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From Coulomb Branches to HOMFLY-PT Homology

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From Coulomb Branches to HOMFLY-PT Homology

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Abstract

We study the topological A-twist of 3d $\mathcal{N} = 4$ Yang-Mills gauge theories, with an eye towards geometric representation theory and knot theory. We present an explicit, geometric category describing $\frac{1}{2}$-BPS vortex line operators in these theories, as well as collisions of local operators bound to them in terms of convolution techniques generalizing the work of Braverman-Finkelberg-Nakajima on Coulomb branches of vacua. Given a suitable Dirichlet boundary, we show that local operators bound to these vortex line operators can be represented as linear operators between the Borel-Moore homologies of generalized affine Springer fibers, vastly generalizing classical work on affine Springer representations of Hecke algebras and affine Weyl groups. We end with an application to knot theory. We apply 3d mirror symmetry to a recent construction in B-twisted 3d $\mathcal{N} = 4$ gauge theory of HOMFLY-PT knot homology due to Oblomkov-Rozansky to obtain a mirror construction in the A-twist. The mirror construction exactly reproduces a different realization of HOMFLY-PT homology for positive algebraic links due to Oblomkov-Rasmussen-Shende, providing a robust check of our proposed mirror construction.

Contents

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Introduction

Since their initial discovery [1–5], symmetries that exchange bosons and fermions have been fundamental to our understanding of quantum field theory. Of particular importance is super-Poincaré symmetry, or simply "supersymmetry," which extends the ordinary, bosonic Poincaré symmetry of spacetime to include fermionic symmetry generators Q_α that anti-commute to ordinary translations and transform as spinors under rotations. Supersymmetry can be viewed as an mild exception to the Coleman-Mandula theorem [6, 7] and has been seen to imply a myriad of physically desirable consequences ranging from non-renormalization theorems [8–13] and exactly calculable quantities [14–21], to remedying phenomenological problems in the Standard Model [22–26] and string theory [27–29]. Moreover, supersymmetric quantum field theories make contact with far-reaching subjects in mathematics, including, *e.g.*, enumerative geometry [30–32], geometric representation theory [33–40], and low dimensional topology [41–46].

This thesis will focus on 3d quantum field theories with $\mathcal{N} = 4$ supersymmetry. This class of field theories sits at the remarkably fertile interface between 2d and 4d supersymmetric quantum field theories. For example, these theories admit two distinguished classes of supersymmetric boundary conditions [47] that preserve half of the full supersymmetry algebra, often called $\frac{1}{2}$-BPS due to its relation to the Bogomol'nyi-Prasad-Sommerfield (BPS) bound [48, 49]: one class of boundary conditions is topological in nature, furnishing 2d $\mathcal{N} = (2, 2)$ supersymmetry, and can be used to understand the physics of symplectic duality [40] and aspects of geometric representation [50, 51]; a second class is holomorphic, furnishing 2d $\mathcal{N} = (0,4)$ supersymmetry, and boundary local operators admit the structure of a vertex operator algebra (VOA) [52, 53], playing a role completely analogous to Wess-Zumino-Witten (WZW) models in the classical Chern-Simons/WZW correspondence [41].

On the other hand, 3d $\mathcal{N} = 4$ degrees of freedom can be used to dress $\frac{1}{2}$-BPS interfaces between 4d $\mathcal{N} = 4$ super Yang-Mills theories. The action of S-duality, an $\mathcal{N} = 4$ analog of electric-magnetic duality [54], extends to such $\frac{1}{2}$-BPS interfaces and those with 3d $\mathcal{N} = 4$ degrees of freedom play a fundamental role [55–57]. Moreover, line operators in the bulk 4d theory can be sent towards an interface, thereby realizing an 'action' of 4d line operators on line operators in the boundary 3d theory, analogous to the action of BPS 't Hooft and Wilson lines on categories of boundary conditions for 2d sigma models to Hitchin moduli spaces $\mathcal{M}_H(\Sigma, G)$ [33]; see, *e.g.*, [58] for a discussion of this action in the context of geometric Langlands and number theory, as well as the upcoming paper [59] for a discussion of how they relate to the corner VOAs of [60, 61].

Perhaps the most important feature of 3d $\mathcal{N} = 4$ theories, at least from the perspective of this thesis, is that they can be topologically twisted to obtain 3d topological quantum field theories (TQFTs). The notion of twisting a theory was first introduced by Witten in the study of Donaldson theory [62]. Roughly speaking, to be able to twist a theory requires that it has a (scalar) fermionic symmetry Q such that $Q^2 = 0$; the "Q-twist" then corresponds to restricting ones attention to (local and extended) operators $\mathcal{O}$ that are invariant under Q or "Q-closed," *i.e.* $Q\mathcal{O} = 0$.

Any operator $\mathcal{O}$ of the form $\mathcal{O} = Q\mathcal{O}'$ for some other operator $\mathcal{O}'$ is called "Q-exact" and is trivial in correlation functions with any Q-closed operators $\mathcal{O}_1, ..., \mathcal{O}_n$ because

$$\langle \mathcal{O}\mathcal{O}_1...\mathcal{O}_n\rangle = \langle (Q\mathcal{O}')\mathcal{O}_1...\mathcal{O}_n\rangle = \langle Q(\mathcal{O}'\mathcal{O}_1...\mathcal{O}_n)\rangle = 0\,, \qquad (0.0.1)$$

and so it suffices to consider Q-closed operators modulo Q-exact operators, *i.e.* Q-cohomology. The Q-twist is called a topological twist when the stress tensor $T_{\mu\nu}$ of the underlying theory is Q-exact; this implies that insertions of all operators are independent of insertion points and that the theory can be put on an arbitrary background manifold compatible with Q.[1]

This thesis will focus on 3d $\mathcal{N} = 4$ super Yang-Mills theories of vector multiplets for a compact gauge group G and hypermultiplets transforming in a representation T^*R, with R a unitary representation of G and $T^*R \cong R \oplus \overline{R}$ for $\overline{R}$ the conjugate representation. These theories admit essentially two distinct topological supercharges of three-dimensional $\mathcal{N} = 4$ supersymmetry algebra Q_A, leading to the topological "A-twist," and Q_B, leading to the topological "B-twist." The A-twist is a dimensional reduction of the Donaldson twist of [62] (at least for pure gauge theory) and is used to define Seiberg-Witten invariants of 3-manifolds [65]. Aspects of the corresponding 3d TQFT were studied in [66] and Q_A-closed local operators are identified with holomorphic functions on the Coulomb branch of vacua $\mathcal{M}_C$ [67], where the hypermultiplet fields are required to have vanishing expectation value while the vector multiplet fields are unconstrained. The B-twist is intrinsically three-dimensional and leads to Rozansky-Witten theory [43] for $\mathcal{N} = 4$ σ-models; the full 3d TQFT structure of Rozansky-Witten theory was studied in depth by Kapustin-Rozansky-Saulina [68]. Local operators in the B-twist are identified with holomorphic functions on the Higgs branch of vacua $\mathcal{M}_H$, where the hypermultiplets have unconstrained expectation values and the vector multiplets must vanish.

For the 3d $\mathcal{N} = 4$ theories studied in this thesis, the Higgs and Coulomb branches $\mathcal{M}_H, \mathcal{M}_C$ are hyperkähler manifolds and are key ingredients of understanding the theory's full moduli space of vacua, which itself is a (possibly singular) union of submanifolds of the form $\mathcal{S}_H \times \mathcal{S}_C$, where $\mathcal{S}_H \subseteq \mathcal{M}_H, \mathcal{S}_C \subseteq \mathcal{M}_C$ are hyperkähler submanifolds of the Higgs and Coulomb branches. While the metrics on $\mathcal{M}_H, \mathcal{M}_C$ depend on the (dimensionful) 3d gauge couplings, their holomorphic structures are entirely independent of the gauge couplings due to one of the aforementioned non-renormalization theorems [13].

[1]In contexts where the fermionic symmetry Q is a nilpotent supercharge of a supersymmetric quantum field theory, the theory must have enough unbroken R-symmetry so that a background R-symmetry bundle can be introduced so that Q is a scalar under a combined Lorentz and R-symmetry transformation. For many local computations, *e.g.* collisions of local operators, it is possible to work with a strictly weaker notions of a topological twist where anti-commutators with Q include all translation generators without regard for the background R-symmetry bundle, *cf.* [40, 63, 64].

For $\mathcal{N} = 4$ super Yang-Mills theory with gauge group G and hypermultiplets transforming in a representation T^*R, the classical Higgs branch is simply a hyperkähler quotient:

$$\mathcal{M}_H = T^*R /\!\!/\!\!/ G := \{\vec{\mu} = 0\}/G\,, \tag{0.0.2}$$

where $\vec{\mu} : T^*R \to (\mathfrak{g}^*)^3$ is the triplet of hyperkähler moment maps for the action of G on T^*R. Famously, the Higgs branch is protected from quantum corrections and the classical result is exact [18]. The Coulomb branch, however, is unprotected and generally receives both perturbative and non-perturbative corrections. Nonetheless, holomorphic functions on these Coulomb branches can be counted using the "monopole formula" of [69]. Recently, an explicit physical description of Coulomb branches in this class of gauge theories was realized using an "abelianization" procedure [67, 70]. Several mathematically precise realizations of these Coulomb branches have been proposed [39, 71–73], and all agree with physical expectations where comparisons are possible.

The landscape of 3d $\mathcal{N} = 4$ theories admits a duality called "3d mirror symmetry" [74–76] that exchanges two (potentially different) theories and acts non-trivially on the 3d $\mathcal{N} = 4$ R-symmetry group. In fact, 3d mirror symmetry of abelian gauge theories can be interpreted as a field-space Fourier transform [77]. If two theories $\mathcal{T}$ and $\mathcal{T}^!$ are exchanged by 3d mirror symmetry, then the A-twist of $\mathcal{T}$ is equivalent to the B-twist of $\mathcal{T}'$ and vise versa, thus identifying $\mathcal{M}_C[\mathcal{T}] \simeq \mathcal{M}_H[\mathcal{T}']$ and $\mathcal{M}_H[\mathcal{T}] \simeq \mathcal{M}_C[\mathcal{T}']$, much like the more familiar mirror symmetry of 2d $\mathcal{N} = (2, 2)$ theories [78]. A large class of 3d $\mathcal{N} = 4$ theories, and supersymmetric defects admitted by them, can be described via brane constructions in Type IIB superstring theory [79, 80]. In this context, 3d mirror symmetry is realized geometrically as S-duality of the corresponding brane configuration.

The 3d TQFT structure admitted by the A-twist (or B-twist) implies that correlation functions of loop operators (compatible with the twist) only depends on the 1-dimensional support of the operator up to smooth deformations. Given a knot K in $\mathbb{R}^3$, the expectation value of a line operator $\mathcal{L}$ with support K is a topological invariant of K "colored by $\mathcal{L}$." In fact, the TQFT gives us topological invariants of (framed) knots and links, with a choice of coloring by line operators $\mathcal{L}_i$ for each connected component, in general (framed) 3-manifolds, much like the more familiar case of Chern-Simons theory and Witten-Reshetikhin-Turaev (WRT) invariants [41, 81, 82]. Line operators play a central role in Chern-Simons theory and interpreting WRT-invariants, and much of their essential physics can be encapsulated using the notion of a category, *cf.* categories of boundary conditions in 2d TQFTs [83].[2] In particular, BPS line operators compatible with the A- and B-twists will realize categories $\mathcal{C}_A$ and $\mathcal{C}_B$ that control the WRT-like invariants for the corresponding 3d TQFTs. One crucial difference between the 3d TQFTs arising from A- and B-twists of 3d $\mathcal{N} = 4$ theories and Chern-Simons theories (with compact, semisimple gauge group) is that the tensor categories arising in the former are not necessarily semisimple[3] and result in "non-semisimple TQFTs" [87]. See, *e.g.*, [88] for a recent discussion on Rozansky-Witten theory and the upcoming paper [59] for a class of 3d $\mathcal{N} = 4$ theories realizing a mathematical construction of non-semisimple TQFTs based on unrolled quantum groups [89, 90].

In this thesis, we approach the problem of identifying homological link invariants, *i.e.* invariants valued in graded vector spaces realized as the (co)homology of some chain complex, in 3d $\mathcal{N} = 4$ gauge theory from a somewhat different perspective, described in detail in the upcoming paper [91]. Of particular interest is the polynomial invariant developed inde-

[2] A category is a collection of "objects" $\{X, Y, ...\}$ together with the data of "morphisms" (or "1-morphisms") $\{f : X \to Y\}$ between any two objects X, Y and a rule for composing morphisms $f : X \to Y, g : Y \to Z$ to get other morphisms $gf : X \to Z$. More generally, a higher category is the data of a category together with the data of "2-morphisms" between any two morphisms $f : X \to Y, f' : X \to Y$ as well as composition thereof, and "3-morphisms" between any two 2-morphisms, and so on; it is called a k-category of this process stops at k-morphisms, so a 1-category is simply a category.

Local and extended operators in a general TQFT admit a concise description using higher categories. k-dimensional extended operators in a d-dimensional TQFT furnish a k-category whose 1-morphisms are $k-1$-dimensional interfaces between k-dimensional extended operators. The composition of 1-morphisms is induced by colliding $k-1$-dimensional interfaces. Similarly, the 2-morphisms are the possible $k-2$-dimensional interfaces between the $k-1$-dimensional interfaces, and so on. Concretely, dimension 1 extended operators, *i.e.* line operators, furnish a 1-category (an honest category) where a choice of object corresponds to a choice of line operator and a morphism between two line operators corresponds to a local operator that can interpolate between them. The higher categories of extended operators often possess additional structures, such as the collision and braiding of line operators, see, *e.g.*, [84, 85] for more details.

[3] As mentioned above, the categories $\mathcal{C}_A$ and $\mathcal{C}_B$ admit a description in terms of modules for the boundary VOAs of [52], in complete analogy with the Chern-Simons/WZW correspondence introduced in [41]. When the Chern-Simons gauge group is a compact, semisimple Lie group, the corresponding WZW model is a rational conformal field theory (CFT) [86]. On the other hand, the VOAs appearing in [52] are logarithmic CFTs.

pendently by Hoste, Lickorish-Millett-Ocneanu, Freyd-Yetter, and Przytycki-Traczyk [92,93], often called the HOMFLY-PT polynomial, and categorifications thereof. Recent work of Oblomkov-Rozansky [94, 95] constructed such a categorification and their construction can be interpreted in terms of 3d TQFT [96,97] as a certain supersymmetric Hilbert space in the B-twist of a 3d $\mathcal{N} = 4$ gauge theory: if the link K arises as the closure of an n strand braid $K = \overline{\beta}, \beta \in \mathrm{Br}_n$, one must consider the B-twist of the 3d $\mathcal{N} = 4$ rank n Atiyah-Drinfeld-Hitchen-Manin (ADHM) quiver gauge theory [98]. This theory is famously self-mirror [74,75] with both the Higgs and Coulomb branches identified with the moduli space of n abelian instantons on $\mathbb{C}^2$, mathematically realized as the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^2)$.

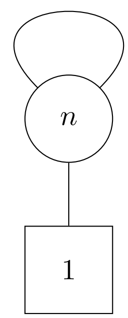


Figure 1: The rank n ADHM quiver. The corresponding 3d $\mathcal{N} = 4$ gauge theory has gauge group $U(n)$ coupled to a single fundamental hypermultiplet and a single adjoint hypermultiplet.

By passing the various ingredients used in the physical realization of the Oblomkov-Rozansky construction through 3d mirror, we arrive at yet another physical realization of HOMFLY-PT homology in the *same* 3d $\mathcal{N} = 4$ gauge theory, but now in the A-twist. For a special class of links, called positive algebraic links, the proposed A-twist construction admits an explicit algebraic realization in terms of generalized affine Springer theory [50, 51] and, in particular, realizes another (conjectural) description of HOMFLY-PT homology for the same class of links due to Oblomkov-Rasmussen-Shende [99] using the algebraic geometry of plane curve singularities. The work of Gorsky-Oblomkov-Rasmussen-Shende [100] shows that positive torus knots admit a compatible description in terms of the representation theory of rational Cherednik algebras; this latter description is nicely captured by the A-twisted gauge theory.

We now outline the structure of this thesis.

In Chapter 1, we review aspects of 3d $\mathcal{N} = 4$ super Yang-Mills theories coupled to hypermultiplets, focusing on their Coulomb branches and the topological A-twist. In Section 1.1 we describe the two types of supermultiplets appearing in these theories, namely, vector multiplets and hypermultiplets, and their variations under supersymmetry. In Section 1.2 we cover general expectations of the Coulomb branches of these theories as well as the description of the Coulomb-branch chiral ring via abelianization [67]. Section 1.3 discusses the various twists admitted by 3d $\mathcal{N} = 4$ and how an analysis of the A-twist naturally leads to the mathematical construction of the Coulomb-branch chiral ring due to [71, 72]. Finally, in Section 1.4, we consider two related examples.

The first is an example of a quiver gauge theory based off of the $(2, 3)$ star quiver, *cf.* [101]. This theory is 3d mirror to a circle reduction of the 4d $\mathcal{N} = 2$ theory of class $\mathcal{S}$ of type A_1 for the 3-punctured sphere $\Sigma_{0,3}$, also called the trinion theory T_2 [102–109]. In particular, the Coulomb branch of the $(2, 3)$ star quiver should reproduce the Higgs branch of the corresponding theory of class $\mathcal{S}$. The theory T_2 is a theory of 8 half-hypermultiplets with a Higgs branch $\mathbb{C}^8$; this remarkably simple Higgs branch appears in a highly non-trivial way from the perspective of the $(2, 3)$ star quiver.

The second example is the rank $n = 2$ ADHM quiver gauge theory. This example serves to familiarize the reader with the (quantized) Coulomb-branch chiral rings that appear later in this thesis, in particular the higher rank ADHM quivers, and moreover realizes the 3d mirror of a circle reduction of the Class $\mathcal{S}$ theory of type A_1 associated to the 1-punctured torus $\Sigma_{1,1}$. The latter description implies that the (quantized) Coulomb branch of the rank 2 ADHM quiver theory can be obtained from the (quantized) Coulomb branch of the $(2, 3)$ star quiver by means of a (quantum) symplectic reduction,[4] which we check explicitly.

[4] Theories of class $\mathcal{S}$ behave topologically with respect to cutting/gluing the underlying surface. For example, consider gluing the surface $\Sigma_{g,\mathbf{k}}$ and $\Sigma_{g',\mathbf{k}'}$ at a puncture to obtain the surface $\Sigma_{g+g',\mathbf{k}+\mathbf{k}'-2}$. Each puncture on the surface $\Sigma_{g,\mathbf{k}}$ corresponds to a factor in the flavor symmetry group of the corresponding theory of class $\mathcal{S}$ and the gluing of surfaces at punctures corresponds to gauging the corresponding flavor symmetry. At the level of Higgs branches, this implies that the Higgs branch associated to the surface $\Sigma_{g+g',\mathbf{k}+\mathbf{k}'-2}$ can be obtained

Chapter 2 considers $\frac{1}{2}$-BPS line operators admitted by the above $\mathcal{N} = 4$ super Yang-Mills theories that are compatible with the topological A-twist. In Section 2.1, we describe several abstract manipulations with line operators in a 3d TQFT, paying particular attention to the role played by boundary conditions in understanding the category of line operators. Then, in Section 2.2, we introduce a large collection of $\frac{1}{2}$-BPS vortex-line operators compatible with the A-twist and propose a concrete, geometric category that models the corresponding category of line operators $\mathcal{C}_A$. Section 2.3 develops explicit tools for performing computations in the proposed category of line operators $\mathcal{C}_A$ and shows how certain Dirichlet boundary conditions naturally lead to constructions in generalized affine Springer theory. Finally, Section 2.4 discusses a series of specific examples of $\frac{1}{2}$-BPS vortex-line operators.

The first examples describe a general phenomenon where $\frac{1}{2}$-BPS vortex-line operators that are defined only in terms of breaking gauge symmetry (and not allowing singular behavior of the hypermultiplets) are somewhat trivial. We show this in a concrete example for the rank 2 ADHM quiver theory, and illustrate how the category of line operators is represented for a specific choice of Dirichlet boundary condition.

The last example considers a different vortex-line operator in the rank 2 ADHM quiver gauge theory with the same pattern of gauge symmetry breaking, but allows the hypermultiplets to have a controlled singular behavior. We represent this line operator using the same Dirichlet boundary condition and compare the algebra of local operators to known results.

Chapter 3 is dedicated to describing two incarnations of HOMFLY-PT homology from the perspective of 3d $\mathcal{N} = 4$ gauge theory. Section 3.1 introduces the HOMFLY-PT invariant as well as the Oblomkov-Rozansky construction [94, 95] of a categorification thereof. We also describe the (conjectural) descriptions of HOMFLY-PT homology for (positive) algebraic links due to Oblomkov-Rasmussen-Shende [99] and for (positive) torus knots due to Gorsky-Oblomkov-Rasmussen-Shende [100]. In Section 3.2, we translate the Oblomkov-Rozansky construction through 3d mirror symmetry and show how the corresponding mirror construc-

by symplectic reduction of the product of Higgs branches for $\Sigma_{g,\mathbf{k}}$ and $\Sigma_{g',\mathbf{k}'}$. The compatibility between Higgs branches of theories of class $\mathcal{S}$, which are themselves holomorphic-symplectic variety, and the cutting/gluing of the surface realizes a "2d TQFT valued in holomorphic symplectic varieties" of [110].

tion reproduces the Oblomkov-Rasmussen-Shende construction for (positive) algebraic links, and speculate on how more general links can be realized. Finally, in Section 3.3 we turn our focus to (positive) torus knots and describe the physics of the Gorsky-Oblomkov-Rasmussen-Shende construction of their HOMFLY-PT homology.

Much of the content of this thesis is adapted from the papers [51, 101, 111] and from the upcoming works [91, 112]. Section 1.1 and Section 1.3 are adapted from [111]. Section 1.2 is adapted from [101], as is the example provided in Section 1.4.1. The example appearing in Section 1.4.2 did not appear in [101], but has a well known Coulomb branch realizing the Hilbert scheme of 2 points on $\mathbb{C}^2$ and is quantized by the rational Cherednik algebra for $\mathfrak{gl}(2, \mathbb{C})$ [113]. Its realization as the (quantum) symplectic reduction the (quantized) Coulomb branch of the (2,3) star quiver theory is expected from their relations to theories of class $\mathcal{S}$ [105, 110] and is described mathematically in [114, 115].

Section 2.1 is adapted from the upcoming work [112]. Section 2.2 and Section 2.3 are adapted from [111]. The discussion of Dirichlet boundary conditions and their connections to generalized affine Springer theory in Section 2.3.4 are new and based off of the mathematical works [50, 51] and the earlier physical work [40]. The examples in Sections 2.4.1 and 2.4.2 are conceptually similar to the examples provided in [111] but consider different theories and use Dirichlet boundary conditions, as opposed to the vacuum boundary conditions used in [111].

Section 3.1, Section 3.2.1, Section 3.2.3, and Section 3.3.2 are adapted from the upcoming work [91]. Section 3.2.2 and Section 3.3.1 are adapted from [51].

Chapter 1

Coulomb Branches and the Topological A-twist

Three-dimensional quantum field theories with $\mathcal{N} = 4$ supersymmetry, in particular those built from gauging flavor symmetries of hypermultiplets with $\mathcal{N} = 4$ vector multiplets, often possess extremely rich moduli spaces of vacua. The full moduli space of vacua is a (possibly singular) union of "branches" where gauge symmetry is partially broken and scalars from both types of multiplets gain non-trivial vacuum expectation values. $\mathcal{N} = 4$ supersymmetry places strict requirements on these branches and, in particular, ensures that each is a hyperkähler manifold [116]. There are two distinguished branches $\mathcal{M}_H$ and $\mathcal{M}_C$ of the full moduli space of vacua $\mathcal{M}$, called the "Higgs" and "Coulomb" branches, respectively. In the absence of mass and FI deformations, each branch of the full moduli space takes the form $\mathcal{S}_H \times \mathcal{S}_C$, where $\mathcal{S}_H$ is a hyperkähler submanifold of $\mathcal{M}_H$, and similarly for $\mathcal{S}_C$.

The Higgs branch $\mathcal{M}_H$ is the subspace of vacua where the vector multiplet scalars vanish and is parameterized by the expectation values of the hypermultiplet scalars (up to gauge transformations). Classically, the Higgs branch of super Yang-Mills with gauge group G and hypermultiplet scalars transforming in a quaternionic representation $\mathcal{R}$ of G is the hyperkähler quotient

$$\mathcal{M}_H = \mathcal{R}/\!/\!/G = \{\vec{\mu} = 0\}/G\,, \tag{1.0.1}$$

where $\vec{\mu} = (\mu_1, \mu_2, \mu_3)$ is the triplet of hyperkähler moment maps. The Higgs branch is protected from both perturbative and non-perturbative quantum corrections [18], so this is also the quantum moduli space.

The Coulomb branch $\mathcal{M}_C$ is the subspace of vacua where the hypermultiplet scalars vanish, and is not protected from quantum corrections. Classically, at a generic point of the Coulomb branch, the vector multiplet scalars take values in a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ of the gauge Lie algebra and break the gauge group to the corresponding maximal torus $T \subset G$. In $\mathbb{R}^3$, the 2-form field strength $F = dA$ of an abelian gauge field A can equivalently be thought of as a 1-form field strength f with 0-form (scalar) gauge field γ called the "dual photon." The fact that F is the field strength of a gauge field for a compact group T, the scalar γ must be periodic. Put together, the classical Coulomb branch is parameterized by the expectation values of the triplet of vector multiplet scalars $\vec{\phi}$ and the dual photon γ:

$$\mathcal{M}_C^{\text{classical}} \overset{\text{generically}}{\sim} (\mathbb{R}^3 \times S^1)^{\text{rank}(G)}/\text{Weyl}(G)\,, \tag{1.0.2}$$

where $\text{Weyl}(G)$ is the Weyl group of G.

The quantum-corrected Coulomb branch $\mathcal{M}_C$ receives both perturbative and non-perturbative corrections. Upon choosing a complex structure, $\mathcal{M}_C$ can be described as a complex integrable system over the vector space $\mathfrak{t}_{\mathbb{C}}/\text{Weyl}(G)$ parameterizing the expectation values of (gauge-invariant) polynomials in the complex scalar φ (determined by the choice of complex structure), *cf.* the Seiberg-Witten integrable system from 4d $\mathcal{N} = 2$ theories [117–119]. The fibers of this map over a generic point on the base $\mathfrak{t}_{\mathbb{C}}/\text{Weyl}(G)$ are dual complex tori $T_{\mathbb{C}}^{\vee}$ parameterized by expectation values of BPS monopole operators, which are inherently non-perturbative objects. Initial studies of the quantum moduli space was restricted to abelian theories or simple non-abelian theories, where these Coulomb branches were related to Higgs branches of a (potentially) different theory via the phenomenon of "3d mirror symmetry" [74–77] and in some instances moduli spaces of monopoles and instantons [79, 120].

Even though the full hyperkähler geometry can be quite intricate and depends on the (dimensionful) gauge couplings of the theory, the complex-symplectic geometry is independent of these parameters due to a holomorphy argument similar to [13]; a 3d gauge coupling has no natural complexification and hence can never appear in an effective superpotential or chiral ring. As a result, it is possible to explicitly enumerate the holomorphic functions on these Coulomb branches [69], *i.e.* the Coulomb-branch chiral ring, which inspired an explicit description of Coulomb branches for a wide range of non-abelian gauge theories [67, 70] using an "abelianization" procedure. Several mathematical incarnations of these Coulomb branches have been proposed [39, 71–73], and all agree with physical expectations where comparisons are possible.

Higgs and Coulomb branches also admit interpretations in terms of 3d TQFT by means of topologically twisting [62] the underlying physical theory. As mentioned in the introduction, the 3d $\mathcal{N} = 4$ supersymmetry algebra admits two (families of) distinct topological super charges Q_A and Q_B [121, 122]; we call the corresponding topological twists the "A-twist" and the "B-twist," respectively. Moreover, the algebra of local operators in the A-twist (resp. B-twist) can be identified with the Coulomb-branch (resp. Higgs-branch) chiral ring, *i.e.* with holomorphic functions on the Coulomb branch (resp. Higgs branch). This TQFT perspective explains many features of these moduli spaces, *e.g.* the independence of the chiral rings on the gauge couplings, and will serve as an organizing principle for later chapters.

The organization of the present chapter is as follows. First, Section 1.1 reviews the basic ingredients present in 3d $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theories, including the elementary supersymmetry multiplets mentioned above and how the 3d $\mathcal{N} = 4$ supersymmetry algebra is realized on them. Then, Section 1.2 reviews the physical aspects of the Coulomb branches of these supersymmetric Yang-Mills theories in more detail and their explicit realization in terms of the abelianization procedure of [67, 70]. Section 1.3 reviews the notion of topological twists and connects the physical analysis of Coulomb branches described in Section 1.2 to the topological A-twist and to recent the mathematical construction of Coulomb branches due to Braverman-Finkelberg-Nakajima (BFN) [71,72]. Finally, Section 1.4 is dedicated to two examples of quiver gauge theories that appear as 3d mirrors of (circle reductions of) theories of class $\mathcal{S}$ for A_1 [102,123,124]: in Section 1.4.1 we consider the 3-legged rank 2 star quiver, *cf.* [101], corresponding to the 3-punctured sphere $\Sigma_{0,3}$, and in Section 1.4.2 we consider the rank 2 ADHM quiver, corresponding to the 1-punctured torus $\Sigma_{1,1}$.

Section 1.1 and Section 1.3 are adapted from [111]. Section 1.2 is adapted from [101], as is the example provided in Section 1.4.1. The example appearing in Section 1.4.2 did not appear in [101], but has a well known Coulomb branch realizing the Hilbert scheme of 2 points on $\mathbb{C}^2$ and is quantized by the rational Cherednik algebra for $\mathfrak{gl}(2,\mathbb{C})$ [113]. Its realization as the quantum symplectic reduction the Coulomb branch of the 3-legged rank 2 star quiver theory is expected from their relations to theories of class $\mathcal{S}$ [105,110] and is described mathematically in [114,115].

1.1 Review of 3d $\mathcal{N}=4$

In this section we review various structural aspects of 3d $\mathcal{N}=4$ supersymmetry. We start with a review of the $\mathcal{N}=4$ supersymmetry algebra in Section 1.1.1. In Section 1.1.2 and Section 1.1.3 we introduce two basic multiplets called vector multiplets and hypermultiplets, respectively, and the action of $\mathcal{N}=4$ supersymmetry on them. There are two other basic types of multiplets called twisted vector multiplets and twisted hypermultiplets, as well as theories with $\mathcal{N}\geq 4$ supersymmetry that include Chern-Simons fields, *e.g.* [57,125–127], but we will not consider them in the following.

Actions that are invariant under these transformations can be quite unwieldy, but can be obtained, *e.g.*, by dimensional reduction of 4d $\mathcal{N}=2$ theories [128]. In much of the following it is sufficient to know the supersymmetry transformations themselves without mention of the action. Strictly speaking, however, the supersymmetry transformations necessarily act on-shell, so one needs the action to check that the putative supersymmetry transformations satisfy the necessary relations.

1.1.1 Supersymmetry algebra

The 3d $\mathcal{N}=4$ supersymmetry algebra is generated by 8 supercharges $Q_\alpha^{a\dot{a}}$ and is of the form

$$\{Q_\alpha^{a\dot{a}}, Q_\beta^{b\dot{b}}\} = \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}P_{\alpha\beta} - i\epsilon_{\alpha\beta}\left(\epsilon^{ab}m^{\dot{a}\dot{b}} + \epsilon^{\dot{a}\dot{b}}t^{ab}\right). \tag{1.1.1}$$

Here $\alpha \in \{+,-\}$ are spinor indices for the Euclidean spin group $SU(2)_E$. Upper indices transform in the fundamental $\square$ representation of $SU(2)_E$ and lower indices in the anti-fundamental $\overline{\square}$. The isomorphism between the fundamental and anti-fundamental representations of $SU(2)$ is implemented by the epsilon tensor and its inverse

$$X^\alpha = \epsilon^{\alpha\beta}X_\beta\,, \qquad\qquad X_\alpha = \epsilon_{\alpha\beta}X^\beta\,, \tag{1.1.2}$$

with

$$\epsilon^{+-} = \epsilon_{-+} = 1. \tag{1.1.3}$$

Lower-case Latin indices $a, \dot{a}$ transform under the $SU(2)_H$ and $SU(2)_C$ R-symmetries, respectively, and have the same conventions as Euclidean spinor indices. The mass and FI parameters $m^{\dot{a}\dot{b}}$ and t^{ab} are central charges in the symmetric tensor representation $\mathrm{Sym}^2(\square)$ of $SU(2)_C$ and $SU(2)_H$, respectively,

$$m^{\dot{a}\dot{b}} = m^{(\dot{a}\dot{b})}, \qquad\qquad t^{ab} = t^{(ab)}. \tag{1.1.4}$$

In the transformation laws for fundamental fields presented below they will be realized by the action of some $m^{\dot{a}\dot{b}} \in \mathfrak{f}$, $t^{ab} \in \mathfrak{f}_t$, where $\mathfrak{f}$ is a Cartan subalgebra of the group of global symmetries acting on hypermultiplets and $\mathfrak{f}_t$ is the algebra of topological symmetries.

The isomorphism between the symmetric tensor representation of $SU(2)_E$ and the adjoint representation (a spacetime vector) is implemented by the sigma matrices

$$\sigma_\mu^{\alpha\beta} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\} . \tag{1.1.5}$$

Here $\mu \in 1,2,3$ indexes a basis for the adjoint representation. For $SU(2)_C$ and $SU(2)_H$, the isomorphism is implemented by identical sigma matrices that we denote $\sigma_{\dot I}^{\dot a\dot b}$ and σ_I^{ab}, respectively. Lowering indices, we also have

$$\begin{aligned} (\sigma^\mu)^\alpha{}_\beta &= \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \\ \sigma^\mu_{\alpha\beta} &= \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}. \end{aligned} \tag{1.1.6}$$

The traceless Hermitian matrices $(\sigma^\mu)^\alpha{}_\beta$ are the usual Pauli matrices. In this form they will often simply be denoted 'σ^μ' in matrix notation, and they satisfy the algebra

$$\sigma^\mu\sigma^\nu = \delta^{\mu\nu}\mathbb{1} + i\epsilon^{\mu\nu\lambda}\sigma_\lambda \, . \tag{1.1.7}$$

Adjoint $SU(2)$ indices are lowered and raised with the metric $\delta_{\mu\nu}$ (similarly: δ_{IJ}, $\delta_{\dot I\dot J}$); and the totally antisymmetric tensor is denoted by $\epsilon^{\mu\nu\lambda}$ where

$$\epsilon^{123} = \epsilon_{123} = 1 \, . \tag{1.1.8}$$

Some useful identities for manipulating sigma matrices in these conventions are

$$\begin{gathered} [\sigma^\mu, \sigma^\nu] = 2i\epsilon^{\mu\nu\lambda}\sigma_\lambda \, , \qquad \mathrm{Tr}(\sigma^\mu\sigma^\nu) = 2\delta^{\mu\nu} \, , \qquad \mathrm{Tr}(\sigma^\mu\sigma^\nu\sigma^\lambda) = 2i\epsilon^{\mu\nu\lambda} \, , \\ (\sigma^\mu)^\alpha{}_\beta(\sigma_\mu)^\gamma{}_\delta = 2\delta^\alpha{}_\delta\delta_\beta{}^\gamma - \delta^\alpha{}_\beta\delta^\gamma{}_\delta \, , \qquad (\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} = 2\epsilon_{\beta(\gamma}\epsilon_{\delta)\alpha} \, . \end{gathered} \tag{1.1.9}$$

We will often use the isomorphism σ implicitly, writing vectors as bi-spinors and vice versa. Given any (co)vector v_μ we set

$$v_{\alpha\beta} := \sigma^\mu_{\alpha\beta} v_\mu \,, \qquad \text{or} \qquad v_\mu := -\frac{1}{2}\sigma^{\alpha\beta}_\mu v_{\alpha\beta} \,. \tag{1.1.10}$$

For instance, the momentum operator $P_\mu = -i\partial_\mu$ as a bi-spinor is

$$P_{\alpha\beta} = \begin{pmatrix} -2P_{\overline{z}} & P_t \\ P_t & 2P_z \end{pmatrix}, \qquad \partial_{\alpha\beta} = \begin{pmatrix} -2\partial_{\overline{z}} & \partial_t \\ \partial_t & 2\partial_z \end{pmatrix}. \tag{1.1.11}$$

Similarly, letting $m^{\dot{I}}, t^I$ denote the mass/FI parameters in the adjoint representations of $SU(2)_C$ and $SU(2)_H$, respectively, and defining real and complex combinations as

$$\begin{aligned} m_{\mathbb{C}} &= \tfrac{1}{2}(m_1 - im_2)\,, & m_{\mathbb{R}} &= -m_3\,, \\ t_{\mathbb{C}} &= \tfrac{1}{2}(t_1 - it_2)\,, & t_{\mathbb{R}} &= -t_3\,, \end{aligned} \tag{1.1.12}$$

we find that

$$m^{\dot{a}\dot{b}} = \begin{pmatrix} 2m_{\mathbb{C}} & m_{\mathbb{R}} \\ m_{\mathbb{R}} & -2\overline{m}_{\mathbb{C}} \end{pmatrix}, \qquad t^{ab} = \begin{pmatrix} 2t_{\mathbb{C}} & t_{\mathbb{R}} \\ t_{\mathbb{R}} & -2\bar{t}_{\mathbb{C}} \end{pmatrix}. \tag{1.1.13}$$

In terms of $P_t, P_z, P_{\overline{z}}$, the SUSY algebra takes the form

$$\begin{gathered} \{Q^{a\dot{a}}_+, Q^{b\dot{b}}_+\} = -2\epsilon^{ab}\epsilon^{\dot{a}\dot{b}} P_{\overline{z}}\,, \qquad \{Q^{a\dot{a}}_-, Q^{b\dot{b}}_-\} = 2\epsilon^{ab}\epsilon^{\dot{a}\dot{b}} P_z\,, \\ \{Q^{a\dot{a}}_+, Q^{b\dot{b}}_-\} = \epsilon^{ab}\epsilon^{\dot{a}\dot{b}} P_t - i\epsilon^{ab} m^{\dot{a}\dot{b}} - i\epsilon^{\dot{a}\dot{b}} t^{ab}\,. \end{gathered} \tag{1.1.14}$$

1.1.2 Vector multiplets

An off-shell 3d $\mathcal{N} = 4$ vector multiplet consists of the fields

$$A_\mu\,, \qquad \phi^{\dot{a}\dot{b}}\,, \qquad \lambda^{a\dot{a}}_\alpha\,, \qquad D^{ab}\,. \tag{1.1.15}$$

Here A_μ is a connection 1-form; $\phi^{(\dot a\dot b)}$ is a scalar field in the adjoint representation of the $SU(2)_C$ R-symmetry, with real component σ and complex component φ

$$\phi^{\dot a\dot b} = \begin{pmatrix} 2\varphi & \sigma \\ \sigma & -2\overline{\varphi} \end{pmatrix}; \tag{1.1.16}$$

$\lambda_\alpha^{a\dot a}$ is a complex gaugino in the bi-fundamental of $SU(2)_H \times U(2)_C$; and $D^{(ab)}$ is an auxiliary field in the adjoint of $SU(2)_H$. For gauge group G, all the fields in (1.1.15) transform additionally in the Lie algebra $\mathfrak{g}$ (or the complexified lie algebra $\mathfrak{g}_{\mathbb{C}}$, in the case of φ and the gauginos λ).

We will work with "physics conventions," in which the real Lie algebra $\mathfrak{g}$ is generated by *Hermitian* matrices. This has the advantage that "real" masses $m_{\mathbb{R}}$ and FI parameters $t_{\mathbb{R}}$ will actually take real values. It has a familiar disadvantage that an extra factor of i appears in Lie algebra structure constants: $[T^a, T^b] = i f^{ab}{}_c T^c$, and in covariant derivatives. The G-covariant derivative takes the form

$$d_A = d - iA\,, \tag{1.1.17}$$

and the field strength is

$$F = i[d_A, d_A] = dA - iA \wedge A\,. \tag{1.1.18}$$

In three dimensions the field strength may be dualized to a vector $(*F)_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda}F^{\nu\lambda}$, or a traceless Hermitian bispinor

$$F^\alpha{}_\beta = 2(\sigma^\mu)^\alpha{}_\beta(\star F)_\mu = -i(\sigma^{\mu\nu})^\alpha{}_\beta F_{\mu\nu}\,, \tag{1.1.19}$$

where, in matrix notation, $\sigma^{\mu\nu} := \frac{1}{2}[\sigma^\mu, \sigma^\nu] = i\epsilon^{\mu\nu\lambda}\sigma_\lambda$. Explicitly,

$$F^\alpha{}_\beta = 4i \begin{pmatrix} -F_{z\bar{z}} & F_{zt} \\ -F_{\bar{z}t} & F_{z\bar{z}} \end{pmatrix}, \qquad \text{or} \qquad F_{\alpha\beta} = 4i \begin{pmatrix} F_{\bar{z}t} & -F_{z\bar{z}} \\ -F_{z\bar{z}} & F_{zt} \end{pmatrix}. \tag{1.1.20}$$

Note that $F_{\alpha\beta}$ is symmetric. The Bianchi identity $d_A F = 0$ then reads

$$(d_A)_{\gamma[\alpha} F^\gamma{}_{\beta]} = 0\,, \qquad \text{or} \qquad d_A^{\alpha\beta} F_{\alpha\beta} = 0\,. \tag{1.1.21}$$

Finally, we can state the transformation rules for the 3d $\mathcal{N} = 4$ vector multiplet:

$$\begin{aligned} Q_\alpha^{a\dot{a}} A_{\beta\gamma} &= \lambda^{a\dot{a}}_{(\beta}\epsilon_{\gamma)\alpha}\,, \qquad\qquad Q_\alpha^{a\dot{a}}\phi^{\dot{b}\dot{c}} = i\lambda_\alpha^{a(\dot{b}}\epsilon^{\dot{c})\dot{a}}\,, \\ Q_\alpha^{a\dot{a}}\lambda_\beta^{b\dot{b}} &= \frac{1}{2}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}F_{\alpha\beta} - \epsilon^{ab}(d_A)_{\alpha\beta}\phi^{\dot{a}\dot{b}} - i\epsilon_{\alpha\beta}\epsilon^{\dot{a}\dot{b}}D^{ab} + \frac{1}{2}\epsilon_{\alpha\beta}\epsilon^{ab}[\phi^{\dot{a}}{}_{\dot{c}}, \phi^{\dot{c}\dot{b}}]\,, \\ Q_\alpha^{a\dot{a}} D^{bc} &= -(d_A)_\alpha{}^\beta\epsilon^{a(b}\lambda_\beta^{c)\dot{a}} - [\phi^{\dot{a}}{}_{\dot{b}}, \epsilon^{a(b}\lambda_\alpha^{c)\dot{b}}]\,. \end{aligned} \tag{1.1.22}$$

One may check that the algebra of supersymmetries acting on the fields satisfies

$$\{Q_\alpha^{a\dot{a}}, Q_\beta^{b\dot{b}}\} = \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}P_{\alpha\beta} - i\epsilon_{\alpha\beta}\left(\epsilon^{ab}\phi^{\dot{a}\dot{b}} + \frac{1}{g^2}\epsilon^{\dot{a}\dot{b}}D^{ab}\right). \tag{1.1.23}$$

Here $\phi^{\dot{a}\dot{b}}$ acts on fields as an infinitesimal $\mathfrak{g}$-gauge transformation. Similarly, the D-term acts as an infinitesimal topological symmetry; explicitly, it "acts" as zero on ϕ and λ, but acts as a translation of the dual photon γ, which satisfies

$$\frac{1}{2g^2}\mathrm{Tr}(F_{\alpha\beta}) = \partial_{\alpha\beta}\gamma\,. \tag{1.1.24}$$

Note that upon using the equation of motion

$$\frac{1}{g^2}D^{ab} = t^{ab} \qquad \text{(in the absence of matter)}\,, \tag{1.1.25}$$

and restricting to gauge-invariant combinations of vector multiplet fields (on which ϕ acts as zero), the algebra (1.1.23) reduces to the general form 1.1.1. Mass parameters could also be introduced, as scalars in background vector multiplets associated to a flavor symmetry; in (1.1.23) this amounts to replacing $\phi \rightsquigarrow m$.

1.1.3 Hypermultiplets

A hypermultiplet contains an $SU(2)_H$ doublet of complex scalar fields and an $SU(2)_C$ doublet of complex fermions. It's convenient to introduce an additional $SU(2)'$ spinor index $A \in \{1,2\}$, writing the scalars as X^{aA} subject to a reality condition

$$(X^{aA})^* = X_{aA}\,. \tag{1.1.26}$$

This makes manifest the full $SO(4) \simeq SU(2)_H \times SU(2)'$ symmetry of the four real scalars in the hypermultiplet. With respect to a 3d $\mathcal{N}=2$ subalgebra, the fields $X^{+1} = X$ and $X^{+2} = Y$ are chiral, whereas $X^{-2} = \overline{X}$ and $X^{-1} = -\overline{Y}$ are anti-chiral. Altogether, we have

$$X^{aA} = \begin{pmatrix} X & Y \\ -\overline{Y} & \overline{X} \end{pmatrix}. \tag{1.1.27}$$

Similarly, we write the fermions as $\psi_\alpha^{\dot{a}A}$. In Lorentzian signature they would obey a reality constraint $(\psi_\alpha^{\dot{A}})^\dagger \sim \psi_{\alpha\dot{a}A}$; but in Euclidean signature the components of $\psi_\alpha^{\dot{a}A}$ are independent, and $\overline{\psi}$ does not appear in the action or integration measure. The supersymmetry transformations for a single free hypermultiplet are simply

$$Q_\alpha^{a\dot{a}} X^{bA} = i\epsilon^{ab}\psi_\alpha^{\dot{a}A}\,, \qquad Q_\alpha^{a\dot{a}}\psi_\beta^{\dot{b}A} = \epsilon^{\dot{a}\dot{b}}\partial_{\alpha\beta}X^{aA}\,. \tag{1.1.28}$$

The $SU(2)'$ indices are raised and lowered by antisymmetric tensors Ω^{AB} and Ω_{AB}. We'll use the convention $\Omega^{12} = \Omega_{21} = 1$. The tensor Ω^{AB} (resp. Ω_{AB}) has a geometric interpretation as the holomorphic Poisson structure (resp. symplectic structure) on the "target space" $T^*\mathbb{C}$ of the theory of a free hypermultiplet.

