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INTERNAL POINTS OF CONVEX SETS

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An interior point of a set K in a topological linear space is always an internal point of the set [1; V, 2.1(b)], as follows immediately from the definitions. Klee gives an example [3; p. 450] which shows that the converse may fail for certain subspaces of the l_p spaces, even though the set K is convex. It is the purpose of this note to describe as best we can the class of topological linear spaces for which the converse fails. We show that it fails for a large class of infinite dimensional topological linear spaces. The class includes all normed linear spaces and even all pre- F -spaces. Notation and terminology follow [1] unless otherwise indicated. If A is a set, $\text{card } A$ designates the cardinality of A .

THEOREM 1. *Suppose that a topological linear space E has a neighbourhood basis $\mathcal{U}$ at the origin O such that $\text{card } \mathcal{U}$ is less than or equal to the dimension of E . Then there is a symmetric convex set K in E having O as an internal point and having no interior points.*

Proof. We suppose first that the scalars are real. Let

$$\{h_a\}_{a \in A} \cup \{h_U\}_{U \in \mathcal{U}}$$

be a Hamel basis H for E such that h_U is in U for each neighbourhood U in $\mathcal{U}$. The existence of a linearly independent family $\{h_U\}$ with h_U in U for each U in $\mathcal{U}$ follows from the absorbing property of each neighbourhood and the condition on $\text{card } \mathcal{U}$. Let

$$K = \text{co}(\{h_a\} \cup \{-h_a\} \cup \{h_U\} \cup \{-h_U\}).$$

Then K is convex and symmetric. To see that the origin is not an interior point, we note first that no vector ch , where h is a member of the Hamel basis and $c > 1$, is in K . Suppose instead that ch is in K . It is easy to show that this is true if ch has the form $ch = \sum_{i=1}^n a_i h_i$, where $|a_i| \leq 1$, $\sum |a_i| = 1$, and $h_1, \dots, h_n$ is some finite subset of H . Since h and the h_i are members of the Hamel basis, this can happen only if $h = h_j$, some j ; $a_i = 0$, $i \neq j$, and $c = a_j$. But this is impossible because $c > 1$ and $|a_j| \leq 1$.

Since h_U is in U , ch_U is in U for sufficiently small $c > 1$. But ch_U is not in K so O is not an interior point of K . There are no interior points of K for if p were such a point, $-p$ would also be interior and therefore the line segment joining $-p$ and p would consist of interior points. Thus O would be an interior point, a contradiction.

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To see that the origin is an internal point, let x be any non-zero vector in E . Then $x = \sum_{i=1}^m a_i h_i$ for some finite subset $h_1, \dots, h_m$ of H . Also, $1/\epsilon = \sum_{i=1}^m |\hat{a}_i| > 0$ since $x \neq 0$. Thus ϵx is in K . Now $-\epsilon x$ is also in K so the line segment joining them is too. This shows O is an internal point of K .

If the scalars are complex, we consider E as a real vector space of "twice" the dimension, i.e. choose a Hamel basis for E over the complex numbers, then work with $H \cup iH$, which is a Hamel basis for E over the reals. The proof is then the same as for the case of real scalars.

COROLLARY 2. *Every infinite dimensional normed linear space contains a convex set K and a point p in K such that p is an internal point but not an interior point.*

COROLLARY 3. *A normed linear space (more generally, any subspace of an F -space, i.e. a pre- F -space) is finite dimensional if the concepts of internal point and interior point coincide for convex sets.*

Proof. By Corollary 2 and the fact that an interior point is always an internal point, it suffices to prove that an internal point of a convex set K in a finite dimensional topological linear space is an internal point of K . Since all Hausdorff topological linear spaces of dimension n are equivalent, it suffices to establish the assertion for the Euclidean topology.

Suppose first that the scalars are real. Let K be a convex subset of E^n [1; IV, 2.1] and let p be an internal point of K . Then if $e_1, \dots, e_n$ is a basis for E^n , there are positive numbers $\epsilon_1, \dots, \epsilon_n$ such that

$$\{p + \epsilon_1 e_1, \dots, p + \epsilon_n e_n\} \subset K.$$

Then $\mathcal{L}(K)$ (see [2; pp. 14ff] for definition and properties) equals E^n so K has an interior point [2; page 16, Theorem 4]. Therefore the interior and internal points of K coincide [1; V, 2.1(c)].

The proof extends to the complex case via the device used at the end of the proof of Theorem 1.

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