid-like diagram β , we say that $\text{HHH}(\beta)$ is parity if it is supported in only even T degrees. We will also say β itself is parity if $\text{HHH}(\beta)$ is parity.

Fact 6.2 ([16]). If both complexes on the right-hand side of step (d) are parity, then the left-hand side is also parity, and we can replace the map with addition (corresponding to a direct sum of modules).

Essentially, we do our calculations assuming that the right-hand side of (4) is parity. If we can successfully reduce β down to complexes that are known to be parity, then we have shown that β and all of the intermediate complexes were also parity, and we can compute $\text{HHH}(\beta)$ along the way. We will often take the a -degree 0 component of HHH , denoted $\text{HHH}^{a=0}$.

Theorem 6.3. *For all $d_2 \geq d_1$ $\text{HHH}^{a=0}(\beta_{d_1, d_2})$ is parity.*

Proof. To make the calculation simpler, we translate the braid pictures into equations, so for example, (d) translates to

$$K_n \cdot \text{JM}_{n+1} = t^{-n} K_{n+1} + qt^{-n} K_n,$$

where JM_i is the Jucys-Murphy braid on i strands. In general when we write K_i, JM_i , or FT_i (for a full twist), it is always on the first i strands, and multiplication represents vertical stacking from bottom to top (or equivalently a tensor product of Soergel bimodules).

We can resolve the first FT_2 by the process:

$$\text{FT}_2 = K_1 \cdot \text{FT}_2 = K_1 \cdot \text{JM}_2 = t^{-1} K_2 + qt^{-1} K_1.$$

Since K_2 absorbs any crossings on the first two strands, this leaves us with

$$\beta_{d_1, d_2} = t^{-1} K_2 \cdot (\text{FT}_3)^{d_1} + qt^{-1} \beta_{d_1, d_2-1}.$$

Now we resolve $\alpha_{d_1} = K_2 \cdot (\text{FT}_3)^{d_1}$ by first writing FT_3 as the product $\text{JM}_2 \cdot \text{JM}_3$. The JM_2 gets absorbed by K_2 , so we get

$$K_2 \cdot (\text{FT}_3)^{d_1} = K_2 \cdot \text{JM}_3 \cdot (\text{FT}_3)^{d_1-1} = t^{-2} K_3 + qt^{-2} K_2 (\text{FT}_3)^{d_1-1},$$

where the K_3 has absorbed the rest of the FT_3 's. When we close up K_3 , we get t^3 , and $\alpha_0 = K_2$ closes up to $\frac{t}{1-q}$. So we get a simple recursion:

$$\text{HHH}^{a=0}(\alpha_{d_1}) = t + qt^{-2} \text{HHH}^{a=0}(\alpha_{d_1-1}),$$

with $\text{HHH}^{a=0}(\alpha_0) = \frac{t}{1-q}$. We can write this in a closed form as

$$\text{HHH}^{a=0}(\alpha_{d_1}) = t \left[\frac{1 - (qt^{-2})^{d_1}}{1 - qt^{-2}} + \frac{(qt^{-2})^{d_1}}{q - 1} \right].$$

So overall we have the recursive relation

$$(9) \quad \text{HHH}^{a=0}(\beta_{d_1, d_2}) = \left[\frac{1 - (qt^{-2})^{d_1}}{1 - qt^{-2}} + \frac{(qt^{-2})^{d_1}}{q - 1} \right] + qt^{-1} \text{HHH}^{a=0}(\beta_{d_1, d_2-1}),$$

with $\beta_{d_1, 0} = (\text{FT}_3)^{d_1} = T(3d_1, 3)$, which is parity (with known homology) by [8]. \square

To compare this to $H_*(\text{Sp}_\gamma)$, it can be checked by direct computation that the Hilbert series $H(d_1, d_2)$ of Sp_γ satisfies essentially the same recursion (9). But we can also apply a theorem of Gorsky and Hogancamp (Proposition 5.5 in [14]). Here HY is the y -ified Khovanov Rozansky homology defined in [14].