For a collection of N hypermultiplets, the extra symmetry $SU(2)'$ is extended to $USp(N)$, and the index A takes values $A = 1, \ldots, 2N$. It is raised and lowered by the tensors

$$\Omega^{AB} = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}, \qquad \Omega_{AB} = \begin{pmatrix} 0 & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix} \tag{1.1.29}$$

(where $\mathbb{1}_N$ denotes the $N \times N$ identity matrix), which now play the role of holomorphic Poisson/symplectic tensors on $T^*\mathbb{C}^N$. The reality constraint on scalars continues to take the form (1.1.26). We will typically split the scalars into chiral halves, generalizing (1.1.27),

$$\begin{array}{ll} X^i = X^{+,i}\,, & Y_i = X^{+,N+i}\,, \\ \overline{X}_i = X^{-,N+i}\,, & \overline{Y}^i = -X^{-,i} \end{array} \qquad i = 1, ..., N\,, \tag{1.1.30}$$

with $X^i, \overline{Y}^i$ transforming in the fundamental representation of $U(N)$, and $Y_i, \overline{X}_i$ in the dual.

We may couple a collection of N hypermultiplets to a G gauge symmetry by identifying G with a subgroup of $USp(N)$. (Equivalently, we specify how the hypermultiplets transform in a unitary symplectic representation of G.) The on-shell SUSY transformations then become

$$Q_\alpha^{a\dot{a}} X^{bA} = i\epsilon^{ab}\psi_\alpha^{\dot{a}A}\,, \qquad Q_\alpha^{a\dot{a}}\psi_\beta^{\dot{b}B} = \left(\epsilon^{\dot{a}\dot{b}}(d_A)_{\alpha\beta} + \epsilon_{\alpha\beta}\phi^{\dot{a}\dot{b}}\right)\cdot X^{aB} \tag{1.1.31}$$

where d_A is the G-covariant derivative and $\phi \cdot X$ denotes an infinitesimal gauge transformation generated by ϕ in the appropriate unitary symplectic representation of G.

1.1.4 Moment maps

In a gauge theory with hypermultiplet matter, the equations of motion set the auxiliary field D^{ab} in the vector multiplet to

$$D^{ab} = \mu^{ab} + t^{ab}\,, \tag{1.1.32}$$

where

$$\mu^{ab} = \begin{pmatrix} 2\mu & \mu_{\mathbb{R}} \\ \mu_{\mathbb{R}} & -2\overline{\mu} \end{pmatrix} \tag{1.1.33}$$

is the triplet of hyperkähler moment maps. Recall that the moment maps take values in the dual of the Lie algebra, $\mu_{\mathbb{R}} \in \mathfrak{g}^*$ and $\mu \in \mathfrak{g}^*_{\mathbb{C}}$. We can describe them explicitly as follows. Let $\{(\tau_k)^A{}_B\}_{k=1}^{\mathrm{rank}\, G}$ denote a basis of generators of $\mathfrak{g}$, as elements of $\mathfrak{usp}(N)$. Then for each generator τ_k,

$$\langle \tau_k, \mu^{ab} \rangle = -X^a{}_A (\tau_k)^A{}_B X^{bB} \,. \tag{1.1.34}$$

In this thesis we will always assume that hypermultiplets transform in a representation of the form $R \oplus R^*$, where R is a unitary representation of G with $\dim_{\mathbb{C}} R = N$, and R^* its dual. In this case, G acts as a subgroup of $U(N)$, and the moment maps may similarly be interpreted as elements of $\mathfrak{u}(N)^*$ or $\mathfrak{u}(N)^*_{\mathbb{C}}$. Letting $\{(T_k)^i{}_j\}_{k=1}^{\mathrm{rank}\, G}$ denote the generators of $\mathfrak{g}$, as elements of $\mathfrak{u}(N)$, we have

$$\tau_k = \begin{pmatrix} T_k & 0 \\ 0 & -T_k \end{pmatrix} \in \mathfrak{usp}(N) \,. \tag{1.1.35}$$

The general expression (1.1.34) for the moment maps simplifies to

$$-\langle T_k, \mu \rangle = Y_i (T_k)^i{}_j X^j \,, \qquad -\langle T_k, \mu_{\mathbb{R}} \rangle = \overline{X}_i (T_k)^i{}_j X^j - Y_i (T_k)^i{}_j \overline{Y}^j \,. \tag{1.1.36}$$

For instance, if $G = U(1)$ acts on a single hypermultiplet, with charge generator

$$T = 1 \in \mathfrak{u}(1) \,, \qquad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{usp}(1) \,, \tag{1.1.37}$$

so that X, Y have charges ± 1, then the moment maps are familiar expressions

$$\mu = XY \,, \qquad \mu_{\mathbb{R}} = |X|^2 - |Y|^2 \,. \tag{1.1.38}$$

1.2 Coulomb branches

We now move to Coulomb branches of the 3d $\mathcal{N} = 4$ gauge theories discussed in Section 1.1. The Coulomb branches of these 3d $\mathcal{N} = 4$ gauge theories have long been an object of physical and mathematical interest. Early physical studies [129, 130] led to the discovery of 3d mirror symmetry [74–76], and related the Coulomb branch of ADE quiver gauge theories to moduli spaces of monopoles and instantons [79, 120]. Unfortunately, non-perturbative corrections make the Coulomb branch difficult to analyze directly in non-abelian gauge theory. (Calculations of instanton corrections in simple non-abelian theories were carried out in *e.g.* [131, 132], but quickly became impractical.) This difficulty was recently circumvented in a surprising confluence of physical [67, 69, 70, 124] and mathematical [39, 71–73, 133] work, based on ideas from algebra, representation theory, and topological quantum field theory.

General properties of any Coulomb branch are discussed in Section 1.2.1. In Section 1.2.2 we introduce the abelianized Coulomb branch of [67]. This abelianized Coulomb branch is then related to the honest Coulomb branch $\mathcal{M}_C$ using recent ideas of Webster [39] in Section 1.2.3. Finally, in Section 1.2.4 we mention how flavor symmetries and R-symmetries are realized on Coulomb branches.

1.2.1 Generalities

Recall that the Coulomb branch of a 3d $\mathcal{N} = 4$ gauge theory is a component of the moduli space of vacua on which all hypermultiplet VEVs vanish, and on which vector multiplet scalars generically acquire diagonal VEVs, breaking the gauge symmetry G to its maximal torus T. The Coulomb branch is a noncompact hyperkähler manifold [116, 130], possibly singular, of dimension

$$\dim_{\mathbb{C}} \mathcal{M}_C = 2\,\mathrm{rank}(G)\,. \tag{1.2.1}$$

In a 3d gauge theory, the Coulomb branch has an exact $SU(2)_C$ metric isometry that rotates its $\mathbb{CP}^1$ of complex structures. Thus it essentially looks *the same* in every complex structure. This $SU(2)_C$ shows up classically as a rotation of the triplet of $\mathfrak{g}$-valued scalar fields in the vector multiplet.

In any fixed complex structure, the Coulomb branch is a holomorphic symplectic manifold, *i.e.* a Kähler manifold, possibly singular, whose smooth part is endowed with a non-degenerate holomorphic two-form Ω. For every choice of complex structure, there is a chiral ring of $\frac{1}{2}$-BPS local operators whose VEVs are holomorphic functions on the Coulomb branch. We simply denote this ring

$$\mathbb{C}[\mathcal{M}_C]\,, \tag{1.2.2}$$

suppressing the dependence on complex structure. The holomorphic symplectic form Ω endows the chiral ring with a Poisson bracket, thus turning $\mathbb{C}[\mathcal{M}_C]$ into a Poisson algebra. Physically, the Poisson bracket of operators may be computed by topological descent [85].

Fibration: scalars and monopoles

In a fixed complex structure, the Coulomb branch moreover has the structure a complex integrable system.[1] Specifically, the Coulomb branch is a singular fibration

$$\begin{array}{rl} T^\vee_{\mathbb{C}} \dashrightarrow & \mathcal{M}_C \\ & \downarrow \pi \\ & \mathfrak{t}_{\mathbb{C}}/W\,, \end{array} \tag{1.2.3}$$

where $\mathfrak{t}_{\mathbb{C}}$ denotes the complexified Cartan subalgebra of G, W the Weyl group, and $T^\vee_{\mathbb{C}}$ the complexified dual of the maximal torus. Roughly speaking, the base $\mathfrak{t}_{\mathbb{C}}/W \simeq \mathbb{C}^{\mathrm{rank}(G)}$ is parameterized by the 'diagonal' expectation value of a complex vector multiplet scalar

$$\varphi \in \mathfrak{t}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}\,. \tag{1.2.4}$$

The complex scalar φ combines two of the three real vector multiplet scalars, as dictated by the choice of complex structure. (In Eq. (1.1.16), we implicitly chose a complex structure

[1] This integrable system is a degeneration of the Seiberg-Witten integrable system familiar from 4d $\mathcal{N}=2$ gauge theory [117–119].

with $\varphi = \phi^{(\dot{+}\dot{+})} = \frac{1}{2}(\phi_1 - i\phi_2)$ and $\sigma = -\phi_3$.) Classically, it is forced to take a diagonal VEV due to vacuum equations $[\varphi, \varphi^\dagger] = 0$. Global coordinates on the base come from G-invariant polynomials (Casimir operators) in φ, which are the true gauge-invariant operators in a non-abelian theory.

We call a point φ on the base *generic* if 1) it breaks gauge symmetry down to a maximal abelian subgroup $T \cong U(1)^{\text{rank}G}$ (making all W-bosons massive) and 2) gives a nonzero effective mass to every hypermultiplet. Algebraically, these criteria mean that, respectively

$$\begin{array}{c} M_\alpha := \langle \alpha, \varphi \rangle \neq 0 \\ \forall\, \alpha \in \text{roots}(G) \end{array} \quad \text{and} \quad \begin{array}{c} M_\lambda := \langle \lambda, \varphi \rangle \neq 0 \\ \forall\, \lambda \in \text{weights}(R) \end{array} \,. \tag{1.2.5}$$

Mathematically, one would say that a generic point of $\mathfrak{t}_{\mathbb{C}}/W$ is in the complement of all weight and root hyperplanes.

The fiber of the integrable system (1.2.3) above any generic point of the base is a complex dual torus $T_{\mathbb{C}}^\vee \simeq (\mathbb{C}^*)^{\text{rank}(G)}$. It is a holomorphic Lagrangian torus with respect to the holomorphic symplectic structure. The coordinates on the fibers are VEVs of chiral monopole operators. Locally, near a generic point on the base where G is broken to T, one may define $\frac{1}{2}$-BPS abelian monopole operators as (*cf.* [130, 134, 135])

$$v_A \sim e^{\frac{1}{g^2}(A, \sigma + i\gamma)} \tag{1.2.6}$$

where g is the gauge coupling, $A \in \mathfrak{t}$ is a cocharacter (satisfying $e^{2\pi i A} = I$), $\sigma \in \mathfrak{t}$ is the third real vector multiplet scalar, $\gamma \in \mathfrak{t}$ are the dual photons (with periodicity $2\pi g^2$), and $(\ ,\ )$ is the Cartan-Killing form. The OPE of monopole operators satisfies $v_A v_B \sim v_{A+B}$, for any cocharacters A and B, so their VEVs are just right to produce global functions on $T_{\mathbb{C}}^\vee \simeq (\mathbb{C}^*)^{\text{rank}(G)}$.

The way that the $T_{\mathbb{C}}^\vee$ fibers vary over the base of the Coulomb branch depends qualitatively on locations of the root and weight hyperplanes. Roughly speaking,

- The fibers blow up (their volume diverges) above root hyperplanes, where W-bosons become massless and gauge symmetry is enhanced.

- The fibers collapse above weight hyperplanes, where hypermultiplets become massless.

The precise hyperkähler metric on the fibration acquires non-perturbative quantum corrections that are extremely difficult to compute directly.

TQFT and non-renormalization

Nevertheless, if one ignores the hyperkähler metric and focuses on $\mathcal{M}_C$ as a complex symplectic manifold, many computations become tractable. In particular, the computation of the chiral ring $\mathbb{C}[\mathcal{M}_C]$ and its Poisson structure (as well as its deformation quantization) reduces to a *relatively* simple algebra problem.

There are two ways to think about this simplification. In [67] it was argued that the chiral ring of a 3d $\mathcal{N}=4$ gauge theory is independent of the gauge coupling, and thus cannot receive non-perturbative quantum corrections, or perturbative corrections beyond one loop.

Alternatively, one may recognize that the chiral ring $\mathbb{C}[\mathcal{M}_C]$ belongs to a topological subsector of the 3d gauge theory. Specifically, the chiral-ring operators are in the cohomology of a topological supercharge Q, which was discussed long ago in [136], and may equivalently be characterized as (*cf.* [70, 71, 85])

- the 3d reduction of the 4d $\mathcal{N}=2$ Donaldson supercharge
- one of the scalars under a diagonal subgroup of $SU(2)_{\text{Lorentz}} \times SU(2)_H$ (where $SU(2)_H$ is the R-symmetry that rotates hypermultiplet scalars)
- the "twisted Rozansky-Witten" supercharge, as it plays the same role on the Coulomb branch that the Rozansky-Witten twist plays for the Higgs branch .

Then the product of chiral-ring operators is topologically protected, and may be computed using standard topological quantum field theory (TQFT) methods. Perhaps surprisingly, the Poisson bracket and deformation quantization (via Omega background) of chiral-ring operators are also topological in nature [85].

The TQFT perspective motivated the initial mathematical work [71, 72] on Coulomb branches. In Section 1.3, we explain how this mathematical characterization of Coulomb-branch operators relates to the physics of 3d $\mathcal{N} = 4$ theories. The TQFT perspective has some important computational consequences, which we draw on in what follows.

1.2.2 The abelianized algebra $\mathcal{A}$

The TQFT derivation of the ring $\mathbb{C}[\mathcal{M}_C]$ (in Section 1.3) proceeds via reduction to 1d quantum mechanics, where $\mathbb{C}[\mathcal{M}_C]$ is identified as the equivariant cohomology (or more technically, Borel-Moore homology) of a certain moduli space. Fixed-point localization embeds the chiral ring into a much simpler "abelianized" algebra $\mathcal{A}$,

$$\mathbb{C}[\mathcal{M}_C] \hookrightarrow \mathcal{A}\,. \tag{1.2.7}$$

Physically speaking, one may think of $\mathcal{A}$ as a local algebra of operators near generic points on the Coulomb branch, where the gauge theory is effectively abelian; this is how the abelian algebra $\mathcal{A}$ arose in [67].[2] Similarly, in an Omega background both $\mathbb{C}[\mathcal{M}_C]$ and $\mathcal{A}$ are deformation-quantized, and one finds an embedding of associative algebras

$$\mathbb{C}_\varepsilon[\mathcal{M}_C] \hookrightarrow \mathcal{A}_\varepsilon\,. \tag{1.2.8}$$

All the computations in this chapter will take place in $\mathcal{A}$ or $\mathcal{A}_\varepsilon$. We review their structure here. Since $\mathcal{A}$ can be recovered from $\mathcal{A}_\varepsilon$ by sending $\varepsilon \to 0$, it would be sufficient to describe $\mathcal{A}_\varepsilon$. However, some relations are simpler and more intuitive for $\mathcal{A}$, so we shall start with the commutative case.

The algebra $\mathcal{A}$ can be defined as the local chiral ring, in the neighborhood of a generic point φ on the base of the Coulomb branch, in the sense of (1.2.5). To make this precise, we denote the loci on the base of the Coulomb branch were W-bosons and hypermultiplets

[2] This perspective is directly analogous to abelianization/non-abelianization in 4d $\mathcal{N} = 2$ theories [137,138], and to localization computations of algebras of line/loop operators therein, *cf.* [20,139,140].

become massless as

$$\Delta = \bigcup_{\text{roots } \alpha} \{M_\alpha(\varphi)=0\} \;\subset \mathfrak{t}_{\mathbb{C}}\,, \qquad \Delta_R = \bigcup_{\text{weights } \lambda \text{ of } R} \{M_\lambda(\varphi)=0\} \;\subset \mathfrak{t}_{\mathbb{C}}\,. \tag{1.2.9}$$

Then we define

$$\mathcal{M}_C^{\text{abel}} = \pi^{-1}\big((\mathfrak{t}_{\mathbb{C}}\backslash\Delta\cup\Delta_R)/W\big)\subset\mathcal{M}_C \tag{1.2.10}$$

as the open subset of the Coulomb branch sitting above the complement of Δ and Δ_R in the fibration (1.2.3); and define $\widetilde{\mathcal{M}}_C^{\text{abel}}$ to be the trivial W-cover of $\mathcal{M}_C^{\text{abel}}$ (undoing the quotient by the Weyl group on the base). Then

$$\mathcal{A} := \mathbb{C}\big[\widetilde{\mathcal{M}}_C^{\text{abel}}\big]\,. \tag{1.2.11}$$

This definition of $\mathcal{A}$ makes it obvious that there is an embedding (1.2.7), since any global function on $\mathcal{M}_C$ defines a W-invariant local function on $\widetilde{\mathcal{M}}_C^{\text{abel}}$.

The algebra $\mathcal{A}$ has two types of generators:

1. Rational functions in the components of the abelian complex scalar $\varphi\in\mathfrak{t}_{\mathbb{C}}$, whose denominators vanish only on Δ and Δ_R.

   In other words, there are polynomials in φ and in the inverted generators $(M_\alpha)^{-1}, (M_\lambda)^{-1}$.

2. Abelian monopole operators v_A as in (1.2.6), for every cocharacter

$$A\in \mathrm{Hom}(U(1),T)\simeq \mathbb{Z}^{\mathrm{rank}(G)}\,. \tag{1.2.12}$$

These operators satisfy relations that are essentially the expected product relations $v_Av_B\sim v_{A+B}$ among monopole operators, with one-loop corrections from hypermultiplets and W-bosons.

To write down the relations, we first recall that there is a natural integer-valued product

$$\langle\lambda, A\rangle\in\mathbb{Z} \tag{1.2.13}$$

between weights λ and cocharacters. Then the classical relation $v_A v_{-A} = 1$ among abelian monopole operators is corrected by hypermultiplets and W-bosons to

$$v_A v_{-A} = \frac{\prod_{\text{weights } \lambda \text{ of } R} \big(M_\lambda\big)^{|\langle\lambda,A\rangle|}}{\prod_{\text{roots } \alpha \text{ of } G} \big(M_\alpha\big)^{|\langle\alpha,A\rangle|}}\,; \tag{1.2.14}$$

and more generally

$$v_A v_B = v_{A+B}\,\frac{\prod_{\substack{\text{weights } \lambda \text{ of } R\\ \text{s.t. } \langle\lambda,A\rangle\langle\lambda,B\rangle<0}} \big(M_\lambda\big)^{\min(|\langle\lambda,A\rangle|,|\langle\lambda,B\rangle|)}}{\prod_{\substack{\text{roots } \alpha \text{ of } G\\ \text{s.t. } \langle\lambda,A\rangle\langle\lambda,B\rangle<0}} \big(M_\alpha\big)^{\min(|\langle\alpha,A\rangle|,|\langle\alpha,B\rangle|)}}\,. \tag{1.2.15}$$

The abelianized algebra $\mathcal{A}$ simply contains polynomials in φ, $1/M_\alpha$, and v_A, modulo these relations:[3]

$$\boxed{\mathcal{A} = \mathbb{C}\big[\varphi, \{M_\alpha^{-1}\}_{\alpha\in\text{roots}}, \{M_\lambda^{-1}\}_{\lambda\in\text{wts(R)}}, \{v_A\}_{A\in\text{cochars}}\big]\Big/\big(\text{relations (1.2.15)}\big)}\,. \tag{1.2.16}$$

Quantization

The A-twist of 3d $\mathcal{N} = 4$ gauge theories is compatible with an Omega deformation. Abstractly, the Omega background (or "Omega deformation") [19] of a 3d cohomological theory in Euclidean spacetime $\mathbb{R}^3 \simeq \mathbb{C}\times\mathbb{R}$ is a deformation of the entire theory, depending on a complex parameter ε, such that the supercharge Q_ε obeys

$$Q_\varepsilon^2 = \varepsilon V\,, \tag{1.2.17}$$

where V is the generator of $U(1)$ rotations on $\mathbb{C}$. Roughly speaking, Q-cohomology is replaced by equivariant Q_ε-cohomology, with respect to spacetime rotations about an axis.

It is known from many different perspectives (*cf.* [141–144]) that the Omega deformation

[3] Technically, there are also the obvious relations $\langle\alpha,\varphi\rangle\cdot M_\alpha^{-1} = 1$, $\langle\lambda,\varphi\rangle\cdot M_\lambda^{-1} = 1$ that follow from the definitions of M_α, M_λ.

induces a deformation quantization of the algebra of local operators. Local operators in the Omega background must lie on the fixed axis of the above $U(1)$ rotations. Since they can no longer move past each other, their collision need not be commutative. It was recently argued in [85] on general topological grounds that the commutator is determined to first order by the E_3 bracket $\{\ ,\ \}_{\varepsilon=0}$ in the undeformed operator algebra,

$$[\mathcal{O}, \mathcal{O}'] = \varepsilon\{\mathcal{O}, \mathcal{O}'\}_{\varepsilon=0} + O(\varepsilon^2)\,. \tag{1.2.18}$$

The quantized algebra $\mathcal{A}_\varepsilon$ is can be described just as above. It is generated by

1. The components of φ, and ε.

2. The inverted masses $(M_\alpha + n\varepsilon)^{-1}$ and $(M_\lambda + n\varepsilon)^{-1}$ for all $n \in \mathbb{Z}$.

   (The shifted quantities $M_\alpha + n\varepsilon$ may be understood physically as complex masses of all the various *modes* of W-bosons in the presence of an Omega background, noting that the Omega background couples to angular momentum. Similarly, $M_\lambda + n\varepsilon$ are masses of the modes of hypermultiplets.)

3. The abelian monopole operators v_A.

The parameter ε is central; and the components of φ (and the $(M_{\alpha,\lambda} + n\varepsilon)^{-1}$) all commute with each other. Otherwise, the generators satisfy two basic sets of relations:

First, note that the components of φ can all be picked out by contraction with weights, *e.g.* $\langle\lambda, \varphi\rangle$. All linear functions in φ arise this way. The commutator of any such linear function and a monopole operator is

$$[\langle\lambda, \varphi\rangle, v_A] = \varepsilon\langle\lambda, A\rangle v_A\,. \tag{1.2.19}$$

For example, if $G = U(N)$, one would customarily write $\varphi = \mathrm{diag}(\varphi_1, ..., \varphi_N)$. Both weights $\lambda = (\lambda_1, ..., \lambda_N)$ and cocharacters $A = (A_1, ..., A_N)$ are elements of a lattice $\mathbb{Z}^N$. The entries of φ are picked out by contractions $\langle(0,...,\overset{i}{1},...,0),\varphi\rangle=\varphi_i$, so (1.2.19) says

$$[\varphi_i, v_A] = \varepsilon A_i v_A\,. \tag{1.2.20}$$

It follows from (1.2.19) that the inverted masses also satisfy (*e.g.*)

$$v_A\,\frac{1}{M_\alpha+n\varepsilon} = \frac{1}{M_\alpha+(n-\langle\alpha,A\rangle)\varepsilon}\,v_A\ . \tag{1.2.21}$$

Second, the product of two abelianized monopole operators is given by an appropriately ordered and shifted version of (1.2.15):

$$v_Av_B = \frac{\displaystyle\prod_{\substack{\lambda\in\text{weights}(R)\,\text{s.t.}\\ |\langle\lambda,A\rangle|\le|\langle\lambda,B\rangle|\\ \langle\lambda,A\rangle\langle\lambda,B\rangle<0}}[M_\lambda+\tfrac{\varepsilon}{2}]^{-\langle\lambda,A\rangle}}{\displaystyle\prod_{\substack{\alpha\in\text{roots}(G)\,\text{s.t.}\\ |\langle\alpha,A\rangle|\le|\langle\alpha,B\rangle|\\ \langle\alpha,A\rangle\langle\alpha,B\rangle<0}}[M_\alpha]^{-\langle\alpha,A\rangle}}\,v_{A+B}\,\frac{\displaystyle\prod_{\substack{\lambda\in\text{weights}(R)\,\text{s.t.}\\ |\langle\lambda,A\rangle|>|\langle\lambda,B\rangle|\\ \langle\lambda,A\rangle\langle\lambda,B\rangle<0}}[M_\lambda+\tfrac{\varepsilon}{2}]^{\langle\lambda,B\rangle}}{\displaystyle\prod_{\substack{\alpha\in\text{roots}(G)\,\text{s.t.}\\ |\langle\alpha,A\rangle|>|\langle\alpha,B\rangle|\\ \langle\alpha,A\rangle\langle\alpha,B\rangle<0}}[M_\alpha]^{\langle\alpha,B\rangle}}\,, \tag{1.2.22}$$

where

$$[a]^b := \begin{cases}\displaystyle\prod_{k=0}^{b-1}(a+k\varepsilon) & b>0\\ \displaystyle\prod_{k=1}^{|b|}(a-k\varepsilon) & b<0\\ 1 & b=0\end{cases} \tag{1.2.23}$$

is a quantum-corrected power. These relations were derived using abelian mirror symmetry in [67], but also follow from an equivariant cohomology (TQFT) computation [70, 71].

Altogether, the quantized algebra is

$$\boxed{\mathcal{A}_\varepsilon = \mathbb{C}\big[\varphi,\{(M_\alpha+n\varepsilon)^{-1}\},\{(M_\lambda+n\varepsilon)^{-1}\},\{v_A\}\big]\big/\big(\text{rel's (1.2.19),(1.2.22)}\big)}\,, \tag{1.2.24}$$

where $\alpha\in\text{roots}(G)$, $\lambda\in\text{weights}(R)$, $n\in\mathbb{Z}$, and $A\in\text{cochars}(T)$.

1.2.3 The image of $\mathbb{C}[\mathcal{M}_C]$ and the algebra $\mathcal{W}_\varepsilon$

Once the Coulomb-branch chiral ring $\mathbb{C}[\mathcal{M}_C]$ (resp $\mathbb{C}_\varepsilon[\mathcal{M}_C]$) is mapped to the abelianized algebra $\mathcal{A}$ (resp. $\mathcal{A}_\varepsilon$), many computations become straightforward. In particular, expected relations among elements of $\mathbb{C}_\varepsilon[\mathcal{M}_C]$ can be checked using the simple relations (1.2.15) in $\mathcal{A}_\varepsilon$. Nevertheless, the precise image of $\mathbb{C}_\varepsilon[\mathcal{M}_C]$ in $\mathcal{A}_\varepsilon$ can be tricky to identify.

A few structural properties of the embedding map were discussed in [67]. For example:

- The image of $\mathbb{C}_\varepsilon[\mathcal{M}_C]$ must sit in the Weyl-invariant subalgebra $\mathcal{A}_\varepsilon^W \subset \mathcal{A}_\varepsilon$, since local operators in the full non-abelian gauge theory are gauge invariant.

- In $\mathcal{A}_\varepsilon$ one finds arbitrarily large negative powers of the masses $M_{\alpha,\lambda} + n\varepsilon$. In the case of W-boson masses, this is unavoidable, due to denominators in the products $v_A v_{-A}$.

  In contrast, the image of $\mathbb{C}_\varepsilon[\mathcal{M}_C]$ in $\mathcal{A}_\varepsilon$ cannot contain any of the elements $\frac{1}{M_{\alpha,\lambda}+n\varepsilon}$ themselves, since operators in $\mathbb{C}_\varepsilon[\mathcal{M}_C]$ must define (as $\varepsilon \to 0$) global functions on the Coulomb branch that extend smoothly across the discriminant locus.

It is also known how a basis for $\mathbb{C}[\mathcal{M}_C]$ as an *infinite-dimensional vector space* should be indexed [69]. Physically, one expects that the elements of $\mathbb{C}[\mathcal{M}_C]$ are monopole operators $V_{A,p(\varphi)}$ labeled by *dominant* cocharacters A (equivalently, by Weyl orbits in the cocharacter lattice) and dressed by polynomials $p(\varphi)$ of $\varphi \in \mathfrak{t}_\mathbb{C}$ that are invariant under the stabilizer W_A of A in the Weyl group. For example, if $A = 0$, the "dressing factors" are just standard Weyl-invariant polynomials $\mathbb{C}[\varphi]^W$. Formally, we might write

$$\mathbb{C}[\mathcal{M}_C] \overset{\text{as a vector space}}{\simeq} \bigoplus_{\text{dominant } A} \overbrace{\mathbb{C}[\varphi]^{W_A}}^{\text{dressing factors}} \langle V_A \rangle \,. \tag{1.2.25}$$

It is unclear whether these structural properties alone can determine how elements of $\mathbb{C}[\mathcal{M}_C]$ (or $\mathbb{C}_\varepsilon[\mathcal{M}_C]$) embed in $\mathcal{A}$ (or $\mathcal{A}_\varepsilon$). However, much stronger constraints on the embedding come from the mathematical/TQFT perspective. In fact, the TQFT construction of the chiral ring gives — in principle — a complete answer to the embedding problem. Namely, elements of $\mathbb{C}_\varepsilon[\mathcal{M}_C]$ are identified with equivariant cohomology classes on a certain moduli space; and the embedding $\mathbb{C}_\varepsilon[\mathcal{M}_C] \hookrightarrow \mathcal{A}_\varepsilon$ just expresses these classes in terms of equivariant fixed points.

It can still be very difficult to explicitly analyze equivariant cohomology classes in practice. Fortunately, Webster [39] recently outlined a combinatorial calculus that accomplishes this task for Coulomb branches. We will discuss the physical meaning of Webster's calculus in [112]. In the current chapter, we take a pragmatic approach and use one simple consequence of Webster's combinatorics: the image of $\mathbb{C}_\varepsilon[\mathcal{M}_C]$ in $\mathcal{A}_\varepsilon$ must always contain a particular subalgebra $\mathcal{W}_\varepsilon$ (defined momentarily),

$$\mathcal{W}_\varepsilon \subseteq \mathbb{C}_\varepsilon[\mathcal{M}_C] \subset \mathcal{A}_\varepsilon\,. \tag{1.2.26}$$

The algebra $\mathcal{W}_\varepsilon$ is defined as follows. One begins with a subalgebra of $\mathcal{A}_\varepsilon$ generated by polynomials in φ and by rescaled monopole operators

$$u_A \;:=\; \prod_{\substack{\alpha\in\text{roots}(G)\\ \text{s.t. }\langle\alpha,A\rangle<0}} [M_\alpha]^{-\langle\alpha,A\rangle}\, v_A \;\;=\; \prod_{\substack{\alpha\in\text{roots}(G)\\ \text{s.t. }\langle\alpha,A\rangle<0}} v_A\, [M_\alpha]^{\langle\alpha,A\rangle}\,. \tag{1.2.27}$$

These u_A monopole operators, carrying additional factors associated to the W-boson masses, have the nice property that their products never generate denominators: we simply have

$$u_A u_B = \prod_{\substack{\lambda\in\text{weights}(R)\,\text{s.t.}\\ |\langle\lambda,A\rangle|\leq|\langle\lambda,B\rangle|\\ \langle\lambda,A\rangle\langle\lambda,B\rangle<0}} [M_\lambda+\tfrac{\varepsilon}{2}]^{-\langle\lambda,A\rangle}\;\; u_{A+B} \prod_{\substack{\lambda\in\text{weights}(R)\,\text{s.t.}\\ |\langle\lambda,A\rangle|>|\langle\lambda,B\rangle|\\ \langle\lambda,A\rangle\langle\lambda,B\rangle<0}} [M_\lambda+\tfrac{\varepsilon}{2}]^{\langle\lambda,B\rangle}\,, \tag{1.2.28}$$

with one-loop corrections from the hypermultiplets alone. Otherwise, the usual relations

$$\big[\langle\lambda,\varphi\rangle, u_A\big] = \varepsilon\langle\lambda,A\rangle u_A \tag{1.2.29}$$

continue to hold for any weight λ and cocharacter A.

In addition, for each root α, let $s_\alpha \in W$ denote the corresponding simple reflection.

Recall that the Weyl group is generated by the s_α's. We may adjoin the s_α to the algebra of φ's and u_A's, in such a way that the s_α's satisfy the standard Weyl-group relations among themselves, and natural commutation relations

$$s_\alpha u_A = u_{A^\alpha}\, s_\alpha\,, \qquad s_\alpha\, f(\varphi) = f(\varphi^\alpha)\, s_\alpha\,, \tag{1.2.30}$$

where A^α is the reflected cocharacter, and φ^α is the reflected element of $\mathfrak{t}_\mathbb{C}$. Finally, for each α, introduce the BGG-Demazure operator[4]

$$\theta_\alpha = \frac{1}{M_\alpha}(s_\alpha - 1)\,. \tag{1.2.31}$$

The algebra $\mathcal{W}_\varepsilon$ is defined as the Weyl-invariant part of an algebra generated by 1) polynomials in φ; 2) the u_A monopole operators; and 3) the BGG-Demazure operators:

$$\boxed{\mathcal{W}_\varepsilon = \mathbb{C}\big[\varphi, \{u_A\}_{A\in \text{cochars}}, \{\theta_\alpha\}_{\alpha\in\text{roots}}\big]^W \;\subset \mathcal{A}_\varepsilon}\,. \tag{1.2.32}$$

The relations, which we leave implicit, are of the form (1.2.28), (1.2.29), (1.2.30). Notice that once Weyl-invariance is imposed, all the s_α's are all projected out, so $\mathcal{W}_\varepsilon$ does become an actual subalgebra of $\mathcal{A}_\varepsilon$.

Practically speaking, the role of the Demazure operators θ_α is to introduce *a few* denominators $\frac{1}{M_\alpha}$, in a controlled way, so that the structural properties of the Coulomb branch discussed above are actually satisfied.

1.2.4 Flavor symmetry and R-symmetry

We finally comment briefly on symmetries of 3d $\mathcal{N}=4$ theories.

Flavor symmetries act either on the Higgs branch or on the Coulomb branch, as tri-Hamiltonian isometries. The symmetry group F acting on the Higgs branch is easy to identify

[4]The "BGG" stands for Bernstein, Gelfand, and Gelfand. The operators θ_α generate the G-equivariant cohomology of the flag variety (known as the nil-Hecke algebra in representation theory), which is a large clue to their physical meaning. Another, related, clue is the appearance of the θ_α in the work of Gukov and Witten on surface operators in 4d [145]. We will tie these clues together in [112].

in a gauge theory, as the normalizer of G in $USp(R)$

$$F = N_{USp(R)}(G)/G\,, \tag{1.2.33}$$

i.e. the group that acts on hypermultiplets independently of G. In general, complex mass parameters associated to the Higgs flavor symmetry (scalars in the F vector multiplet) can deform the Coulomb-branch chiral ring.

In the UV, the Coulomb-branch flavor group K is the Pontryagin dual of $\pi_1(G)$

$$K = \mathrm{Hom}(\pi_1(G), U(1)) \simeq U(1)^{\mathrm{rank(Z(G))}}\,, \tag{1.2.34}$$

which is an abelian group with the same rank as the center of G. In the IR the group K may undergo a non-abelian enhancement, controlled by the "balanced" nodes in a given quiver [55, 56, 135], *i.e.* nodes $\textcircled{N_c}$ that are coupled to exactly $N_f = 2N_c$ hypermultiplets.[5]

Since the chiral ring $\mathbb{C}[\mathcal{M}_C]$ is insensitive to RG flow, the fully enhanced IR symmetry group K will act on it. More so, since $\mathbb{C}[\mathcal{M}_C]$ is a holomorphic object, the complexification $K_\mathbb{C}$ will actually act. This action is generated by the complex moment map operators $\mu = \mu_\mathbb{C} \in \mathbb{C}[\mathcal{M}_C] \otimes \mathrm{Lie}(K)^*$, which are related to the K currents by supersymmetry.

The $K_\mathbb{C}$ action extends to the quantized chiral ring $\mathbb{C}_\varepsilon[\mathcal{M}_C]$, where it is generated by taking commutators (rather than Poisson brackets) with moment maps. Explicitly, if $T \in \mathrm{Lie}K_\mathbb{C}$ is a generator of the (complexified) Lie algebra, and we denote by $\mu_T = \langle T, \mu\rangle \in \mathbb{C}_\varepsilon[\mathcal{M}_C]$ the contraction of T and μ, there must be commutation relations

$$[\mu_T, \mu_{T'}] = \varepsilon\, \mu_{[T,T']}\,, \tag{1.2.35}$$

[5] It is worth noting that there can be yet further enhancement beyond the naive consideration of balanced nodes. For example, in the theory $\mathcal{T}_{2,3}$ discussed below there is an obvious $SU(2)^3$ Coulomb-branch flavor symmetry. However, this theory is 3d mirror to a theory of 8 free half-hypermultiplets with Higgs-branch flavor symmetry $USp(4)$, which should be equal to the Coulomb-branch flavor symmetry of $\mathcal{T}_{2,3}$. Indeed, the Coulomb branch of $\mathcal{T}_{2,3}$ is $T^*\mathbb{C}^4 \simeq \mathbb{C}^8$ which has a full $USp(4)$ worth of hyperkähler isometries.

and the infinitesimal T action on any other operator $\mathcal{O}$ is

$$T\cdot\mathcal{O}=\tfrac{1}{\varepsilon}[\mu_T,\mathcal{O}]\,. \tag{1.2.36}$$

In addition to flavor symmetries, 3d $\mathcal{N}=4$ gauge theories with linear matter also have an $SU(2)_C\times SU(2)_H$ R-symmetry. The two factors act on the Coulomb and Higgs branches, respectively, but in a way that rotates the hyperkähler $\mathbb{CP}^1$'s of complex structures rather than as tri-holomorphic isometries. The $SU(2)_C$ acting on the Coulomb branch is important to us. Any fixed complex structure on the Coulomb branch is preserved by a $U(1)_R$ subgroup of $SU(2)_C$, which induces into a complexified $\mathbb{C}^*$ action on the chiral ring $\mathbb{C}[\mathcal{M}_C]$. The $\mathbb{C}^*$ action extends to the quantized $\mathbb{C}_\varepsilon[\mathcal{M}_C]$, in such a way that the quantization parameter ε and all moment maps canonically have charge[6]

$$[\mu]=[\varepsilon]=1\,. \tag{1.2.37}$$

In the abelianized chiral ring $\mathcal{A}_\varepsilon$, the complex φ scalars also necessarily have $[\varphi]=1$. It then follows from monopole products (1.2.22) (or in fact the simpler commutative (1.2.14)) that

$$[v_A]=\frac{1}{2}\Big(\sum_{\text{weights }\lambda\text{ or }R}|\langle\lambda,A\rangle|-\sum_{\text{roots }\alpha\text{ of }G}|\langle\alpha,A\rangle|\Big)\,. \tag{1.2.38}$$

This is consistent with physical expectations for monopole charges [55, 56, 135].

If a 3d $\mathcal{N}=4$ gauge theory flows to a CFT, the $\mathbb{C}^*$ charges of chiral-ring operators coincide with their conformal dimensions, and must therefore be strictly positive.

1.3 Twists of 3d $\mathcal{N}=4$ and the BFN construction

We now turn to the subject of topological twists the 3d $\mathcal{N}=4$ theories discussed in Section 1.1. In Section 1.3.1 we review the notion of twisting a supersymmetric gauge theory and in Section 1.3.1 we review the possible twists admitted by 3d theories with $\mathcal{N}=4$ supersymmetry. There

[6]We are working in conventions where the minimal charge of a $\mathbb{C}^*$ representation is $\frac{1}{2}$.

are two distinct topological twists of 3d $\mathcal{N}=4$ gauge theories, which we will refer to as the A- and B-twists. The supercharges that define these respective twists in flat space — in the usual sense that topologically twisting the theory amounts to working in the cohomology of a particular supercharge — are

$$Q_A := Q_+^{+\dot{+}} + Q_-^{-\dot{+}}\,, \qquad Q_B := Q_+^{+\dot{+}} + Q_-^{+\dot{-}}\,. \tag{1.3.1}$$

The A-twist is a dimensional reduction of the 4d Donaldson-Witten twist [146], and is involved in the definition of Seiberg-Witten invariants of 3-manifolds [65]. Some families of A-twisted 3d sigma-models were studied in [66, 147]. The B-twist is intrinsically three-dimensional. It was first identified by Blau and Thompson [136] in pure 3d $\mathcal{N}=4$ gauge theories, and then studied by Rozansky and Witten [43] in 3d $\mathcal{N}=4$ sigma-models (which could be thought of as 3d $\mathcal{N}=4$ gauge theories on their Higgs branches). The extended TQFT defined by the B-twist of a sigma-model was described by Kapustin-Rozansky-Saulina [68]. The fact that the A- and B-twists are the *only* topological twists in 3d $\mathcal{N}=4$ theories follows from a basic algebraic classification of nilpotent supercharges whose commutators contain all translation [121, 122].

For the purpose of this thesis, we will be particularly interested in the topological A-twist of the gauge theories described in Section 1.1. Bulk local operators in this topological twist can be preserved by as many as four independent supercharges, namely the supercharges

$$\{Q_\alpha^{a\dot{+}}\}_{\alpha,a=\pm}\,. \tag{1.3.2}$$

The corresponding spaces of $\frac{1}{2}$-BPS local operators is nothing other than the Coulomb-branch chiral ring [13, 67, 69, 148–150]. In reanalyzing the Coulomb branches of these theories, and in preparation for the discussion of A-type line operators in the Chapter 2, we find it useful to rewrite the theory as an effective 1d theory of maps into the solutions to certain BPS equations, from which it is easy to identify the algebra of local operators. In Section 1.3.2 we discuss the relevant BPS equations and in Section 1.3.3 we relate this construction to that of

Braverman-Finkelberg-Nakajima construction [71, 72].

1.3.1 Twisting

We work almost exclusively in flat, Euclidean $\mathbb{R}^n$. Suppose we have a QFT on $\mathbb{R}^n$ with a $\mathbb{Z}$-valued fermion number[7] and a fermionic symmetry generated by a charge Q[8], such that

- Q has fermion number 1: we write $|Q| = 1$
- Q is nilpotent: $Q^2 = 0$
- translations are Q-exact: $P_\mu = \{Q, Q_\mu\}$ for some other symmetries Q_μ with $|Q_\mu| = -1$, and this extends to the corresponding currents, $T_{\mu\nu} = \{Q, S_{\mu\nu}\}$ for some $S_{\mu\nu}$.

We will call such a Q is a "topological" supercharge. More generally, one could ask for only a portion of translations to be exact; the exact translations can be organized so that some directions are topological (the translations are exact) while others are holomorphic (the anti-holomorphic translations are exact). We call such a supercharge "mixed" or "holomorphic-topological."

Since $Q^2 = 0$, it makes sense to consider the Q-cohomology of various local and extended operators in the QFT. From a physical perspective, it is more natural to simply restrict attention to Q-closed (*i.e.* Q-invariant) objects. This is actually *equivalent* to working in cohomology. Namely, once one decides to consider only Q-closed local operators, line operators, boundary conditions, vacua, etc., the insertion of any Q-exact operator in a correlation function automatically evaluates to zero; schematically,

$$\langle Q(\mathcal{O})\mathcal{O}' \cdots \rangle = \langle Q(\mathcal{O}\mathcal{O}' \cdots)\rangle = 0\,. \tag{1.3.3}$$

[7]The discussion in this section would work perfectly well with a $\mathbb{Z}_2$-valued fermion number. However, in our 3d $\mathcal{N} = 4$ applications we will always have a $\mathbb{Z}$ enhancement of the fermion number, coming from an R-symmetry, so we work in this context.

[8]More precisely, we also need Q to transform as a scalar under a suitable Lorentz group. In the contexts where Q arises from a supersymmetry, the suitable Lorentz group is defined as a subgroup of the product of the (usual) Lorentz group and the R-symmetry group.

(The first equality holds if $\mathcal{O}' \cdots$ are all Q-closed, and the second equality holds because Q is a symmetry.) Since two operators are physically distinguishable only insofar as they are measured by correlation functions, we find that, in the sector of the theory containing only Q-closed operators, the Q-exact operators are automatically equivalent to zero. Thus, only Q-cohomology classes are measured.

We will refer to the Q-cohomology of a theory as its "Q-twist." We note that the requirements above for a topological supercharge are weaker than the standard notion of a topological twist [62, 146]. Namely, a standard topological twist requires Q to be defined on arbitrary curved spacetimes, and leads to metric-independent correlation functions due to Q-exactness of $T_{\mu\nu}$; whereas the above only requires Q to exist on flat $\mathbb{R}^3$. When Q can be defined on arbitrary spacetimes (and $T_{\mu\nu}$ is always Q-exact), we will say that the Q-twist has the structure of a full TQFT.

Nilpotence varieties

A natural source of theories that admit the above nilpotent fermionic symmetries Q are those with supersymmetry. In particular, given an algebra of supersymmetries one can consider the moduli space of (non-zero) nilpotent elements, called the "nilpotence variety" [122]. Based on the discussion above, this moduli can equivalently be thought of as the moduli space of possible twists admitted by the theory [121].

The nilpotence variety is naturally a complex projective variety and admits a stratification by type of supercharge, *e.g.* by the dimension of the image of $\{Q, -\}$. Consider the case of $\mathcal{N}=4$ supersymmetry in 3d (without central charges); the supersymmetry algebra of interest takes the form of Eq. (1.1.1) (with m, t set to zero). A general point of the nilpotence variety can be written as $Q = q^{\alpha}_{a\dot{a}} Q^{a\dot{a}}_{\alpha}$, where $q^{\alpha}_{a\dot{a}}$ are naturally projective coordinates on $\mathbb{P}^7$ subject to the three quadratic relations

$$\epsilon^{ab}\epsilon^{\dot{a}\dot{b}} q^{(\alpha}_{a\dot{a}} q^{\beta)}_{b\dot{b}} = 0. \tag{1.3.4}$$

These equations cut out two copies of $\mathbb{P}^3 \times \mathbb{P}^1$ in $\mathbb{P}^7$ that intersect along a $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ [122].