Theorem 6.4 ([14]). *Assume that $\beta = \text{JM}_1^{d_n} \dots \text{JM}_n^{d_1}$, $d_n \geq d_{n-1} \geq \dots \geq d_1$, and $\text{HHH}^{a=0}(\beta)$ is parity. Then:*

$$(1) \quad \text{HY}^{a=0}(\beta) = \text{HHH}^{a=0}(\beta) \otimes \mathbb{C}[y_1, \dots, y_n] \text{ and } \text{HHH}^{a=0}(\beta) = \text{HY}^{a=0}(\beta)/(y)$$

(2) $I(d_1, \dots, d_n) \subseteq \text{HY}^{a=0}(\beta) \subseteq J'(d_1, \dots, d_n)$, where I is the product

$$I(d_1, \dots, d_n) = J'(1, \dots, 1)^{d_1} \cdot J'(0, 1, \dots, 1)^{d_2-d_1} \cdot \dots \cdot J'(0, \dots, 0, 1)^{d_{n-1}-d_{n-2}}.$$

In our case β can be expressed exactly as above, and Theorem 6.3 along with Corollary 4.5 implies that for $n = 3$:

$$\text{HY}^{a=0}(L_\gamma) = J'(d_1, d_2).$$

Note the analogy between the relationship of HY to HHH in statement (1) of Theorem 6.4, and the relationship of $H_*^T(\text{Sp}_\gamma)$ to $H_*(\text{Sp}_\gamma)$ due to Fact 2.7. The result is the following weaker version of Conjecture 1.2.

Theorem 6.5. *For $n = 3$ and γ as in 1,*

$$\text{HY}^{a=0}(L_\gamma) \otimes_{\mathbb{C}[\mathbf{x}]} \mathbb{C}[\mathbf{x}, \mathbf{x}^\pm] = \Delta H_*^T(\text{Sp}_\gamma)$$

and

$$\text{HHH}^{a=0}(L_\gamma) \otimes_{\mathbb{C}[\mathbf{x}]} \mathbb{C}[\mathbf{x}, \mathbf{x}^\pm] = H_*(\text{Sp}_\gamma).$$

In order to show Theorem 6.5, we used Corollary 4.5, but one could instead show that the link splitting map defined in [14] is canonical for β as above, analogous to Proposition 6.11 in [14]. This means that Theorem 6.5 could be generalized to higher n solely by showing that β is parity, such as with more explicit recursions. We are optimistic that this can be done for $n = 4$ and possibly for higher n .

APPENDIX A. GENERALIZED (q, t) -CATALAN NUMBERS AND THE
FUNDAMENTAL DOMAIN FOR $n = 3$
BY EUGENE GORSKY AND JOSHUA P. TURNER

In [6] Zongbin Chen introduced a notion of the fundamental domain for an unramified affine Springer fiber, which captures the behavior of cells in Sp_γ under translations by the lattice Λ . More precisely, for $n \leq 4$ (and conjecturally in general) Sp_γ admits a cell decomposition with cells parametrized by the lattice Λ . There is one torus fixed point in each cell. In general, the dimension of a cell is a complicated piecewise-linear function on Λ which stabilizes outside of the fundamental domain \mathcal{P} . The cells corresponding to points in Λ outside \mathcal{P} can be obtained by translation of cells corresponding to points in \mathcal{P} .

At the same time, by Theorem 1.5 the (non-equivariant) homology of Sp_γ as a module over Λ is captured by $\mathcal{J}/(y)\mathcal{J}$ as a module over $\mathbb{C}[x_1^\pm, \dots, x_n^\pm]$, so the cells in \mathcal{P} should correspond to the generators of \mathcal{J} , and (following Conjecture 1.10) to the generalized (q, t) -Catalan numbers $F(d_n, \dots, d_1)$.

In this appendix we explore the definition and some general properties of \mathcal{P} and establish its precise relation with the generalized (q, t) -Catalan numbers for $n = 3$. We hope to generalize this to higher n in future work.

A.1. The Fundamental Domain. We define the action of S_n on \mathbb{R}^n by $\sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. Note that for the basis vectors \mathbf{e}_i we have $\sigma \mathbf{e}_i = \mathbf{e}_{\sigma^{-1}(i)}$.

We start with the matrix $\gamma = \text{diag}(\gamma_1, \dots, \gamma_n)$ as in (1), where γ_i are pairwise distinct monomials with order d_i . We will always assume that $d_1 \leq \dots \leq d_n$. Define $d_{ij} = \min(d_i, d_j)$ for $i \neq j$, noting that d_{ij} is the order of $\gamma_i - \gamma_j$.

Definition A.1. (Compare with [6, Proposition 2.8]) We define the polytope $\mathcal{P}(d_1, \dots, d_n)$ as the convex hull of the points $p_\sigma = \sigma(b_{1,\sigma}, \dots, b_{n,\sigma})$ where:

$$b_{i,\sigma} = \sum_{j < i} d_{\sigma^{-1}(i), \sigma^{-1}(j)}, \quad \sigma \in S_n.$$

Example A.2. For $n = 2$ we get two points $p_e = (0, d_1)$ and $p_{(1\ 2)} = (d_1, 0)$, and \mathcal{P} is the segment connecting them.

Example A.3. For $n = 3$ we get 6 points shown in the following table

σ	$(b_{1,\sigma}, b_{2,\sigma}, b_{3,\sigma})$	p_σ
e	$(0, d_1, d_1 + d_2)$	$(0, d_1, d_1 + d_2)$
$(1\ 2)$	$(0, d_1, d_1 + d_2)$	$(d_1, 0, d_1 + d_2)$
$(1\ 3)$	$(0, d_2, 2d_1)$	$(2d_1, d_2, 0)$
$(2\ 3)$	$(0, d_1, d_1 + d_2)$	$(0, d_1 + d_2, d_1)$
$(1\ 2\ 3)$	$(0, d_1, d_1 + d_2)$	$(d_1, d_1 + d_2, 0)$
$(1\ 3\ 2)$	$(0, d_2, 2d_1)$	$(2d_1, 0, d_2)$

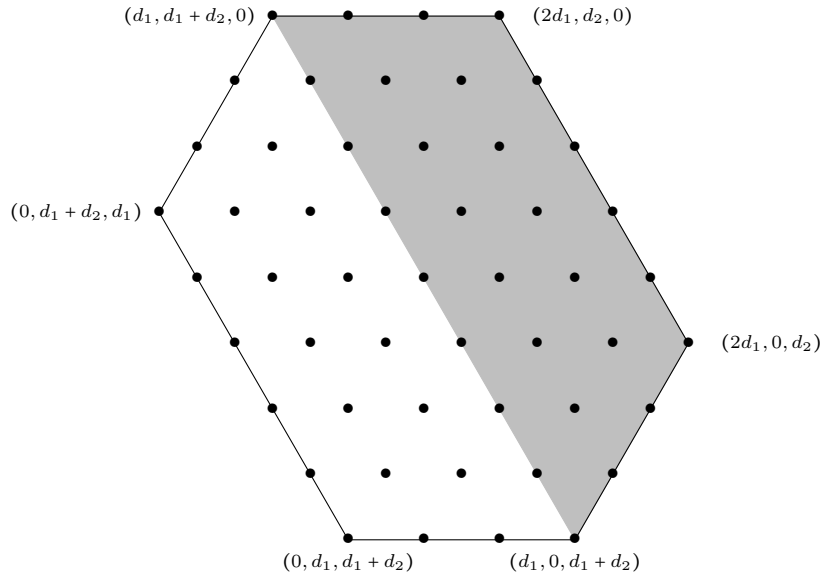


FIGURE 1. Fundamental domain for $(d_1, d_2) = (3, 5)$

Example A.4. Suppose that $d_i = d$ for all i . Then $b_{i,\sigma} = d(i - 1)$,

$$p_\sigma = (d(\sigma(1) - 1), \dots, d(\sigma(n) - 1)),$$

and \mathcal{P} is the standard $(n - 1)$ -dimensional permutahedron dilated d times.

We now establish some general properties of \mathcal{P} .

Lemma A.5. *The polytope $\mathcal{P}(d_1, \dots, d_n)$ is contained in the hyperplane $\sum x_i = \sum_{i < j} d_{ij}$.*

Proof. For any σ we have $\sum_i b_{i,\sigma} = \sum_{j < i} d_{\sigma^{-1}(i), \sigma^{-1}(j)} = \sum_{i < j} d_{ij}$. □

Proposition A.6. Let \mathbf{e}_i denote the i -th basis vector. Then $\mathcal{P}(d_1, \dots, d_n)$ is the Minkowski sum of $\binom{n}{2}$ segments connecting $d_{ij}\mathbf{e}_i$ and $d_{ij}\mathbf{e}_j$.