Up to (discrete and continuous) symmetries, we can always choose

$$q^{+}_{a\dot{a}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad q^{-}_{a\dot{a}} = \begin{pmatrix} 0 & c \\ c' & 0 \end{pmatrix} \tag{1.3.5}$$

where $cc' = 0$. If both c and c' vanish, the resulting Q is a mixed supercharge:

$$Q_{HT} = Q^{+\dot{+}}_{+} \quad (c = 0, c' = 0) \tag{1.3.6}$$

and we call the corresponding twist the "holomorphic-topological twist" or simply the "HT-twist." On the other hand, if either c or c' is non-zero we can use a (complexified) R-symmetry rotation to scale it to 1, thus we obtain two distinct topological supercharges:

$$\begin{aligned} Q_A &= \delta^{\alpha}_{a} Q^{a\dot{a}}_{\alpha} = Q^{+\dot{+}}_{+} + Q^{-\dot{+}}_{-} \quad (c = 0, c' = 1) \\ Q_B &= \delta^{\alpha}_{\dot{a}} Q^{a\dot{a}}_{\alpha} = Q^{+\dot{+}}_{+} + Q^{+\dot{-}}_{-} \quad (c = 1, c' = 0) \end{aligned} \tag{1.3.7}$$

We call the corresponding topological twists the "A-twist" and "B-twist."[9]

The A-twist is a dimensional reduction of the 4d Donaldson-Witten twist [146], and is involved in the definition of Seiberg-Witten invariants of 3-manifolds [65]. Some families of A-twisted 3d sigma-models were studied in [66, 147]. The B-twist is intrinsically three-dimensional. It was first identified by Blau and Thompson [136] in pure 3d $\mathcal{N} = 4$ gauge theories, and then studied by Rozansky and Witten [43] in 3d $\mathcal{N} = 4$ sigma-models (which could be thought of as 3d $\mathcal{N} = 4$ gauge theories on their Higgs branches). The extended TQFT defined by the B-twist of a sigma-model was described by Kapustin-Rozansky-Saulina [68].

The HT-twist is more general than the A- and B-twists and is also enjoyed by 3d $\mathcal{N} = 2$ theories, recently studied in [151, 152], and can be defined on 3 manifolds that admit a "transverse holomorphic foliation," *i.e.* a local splitting of the 3d manifold as the product of

[9] These topological twists often appear under the names "H-twist" and the "C-twist," respectively, corresponding the $SU(2)$ subgroup of the $SU(2)_H \times SU(2)_C$ R-symmetry group used to make the supercharge a scalar. We use the name A-twist and B-twist as they reduce to these twists for a certain 2d $\mathcal{N} = (2, 2)$ subalgebra of the 3d $\mathcal{N} = 4$ algebra.

a Riemann surface and a line. The algebra of local operators in the HT-twist has the structure of a commutative chiral algebra (*i.e.* correlation functions of Q_{HT}-closed operators depend holomorphically on the locations of operator insertions and are non-singular as insertions collide) and admits an odd Poisson bracket.

1.3.2 The topological A-twist and SQM$_A$

A useful perspective for understanding the topological A-twist of 3d $\mathcal{N}=4$ gauge theories is to write the full 3d $\mathcal{N}=4$ theory as an effective 1d super quantum mechanical theory whose target space is the solution space to a suitable set of BPS equations.

This rewriting of the theory will preserve 1d $\mathcal{N}=4$ supersymmetry, and there are essentially two inequivalent choices of 1d $\mathcal{N}=4$ subalgebras, which we will call SQM$_A$ and SQM$_B$. These 1d superalgebras are simply the largest 1d supersymmetry algebras containing the corresponding topological supercharges Q_A and Q_B, respectively, and are compatible with the choice of splitting $\mathbb{R}^3 \simeq \mathbb{C}_{z,\bar{z}} \times \mathbb{R}_t$ with $z = x_1 + ix_2$, $t = x_3$. In this thesis, we are only interested in the topological A-twist and SQM$_A$, but mention the algebra SQM$_B$ for completeness.

The 1d $\mathcal{N}=4$ algebra SQM$_A$ is generated by the four supercharges

$$Q_A^{\dot{a}} = \delta^{\alpha}{}_{a} Q_{\alpha}^{a\dot{a}} \,, \qquad \widetilde{Q}_A^{\dot{a}} = (\sigma^3)^{\alpha}{}_{a} Q_{\alpha}^{a\dot{a}} \,, \tag{1.3.8}$$

which satisfy

$$\{Q_A^{\dot{a}}, \widetilde{Q}_A^{\dot{b}}\} = 2\epsilon^{\dot{a}\dot{b}}(P_t - it_{\mathbb{R}}) \,, \qquad \{Q_A^{\dot{a}}, Q_A^{\dot{b}}\} = \{\widetilde{Q}_A^{\dot{a}}, \widetilde{Q}_A^{\dot{b}}\} = 2im^{(\dot{a}\dot{b})} \,. \tag{1.3.9}$$

Clearly this 1d subalgebra preserves the full 3d $SU(2)_C$ R-symmetry, but breaks $SU(2)_H$ to a diagonal $U(1)_H$ subgroup.[10] For completeness, we note that there is actually a $\mathbb{CP}^1$ family of SQM$_A$ algebras, parameterized by the choices of unbroken $U(1)_H$'s inside $SU(2)_H$. The

[10] In [80], the SQM$_A$ algebra was denoted SQM$_V$, because it turns out to be preserved by vortex-line operators. In [70], it was similarly shown that SQM$_A$ is the subalgebra preserved by dynamical $\frac{1}{2}$-BPS vortex excitations.

different choices lead to different combinations of $t_{\mathbb{R}}$ and $t_{\mathbb{C}}, \bar{t}_{\mathbb{C}}$ appearing in the $\{Q^{\dot{a}}_A, \widetilde{Q}^{\dot{b}}_A\}$ commutation relation. Equivalently, in a 3d $\mathcal{N}=4$ gauge theory, different choices of SQM_A algebra correlate with different choices of complex structure on the Higgs branch. We will work with (1.3.8), and thus fix a choice of complex structure on the Higgs branch once and for all.

Similarly, the 1d $\mathcal{N}=4$ algebra SQM_B is generated by the four supercharges

$$Q^a_B = \delta^\alpha{}_{\dot{a}} Q^{a\dot{a}}_\alpha\,, \qquad \widetilde{Q}^a_B = (\sigma^3)^\alpha{}_{\dot{a}} Q^{a\dot{a}}_\alpha\,, \tag{1.3.10}$$

which satisfy

$$\{Q^a_B, \widetilde{Q}^b_B\} = 2\epsilon^{ab}(P_t - im_{\mathbb{R}})\,, \qquad \{Q^a_B, Q^b_B\} = \{\widetilde{Q}^a_B, \widetilde{Q}^b_B\} = 2it^{(ab)}\,. \tag{1.3.11}$$

This subalgebra preserves an $SU(2)_H \times U(1)_C$ subgroup of the bulk R-symmetry. It is again part of a $\mathbb{CP}^1$ family, parameterized by different choices of $U(1)_C$ inside $SU(2)_C$, or different choices of complex structure on the Coulomb branch (we fix this choice once and for all).[11]

BPS equations for SQM_A

We now review the BPS equations for SQM_A and their moduli space of solutions. The analysis is very similar to that in [153, 154] for 4d $\mathcal{N}=2$, and [32, 70, 155] for 3d $\mathcal{N}=4$. From the perspective of the full 3d theory, solutions to these equations correspond to $\frac{1}{2}$-BPS configurations. In particular, the local operators built from these solutions in Section 1.3.2 (as well as the line operators in Section 2.3) will be $\frac{1}{2}$-BPS.

As usual, we consider a 3d $\mathcal{N}=4$ gauge theory whose vector multiplet contains scalars $\varphi \in \mathfrak{g}_{\mathbb{C}}$, $\sigma \in \mathfrak{g}$ and whose hypermultiplets contain pairs of complex scalars $X \in R$, $Y \in R^*$. The full supersymmetry transformations of these fields were summarized in Section 1.1. By setting to zero the variations of gauginos and hypermultiplet fermions under the four supercharges $Q^{\dot{a}}_A = \delta^\alpha{}_a Q^{a\dot{a}}_\alpha$, $\widetilde{Q}^{\dot{a}}_A = (\sigma^3)^\alpha{}_a Q^{a\dot{a}}_\alpha$ that generate SQM_A, we find bosonic BPS

[11] In [80], the SQM_B algebra was denoted SQM_W, because it turns out to be preserved by Wilson lines.

equations

$$[D_t, D_z] = [D_t, D_{\bar{z}}] = 0\,, \qquad D_t X = D_t Y = D_t\varphi = D_t\sigma = 0\,, \tag{1.3.12a}$$

$$\begin{gathered}[\sigma,\varphi] = [\sigma,\varphi^\dagger] = [\varphi,\varphi^\dagger] = 0\,,\\ \sigma\cdot X = \sigma\cdot Y = 0\,, \quad \varphi\cdot X = \varphi\cdot Y = 0\,, \qquad \varphi^\dagger\cdot X = \varphi^\dagger\cdot Y = 0\,,\end{gathered} \tag{1.3.12b}$$

$$D_z\sigma = D_z\varphi = D_z\varphi^\dagger = 0\,, \qquad D_{\bar{z}}\sigma = D_{\bar{z}}\varphi = D_{\bar{z}}\varphi^\dagger = 0\,, \tag{1.3.12c}$$

$$\boxed{F_{z\bar{z}} = \mu_{\mathbb{R}}\,, \qquad D_{\bar{z}}X = D_{\bar{z}}Y = 0\,, \qquad \mu_{\mathbb{C}} = 0\,.} \tag{1.3.12d}$$

Here $D_t, D_z, D_{\bar{z}}$ are covariant derivatives with respect to the G-connection A; and in (1.3.12b) the schematic expressions $\sigma\cdot X, \varphi\cdot X, \sigma\cdot Y$, etc. denote the infinitesimal action of the $\sigma\in\mathfrak{g}$, $\varphi\in\mathfrak{g}_{\mathbb{C}}$ on the representation $X\in R$ and $Y\in R^*$. (More explicitly, one could write $\rho_R(\sigma)X = 0$, $\rho_{R^*}(\sigma)Y = 0$, etc.)

Observe that the first set of equations (1.3.12a) guarantees that all fields are covariantly constant in time, as we would expect for BPS equations in quantum mechanics. This allows us to restrict our analysis of solutions to the transverse plane $\mathbb{C}_{z,\bar{z}}$, knowing that solutions can then be extended along $\mathbb{R}_t$ in a unique way.

The equations (1.3.12b) restrict σ,φ to lie in a common Cartan subalgebra and that X and Y should be fixed under the infinitesimal action of these fields. The third set (1.3.12c) requires σ,φ to be covariantly constant in the $\mathbb{C}_{z,\bar{z}}$ plane as well. So far, these are standard BPS vacuum equations for 3d $\mathcal{N}=4$ SUSY.

The final boxed set of equations (1.3.12d) are the interesting ones: they are generalized vortex equations in the $\mathbb{C}_{z,\bar{z}}$ plane [153, 156–159], requiring X and Y to be covariantly holomorphic (not constant), and sourcing the magnetic flux $F_{z\bar{z}}$ with the real moment map.

Mass and FI parameters can be included in the BPS equations in a standard way. Namely, the FI parameters deform the moment maps $(\mu_{\mathbb{R}}, \mu_{\mathbb{C}}) \rightsquigarrow (\mu_{\mathbb{R}}+t_{\mathbb{R}}, \mu_{\mathbb{C}}+t_{\mathbb{C}})$ while masses $m_{\mathbb{R}}\in\mathfrak{f}$, $m_{\mathbb{C}}\in\mathfrak{f}_{\mathbb{C}}$ (valued in a common Cartan subalgebra of the flavor symmetry) enter the same way as σ,φ. We will come back to them later in Section 2.3.

Supersymmetric Hilbert spaces

In [70], a three-step procedure was proposed for computing the supersymmetric Hilbert space in the A-twist on $\mathbb{C}_{z,\bar{z}}$ in the presence of a vacuum boundary condition $\mathcal{B}_\nu$ as $|z| \to \infty$, denoted $\mathcal{H}_A(\mathcal{B}_\nu)$, as well as the action of local operators on them. Alternatively, we could consider a $D_{z,\bar{z}} \times \mathbb{R}_t$ spacetime with a boundary condition $\mathcal{B}$ at $\partial D_{z,\bar{z}}$, this is the "cylinder setup" described in [70, 111].[12]

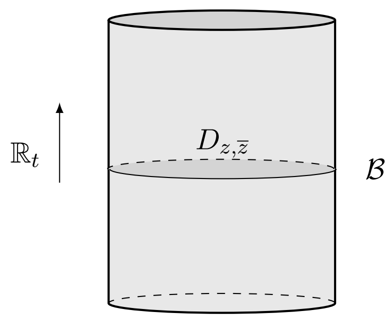


Figure 1.1: An illustration of the "cylinder setup" of [70, 111].

The basic idea is to

1) Rewrite the 3d $\mathcal{N} = 4$ theory on $\mathbb{C}_{z,\bar{z}} \times \mathbb{R}_t$ (or $D_{z,\bar{z}} \times \mathbb{R}_t$) as a 1d SQM$_A$ quantum mechanics on $\mathbb{R}_t$. This 1d theory has an infinite-dimensional target, roughly consisting of maps from $\mathbb{C}_{z,\bar{z}}$ to the original 3d target, subject to the boundary conditions $\mathcal{B}$. Additional degrees of freedom supported on $\mathcal{B}$ may further enhance the target of this quantum mechanics.

2) Solve the BPS equations (1.3.12) along $\mathbb{C}_{z,\bar{z}}$, compatible with the boundary condition $\mathcal{B}$, to localize the theory from (1) to an effective 1d SQM$_A$ sigma-model with a vastly smaller target $\mathcal{M}(\mathcal{B})$.

[12] Strictly speaking, $\mathcal{B}_\nu$ doesn't impose boundary condition so much as specifying asymptotic behavior. For $\mathcal{B}$, we need an honest, $\frac{1}{2}$-BPS (in particular, Q_A-preserving), finite-distance boundary condition. Thankfully, there are "Lefschetz thimble" boundary conditions $\overline{\mathcal{B}}_\nu$ discussed in [40] (generalizing the classical 2d $\mathcal{N} = (2, 2)$ constructions of [160]) such that the finite-distance boundary condition $\overline{\mathcal{B}}_\nu$ that mimics the effect of asymptoting to the vacuum ν.

3) Compute the Q_A-cohomology of the Hilbert space of the effective 1d quantum mechanics (*a.k.a.* the space of SUSY ground states) by taking cohomology,

$$\mathcal{H}_A(\mathcal{B}) \simeq H^\bullet(\mathcal{M}(\mathcal{B}))\,. \tag{1.3.13}$$

Note that the supercharge Q_A acts as an "A-type", or "de-Rham-type" supercharge in the 1d SQM$_A$ quantum mechanics that describes this effective 1d system. Thus, just as in Witten's classic work [161], the Q_A-cohomology of the full Hilbert space of states should be given by a form of de Rham cohomology. (We will comment further on the precise cohomology being used in Section 1.3.2 below.) Step (2) is based on the premise that taking cohomology gives equivalent results before or after localizing to the solutions of BPS equations.

A nice simplification arises in this framework. Cohomology is intrinsically topological, and cannot have local dependence on the Kähler structure of $\mathcal{M}(\mathcal{B})$ in Step (2). Thus we expect to be able to compute the SUSY Hilbert space (1.3.13) by using an algebraic description of $\mathcal{M}(\mathcal{B})$.

By "rewriting a 3d theory as a 1d theory," we mean to reinterpret all the fields of the 3d theory on $\mathbb{C}_{z,\overline{z}} \times \mathbb{R}_t$ as fields on $\mathbb{R}_t$ valued in functions (or sections of various bundles) on $\mathbb{C}_{z,\overline{z}}$. Given a 3d gauge group G and representation $R \oplus R^*$, the 3d $\mathcal{N} = 4$ multiplets decompose under the 1d SQM$_A$ subalgebra[13] as follows:

- The 3d hypermultiplets split into pairs of 1d chiral multiplets, with bottom components X and Y. More precisely, the bottom components are maps $X(z,\overline{z})$, $Y(z,\overline{z})$ from the $\mathbb{C}_{z,\overline{z}}$ plane into the original target space $R \oplus R^*$ of the 3d theory.

- The 1d gauge group consists of all G-valued gauge transformations $g(z,\overline{z})$ on the $\mathbb{C}_{z,\overline{z}}$ plane. We will denote this infinite-dimensional group as $\mathcal{G}$.

- The 3d vector multiplet splits into 1) a 1d vector multiplet for the gauge group $\mathcal{G}$,

[13]Note that the 1d $\mathcal{N} = 4$ multiplets used here are sometimes denoted "$\mathcal{N} = (2,2)$" multiplets in the literature. This is because they are the multiplets one obtains by reducing 2d $\mathcal{N} = (2,2)$ chiral and vector multiplets to 1d.

containing the connection A_t and the triplet of scalars $\sigma, \varphi, \varphi^\dagger$; and 2) a 1d chiral multiplet with bottom component $A_{\overline{z}}$.

The supersymmetric Lagrangian for this 1d $\mathcal{N} = 4$ quantum mechanics includes an important superpotential term

$$W = \int_{\mathbb{C}_{z,\overline{z}}} d^2 z \, \mathrm{Tr}\, X D_{\overline{z}} Y \,, \tag{1.3.14}$$

which captures the kinetic terms for X and Y on $\mathbb{C}_{z,\overline{z}}$. Note that the superpotential involves the chiral multiplet $A_{\overline{z}}$ (in $D_{\overline{z}} = \partial_{\overline{z}} - iA_{\overline{z}}$) as well as X and Y.

In this 1d $\mathcal{N} = 4$ quantum mechanics, the $\frac{1}{2}$-BPS equations (1.3.12) may now be interpreted as familiar equations for SUSY vacua (*i.e.* full-BPS equations). In particular, the F-term equations coming from W reproduce most of (1.3.12d):

$$\frac{\delta W}{\delta X} = D_{\overline{z}} Y = 0 \,, \qquad \frac{\delta W}{\delta Y} \sim D_{\overline{z}} X = 0 \,, \qquad \frac{\delta W}{\delta A_{\overline{z}}} \sim \mu_{\mathbb{C}} = 0 \,. \tag{1.3.15}$$

The remaining vortex equation $F_{z\overline{z}} - \mu_{\mathbb{R}} = 0$ appears as a D-term in the supersymmetric quantum mechanics.

Regardless of their interpretation, a common technique for analyzing the vortex equations (1.3.12d) involves trading the real D-term equation $F_{z\overline{z}} = \mu_{\mathbb{R}}$ for a complexification of the gauge group. In mathematics, this is often called a Kobayashi-Hitchin correspondence (with a prototypical realization in the Donaldson-Uhlenbeck-Yau Theorem [162, 163]). This ultimately allows a complex-analytic or (even better) an algebraic description of the moduli space of solutions. We briefly recall the basic ideas, aiming to provide intuition rather than mathematical rigor.

Recall that the first three vortex equations $D_{\overline{z}} X = D_{\overline{z}} Y = \mu_{\mathbb{C}} = 0$ are critical-point equations for the superpotential W in (1.3.14), whereas $F_{z\overline{z}} = \mu_{\mathbb{R}}$ is a real D-term constraint for the infinite-dimensional gauge group $\mathcal{G}$ of all G-valued gauge transformations the $\mathbb{C}_{z,\overline{z}}$ plane. The space of solutions to the vortex equations on $\mathbb{C}_{z,\overline{z}}$ is thus a real symplectic quotient

$$\begin{aligned}\mathcal{M} &= \{A, X, Y \text{ s.t. } \delta W = 0\}/\!/\mathcal{G} \\ &= \{A, X, Y \text{ s.t. } \delta W = 0 \text{ and } F_{z\bar{z}} = \mu_{\mathbb{R}}\}/\mathcal{G}\,.\end{aligned} \tag{1.3.16}$$

By comparison with the finite-dimensional setting [164, 165], we *expect* to be able to ignore the D-term constraint while at the same time complexifying the gauge group $\mathcal{G} \rightsquigarrow \mathcal{G}_{\mathbb{C}}$, and possibly imposing some stability conditions. Roughly, we should have

$$\mathcal{M} \approx \{A, X, Y \text{ s.t. } \delta W = 0\}/\mathcal{G}_{\mathbb{C}}\,, \tag{1.3.17}$$

where $\mathcal{G}_{\mathbb{C}}$ is the group of all $G_{\mathbb{C}}$-valued gauge transformations the $\mathbb{C}_{z,\bar{z}}$ plane.

In (1.3.17), we can further use complexified gauge transformations to gauge-fix $A_{\bar{z}} = 0$, so that the covariant derivative $D_{\bar{z}} = \partial_{\bar{z}}$ becomes an ordinary derivative. We are left with a residual gauge group consisting of holomorphic gauge transformations $\mathcal{G}_{\mathbb{C}}^{\text{hol}} = \{g(z) \in \mathcal{G}_{\mathbb{C}} \text{ s.t. } \partial_{\bar{z}} g = 0\}$, and a complex-analytic moduli space

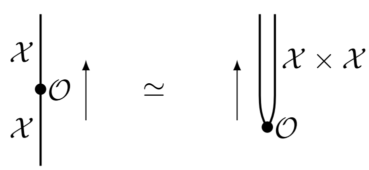
$$\mathcal{M} \approx \left\{\begin{array}{c}\text{holomorphic } G_{\mathbb{C}} \text{ bundle } E \text{ on } \mathbb{C}_{z,\bar{z}} \\ \text{with holo. sections } X(z), Y(z) \text{ of an associated } R \oplus R^* \text{ bundle} \\ \text{such that } \mu_{\mathbb{C}}(X, Y) = 0\end{array}\right\} \Big/ \mathcal{G}_{\mathbb{C}}^{\text{hol}}\,. \tag{1.3.18}$$

Making the equivalence of (1.3.16) and (1.3.18) precise can be a subtle and difficult endeavor. One must specify suitable ($\frac{1}{2}$-BPS) boundary conditions, *e.g.* a vacuum boundary condition $\mathcal{B}_\nu$ as $|z| \to \infty$, as well as stability conditions for the $G_{\mathbb{C}}$-bundles and holomorphic sections appearing in (1.3.18). Some of the mathematical history of this endeavor, starting with [166, 167] for abelian G, was reviewed in the introduction. A holomorphic/algebraic formulation of these moduli spaces with vacuum boundary condition $\mathcal{B}_\nu$ was established relatively recently by [168].

Local operators and convolution

The action of local operators $\mathcal{O}$ on Hilbert spaces $\mathcal{H}_A(\mathcal{B})$ acquires a natural description in terms of a *convolution product* in cohomology. Earlier, such convolution products were used to define the OPE in the Coulomb-branch chiral ring defined by Braverman-Finkelberg-Nakajima [71,72] (see also [101, App. A] for related discussion).

The convolution product is simply an implementation of a state-operator correspondence in A-type quantum mechanics. Let's briefly review this idea. To keep things simple, consider A-type (de Rham type) $\mathcal{N}=2$ quantum mechanics with a smooth, compact target $\mathcal{X}$ and nilpotent supercharge Q.[14] The Q-cohomology of the Hilbert space is $\mathcal{H}=H^\bullet(\mathcal{X})$. The state-operator correspondence in topological quantum mechanics says that the (Q-cohomology of the) space of local operators Ops at a point is isomorphic to the Hilbert space on the sphere S^0 linking the point. Since S^0 is just two points (with opposite orientations), our theory on $S^0\times\mathbb{R}$ is just two non-interacting copies of the theory on $\mathbb{R}$,



(1.3.19)

In other words, it's quantum mechanics with target $\mathcal{X}\times\mathcal{X}$. We deduce that local operators are

$$H^\bullet_Q(\mathrm{Ops})=H^\bullet(\mathcal{X}\times\mathcal{X})\,. \tag{1.3.20}$$

From one perspective, this result is hardly surprising. Using the Künneth formula and Hodge duality, we have an isomorphism

$$H^\bullet_Q(\mathrm{Ops})\simeq H^\bullet(\mathcal{X})\otimes H^\bullet(\mathcal{X})\overset{\text{Hodge}}{\simeq} H^\bullet(\mathcal{X})\otimes H^\bullet(\mathcal{X})^*\simeq \mathrm{End}_{\mathbb{C}}(H^\bullet(\mathcal{X}))\,. \tag{1.3.21}$$

[14] We are ultimately interested in analyzing a 1d $\mathcal{N}=4$ theory. However, the relevant structure of convolution products shows up already for 1d $\mathcal{N}=2$, which is why we consider a more general $\mathcal{N}=2$ setup here. The fact that we actually have 1d $\mathcal{N}=4$ SUSY leads to additional features, such as the ability to compute Hilbert spaces via fixed-point localization.

This is just the full set of linear transformations acting on the complex vector space $H^\bullet(\mathcal{X})$.

However, there is also an intrinsic geometric description of the OPE on $H^\bullet_Q(\text{Ops})$ and its action on $\mathcal{H}$ that avoids (or rather, repackages) Hodge duality, and which we will generalize momentarily. To see the action of $H^\bullet_Q(\text{Ops})$ on $\mathcal{H}$, we use the two maps from $\mathcal{X} \times \mathcal{X}$ to $\mathcal{X}$, coming from projection onto the first and second factors

$$\begin{array}{ccc} & \mathcal{X}_2 \times \mathcal{X}_1 & \\ \pi_2 \swarrow & & \searrow \pi_1 \\ \mathcal{X}_2 & & \mathcal{X}_1 \end{array} \tag{1.3.22}$$

This is called a convolution diagram. Given a local operator $\mathcal{O} \in H^\bullet(\mathcal{X}_2 \times \mathcal{X}_1)$, we can define an action on $\mathcal{H} = H^\bullet(\mathcal{X}_1) = H^\bullet(\mathcal{X}_2)$ by

$$\mathcal{O} : \begin{array}{ccc} H^\bullet(\mathcal{X}_1) & \longrightarrow & H^\bullet(\mathcal{X}_2) \\ v & \mapsto & (\pi_2)_*(\mathcal{O} \wedge \pi_1^*(v))\,. \end{array} \tag{1.3.23}$$

The Hodge duality above has been repackaged in the push-forward map $(\pi_2)_*$, which involves an integration along the fibers of π_2, *i.e.* an integration along $\mathcal{X}_1$.[15]

Similarly, the product (the algebra structure) on $H^\bullet_Q(\text{Ops})$ comes geometrically from considering three copies of $\mathcal{X}$, and projections onto pairs of factors

$$\begin{array}{ccc} & \mathcal{X}_3 \times \mathcal{X}_2 \times \mathcal{X}_1 & \\ \pi_{31} \swarrow & \downarrow \pi_{32} & \searrow \pi_{21} \\ \mathcal{X}_3 \times \mathcal{X}_1 & \mathcal{X}_3 \times \mathcal{X}_2 & \mathcal{X}_2 \times \mathcal{X}_1 \end{array} \tag{1.3.24}$$

Given any $\mathcal{O} \in H^\bullet(\mathcal{X}_2 \times \mathcal{X}_1)$ and $\mathcal{O}' \in H^\bullet(\mathcal{X}_3 \times \mathcal{X}_2)$, there is now a "convolution product"

[15] The whole story may be even more familiar in standard, non-supersymmetric quantum mechanics. Consider a particle moving on $\mathcal{X} = \mathbb{R}$, with Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. Linear operators $\mathcal{O} : \mathcal{H} \to \mathcal{H}$ can be represented by their integral kernel $K_{\mathcal{O}}(x,y)$ (in a manner made precise by the Schwartz kernel theorem) so that $\mathcal{O}|x\rangle = \int dx\, K_{\mathcal{O}}(x,y)|y\rangle$. This is the convolution product of (1.3.22), in the infinite-dimensional setting. Similarly, the product of two operators is represented as convolution of their kernels $K_{\mathcal{O}'\cdot\mathcal{O}}(x,y) = \int dz\, K_{\mathcal{O}'}(x,z)K_{\mathcal{O}}(z,y)$, analogous to (1.3.24).

defined by

$$\mathcal{O}' \cdot \mathcal{O} = (\pi_{31})_* \big(\pi_{32}^*(\mathcal{O}') \wedge \pi_{21}^*(\mathcal{O})\big) \tag{1.3.25}$$

Ignoring the indices and identifying $H^\bullet(\mathcal{X}_2 \times \mathcal{X}_1) = H^\bullet(\mathcal{X}_3 \times \mathcal{X}_2) = H^\bullet(\mathcal{X}_3 \times \mathcal{X}_1) = H^\bullet_Q(\text{Ops})$, we see that this is a product $H^\bullet_Q(\text{Ops}) \times H^\bullet_Q(\text{Ops}) \to H^\bullet_Q(\text{Ops})$. Working through the various push-forwards and pull-backs involved, one can show that it is the *same* as the more naive product resulting from the identification $H^\bullet_Q(\text{Ops}) \simeq \text{End}_{\mathbb{C}}(H^\bullet(\mathcal{X}))$ in (1.3.21).

Note that the same sort of analysis could have been used to describe local operators at $\frac{1}{2}$-BPS *junctions* of $\mathcal{N} = 2$ quantum mechanics theories, with different targets. At a junction of SQM with target $\mathcal{X}_1$ and SQM with target $\mathcal{X}_2$, the local operators are $H^\bullet_Q(\text{Ops}(\mathcal{X}_1, \mathcal{X}_2)) = H^\bullet(\mathcal{X}_2 \times \mathcal{X}_1)$. They act on states in the Hilbert space $H^\bullet(\mathcal{X}_1)$ to produce states in $H^\bullet(\mathcal{X}_2)$. Geometrically, the action comes from the same convolution diagram (1.3.22), now with $\mathcal{X}_1$ and $\mathcal{X}_2$ interpreted as (potentially) different spaces.

The OPE coming from collision of two different junctions — say between SQM with targets $\mathcal{X}_1, \mathcal{X}_2$ and SQM with targets $\mathcal{X}_2, \mathcal{X}_3$ — is also encoded geometrically in the convolution diagram (1.3.24), with $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ interpreted as potentially different spaces.

Returning to 3d, the algebra of local operators would naively be realized via convolution through a space that roughly looks like $\mathcal{M}(\mathcal{B}) \times \mathcal{M}(\mathcal{B})$. However, this is much too large from the perspective of operators in the 3d theory, as it includes operators with arbitrary support on $\mathbb{C}_{z,\bar{z}}$. Instead, we consider a smaller space where the two BPS configurations agree (up to gauge transformation) away from the location of the putative local operator, say $z = 0$, *i.e.* pairs of BPS configurations that differ from one another via a (possibly singular) gauge transformation:

$$\mathcal{M}_{\text{rav}}(\mathcal{B}) = \left\{ \begin{array}{c} \text{pairs of solutions to the SQM}_A \text{ BPS equations} \\ \text{each compatible with } \mathcal{B} \\ \text{equivalent to each other away from } z=0 \end{array} \right\}$$
$$= \left\{ \begin{array}{c} \text{pairs } (E', X', Y'; E, X, Y) \text{ as in (1.3.18)} \\ \text{compatible with } \mathcal{B} \\ \text{and a bundle morphism } g : E \to E' \\ \text{with } (X', Y') = g.(X, Y) \\ \text{such that } g \text{ is an isomorphism away from } z=0 \end{array} \right\} \Big/ \mathcal{G}^{\text{hol}}_{\mathbb{C}} \times \mathcal{G}^{\text{hol}}_{\mathbb{C}} \,. \tag{1.3.26}$$

Mathematically, the construction in (1.3.26) is called a fiber product; the product is "fibered over" the moduli space of solutions to the BPS equations on the punctured plane $\mathbb{C}^*_{z,\bar{z}}$ $\mathcal{M}^*(\mathcal{B})$:

$$\mathcal{M}_{\text{rav}}(\mathcal{B}) = \mathcal{M}(\mathcal{B}) \times_{\mathcal{M}^*(\mathcal{B})} \mathcal{M}(\mathcal{B})\,. \tag{1.3.27}$$

Note that the fiber product is a subspace of the ordinary product, $\mathcal{M}_{\text{rav}}(\mathcal{B}) \subseteq \mathcal{M}(\mathcal{B}) \times \mathcal{M}(\mathcal{B})$.[16]

Cohomology classes $\mathcal{O} \in H^\bullet\big(\mathcal{M}_{\text{rav}}(\mathcal{B})\big)$ have natural convolution products and/or actions. The most direct way to see this is to note that there are convolution diagrams involving $\mathcal{M}_{\text{rav}}(\mathcal{B})$ alone:

$$\begin{array}{ccc} & \mathcal{M}(\mathcal{B}) \times_{\mathcal{M}^*(\mathcal{B})} \mathcal{M}(\mathcal{B}) = \mathcal{M}_{\text{rav}}(\mathcal{B}) & \\ \pi_2 \swarrow & & \searrow \pi_1 \\ \mathcal{M}(\mathcal{B}) & & \mathcal{M}(\mathcal{B}) \end{array} \tag{1.3.28}$$

where the maps are given by forgetting $(g; E, X, Y)$ and $(E', X', Y'; g)$. This diagram is used in computing the action $H^\bullet\big(\mathcal{M}_{\text{rav}}(\mathcal{B})\big) : H^\bullet\big(\mathcal{M}(\mathcal{B})\big) \to H^\bullet\big(\mathcal{M}(\mathcal{B})\big)$ which schematically takes the form of pushing-forward (via π_2) the intersection of a class in $H^\bullet\big(\mathcal{M}_{\text{rav}}(\mathcal{B})\big)$ with the pullback (via π_1) of a class in $H^\bullet\big(\mathcal{M}(\mathcal{B})\big)$, *cf.* Eq. (1.3.23). This can be realized pictorially

[16]We use the notation "rav" because, schematically, $\mathcal{M}_{\text{rav}}(\mathcal{B})$ is the space of solutions to BPS equations on $\mathbb{C}_{z,\bar{z}} \cup_{\mathbb{C}^*_{z,\bar{z}}} \mathbb{C}_{z,\bar{z}}$, *i.e.* on two copies of the $\mathbb{C}_{z,\bar{z}}$ plane, identified over the punctured plane $\mathbb{C}^*_{z,\bar{z}}$. The union of planes $\mathbb{C}_{z,\bar{z}} \cup_{\mathbb{C}^*_{z,\bar{z}}} \mathbb{C}_{z,\bar{z}}$ looks like a "raviolo."

as in Figure 1.2.

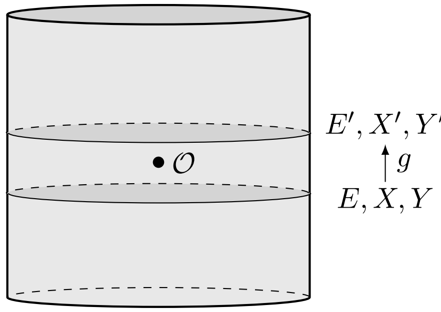


Figure 1.2: An illustration of the convolution action in Eq. (1.3.28).

Similarly, there is a convolution diagram

$$\begin{array}{ccc} & \mathcal{M}(\mathcal{B}) \times_{\mathcal{M}^*(\mathcal{B})} \mathcal{M}(\mathcal{B}) \times_{\mathcal{M}^*(\mathcal{B})} \mathcal{M}(\mathcal{B}) & \\ \pi_{31} \swarrow & \downarrow \pi_{32} & \searrow \pi_{21} \\ \mathcal{M}_{\text{rav}}(\mathcal{B}) & \mathcal{M}_{\text{rav}}(\mathcal{B}) & \mathcal{M}_{\text{rav}}(\mathcal{B}) \end{array} \tag{1.3.29}$$

computing a product $H^\bullet\big(\mathcal{M}_{\text{rav}}(\mathcal{B})\big) \otimes H^\bullet\big(\mathcal{M}_{\text{rav}}(\mathcal{B})\big) \to H^\bullet\big(\mathcal{M}_{\text{rav}}(\mathcal{B})\big)$ from collision of junctions. This product is represented pictorially in Figure 1.3.

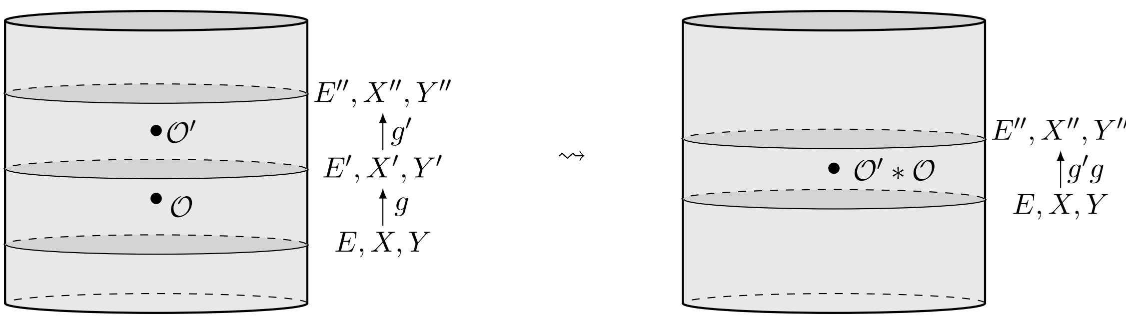


Figure 1.3: An illustration of the convolution product in Eq. (1.3.29).

Mathematically, restricting convolution algebras to fiber products rather than ordinary direct products, as we did here, is a common operation. A thorough discussion of such products and their use in geometric representation theory is contained in [169]. We saw above that convolution coming from ordinary products was a little boring: it just reproduced the full algebra of linear transformations (a matrix algebra) acting on a vector space. In contrast, convolution with fiber products can define interesting and highly nontrivial subalgebras.

Borel-Moore homology

We have been vague so far about precisely *which* (co)homology theory we should be using to compute the spaces of supersymmetric states and local operators. The moduli spaces in question split into finite-dimensional components, which helps. However, the components are typically noncompact and singular. Physically, we should ask ourselves how to interpret supersymmetric quantum mechanics with a noncompact and singular target space. We consider these potential problems one at a time.

To handle noncompactness, we introduce equivarance. This standard technique is familiar from classic work on localization of supersymmetric path integrals [19, 170–173]. Given A-type SQM with a Riemannian target $\mathcal{X}$, and an abelian isometry group T that acts on $\mathcal{X}$ (a flavor symmetry), one may turn on twisted masses for T. The twisted masses m take values in the complexified Lie algebra $\mathfrak{t}_{\mathbb{C}}$. Physically, they introduce a scalar potential $|mV|^2$, where $V \in \mathfrak{t}^* \otimes T\mathcal{X}$ is the vector field generating the T action. For generic m, the potential localizes physical wavefunctions to a neighborhood of the fixed locus of T. In particular, if $\mathcal{X}$ happens to be noncompact but the T action has a compact fixed locus, low-energy wavefunctions will decay exponentially near infinity. The SUSY ground states are well defined, and become identified with classes in T-equivariant cohomology.

In the case at hand, we introduce twisted masses $m_{\mathbb{C}}$ for a torus of the Higgs-branch flavor symmetry T_F and ε for the diagonal $U(1)_\varepsilon$ subgroup of $U(1)_E \times U(1)_H$ that includes rotations in the z-plane and leaves the supercharge Q_A invariant. The former twisted masses are simply the complex masses of the bulk 3d $\mathcal{N} = 4$ theory, and the latter amounts to turning on the A-type Omega background. Then we work with the equivariant cohomologies

$$H^\bullet_{T_F \times U(1)_\varepsilon}\big(\mathcal{M}(\mathcal{B})\big)\,, \qquad H^\bullet_{T_F \times U(1)_\varepsilon}\big(\mathcal{M}_{\rm rav}(\mathcal{B})\big)\,. \tag{1.3.30}$$

In the examples we study, the $T_F \times U(1)_\varepsilon$ actions actually has isolated fixed points. Then, by localization [174, 175] we will be able to describe the full equivariant cohomology in terms of appropriate linear combinations of fixed-point classes, *i.e.* delta-function classes at the fixed points. We also note that our moduli spaces $\mathcal{M}(\mathcal{B}), \mathcal{M}_{\rm rav}(\mathcal{B})$ are Kähler, and the $T_F \times U(1)_\varepsilon$ metric isometry extends holomorphically to a complex isometry $T_{F,\mathbb{C}} \times \mathbb{C}^*_\varepsilon$, with the same fixed-point set. In an algebraic context, we will consider $H^\bullet_{T_{F,\mathbb{C}}\times\mathbb{C}^*_\varepsilon}\big(\mathcal{M}(\mathcal{B})\big)$, $H^\bullet_{T_{F,\mathbb{C}}\times\mathbb{C}^*_\varepsilon}\big(\mathcal{M}_{\rm rav}(\mathcal{B})\big)$ instead of (1.3.30); however, the two are completely equivalent.

Mathematically, the spaces (1.3.30) are modules for the polynomial algebra $\mathbb{C}[m_\mathbb{C}, \varepsilon]$ of equivariant parameters. Since the $T_F \times U(1)_\varepsilon$ (or $T_{F,\mathbb{C}} \times \mathbb{C}^*_\varepsilon$) action has fixed points, they are free modules: no constraints are imposed on $m_\mathbb{C}, \varepsilon$. Moreover, physically, $m_\mathbb{C}$ and ε are fixed complex numbers, so $\mathbb{C}[m_\mathbb{C}, \varepsilon] \simeq \mathbb{C}$. Thus, there is no interesting structure in the $\mathbb{C}[m_\mathbb{C}, \varepsilon]$ action, and we will usually leave it implicit.

We may go a step further. Since the moduli spaces take the form $\mathcal{M}(\mathcal{B}) = \mathcal{G}^{\rm hol}_\mathbb{C} \backslash \widetilde{\mathcal{M}}(\mathcal{B})$, there is an equivalence with $\mathcal{G}^{\rm hol}_\mathbb{C}$-equivariant cohomology of $\widetilde{\mathcal{M}}(\mathcal{B})$,

$$H^\bullet_{T_{F,\mathbb{C}}\times\mathbb{C}^*_\varepsilon}\big(\mathcal{M}(\mathcal{B})\big) \simeq H^\bullet_{\mathcal{G}^{\rm hol}_\mathbb{C}\times T_{F,\mathbb{C}}\rtimes\mathbb{C}^*_\varepsilon}\big(\widetilde{\mathcal{M}}(\mathcal{B})\big)\,. \tag{1.3.31}$$

The RHS is naturally a module for $\mathbb{C}[\varphi]^{{\rm Weyl}(G)}$, the polynomial algebra in equivariant parameters for the constant gauge transformations in $\mathcal{G}^{\rm hol}_\mathbb{C}$, invariant under the Weyl group. Physically, the φ's are the bulk vector multiplet scalars. When $\mathcal{G}^{\rm hol}_\mathbb{C}$ action on $\widetilde{\mathcal{M}}(\mathcal{B})$ is free (it does *not* have fixed points), the corresponding action of $\mathbb{C}[\varphi]^{{\rm Weyl}(G)}$ on equivariant cohomology is interesting. It is literally the action of the Coulomb-branch φ operators in the space of boundary operators.

Similarly, raviolo spaces are of the form $\mathcal{M}_{\rm rav}(\mathcal{B}) = \mathcal{G}^{\rm hol\prime}_\mathbb{C} \backslash \widetilde{\mathcal{M}}_{\rm rav}(\mathcal{B}) / \mathcal{G}^{\rm hol}_\mathbb{C}$, so equivariant cohomology can be lifted

$$H^\bullet_{T_{F,\mathbb{C}}\times\mathbb{C}^*_\varepsilon}\big(\mathcal{M}_{\rm rav}(\mathcal{B})\big) \simeq H^\bullet_{\mathcal{G}^{\rm hol\prime}_\mathbb{C}\times\mathcal{G}^{\rm hol}_\mathbb{C}\times T_{F,\mathbb{C}}\rtimes\mathbb{C}^*_\varepsilon}\big(\widetilde{\mathcal{M}}_{\rm rav}(\mathcal{B})\big)\,. \tag{1.3.32}$$

The RHS is a module for $\mathbb{C}[\varphi', \varphi]^{{\rm Weyl}(G')\times{\rm Weyl}(G)}$, where φ' and φ are the vector multiplet scalars acting above and below the junction. Neither $\mathcal{G}^{\rm hol\prime}_\mathbb{C}$ nor $\mathcal{G}^{\rm hol}_\mathbb{C}$ has fixed points, so both

φ' and φ are set to constants.

Equivariant cohomology takes care of noncompactness. However, we must still deal with the fact that the moduli spaces $\mathcal{M}(\mathcal{B})$ and $\mathcal{M}_{\text{rav}}(\mathcal{B})$ may be singular. In [70], this issue was deftly avoided, because (in the examples studied there) the bulk Coulomb-branch chiral ring was generated by monopole operators of minuscule charge, which could be captured by subspaces of $\mathcal{M}_{\text{rav}}$, which turned out to be smooth. In the presence of nontrivial line operators, the spaces $\mathcal{M}_{\text{rav}}$ are almost *never* smooth. So there is a genuine and practical difficulty to overcome.

The singularities of $\mathcal{M}_{\text{rav}}(\mathcal{B})$ are an artifact of our simplifications from Section 1.3.2. In particular, they arise from restricting to solutions of BPS equations in Step 2, which propagates to the definition of $\mathcal{M}_{\text{rav}}(\mathcal{B})$ as a fiber product. A physically rigorous analysis would return to the very-infinitely-dimensional space of *all* field configurations in the presence of a monopole singularity, and then impose BPS equations by turning on a suitable potential. We will shortcut such an analysis with a well-motivated guess: when $\mathcal{M}_{\text{rav}}(\mathcal{B})$ or $\mathcal{M}(\mathcal{B})$ spaces turn out to be singular, we will take their (renormalized) equivariant *Borel-Moore homology*.

This is a topological homology theory that is Poincaré-dual to cohomology with compact support; a thorough review, relevant for convolution constructions, is contained in [169]. Notably, Borel-Moore homology was used in the Braverman-Finkelberg-Nakajima constructions of Coulomb-branch chiral rings [71, 72], which we review below.

Motivated in particular by [50,51,71,72], we will use equivariant Borel-Moore whenever we encounter singular spaces. This is how the "co"homologies used to describe supersymmetric Hilbert space and algebra of local operators in this thesis are to be interpreted. Important examples of spaces with unavoidable singularities will appear in Section 2.4.

There is another option for handling singularities and noncompactness, which might be deemed equally reasonable from a physical perspective: instead of equivariant Borel-Moore homology, one might use equivariant intersection cohomology. Intersection cohomology is intimately related to L^2 cohomology [176, 177]. More practically, mathematical constructions using intersection cohomology have been known to match physical expectations in many setups similar to ours. This includes the identification of the Satake category [178] (generated by intersection cohomology sheaves) with 't Hooft lines of 4d super-Yang-Mills [33, Sec. 10]. More directly: for particular classes of 3d $\mathcal{N}=4$ theories whose bulk Coulomb-branch chiral rings are expected to be finite W-algebras, the spaces $\mathcal{M}_{\text{rav}}(\mathcal{B}_\nu)$ appeared in [179, 180]; it was shown there that their equivariant cohomology reproduces the desired W-algebras. (This mathematical work was an important inspiration for [70].)

In our actual examples, we will only encounter relatively mild singularities, modeled locally on transverse intersections such as $\{xy=0\}\subset\mathbb{C}^2$. Equivariant intersection cohomology and equivariant Borel-Moore homology give exactly the same answers in these cases. (Both are computed using a normalization of the singularity, *e.g.* pulling $\{xy=0\}$ apart to $\{x=0\}\sqcup\{y=0\}$.) Thus, so far, both seem equally good for matching physical expectations. Strictly speaking, the pull-back maps (2.3.24) and infinite sums (2.3.25) that appear in the representations of monopole operators only make sense in Borel-Moore homology, so the latter may well be a better mathematical model to use.

1.3.3 The BFN construction

Given the above physical description of local operators, it is possible to recover the recent construction of the algebra of local operators in the A-twist, *i.e.* (holomorphic) functions on the Coulomb branch $\mathcal{M}_C$, due to Braverman-Finkelberg-Nakajima (BFN) [71, 72].

We start with the cylinder setup, *i.e.* on $D\times\mathbb{R}_t$ with a boundary condition at ∂D. Since the theory is topological, we can deform the $D\times R_t$ spacetime to a half-spacetime $\mathbb{C}\times\mathbb{R}_{t\geq 0}$ with boundary condition at $t=0$. Under this deformation, a state on D gets identified with a state on a hemisphere surrounding, say, the point $z=0$ on the boundary. Again, the size of this hemisphere is arbitrary so it suffices to work in an infinitesimal neighborhood of the point $z=0$. An algebraic model for this infinitesimal hemisphere anchored to the boundary is as an infinitesimal or "formal" disk $\mathbb{D}$ (in the bulk) identified with another formal disk $\mathbb{D}$ (on the boundary) along their boundaries, a "formal" punctured disk $\mathbb{D}^*$.

States on this infinitesimal hemisphere can be identified with homology classes of a moduli

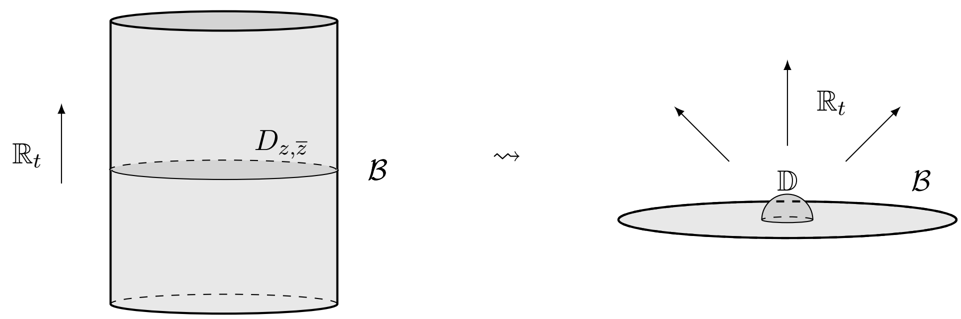


Figure 1.4: Deformation of the cylinder setup described in Section 1.3.2 to the half-space setup of the present section.

space $\mathcal{M}_{\mathbb{D}}(\mathcal{B})$, where

$$\mathcal{M}_{\mathbb{D}}(\mathcal{B}) = \left\{ \begin{array}{c} \text{solutions to SQM}_A \text{ BPS equations on a formal disk } \mathbb{D} \\ \text{compatible with } \mathcal{B} \text{ on its boundary } \partial\mathbb{D} = \mathbb{D}^* \end{array} \right\} \tag{1.3.33}$$

In general, this space is an infinite-dimensional stack, and it seems from the mathematics literature that one correct way to interpret $H_\bullet\big(\mathcal{M}_{\mathbb{D}}(\mathcal{B})\big)$ is via equivariant Borel-Moore homology.

We now introduce some standard algebraic notation that will be useful in the remainder of this thesis. The holomorphic functions on the formal disk $\mathbb{D}$ are formal Taylor series. The ring of formal Taylor series is denoted

$$\mathcal{O} = \mathbb{C}[\![z]\!]\,. \tag{1.3.34}$$

The group of complexified, holomorphic gauge transformations (preserved in holomorphic gauge) on $\mathbb{D}$ is $\mathcal{G}_{\mathbb{C}}^{\text{hol}} = G(\mathcal{O})$, where

$$G(\mathcal{O}) := \{\text{the algebraic group } G_{\mathbb{C}} \text{ defined over formal Taylor series } \mathcal{O}\} \tag{1.3.35}$$

is an algebraic version of the positive loop group. In the case of $G = U(n)$, the group $G(\mathcal{O})$ simply consists of invertible $n \times n$ matrices whose entries are formal Taylor series in z.

Similarly, the holomorphic functions in an infinitesimal punctured neighborhood of the origin $\mathbb{D}^*$ — with a possible meromorphic singularity at the origin — are formal Laurent series, denoted

$$\mathcal{K} = \mathbb{C}(\!(z)\!)\,. \tag{1.3.36}$$

The ring $\mathcal{K}$ is an algebraic version of the loop space $L\mathbb{C}$. Moreover, the group of complexified, holomorphic gauge transformations on $\mathbb{D}^*$ is $G(\mathcal{K})$, which is defined over $\mathcal{K}$ the same way as $G(\mathcal{O})$ is defined over $\mathcal{O}$.[17] Informally, elements of $G(\mathcal{K})$ are often called singular gauge transformations.

In order to recover the BFN construction, we consider a boundary condition $\mathcal{B}_R$ that preserves a 2d $\mathcal{N} = (2,2)$ subalgebra of 3d $\mathcal{N} = 4$ (including Q_A), defined by

- Setting the hypermultiplet scalars Y (which are valued in R^*) to zero at the boundary, and extending this to the entire hypermultiplet in a way that preserves 2d $\mathcal{N} = (2,2)$ supersymmetry. In particular, $X \in R$ will get a Neumann-like boundary condition, so that the values of X are unconstrained at the boundary.

- Preserving gauge symmetry at the boundary, meaning Neumann boundary conditions for the 3d gauge field, extended to the entire 3d vector multiplet in a way that preserves 2d $\mathcal{N} = (2,2)$ SUSY. In particular, the complex scalars φ also receive a Neumann-like boundary condition, so their values at the boundary are unconstrained.

See [40, 111] for further details. Note that this choice of boundary condition forces $Y = 0$ on the entire formal disk $\mathbb{D}$ since it is a holomorphic section that vanishes for all $z \neq 0$. In

[17]Naively, one may want to consider here the group of *holomorphic* gauge transformations in an infinitesimal punctured neighborhood of $z = 0$. However, there is now a big difference between holomorphic and algebraic: the former contain gauge transformations with essential singularities, whereas the latter only contain meromorphic gauge transformations. We refer the reader to a careful discussion in [181] on how to interpret the distinction physically, and why a restriction to algebraic gauge transformations is sensible.

particular, we have the following description of the moduli space:

$$\mathcal{M}_{\mathbb{D}}(\mathcal{B}_R) = \left\{ \begin{array}{c} E, X \text{ such that } E \text{ is an algebraic } G_{\mathbb{C}} \text{ bundle on a formal disc} \\ \text{and } X \text{ is an algebraic section of an associated } R\text{-bundle} \end{array} \right\} \\ = R(\mathcal{O})/G(\mathcal{O}), \tag{1.3.37}$$

where the points of $R(\mathcal{O})$ are the $\mathcal{O}$-valued points of R, *i.e.* $R(\mathcal{O})$ is R-valued formal series in z. Note that, since $R(\mathcal{O})$ is contractible and $G(\mathcal{O})$ contracts to G, the vector space of local operators on the boundary condition $\mathcal{B}_R$ is isomorphic to $H^\bullet_G(\text{point}) \cong \mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{t}]^W$, where W is the Weyl group of G.

We can similarly construct the analog of the raviolo space given in Eq. (1.3.26):

$$\begin{aligned} \mathcal{M}_{\text{rav}}(\mathcal{B}_R) &= \left\{ \begin{array}{c} \text{pairs of solutions to the SQM}_A \text{ BPS equations} \\ \text{each compatible with } \mathcal{B}_R \\ \text{equivalent to each other away from } z=0 \end{array} \right\} \\ &= \left\{ \begin{array}{c} E', X'; g; E, X \text{ s.t. } E', X' \text{ and } E, X \text{ are as in (1.3.37)} \\ \text{and } g(z) \in G(\mathcal{K}) \text{ is an isomorphism away from } z=0 \end{array} \right\} /G(\mathcal{O})' \times G(\mathcal{O}), \\ &= \begin{array}{c} G(\mathcal{O})' \backslash R(\mathcal{O}) \times G(\mathcal{K}) \times R(\mathcal{O})\big|_{(*)} / G(\mathcal{O}) \\ \quad\; X' \qquad\quad g \qquad\quad X \end{array}. \end{aligned} \tag{1.3.38}$$

where the constraint $(*)$ requires $X' = gX$. We realize local operators as homology classes of $\mathcal{M}_{\text{rav}}(\mathcal{B}_R)$ and realize its algebra structure as well as its action on the Borel-Moore homology of $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_R)$ using the convolution diagrams described in Section 1.3.2.

The above physical setup reproduces the BFN construction of the Coulomb-branch chiral ring [71, 72]. More precisely, Braverman-Finkelberg-Nakajima consider the "space of triples"

$$\begin{aligned} \mathcal{R}_{G,R} &= \big\{(g, X) \in G(\mathcal{K}) \times R(\mathcal{O}) | gX \in R(\mathcal{O})\big\}/G(\mathcal{O}) \\ &= R(\mathcal{O}) \times G(\mathcal{K}) \times R(\mathcal{O})|_{(*)}/G(\mathcal{O}), \end{aligned} \tag{1.3.39}$$

and the define the ring of functions on the Coulomb-branch chiral ring as it ($G(\mathcal{O})$-equivariant) Borel-Moore homology of $\mathcal{R}_{G,R}$:

$$\mathbb{C}[\mathcal{M}_C] = H_\bullet^{G(\mathcal{O})}(\mathcal{R}_{G,R}) \simeq H_\bullet\big(\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_R)\big)\,. \tag{1.3.40}$$

For pure gauge theory, with $R = 0$, the boundary condition $\mathcal{B}_R$ is canonical — the only choice made is to put Neumann b.c. on the gauge fields. In this case the raviolo space above reduces to

$$\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_0) = G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}) = G(\mathcal{O})\backslash \mathrm{Gr}_G\,, \tag{1.3.41}$$

where Gr_G is the affine Grassmannian; and the corresponding representation of bulk local operators becomes

$$H_\bullet\big(\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_0)\big) = H_\bullet^{G(\mathcal{O})}(\mathrm{Gr}_G)\,. \tag{1.3.42}$$

This famous convolution algebra was studied by [182] and proposed by Teleman [183] to be the Coulomb branch chiral ring of pure gauge theory.

Deformations by complex masses

As discussed in Section 1.2, the physical Coulomb branch can be deformed by introducing complex mass parameters for a torus of the flavor symmetry group T_F. Using the above description of the 3d theory as an effective 1d super quantum mechanical theory, the complex mass parameters in 3d are similarly complex mass parameters in 1d. Including these masses requires that homology classes are further equivariant with respect to this flavor torus action.

Explicitly, the moduli space $\mathcal{R}_{G,R}$ admits an action of the group $T_F(\mathcal{O})$ that simply acts via $t : [g, X] \mapsto [tgt^{-1}, t.X]$. (The product tgt^{-1} and the action $t.X$ is through in the larger group $GL(R, \mathcal{O})$, after choosing a representative for t.) Since $T_F(\mathcal{O})$ is a subgroup of (a quotient of) the normalizer of the $G(\mathcal{O})$-action on $R(\mathcal{O})$ (inside $GL(R, \mathcal{O})$), this yields a well defined action of $T_F(\mathcal{O})$ on $\mathcal{R}_{G,R}$ that commutes with the action of $G(\mathcal{O})$. Thus, the deformed algebra is simply $H^\bullet_{G(\mathcal{O})\times T_F(\mathcal{O})}(\mathcal{R}_{G,R})$. Since $T_F(\mathcal{O})$ contracts to T_F, taking equivariance with respect to the $T_F(\mathcal{O})$ agrees with that of T_F.

An alternative description of this construction is as follows. Let

$$1 \to G \to \widehat{G} \to T_F \to 1 \tag{1.3.43}$$

be an exact sequence of groups such that the G action on R extends to an action of $\widehat{G}$. We denote $\widehat{G}^{\mathcal{O}}(\mathcal{K})$ the preimage of $T_F(\mathcal{O})$ inside $\widehat{G}(\mathcal{K})$. It follows that $G(\mathcal{K})/G(\mathcal{O}) \cong \widehat{G}^{\mathcal{O}}(\mathcal{K})/\widehat{G}(\mathcal{O})$ and, moreover,

$$\widehat{\mathcal{R}}_{G,R} = R(\mathcal{O}) \times \widehat{G}^{\mathcal{O}}(\mathcal{K}) \times R(\mathcal{O})|_{(*)}/\widehat{G}(\mathcal{O}) \cong \mathcal{R}_{G,R}. \tag{1.3.44}$$

which makes manifest the action of $\widehat{G}(\mathcal{O})$. We can then define the deformed algebra as the $\widehat{G}(\mathcal{O})$-equivariant homology of $\widehat{\mathcal{R}}_{G,R} \cong \mathcal{R}_{G,R}$.

Quantizing with an Omega background

It is equally straightforward to introduce an Omega background and thereby quantize the algebra $\mathbb{C}[\mathcal{M}_C]$. From this perspective, the Omega background simply corresponds to introducing complex mass parameters for the symmetry of the moduli space $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_R)$ generated by $\delta z = \varepsilon z$, called "loop rotation," together with scaling $X(z)$ with weight $\frac{1}{2}$ (as X is a spinor in the A-twist). Note that the action of loop rotation *does not* commute with the action of $\widehat{G}(\mathcal{O})$. Nonetheless, $\widehat{G}(\mathcal{O}) \rtimes \mathbb{C}^\times_\varepsilon$ contracts to the group $G \times T_F \times \mathbb{C}^\times_\varepsilon$ and equivariance can be taken with respect to either group. The fully deformed and quantized algebra can then be written as

$$\mathbb{C}_\varepsilon[\mathcal{M}_C] = H_\bullet^{\widehat{G}(\mathcal{O}) \rtimes \mathbb{C}^\times_\varepsilon}(\mathcal{R}_{G,R}) \simeq H_\bullet^{G \times T_F \times \mathbb{C}^\times_\varepsilon}(\mathcal{R}_{G,R}). \tag{1.3.45}$$

Relation to abelianization

There are several highly nontrivial aspects of the formal definition (1.3.45). The space $\mathcal{R}_{G,R}$ is not contractible; it has highly nontrivial topology, as it must in order for (1.3.45) to contain monopole operators. Both the cohomology classes in (1.3.45) and their convolution product can be difficult to describe explicitly.

A useful tool in equivariant homology is fixed-point localization. Letting $T \subset G$ denote a maximal torus as usual, one finds that the $T \times T_F \times U(1)_\varepsilon$ fixed points of $\mathcal{R}_{G,R}$ are isolated and actually quite easy to describe: they are points (E, X), (E', X') where $X = X' = 0$ are zero-sections, E is trivial, and E' is obtained from E by a gauge transformation

$$g(z) = z^A \,, \qquad A \in \mathrm{cochar}(T)\,. \tag{1.3.46}$$

(Here "z" is the local coordinate on $\mathbb{C}_z$, and we are using $A \in \mathrm{Hom}(U(1), T)$ to define a meromorphic gauge transformation. For example, if $G = U(N)$, z^A means $\mathrm{diag}(z^{A_1}, ..., z^{A_N})$.) Thus the fixed points are labeled by cocharacters — just right to correspond to abelian monopole operators!

Let $\mathcal{F}$ denote the fixed point set of the $T \times T_F \times U(1)_\varepsilon$ action on $\mathcal{R}_{G,R}$. We just explained that $\mathcal{F} \simeq \mathrm{cochar}(T)$ is isomorphic to the cocharacter lattice. The equivariant homology of the fixed point set just contains a copy of $H_\bullet^{T\times T_F\times U(1)_\varepsilon}(\text{point}) = \mathbb{C}[\varphi, m_\mathbb{C}, \varepsilon]$ for every point in $\mathcal{F}$, *i.e.* $H_\bullet^{T\times T_F\times U(1)_\varepsilon}(\mathcal{F}) \simeq \mathbb{C}\big[\varphi, m_\mathbb{C}, \varepsilon, \{v_A\}_{A\in\mathrm{cochar}(T)}\big]$. Its "localized" version inverts all weights $\langle\lambda, \varphi\rangle + n\varepsilon$ (for any λ in the weight lattice of G),

$$H_\bullet^{T\times T_F\times U(1)_\varepsilon}(\mathcal{F})^{\mathrm{loc}} \simeq \mathbb{C}\big[\varphi, m_\mathbb{C}, \varepsilon, \{v_A\}_{A\in\mathrm{cochar}(T)}, \tfrac{1}{\langle\lambda,\varphi\rangle+n\varepsilon}\big]\,, \tag{1.3.47}$$

from which we see that our abelianized algebra $\mathcal{A}_\varepsilon$ from (1.2.24) sits inside

$$\mathcal{A}_\varepsilon \subset H_\bullet^{T\times T_F\times U(1)_\varepsilon}(\mathcal{F})^{\mathrm{loc}}\,. \tag{1.3.48}$$

The only difference between $\mathcal{A}_\varepsilon$ and $H_\bullet^{T\times T_F\times U(1)_\varepsilon}(\mathcal{F})^{\mathrm{loc}}$ (as vector spaces) is that in $\mathcal{A}_\varepsilon$ we only inverted roots $M_\alpha + n\varepsilon$ and weights $M_\lambda + n\varepsilon$ where $\lambda \in \mathrm{weights}(R)$; whereas the localized homology indiscriminately inverts all weights. The localization theorem provides the map

$$\mathbb{C}_\varepsilon[\mathcal{M}_C] = H_\bullet^{(G(\mathcal{O})\times T_F(\mathcal{O}))\rtimes\mathbb{C}^\times_\varepsilon}(\mathcal{R}_{G,R}) \;\hookrightarrow\; H_\bullet^{T\times T_F\times\mathbb{C}^\times_\varepsilon}(\mathcal{F})^{\mathrm{loc}}\,. \tag{1.3.49}$$

When this is carefully interpreted using Borel-Moore homology, one finds that the image actually lies inside $\mathcal{A}_\varepsilon$,

$$\mathbb{C}_\varepsilon[\mathcal{M}_C] \hookrightarrow \mathcal{A}_\varepsilon \,. \tag{1.3.50}$$

In other words, only the roots $M_\alpha + n\varepsilon$ need to be inverted.

It is hardly obvious mathematically that the maps (1.3.49), (1.3.50) are embeddings of algebras (under the convolution product) rather than just vector spaces. The compatibility of fixed-point localization with the convolution product was proved by [72].

1.4 Examples

We now consider some chiral-ring computations in earnest. In Section 1.4.1, we consider the example of the $(2,3)$ star-shaped quiver discussed in [101]. This 3d $\mathcal{N}=4$ theory is 3d mirror to a dimensional reduction of the 4d A_1 Trinion theory, also known as T_2, which is the corresponding theory of Class $\mathcal{S}$ associated to a pair of pants $\Sigma_{0,3}$ [103–109].

A general feature of theories of Class $\mathcal{S}$ is that the theories associated to surfaces $\Sigma_{g,\mathbf{k}}$ and $\Sigma_{g',\mathbf{k}'}$ can be "glued" together along punctures to form the theory associated to the surface $\Sigma_{g+g',\mathbf{k}+\mathbf{k}'-2}$. In terms of the field theories, this process is simply gauging the diagonal G subgroup of the $G\times G$ flavor symmetry associated to the chosen punctures. For the Higgs branch of these theories, or the Coulomb branch of the related 3d theory, this process is realized by taking a symplectic reduction of the product space by the anti-diagonal copy of $G_\mathbb{C}$ in the $G_\mathbb{C}\times G_\mathbb{C}$ hyperkähler isometry groups of the two punctures.

With the above in mind, there is a second theory that we shall investigate which can be obtained by gluing with T_2. In Section 1.4.2, we consider the rank 2 ADHM quiver gauge theory [98] and it's relation to quantum symplectic reduction of A_1 Trinion theory, realizing the gluing of a pair of pants to get a once-punctured torus $\Sigma_{1,1}$. The Coulomb branch in this case can be identified with the Hilbert scheme of 2 points on $\mathbb{C}^2$, denoted $\mathrm{Hilb}^2(\mathbb{C}^2)$, and quantizes to the (spherical subalgebra of the) $\mathfrak{gl}(2,\mathbb{C})$ rational Cherednik algebra [113], denoted $\overline{\mathcal{H}}_2^{\mathrm{sph}}$.

1.4.1 $\mathcal{T}_{2,3}$ star-quiver

We begin with a three-legged quiver $\mathcal{T}_{2,3}$, given in Figure 1.5.

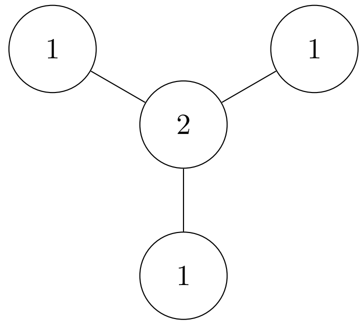


Figure 1.5: The $(N, \mathbf{k}) = (2, 3)$ star quiver.

This theory is 3d mirror to (the S^1 reduction of) of the basic A_1 trinion theory of Class $\mathcal{S}$, *i.e.* a theory of free half-hypermultiplets in the tri-fundamental representation of the $SU(2)^3$ flavor symmetry [123].[18] Correspondingly, we expect to find a simple 3d Coulomb branch

$$\mathcal{M}_C \simeq \mathbb{C}^8 \,. \tag{1.4.1}$$

The way this arises from a 3d perspective turns out to be rather nontrivial.

Naively, the gauge group of $\mathcal{T}_{2,3}$ is $U(2) \times U(1)^3$. The hypermultiplets sit in three fundamental representations of $U(2)$, each charged under a separate $U(1)$. As discussed in Section 1.2.1, the diagonal $U(1)_{\text{diag}}$ subgroup of $U(2) \times U(1)^3$ acts trivially on the hypermultiplets, so the true gauge group is actually a quotient

$$G = \left[U(2) \times U(1)^3\right] / U(1)_{\text{diag}} \,. \tag{1.4.2}$$

[18] The theory of eight free (half-)hypermultiplets, the 3d mirror of $\mathcal{T}_{2,3}$, actually has a larger Higgs flavor symmetry group than this naive $SU(2)^3$. Indeed, the full symmetry group is $USp(4)$, corresponding to the hyperkähler isometries of $T^*\mathbb{C}^4 \simeq \mathbb{C}^8$. The 36 generators of (the complexification of) $USp(4)$ fit into a (complex) moment map built out of the independent bilinears in the coordinates of $T^*\mathbb{C}^4$. This enhancement is not a general feature and only appears due to the free nature of the dual theory.

Correspondingly, the cocharacter lattice that will label monopole charges is

$$\text{cochar}(T) = \text{Hom}(U(1), T) = \mathbb{Z}^5/\mathbb{Z}_{\text{diag}}\,, \tag{1.4.3}$$

which we may understand as 5-tuples of integers

$$A = (\overbrace{A_1, A_2;}^{U(2)} \overset{U(1)_1}{B_1}\,,\ \overset{U(1)_2}{B_2}\,,\ \overset{U(1)_3}{B_3}\,)\ \in\ \text{cochar}(U(2)) \times \text{cochar}(U(1)^3) \tag{1.4.4}$$

modulo the 1-dimensional sublattice generated by $A_{\text{diag}} = (1,1;1,1,1)$. In other words, two cocharacters $A, A' \in \mathbb{Z}^5$ are equivalent if they differ by an integer multiple of A_{diag}. Dually, the weight lattice of G may be identified with 5-tuples of integers that sum to zero

$$\begin{aligned} \text{weights}(G) &= \text{Hom}(T, U(1)) \\ &= \big\{\lambda = (\lambda_1, \lambda_2; \lambda_1', \lambda_2', \lambda_3') \in \mathbb{Z}^5 \text{ s.t. } \lambda_1 + \lambda_2 + \lambda_1' + \lambda_2' + \lambda_3' = 0\big\}\,. \end{aligned} \tag{1.4.5}$$

Note that there is a well-defined product $\langle\ ,\ \rangle : \text{weights}(G) \times \text{cochar}(T) \to \mathbb{Z}$. In particular, $\langle \lambda, A_{\text{diag}}\rangle = 0$ for any weight λ. The matter representation may now be written as $\mathcal{R} = R \oplus R^*$, with weights of R chosen to be

$$\text{weights}(R)\ =\ \left\{ \begin{matrix} (1,0;-1,0,0) & (1,0;0,-1,0) & (1,0;0,0,-1) \\ (0,1;-1,0,0) & (0,1;0,-1,0) & (0,1;0,0,-1) \end{matrix} \right\}. \tag{1.4.6}$$

The $\mathcal{A}_\varepsilon$ and $\mathcal{W}_\varepsilon$ algebras

Our first step in constructing the Coulomb branch is to identify the abelianized algebra $\mathcal{A}_\varepsilon$ from Section 1.2.2, which contains all putative Coulomb-branch operators. We work from the outset with its quantized version. As described in Section 1.2.2, there are three types of generators:

1. Polynomials in Omega background parameter ε, the complex scalars φ_1^a, $a = 1, 2, 3$ corresponding to the $U(1)$ factors in G, and the diagonal components $(\varphi_{21}, \varphi_{22})$ of the

complex scalar corresponding to the $U(2)$ factor.

Due to the $U(1)_{\text{diag}}$ quotient, we should restrict to polynomials that are invariant under a simultaneous translation of all the φ's. It is natural to think of such polynomials as generated by weights of G, *i.e.* by the linear functions

$$\langle \lambda, \varphi \rangle = \lambda_1 \varphi_{21} + \lambda_2 \varphi_{22} + \lambda'_1 \varphi_1^1 + \lambda'_2 \varphi_1^2 + \lambda'_3 \varphi_1^3 \qquad \lambda \in \text{weights}(G)\,. \tag{1.4.7}$$

The constraint $\lambda_1 + \lambda_2 + \lambda'_1 + \lambda'_2 + \lambda'_3 = 0$ guarantees that $\langle \lambda, \varphi \rangle$ is invariant under translations.

2. The inverted masses $(M_\alpha + n\varepsilon)^{-1}$ for all roots α of G and all $n \in \mathbb{Z}$. Here the only nonzero roots are $\alpha = \pm(1, -1; 0, 0, 0)$, corresponding to the $U(2)$ factor, so we adjoin elements of the form
$$\frac{1}{\varphi_{21} - \varphi_{22} + n\varepsilon}\,. \tag{1.4.8}$$
Similarly, we adjoin inverted hypermultiplet masses $(M_\lambda + n\varepsilon)^{-1}$ for all weights (1.4.6).

3. The abelian monopole operators v_A labeled by cocharacters $A \in \text{cochar}(T)$ as above. All monopole operators with diagonal cocharacter $v_{nA_{\text{diag}}} = v_{(n,n;n,n,n)}$ are central in the algebra, and we impose the relations
$$v_{(n,n;n,n,n)} = 1 \quad \forall\, n\,. \tag{1.4.9}$$

The next intermediary step is to construct the subalgebra $\mathcal{W}_\varepsilon \subset \mathcal{A}_\varepsilon$ from Section 1.2.3. It will help us decide which elements of $\mathcal{A}_\varepsilon$ are actual chiral-ring operators.

To this end, we introduce the rescaled monopole operators u_A as in (1.2.27), whose products contain no denominators. For example, we have

$$\begin{aligned} u_{(\pm1,0;B_1,B_2,B_3)} &= \pm(\varphi_{22} - \varphi_{21}) v_{(\pm1,0;B_1,B_2,B_3)} \\ u_{(0,\pm1;B_1,B_2,B_3)} &= \pm(\varphi_{21} - \varphi_{22}) v_{(0,\pm1;B_1,B_2,B_3)} \end{aligned} \tag{1.4.10}$$

for any B_1, B_2, B_3. etc. We also introduce the single Weyl reflection s that generates the Weyl group $\mathbb{Z}_2$. It satisfies $s^2 = 1$ and acts on monopoles by reflecting their cocharacters:

$$s v_{(A_1,A_2;B_1,B_2,B_3)} = v_{(A_2,A_1;B_1,B_2,B_3)} s \qquad s u_{(A_1,A_2;B_1,B_2,B_3)} = u_{(A_2,A_1;B_1,B_2,B_3)} s. \tag{1.4.11}$$

Similarly, $s\varphi_1^a = \varphi_1^a s$ and $s\varphi_{21} = \varphi_{22} s$. The corresponding BGG-Demazure operator is $\theta = \frac{1}{\varphi_{21}-\varphi_{22}}(s-1)$. Recall that $\mathcal{W}_\varepsilon$ is the Weyl-symmetric part of $\mathbb{C}[\varphi, u_A, \theta]$.

Some important elements of $\mathcal{W}_\varepsilon$, which are assured to belong to the the full chiral ring $\mathbb{C}_\varepsilon[\mathcal{M}_C]$, are[19]

$$[\pm\theta u_{(\pm1,0;B_1,B_2,B_3)}]_W = v_{(\pm1,0;B_1,B_2,B_3)} + v_{(0,\pm1;B_1,B_2,B_3)} \,. \tag{1.4.12}$$

These are the undressed *nonabelian* monopole operators labeled by a fundamental cocharacter on the central $U(2)$ node. The dressed nonabelian monopoles are simply

$$[u_{(\pm1,0;B_1,B_2,B_3)}]_W = u_{(\pm1,0;B_1,B_2,B_3)} + u_{(0,\pm1;B_1,B_2,B_3)} \,. \tag{1.4.13}$$

In addition, $\mathcal{W}_\varepsilon$ contains monopoles charged only under the legs (which are trivially Weyl-invariant)

$$[u_{(0,0;B_1,B_2,B_3)}]_W = u_{(0,0;B_1,B_2,B_3)} = v_{(0,0;B_1,B_2,B_3)} \,, \tag{1.4.14}$$

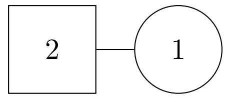
and all Weyl-invariant polynomials in the φ's. These are *all* the operators we will need to generate $\mathbb{C}_\varepsilon[\mathcal{M}_C]$!

Moment maps

The theory $\mathcal{T}_{2,3}$ has an $SU(2)^3$ flavor symmetry acting on its Coulomb branch (described in Section 1.2.4), and a corresponding $SL(2,\mathbb{C})^3$ symmetry in the chiral ring. This symmetry should be generated by three $\mathfrak{sl}(2,\mathbb{C})^*$-valued complex moment maps μ_a, $a = 1, 2, 3$.

[19] Here we use $[...]_W$ to denote a sum over the Weyl group, proportional to the projection of $[...]$ to Weyl-invariant operators.

Each of these moment maps is associated to a leg of the quiver. Each leg



(1.4.15)

looks like a copy of $T[SU(2)]$ theory, and effectively treats the central node as a flavor symmetry. We may therefore import well known results from the chiral ring of $T[SU(2)]$ (studied *e.g.* in [56, 67]) to identify the moment maps.

The raising and lowering operators in the moment maps turn out to be instances of (1.4.14)

$$V_1^{1\pm} := v_{(0,0;\pm1,0,0)}\,, \qquad V_1^{2\pm} := v_{(0,0;0,\pm1,0)}\,, \qquad V_1^{3\pm} = v_{(0,0;0,0\pm1)}\,. \tag{1.4.16}$$

We may check that they satisfy expected $\mathfrak{sl}(2,\mathbb{C})$ commutation relations. A quick application of (1.2.22) yields

$$[V_1^{a\pm}, V_1^{b\pm}] = 0\,, \tag{1.4.17}$$

as well as

$$\begin{aligned}[V_1^{a+}, -V_1^{b-}] &= \delta^{ab}\big[(\varphi_{21}-\varphi_1^a-\varepsilon/2)(\varphi_{22}-\varphi_1^a-\varepsilon/2)-(\varphi_{21}-\varphi_1^a+\varepsilon/2)(\varphi_{22}-\varphi_1^a+\varepsilon/2)\big]\\ &= \varepsilon\delta^{ab}(2\varphi_1^a-\varphi_{21}-\varphi_{22})\,.\end{aligned} \tag{1.4.18}$$

Similarly, (1.2.19) implies

$$[(2\varphi_1^a-\varphi_{21}-\varphi_{22}), \pm V_1^{b\pm}] = \pm 2\delta^{ab}\varepsilon(\pm V_1^{a\pm}) \tag{1.4.19}$$

therefore $\{V_1^{a+}, -V_1^{a-}, 2\varphi_1^a-\varphi_{21}-\varphi_{22}\}_{a=1}^3$ can be identified as three mutually commuting $\mathfrak{sl}(2,\mathbb{C})$ triples. These operators fit into moment maps as

$$\widetilde{\mu}_a := \mu_a - \frac{\varepsilon}{2}\mathbb{1} = \begin{pmatrix} \varphi_1^a-\varphi_{21}/2-\varphi_{22}/2-\varepsilon/2 & V_1^{a+} \\ -V_1^{a-} & \varphi_{21}/2+\varphi_{22}/2-\varphi_1^a-\varepsilon/2 \end{pmatrix}. \tag{1.4.20}$$

The shift by $\frac{\varepsilon}{2}$ is included for later convenience. It does not affect the action generated by the moment map; in particular, letting $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{C})$ be the Cartan element we find $\langle H, \mu_a \rangle = \langle H, \widetilde{\mu}_a \rangle = \mathrm{Tr}(H\mu_a^T) = \mathrm{Tr}(H\widetilde{\mu}_a^T) = 2\varphi_1^a - \varphi_{21} - \varphi_{22}$.

Note that, using the general R-charge formula (1.2.38), we have

$$[\mu_a] = [V_a^{a\pm}] = [2\varphi_1^a - \varphi_{21} - \varphi_{22}] = 1 \tag{1.4.21}$$

as required for moment maps.

Tri-fundamentals

Having identified the moment maps, we may organize the chiral ring into $SL(2,\mathbb{C})^3$ representations. It is easy to check using (1.2.22) that the operator

$$Q^{222} := v_{(1,0;0,0,0)} + v_{(0,1;0,0,0)}\,, \tag{1.4.22}$$

which is of type (1.4.12), is a "tri"-lowest-weight vector. Namely,

$$[-V_1^{a-}, Q^{222}] = 0 \quad \forall\, a = 1,2,3\,. \tag{1.4.23}$$

By acting with raising operators on Q^{222} we then produce an entire eight-dimensional tri-fundamental representation. For example,

$$\begin{aligned} Q^{122} &:= \tfrac{1}{\varepsilon}[V_1^{1+}, Q^{222}] = v_{(1,0;1,0,0)} + v_{(0,1;1,0,0)}\,, \\ &\quad\ \tfrac{1}{\varepsilon}[V_1^{1+}, Q^{122}] = 0\,, \\ Q^{112} &:= \tfrac{1}{\varepsilon}[V_1^{2+}, Q^{122}] = v_{(1,0;1,1,0)} + v_{(0,1;1,1,0)}\,, \\ Q^{111} &:= \tfrac{1}{\varepsilon}[V_1^{3+}, Q^{112}] = v_{(1,0;1,1,1)} + v_{(0,1;1,1,1)}\,, \end{aligned} \tag{1.4.24}$$

etc. The complete list of operators in this representation is summarized in Table 1.1.

Alternatively, we could have observed that $Q_{222} := -(v_{(-1,0;0,0,0)} + v_{(0,-1;0,0,0)})$ is a tri-highest-weight vector, which generates an eight-dimensional tri-antifundamental representa-

Operator	Expression 1	Expression 2	Expression 3
$Q^{222} = Q_{111}$	$(\theta u_{(1,0;0,0,0)})_W$	$v_{(1,0;0,0,0)} + v_{(0,1;0,0,0)}$	$v_{(-1,0;-1,-1,-1)} + v_{(0,-1;-1,-1,-1)}$
$Q^{122} = -Q_{211}$	$(\theta u_{(1,0;1,0,0)})_W$	$v_{(1,0;1,0,0)} + v_{(0,1;1,0,0)}$	$v_{(-1,0;0,-1,-1)} + v_{(0,-1;0,-1,-1)}$
$Q^{212} = -Q_{121}$	$(\theta u_{(1,0;0,1,0)})_W$	$v_{(1,0;0,1,0)} + v_{(0,1;0,1,0)}$	$v_{(-1,0;-1,0,-1)} + v_{(0,-1;-1,0,-1)}$
$Q^{221} = -Q_{112}$	$(\theta u_{(1,0;0,0,1)})_W$	$v_{(1,0;0,0,1)} + v_{(0,1;0,0,1)}$	$v_{(-1,0;-1,-1,0)} + v_{(0,-1;-1,-1,0)}$
$Q^{211} = Q_{122}$	$-(\theta u_{(-1,0;-1,0,0)})_W$	$v_{(1,0;0,1,1)} + v_{(0,1;0,1,1)}$	$v_{(-1,0;-1,0,0)} + v_{(0,-1;-1,0,0)}$
$Q^{121} = Q_{212}$	$-(\theta u_{(-1,0;0,-1,0)})_W$	$v_{(1,0;1,0,1)} + v_{(0,1;1,0,1)}$	$v_{(-1,0;0,-1,0)} + v_{(0,-1;0,-1,0)}$
$Q^{112} = Q_{221}$	$-(\theta u_{(-1,0;0,0,-1)})_W$	$v_{(1,0;1,1,0)} + v_{(0,1;1,1,0)}$	$v_{(-1,0;0,0,-1)} + v_{(0,-1;0,0,-1)}$
$Q^{111} = -Q_{222}$	$-(\theta u_{(-1,0;0,0,0)})_W$	$v_{(1,0;1,1,1)} + v_{(0,1;1,1,1)}$	$v_{(-1,0;0,0,0)} + v_{(0,-1;0,0,0)}$

Table 1.1: Expressions for the eight operators furnishing a tri-fundamental representation of the $SL(2,\mathbb{C})^3$ action on the chiral ring of the $\mathcal{T}_{2,3}$ star quiver. These eight operators generate the entire chiral ring. The first expression of the operator is in terms of the Weyl symmetrized image of a rescaled monopole operator under the BGG-Demazure operator θ. The second two expressions are related to one another by adding a diagonal cocharacter A_{diag}.

tion. However, it is equivalent to the tri-fundamental above. In particular, since cocharacters (1.4.4) that differ by a multiple of A_{diag} are equivalent, we actually have $Q_{222} = -Q^{111}$, and more generally

$$\epsilon_{i_1 j_1}\epsilon_{i_2 j_2}\epsilon_{i_3 j_3} Q^{j_1 j_2 j_3} = Q_{i_1 i_2 i_3}\,. \tag{1.4.25}$$

We also note that the R-charge formula (1.2.38) quickly implies that

$$[Q^{i_1 i_2 i_3}] = [Q_{i_1 i_2 i_3}] = \tfrac{1}{2}\,. \tag{1.4.26}$$

Relations

Using the above expressions for the tri-fundamental operators, it is a straightforward application of (1.2.19) and (1.2.22) to find additional relations satisfied by the Q's and μ's.

For example, there are commutation relations

$$\begin{aligned}
[Q^{122}, Q^{222}] &= [v_{(1,0;1,0,0)} + v_{(0,1;1,0,0)}, v_{(1,0;0,0,0)} + v_{(0,1;0,0,0)}] \\
&= \left[\tfrac{\varphi_{21}-\varphi_1^1+\varepsilon/2}{(\varphi_{21}-\varphi_{22})(\varphi_{22}-\varphi_{21}-\varepsilon)} - \tfrac{\varphi_{21}-\varphi_1^1-\varepsilon/2}{(\varphi_{21}-\varphi_{22})(\varphi_{22}-\varphi_{21}+\varepsilon)} + (\varphi_{21}\leftrightarrow\varphi_{22})\right] v_{(1,1;1,0,0)} \\
&= 0\,;
\end{aligned} \tag{1.4.27}$$

similarly,

$$\begin{aligned}[Q^{111},Q^{222}] &= [v_{(-1,0;0,0,0)}+v_{(0,-1;0,0,0)},v_{(1,0;0,0,0)}+v_{(0,1;0,0,0)}] \\ &= \left[\frac{\prod_{a=1}^{3}\varphi_{21}-\varphi_1^a+\varepsilon/2}{(\varphi_{21}-\varphi_{22})(\varphi_{22}-\varphi_{21}-\varepsilon)}-\frac{\prod_{a=1}^{3}\varphi_{21}-\varphi_1^a-\varepsilon/2}{(\varphi_{21}-\varphi_{22})(\varphi_{22}-\varphi_{21}+\varepsilon)}+(\varphi_{21}\leftrightarrow\varphi_{22})\right] \\ &= -\varepsilon\,;\end{aligned} \tag{1.4.28}$$

and more generally

$$[Q^{i_1i_2i_3},Q^{j_1j_2j_3}] = -\varepsilon\epsilon^{i_1j_1}\epsilon^{i_2j_2}\epsilon^{i_3j_3}\,. \tag{1.4.29}$$

Sending $\varepsilon\to 0$, these recover the Poisson brackets $\{Q^{i_1i_2i_3},Q^{j_1j_2j_3}\}=-\epsilon^{i_1j_1}\epsilon^{i_2j_2}\epsilon^{i_3j_3}$ expected from the duality of $\mathcal{T}_{2,3}$ with free half-hypermultiplets in 4d.

We may also consider contractions between moment maps and Q's. Schematically writing

$$Q^{i_1i_2i_3} = v_{(1,0;B_{i_1i_2i_3})}+v_{(0,1;B_{i_1i_2i_3})}\,, \tag{1.4.30}$$

we use (1.2.22) to find

$$\begin{aligned}(\widetilde{\mu}_1)^{i_1}{}_{i'}Q^{i'i_2i_3} = (\widetilde{\mu}_2)^{i_2}{}_{i'}Q^{i_1i'i_3} &= (\widetilde{\mu}_3)^{i_3}{}_{i'}Q^{i_1i_2i'} \\ &= \frac{\varphi_{21}-\varphi_{22}}{2}\big[v_{(1,0;B_{i_1i_2i_3})}-v_{(0,1;B_{i_1i_2i_3})}\big]\end{aligned}\,. \tag{1.4.31}$$

More generally, for all $n\geq 0$, we may contract with powers of the moment maps to get

$$\begin{aligned}(\widetilde{\mu}_1^n)^{i_1}{}_{i'}Q^{i'i_2i_3} = (\widetilde{\mu}_2^n)^{i_2}{}_{i'}Q^{i_1i'i_3} &= (\widetilde{\mu}_3^n)^{i_3}{}_{i'}Q^{i_1i_2i'} \\ &= \left(\frac{\varphi_{21}-\varphi_{22}}{2}\right)^n\big[v_{(1,0;B_{i_1i_2i_3})}+(-1)^n v_{(0,1;B_{i_1i_2i_3})}\big]\end{aligned}\,. \tag{1.4.32}$$

Note that the RHS of (1.4.31) contains an alternative dressed version of the fundamental nonabelian monopole operators.

Finally, we can recover the moment maps themselves as contractions of Q's,

$$\frac{1}{2}\left(Q^{i_1i_2i_3}Q_{i_1i_2'i_3}+\varepsilon\delta^{i_1}{}_{i_1'}\right) = (\mu_1)^{i_1}{}_{i_1'}\,, \tag{1.4.33}$$

and similarly for $(\mu_2)^{i_2}{}_{i'_2}$ and $(\mu_3)^{i_3}{}_{i'_3}$.

It is straightforward but tedious to show that the $Q^{i_1 i_2 i_3}$ operators generate all of $\mathcal{W}_\varepsilon$. In this case we know from duality with free half-hypermultiplets in 4d that these tri-fundamental operators really generate the entire chiral ring $\mathbb{C}_\varepsilon[\mathcal{M}_C]$. Since $\mathcal{W}_\varepsilon$ is necessarily contained in $\mathbb{C}_\varepsilon[\mathcal{M}_C]$, there is no choice but to have

$$\mathcal{W}_\varepsilon = \mathbb{C}_\varepsilon[\mathcal{M}_C]\,. \tag{1.4.34}$$

1.4.2 Rank 2 ADHM quiver and quantum symplectic reduction of $\mathcal{T}_{2,3}$

In this section we consider the rank 2 ADHM quiver gauge theory [116] and quantum symplectic reduction of the Coulomb branch of $\mathcal{T}_{2,3}$. The rank 2 ADHM gauge theory is a $U(2)$ gauge theory with a fundamental hypermultiplet and an adjoint hypermultiplet ($R = \text{fund}\oplus\text{adjoint}$) and can be described by the quiver in Figure 1.6. Alternatively, this can be described as a $(U(2)\times U(1))/U(1)$ gauge theory with an adjoint hypermultiplet (for the $U(2)$ factor) and a bi-fundamental hypermultiplet; we use this latter perspective to utilize similarities with the $\mathcal{T}_{2,3}$ analysis appearing in Section 1.4.1.

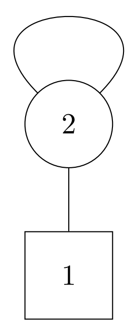


Figure 1.6: The rank 2 ADHM quiver. The corresponding 3d $\mathcal{N}=4$ gauge theory has gauge group $U(2)$ coupled to a single fundamental hypermultiplet and a single adjoint hypermultiplet.

Direct analysis of the quantized Coulomb branch

We can perform the same analysis as in Section 1.4.1 to determine the structure of $\mathcal{A}_\varepsilon$ and $\mathcal{W}_\varepsilon$, it is nearly identical so we only summarize the results. The main difference from $\mathcal{T}_{2,3}$ (and the other $\mathcal{T}_{N,\mathbf{k}}$ star quivers discussed in [101]) is that this theory admits a non-trivial flavor symmetry given by dilating the adjoint hypermultiplets.

The cocharacter lattice of $(U(2)\times U(1))/U(1) \cong U(2)$ is $(\mathbb{Z}^2\times\mathbb{Z})/\mathbb{Z}$, and elements will again be labeled $A = (A_1, A_2; B)$ which are defined up to $(1,1;1)$. We will similarly denote the complex scalars by $\varphi_1, \varphi_{21}, \varphi_{22}$.