Proof. Let $\mathcal{P}'(d_1, \dots, d_n)$ denote the above Minkowski sum. We can write

$$p_\sigma = \sigma \left(\sum_{i < j} d_{\sigma^{-1}(i), \sigma^{-1}(j)} \mathbf{e}_i \right) = \sum_{i < j} d_{\sigma^{-1}(i), \sigma^{-1}(j)} \mathbf{e}_{\sigma^{-1}(i)} = \sum_{\sigma(i) < \sigma(j)} d_{ij} \mathbf{e}_i.$$

Given a permutation σ and $i < j$, we can choose one end of the segment connecting $d_{ij}\mathbf{e}_i$ and $d_{ij}\mathbf{e}_j$ as follows: if $\sigma(i) < \sigma(j)$, we choose \mathbf{e}_i , otherwise we choose \mathbf{e}_j . Clearly, the sum of these points equals p_σ , so $p_\sigma \in \mathcal{P}'(d_1, \dots, d_n)$. Therefore $\mathcal{P}(d_1, \dots, d_n) \subseteq \mathcal{P}'(d_1, \dots, d_n)$.

On the other hand, $\mathcal{P}'(d_1, \dots, d_n)$ is a zonotope with edges parallel to the edges of the standard permutahedron $\mathcal{P}(1, \dots, 1)$. It follows e.g. from [2, Section 9] that the vertices of $\mathcal{P}'(d_1, \dots, d_n)$ are in bijection with the vertices of $\mathcal{P}(1, \dots, 1)$, and are given by p_σ . So $\mathcal{P}'(d_1, \dots, d_n) \subset \mathcal{P}(d_1, \dots, d_n)$. \square

Example A.7. For $n = 3$ we get three segments $[(d_1, 0, 0), (0, d_1, 0)]$, $[(d_1, 0, 0), (0, 0, d_1)]$ and $[(0, d_2, 0), (0, 0, d_2)]$.

Remark A.8. Quite surprisingly, a similar polytope appeared in a recent work of Alishahi, Liu and the first author on Heegaard Floer homology [1].

A.2. Generalized (q, t) -Catalans for $n = 3$. Given $d_1 \leq d_2$, we can consider the Young diagram $\lambda_{d_1, d_2} = (d_1 + d_2, d_1)$. We will draw Young diagrams in French notation, with the corner at $(0, 0)$, see Figure 2. We also consider the line

$$\ell_{d_1, d_2} = \{x + d_2 y = d_1 + 2d_2 + \varepsilon\}$$

where ε is a small positive number. The following lemma shows that λ_{d_1, d_2} is a triangular partition in the sense of [3].

Lemma A.9. The diagram λ_{d_1, d_2} is the largest Young diagram below the line ℓ_{d_1, d_2} . If a diagram μ is strictly below ℓ_{d_1, d_2} , then $\mu \subset \lambda_{d_1, d_2}$.

Proof. Let us describe all integer points (x, y) satisfying $x + d_2 y < d_1 + 2d_2 + \varepsilon$. For $y = 1$ we get $x < d_1 + d_2 + \varepsilon$, so $x \leq d_1 + d_2$. For $y = 2$ we get $x < d_1 + \varepsilon$, so $x \leq d_1$. For $y \geq 3$ we get $x < d_1 + (2 - y)d_2 + \varepsilon < 0$, so there are no integer points (here we used $d_1 \leq d_2$). The result follows. \square

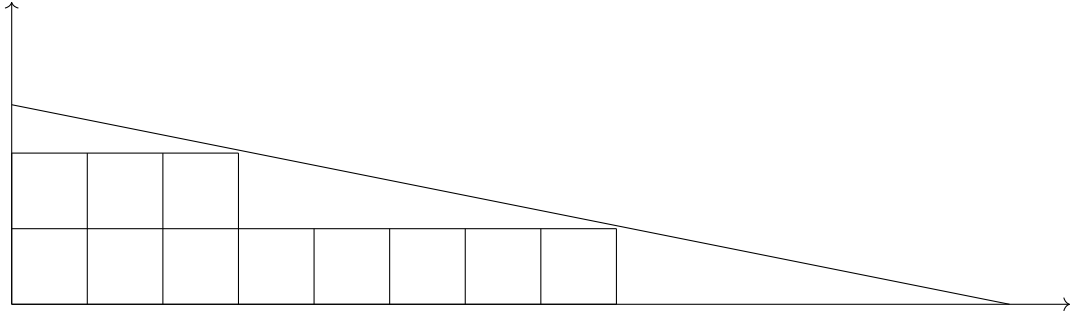


FIGURE 2. The line ℓ_{d_1, d_2} and the diagram λ_{d_1, d_2} for $(d_1, d_2) = (3, 5)$.