The main operators of interest are the analogs of the tri-fundamentals

$$Q^2 = v_{(1,0;0)} + v_{(0,1;0)} \qquad Q^1 = v_{(1,0;1)} + v_{(0,1;1)} = v_{(0,-1;0)} + v_{(-1,0;0)}, \tag{1.4.35}$$

which have R-charge $[Q^i] = \frac{1}{2}$, and the $SL(2,\mathbb{C})$ moment map

$$\widetilde{\mu}_1 := \begin{pmatrix} \varphi_1 - \varphi_{21}/2 - \varphi_{22}/2 - \varepsilon/2 & v_{(0,0;1)} \\ -v_{(0,0;-1)} & \varphi_{21}/2 + \varphi_{22}/2 - \varphi_1 - \varepsilon/2 \end{pmatrix}. \tag{1.4.36}$$

We will denote $E = v_{(0,0;1)}$, $F = -v_{(0,0;-1)}$, $H = 2\varphi_1 - \varphi_{21} - \varphi_{22}$.

The relations satisfied by the above operators can be found just as before. For example, the Q^i generate a (scaled) Weyl algebra

$$\begin{aligned}
[Q^2, Q^1] &= [v_{(-1,0;0)} + v_{(0,-1;0)}, v_{(1,0;0)} + v_{(0,1;0)}] \\
&= \Big[\tfrac{(\varphi_{21}-\varphi_1-\varepsilon/2)(\varphi_{21}-\varphi_{22}+m_{\mathbb{C}}-\varepsilon/2)(\varphi_{22}-\varphi_{21}+m_{\mathbb{C}}+\varepsilon/2)}{(\varphi_{22}-\varphi_{21})(\varphi_{21}-\varphi_{22}-\varepsilon)} \\
&\qquad - \tfrac{(\varphi_{21}-\varphi_1+\varepsilon/2)(\varphi_{21}-\varphi_{22}+m_{\mathbb{C}}+\varepsilon/2)(\varphi_{22}-\varphi_{21}+m_{\mathbb{C}}-\varepsilon/2)}{(\varphi_{21}-\varphi_{22})(\varphi_{22}-\varphi_{21}-\varepsilon)} + (\varphi_{21} \leftrightarrow \varphi_{22}) \Big] \\
&= -2\varepsilon\,.
\end{aligned} \tag{1.4.37}$$

Just as before, Q^i transform as in the fundamental representation of the $SL(2,\mathbb{C})$ generated by E, F, H and so the bilinears

$$Q^+ = \tfrac{1}{2}Q^1Q^1 \qquad Q^0 = -\tfrac{1}{2}(Q^1Q^2 + Q^2Q^1) \qquad Q^- = -\tfrac{1}{2}Q^2Q^2 \tag{1.4.38}$$

transform in the adjoint representation of the above $SL(2,\mathbb{C})$. In fact, the operators $Q^\pm, Q^0$ realize an independent copy of $\mathfrak{sl}(2,\mathbb{C})$. The last relations of interest involve the quadratic Casimirs of these two $\mathfrak{sl}(2,\mathbb{C})$'s:

$$\begin{aligned} H^2 + 2(EF+FE) &= HQ^0 + 2(EQ^- + FQ^+) + (m_\mathbb{C} + \tfrac{1}{2}\varepsilon)(m_\mathbb{C} - \tfrac{1}{2}\varepsilon) \\ (Q^0)^2 + 2(Q^+Q^- + Q^-Q^+) &= -3\varepsilon^2 \end{aligned} \tag{1.4.39}$$

These two relations, together with the commutation relations described above, identify the quantized and deformed Coulomb-branch chiral ring with the (spherical subalgebra of the) rational Cherednik algebra for $\mathfrak{gl}(2,\mathbb{C})$ (for parameters $t = -\varepsilon, c = m_\mathbb{C}$), *cf.* [113, 184].

Quantum symplectic reduction of $\mathcal{T}_{2,3}$

The above chiral ring can also be obtained by quantum symplectic reduction of $\mathcal{T}_{2,3}$ with respect to an anti-diagonal $SL(2,\mathbb{C})$ flavor symmetry. In particular, we consider the quantum symplectic reduction of the algebra generated by the $Q^{i_1i_2i_3}$ with respect to the $SL(2,\mathbb{C})$ action generated by the moment map $\widetilde{\mu} = \widetilde{\mu}_2 - \widetilde{\mu}_3^{\mathrm{T}}$ at the level $\widetilde{\mu} = (m_\mathbb{C} + \frac{1}{2}\varepsilon)\mathbb{1}$ for $m_\mathbb{C} \in \mathbb{C}$. Just like usual symplectic reduction, quantum symplectic reduction is performed by imposing the relation $\widetilde{\mu} = (m_\mathbb{C} + \frac{1}{2}\varepsilon)\mathbb{1}$ on the algebra of $SL(2,\mathbb{C})$-invariant operators.

We can easily identify the above operators and the relations they satisfy. In particular, the fundamental Heisenberg pair Q^i can be identified with the following $SL(2,\mathbb{C})$-invariant operators:

$$\widehat{Q}^1 = Q^{2i_2i_3}\delta_{i_2i_3} \qquad \widehat{Q}^2 = Q^{1i_2i_3}\delta_{i_2i_3} \tag{1.4.40}$$

which satisfy $[\widehat{Q}^2, \widehat{Q}^1] = -2\varepsilon$. Similarly, the $\mathfrak{sl}(2,\mathbb{C})$ triple E, F, H is identified with components of the remaining moment map $\widetilde{\mu}_1$:

$$\widehat{E} = -v_{(0,0;-1,0,0)} \qquad \widehat{F} = v_{(0,0;1,0,0)} \qquad \widehat{H} = \varphi_{21} + \varphi_{22} - 2\varphi_1. \tag{1.4.41}$$

The relations identified above are also straightforward to match. In particular, we find that

$$(\widehat{Q}^0)^2 + 2(\widehat{Q}^+\widehat{Q}^- + \widehat{Q}^-\widehat{Q}^+) = -3\varepsilon \tag{1.4.42}$$

where the bilinears $\widehat{Q}^0, \widehat{Q}^\pm$ are defined just as above. More interestingly, we find

$$\begin{aligned}\widehat{H}^2 + 2(\widehat{E}\widehat{F} + \widehat{F}\widehat{E}) = \widehat{H}\widehat{Q}^0 + 2(\widehat{E}\widehat{Q}^- + \widehat{F}\widehat{Q}^+) + (m_{\mathbb{C}} + \tfrac{1}{2})(m_{\mathbb{C}} - \tfrac{1}{2}) \\ + \tfrac{1}{2}\big(\mathrm{Tr}[(\widetilde{\mu} - (m_{\mathbb{C}} + \tfrac{1}{2}\varepsilon)\mathbb{1})^2] + \varepsilon\mathrm{Tr}[\widetilde{\mu} - (m_{\mathbb{C}} + \tfrac{1}{2}\varepsilon)\mathbb{1}]\big),\end{aligned} \tag{1.4.43}$$

which agrees with our earlier relation upon imposing $\widetilde{\mu} = m_{\mathbb{C}} + \frac{1}{2}\varepsilon\mathbb{1}$.

Chapter 2

BPS Line Operators in A-twisted 3d $\mathcal{N} = 4$ Gauge Theory

BPS line operators in supersymmetric gauge theories hold a wealth of algebraic and geometric structure. Such structure has been most extensively studied in four-dimensional supersymmetric gauge theories, where one encounters BPS Wilson lines [136, 185–189], BPS 't Hooft lines [33, 190–192], and hybrids thereof. A few of the contexts in which these line operators have played a central role during the last decade and a half include the physics of geometric Langlands [33, 34], wall crossing phenomena [138, 193], and the Alday-Gaiotto-Tachikawa (AGT) correspondence [194–197]. It was also realized that the precise spectrum of line operators constitutes part of the very definition of a 4d gauge theory [138, 198].

This chapter focuses on $\frac{1}{2}$-BPS line operators in *three* dimensions, specifically in the A-twist of 3d $\mathcal{N} = 4$ gauge theories. Much as in 4d, line operators in 3d gauge theories come in two basic varieties: Wilson lines ('order' operators) and *vortex* lines (disorder operators). Supersymmetric Wilson lines in pure 3d $\mathcal{N} = 4$ gauge theories were introduced by [136]; and their analogues in sigma-models [43, 68] played a central role in the construction of Rozansky-Witten invariants. Supersymmetric vortex lines are codimension-two disorder operators, modeled on singular limits of the Nielsen-Olesen vortex [199] and its supersymmetric cousins, *e.g.* [153, 159, 200]. They may also be understood as dimensional reductions of supersymmetric surface operators in 4d gauge theories: the basic Gukov-Witten surface operators [34, 181, 201] and their 4d $\mathcal{N} = 2$ analogues [195, 202–205], studied and generalized in many later works — a small sampling includes [206–214] (see [144] for a clear review). Compactifying further to two dimensions, vortex lines become twist fields, which played a fundamental role in T-duality/mirror symmetry [78] and were recently reexamined by [215, 216].

Supersymmetric vortex lines in the 3d $\mathcal{N} = 6$ Aharony-Bergman-Jafferis-Maldacena (ABJM) theory [125] were constructed by [217] (further studied in many works *e.g.* [218–221]); then generalized and studied in 3d $\mathcal{N} = 2$ theories by [222–224] using supersymmetric localization. Further physical aspects of vortex lines in abelian 3d $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories were developed in [225, 226]. A systematic study of $\frac{1}{2}$-BPS vortex lines in 3d $\mathcal{N} = 4$ quiver gauge theories — both abelian and nonabelian — was initiated more recently by Assel and Gomis [80] using IIB brane constructions [79], akin to the constructions of surface operators in [206, 227]. It was shown by [80] that 3d mirror symmetry [74–76] swaps Wilson and vortex lines in quiver gauge theories. The rather nontrivial mirror map was verified with computations of supersymmetric partition functions, generalizing [223, 224, 228, 229].

Our overarching goal in the current chapter is to describe — in both a theoretical and a computationally effective way — the BPS local operators at *junctions* of line operators, and their OPE.[1] We will expand on precisely what this means further below.

The organization of this chapter is as follows. Section 2.1 describes the general structures underlying line operators in any 3d TQFT, including how line operators can be encoded in a category (with certain extra structures) and the role boundary conditions play in realizing this category. Section 2.2 provides a physical description of $\frac{1}{2}$-BPS vortex line operators compatible with the A-twist of 3d $\mathcal{N} = 4$ gauge theories and Section 2.3 describes the vector space of local operators at a junction of two such vortex line operators as well as the collisions thereof. Of particular importance is the role played by Dirichlet boundary conditions

[1] Our use of the term "line operator," as opposed to "loop operator," is meant to emphasize the focus on such local structure.

(described in Section 2.3.4) and how they naturally give rise to notions in generalized affine Springer theory. Finally, Section 2.4 is dedicated to a series of example computations.

Section 2.2 and Section 2.3 are adapted from [111]. The discussion of Dirichlet boundary conditions and their connections to generalized affine Springer theory in Section 2.3.4 are new and based off of the mathematical works [50, 51] and the earlier physical work [40]. The examples in Sections 2.4.1 and 2.4.2 are conceptually similar to the examples provided in [111] but consider different theories and use Dirichlet boundary conditions, as opposed to the vacuum boundary conditions used in [111].

2.1 Line operators and categories

TQFT provides both a practical toolbox and a powerful organizing principal for understanding extended operators in supersymmetric field theories. In the mathematical formalism of TQFT, extended operators are described by higher categories (*cf.* [84]). In particular, the line operators of a 3d TQFT have the structure of a braided tensor category. We review this in some detail below. Classifying line operators and the local operators bound to them then amounts to identifying the appropriate category — giving a definition of its objects and morphisms, as well as any additional structures one is interested in. In the analogous setup of 4d $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric gauge theories, the categories describing BPS Wilson and 't Hooft lines were identified and used to great effect in [33, 154, 230, 231].

Even once the categories of line operators are identified, computations of morphisms (*i.e.* local operators) may still be challenging. This is especially true for the A-twist of the 3d $\mathcal{N} = 4$ theories described in Chapter 1, where (as we explain in Section 2.2) we will encounter a category of D-modules on the *loop space* of the original 3d target. For the present section, however, our perspective will be very general. The concepts here will help us organize our descriptions and computations of A-type line operators in 3d $\mathcal{N} = 4$ theories in the rest of the thesis.

2.1.1 Objects and morphisms

Consider a general 3d TQFT; we will be interested in topological twists of 3d $\mathcal{N} = 4$ gauge theories, but one can consider such a theory is a Chern-Simons theory with compact, semisimple gauge group G with positive integer level without much loss. We first note that the local properties of line operators in this theory, *e.g.* local operators bound to and interpolating between line operators supported on a line in $\mathbb{R}^3$, can be described by a category. A category comes with two basic pieces of data: objects and morphisms. The objects in the category of line operators $\mathcal{C}$ are the line operators admitted by the theory, *i.e.* 1 dimensional extended operators $\mathcal{L}$ supported in the neighborhood of an oriented line in spacetime:

$$\mathrm{Ob}(\mathcal{C}) = \left\{ \begin{array}{c} \text{Dimension 1 extended operators } \mathcal{L} \\ \text{supported in the neighborhood of an oriented line} \end{array} \right\}. \tag{2.1.1}$$

In Chern-Simons theories, such objects are simply Wilson lines and are labeled by representations of the Chern-Simons gauge group. Given a Lagrangian theory, one can engineer line operators via coupling to a 1d quantum mechanical system, *cf.* the orbit method for Chern-Simons theories [41] as well as Wilson lines realized as coupling to 1d $\mathcal{N} = 4$ Fermi multiplets [80]. More generally, we could consider modifying the theory in a small, open neighborhood of the line rather than exactly at the line. For example, one can engineer a line operator by specifying the behavior of the fields in the vicinity of the line, *cf.* vortex lines in 3d [199] and 't Hooft lines in 4d [190]; we will see below that many line operators admit both types of descriptions. For any theory, there is a distinguished line operator $\mathbb{1}_\mathcal{C} \in \mathrm{Ob}(\mathcal{C})$ (the "trivial" line operator) associated to doing nothing to the bulk theory.

A morphism between a pair of line operators $\mathcal{L}, \mathcal{L}' \in \mathrm{Ob}(\mathcal{C})$ is given by a local operator $\mathcal{O}$ that interpolates between them. Just as with line operators, one should broaden the definition of local operators at a point p in spacetime to include those operators supported in a small, open neighborhood p. Alternatively, one can think of local operators as states on a sphere (the link of a point) pierced by the line operators $\mathcal{L}, \mathcal{L}'$ via a state-operator correspondence.

The space of morphisms is then simply the vector space of such local operators:

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{L}') = \big\{\text{Local operators } \mathcal{O} \text{ interpolating between } \mathcal{L} \text{ and } \mathcal{L}'\big\}. \tag{2.1.2}$$

For any line operator $\mathcal{L}$ there is a distinguished local operator $\mathbb{1}_{\mathcal{L}} \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{L})$ (the identity local operator) that interpolates from $\mathcal{L}$ to itself. For Chern-Simons theories, these local operators are monopole operators; these operators are electrically charged due to the Chern-Simons terms and can be dressed by Wilson lines to produce gauge-invariant configurations. In fact, there is a unique monopole operator between two Wilson lines if and only if the corresponding co-adjoint orbits are related by a large gauge transformation; this induces the usual identification between Wilson lines. The vector space $\mathrm{End}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}) := \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ simply encodes all (bulk) local operators. In practice, this is a useful way to recover bulk local operators from the category $\mathcal{C}$.

Given any three lines $\mathcal{L}, \mathcal{L}', \mathcal{L}''$, there is a composition operation

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{L}') \otimes \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}', \mathcal{L}'') \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{L}'') \tag{2.1.3}$$

coming from collision of local operators. In particular, the vector space $\mathrm{End}_{\mathcal{C}}(\mathcal{L}) := \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{L})$ has the structure of an algebra; the algebra $\mathrm{End}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}})$ is simply the algebra local operators of the bulk TQFT. From the perspective of local operators supported in neighborhoods of a point, the junctions of $\mathcal{L}, \mathcal{L}'$ and $\mathcal{L}', \mathcal{L}''$ are simultaneously supported in a larger neighborhood pierced by $\mathcal{L}$ from below and $\mathcal{L}''$ from above. See Figure 2.1.

Disclaimer: A_∞ and homotopy categories

An important disclaimer to all of the above comes from the fact that topological twists of 3d theories are somewhat more subtle than the above discussion about honest 3d TQFTs. First consider local operators bound to a (possibly trivial) line operator $\mathcal{L}$, denoted $\mathrm{Ops}(\mathcal{L})$. This is a vector space admitting an action of Q, *i.e.* a chain complex. If we restrict our attention to Q-closed local operators, there are many local operators that agree in all correlation functions:

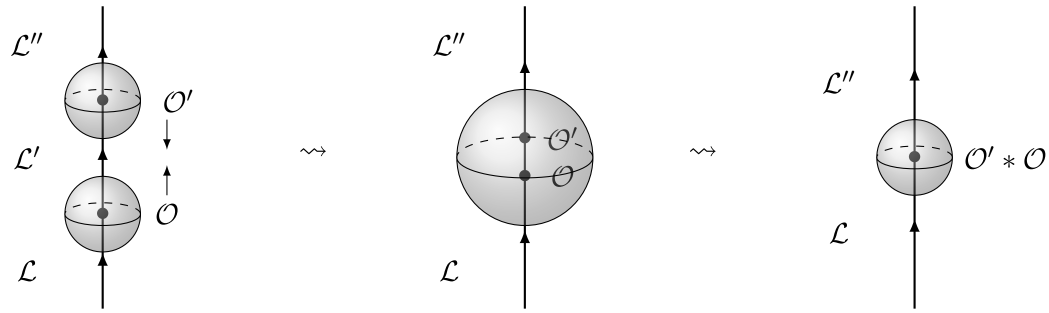


Figure 2.1: An illustration of composing morphisms in the category of line operators $\mathcal{C}$ induced by collision of local operators $\mathcal{O} \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{L}')$ and $\mathcal{O}' \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}', \mathcal{L}'')$ at the junctions of line operators $\mathcal{L}$, $\mathcal{L}'$, and $\mathcal{L}''$.

the local operators $\mathcal{O}$ and $\mathcal{O}+Q\mathcal{O}'$ (for *any* $\mathcal{O}' \in \mathrm{Ops}(\mathcal{L})$) are the equivalent in all correlation functions of Q-closed local operators, and are therefore physically indistinguishable. We are therefore interested in the Q-cohomology of $\mathrm{Ops}(\mathcal{L})$, which we denote $\mathrm{End}_{\mathcal{C}}(\mathcal{L})$.

More importantly, the collision of two Q-closed operators $\mathcal{O}, \mathcal{O}'$ may not have a well defined limit as their insertions collide. Nonetheless, their collision is well defined *up to Q-exact terms*, *i.e.*, there is a Q-closed local operator $\mathcal{O}''$ such that

$$(\mathcal{O} * \mathcal{O}')(p) = \lim_{p' \to p} \mathcal{O}(p)\mathcal{O}'(p') = \mathcal{O}''(p) + Q(...) \tag{2.1.4}$$

The compatibility of the supercharge Q and collisions of Q-closed local operators is encapsulated in the notion of an A_∞-algebra [232]. Note that it is useful to keep the information of the full chain complex $\mathrm{Ops}(\mathcal{L})$ and the homotopies relating collisions of local operators, as opposed to simply passing to $\mathrm{End}_{\mathcal{C}}(\mathcal{L})$. This additional data can often be used to define "higher operations" on $\mathrm{End}_{\mathcal{C}}(\mathcal{L})$ obtained via descent [85]; in the context of local operators in twisted quantum mechanics this leads to Massey products on the (co)homology of the target [233].

The same analysis applies to local operators at the junction of arbitrary line operators $\mathcal{L}, \mathcal{L}'$; the vector space of such operators $\mathrm{Ops}(\mathcal{L}, \mathcal{L}')$ admits an action of Q, and we denote the cohomology of this chain complex by $\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{L}')$. Again, the collision of local operators at two (or more) junctions is well defined up to Q-closed local operators. Just as with categories of boundary conditions in 2d TQFTs, the true category of line operators $\mathcal{C}$ is an A_∞-category, *cf.* [83]. Again, it is often useful to keep track of the full chain complexes $\mathrm{Ops}(\mathcal{L}, \mathcal{L}')$ as they can lead to higher operations on the category.

Somewhat more dramatically, there may be line operators $\mathcal{L}, \mathcal{L}'$ that should be considered equivalent. For example, suppose there exist local operators $\mathcal{O} \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{L}')$ and $\mathcal{O}' \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{L}')$ such

$$\mathcal{O}' * \mathcal{O} = \mathbb{1}_{\mathcal{L}} + Q(...) \qquad \text{and} \qquad \mathcal{O} * \mathcal{O}' = \mathbb{1}_{\mathcal{L}'} + Q(...). \tag{2.1.5}$$

The local operators $\mathcal{O}_1, \mathcal{O}_2$ are said to be homotopy equivalences, and the line operators $\mathcal{L}, \mathcal{L}'$ homotopy equivalent. At the level of partition functions involving other Q-closed operators, the line operators $\mathcal{L}, \mathcal{L}'$ are entirely equivalent. There is a yet weaker notions of equivalence (weak homotopy equivalence) with respect to which one could identify line operators, leading to the notion of a homotopy category as introduced by Quillen [234].

Later in this chapter, we will give a geometric model for $\mathcal{C}$ in A-twisted 3d $\mathcal{N} = 4$ gauge theories. It is important to keep in mind that the equivalence relation on objects has already been imposed in all these geometric descriptions. In a sense, the geometric models are simplified descriptions of the very large set of Q-closed physical line operators. For the remainder of this chapter, we will mostly ignore the subtleties arising from the above equivalences of line operators.

2.1.2 Sums, products, duality, and the identity

There are several elementary operations that can be performed on line operators to build other line operators. Given two line operators $\mathcal{L}$ and $\mathcal{L}'$, we can consider their superposition, denoted $\mathcal{L} \oplus \mathcal{L}'$. A local operator joining the superposition $\mathcal{L} \oplus \mathcal{L}'$ and a third line operator $\mathcal{L}''$ is simply the choice a local operators joining $\mathcal{L}$ and $\mathcal{L}''$ and joining $\mathcal{L}'$ and $\mathcal{L}''$, *i.e.*

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{L} \oplus \mathcal{L}', \mathcal{L}'') = \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{L}'') \oplus \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}', \mathcal{L}''), \tag{2.1.6}$$

and similarly for $\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}'', \mathcal{L} \oplus \mathcal{L}')$. Physically, we can always attach trivial Chan-Paton bundle with fiber V to line operator $\mathcal{L}$, corresponding to "multiplying" the line operator $\mathcal{L}$ by the vector space V to get $V \otimes \mathcal{L} \simeq \mathcal{L}^{\oplus \dim V}$.

A direct sum of lines operators $\mathcal{L} \oplus \mathcal{L}'$ can also be deformed by choosing a local operator $\chi \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{L}') \subset \mathrm{End}_{\mathcal{C}}(\mathcal{L} \oplus \mathcal{L}')$ of fermion number $|\chi| = 1$, and modifying $Q \to Q + \chi$ along the line $\mathcal{L} \oplus \mathcal{L}'$. Mathematically, in the various geometric models of $\mathcal{C}$, the resulting line operator is called the "cone" of χ and produces nontrivial complexes of objects. Due to a dearth of local operators, this operation is not interesting for Chern-Simons theories but can be quite intricate in twists of supersymmetric field theories. In the physics of boundary conditions for 2d TQFTs, which are analogous to line operators in 3d TQFT, this sort of construction leads to familiar bound states of branes [83].

More interestingly, we can collide pairs of line operators by bringing them together in a transverse direction, as in Figure 2.2. Two parallel line operators, separated by a very small distance, are contained in an larger open neighborhood a single line, and thus define a new line operator. In the category $\mathcal{C}$, we get a product operation on objects

$$\begin{array}{ccc} (\mathcal{L}, \mathcal{L}') & \mapsto & \mathcal{L} \otimes \mathcal{L}' \\ \in \mathrm{Ob}(\mathcal{C})^2 & & \in \mathrm{Ob}(\mathcal{C}) \end{array} . \tag{2.1.7}$$

This operation is familiar in Chern-Simons theory where one finds that the collision of two Wilson lines results in a sum of Wilson lines and corresponds to a truncated version of the usual tensor product of representations.

As in the case of local operators, the cohomology class (appropriately interpreted) of this product is invariant under small deformations of $\mathcal{L}$ and $\mathcal{L}'$, *i.e.* it does not matter in which direction $\mathcal{L}$ and $\mathcal{L}'$ are brought together. However, nontrivial monodromy (a higher operation) may arise as we move all the way around the S^1 of possible transverse directions. This monodromy gives rise to an element

$$\beta_{\mathcal{L}, \mathcal{L}'} \in \mathrm{End}_{\mathcal{C}}(\mathcal{L} \otimes \mathcal{L}') \tag{2.1.8}$$

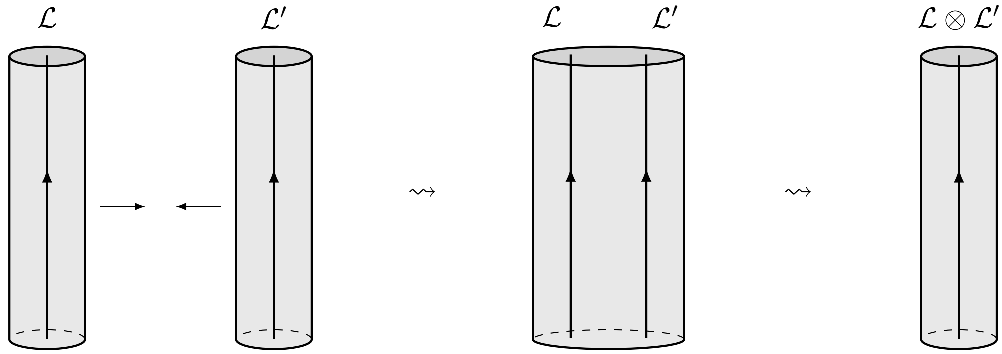


Figure 2.2: An illustration of the tensor product of line operators, induced by collision in a transverse direction.

in the endomorphism algebra of any product. Altogether, the product (2.1.7) and elements $\beta_{\mathcal{L},\mathcal{L}'}$ give $\mathcal{C}$ the structure of a *braided tensor category*.

As mentioned in the previous section, any category $\mathcal{C}$ of line operators always contains a canonical object $\mathbb{1}$, the trivial, or empty line. It is the line-operator analogue of the identity operator in the bulk. The trivial line is obviously an identity for the tensor product,

$$\mathbb{1} \otimes \mathcal{L} = \mathcal{L} \otimes \mathbb{1} = \mathcal{L} \,. \tag{2.1.9}$$

Finally, given a line operator $\mathcal{L}$ there is often is a dual line operator $\mathcal{L}^\vee$ obtained by "inverting" the line operator $\mathcal{L}$. Importantly, local operators at the junction of another line operator $\mathcal{L}'$ and $\mathcal{L}$ are the exact same as local operators at the end of $\mathcal{L}' \otimes \mathcal{L}^\vee$

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}', \mathcal{L}^\vee) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}' \otimes \mathcal{L}, \mathbb{1}) \,. \tag{2.1.10}$$

See Figure 2.3. The trivial line is self-dual $\mathbb{1} = \mathbb{1}^\vee$. The dual to a Wilson line in Chern-Simons theory is simply the Wilson line for the conjugate representation.

2.1.3 Boundary conditions and representations of the category

Another invaluable tool is to consider the junctions of line operators and boundary conditions. In Chern-Simons theory, this allows us to reinterpret the category of Wilson lines as the

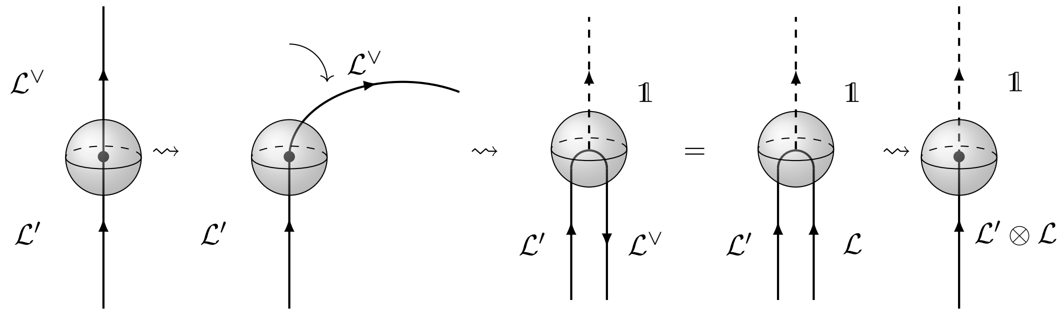


Figure 2.3: An illustration of the relation in Eq. (2.1.10) between local operators at the junction of $\mathcal{L}'$ and $\mathcal{L}^\vee$ and local operators at the end of the composite line operator $\mathcal{L}' \otimes \mathcal{L}$ obtained by colliding $\mathcal{L}'$ and $\mathcal{L}$.

category of modules for a boundary WZW model. When applied to twists of 3d $\mathcal{N} = 4$ theories, this construction views the approach of [70], as well as Sections 1.3.2 and 1.3.3, in a more algebraic and categorical framework. The general idea goes as follows. Given $\mathcal{B}$ and any object $\mathcal{L}$, we consider the space of local operators at a junction of $\mathcal{B}$ and $\mathcal{L}$, denoted $\mathrm{Hom}_{\mathcal{C}}(\mathcal{B}, \mathcal{L})$. We thus obtain a map

$$\rho_{\mathcal{B}} : \begin{array}{ccc} \mathrm{Ob}(\mathcal{C}) & \to & \mathrm{Ob}(\mathrm{Vect}) \\ \mathcal{L} & \mapsto & \mathrm{Hom}_{\mathcal{C}}(\mathcal{B}, \mathcal{L}) \end{array} . \tag{2.1.11}$$

In addition, given any pair of objects $\mathcal{L}, \mathcal{L}'$ in $\mathcal{C}$ and a local operator $\mathcal{O} \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{L}')$ at their junction, $\mathcal{O}$ acts on local operators at the junction between $\mathcal{L}$ and $\mathcal{B}$ by collision "from above." See Figure 2.4. This givens us maps

$$\rho_{\mathcal{B}}(\mathcal{O}) : \begin{array}{ccc} \rho_{\mathcal{B}}(\mathcal{L}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{B}, \mathcal{L}) & \to & \rho_{\mathcal{B}}(\mathcal{L}') = \mathrm{Hom}_{\mathcal{C}}(\mathcal{B}, \mathcal{L}') \\ v & \mapsto & \mathcal{O} * v \end{array} . \tag{2.1.12}$$

Moreover, this is compatible with successive collisions with junctions between other line operators. In particular, the vector space $\mathrm{Hom}_{\mathcal{C}}(\mathcal{B}, \mathcal{L})$ is thus a representation for the associative algebra $\mathrm{End}_{\mathcal{C}}(\mathcal{L})$.[2] We see that for any $\mathcal{B}$, Eq. (2.1.11) and Eq. (2.1.12) yields a functor

[2]It is simultaneously a module for the algebra of local operators $\mathrm{End}(\mathcal{B})$ bound to $\mathcal{B}$ itself, but we will only be interested in the action of $\mathrm{End}_{\mathcal{C}}(\mathcal{L})$ in the following.

$\rho_{\mathcal{B}} : \mathcal{C} \to \text{Vect}$, also called a representation of the category $\mathcal{C}$.[3]

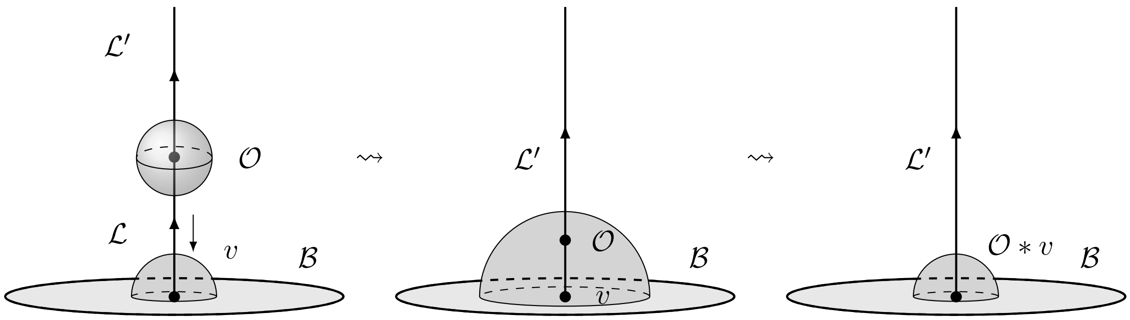


Figure 2.4: Illustration of how a choice of boundary condition $\mathcal{B}$ yields a representation of the category of line operators $\mathcal{C}$.

If the 3d theory is a full TQFT, there are several other extremely useful descriptions of $\rho_{\mathcal{B}}(\mathcal{L})$. By deforming the spacetime metric, we can squeeze the boundary condition $\mathcal{B}$ to the neighborhood of a line as in the right of Figure 2.5. Then $\rho_{\mathcal{B}}(\mathcal{L})$ is interpreted as the space of local operators at an junction of $\mathcal{L}$ and a line operator we label $\text{ch}(\mathcal{B})$[4]; indeed, this realizes the boundary condition $\mathcal{B}$ as an object $\text{ch}(\mathcal{B})$ in $\mathcal{C}$. Alternatively, we can "invert" the boundary condition $\mathcal{B}$, so that it bounds the outside of an infinite solid cylinder, with $\mathcal{L}$ running along its axis, as on the left of Figure 2.5. Then

$$\rho_{\mathcal{B}}(\mathcal{L}) \simeq \mathcal{H}(\mathcal{B}; \mathcal{L}) \tag{2.1.13}$$

becomes identified with the Hilbert space of the theory on a disk D, with $\mathcal{B}$ running along its boundary and $\mathcal{L}$ piercing its center.[5] For the special case $\mathcal{L} = \mathbb{1}$, we simply recover the

[3]This description is very slightly naive, because it ignores the equivalence relation placed on objects of $\mathcal{C}$ (*cf.* Section 2.1.1) and as well as boundary conditions. The correct construction keeps track of some of the chain-level structure on morphism spaces, rather than just their Q-cohomology. Most generally, one should remember the A_∞ structures, and produce a functor of A_∞ categories.

[4]The notation corresponds the realization of this operation as a categorical Chern character map, *cf.* [97]. The usual Chern character map takes a coherent sheaf (or vector bundle) over X, an object in the 1-category $\text{Coh}(X)$, and constructs a cohomology class of X, a vector in the 0-category $H^\bullet(X)$. Similarly, the categorical Chern character map ch sends a boundary condition, an object in the 2-category of boundary conditions Bdy, to a line operator, an object in the 1-category of line operators $\mathcal{C}$.

[5]Again, this is very slightly naive. In situations where the TQFT is realized as a topological twist of a supersymmetric theory, one must carefully follow the boundary condition $\mathcal{B}$ through the deformation. In particular, since the Lorentz group is redefined using the theory's R-symmetry, one must perform non-trivial (local) R-symmetry rotations when deforming spacetime.

Hilbert space on the disk D with the boundary condition $\mathcal{B}$ on ∂D,[6]

$$\rho_{\mathcal{B}}(\mathbb{1}) \simeq \mathcal{H}(\mathcal{B})\,. \tag{2.1.14}$$

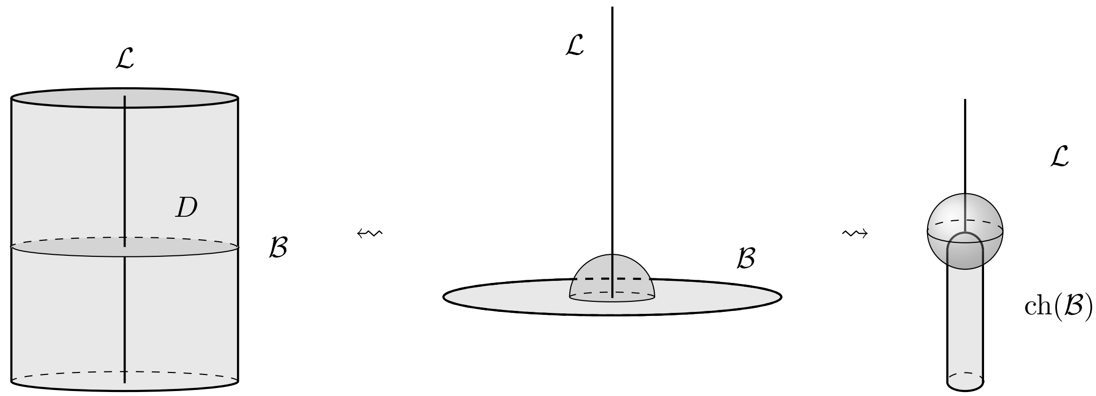


Figure 2.5: Different interpretations of the space $\mathcal{H}(\mathcal{B};\mathcal{L}) = \rho_{\mathcal{B}}(\mathcal{L}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{B},\mathcal{L})$ in a full 3d TQFT. Left: $\mathcal{H}(\mathcal{B};\mathcal{L})$, states on D punctured by $\mathcal{L}$ with $\mathcal{B}$ imposed on ∂D on. Center: $\rho_{\mathcal{B}}(\mathcal{L})$, local operators at a transverse junction of $\mathcal{L}$ and $\mathcal{B}$. Right: $\mathrm{Hom}_{\mathcal{C}}(\mathrm{ch}(\mathcal{B}),\mathcal{L})$, local operators at a junction of the line operator $\mathcal{L}$ and the wrapped boundary condition $\mathrm{ch}(\mathcal{B})$.

Computing the Hilbert spaces (2.1.13), (2.1.14) in actual 3d $\mathcal{N} = 4$ theories turns out to be fairly manageable. Once we put the theory on a disk D with the boundary condition $\mathcal{B}$, it effectively becomes one-dimensional; and the Hilbert space may be computed using standard methods from supersymmetric quantum mechanics. If we choose a boundary condition $\mathcal{B}$ that is large enough, the Hilbert spaces $\mathcal{H}(\mathcal{B};\mathcal{L})$ and the maps among them will give a faithful representation of the category of line operators $\mathcal{C}$. Concretely, "large enough" means that all line operators in $\mathcal{C}$ (or all line operators of interest) can end on $\mathcal{B}$. In the latter parts of this chapter, we introduce such boundary conditions to explicitly compute morphism spaces in the A-twist of 3d $\mathcal{N} = 4$ gauge theories.

An important question is how close the maps in (2.1.11) are to being isomorphisms. We can make a few general remarks. A necessary condition for the functor $\rho_{\mathcal{B}}$ to be faithful —

[6]The inverse of this transformation was used in Section 1.3.3 to understand the algebra of local operators in the A-twist, *i.e.*, of $\mathrm{End}_{\mathcal{C}_A}(\mathbb{1})$.

meaning that all $\rho_{\mathcal{B}}(\mathcal{L})$ are nonzero and all the maps

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{L},\mathcal{L}') \to \mathrm{Hom}_{\mathbb{C}}\big(\rho_{\mathcal{B}}(\mathcal{L}),\rho_{\mathcal{B}}(\mathcal{L}')\big) \tag{2.1.15}$$

are injective — is that all lines $\mathcal{L}$ can end on the boundary condition $\mathcal{B}$. Otherwise, some $\rho_{\mathcal{B}}(\mathcal{L})$ will clearly be zero.[7] If a single boundary condition $\mathcal{B}$ is *not* sufficient to faithfully probe all the line operators of interest, then one could try to analyze the maps (2.1.11) for multiple boundary conditions at once.

The functor (2.1.11) will almost never be full — meaning that all maps

$$\rho_{\mathcal{B}} : \mathrm{Hom}_{\mathcal{C}}(\mathcal{L},\mathcal{L}') \to \mathrm{Hom}_{\mathbb{C}}\big(\rho_{\mathcal{B}}(\mathcal{L}),\rho_{\mathcal{B}}(\mathcal{L}')\big) \tag{2.1.16}$$

are surjective. Indeed, we would not want this! Linear transformations of vector spaces, $\mathrm{Hom}_{\mathbb{C}}\big(\rho_{\mathcal{B}}(\mathcal{L}),\rho_{\mathcal{B}}(\mathcal{L}')\big)$ form a large, boring matrix algebra. The local operators $\mathrm{Hom}_{\mathcal{C}}(\mathcal{L},\mathcal{L}')$ at junctions of lines should be embedded inside in an interesting way.

As previewed in Section 1.3.3, $\mathrm{Hom}_{\mathbb{C}}\big(\rho_{\mathcal{B}}(\mathcal{L}),\rho_{\mathcal{B}}(\mathcal{L}')\big)$ simply consists of all linear operators. This includes not only local operators at the junction of $\mathcal{L}$ and $\mathcal{L}'$, but *e.g.* surface operators wrapping D (at a fixed time) and line operators that wrap the boundary $\partial D \times \{t_0\}$ at fixed time, and look like interfaces along $\mathcal{B}$. Below, we will attempt to characterize the image of $\rho_{\mathcal{B}}$ in a precise way that excludes these other non-local operators.

2.1.4 Omega backgrounds

As mentioned in Section 1.2, the A-twist of 3d $\mathcal{N}=4$ gauge theories is compatible with an Omega background. One can similarly ask for line operators that are compatible with the Omega background; these must lie along the axis rotated by the Omega background. We thus expect the entire category of line operators $\mathcal{C}$ in the A-twist of 3d $\mathcal{N}=4$ gauge theories to get deformed

$$\mathcal{C} \ \rightsquigarrow\ \mathcal{C}_{\varepsilon}\,. \tag{2.1.17}$$

[7] The vacuum boundary conditions used in [70, 111] seem to have this property, at least for the sort of $\frac{1}{2}$-BPS line operators defined in Section 2.2.

The deformation destroys the braided tensor structure of $\mathcal{C}$, since transverse collisions of lines no longer make sense; thus $\mathcal{C}_\varepsilon$ is an ordinary derived (or dg) category.

Concretely, saying that the category gets deformed means that we can track line operators through the deformation and that their morphism spaces

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{L},\mathcal{L}') \;\rightsquigarrow\; \mathrm{Hom}_{\mathcal{C}_\varepsilon}(\mathcal{L},\mathcal{L}') \tag{2.1.18}$$

all get deformed in a consistent way, preserving the associative structure of morphism composition. The simplest example of this is the endomorphism algebra of the trivial operator

$$\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \;\rightsquigarrow\; \mathrm{End}_{\mathcal{C}_\varepsilon}(\mathbb{1})\,, \tag{2.1.19}$$

which gets quantized. For $\mathcal{C}$ the category of A-type line operators, this reproduces the usual quantization of the Coulomb-branch chiral ring.

Note that if we have the boundary condition $\mathcal{B}$ is compatible with an Omega background, we find a representation of the deformed category,

$$\rho_{\mathcal{B}}\;:\;\mathcal{C}_\varepsilon \to \mathrm{Vect}\,. \tag{2.1.20}$$

These representations of the deformed category $\mathcal{C}_\varepsilon$ will play a central role in Sections 2.3.4 and 2.4, as well as Section 3.3.

2.2 $\frac{1}{2}$-BPS vortex lines

In this section we turn to A-type line operators, *i.e.* $\frac{1}{2}$-BPS line operators that are preserved by the 1d $\mathcal{N}=4$ algebra SQM_A.

As reviewed in the Introduction, many aspects of these extended operators have already been studied in the literature — often under the guise of $\frac{1}{2}$-BPS surface operators in 4d $\mathcal{N}=2$ gauge theories, which share much of the same structure. Moreover, surface operators in 4d

$\mathcal{N} = 2$ gauge theories were themselves a generalization of the prototypical Gukov-Witten defects of 4d $\mathcal{N} = 4$ super-Yang-Mills theory, classified in [34, 181].

We know from the literature to expect several different — but largely equivalent — constructions of A-type line operators, as

1) disorder operators, modeled on singular solutions to the BPS equations for the SQM_A subalgebra of 3d $\mathcal{N} = 4$
(the BPS equations are generalized vortex equations, whence we typically refer to A-type line operators as *vortex lines*);
2) coupled 3d-1d systems (coupling bulk 3d fields to 1d SQM_A quantum mechanics, by gauging 1d flavor symmetries and introducing superpotential interactions).

In addition, all A-type line operators should define objects in the category of line operators in the A-twist, so we may also hope for a description as

3) objects of a (dg/A_∞) braided tensor category, with some mathematical definition.

In this chapter, we will largely focus on constructions (1) and (2). We consider a class of line operators characterized by

- A meromorphic singularity in the hypermultiplet scalars at $z = 0$ in the $\mathbb{C}_z$ plane transverse to a line operator. In description (2), these singularities can be engineered by coupling 3d hypers to 1d chiral matter via a superpotential.
- A breaking of gauge symmetry near $z = 0$, compatible with the singular profile of hypermultiplets. In description (2), this breaking can be engineered by gauging flavor symmetries of a 1d sigma model (essentially a coset model) with the 3d gauge group.

It is essential for us to allow higher-order singularities in the matter fields, and breaking of gauge symmetry to higher order around $z = 0$; correspondingly, when coupling to 1d quantum mechanics, we allow higher-order derivative couplings. In the context of geometric Langlands, such singularities were referred to as “wild ramification,” and studied from a physical perspective in [181]. In 3d $\mathcal{N}=4$ gauge theories, A-type line operators defined by higher-order singularities turn out to be the 3d mirrors of ordinary B-type Wilson lines with higher (non-minuscule) charge.

Many standard brane constructions of surface operators in 4d $\mathcal{N}=2$ theories and line operators in 3d $\mathcal{N}=4$ (*e.g.* [80,159,206,227]) actually lead naturally to higher-order singularities. In quiver quantum-mechanics descriptions of these operators, there are higher-derivative couplings present. A simple example of this phenomenon was described in [111, Section 5.3].

We begin in Section 1.3.2 by reviewing the BPS equations for SQM_A, their relation to 1d quantum mechanics, and their associated holomorphic data. Then in Section 2.2.1 we discuss in detail the structure of vortex lines in a theory of free hypermultiplets. Perhaps surprisingly, this turns out to be interesting and nontrivial, and gives us a concrete realization of all three constructions (1)-(3) above! Geometrically, we will associate vortex lines in a theory of n free hypermultiplets with holomorphic Lagrangians in the loop space $L(T^*\mathbb{C}^n)$.

We then consider A-type lines in gauge theories in Section 2.2.3. Roughly speaking, this requires combining meromorphic singularities in hypermultiplet fields with compatible patterns of gauge-symmetry breaking in the neighborhood of a line. We give many examples, and define a general class of A-type line operators in gauge theories whose junctions we will study in the remainder of this thesis.

2.2.1 Free matter

Even a theory with free hypermultiplet matter can have interesting, nontrivial $\frac{1}{2}$-BPS vortex lines. Indeed, they illustrate most of the main features of vortex lines in gauge theories, while avoiding subtleties such as the equivalence of real and holomorphic moduli spaces (1.3.16)-(1.3.18) above. We discuss free hypermultiplets in this section, and then add gauge interactions in Section 2.2.3.

Consider the 3d $\mathcal{N}=4$ theory of a single free hypermultiplet. (In this case, $G=1$ and the hypermultiplet scalars (X,Y) are just valued in $R\oplus R^* = \mathbb{C}\oplus\mathbb{C}$.) The SQM_A BPS equations (1.3.12) simply require X,Y to be constant in time, and holomorphic in the $\mathbb{C}_z$

plane,

$$\partial_{\bar{z}} X = \partial_{\bar{z}} Y = 0\,. \tag{2.2.1}$$

A large family of solutions with a singularity at the origin come from allowing X and Y to have poles of some order, say

$$X(z) = \frac{a}{z^k} + \frac{b}{z^{k-1}} + \dots\,, \qquad Y(z) = \frac{a'}{z^{k'}} + \frac{b'}{z^{k'-1}} + \dots\,. \tag{2.2.2}$$

Given such a solution, we can attempt to define a "disorder" line operator $\mathbb{V}_{k,k'}$ using a standard prescription: we excise the line $\{z=0\}$ from spacetime, and restrict the path integral on $\mathbb{C}^*_z \times \mathbb{R}_t$ to field configurations that approach (2.2.2) near $z=0$.

There is actually some choice in how to interpret (2.2.2). The vortex-line operators $\mathbb{V}_{k,k'}$ that we define in this chapter will *allow* poles of order $\leq k$, $\leq k'$ in X, Y at $z=0$, but do not *require* poles. In other words, we do not fix the coefficients of singular terms, such as $a, b, a', b', \ldots$, above.[8] A qualitative feature of this choice is that the $U(1)_m$ flavor symmetry that rotates X and Y with opposite charge is preserved. As we shall verify later, vortex lines defined in this manner turn out to be naturally dual to B-type Wilson lines.

Lagrangians in the loop space

There is an important additional constraint on the values of k and k' appearing in (2.2.2) that we must discuss. In order for (2.2.2) to be a $\frac{1}{2}$-BPS field configuration, it is not quite sufficient to just satisfy the bosonic BPS equations (2.2.1); we must also consider the fermionic fields. Equivalently, we must make sure that a singularity of the form (2.2.2) makes sense for entire 1d SQM_A multiplets.

From the superpotential (1.3.14), it is clear that the 1d chiral multiplet with bottom component X has an F-term $\partial_z \overline{Y}$. Similarly, the multiplet with bottom component Y has an F-term $-\partial_z \overline{X}$. This structure is ultimately governed by the holomorphic symplectic form $\Omega = dX \wedge dY$ on the 3d target space.

[8] This contrasts with the surface operators defined by Gukov-Witten [34], which did give the adjoint matter fields a first-order pole with fixed residue.

Suppose then that we work on the "punctured" spacetime $\mathbb{C}^*_z \times \mathbb{R}_t$, and expand X and Y into modes as

$$X = \sum_{n\in\mathbb{Z}} x_n z^n = \sum_{n\in\mathbb{Z}} x_n(r,t) r^n e^{in\theta}\,, \qquad Y = \sum_{n\in\mathbb{Z}} y_n z^n = \sum_{n\in\mathbb{Z}} y_n(r,t) r^n e^{in\theta}\,. \tag{2.2.3}$$

The respective F-terms in the X and Y multiplets are

$$\partial_z \overline{Y} = \sum_{n\in\mathbb{Z}} \partial_r \overline{y}_{-n-1} r^{-n-1} e^{in\theta}\,, \qquad -\partial_z \overline{X} = -\sum_{n\in\mathbb{Z}} \partial_r \overline{x}_{-n-1} r^{-n-1} e^{in\theta}\,. \tag{2.2.4}$$

Therefore, the pairs of modes $(r^n x_n, \frac{1}{r^{n+1}}\partial_r \overline{y}_{-n-1})$ all lie in the same multiplet, as do the pairs $(r^n y_n, \frac{1}{r^{n+1}}\partial_r \overline{x}_{-n-1})$. If we think about a putative singularity at $z=0$ as a boundary condition on the modes, we encounter a familiar structure: a "Dirichlet" boundary condition that sets any mode $x_n\big|_{r=0} = 0$ must be accompanied by a "Neumann" boundary condition that leaves its conjugate $y_{-n-1}\big|_{r=0}$ unconstrained.

For example, we would describe the trivial (*i.e.* empty) vortex line $\mathbb{1}$ in this language as the boundary condition

$$\mathbb{1}\;:\quad \tfrac{1}{r^n} x_{-n}\big|_{r=0} = r^{n-1}\partial_r \overline{y}_{n-1}\big|_{r=0} = 0\,,\quad \tfrac{1}{r^n} y_{-n}\big|_{r=0} = r^{n-1}\partial_r \overline{x}_{n-1}\big|_{r=0} = 0 \quad \forall\, n>0\,, \tag{2.2.5}$$

which simply says that all negative modes x_{-n}, y_{-n} vanish at the origin, while all positive modes are unconstrained. In other words, X, Y are regular on $\mathbb{C}_z$.

Alternatively, we could "flip" a mode from Y to X, allowing X to have a first-order pole, while constraining Y to have a first-order zero. Then the boundary condition sets

$$y_0\big|_{r=0} = \tfrac{1}{r}\partial_r \overline{x}_{-1}\big|_{r=0} = 0\,, \tag{2.2.6}$$

which is effectively Dirichlet for y_0 and Neumann for x_{-1}.

There is a natural geometric characterization of the sorts of singularities that are preserved by the SQM_A subalgebra. Let

$$\Omega_L = \frac{1}{2\pi i}\oint dz\, dX \wedge dY = \sum_{n\in\mathbb{Z}} dx_n \wedge dy_{-n-1} \tag{2.2.7}$$

be the holomorphic symplectic form on the *loop space* $L(R\oplus R^*)$ of the original 3d target, parameterized by the modes of X and Y. Then the above analysis of multiplets amounts to saying that $\frac{1}{2}$-BPS singularities must be supported on *holomorphic Lagrangian submanifolds* in the loop space, with respect to Ω_L.[9]

For example, from this geometric perspective, the trivial line operator is the Lagrangian

$$\mathbb{1}\ :\quad \{x_n = y_n = 0\}_{n<0}\,. \tag{2.2.8}$$

The vortex line that we first described in (2.2.2), with free coefficients $a, b, a', b', ...$, corresponds to a holomorphic Lagrangian if and only if $k + k' = 0$. In this case, we get the vortex line

$$\mathbb{V}_k\ :\quad \left\{\begin{array}{ll} x_n = 0 & n < -k \\ y_n = 0 & n < k \end{array}\right\}. \tag{2.2.9}$$

If $k + k' \neq 0$, the singularity (2.2.2) is not $\frac{1}{2}$-BPS.

Flipping modes with 1d chirals

An alternative definition of the vortex line $\mathbb{V}_k$ comes from coupling the 3d theory of a free hypermultiplet to additional purely 1d degrees of freedom — namely, to free 1d chiral multiplets.

Consider, for example, a single 1d $\mathcal{N} = 4$ chiral multiplet q, localized on the line ℓ at $\{z = 0\}$.[10] We will denote the scalar component of this supermultiplet by q. If we couple

[9]Such holomorphic Lagrangian submanifolds appear naturally as $\frac{1}{2}$-BPS boundary conditions for 2d $\mathcal{N} = (4,4)$ sigma-models. They were studied extensively in [33] and many subsequent papers, and are often referred to as (B,A,A) branes. The connection between (B,A,A) branes and line operators in 3d $\mathcal{N} = 4$ theories comes from reduction of the 3d theories along a circle linking the line — which turns the line into a boundary condition for an effective 2d $\mathcal{N} = (4,4)$ theory. We elaborate on this construction in [112].

[10]The sort of 1d $\mathcal{N} = 4$ multiplets that can be coupled to the bulk theory must be of "1d $\mathcal{N} = (2,2)$" type. This is because the bulk multiplets themselves reduce to "1d $\mathcal{N} = (2,2)$" type multiplets under the subalgebra

the 1d chiral to the bulk hypermultiplet fields with a superpotential $W_{1d} = q\,X\big|_{z=0}$, then the total superpotential (including (1.3.14)) becomes

$$W = \int d^2z \big[X\partial_{\overline{z}}Y + qX\delta^{(2)}(z,\overline{z})\big]\,. \tag{2.2.10}$$

The F-term equation $\partial_{\overline{z}}Y = 0$ gets modified to

$$\partial_{\overline{z}}Y + q\delta^{(2)} = 0 \quad\Rightarrow\quad Y = -\frac{q}{z} + \text{regular, holomorphic}\,, \tag{2.2.11}$$

allowing Y to have a pole with (undetermined) coefficient $-q$. Dually, there is a new F-term equation for q, namely

$$\frac{\delta W}{\delta q} = 0 \quad\Rightarrow\quad X\delta^{(2)}(z,\overline{z}) = 0 \quad\Rightarrow\quad X = z\cdot(\text{regular, holomorphic})\,, \tag{2.2.12}$$

which requires X to have a first-order zero. Altogether, coupling to the 1d chiral q provides an equivalent definition of the vortex line $\mathbb{V}_{-1}$.

This sort of procedure, using 1d matter to "flip" a mode from X to Y, is analogous to "flips" of supersymmetric boundary conditions from Neumann to Dirichlet and vice versa. Such flips were introduced in [222], in the context of 3d $\mathcal{N} = 2$ boundary conditions for 4d $\mathcal{N} = 2$ theories, as a generalization of Witten's $SL(2,\mathbb{Z})$ action on boundary conditions [235].

It is easy to generalize the coupling $W_{1d} = qX$ to produce other vortices $\mathbb{V}_k$. If $k < 0$, we can introduce $|k|$ 1d chiral multiplets $q_1, ..., q_{|k|}$ with scalar components $q_1, ..., q_{|k|}$, and a superpotential coupling

$$W_{1d} = \big[q_1X + q_2\partial_zX + \ldots + q_{|k|}\partial_z^{|k|-1}X\big]\big|_{z=0}\,, \tag{2.2.13}$$

SQM$_A$, as discussed in Section ??.

so that the F-terms of the total superpotential effectively set

$$Y=-(|k|-1)!\frac{q_{|k|}}{z^k}-\cdots-\frac{q_2}{z^2}-\frac{q_1}{z}+\text{regular}\,,\qquad X=z^k\cdot(\text{regular})\,. \tag{2.2.14}$$

Note that the 1d chirals q_i must have nontrivial charges under the $U(1)_E$ group of spacetime rotations in the $\mathbb{C}_z$ plane, in order for the coupling (2.2.13) to preserve this symmetry. From the point of view of the 1d SQM along the line, $U(1)_E$ (mixed with the bulk $U(1)_H$ R-symmetry) is an ordinary flavor symmetry.

Dually, to produce $\mathbb{V}_k$ with $k>0$, we could introduce 1d chiral multiplets $q_1,...,q_k$ and a coupling

$$W_{1d}=\big[q_1Y+q_2\partial_zY+\ldots+q_k\partial_z^{k-1}Y\big]\big|_{z=0}\,, \tag{2.2.15}$$

which effectively sets

$$X=(k-1)!\frac{q_k}{z^k}+\cdots+\frac{q_2}{z^2}+\frac{q}{z}+\text{regular}\,,\qquad Y=z^k\cdot(\text{regular})\,. \tag{2.2.16}$$

Multiple hypermultiplets

For trivial gauge group and N hypermultiplets, *i.e.* $R\oplus R^*\simeq\mathbb{C}^N\oplus\mathbb{C}^N$, the family of vortex lines described above generalizes in a straightforward way. Let $(X^i,Y_i)_{i=1}^N$ be the complex hypermultiplet scalars. Then the holomorphic symplectic form on loop space is

$$\Omega_L=\frac{1}{2\pi i}\oint dz\sum_{i=1}^N dX^i\wedge dY_i=\sum_{i=1}^N\sum_{n\in\mathbb{Z}}dx_n^i\wedge dy_{i,-n-1}\,, \tag{2.2.17}$$

and a general $\frac{1}{2}$-BPS vortex should correspond to a holomorphic Lagrangian in the space of modes $x_n^i,y_{i,n}$. The simplest holomorphic Lagrangians are just products of (2.2.9); they define vortex lines

$$\mathbb{V}_{k_1,\ldots,k_N}\;:\quad\left\{\begin{array}{ll}x_n^i=0 & n<-k_i\\ y_{i,n}=0 & n<k_i\end{array}\right\}\,, \tag{2.2.18}$$

for which each X^i is allowed a pole of order k_i (and Y_i is required to have a zero of order k_i) or vice versa. However, many more intricate configurations are possible as well.

As before, any vortex line (2.2.18) can equivalently be engineered by coupling the bulk 3d $\mathcal{N}=4$ theory to free 1d chiral matter, with appropriate $U(1)_E$ charges.

2.2.2 Algebraic reformulation

We now introduce an algebraic reformulation of the above that will be useful in the remainder of this thesis. Above, we encountered the loop space $L(R\oplus R^*)\simeq T^*(LR)$. Its algebraic version is

$$R(\mathcal{K})\oplus R^*(\mathcal{K})\simeq T^*R(\mathcal{K})\,, \tag{2.2.19}$$

where $R(\mathcal{K})=R\otimes\mathcal{K}$ denotes formal Laurent series whose coefficients are elements of R, or (equivalently) vectors in R whose entries are formal Laurent series. Geometrically, we may think of $R(\mathcal{K})\oplus R^*(\mathcal{K})$ as the space of holomorphic sections of a holomorphic $R\oplus R^*$ bundle on an infinitesimal punctured disk.

The holomorphic symplectic form on the algebraic loop space $R(\mathcal{K})\oplus R^*(\mathcal{K})$ is still given by the residue formula (2.2.17). A general $\frac{1}{2}$-BPS vortex-line operator in a theory of free hypermultiplets is labeled by a choice of holomorphic Lagrangian submanifold

$$\mathcal{L}_0\subset T^*R(\mathcal{K})\,. \tag{2.2.20}$$

We can think of this Lagrangian as specifying how sections of a holomorphic $R\oplus R^*$ on an infinitesimal punctured disk are allowed to extend over the origin. In a theory with N hypermultiplets, $R\simeq\mathbb{C}^N$, the simple holomorphic Lagrangians (2.2.18) described above, labeled by an N-tuple of integers $\mathbf{k} = (k_1, ..., k_N)$, may be expressed algebraically as

$$\mathbb{V}_{\mathbf{k}} : \quad \mathcal{L}_0 = \begin{pmatrix} z^{k_1}\mathcal{O} \\ z^{k_2}\mathcal{O} \\ \vdots \\ z^{k_N}\mathcal{O} \end{pmatrix} \oplus \begin{pmatrix} z^{-k_1}\mathcal{O} \\ z^{-k_2}\mathcal{O} \\ \vdots \\ z^{-k_N}\mathcal{O} \end{pmatrix}^T . \tag{2.2.21}$$

An algebro-geometric category

As prefaced in the Introduction, we expect vortex lines preserved by the A-twist to be objects of a braided tensor category. In the case of a 3d theory of free hypermultiplets, the category turns out to have a description in algebraic geometry as

$$\mathcal{C}_A = \text{D-mod}(R(\mathcal{K}))\,, \tag{2.2.22}$$

namely, the derived category of D-modules on the algebraic loop space $R(\mathcal{K})$. The physics and mathematics of this category (and its gauge-theory analogues) will be explored in [112]. For now, we just observe that holomorphic Lagrangians $\mathcal{L}_0 \subset T^*R(\mathcal{K})$, such as (2.2.21), naturally correspond to objects in $\mathcal{C}_A$. The Lagrangian is the micro-local support of a particular D-module.

2.2.3 Adding gauge interactions

We would like to extend the characterizations of vortex lines in theories of free hypermultiplets (Section 2.2.1) to gauge theories. As before, we expect to have several different but highly overlapping descriptions of vortex lines, as

1) singular solutions to the physical BPS equations (1.3.12)
1') singularities in holomorphic or algebraic data, such as (1.3.18)
2) coupled 3d-1d systems
3) objects of a geometrically defined category.

In the case of free hypermultiplets, there was no distinction between (1) and (1'), since the BPS equations were automatically holomorphic. This is no longer true of gauge theories. We saw in Section 1.3.2 that, in gauge theory, rewriting the BPS equations in terms of holomorphic data amounts to replacing an infinite-dimensional symplectic quotient by an infinite-dimensional holomorphic quotient. The precise relation can be quite subtle.

Nevertheless, there are some natural physical expectations for how the correspondence should work. The most practical approach (which we will follow, motivated by [34]) is to use a quantum-mechanics description (2) of a given line operator as a link between real-analytic (1) and holomorphic (1') regimes. In this section we will build up our intuition with several important classes of examples, and then combine them to describe a general class of A-type line operators in gauge theories in Section 2.2.4.

Trivial line

In gauge theory with any G and R, a canonical example of an A-type line operator is given by the trivial line $\mathbb{1}$.

As a 3d-1d coupled system, we would say that $\mathbb{1}$ is defined by doing nothing: coupling the bulk 3d theory to the trivial 1d quantum mechanics with Hilbert space $\mathbb{C}$.

The SQM$_A$ BPS equations are just the standard ones (1.3.12) in the bulk. In the presence of the trivial line, they must have ordinary, nonsingular solutions. In particular, near $z = 0$ the hypermultiplet fields are nonsingular and the gauge group is unbroken.

It is useful to give a holomorphic characterization of the trivial line, at least for purposes of establishing some notation. Since the hypermultiplet scalars are nonsingular, they belong to the subspace

$$(X, Y) \in \mathcal{L}_0 = R(\mathcal{O}) \oplus R^*(\mathcal{O}) \subset R(\mathcal{K}) \oplus R^*(\mathcal{K})\,, \qquad (2.2.23)$$

in the algebraic notation of Section 2.2.2.

Note that the algebraic group $G(\mathcal{O})$ acts naturally on $R(\mathcal{K}) \oplus R^*(\mathcal{K}) \simeq T^*R(\mathcal{K})$ (multiplying a Taylor-series entry of some $g(z) \in G(\mathcal{O})$ with a formal Laurent series in $T^*R(\mathcal{K})$ gives another formal Laurent series). Moreover, $G(\mathcal{O})$ preserves the Lagrangian subspace

$\mathcal{L}_0 \subset T^*R(\mathcal{K})$. Altogether, the trivial line is associated to the holomorphic data

$$\mathbb{1}\ :\quad \mathcal{L}_0 = T^*R(\mathcal{O})\,,\ \mathcal{G}_0 = G(\mathcal{O})\,,\qquad \text{with } \mathcal{G}_0 \text{ preserving } \mathcal{L}_0\,. \tag{2.2.24}$$

Abelian vortex lines and screening

Consider $G = U(1)$ gauge theory with a single hypermultiplet $(X,Y) \in T^*\mathbb{C}$, where X, Y have charges $+1, -1$.

Working with holomorphic data, we can try to define a vortex line the same way as in Section 2.2.1: we allow X to have a pole of order k near $z = 0$, and dually require Y to have a zero of order k, *i.e.*

$$X \in z^{-k}\mathcal{O}\,,\qquad Y \in z^k\mathcal{O}\,. \tag{2.2.25}$$

(Note that the holomorphic-Lagrangian constraint of Section 2.2.1 must still be satisfied.) More succinctly, $(X,Y) \in \mathcal{L}_0 = z^{-k}\mathcal{O} \oplus z^k\mathcal{O}$. This sort of singularity in the hypermultiplets does not require any breaking of gauge symmetry; we can still have full, nonsingular, holomorphic gauge transformations near the origin,

$$\mathcal{G}_0 = GL(1,\mathcal{O}) = \{a + z\mathbb{C}[\![z]\!]\,,\ a \neq 0\}\,. \tag{2.2.26}$$

The vortex lines defined by (2.2.25)–(2.2.26) can actually be screened, by dynamical vortex particles. (This was discussed from a physical, analytic perspective in [226].) From a holomorphic perspective, we can act with a gauge transformation $g(z) = z^k$, which is well defined in a formal *punctured* neighborhood $\mathbb{D}^*$ of $z = 0$, to make X, Y nonsingular:

$$g(z) = z^k\ :\quad z^{-k}\mathcal{O} \oplus z^k\mathcal{O} \mapsto \mathcal{O} \oplus \mathcal{O}\,. \tag{2.2.27}$$

Physically, (2.2.27) corresponds to a "large" gauge transformation in the complement of the line operator, *i.e.* on $\mathbb{C}^*_z$. Line operators related by such gauge transformations are physically equivalent; here we find that the vortex line (2.2.25)–(2.2.26) is equivalent to the trivial line

1.

In order to define nontrivial vortex lines in $G = U(1)$ gauge theory, we must add more hypermultiplets. Thus, let us now consider N fundamental hypers $(X^i, Y_i)_{i=1}^N \in T^*\mathbb{C}^N$, where the charges of X^i, Y_i are all $+1, -1$ as before.

Choosing a vector of integers $\mathbf{k} = (k_1, ..., k_n) \in \mathbb{Z}^N$, we define a putative vortex line in terms of the holomorphic data

$$\mathbb{V}_{\mathbf{k}} \;:\quad (X^i, Y_i) \in \mathcal{L}_0 = \bigoplus_{i=1}^N z^{-k_i}\mathcal{O} \oplus z^{k_i}\mathcal{O}\,, \qquad \mathcal{G}_0 = GL(1, \mathcal{O})\,. \tag{2.2.28}$$

In other words, we allow each X^i to have a pole of order k_i and require that Y_i have a zero of order k_i (or vice versa when $k_i < 0$); and we again leave the gauge group unmodified.

Now these vortex lines are only partly screened. A singular gauge transformation $g = z^m$ (for $m \in \mathbb{Z}$) can be used to shift all integers k_i simultaneously, but not individually. Thus there are equivalences of vortex lines

$$\mathbb{V}_{\mathbf{k}} \sim \mathbb{V}_{\mathbf{k}'} \quad \text{if} \quad \mathbf{k} - \mathbf{k}' = m(1, ..., 1) \;\text{ for }\; m \in \mathbb{Z}\,. \tag{2.2.29}$$

Physical vortex charge becomes an element of the quotient lattice $\mathbf{k} \in \mathbb{Z}^N/\mathbb{Z}$.

Let us also explain how to engineer these vortex lines by coupling to quantum mechanics, providing a more physical definition from which one can recover the holomorphic data above. For simplicity, we focus on $N = 1$ hypermultiplets and ignore screening.

To obtain the vortex line (2.2.25) with $k = 1$, we follow the same procedure as for free matter. Namely, we introduce a 1d chiral multiplet q of gauge charge $+1$, and a superpotential coupling $qY\big|_{z=0}$. The total superpotential, in 1d $\mathcal{N} = 4$ terms, becomes

$$W = \int d^2z\big[-YD_{\bar z}X + qY\delta^{(2)}(z,\bar z)\big]\,, \tag{2.2.30}$$

generalizing (2.2.10).[11] The F-term for Y sets $D_{\bar z}X = q\,\delta^{(2)}$. After complexifying the gauge

[11] We have used an integration by parts to replace $XD_{\bar z}Y \rightsquigarrow -YD_{\bar z}X$, which is more convenient for intro-

group and passing to a holomorphic gauge with $A_{\bar z}=0$, this implies $X=\frac{q}{z}+$(regular, holomorphic), in other words $X\in z^{-1}\mathcal{O}$ near $z=0$. Dually, the F-term for q sets $Y\big|_{z=0}=0$, and the F-term for X sets $D_{\bar z}Y=0$; after passing to holomorphic gauge, these together imply $Y\in z\,\mathcal{O}$ near $z=0$.

The generalization to higher k gets more interesting. Suppose $k=2$. To get $X\in z^{-2}\mathcal{O}$, we introduce a pair of 1d chirals q_1,q_2, and a higher-derivative coupling

$$W=\int d^2z\big[-YD_{\bar z}X+\big(q_1Y+q_2\partial_zY\big)\delta^{(2)}(z,\bar z)\big]\,. \tag{2.2.31}$$

Note that the covariant D_z derivative cannot enter W, because A_z is not a chiral field. One may therefore be worried about gauge invariance. It turns out that (2.2.31) *can* be made invariant under the group of real (physical) gauge transformations $g(z,\bar z)\in U(1)$ near the origin of $\mathbb{C}_z$, if we give q_1 and q_2 a transformation rule

$$q_1\to g\big|_0q_1+\partial_zg\big|_0q_2\,,\qquad q_2\to g\big|_0q_2\,, \tag{2.2.32}$$

for $g(z,\bar z)\in U(1)$. (Here $\big|_0$ is shorthand for evaluation at $z=\bar z=0$.)

To recover the holomorphic data from (2.2.31) we note that in holomorphic gauge the F-terms $\delta W/\delta Y=0$ and $\delta W/\delta q_i=0$ set

$$X=\frac{q_2}{z^2}+\frac{q_1}{z}+\text{regular}\ \in\ z^{-2}\mathcal{O}\,,\qquad Y\big|_0=\partial_zY\big|_0=0\quad\Rightarrow\quad Y\in z^2\mathcal{O}\,, \tag{2.2.33}$$

as desired. Moreover, the gauge transformation (2.2.32) of q_1,q_2 is just right to ensure that, in holomorphic gauge, the polar terms in X transform as expected:

$$X(z)\to g(z)X(z)\quad\text{with } g(z)=g\big|_0+z\partial_zg\big|_0+\ldots \tag{2.2.34}$$

ducing singularities in X (as opposed to Y).

The pattern is now clear. For any $k > 0$, we may introduce 1d chirals $q_1, ..., q_k$ with

$$W = \int d^2z \big[-Y D_{\overline{z}} X + \big(q_1 Y + q_2 \partial_z Y + ... + q_k \partial_z^{k-1} Y\big) \delta^{(2)}(z, \overline{z})\big] \,. \tag{2.2.35}$$

This is gauge-invariant if the $(q_1, ..., q_k)$ are given an appropriate linear gauge transformation that involves the first $k-1$ derivatives of g at $z = \overline{z} = 0$. In holomorphic gauge, the F-terms will restrict $Y \in z^k \mathcal{O}$, and allow $X \in z^{-k}\mathcal{O}$ as desired.

Similarly, for a $U(1)$ gauge theory with $N \geq 1$ hypermultiplets, we can engineer the vortex-line operators $\mathbb{V}_{\mathbf{k}}$ from (2.2.28) by coupling to a collection of $|k_1| + |k_2| + ...|k_N|$ 1d chiral multiplets, and using them to "flip" the required modes from X to Y or vice versa. The gauge group near the origin will remain unmodified (in other words, $\mathcal{G}_0 = G(\mathcal{O})$) as long as the 1d chirals are given an appropriate gauge transformations, involving derivatives of g.

Pure gauge theory

Next, we recall (and generalize) ways to define an A-type line operator in terms of gauge-symmetry breaking. To avoid additional constraints related to hypermultiplets, we focus on pure gauge theory (meaning general G and $R = 0$). The main interesting examples require G to be non-abelian.

A class of line operators associated to gauge-symmetry breaking that is now quite standard was introduced in [34] and generalized (as surface operators) in [195, 206]. An operator in this class is characterized by choosing a Levi subgroup $\mathbb{L} \subset G$, which becomes the unbroken physical gauge group at $z = 0$. In additional, there are some continuous parameters involved. For A-type line operators in a 3d $\mathcal{N} = 4$ theory, a relevant continuous parameter is the holonomy α of the gauge connection around an infinitesimal loop linking the line operator. This holonomy must be $\mathbb{L}$-invariant, and can be conjugated to take values in the real torus T of G (modulo the Weyl group of $\mathbb{L}$). Unlike the case of surface operators in 4d $\mathcal{N} = 2$ theories, the parameter α does *not* get complexified.

In holomorphic terms, the data $(\mathbb{L}, \alpha)$ gets replaced by a single parabolic subgroup $P \subset G_{\mathbb{C}}$. The subgroup P is a minimal parabolic subgroup containing $\mathbb{L}$, and is the subgroup of $G_{\mathbb{C}}$ preserved by the line operator after passing to holomorphic gauge. As explained carefully in [34], the continuous parameter α determines *which* P to take. In general, there are finitely many discrete choices of P's, corresponding to finitely many chambers in which α can lie. For example, if $G = U(2)$ and $\mathbb{L} = T = U(1)^2$, the generic holonomy is

$$\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \in \mathfrak{t}/\Lambda_{\mathrm{cochar}} \simeq T\,. \tag{2.2.36}$$

There are two possible parabolic subgroups containing T, namely the lower and upper Borels

$$P = B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\,, \quad \text{or} \quad P = B_- = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \qquad (B, B_- \subset GL(2,\mathbb{C}))\,. \tag{2.2.37}$$

If α_i are small and $\alpha_1 > \alpha_2$ then the holomorphic data contains $P = B$; whereas if $\alpha_2 > \alpha_1$ the holomorphic data contains $P = B_-$.

Notably, most of the information in α gets lost in the translation to holomorphic data. In later sections, we will calculate spaces of local operators bound to A-type lines by taking Q_A-cohomology of certain moduli spaces of solutions to BPS equations. These calculations depend only on the holomorphic data. Stated more generally, the A-twist of 3d $\mathcal{N} = 4$ gauge theory is locally insensitive to real continuous parameters.

We may reformulate and generalize the holomorphic data in algebraic terms. The group of holomorphic gauge transformations in an infinitesimal neighborhood of the origin is $G(\mathcal{O})$, as in (1.3.35). Breaking $G_{\mathbb{C}}$ to a parabolic P right at the origin $z = 0$ means that, in an infinitesimal neighborhood, we break $G(\mathcal{O})$ to

$$\mathcal{I}_P = \{g(z) \in G(\mathcal{O}) \text{ s.t. } g(0) \in P\} \tag{2.2.38}$$

This is called a "parahoric" subgroup of $G(\mathcal{O})$. When $P = B$ is a Borel, then $\mathcal{I}_B$ is called an

"Iwahori" subgroup. For example, if $G = U(2)$ and $P = B$ as in (2.2.37), we have

$$\mathcal{I}_B = \left\{ g(z) \in GL(2,\mathcal{O}) \text{ s.t. } g(z) = \begin{pmatrix} a(z) & b(z) \\ z\,c(z) & d(z) \end{pmatrix} \right\} . \tag{2.2.39}$$

So far, (2.2.38) describes a "zeroth-order" breaking of gauge symmetry on the support of a line operator. We would also like to consider higher-order symmetry breaking, in a neighborhood of $z = 0$. In algebraic terms, this is easily characterized by choosing a more general subgroup

$$\mathcal{G}_0 \subseteq G(\mathcal{O}) \tag{2.2.40}$$

to remain unbroken.[12] For example, if $G = U(2)$ we could take any

$$\mathcal{G}_0 = \mathcal{I}_B^k := \left\{ g(z) \in GL(2,\mathcal{O}) \text{ s.t. } g(z) = \begin{pmatrix} a(z) & b(z) \\ z^k\,c(z) & d(z) \end{pmatrix} \right\} , \quad k \geq 0\,. \tag{2.2.41}$$

As in the case of zeroth-order symmetry breaking, the algebra/holomorphic data (2.2.40) should be supplemented by additional real parameters, when describing a breaking of the real gauge group G and a singularity of the real physical fields. For example, there may be higher-order poles in the real gauge connection. Such parameters were discussed in [181] in the context of wild ramification. We will not need them for computations in the A-twist.

Finally, we recall from [34] that line operators characterized by a breaking of gauge symmetry have a natural construction by coupling to quantum mechanics. For zeroth order breaking, we may construct the line operator labeled by $(\mathbb{L}, \alpha)$ — or holomorphically by P

[12] In this thesis, we will only consider symmetry breaking up to some finite order around $z = 0$, which means that $\mathcal{G}_0$ has finite codimension inside $G(\mathcal{O})$. In principle one could consider subgroups of infinite codimension as well. Choices of $\mathcal{G}_0$ with infinite codimension are relevant for line-like operators constructed by wrapping boundary conditions on a circle, and will be discussed further in [112].

One may generalize in yet another direction, and choose the group $\mathcal{G}_0$ of holomorphic gauge transformations near the origin to be a subgroup of the full algebraic loop group $G(\mathcal{K})$, rather than a subgroup of $G(\mathcal{O})$. This is possible because, once the origin is excised from the plane $\mathbb{C}_z$, all "singular gauge transformations" in $G(\mathcal{K})$ become available. We will not need such choices in this thesis, but they will be part of the general categorical setup of [112].

— by introducing a 1d SQM_A SQM sigma-model with Kähler target

$$\mathcal{X} = G/\mathbb{L} \simeq G_{\mathbb{C}}/P\,. \tag{2.2.42}$$

The space $\mathcal{X}$ is a homogeneous G-space, with a left G-action that manifests as a flavor symmetry in the 1d quantum mechanics. This 1d theory is coupled to the 3d $\mathcal{N}=4$ bulk by gauging the G flavor symmetry with the bulk gauge symmetry. Since the stabilizer of any point of $\mathcal{X}$ is (conjugate to a copy of) $\mathbb{L}$, the effect is to break G to $\mathbb{L}$.

The continuous parameters α enter as real Kähler parameters in the 1d sigma-model to $\mathcal{X}$. This makes it quite clear that the A-twist (whose supercharge acts as de Rham differential in 1d) will be locally insensitive to them. Many examples of line operators of this type are discussed in [80], by realizing the coset space $\mathcal{X}$ as a 1d gauged linear sigma model (GLSM). In the 1d GLSM's, the parameters α entered as real FI parameters.

More generally, we expect to be able to realize a line operator with holomorphic data $\mathcal{G}_0$ by coupling to a 1d sigma-model with target

$$\mathcal{X} = G(\mathcal{O})/\mathcal{G}_0\,. \tag{2.2.43}$$

Coupling to the 3d bulk is again done by gauging the 1d flavor symmetry. In this case, however, the flavor symmetry group is $\mathcal{G}(\mathcal{O})$, acting by left multiplication on $\mathcal{X}$; or, in real/physical terms, the symmetry group is the group $\mathcal{G}$ of gauge transformations on the disk that appeared in Section 1.3.2. Though it may look exotic, gauging this infinite-dimensional flavor symmetry is a perfectly reasonable operation! As discussed in Section 1.3.2, when we rewrite the 3d $\mathcal{N}=4$ bulk theory as 1d SQM_A quantum mechanics, the decomposition of the bulk G gauge multiplet contains a 1d vector multiplet for the infinite-dimensional gauge group $\mathcal{G}$. This 1d $\mathcal{G}$ vector multiplet can be used canonically to gauge the $\mathcal{G}$ flavor symmetry of quantum-mechanics with target $\mathcal{X}$.

As mentioned briefly in Footnote 12, we will only consider symmetry breaking up to some finite order around $z = 0$, which implies that $\mathcal{X} = G(\mathcal{O})/\mathcal{G}_0$ is a finite-dimensional space. Thus, *most* of the infinite-dimensional flavor symmetry group $G(\mathcal{O})$ (or $\mathcal{G}$ in the real case) acts trivially on $\mathcal{X}$. In turn, the coupling between 1d and 3d theories induced by gauging will only involve a finite number of derivatives. For example, if we chose $\mathcal{G}_0 = \mathcal{I}_P$ as in (2.2.38), we would find

$$\mathcal{X} = G(\mathcal{O})/\mathcal{I}_P = G_{\mathbb{C}}/P\,, \tag{2.2.44}$$

and recover the well-known setup (2.2.42), and a coupling with no derivatives at all.

2.2.4 General A-type line operators

For a 3d $\mathcal{N} = 4$ theory with general gauge group G and hypermultiplet representation T^*R, we may combine the various ingredients described above to define vortex-line operators.

In terms of holomorphic/algebraic data, we characterize a vortex line by choosing

1) a holomorphic Lagrangian subspace $\mathcal{L}_0 \subset T^*R(\mathcal{K})$, encoding the meromorphic singularity in the hypermultiplet scalars

2) a subgroup $\mathcal{G}_0 \subseteq G(\mathcal{O})$ of the group of holomorphic gauge transformations in an infinitesimal neighborhood of $z = 0$, encoding the breaking of gauge symmetry.

These two choices must be compatible, in the sense that $\mathcal{G}_0$ must preserve $\mathcal{L}_0$. Moreover, as we saw in Section 2.2.3, there are redundancies in this data, as some vortex-line operators can be related by "screening." In algebraic terms, two pairs of data $(\mathcal{L}_0, \mathcal{G}_0)$ and $(\mathcal{L}'_0, \mathcal{G}'_0)$ are physically equivalent if there exists an element $g(z) \in G(\mathcal{K})$ such that

$$\text{screening equivalence}: \quad (g \cdot \mathcal{L}_0\,,\ g\,\mathcal{G}_0\,g^{-1}) = (\mathcal{L}'_0, \mathcal{G}'_0)\,. \tag{2.2.45}$$

When defining an A-type line operator in the full, physical 3d $\mathcal{N} = 4$ theory, this data should be accompanied by additional real parameters, associated to a $\mathcal{G}_0$-invariant singularity in the holomorphic connection A_z. We will not need them for analyses in the A-twist. In the quantum-mechanics definition of vortex-line operators (further below), the real parameters are Kähler parameters of $G(\mathcal{O})/\mathcal{G}_0$.

Example: U(2) with matter

Let us give a simple example of line operators in the general class above, in the case of non-abelian gauge theory with matter. We take $G = U(2)$ and $R = \mathbb{C}^2$ the fundamental representation.

Since $G_{\mathbb{C}} = GL(2,\mathbb{C})$, the group of holomorphic gauge transformations in an infinitesimal neighborhood of $z = 0$ is

$$G(\mathcal{O}) = GL(2,\mathcal{O}) = \left\{ g(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} \text{ s.t. } a,b,c,d \in \mathcal{O}\,,\ \det g\big|_{z=0} \neq 0 \right\}. \tag{2.2.46}$$

Suppose that we require the hypermultiplets $X = \left(\begin{smallmatrix} X^1 \\ X^2 \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} Y_1 \\ Y_2 \end{smallmatrix}\right)^T$ to take the form

$$(X,Y) \in \mathcal{L}_0 = \begin{pmatrix} z^{-k_1}\mathcal{O} \\ z^{-k_2}\mathcal{O} \end{pmatrix} \oplus \begin{pmatrix} z^{k_1}\mathcal{O} \\ z^{k_2}\mathcal{O} \end{pmatrix}^T . \tag{2.2.47}$$

for some $k_1, k_2 \geq 0$.

If $k_1 = k_2$, the holomorphic Lagrangian subspace $\mathcal{L}_0$ is preserved by the full $G(\mathcal{O})$ gauge symmetry, so we may simply choose $\mathcal{G}_0 = G(\mathcal{O})$ to define a vortex-line operator.

If $k_1 \neq k_2$, the gauge group must be broken. A simple case is $(k_1, k_2) = (1, 0)$. Then we are looking at $X^1 \in z^{-1}\mathcal{O}$, $X^2 \in \mathcal{O}$. The largest subgroup of $G(\mathcal{O})$ that preserves this singularity is the standard Iwahori $\mathcal{I}_B$ from (2.2.39), containing elements of the form

$$g(z) = \begin{pmatrix} a(z) & b(z) \\ z\,c(z) & d(z) \end{pmatrix}, \quad a,b,c,d \in \mathcal{O}\,. \tag{2.2.48}$$

Then we can choose $\mathcal{G}_0 = \mathcal{I}_B$ together with (2.2.47) to define a vortex line.

Many other interesting options are possible. For example, if $k_1 > k_2$, a maximal subgroup of $G(\mathcal{O})$ that preserves the meromorphic singularity $X^1 \in z^{-k_1}\mathcal{O},\ X^2 \in z^{-k_2}\mathcal{O}$ is the "higher" Iwahori subgroup (sometimes called a higher congruence subgroup) $\mathcal{I}_B^{k_1-k_2}$ from (2.2.41), containing elements of the form

$$g(z) = \begin{pmatrix} a(z) & b(z) \\ z^{k_1-k_2}c(z) & d(z) \end{pmatrix} . \tag{2.2.49}$$

For fixed $k_1 \geq k_2$, we can define a vortex-line operator by supplementing (2.2.47) with $\mathcal{G}_0 = \mathcal{I}_B^k$ for any $k \geq k_1 - k_2$.

Coupling to quantum mechanics

The vortex-line operators characterized by holomorphic data $\mathcal{L}_0$ and $\mathcal{G}_0$ can be systematically engineered by coupling the 3d $\mathcal{N}=4$ theory to a 1d $\mathcal{N}=4$ sigma-model (with multiplets of "1d $\mathcal{N}=(2,2)$" type). The procedure for doing so combines the quantum-mechanics construction of singularities in free-matter theories (Section 2.2.1) and in pure gauge theories (Section 2.2.3).

Many examples of this construction are known in the literature, usually involving 1d GLSM's and brane constructions (*e.g.* many appear in [80]). Here we give a general geometric description.

We consider general G and R, but assume for simplicity that $\mathcal{L}_0 \subset R(\mathcal{K}) \oplus R^*(\mathcal{K})$ is a subspace of the form

$$\mathcal{L}_0 \simeq (z^{-k_1}\mathcal{O}, z^{-k_2}\mathcal{O}, ..., z^{-k_N}\mathcal{O})^T \oplus (z^{k_1}\mathcal{O}, z^{k_2}\mathcal{O}, ..., z^{k_N}\mathcal{O})\,, \tag{2.2.50}$$

for some integers $\mathbf{k} = (k_1, ..., k_N)$.

Let us ignore the gauge group for the moment. We learned in Section 2.2.1 that the singularity (2.2.50) can be engineered by introducing $|\mathbf{k}| := |k_1| + |k_2| + ... + |k_N|$ 1d chiral multiplets q_i, and a superpotential

$$W = \int d^2z\, X D_{\overline{z}} Y + W_0(q; X, \partial_z X, ...; Y, \partial_z Y, ...)\big|_{z=\overline{z}=0}\,, \tag{2.2.51}$$

where W_0 contains quadratic couplings between the q's and appropriate ∂_z derivatives of the bulk hypermultiplets X and Y. These quadratic couplings effectively "flip" non-negative modes of X into negative modes of Y and vice versa, to recover $\mathcal{L}_0$.

Formally, we may think of $W_0\big|_{z=\overline{z}=0}$ as a function

$$W_0\big|_{z=\overline{z}=0} : V \times (R(\mathcal{O}) \oplus R^*(\mathcal{O})) \to \mathbb{C}\,, \tag{2.2.52}$$

where $R(\mathcal{O}) \oplus R^*(\mathcal{O})$ is parameterized by ∂_z derivatives of X and Y, and

$$V = \bigoplus_{i=1}^{N} \begin{cases} z^{-k_i}\mathbb{C} & k_i > 0 \\ z^{k_i}\mathbb{C} & k_i < 0 \end{cases} \;\simeq\; \mathbb{C}^{|k_1|+...+|k_N|} \tag{2.2.53}$$

is the finite-dimensional vector space parameterized by the q's.

Now, the fact that $\mathcal{L}_0$ is only invariant under $\mathcal{G}_0$ rather than all of $G(\mathcal{O})$ means that the superpotential $W_0\big|_{z=\overline{z}=0}$ is invariant only under $\mathcal{G}_0$. Thus, a coupled 3d-1d system with total superpotential (2.2.51) only makes sense if we break gauge symmetry explicitly near $z = 0$. We would rather like to break gauge symmetry through a coupling to a 1d sigma-model.

In pure gauge theory, we broke gauge symmetry by coupling to the coset space $\mathcal{X} = G(\mathcal{O})/\mathcal{G}_0$. In the presence of matter, we enhance this construction as follows. The vector space V is a finite-dimensional representation of group $\mathcal{G}_0$.[13] It can therefore be used to define a holomorphic, homogeneous, associated vector bundle $\mathcal{E}$ over $\mathcal{X}$,

$$\mathcal{E} = (\mathcal{G}(\mathcal{O}) \times V)/\mathcal{G}_0\,, \tag{2.2.54}$$

[13] Explicitly, the 1d chirals q_i discussed above correspond to the negative modes appearing in $\mathcal{L}_0$. They transform linearly under an element $g(z) \in \mathcal{G}_0$, in a way that depends on g and its ∂_z derivatives at $z = 0$. See, for example, (2.2.32).

whose points are pairs $(g, q) \in \mathcal{G}(\mathcal{O}) \times V$ modulo the equivalence relation

$$(gh, q) \sim (g, hq) \quad \forall\, h \in \mathcal{G}_0\,. \tag{2.2.55}$$

The map $\mathcal{E} \to \mathcal{X}$ just forgets q; so all fibers of $\mathcal{E}$ are isomorphic to V. The $G(\mathcal{O})$ action on $\mathcal{X}$ lifts to the total space of the bundle, with an element $g' \in G(\mathcal{O})$ sending $(g, q) \mapsto (g'g, q)$.

In order to engineer our desired vortex line by coupling to quantum mechanics in a gauge-invariant way, we introduce a 1d $\mathcal{N} = 4$ sigma-model whose target is the total space of $\mathcal{E}$. We couple to the 3d bulk theory (also rewritten as a 1d $\mathcal{N} = 4$ theory) by

- Gauging the flavor symmetry of the sigma-model with the bulk gauge symmetry (exactly as in (2.2.43)).
- Introducing a $G(\mathcal{O})$-invariant superpotential $\int d^2z\, X D_{\bar{z}} Y + \widetilde{W}_0$, where $\widetilde{W}_0 : \mathcal{E} \times (R(\mathcal{O}) \oplus R^*(\mathcal{O})) \to \mathbb{C}$ is defined by

$$\widetilde{W}_0\big((g, q); X; Y\big) = W_0(q; g^{-1} \cdot X; g^{-1} \cdot Y)\big|_{z=\bar{z}=0}\,. \tag{2.2.56}$$

Here on the RHS we suppressed potential ∂_z derivatives of X and Y in order to simplify the notation. We also schematically write $g^{-1} \cdot X$, $g^{-1} \cdot Y$ to denote the action of $g(z)^{-1} \in G(\mathcal{O})$ on X and Y.