Following [3], we define two statistics on subdiagrams of λ_{d_1, d_2} .

Definition A.10. Given $\mu \subset \lambda_{d_1, d_2}$, we define $\text{area}(\mu) = |\lambda_{d_1, d_2}| - |\mu|$ and

$$\text{dinv}(\mu) = \left\{ \square \in \mu : \frac{a(\square)}{\ell(\square) + 1} \leq d_2 < \frac{a(\square) + 1}{\ell(\square)} \right\}$$

Here $a(\square)$ and $\ell(\square)$ are respectively the arm and the leg of a box \square in μ .

Note that d_2 in the definition of dinv is negative reciprocal to the slope of the line ℓ_{d_1, d_2} .

Theorem A.11. *The map $\phi : \mu \mapsto (\text{area}(\mu), \text{dinv}(\mu))$ yields a bijection between the subdiagrams $\mu \subset \lambda_{d_1, d_2}$ and the integer points in the trapezoid*

$$(10) \quad \{d_1 + d_2 \leq x + y \leq 2d_1 + d_2, 2x + y \leq 2d_1 + d_2, x + 2y \leq 2d_1 + d_2\}.$$

As a consequence, we get the generalized (q, t) -Catalan number

$$(11) \quad \sum_{\mu \subset \lambda_{d_1, d_2}} q^{\text{area}(\mu)} t^{\text{dinv}(\mu)} = F(d_1, d_2).$$

Equation (11) is a special case of the main result of [4], but we give a more direct proof here generalizing [17, Theorem 4.1].

Proof. Let us write $\mu = (a + b, a)$, then $0 \leq a \leq d_1$ and $0 \leq b \leq d_1 + d_2 - a$. We have the following cases (see Figure 3):

(a) If $a + b \leq d_2$ then $\text{dinv}(\mu) = a + b$, so

$$\phi(\mu) = (2d_1 + d_2 - 2a - b, a + b) = (x, y).$$

We have $a = 2d_1 + d_2 - x - y, b = 2y + x - 2d_1 - d_2$, so the inequalities $0 \leq a, a \leq d_1, 0 \leq b$ and $a + b \leq d_2$ respectively translate to the inequalities $x + y \leq 2d_1 + d_2, x + y \geq d_1 + d_2, 2y + x \geq 2d_1 + d_2$ and $y \leq d_2$.

(b) If $a + b > d_2, b \leq d_2$ then $\text{dinv}(\mu) = 2a + 2b - d_2$, so

$$\phi(\mu) = (2d_1 + d_2 - 2a - b, 2a + 2b - d_2) = (x, y).$$

We have $y = d_2 \pmod{2}$, and

$$a = \frac{4d_1 + d_2 - 2x - y}{2}, \quad b = x + y - 2d_1$$

The inequalities $0 \leq a, a \leq d_1, 0 \leq b, b \leq d_2$ and $a + b > d_2$ respectively translate to the inequalities

$$2x + y \leq 4d_1 + d_2, 2x + y \geq 2d_1 + d_2, x + y \geq 2d_1, x + y \leq 2d_1 + d_2, y > d_2.$$

The second, fourth and fifth inequalities define a triangle \mathbf{T} with vertices $(0, 2d_1 + d_2), (d_1, d_2)$ and $(2d_1, d_2)$ with the bottom side removed. The other two inequalities are satisfied on this triangle. In other words, in this case the image of ϕ is the set of all integer points in the triangle \mathbf{T} satisfying $y = d_2 \pmod{2}$.