To check that $\widetilde{W}_0$ is well defined on the quotient space $\mathcal{E}$, note that

$$\begin{aligned} \widetilde{W}_0((gh^{-1}, hq); X; Y) &= W_0(hq; hg^{-1}X; hg^{-1}Y)\big|_{z=\bar{z}=0} \\ &= W_0(q; g^{-1} \cdot X; g^{-1} \cdot Y)\big|_{z=\bar{z}=0} = \widetilde{W}_0((g, q); X; Y) \end{aligned} \tag{2.2.57}$$

due to $\mathcal{G}_0$-invariance of W_0. Moreover, $\widetilde{W}_0$ is invariant under the left action of $G(\mathcal{O})$, since

$$\begin{aligned} \widetilde{W}_0((g'g, q); g' \cdot X; g' \cdot Y) &= W_0(q; g^{-1}g'^{-1}g' \cdot X; \ldots)\big|_{z=\bar{z}=0} \\ &= W_0(q; g^{-1} \cdot X, g^{-1} \cdot Y)\big|_{z=\bar{z}=0} = \widetilde{W}_0((g, q); X; Y) \end{aligned}\,. \tag{2.2.58}$$

Finally, we emphasize that the gauge-fixed form of (2.2.56) looks just like the simpler (2.2.51). In holomorphic terms, we use the bulk $G(\mathcal{O})$ action to bring any point $(g, q) \in \mathcal{E}$ to $(1, q)$. The stabilizer of $(1, q)$ is $\mathcal{G}_0$, and the superpotential over this point is manifestly $\widetilde{W}_0((1, q); X; Y) = W_0(q; X; Y)\big|_{z=\bar{z}=0}$. From (2.2.51), we recover the original Lagrangian $\mathcal{L}_0$.

Category

In [112], we will propose that the category of line operators in the A-twist of a 3d $\mathcal{N} = 4$ gauge theory is

$$\mathcal{C}_A = \text{D-mod}_{G(\mathcal{K})}(R(\mathcal{K}))\,. \tag{2.2.59}$$

This is the derived category of D-modules on the loop space $R(\mathcal{K})$, equivariant for the loop group $G(\mathcal{K})$; it generalizes (2.2.22) to gauge theories. It turns out that $\frac{1}{2}$-BPS A-type line operators characterized by the algebraic data $(\mathcal{L}_0, \mathcal{G}_0)$ naturally define objects in (2.2.59). There are much more general objects in (2.2.59) as well, which will be explored in [112].

A version of the category (2.2.59) recently appeared in work of Costello-Creutzig-Gaiotto on chiral boundary conditions for 3d $\mathcal{N} = 4$ theories [53]. There, it was the category of modules for a boundary VOA. These modules are naturally associated to bulk line operators that end on the boundary, much as in the classic relation between 3d Chern-Simons and WZW [41, 236].

Mass parameters and quantization

As we shall see in the examples below, it is possible to deform the above vortex lines by turning on complex masses and/or an Omega background. After rewriting our 3d $\mathcal{N} = 4$ gauge theories as 1d SQM_A quantum mechanics, both of these deformations are interpreted as turning on twisted masses for flavor symmetries, *cf.* [70, Sec 2.5]. In particular, rotations of the $\mathbb{C}_z$ plane, which are involved in the Omega background, simply become symmetries of the target space of the quantum mechanics.

Such twisted masses do not affect the vortex-line operators *per se*. Rather, they deform the spaces of local operators at junctions of vortex lines, and the algebraic structure of local operators coming from collision. Local operators will be the subject of the next section. We shall see, just like in [70] and Chapter 1, that complex masses and the Omega background deform homology to equivariant homology in various constructions.

It is also worth noting that, in the presence of an Omega background, turning on quantized mass parameters $m_{\mathbb{C}} = \lambda\varepsilon$ (where λ is an integral cocharacter of the 3d Higgs-branch flavor symmetry F) is equivalent to introducing a flavor vortex for a subgroup $U(1)_\lambda \subseteq F$. This is mirror to the phenomenon of abelian Wilson lines being equivalent to quantized FI parameters. See [40, 111] for further discussion.

2.3 Junctions of vortex lines

Given a pair $\mathcal{L}, \mathcal{L}'$ of $\frac{1}{2}$-BPS A-type line operators in a 3d $\mathcal{N} = 4$ gauge theory, we would like to be able to compute the Q_A-cohomology of the space of local operators at their junction. In categorical terms, we seek $\mathrm{Hom}_A(\mathcal{L}, \mathcal{L}') := \mathrm{Hom}_{\mathcal{C}_A}(\mathcal{L}, \mathcal{L}')$. We would also like to find the OPE induced from collision of junctions, *i.e.* composition of Homs.

Even the simplest case, where $\mathcal{L} = \mathcal{L}' = \mathbb{1}$ are both the trivial line, the algebra

$$\mathrm{End}_A(\mathbb{1}) = \mathrm{Hom}_A(\mathbb{1}, \mathbb{1}) \supseteq \mathbb{C}[\mathcal{M}_C] \tag{2.3.1}$$

contains the Coulomb-branch chiral ring. The ring $\mathbb{C}[\mathcal{M}_C]$ includes monopole operators, whose OPE's famously receive perturbative and (in non-abelian gauge theories) nonperturbative quantum corrections, making them difficult to compute with a semi-classical approach.

Fortunately, the last few years have seen remarkable progress in developing exact, TQFT-based methods to compute the Coulomb-branch chiral ring, *e.g.* [39,53,67,69–72,101,237–241]. Many of these methods can be adapted to exact computations of local operators at junctions of more general vortex lines as well. This was already done in limited contexts in [39, 40, 237].

In this section, we adapt the approach of [40, 70, 111] to compute spaces $\mathrm{Hom}_A(\mathcal{L}, \mathcal{L}')$ and their OPE. Physically, this requires choosing a $\frac{1}{2}$-BPS boundary condition $\mathcal{B}$ and a half-space with line operators perpendicular to the boundary, with $\mathcal{B}$ wrapping the boundary as described in Section 2.1.3.

We will quickly restrict our focus to $\mathcal{N} = (2,2)$ Dirichlet boundary conditions $\mathcal{B}$ that fully break the gauge group at the boundary. Such boundary conditions — when available — allow for *relatively* simple computations of spaces of local operators. Even so, mathematically, we will need to employ equivariant intersection cohomology or Borel-Moore homology. Many other interesting boundary conditions can be studied.

The algebraic definitions of moduli spaces in this section — in particular, their equivalence with analytic definitions, via a Kobayashi-Hitchin correspondence — are almost all conjectural.

2.3.1 Half-space setup

We now implement the abstract discussion of Section 2.1.3 for the case of the A-twist of our 3d $\mathcal{N} = 4$ theories. Just like line operators, BPS boundary conditions for 3d $\mathcal{N} = 4$ theories are classified by the 2d SUSY subalgebras that they preserve. We are interested in $\frac{1}{2}$-BPS boundary conditions $\mathcal{B}$ that preserve 2d $\mathcal{N} = (2,2)$ SUSY and $U(1)_C \times U(1)_H$ R-symmetry. (These were studied in [68] for 3d $\mathcal{N} = 4$ sigma-models, and in [40, 242] for gauge theories.) Such boundary conditions are compatible with both the A and B twists of the bulk.

Suppose that a bulk 3d $\mathcal{N} = 4$ theory has gauge group G and hypermultiplets in a representation T^*R. We saw in Section 2.2 that a line operator $\mathcal{L}$ (as seen by the A-twist) is characterized by algebraic data consisting of 1) a subgroup $\mathcal{G}_0 \subseteq G(\mathcal{O})$ of algebraic gauge transformations in an infinitesimal neighborhood of $z = 0$; and 2) a $\mathcal{G}_0$-invariant Lagrangian subspace $\mathcal{L}_0$ of the algebraic loop space $T^*R(\mathcal{K})$.

Returning to the half-space setup, we consider the bulk 3d theory on $\mathbb{D} \times \mathbb{R}_{t\geq 0}$ with line operator(s) at the origin of $\mathbb{D}$ and our boundary condition $\mathcal{B}$ at $t = 0$. A large class of boundary conditions $\mathcal{B}$, described in detail in [40], admit an algebraic characterization. The boundary conditions we will use in the remainder of this thesis are similar to those considered in [40]: they are labeled a choice of meromorphic profile for (half of) the hypermultiplet scalars that fully breaks the bulk gauge symmetry at the boundary. Nevertheless, it is important to mention that there are *many* other choices of boundary conditions available, see *e.g.* Section 1.3.3 and the papers [40, 70].

In order to algebraically describe these boundary conditions, we choose an element $X_\partial \in R(\mathcal{K})$ such that it's stabilizer in $G(\mathcal{K})$ is trivial, and consider the following boundary condition $\mathcal{B}_{X_\partial}$:

1. Dirichlet boundary conditions setting X to X_∂ at the boundary and Neumann boundary conditions leaving Y unconstrained, and extending to the remainder of the hypermultiplet in a way that preserves 2d $\mathcal{N} = (2,2)$ supersymmetry.

2. Dirichlet boundary conditions for the 3d gauge fields fully breaking the gauge symmetry at the boundary, Dirichlet boundary conditions setting the complex scalar φ to zero, and extending to the remainder of the vector multiplet in a way preserving 2d $\mathcal{N} = (2,2)$ supersymmetry.

Just as before, we can describe the space of states $\mathcal{H}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L})$ on an infinitesimal hemisphere anchored to the boundary with boundary condition $\mathcal{B}_{X_\partial}$, or equivalently the space of boundary local operators for $\mathcal{B}_{X_\partial}$, as the (Borel-Moore) homology of the moduli space $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L})$ of solutions to the SQM$_A$ BPS equations on the formal disk compatible with $\mathcal{B}_{X_\partial}$ on the boundary of the formal disk and the line operator $\mathcal{L}$ at $0 \in \mathbb{D}$

$$\mathcal{H}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L}) = H_\bullet\big(\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L})\big)\,. \tag{2.3.2}$$

Let us start with a description of the moduli space $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L})$. A point on this moduli space corresponds to holomorphic $G_{\mathbb{C}}$ bundle E on the formal disk $\mathbb{D}$ together with sections $X(z), Y(z)$ of the associated T^*R bundle. The sections $X(z), Y(z)$ must satisfy the moment map constraint $\mu(X,Y) = 0$ on the whole formal disk $\mathbb{D}$ and, moreover, be compatible with the boundary condition $\mathcal{B}_{X_\partial}$ on the boundary formal punctured disk $\partial\mathbb{D} = \mathbb{D}^*$.

A configuration $X(z), Y(z)$ is compatible to $\mathcal{B}_{X_\partial}$ on $\mathbb{D}^*$ if X is gauge-equivalent (over $\mathbb{D}^*$) to X_∂, *i.e.* $X = gX_\partial$ for some gauge transformation $g \in G(\mathcal{K})$. Thus, the hypermultiplet field configurations that are compatible with $\mathcal{B}_{X_\partial}$ are precisely those in the orbit

$$X, Y \in G(\mathcal{K}) \cdot T^*_{X_\partial} R(\mathcal{K}) \subset T^* R(\mathcal{K})\,. \tag{2.3.3}$$

Of course, such configurations must still satisfy the complex moment map constraint $\mu(X, Y) = 0$, so we are interested in the $G(\mathcal{K})$-orbit of $T^*_{X_\partial} R(\mathcal{K}) \cap \mu^{-1}(0)$. We will denote this space as

$$\widetilde{\mathcal{M}}_{X_\partial} = G(\mathcal{K}) \cdot \left(T^*_{X_\partial} R(\mathcal{K}) \cap \mu^{-1}(0)\right). \tag{2.3.4}$$

This is simply the moduli space of solutions to the SQM$_A$ BPS equations on $\mathbb{D}^*$. Under the assumption that the $G(\mathcal{K})$ stabilizer of X_∂ is trivial, we see that $\widetilde{\mathcal{M}}_{X_\partial}$ is a trivial bundle over $G(\mathcal{K}) \cdot X_\partial \cong G(\mathcal{K})$ with fiber identified with the $G(\mathcal{K})$ translate of the vector space $\{Y \in R^*(\mathcal{K}) : \mu(X_\partial, Y) = 0\}$.

Near the origin, we further require the hypermultiplets lie in $\mathcal{L}_0$ so that this configuration is compatible with the line operator $\mathcal{L}$. We implement the constraint defining the line operator by taking the intersection:

$$\widetilde{\mathcal{M}}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L}) = \mathcal{L}_0 \cap \widetilde{\mathcal{M}}_{X_\partial}\,. \tag{2.3.5}$$

This moduli space captures those field configurations on that are simultaneously compatible with the boundary condition $\mathcal{B}_{X_\partial}$ and the line operator $\mathcal{L}$. Finally, we must quotient by the group $\mathcal{G}_0$ of gauge transformations preserved by $\mathcal{L}$.

Let us put everything together. Given a line operator $\mathcal{L}$ (described by the algebraic data $\mathcal{G}_0, \mathcal{L}_0$) and the above Dirichlet boundary condition $\mathcal{B}_{X_\partial}$, the algebraic moduli space of solutions to the SQM$_A$ BPS equations on the formal disk $\mathbb{D}$ is

$$\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L}) = \mathcal{G}_0 \backslash \widetilde{\mathcal{M}}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L})\,, \qquad \widetilde{\mathcal{M}}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L}) := \mathcal{L}_0 \cap \widetilde{\mathcal{M}}_{X_\partial}\,. \tag{2.3.6}$$

In general, a space such as $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L}) = \mathcal{G}_0\backslash\widetilde{\mathcal{M}}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L})$ should be interpreted as a derived stack. However, the fact that X_∂ fully breaks gauge symmetry ensures that $\mathcal{G}_0$ acts freely in (2.3.6), and that $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L})$ has the much simpler structure of a (potentially singular) variety. In fact, as will be discussed in Section 2.3.2, it is a disjoint union of finite-dimensional varieties.

It is instructive to recast the space (2.3.6) geometrically. For a constant profile, *i.e.* $X_\partial \in R \subset R(\mathcal{K})$, we expect a geometric description given by a moduli space of maps from $\mathbb{CP}^1$, thought of as a compactification of an honest disk D, to the Higgs branch. This moduli space must be considered modulo isomorphisms, *i.e.* gauge transformations:

$$\mathcal{M}(\mathcal{B}_{X_\partial};\mathcal{L}) = \left\{\begin{array}{c} E,(X,Y) \text{ such that } E \text{ is a principal algebraic } G_{\mathbb{C}} \text{ bundle on } D \\ \text{with structure reduced to } \mathcal{G}_0 \text{ near } z=0 \text{ and trivialized at } z=\infty, \\ \text{and } (X(z),Y(z) \text{ is a section of an associated } T^*R \text{ bundle} \\ \text{satisfying } \mu(X,Y)=0 \text{ with } (X,Y)\in\mathcal{L}_0 \text{ near } z=0 \\ \text{and } (X,Y)\in G_{\mathbb{C}}\cdot(T^*_{X_\partial}R\cap\mu^{-1}(0)) \text{ at } z=\infty \end{array}\right\}\Big/\text{iso} \tag{2.3.7}$$

We again emphasize that the moduli spaces (2.3.6), (2.3.7) generalize spaces studied in [70,179,180] (for the case $\mathcal{L} = \mathbb{1}$). They are generalized vortex moduli spaces, encountered in many places in math and physics, as reviewed at the start of the Chapter and in Section 2.2. The proposed (yet unproven) equivalence of (2.3.6), (2.3.7) with physical solutions to the vortex equations is a natural extension of [168] to incorporate a potential singularity at $z = 0$. See [50] for more details when $\mathcal{L} = \mathbb{1}$ is the trivial line operator.

The "raviolo space" $\mathcal{M}_{\text{rav}}$, used for computing local operators at a junction of lines, should admit a similar algebro-geometric description. Given a pair of line operators $\mathcal{L}', \mathcal{L}$, we propose that

$$\mathcal{M}_{\text{rav}}(\mathcal{B}_{X_\partial};\mathcal{L},\mathcal{L}') = {}_{\mathcal{G}_0{}'}\backslash\widetilde{\mathcal{M}}_{\text{rav}}(\mathcal{B}_{X_\partial};\mathcal{L},\mathcal{L}')\big/_{\mathcal{G}_0}\,, \tag{2.3.8}$$

$$\widetilde{\mathcal{M}}_{\text{rav}}(\mathcal{B}_{X_\partial};\mathcal{L},\mathcal{L}') := \big(\underset{X',Y'}{\mathcal{L}_0' \cap \widetilde{\mathcal{M}}_{X_\partial}}\big) \times \underset{g}{G(\mathcal{K})} \times \big(\underset{X,Y}{\mathcal{L}_0 \cap \widetilde{\mathcal{M}}_{X_\partial}}\big)\big|_{(*)}$$

with a constraint $(*)$ requiring $X' = gX, Y' = Yg^{-1}$. The element

$$g(z) \in G(\mathcal{K}) \tag{2.3.9}$$

is a gauge transformation valued in Laurent series that relates X, Y on the "bottom" disk with X', Y' on the "top" disk, away from $z = 0$, *cf.* Figure 1.2. The remaining gauge transformations $(g_0', g_0) \in \mathcal{G}_0' \times \mathcal{G}_0$ on the top and bottom disks act on the algebraic data as

$$X', Y', g, X, Y \;\mapsto\; g_0'X',\, Y'g_0'^{-1},\, g_0'gg_0^{-1},\, g_0X,\, Yg_0^{-1}\,. \tag{2.3.10}$$

Coupling to quantum mechanics

In Section 2.2 we also reviewed how A-type line operators could be engineered by coupling to SQM_A quantum mechanics. Such a definition can also be incorporated fairly easily into the algebraic moduli spaces above, either replacing singularity data given by $\mathcal{G}_0, \mathcal{L}_0$, or further enhancing it.

We'll just describe the case where a line operator is *entirely* defined by coupling to 1d degrees of freedom, with no other singularity present in the bulk fields. Suppose that we define $\mathcal{L}$ by introducing a 1d sigma-model with Kähler target $\mathcal{E}$ as in Section 2.2.4, thought of as an algebraic variety with complexified flavor symmetry $G(\mathcal{O})$. (All but a finite part of $G(\mathcal{O})$ is assumed to act trivially.) In an algebraic formulation, the sigma-model is coupled to the bulk by gauging $G(\mathcal{O})$.

We may also introduce an algebraic $G(\mathcal{O})$-invariant superpotential $\widetilde{W}_0 : \mathcal{E} \times T^*R(\mathcal{O}) \to \mathbb{C}$, as in (2.2.56). Let $W = \int d^2z X D_{\overline{z}} Y + \widetilde{W}_0$, and note that the critical locus $\delta W = 0$ is algebraic. Explicitly, if α are local coordinates on $\mathcal{E}$, and x_n, y_n are the modes of X and Y, then the critical locus is equivalent to

$$\delta W = 0 \ : \quad \mu_{\mathbb{C}}(X,Y) = 0\,; \qquad \frac{\partial \widetilde{W}_0}{\partial \alpha} = 0\,, \quad \frac{\partial \widetilde{W}_0}{\partial y_n} = x_{-n-1}\,, \quad \frac{\partial \widetilde{W}_0}{\partial x_n} = -y_{-n-1} \quad (n \geq 0)\,. \tag{2.3.11}$$

The equations involving $\frac{\partial \widetilde{W}_0}{\partial y_n}$ and $\frac{\partial \widetilde{W}_0}{\partial x_n}$ are not really constraints on the space $\mathcal{E} \times T^*R(\mathcal{O})$, since the negative modes x_{-n-1} and y_{-n-1} are not part of $T^*R(\mathcal{O})$ to begin with. Instead, one can view (2.3.11) as equations on $\mathcal{E} \times T^*R(\mathcal{K})$. For example, in the extreme case of vanishing superpotential $\widetilde{W}_0 = 0$, last two equations in (2.3.11) set all negative modes to zero, so that the critical locus $\delta W = 0$ is precisely $\mathcal{E} \times T^*R(\mathcal{O})$ inside $\mathcal{E} \times T^*R(\mathcal{K})$.

In the presence of a line operator $\mathcal{L}$ with quantum-mechanics data $\mathcal{E}, W_0$, and a Dirichlet boundary condition $\mathcal{B}_{X_\partial}$, we expect that the moduli space $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L})$ can be described as

$$\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L}) = G(\mathcal{O})\big\backslash \widetilde{\mathcal{M}}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L})\,, \tag{2.3.12}$$

$$\widetilde{\mathcal{M}}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L}) := \mathcal{E} \times \widetilde{\mathcal{M}}_{X_\partial}\big|_{\delta W = 0}\,.$$

Similarly, given a pair of line operators with data $\mathcal{E}', W_0'$ and $\mathcal{E}, W_0$, the raviolo space is

$$\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_{X_\partial};\mathcal{L},\mathcal{L}') = {}_{G(\mathcal{O})}\big\backslash \widetilde{\mathcal{M}}_{\mathrm{rav}}(\mathcal{B}_{X_\partial};\mathcal{L},\mathcal{L}')\big/_{G(\mathcal{O})}\,, \tag{2.3.13}$$

$$\widetilde{\mathcal{M}}_{\mathrm{rav}}(\mathcal{B}_{X_\partial};\mathcal{L},\mathcal{L}') = \big(\mathcal{E}' \times \widetilde{\mathcal{M}}_{X_\partial}\big) \times G(\mathcal{K}) \times \big(\mathcal{E} \times \widetilde{\mathcal{M}}_{X_\partial}\big)\Big|_{(*)}$$

with constraints $(*)$ given by $\delta W' = \delta W = 0$ and $(X',Y') = (gX, Yg^{-1})$.

2.3.2 Vortex number

A key feature of the Dirichlet boundary condition $\mathcal{B}_{X_\partial}$ is that the spaces $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L})$ and $\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_{X_\partial};\mathcal{L},\mathcal{L}')$ break up into a disjoint union of finite-dimensional components. This is the main reason we use them here. It makes the homology of these moduli spaces much easier to analyze by elementary methods. It also endows the homology with an additional grading.

In the case of $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L})$, components are labeled by *vortex number* $\mathfrak{n}\in\pi_1(G)$,

$$\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L}) = \bigsqcup_{\mathfrak{n}\in\pi_1(G)} \mathcal{M}^{\mathfrak{n}}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L})\,, \tag{2.3.14}$$

and correspondingly the homology $\mathcal{H}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L}) = \bigoplus_{\mathfrak{n}\in\pi_1(G)} H_\bullet\big(\mathcal{M}^{\mathfrak{n}}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L})\big)$ is graded by $\pi_1(G)$. Physically, vortex number is usually interpreted as a first Chern class, and expressed as an integral of the curvature of the G-bundle on the disk

$$\mathfrak{n} = \frac{1}{2\pi}\int_{\mathbb{D}} \mathrm{Tr}\, F\,. \tag{2.3.15}$$

Topologically, vortex number arises because the group $G(\mathcal{K})$ that appears in (2.3.6) is a version of the loop group $LG_{\mathbb{C}}$, which has connected components labeled by $\pi_1(G)$. Viewed as an algebraic ind-scheme, $G(\mathcal{K})$ is stratified, rather than disconnected, with strata labeled by elements $\mathfrak{n}\in\pi_1(G)$. However, after passing to the quotient by $\mathcal{G}_0$ in (2.3.6), one again finds connected components labeled by $\mathfrak{n}\in\pi_1(G)$.

The most direct way to understand vortex number algebraically is as a *degree*. The basic example (and the only one relevant for us) is $G=U(N)$. In this case $G_{\mathbb{C}} = GL(N,\mathbb{C})$, and $G(\mathcal{K})$ is the group of invertible $N\times N$ matrices whose entries are formal Laurent series in z. Given any $g(z)\in G(\mathcal{K})$, the determinant

$$\det g(z) = a_{\mathfrak{n}} z^{\mathfrak{n}} + a_{\mathfrak{n}+1} z^{\mathfrak{n}+1} + \ldots \qquad \in \mathbb{C}(\!(z)\!) \tag{2.3.16}$$

is a nonzero formal series, and has a well-defined degree $\mathfrak{n}\in\mathbb{Z}\simeq\pi_1(U(N))$ given by the highest power of z that appears with nonzero coefficient.

When a line operator $\mathcal{L}$ breaks gauge symmetry near the origin from $G(\mathcal{O})$ to $\mathcal{G}_0$, the notion of vortex number and the corresponding decomposition (2.3.14) may be refined. We will see this happening in non-abelian examples. Nevertheless, there is *always* a decomposition by at least the vortex numbers $\mathfrak{n}\in\pi_1(G)$, which is what we are discussing here.

In a similar way, the raviolo spaces used to construct local operators at junctions break up into connected components labeled by pairs of vortex numbers

$$\mathcal{M}_{\text{rav}}(\mathcal{B}_{X_\partial};\mathcal{L},\mathcal{L}') = \bigsqcup_{\mathfrak{n}',\mathfrak{n}\in\pi_1(G)} \mathcal{M}_{\text{rav}}^{\mathfrak{n}',\mathfrak{n}}(\mathcal{B}_{X_\partial};\mathcal{L},\mathcal{L}')\,. \tag{2.3.17}$$

In the algebraic formulation of (2.3.8), $\mathfrak{n}'$ and $\mathfrak{n}$ are the degrees of the two $G(\mathcal{K})$ elements used to relate the sections X' and X (respectively) to X_∂.

The decomposition (2.3.17) implies that the homology of $\mathcal{M}_{\text{rav}}(\mathcal{B}_{X_\partial};\mathcal{L},\mathcal{L}')$ will be graded by pairs of vortex numbers $\mathfrak{n}',\mathfrak{n}$. The difference $\mathfrak{n}'-\mathfrak{n}$ corresponds to the physical monopole charge of a local operator. It is the charge under the $U(1)_t$ topological flavor symmetries dual to the center of the group G.

2.3.3 Summary and interpretation

The final approach used to compute local operators at junctions of lines looks as follows.

Given a 3d $\mathcal{N}=4$ gauge theory with data G, R, we choose a Dirichlet boundary condition $\mathcal{B}_{X_\partial}$ that completely breaks the gauge symmetry. As described in Section 2.1.3, such a choice allows us to represent the category of line operators. We expect that for sufficiently nice X_∂ the representation — in particular, the maps $\rho_{\mathcal{B}_{X_\partial}}$ on spaces of local operators — will be injective for a large set of line operators.

For every line operator $\mathcal{L}$ defined by algebraic data $\mathcal{G}_0,\mathcal{L}_0$, we construct the algebraic moduli space $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L})$ as in (2.3.6). If $\mathcal{L}$ is defined by coupling to quantum mechanics, we can use the definition (2.3.12) instead. In our examples, $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L})$ will break up into infinitely many finite-dimensional components, labeled by vortex numbers $\mathfrak{n}$. We take Borel-Moore homology to (conjecturally) construct the Q_A-cohomology of the vector space of boundary local operators,

$$\mathcal{H}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L}) = H_\bullet\big(\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L})\big) = \bigoplus_{\mathfrak{n}} H^\bullet\big(\mathcal{M}_{\mathbb{D}}^{\mathfrak{n}}(\mathcal{B}_{X_\partial};\mathcal{L})\big)\,. \tag{2.3.18}$$

The vector space is graded by vortex number.

For every pair of line operators $\mathcal{L}, \mathcal{L}'$, we construct the raviolo space $\mathcal{M}_{\text{rav}}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}')$, using algebraic data as in (2.3.8) or quantum-mechanics data as in (2.3.13) (or some combination thereof). Again, the raviolo spaces break up into finite-dimensional components labeled by pairs of vortex numbers. We expect the Q_A-cohomology $\text{Hom}_A(\mathcal{L}, \mathcal{L}')$ of the space of local operators at a junction of lines to be represented by the homology

$$H_\bullet\big(\mathcal{M}_{\text{rav}}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}')\big) = \bigoplus_{\mathfrak{n}', \mathfrak{n}} H_\bullet\big(\mathcal{M}_{\text{rav}}^{\mathfrak{n}', \mathfrak{n}}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}')\big)\,. \tag{2.3.19}$$

The product of local operators at junctions (*a.k.a.* composition of Hom's) and their action on the vector spaces $\mathcal{H}_{\mathbb{D}}(\mathcal{B}_{X_\partial}, \mathcal{L})$ are both given by convolution, as in Section 1.3.3.

Identifying monopole operators

The sort of local operators we expect to find at junctions of A-type lines are a generalization of operators in the bulk Coulomb-branch chiral ring. In particular, we should see operators formed out of bulk vector multiplet scalars, as well as monopole operators. The vector multiplet scalars φ will appear in a straightforward way as equivariant parameters (see Section 1.3.2). We recall how monopole operators are identified, following [33, Sec 10] and [70, 72].

A physical monopole operator is labeled by a "monopole charge" A. Mathematically, this is an element of the cocharacter lattice $A \in \text{cochar}(G) \simeq \text{Hom}(\mathbb{C}^*, T_{\mathbb{C}})$. The charge A thus determines a group homomorphism from $\mathbb{C}^*$ to the maximal torus $T_{\mathbb{C}} \subseteq G_{\mathbb{C}}$. (In the physical definition of a monopole operator, A is literally used to embed a fundamental Dirac singularity for $U(1)$ into gauge theory with group G.) Let $z^A \in G(\mathcal{K})$ denote the image of $z \in \mathcal{K}$ under this homomorphism. For example, if $G = U(N)$, cocharacters are N-tuples of integers $A = (A_1, ..., A_N) \in \mathbb{Z}^N$, and

$$z^A = \text{diag}(z^{A_1}, z^{A_2}, ..., z^{A_N}) \in G(\mathcal{K})\,. \tag{2.3.20}$$

At a junction of lines $\mathcal{L}$ and $\mathcal{L}'$, we may use any element

$$wz^A \in G(\mathcal{K})\,, \qquad w \in \text{Weyl}(G) \tag{2.3.21}$$

to try to define a monopole operator. Note, these are elements of the extended affine Weyl group $\text{Weyl}(G) \ltimes \text{cochar}(G) \simeq W_{\text{aff}}(G) \simeq \text{Weyl}(G(\mathcal{K}))$. When $\mathcal{L} = \mathcal{L}' = \mathbb{1}$, only the orbit of wz^A under $\text{Weyl}(G) \times \text{Weyl}(G)$ acting on the left and right matters; this action can be used to remove w and to conjugate A to a dominant cocharacter ($A_1 \geq A_2 \geq ... \geq A_N$), whence one usually says that the charges of bulk monopole operators are dominant cocharacters. However, if $\mathcal{L}$ and $\mathcal{L}'$ break the bulk gauge symmetry to $\mathcal{G}_0$ and $\mathcal{G}_0'$, respectively, we may only act on (2.3.21) with $\text{Weyl}(\mathcal{G}_0) \times \text{Weyl}(\mathcal{G}_0')$. Then monopole charges take values in

$$\text{for } \text{Hom}_A(\mathcal{L}, \mathcal{L}'): \quad \text{Weyl}(\mathcal{G}_0')\backslash\text{Weyl}(G(\mathcal{K}))/\text{Weyl}(\mathcal{G}_0)\,. \tag{2.3.22}$$

Now consider a space $\mathcal{M}_{\text{rav}}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}')$. In the algebraic formulation (2.3.8), points of $\mathcal{M}_{\text{rav}}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}')$ are labeled in part by singular gauge transformations $g(z) \in G(\mathcal{K})$. We expect that a putative monopole operator $M_{w,A}$ of "charge" $(w, A) \in \text{Weyl}(G) \ltimes \text{cochar}(G) \simeq \text{Weyl}(G(\mathcal{K}))$ is represented by the fundamental class of a subvariety of $\mathcal{M}_{\text{rav}}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}')$ consisting of all points that are gauge-equivalent to a configuration with $g(z) = wz^A$. In other words, given the map that forgets the hypermultiplets

$$\mathcal{M}^{\text{rav}}(\mathcal{B}_{X_\partial}; \mathcal{L}', \mathcal{L}) \xrightarrow{\pi_g} \mathcal{G}_0'\backslash G(\mathcal{K})/\mathcal{G}_0\,, \tag{2.3.23}$$

a monopole operator $M_{w,A}$ should correspond to the pullback

$$M_{w,A} \;\sim\; \pi_g^*[wz^A] \in H_\bullet\big(\mathcal{M}_{\text{rav}}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}')\big)\,, \tag{2.3.24}$$

where $[wz^A]$ is the fundamental class of the closure of the double-orbit $\mathcal{G}_0' \cdot wz^A \cdot \mathcal{G}_0$ in $\mathcal{G}_0'\backslash G(\mathcal{K})/\mathcal{G}_0$. Similarly, we expect "dressed" monopole operators corresponding to Chern classes of line bundles over the wz^A orbit.

Note that the formula (2.3.24) does *not* imply that a junction of line operators $\mathcal{L}, \mathcal{L}'$ will have monopole operators of all possible charges! It may well be that, for given (w, A), the pull-back $\pi_g^*[wz^A]$ is zero. For example, this would happen if the condition imposed on hypermultiplet fields by $\mathcal{L}'_0$ and $\mathcal{L}_0$ made it impossible to have points with $g(z)$ in the orbit of wz^A in the space $\mathcal{M}_{\rm rav}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}')$.

When we decompose $\mathcal{M}_{\rm rav}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}') = \bigsqcup_{\mathfrak{n}',\mathfrak{n}} \mathcal{M}^{\mathfrak{n}',\mathfrak{n}}_{\rm rav}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}')$ into components labeled by vortex numbers, $\pi_g^*[wz^A]$ can only be supported on components with $\mathfrak{n}' - \mathfrak{n}$ equal to the topological type of A (as a cycle in $\pi_1(G)$). We write this relation as $\mathfrak{n}' - \mathfrak{n} \sim A$, noting that it only depends on the Weyl$(\mathcal{G}'_0)\times$Weyl$(\mathcal{G}_0)$ orbit of wz^A. Then we expect a dressed or undressed monopole operator of charge (w, A) to be represented as a diagonal sum, schematically

$$M_{w,A} = \sum_{\mathfrak{n}'-\mathfrak{n}\sim A} M^{\mathfrak{n}',\mathfrak{n}}_{w,A}\,, \qquad M^{\mathfrak{n}',\mathfrak{n}}_{w,A} \in H_\bullet\big(\mathcal{M}^{\mathfrak{n}',\mathfrak{n}}_{\rm rav}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}')\big) \tag{2.3.25}$$

Idempotents

We expect that the map $\mathrm{Hom}_A(\mathcal{L}, \mathcal{L}') \to H_\bullet\big(\mathcal{M}_{\rm rav}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}')\big)$ should always be surjective. This would let us relate any class in $H_\bullet\big(\mathcal{M}_{\rm rav}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}')\big)$ to a physical operator. The surjectivity statement requires a slightly technical modification when dealing with boundary conditions that allow for a well-defined notion of vortex number. Namely, due to the decomposition of $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L})$ into disjoint components $\mathcal{M}^{\mathfrak{n}}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L})$, there are extra operators acting on (and among) the cohomologies $H_\bullet\big(\mathcal{M}^{\mathfrak{n}}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L})\big)$, which have nothing to do with local operators at junctions of lines. These extra operators are projections to summands of (2.3.18) with fixed $\mathfrak{n}$. If $\mathcal{L}$ breaks gauge symmetry, so that vortex number is refined, then even more projections will appear.

Mathematically, these projections are "orthogonal idempotents" $e_{\mathfrak{n}}$. They are represented as classes

$$e_{\mathfrak{n}} = \pi^*[1] \cap H_\bullet\big(\mathcal{M}^{\mathfrak{n},\mathfrak{n}}_{\rm rav}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L})\big) \tag{2.3.26}$$

for each fixed $\mathfrak{n}$, and they satisfy $e_{\mathfrak{n}} e_{\mathfrak{n}'} = \delta_{\mathfrak{n},\mathfrak{n}'} e_{\mathfrak{n}}$.

There are two ways to correct the surjectivity statement to account for these spurious operations. One option (*cf.* [70, Sec 4.4.1]) is to enhance $\mathrm{Hom}_A(\mathcal{L}, \mathcal{L}')$ on the LHS, by throwing in all possible idempotents, acting by multiplication on both the left and right. With this enhancement of the LHS, surjectivity should be regained.

Alternatively, we may focus on operators in $H_\bullet\big(\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}')\big)$ that act in a way that is independent of decomposition by vortex number. (In particular, we want operators whose convolution products are independent of vortex number.) We would expect such operators to come from actual elements of $\mathrm{Hom}_A(\mathcal{L}, \mathcal{L}')$. They will necessarily be represented as infinite diagonal sums over graded components $H_\bullet\big(\mathcal{M}^{\mathfrak{n}',\mathfrak{n}}_{\mathrm{rav}}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L})\big)$, just like in (2.3.25).

2.3.4 Dirichlet boundary conditions and generalized affine Springer theory

Throughout this section, we have considered Dirichlet boundary conditions $\mathcal{B}_{X_\partial}$ specified a holomorphic profile X_∂ that fully broke gauge symmetry at the boundary. Local operators at junctions of line operators $\mathcal{L}$ and $\mathcal{L}'$ were then represented by certain linear operators on the homologies of moduli spaces $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L})$ arising via convolution through the raviolo spaces $\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}')$.

On the other hand, the construction in Section 1.3.3 realized built the Coulomb branch out of correspondences between solutions to BPS equations that were compatible with a different boundary condition $\mathcal{B}_R$ that set half the hypermultiplet scalars fields to zero ($Y = 0$) and *p*reserved gauge symmetry at the boundary. As described in Section 1.3.3, the boundary condition $\mathcal{B}_R$ is *almost* canonical: it only depends on a splitting of the hypermultiplet representation as $R \oplus R^*$. The moduli spaces appearing in Section 2.3.1, however, were highly dependent on the choice of X_∂. In this sense, the moduli spaces $\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_R; \mathcal{L}, \mathcal{L}')$ and the homology classes built from it are more intrinsic than those appearing in Section 2.3.1. Thankfully, the two discussions are compatible.

It is straightforward to generalize the construction of Section 1.3.3 to local operators at the junction of two line operator $\mathcal{L}$ and $\mathcal{L}'$ given by the geometric data $(\mathcal{G}_0, \mathcal{L}_0)$ and $(\mathcal{G}_0, \mathcal{L}_0')$, respectively. Upon setting $Y = 0$, the Lagrangian $\mathcal{L}_0 \subset T^*R(\mathcal{K})$ becomes a representation R_0 of $\mathcal{G}_0$, and similarly for the Lagrangian $\mathcal{L}'_0$. We then construct the following raviolo space:

$$\mathcal{M}_{\text{rav}}(\mathcal{B}_R; \mathcal{L}, \mathcal{L}') = \begin{array}{c} \mathcal{G}'_0 \backslash R'_0 \times G(\mathcal{K}) \times R_0\big|_{(*)} / \mathcal{G}_0 \\ X' \qquad g \qquad X \end{array} \tag{2.3.27}$$

where, again, the constraint $(*)$ requires that $X' = gX$. We then identify local operators at the junction of $\mathcal{L}$ and $\mathcal{L}'$ with (suitably defined) homology classes of $\mathcal{M}_{\text{rav}}(\mathcal{B}_R; \mathcal{L}, \mathcal{L}')$.

We choose an element $X_\partial \in R(\mathcal{K})$ such that it's stabilizer in $G(\mathcal{K})$ is trivial, and consider the boundary condition $\mathcal{B}_{X_\partial}$. (See 2.3.1 for a more detailed description of the boundary condition.) From this data, we constructed moduli spaces of solutions on the formal disk $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L}), \mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L}')$ and the raviolo $\mathcal{M}_{\text{rav}}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}')$. The moduli spaces $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L})$ can be recast using the fact that the stabilizer of X_∂ in $G(\mathcal{K})$ is trivial. When X_∂ has a trivial stabilizer in $G(\mathcal{K})$, the allowed values of Y and Y' form a (trivial) vector bundle over the space with $Y = Y' = 0$. We expect that no information is lost in retracting to the locus with $Y = Y' = 0$. For example, after the retraction, the moduli space $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L})$ takes a particularly simple form:

$$\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L}) \overset{\text{retracts to}}{\to} \big(R_0 \cap G(\mathcal{K}).X_\partial\big)/\mathcal{G}_0\,. \tag{2.3.28}$$

By abuse of notation, we will call the retracted space with $Y = Y' = 0$ by the same name. In particular, this realizes (the retracted moduli space) $\mathcal{M}_{\text{rav}}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}')$ as a subspace of $\mathcal{M}_{\text{rav}}(\mathcal{B}_R; \mathcal{L}, \mathcal{L}')$. We thus have three natural maps that realize the desired action:

$$\begin{array}{ccc} & \mathcal{M}_{\text{rav}}(\mathcal{B}_R; \mathcal{L}, \mathcal{L}') & \\ & \uparrow \iota & \\ & \mathcal{M}_{\text{rav}}(\mathcal{B}_{X_\partial}; \mathcal{L}, \mathcal{L}') & \\ {}^{\pi_2}\swarrow & & \searrow^{\pi_1} \\ \mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L}') & & \mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L}) \end{array} \tag{2.3.29}$$

The map ι is the inclusion map and π_1, π_2 are the maps that forget $(X'; g)$ and $(g; X)$, respectively. Schematically, the collision of the junction between $\mathcal{L}$ and $\mathcal{L}'$ with the boundary should be realized as

$$\mathcal{O}.v = (\pi_2)_*\big(\iota^*\mathcal{O} \cap (\pi_1)^* v\big) \in H^\bullet(\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L}')). \tag{2.3.30}$$

for $\mathcal{O} \in H_\bullet(\mathcal{M}_{\text{rav}}(\mathcal{B}_R; \mathcal{L}, \mathcal{L}'))$ and $v \in H_\bullet(\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L})$

Since the stabilizer of X_∂ is trivial, the space $R_0 \cap G(\mathcal{K}).X_\partial$ can be identified with the space of gauge transformations $g \in G(\mathcal{K})$ such that $X_\partial \in gR_0$:

$$R_0 \cap G(\mathcal{K}).X_\partial \simeq \{g \in G(\mathcal{K}) | X_\partial \in gR_{\mathcal{G}_0}\}\,. \tag{2.3.31}$$

Once we quotient by the action of bulk gauge transformations $\mathcal{G}_0$, we see that $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}; \mathcal{L})$ is a subspace (really, a sub-ind-scheme) of the partial affine flag variety $\text{Fl}_{\mathcal{G}_0} = G(\mathcal{K})/\mathcal{G}_0$:

$$\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial}, \mathcal{L}) \subset \text{Fl}_{\mathcal{G}_0}. \tag{2.3.32}$$

These spaces are a natural generalization of the "classical" affine Springer fibers [243], which takes $\mathcal{G}_0$ to be a parahoric subgroup and $R_{\mathcal{G}_0}$ to be it's adjoint representation, which we call "generalized affine Springer fibers."[14] The "classical" affine Springer fibers lead to (affine) Springer representations affine Weyl groups [244] and of various Cherednik algebras [245–247]. The above is a natural generalization to the setting of generalized affine Springer fibers, recently constructed in [50] for Coulomb branches and [51] to more general $\frac{1}{2}$-BPS vortex-line operators.

[14]The generalized affine Springer fiber depends on many parameters, including the full loop group $G(\mathcal{K})$, the representation R, the subgroup picked out by the line operator $\mathcal{G}_0$ as well as the representation R_0. For simplicity of notation, we only denote the latter dependence.

Deformation and quantization

The above construction can be deformed to include complex mass parameters and an Omega background just as with the construction in Section 1.3.3, thereby realizing an action of the deformed and quantized algebras of local operators at junctions of line operators.

The construction of Section 1.3.3 generalizes as expected. Given the flavor group T_F encoded in an exact sequence of groups

$$1 \to G \to \widehat{G} \to T_F \to 1 \tag{2.3.33}$$

such that the G action on R extends to a $\widehat{G}$ action. we denote by $\widehat{G}^{\mathcal{O}}(\mathcal{K})$ the preimage of $T_F(\mathcal{O}) \subset T_F(\mathcal{K})$ under the induced map $\widehat{G}(\mathcal{K}) \to T_F(\mathcal{K})$. Similarly, we denote by $\widehat{\mathcal{G}_0}$ the largest subgroup of $\widehat{G}(\mathcal{O})$ that both surjects onto $T_F(\mathcal{O})$ and intersects with $G(\mathcal{O})$ to yield $\mathcal{G}_0$. For example, if $\widehat{G} = G \times T_F$, the subgroup $\widehat{\mathcal{G}_0}$ is simply $\mathcal{G}_0 \times T_F(\mathcal{O})$. The subgroup $\widehat{\mathcal{G}_0}$ is defined so that

$$G(\mathcal{K})/\mathcal{G}_0 \cong \widehat{G}^{\mathcal{O}}(\mathcal{K})/\widehat{\mathcal{G}_0}, \tag{2.3.34}$$

just as we saw in Section 1.3.3 with $\mathcal{G}_0 = G(\mathcal{O})$. Local operators at the junction between $\mathcal{L}$ and $\mathcal{L}'$ are then realized as homology classes of

$$\widehat{\mathcal{M}}_{\rm rav}(\mathcal{B}_R; \mathcal{L}, \mathcal{L}') = \widehat{\mathcal{G}_0}' \rtimes \mathbb{C}^*_\varepsilon \backslash R'_0 \times (\widehat{G}^{\mathcal{O}}(\mathcal{K}) \rtimes \mathbb{C}^*_\varepsilon) \times R_0\big|_{(*)}/\widehat{\mathcal{G}_0} \rtimes \mathbb{C}^*_\varepsilon \tag{2.3.35}$$

or, equivalently, as $\widehat{\mathcal{G}_0}' \rtimes \mathbb{C}^*_\varepsilon$-equivariant homology classes of

$$\widehat{\mathcal{R}}_{\mathcal{G}'_0, R'_0; \mathcal{G}_0, R_0} = R'_0 \times (\widehat{G}^{\mathcal{O}}(\mathcal{K}) \rtimes \mathbb{C}^*_\varepsilon) \times R_0\big|_{(*)}/\widehat{\mathcal{G}_0} \rtimes \mathbb{C}^*_\varepsilon. \tag{2.3.36}$$

Collision of these operators in $H^\bullet(\widehat{\mathcal{M}}_{\rm rav}(\mathcal{B}_R; \mathcal{L}, \mathcal{L}'))$ and $H^\bullet(\widehat{\mathcal{M}}_{\rm rav}(\mathcal{B}_R; \mathcal{L}', \mathcal{L}''))$ obtained by the usual convolution diagrams.