(c) If $b > d_2$ then $\text{dinv}(\mu) = 2a + d_2 + 1$, so

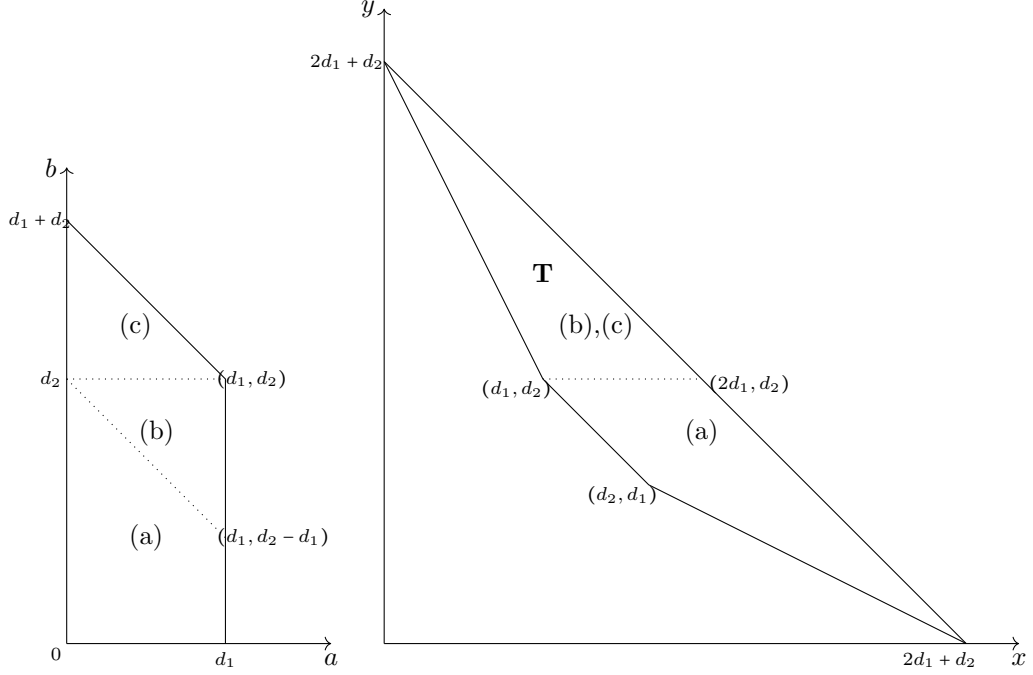
$$\phi(\mu) = (2d_1 + d_2 - 2a - b, 2a + d_2 + 1) = (x, y).$$

We have $y = d_2 + 1 \pmod{2}$, and

$$a = \frac{y - d_2 - 1}{2}, \quad b = 2d_1 + 2d_2 + 1 - x - y.$$

The inequalities $0 \leq a, a \leq d_1, b > d_2$ and $a + b \leq d_1 + d_2$ respectively translate to the inequalities

$$y \geq d_2 + 1, y \leq 2d_1 + d_2 + 1, x + y < 2d_1 + d_2 + 1, 2x + y \geq 2d_1 + d_2 + 1.$$

FIGURE 3. The bijection ϕ

Similarly to (b), the image of ϕ in this case is the set of all integer points in the triangle \mathbf{T} satisfying $y = d_2 + 1 \pmod 2$. \square

Finally, we compare the above combinatorial results with the fundamental domain. Observe that for $n = 3$ the fundamental domain $\mathcal{P}(d_1, d_2)$ is a hexagon with an axis of symmetry, which cuts it into two equal halves (see Figure 1).

Theorem A.12. *The integer points in a half of $\mathcal{P}(d_1, d_2)$ (including boundary) are in bijection with the generators of the ideal $\mathcal{J}(d_1, d_2)$.*

Proof. We construct the desired bijection in several steps:

1) The integer points in a half of $\mathcal{P}(d_1, d_2)$ are in bijection with the subdiagrams of λ_{d_1, d_2} . Indeed, we can write such points as $(d_1, 0, d_1 + d_2) + a(1, 0, -1) + b(0, 1, -1)$. It is easy to see by comparing Figures 1 and 3 that $0 \leq a \leq d_1$ and $0 \leq b \leq d_1 + d_2 - a$, and hence (a, b) define a subdiagram $\mu = (a + b, a)$.

2) By Theorem A.11, we have a bijection ϕ between the subdiagrams of λ_{d_1, d_2} and the points in the trapezoid (10).

3) By Proposition 4.12, there is a bijection between the generators of $\mathcal{J}(d_1, d_2)$ and the points in the trapezoid (10). \square

We expect that the bijection in Theorem A.12 is far more than a combinatorial coincidence. In particular, by tracing through the bijections we see that div_γ defines a piecewise linear function on the fundamental domain $\mathcal{P}(d_1, d_2)$, and we expect this function to be closely related to the dimension of cells in an appropriately chosen cell decomposition of Sp_γ . The corresponding (equivariant) homology classes of the cells would then correspond to some elements of $\mathcal{J}(d_1, d_2)$, and we expect that these

would indeed generate the ideal. We plan to study these questions and generalize them to $n > 3$ in future work.

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