The generalized affine Springer fibers are similarly defined

$$\widehat{\mathcal{M}}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L}) = \big(R_0 \cap (\widehat{G}(\mathcal{K}) \rtimes \mathbb{C}^*_\varepsilon).X_\partial\big)/\widehat{\mathcal{G}_0} \rtimes \mathbb{C}^*_\varepsilon \simeq \mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L}). \tag{2.3.37}$$

Let L denote the stabilizer of X_∂ inside the flavor and loop-rotation extended group $\widehat{G}^{\mathcal{O}}(\mathcal{K}) \rtimes \mathbb{C}^*_\varepsilon$, and we further assume that $L \subset \widehat{\mathcal{G}_0} \rtimes \mathbb{C}^*_\varepsilon$. The action of L on R_0 then induces an action of L on $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial};\mathcal{L})$ and the above construction realizes an action of the algebra $H^\bullet(\widehat{\mathcal{M}}_{\text{rav}}(\mathcal{B}_R;\mathcal{L},\mathcal{L}))$ via convolution.

2.4 Examples

In the final section of the chapter we consider concrete examples of $\frac{1}{2}$-BPS vortex line operators in the rank 2 ADHM quiver gauge theory described in Section 1.4.2, *i.e.*, 3d $\mathcal{N}=4$ gauge theory with gauge group $G = U(2)$ and hypermultiplets transforming in a representation T^*R for $R = \mathbb{C}^2 \oplus \mathbb{C}^{2\times 2}$ (a single fundamental hypermultiplet and a single adjoint hypermultiplet).

In Section 2.4.1 we consider the example of an "abelianizing" or "Iwahori" A-type line operator $\mathbb{V}_{\mathcal{I}}$, which can be defined for any non-abelian gauge theory as in Section 2.2.3. The line operator $\mathbb{V}_{\mathcal{I}}$ breaks the gauge group $U(2)$ to its maximal torus $U(1)^2$ along a line, and may be accompanied by a monodromy defect for the connection; but it does *not* introduce any singularity in the hypermultiplet fields. In terms of algebraic data, $\mathbb{V}_{\mathcal{I}}$ breaks $G(\mathcal{O}) = GL(2,\mathcal{O})$ to the Iwahori subgroup $\mathcal{I} = \mathcal{I}_B$ from (2.2.39), while retaining the standard Lagrangian,

$$\mathbb{V}_{\mathcal{I}}: \quad \mathcal{G}_0 = \mathcal{I}\,, \qquad R_0 = R(\mathcal{O}) = \mathcal{O}^2 \oplus \mathfrak{gl}(2,\mathcal{O})\,. \tag{2.4.1}$$

Alternatively, $\mathbb{V}_{\mathcal{I}}$ may be defined by introducing a 1d SQM$_A$ sigma-model whose target is the flag manifold $\mathcal{X} = G/T \simeq G(\mathcal{O})/\mathcal{I} \simeq \mathbb{CP}^1$, and coupling it to the vector multiplets of the bulk 3d theory by gauging its flavor symmetry.

The general analysis of [111] shows that this line operator is simply a direct sum of trivial line operators, and correspondingly $\text{End}^\varepsilon_A(\mathbb{V}_{\mathcal{I}})$ is a matrix algebra over the Coulomb branch algebra $\mathrm{End}^{\varepsilon}_{A}(\mathbb{1})$. Similarly, elements of $\mathrm{Hom}^{\varepsilon}_{A}(\mathbb{1},\mathbb{V}_{\mathcal{I}})$, and $\mathrm{Hom}^{\varepsilon}_{A}(\mathbb{V}_{\mathcal{I}},\mathbb{1})$ can be interpreted as vectors and covectors with coefficients in $\mathrm{End}^{\varepsilon}_{A}(\mathbb{1})$, respectively. We will implement the computational methods of Section 2.3 to compute find $\mathrm{End}^{\varepsilon}_{A}(\mathbb{V}_{\mathcal{I}})$, $\mathrm{Hom}^{\varepsilon}_{A}(\mathbb{V}_{\mathcal{I}},\mathbb{1})$, and $\mathrm{Hom}^{\varepsilon}_{A}(\mathbb{1},\mathbb{V}_{\mathcal{I}})$. The *way* in which these spaces get arranged into matrix algebras or (co)vectors turns out to be highly nontrivial, and will provide a good test of our methods.

The computation of $\mathrm{End}^{\varepsilon}_{A}(\mathbb{V}_{\mathcal{I}})$ reveals an additional piece of structure. We find that $\mathrm{End}^{\varepsilon}_{A}(\mathbb{V}_{\mathcal{I}})$ most directly takes the form of an abelianized version of the bulk Coulomb-branch algebra (closely related to the abelianization construction of [67] and Chapter 1), tensored with a copy of the *nil-Hecke algebra* $\mathbf{H}_2$ for $GL(2)$ [248]. Abstractly, the nil-Hecke algebra may be defined as the G-equivariant (co)homology of a product of flag varieties. Here $G/B \simeq \mathbb{CP}^1$, and

$$\mathbf{H}_2 = H^{\bullet}_{GL(2)}(\mathbb{CP}^1 \times \mathbb{CP}^1)\,, \tag{2.4.2}$$

with a product from convolution. Explicit relations for $\mathbf{H}_2$ will be given below. In Section 2.4.1 we relate this presentation of $\mathrm{End}^{\varepsilon}_{A}(\mathbb{V}_{\mathcal{I}})$ with the expected matrix algebra.

We note that the structure of $\mathrm{End}^{\varepsilon}_{A}(\mathbb{V}_{\mathcal{I}})$, both as a 2×2 matrix algebra over $\mathbb{C}_{\varepsilon}[\mathcal{M}_C]$, and as a semidirect product of an abelianized $\mathbb{C}_{\varepsilon}[\mathcal{M}_C]$ with the nil-Hecke algebra, was discussed by Webster in [39]. The analysis of [39] was performed in a BFN-like setup (as in Section 1.3.3) rather than by choosing a Dirichlet boundary condition $\mathcal{B}_{X_\partial}$ as we do here. Of course, the structure of the line operator $\mathbb{V}_{\mathcal{I}}$ should be independent of how it is probed by boundary conditions. Happily, the final results of our computation here agree with [39].

We also recall that a version of the nil-Hecke algebra (in fact, a categorification of thereof) appeared in physics in the work of Gukov and Witten [34]. They considered a surface operator in 4d $\mathcal{N}=4$ SYM that broke gauge symmetry $G \to T$. These operators were not trivial as in (2.4.1), because in 4d $\mathcal{N}=4$ SYM the breaking of gauge symmetry is accompanied by a singularity in the adjoint-valued matter fields. Nevertheless, the breaking of gauge symmetry was sufficient to introduce a copy of the nil-Hecke algebra, sitting inside a larger affine Hecke algebra that described line operators bound to the surface.

Finally, in Section 2.4.2 we study a non-trivial $\frac{1}{2}$-BPS vortex line operator $\mathbb{V}_{\text{RCA}}$ given by the following algebraic data:

$$\mathbb{V}_{\text{RCA}} \; : \quad \mathcal{G}_0 = \mathcal{I}\,, \qquad R_0 = \mathcal{O}^2 \oplus \text{Lie}(\mathcal{I})\,. \tag{2.4.3}$$

We verify the results of [249] by identifying local operators bound to this line operator that generate the *full* rational Cherednik algebra for $\mathfrak{gl}(2,\mathbb{C})$, denoted $\overline{\mathcal{H}}_2$, as opposed to its spherical subalgebra realized by local operators in the bulk, denoted $\overline{\mathcal{H}}_2^{\text{sph}}$, and described in Section 1.4.2.

$$\text{End}^{\varepsilon}_A(\mathbb{V}_{\text{RCA}}) \simeq \overline{\mathcal{H}}_2 \qquad \text{vs} \qquad \text{End}^{\varepsilon}_A(\mathbb{1}) \simeq \overline{\mathcal{H}}_2^{\text{sph}} \tag{2.4.4}$$

Explicit relations for the rational Cherednik algebra will be presented below. In the course of our analysis, we identify a Dirichlet boundary condition (or, mathematically, a generalized affine Springer fiber) that realizes the polynomial representation of the rational Cherednik algebra.

This line operator illustrates a particularly simple example of the dramatic consequences of forcing zeroes in the hypermultiplets. From Section 2.3, local operators bound to both $\mathbb{V}_{\mathcal{I}}$ and $\mathbb{V}_{\text{RCA}}$ can be obtained from pulling back homology classes on the affine flag variety $GL(2,\mathcal{K})/\mathcal{I}$. Nonetheless, we will see that the same homology classes on $GL(2,\mathcal{K})/\mathcal{I}$ will realize markedly different operators.

Our analysis utilizes a particularly simple Dirichlet boundary condition and is relevant to the physical construction of HOMFLY-PT knot homology [92, 93, 250, 251] appearing in Chapter 3 and the upcoming work [91]. The rational Cherednik algebra for $\mathfrak{gl}(n,\mathbb{C})$, and the representation theory thereof, will serve as a robust check that the corresponding analysis is correct. Somewhat more precisely, from work of Gorsky-Oblomkov-Rasmussen-Shende [100] it follows that the HOMFLY-PT homology of torus knots can be obtained from special representations of the rational Cherednik algebra for $\mathfrak{gl}(n,\mathbb{C})$ that arise from certain the geometry of generalized affine Springer fibers [99]. In Section 3.2, we show that the proposed physical construction of HOMFLY-PT knot homology exactly reproduces the desired representations and generalized affine Springer fibers for torus knots. The representation arising in Section 2.4.2 corresponds to the $(2,k)$ torus knot in the limit $k \to \infty$.

2.4.1 The Iwahori line

Consider first the example of the Iwahori line operator $\mathbb{V}_{\mathcal{I}}$ in the rank 2 ADHM quiver theory, *i.e.* the 3d $\mathcal{N}=4$ super Yang-Mills theory with gauge group $G=U(2)$ with a single fundamental hypermultiplet and a single adjoint hypermultiplet, $R=\mathbb{C}^2\oplus\mathbb{C}^{2\times 2}$. Explicitly, we denote the hypermultiplet scalars $(X,I,Y,J)\in T^*R$ as

$$X^a{}_b\,, \quad I^a\,, \quad Y^a{}_b\,, \quad J_b \tag{2.4.5}$$

where $a,b=1,2$ are indices for the fundamental/antifundamental representation of $G=U(2)$. Complexified gauge transformations $g\in G_{\mathbb{C}}=GL(2,\mathbb{C})$ act as $X\to gXg^{-1}$, $I\to gI$, $Y\to gYg^{-1}$, $J\to Jg^{-1}$. The algebraic data describing the Iwahori line operator $\mathbb{V}_{\mathcal{I}}$ is

$$\mathcal{G}_0=\mathcal{I}\,, \qquad \mathcal{L}_0=R(\mathcal{O})\oplus R^*(\mathcal{O})\,, \tag{2.4.6}$$

where the Iwahori subgroup is

$$\mathcal{I}=\left\{g(z)=\begin{pmatrix} a(z) & b(z)\\ z\,c(z) & d(z)\end{pmatrix}\in G(\mathcal{O})\right\}=\left\{g\in G(\mathcal{O})\;\middle|\;g(0)\in B\right\}. \tag{2.4.7}$$

Equivalently, $\mathbb{V}_{\mathcal{I}}$ is defined by coupling to SQM$_A$ quantum mechanic with target $\mathcal{X}=\mathbb{CP}^1$ by gauging the $U(2)$ flavor symmetry (note only $PSU(2)$ acts nontrivially); or, algebraically, by gauging $G_{\mathbb{C}}=GL(2,\mathbb{C})$.

In the remainder of this subsection, we will compute $\mathrm{End}^{\varepsilon}_A(\mathbb{V}_{\mathcal{I}})$, $\mathrm{Hom}^{\varepsilon}_A(\mathbb{1},\mathbb{V}_{\mathcal{I}})$, and $\mathrm{Hom}^{\varepsilon}_A(\mathbb{V}_{\mathcal{I}},\mathbb{1})$ using the techniques of Section 2.3. As such, we need to choose Dirichlet boundary conditions so that the boundary values X_∂, I_∂ fully break gauge and will assume that the representation of these spaces coming from the boundary condition $\mathcal{B}_{X_\partial,I_\partial}$ is faithful. The fact that we eventually recover all the expected structure of a matrix algebra on $\mathbb{V}_{\mathcal{I}}$

confirms that the assumption is reasonable.

For the boundary values X_∂, I_∂ we take

$$\mathcal{B}_{X_\partial,I_\partial}: \quad X_\partial = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad I_\partial = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.4.8}$$

It is clear that this choice fully breaks the gauge symmetry but preserves the $U(1)$ flavor symmetry that rotates the adjoint scalars as well as the spatial rotations of $\mathbb{D}$ (loop rotations) so long they are compensated by a constant gauge transformation. (The latter point is non-trivial because the hypermultiplets scalars transform as spinors with respect to the modified Lorentz group used in the A-twist.)

Boundary local operators

Given the Dirichlet boundary condition $\mathcal{B}_{X_\partial,I_\partial}$, the space of boundary local operators $\rho_{\mathcal{B}_{X_\partial,I_\partial}}(\mathbb{1})$ at the junction of the trivial line operator $\mathbb{1}$ and $\mathcal{B}_{X_\partial,I_\partial}$ is realized as the (equivariant) homology of the moduli space $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial}, \mathbb{1})$. Similarly for the space of boundary local operators $\rho_{\mathcal{B}_{X_\partial,I_\partial}}(\mathbb{V}_{\mathcal{I}}$. Local operators bound to $\mathbb{V}_{\mathcal{I}}$ itself as well as local operators at junctions between $\mathbb{V}_{\mathcal{I}}$ and $\mathbb{1}$ will then be represented as linear maps among these homologies.

First, consider the space of local operators at the junction of the trivial line operator $\mathbb{1}$ and the boundary condition $\mathcal{B}_{X_\partial,I_\partial}$. From Section 2.3.1, we find that the corresponding moduli space is

$$\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial}; \mathbb{1}) = G(\mathcal{O}) \Big\backslash \Big(R(\mathcal{O}) \cap \big[G(\mathcal{K}) \cdot \big(X_\partial, I_\partial\big)\big] \Big). \tag{2.4.9}$$

We can always use $G(\mathcal{O})$ to make the $G(\mathcal{K})$ element lower triangular with monomial entries:

$$g' = \begin{pmatrix} z^{A_1} & 0 \\ z^{A_2} p & z^{A_2} \end{pmatrix}. \tag{2.4.10}$$

Moreover, we can always use a gauge transformation to ensure p has no terms of degree larger than $A_2 - A_1 - 1$. There are no nontrivial $G(\mathcal{O})$ transformations that fix this form and hence this parameterizes a cell in the moduli space $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial, I_\partial}; \mathbb{1})$, which lies in vortex number $\mathfrak{n} = A_1 + A_2$. With this in mind, we have

$$g'.(X_\partial, I_\partial) = (g'.X_\partial.g'^{-1}, g'.I_\partial) = \left(\begin{pmatrix} -p & -z^{A_2-A_1}p^2 \\ z^{A_1-A_2} & p \end{pmatrix}, \begin{pmatrix} 0 \\ z^{A_2} \end{pmatrix} \right). \tag{2.4.11}$$

Belonging to $R(\mathcal{O})$ implies that $A_2, A_1 - A_2 \geq 0$ and $p, z^{A_2-A_1}p^2 \in \mathbb{C}[[z]]$, thus p must be a polynomial with no terms of degree less than $\frac{A_1-A_2}{2}$. Thus, the above cell is $\lfloor \frac{A_1-A_2}{2} \rfloor$ dimensional.

The cells with the same vortex number $\mathfrak{n} = A_1 + A_2$ can potentially close onto one another. Indeed, if we start in the cell labeled by (A_1, A_2) and take the limit that the coefficient of $z^{A_1-A_2-1}$ in p goes to ∞, with the ratio of coefficients $p_d/p_{A_1-A_2-1}$ fixed, we land on the cell labeled by $(A_1 - 1, A_2 + 1)$. These exactly reproduce the attaching maps for projective space, so we conclude the following

$$\mathcal{M}^{\mathfrak{n}}_{\mathbb{D}}(\mathcal{B}_{X_\partial, I_\partial}; \mathbb{1}) = \mathbb{CP}^{\lfloor \frac{\mathfrak{n}}{2} \rfloor}, \tag{2.4.12}$$

or, equivalently,

$$\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial, I_\partial}; \mathbb{1}) = \overbrace{\{\text{point}\}}^{\mathfrak{n}=0} \sqcup \overbrace{\{\text{point}\}}^{\mathfrak{n}=1} \sqcup \overbrace{\mathbb{CP}^1}^{\mathfrak{n}=2} \sqcup \overbrace{\mathbb{CP}^1}^{\mathfrak{n}=3} \sqcup \overbrace{\mathbb{CP}^2}^{\mathfrak{n}=4} \sqcup \overbrace{\mathbb{CP}^2}^{\mathfrak{n}=5} \sqcup \ldots. \tag{2.4.13}$$

With the moduli space $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial, I_\partial}; \mathbb{1})$ in hand, we compute its equivariant (Borel-Moore) homology in order to determine the vector space of boundary local operators. Our main tool for this computation will be the classic Atiyah-Bott localization procedure [174]. The cell decomposition in Eq. (2.4.10) is particularly tailored for this localization computation. Consider a $T_{F,\mathbb{C}} = \mathbb{C}^*_F$ (flavor) and $\mathbb{C}^*_\varepsilon$ (loop rotations) transformation with parameters $m_{\mathbb{C}}$ and ε. In order to maintain the form given above, we must apply a compensating, torus-valued gauge transformation with parameters φ_1, φ_2 satisfying

$$\varphi_1 = (A_1+1)\varepsilon - m_{\mathbb{C}} \qquad \varphi_2 = (A_2+\tfrac{1}{2})\varepsilon\,. \tag{2.4.14}$$

Under such a transformation, the coefficient of p multiplying z^d transforms as

$$\delta p_d \sim \big(m - (d+\tfrac{1}{2})\varepsilon\big)p_d\,. \tag{2.4.15}$$

For generic values of $m_{\mathbb{C}}, \varepsilon$, the combined action of the flavor torus $\mathbb{C}^*_F$ and loop rotation $\mathbb{C}^*_\varepsilon$ has a unique fixed point at $p=0$. Thus, the subspace of boundary local operators with vortex number $\mathfrak{n}$, *i.e.* the equivariant homology of $\mathcal{M}^{\mathfrak{n}}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1})$, is generated by $\lfloor \frac{\mathfrak{n}}{2} \rfloor$ fixed-point classes labeled by $A = A_1, A_2$ with $A_1 \geq A_2 \geq 0$; we denote the corresponding homology class $|A\rangle$:

$$H^{\mathbb{C}^*_F\times\mathbb{C}^*_\varepsilon}_\bullet(\mathcal{M}^{\mathfrak{n}}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1})) \simeq \bigoplus_{\substack{A_1\geq A_2\geq 0\\ A_1+A_2=\mathfrak{n}}} \mathbb{C}\,|A\rangle\ . \tag{2.4.16}$$

(We assume that $m_{\mathbb{C}}, \varepsilon$ take generic values, and invert them at will.) The equivariant parameters φ_a, representing operators formed from complex vector multiplet scalars, act as

$$\varphi_1\,|A\rangle = \big((A_1+1)\varepsilon - m_{\mathbb{C}})\big)\,|A\rangle \qquad \varphi_2\,|A\rangle = (A_2+\tfrac{1}{2})\varepsilon\,|A\rangle\ , \tag{2.4.17}$$

In particular, $\mathrm{Tr}\,\varphi = \varphi_1+\varphi_2$ measures vortex number $\mathfrak{n}$.

Generalizing the moduli space and homology to a Iwahori line $\mathbb{V}_{\mathcal{I}}$ is fairly straightforward. From (2.3.6) and (2.3.12) we now have

$$\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{V}_{\mathcal{I}}) = G(\mathcal{O})\Big\backslash\Big(\mathbb{CP}^1 \times \big[R(\mathcal{O}) \cap G(\mathcal{K})\cdot(X_\partial,I_\partial)\big]\Big) \tag{2.4.18}$$

$$= \mathcal{I}\Big\backslash\Big(R(\mathcal{O}) \cap G(\mathcal{K})\cdot(X_\partial,I_\partial)\Big)\,. \tag{2.4.19}$$

In the first description, the moduli space consists $X(z), I(z)$ as in above, exactly as for the trivial line (because these constraints come from $\mathcal{B}_{X_\partial,I_\partial}$); together with a choice of point

$p = \binom{p_1}{p_2} \in \mathbb{CP}^1$. Gauge transformations $g(z) \in G(\mathcal{O})$ act as

$$p \mapsto g(0)p\,, \qquad X(z) \mapsto g(z)X(z)g(z)^{-1}\,, \qquad I(z) \mapsto g(z)I(z)\,. \tag{2.4.20}$$

In the second description, we have gauge-fixed $p = \binom{1}{0}$, thereby breaking the gauge group to the Iwahori subgroup, *i.e.* $g(z) \in G(\mathcal{O})$ satisfying $g(0) \in B = \text{stab}_G(p)$.

Just as in the case of the trivial line, the moduli space decomposes into connected components $\mathcal{M}^{\mathfrak{n}}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{V}_{\mathcal{I}})$ labeled by vortex number $\mathfrak{n} \geq 0$. More so, it is clear that $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{V}_{\mathcal{I}})$ is a $\mathbb{CP}^1$ fibration over $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1})$ (the map from $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{V}_{\mathcal{I}})$ to $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1})$ is given by forgetting $p \in \mathbb{CP}^1$). Thus we expect each $\mathcal{M}^{\mathfrak{n}}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{V}_{\mathcal{I}})$ to have cells that are *products* of the above cells, and two standard cells on $\mathbb{CP}^1$ (the copy of $\mathbb{C}$ containing the north pole and the south pole). Abstractly, the cells on $\mathbb{CP}^1 = G_{\mathbb{C}}/B$ are labeled by elements of the Weyl group $\sigma \in \text{Weyl}(G) = \mathbb{Z}_2 = \{1, w\}$.

Again, there is a unique fixed point of $\mathbb{C}^*_F \times \mathbb{C}^*_\varepsilon$ (up to a compensating torus-valued gauge transformation) at the origin of each cell; each yields a fixed point class in equivariant homology that we denote $|A,\sigma\rangle$ for $\sigma = 1, w$. Following the above analysis, we find that the action of the equivariant parameters φ_a is given by

$$\varphi_1 |A,1\rangle = \big((A_1+1)\varepsilon - m\big)|A,1\rangle \qquad \varphi_2 |A,1\rangle = (A_2 + \tfrac{1}{2})\varepsilon\, |A,1\rangle\ , \tag{2.4.21}$$

and

$$\varphi_1 |A,w\rangle = (A_2 + \tfrac{1}{2})\varepsilon\, |A,w\rangle \qquad \varphi_2 |A,w\rangle = \big((A_1+1)\varepsilon - m\big)|A,w\rangle\ , \tag{2.4.22}$$

We therefore find that the homology is given by

$$\begin{aligned} H^{\mathbb{C}^*_F\times\mathbb{C}^*_\varepsilon}(\mathcal{M}^{\mathfrak{n}}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{V}_{\mathcal{I}}) &\simeq \bigoplus_{\substack{A_1\geq A_2\geq 0\\ A_1+A_2=\mathfrak{n}\\ \sigma\in\{1,w\}}} \mathbb{C}\,|A,\sigma\rangle \\ &\simeq H^{\mathbb{C}^*_F\times\mathbb{C}^*_\varepsilon}(\mathcal{M}^{\mathfrak{n}}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1}) \otimes_{\mathbb{C}[\varphi]} H^{\bullet}_{\mathbb{C}^*}(\mathbb{CP}^1)\,. \end{aligned} \tag{2.4.23}$$

Here $\mathbb{C}[\varphi] = \mathbb{C}[\varphi_1, \varphi_2]$ is the ring of polynomials in gauge equivariant parameters, and tensoring over it on the RHS means that its actions on $\mathcal{H}(\mathcal{B}_{X_\partial, I_\partial}; \mathbb{1})$ and $H^*_{\mathbb{C}^*}(\mathbb{CP}^1)$ are compatible.

Local operators in the bulk

We are ready to begin describing local operators in this example.

In [70, 179, 180], it was shown that the space of bulk local operators $\mathrm{End}^\varepsilon_A(\mathbb{1}) = \mathbb{C}_\varepsilon[\mathcal{M}_C]$ is faithfully represented in the homology of the raviolo space $\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_\nu; \mathbb{1}, \mathbb{1})$ for a vacuum boundary condition. We will recover a version of this result momentarily, instead using the above half-space construction and the Dirichlet boundary condition $\mathcal{B}_{X_\partial, I_\partial}$. The algebra structure on local operators, and their compatible action

$$\begin{aligned} H^{\mathbb{C}^*_F \times \mathbb{C}^*_\varepsilon}_\bullet\big(\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_{X_\partial, I_\partial}; \mathbb{1}, \mathbb{1})\big) : H^{\mathbb{C}^*_F \times \mathbb{C}^*_\varepsilon}_\bullet\big(\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial, I_\partial}; \mathbb{1})\big) \\ \to H^{\mathbb{C}^*_F \times \mathbb{C}^*_\varepsilon}_\bullet\big(\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial, I_\partial}; \mathbb{1})\big) \end{aligned} \tag{2.4.24}$$

both come from convolution, as discussed in Section 1.3.2. When the representation (2.4.24) is faithful, the algebra structure of local operators may be fully reconstructed from it.

The bulk algebra $\mathrm{End}^\varepsilon_A(\mathbb{1}) \simeq \mathbb{C}_\varepsilon[\mathcal{M}_C]$, quantized in the Omega background, was described in Section 1.4.2. We found that $\mathrm{End}^\varepsilon_A(\mathbb{1})$ is generated (as an algebra) by six operators Q^1, Q^2, E, F, H. Our 3d $\mathcal{N} = 4$ theory has a topological $U(1)_{\mathrm{top}}$ flavor symmetry acting on the Coulomb branch, whose charge is monopole number. As we saw, this symmetry is enhanced in the infrared to $SU(2)_{\mathrm{top}}$, whose complexification acts on the Coulomb-branch chiral ring. The operators E, F, H are the components of the moment map that generates this action, and the Q^i operators transform in a fundamental representation thereof.

Physically, the operator $H = -(\varphi_1 + \varphi_2) = -\mathrm{Tr}\varphi$ is the (negative of the) trace of the vector multiplet scalar, and Q^i are fundamental non-abelian monopole operators of charge ± 1, defined by the minuscule cocharacters $(1, 0)$ and $(0, -1)$, respectively. The operators E, F are monopoles operators corresponding to the cocharacter $\pm(1, 1)$. We also note that, in units where ε has charge $+1$ under the $U(1)_C$ R-symmetry, Q^i have R-charge $+\frac{1}{2}$ and E, F, H have R-charge $+1$.

We now describe the algebra $\mathrm{End}^{\varepsilon}_{A}(\mathbb{1})$ in terms of its representation in the equivariant homology of $\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1},\mathbb{1})$ — since this is the approach that will generalize. Recall that the basic monopole operators are labeled by dominant cocharacters, or simply cocharacters λ modulo permutation, corresponding to their topological charge for the a maximal torus of the gauge group. In the algebraic description developed over the past sections, they correspond to equivariant homology classes of $\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1},\mathbb{1})$ that arise as pull-backs via the map

$$\pi_g : \mathcal{M}_{\mathrm{rav}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1},\mathbb{1}) \to G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}) \tag{2.4.25}$$

that forgets the hypermultiplets X',I' and X,I. For $G_{\mathbb{C}} = GL(2,\mathbb{C})$, the equivariant homology classes of interest arise from orbits of $z^\lambda \in G(\mathcal{K}) = GL(2,\mathcal{K})$ with respect to the action of $G(\mathcal{O})' \times G(\mathcal{O}) = GL(2,\mathcal{O})' \times GL(2,\mathcal{O})$ on the left and right. Let $\mathcal{O}_\lambda$ denote this double orbit

$$\mathcal{O}_\lambda := \overline{GL(2,\mathcal{O})'\, z^\lambda \, GL(2,\mathcal{O})} \subset GL(2,\mathcal{K})\,. \tag{2.4.26}$$

The orbit only depends on the Weyl-conjugacy class of λ. The basic monopole operator of charge λ then corresponds to the pullback of the fundamental class of this orbit (or, better, its closure) $[GL(2,\mathcal{O})'\backslash\mathcal{O}_\lambda/GL(2,\mathcal{O})]$ to the equivariant homology of $\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1},\mathbb{1})$ with respect to π_g. The appropriate way to think about the homology class $[GL(2,\mathcal{O})'\backslash\mathcal{O}_\lambda/GL(2,\mathcal{O})]$ is as the $GL(2,\mathcal{O})$ equivariant homology class corresponding to (the closure of) $GL(2,\mathcal{O})'\backslash\mathcal{O}_\lambda \subset \mathrm{Gr}_{GL(2,\mathbb{C})}$. The spaces $GL(2,\mathcal{O})'\backslash\mathcal{O}_\lambda$ are often denoted $\mathrm{Gr}^{\leq\lambda}_{GL(2,\mathbb{C})}$ and are closed for λ minuscule.

Bulk local operators, represented as equivariant homology classes of $\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1},\mathbb{1})$, act on boundary local operators, represented as equivariant homology classes of $\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1})$, by pulling-back via π, capping with a homology class of $\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1},\mathbb{1})$, and then pushing forward with π', where π' and π forget $(g;X,I)$ and $(X',I';g)$, respectively,

$$\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1}) \xleftarrow{\pi'} \mathcal{M}_{\mathrm{rav}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1},\mathbb{1}) \xrightarrow{\pi} \mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1})\,. \tag{2.4.27}$$

In order to understand the action of homology classes of $\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1},\mathbb{1})$, we again rely on equivariant localization. The main point is that we can use the abelianized/localized description of the Coulomb branch of [67] in a fashion that is compatible with the fixed-point localization used above. See [51] for more details.

The algebra is generated by monopole operators associated to minuscule cocharacters λ and their dressed versions [252], which have a known localization. In particular, we consider the double orbit $\mathcal{O}_\lambda$ for λ a minuscule cocharacter, or, equivalently, the $GL(2,\mathcal{O})$ equivariant homology class $\big[\mathrm{Gr}^{\leq\lambda}_{GL(2,\mathbb{C})}\big]$. The fixed points in $\mathrm{Gr}^{\leq\lambda}_{GL(2,\mathbb{C})}$ are labeled by cocharacters in the Weyl orbit of λ and we can write, c.f. [72, Proposition 6.6],

$$\big[\mathrm{Gr}^{\leq\lambda}_{GL(2,\mathbb{C})}\big] = \sum_{w\in W/W_\lambda} \frac{[z^{w.\lambda}]}{e(T_{w.\lambda}\mathrm{Gr}^{\leq\lambda}_{GL(2,\mathbb{C})})}\,, \tag{2.4.28}$$

where W_λ is the stabilizer of λ in the Weyl group $W = S_2$, and $e(T_{w.\lambda}\mathrm{Gr}^{\leq\lambda}_{GL(2,\mathbb{C})})$ is the $\mathbb{C}^*_{\varphi_1}\times\mathbb{C}^*_{\varphi_2}\times\mathbb{C}^*_{\varepsilon}$ equivariant Euler character of the tangent space to $[z^{w.\lambda}]$, and is a polynomial in the φ_a and ε. The dressed versions have an identical form, but allow for a W_λ-invariant polynomial of $\varphi_a, m_{\mathbb{C}}, \varepsilon$ in the numerator.

Since we know the action of the complex scalars φ_a, it suffices to understand the action of the various fixed-point classes $[z^\lambda]$. For the fixed point class $|A\rangle$ we have, *cf.* (4.45), (4.48) in [70],

$$[z^\lambda]\,|A\rangle := \pi'_*([z^\lambda]\cap\pi^*\,|A\rangle) = \begin{cases} e(\lambda;A)\,|\lambda+A\rangle & z^{\lambda+A}\cdot(X_\partial,I_\partial)\in\mathcal{M}_{\mathbb{D}}(\mathcal{B},\mathcal{L}') \\ 0 & \text{else}\end{cases}\,, \tag{2.4.29}$$

where $e(\lambda;A)$ is an excess intersection factor that measures the equivariant weights of the tangent vectors to fixed point associated to $|A\rangle$, viewed as an element of $R(\mathcal{O})$, that are no longer tangent to $R(\mathcal{O})$ after applying the gauge transformation z^λ. In fact, the factor $e(\lambda;A)$ *automatically* vanishes if $z^{\lambda+A}\cdot(X_\partial,I_\partial)\notin\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1})$, reflecting the fact that the fiber over $z^A\cdot(X_\partial,I_\partial)$ in the raviolo space does *not* include z^λ.

For the trivial cocharacter $\lambda = (0,0)$, one finds that the fundamental class $\big[\mathrm{Gr}^{\leq(0,0)}_{GL(2,\mathbb{C})}\big]$ is the identity operator $1 \in \mathrm{End}^{\varepsilon}_{A}(\mathbb{1})$. The operator $H = -\mathrm{Tr}\,\varphi$ is a dressed version of this trivial monopole, and acts on the fixed point class $|A\rangle$ as

$$H\,|A\rangle = \big(m_{\mathbb{C}} - (A_1 + A_2 + \tfrac{3}{2})\varepsilon\big)\,|A\rangle\ . \tag{2.4.30}$$

The operators Q^2,Q^1 arise from the fundamental and anti-fundamental cocharacters $\lambda = (1,0)$ and $\lambda = (0,-1)$. For Q^2 we have

$$Q^2 = \big[\mathrm{Gr}^{\leq(1,0)}_{GL(2,\mathbb{C})}\big] = \frac{[z^{(1,0)}] - [z^{(0,1)}]}{\varphi_2 - \varphi_1}\,. \tag{2.4.31}$$

Under the gauge transformation $z^{(1,0)}$, one finds

$$[z^{(1,0)}]\,|A\rangle = (2m_{\mathbb{C}} - (A_1 - A_2 + 1)\varepsilon)\,|A + (1,0)\rangle\ , \tag{2.4.32}$$

and similarly for $z^{(0,1)}$

$$[z^{(0,1)}]\,|A\rangle = (A_1 - A_2)\varepsilon\,|A + (0,1)\rangle\ . \tag{2.4.33}$$

Note that $[z^{(0,1)}]\,|A\rangle = 0$ if and only if $A_1 = A_2$, *i.e.*, if $A + (0,1)$ does *not* label a fixed point. Putting the above together, we find that

$$Q^2\,|A\rangle = \frac{(2m_{\mathbb{C}} - (A_1 - A_2 + 1)\varepsilon)\,|A + (1,0)\rangle + (A_2 - A_1)\varepsilon\,|A + (0,1)\rangle}{m_{\mathbb{C}} - (A_1 - A_2 + \frac{1}{2})\varepsilon} \tag{2.4.34}$$

From this action, it immediately follows that

$$[H, Q^2] = -\varepsilon Q^2\,. \tag{2.4.35}$$

Completely analogously, we can write

$$Q^1 = \big[\mathrm{Gr}^{\leq(0,-1)}_{GL(2,\mathbb{C})}\big] = \frac{[z^{(0,-1)}] - [z^{(-1,0)}]}{\varphi_2 - \varphi_1}. \tag{2.4.36}$$

Following the above, we find that the action of Q^1 on the fixed point $|A\rangle$ is given by

$$\begin{aligned} Q^1 |A\rangle = &\frac{(2m_{\mathbb{C}} - (A_1 - A_2 + 1)\varepsilon)A_2\varepsilon\, |A + (0,-1)\rangle}{m_{\mathbb{C}} - (A_1 - A_2 + \frac{1}{2})\varepsilon} \\ &- \frac{(A_1 - A_2)\varepsilon\big((A_1 + \frac{1}{2})\varepsilon - m_{\mathbb{C}}\big)\, |A + (-1,0)\rangle}{m_{\mathbb{C}} - (A_1 - A_2 + \frac{1}{2})\varepsilon} . \end{aligned} \tag{2.4.37}$$

For generic m, the first term vanishes if and only if $A_2 = 0$ and the second term vanishes if and only if $A_1 = A_2$, again this signifies that the putative gauge transformation does not yield an element of $R(\mathcal{O})$. Moreover, it is straightforward to see that

$$[H, Q^1] = \varepsilon Q^1 \tag{2.4.38}$$

as linear operators. Moreover, it is straightforward, albeit tedious, to check that

$$[Q^2, Q^1] = -2\varepsilon. \tag{2.4.39}$$

The analyses for $\lambda = (1,1)$ and $\lambda = (-1,-1)$ are identical to the above. Each yields is a fibration, with fiber a single point, and we have

$$F = -\big[\mathrm{Gr}^{\leq(1,1)}_{GL(2,\mathbb{C})}\big]\,, \qquad E = \big[\mathrm{Gr}^{\leq(-1,-1)}_{GL(2,\mathbb{C})}\big]\,. \tag{2.4.40}$$

These operators act on $|A\rangle$ as

$$F\,|A\rangle = -\,|A + (1,1)\rangle \qquad E\,|A\rangle = \big((A_1 + \tfrac{1}{2})\varepsilon - m_{\mathbb{C}}\big)A_2\varepsilon\,|A - (1,1)\rangle\,. \tag{2.4.41}$$

It immediately follows that

$$[H, E] = 2\varepsilon E \qquad [H, F] = -2\varepsilon F \qquad [E, H] = \varepsilon H. \tag{2.4.42}$$

Again, it is straightforward to check the remaining relations discussed in Section 1.4.2 are satisfied.

Local operators on the Iwahori line

We now turn to the local operators $\mathrm{End}^{\varepsilon}_{A}(\mathbb{V}_{\mathcal{I}})$ bound to the Iwahori line, which are realized as elements in the equivariant homology of the raviolo space $\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{V}_{\mathcal{I}},\mathbb{V}_{\mathcal{I}})$. Again, we employ a localized description of the algebra. Many of the details are the same as for the trivial line $\mathbb{1}$, but now local operators are realized by pulling back classes on $\mathcal{I}'\backslash GL(2,\mathcal{K})/\mathcal{I}$. We will focus on small monopole number $|\mathfrak{m}|\leq 1$, starting with operators of monopole number zero.

The most basic local operators of monopole number zero come from orbits of the identity. Let

$$\mathcal{I}'\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\mathcal{I}\subset GL(2,\mathcal{K}) \tag{2.4.43}$$

be the double Iwahori orbit of the identity; this again represents the identity operator 1. Similarly, we also find operators corresponding to the bulk vector multiplet scalars, which may be considered dressed versions of the identity. In the bulk, we had to take G-invariant combinations $\mathrm{Tr}(\varphi)=\varphi_1+\varphi_2$ and $\mathrm{Tr}(\varphi^2)$. On the Iwahori line, we have access to φ_1 and φ_2 independently. We already know from the above that their action on a fixed-point basis of $H_\bullet^{\mathbb{C}^*_F\times\mathbb{C}^*_\varepsilon}(\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial},\mathbb{V}_{\mathcal{I}}))$ is given by

$$\varphi_1\,|A,1\rangle=\big((A_1+1)\varepsilon-m_{\mathbb{C}}\big)\,|A,1\rangle\qquad \varphi_2\,|A,1\rangle=(A_2+\tfrac{1}{2})\varepsilon\,|A,1\rangle\ , \tag{2.4.44}$$

and

$$\varphi_1\,|A,w\rangle=(A_2+\tfrac{1}{2})\varepsilon\,|A,w\rangle\qquad \varphi_2\,|A,w\rangle=\big((A_1+1)\varepsilon-m_{\mathbb{C}}\big)\,|A,w\rangle\ . \tag{2.4.45}$$

More interestingly, we may consider the orbit closure $\overline{\mathcal{I}'\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\mathcal{I}}\subset G(\mathcal{K})$. This reduces to a $\mathbb{CP}^1$ inside $\mathcal{I}'\backslash GL(2,\mathcal{K})$, and there are torus fixed points at the north and south poles. We denote the pulled-back fixed point classes [1] and $[w]$.

We again determine the convolution product among these classes by computing the action of $\varphi_2-\varphi_1$, [1], and $[w]$ on the fixed-point basis of $H_\bullet^{\mathbb{C}^*_F\times\mathbb{C}^*_\varepsilon}(\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial,I_\partial},\mathbb{V}_{\mathcal{I}}))$, which is a straightforward repetition of the above analysis. We find that the pullback via π_g of the fundamental class of this $\mathbb{CP}^1$, which we denote θ, acts as

$$\theta\,|A,\sigma\rangle := \left(\frac{[1]-[w]}{\varphi_2-\varphi_1}\right)|A,\sigma\rangle = (-1)^{|\sigma|}\frac{|A,1\rangle + |A,w\rangle}{m_{\mathbb{C}} - (A_1 - A_2 + \frac{1}{2})\varepsilon}\,, \tag{2.4.46}$$

where $|\sigma|$ denotes the parity of the permutation σ. We therefore find that

$$\theta(\varphi_2-\varphi_1)\,|A,\sigma\rangle = |A,\sigma\rangle + |A,w\sigma\rangle \tag{2.4.47}$$

so that $s := 1-\theta(\varphi_2-\varphi_1)$ acts as a Weyl reflection:

$$s\,|A,\sigma\rangle = |A,w\sigma\rangle\ . \tag{2.4.48}$$

It is also useful to note that $s\,(\varphi_2-\varphi_1) = -(\varphi_2-\varphi_1)\,s$ (where now the two sides denote the convolution product); intuitively, this is because the gauge transformation $g = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ associated to $s=[w]$ swaps the equivariant parameters φ_1,φ_2, whose difference appears in $(\varphi_2-\varphi_1)$.

The operators 1, $\varphi := \varphi_2-\varphi_1$, s, and θ found above generate a copy of the nil-Hecke algebra $\mathbf{H}_2$ for $SL(2)$ [248]. Indeed, it is easy to verify the standard algebra relations

$$\begin{aligned} s^2 = 1\,,\qquad \theta^2 = 0\,,\qquad s\partial = -\theta s = \theta\,,\\ s\varphi = -\varphi s\,,\qquad \{\theta,\varphi\} = 2\,,\qquad [\theta,\varphi] = -2s\,, \end{aligned} \tag{2.4.49}$$

where *all* products come from convolution. Abstractly, the nil-Hecke algebra is obtained from the polynomial algebra $\mathbb{C}[\varphi]$ by first adjoining the Weyl reflection s and then the "BGG-Demazure operator" θ from Section 1.2.3.

Geometrically, the nil-Hecke algebra is defined as the equivariant (co)homology

$$\mathbf{H}_2 = H^*_{SL(2,\mathbb{C})}(\mathbb{CP}^1\times\mathbb{CP}^1)\,, \tag{2.4.50}$$

with its natural convolution product. By writing $\mathbb{CP}^1 = SL(2,\mathbb{C})/B$, one also obtains the equivalent description $\mathbf{H}_2 \simeq H^*(B\backslash SL(2,\mathbb{C})/B)$. As discussed in the introduction to this section, we expected the nil-Hecke algebra to appear on the Iwahori line due to the appearance of $\mathbb{CP}^1 \times \mathbb{CP}^1$ in the raviolo space

$$\mathcal{M}_{\text{rav}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{V}_\mathcal{I},\mathbb{V}_\mathcal{I}) \simeq GL(2,\mathcal{O})'\backslash\big(\mathbb{CP}^1 \times \widetilde{\mathcal{M}}_{\text{rav}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1},\mathbb{1}) \times \mathbb{CP}^1\big)/GL(2,\mathcal{O})\,. \quad (2.4.51)$$

Next, let's identify the basic operators of monopole number 1. We will find that they have the structure of a product of the nil-Hecke algebra $\mathbf{H}_2$ and an abelianized monopole algebra, along the lines of [39,67]. The most basic such classes come from $\overline{\mathcal{I}'\left(\begin{smallmatrix} z & 0\\ 0 & 1\end{smallmatrix}\right)\mathcal{I}} \subset GL(2,\mathcal{K})$. This cycle yields a $\mathbb{CP}^1$ bundle over $\mathbb{CP}^1$; more precisely, it is the projectivization of the rank-two bundle $\mathcal{O}(1)\oplus\mathcal{O}(-1)\to\mathbb{CP}^1$. See, *e.g.* [111, Section 7.4.4] for more details. There are four fixed points

$$\sigma z^{e_a} = \begin{pmatrix} z & 0\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix} 1 & 0\\ 0 & z\end{pmatrix}, \quad \begin{pmatrix} 0 & z\\ 1 & 0\end{pmatrix}, \quad \begin{pmatrix} 0 & 1\\ z & 0\end{pmatrix}, \quad (2.4.52)$$

which we can label by an element $\sigma \in \{1, w\}$ of the Weyl group and a cocharacter $e_1 = (1,0)$ or $e_2 = (0,1)$. We denote the pulled-back fixed-point classes $[\sigma z^{e_a}]$.

Let us define

$$u_a^+ := [z^{e_a}]\,. \quad (2.4.53)$$

These are "abelianized" monopole operators discussed in [39, 101], slight renormalizations of the abelianized monopole operators of [67, 70]. They act on states as

$$u_1^+\,|A,1\rangle = (2m_\mathbb{C} - (A_1 - A_2 + 1)\varepsilon)\,|A+(1,0),1\rangle \quad (2.4.54)$$

$$u_1^+\,|A,w\rangle = (A_1 - A_2)\varepsilon\,|A+(0,1),w\rangle \quad (2.4.55)$$

$$u_2^+\,|A,1\rangle = (A_1 - A_2)\varepsilon\,|A+(0,1),1\rangle \quad (2.4.56)$$

$$u_2^+\,|A,w\rangle = (2m_\mathbb{C} - (A_1 - A_2 + 1)\varepsilon)\,|A+(1,0),w\rangle \quad (2.4.57)$$

$$(2.4.58)$$

Just as in [111], all other monopole operators coming from this orbit can all be expressed as convolution products of u_a^+ with elements of the nil-Hecke algebra. Some other simple relations among operators of monopole numbers $\mathfrak{m} = 0, 1$ are

$$su_a^+ = u_{w(a)}^+ s \qquad u_a^+ \varphi_b = (\varphi_b - \delta_{ab}\varepsilon)u_a^+. \tag{2.4.59}$$

Now let's move to the negative monopole sector and consider the $\overline{\mathcal{I}' \left(\begin{smallmatrix} 1 & 0 \\ 0 & z^{-1} \end{smallmatrix}\right) \mathcal{I}} \subset GL(2, \mathcal{K})$. This orbit is obtained by simply multiplying the elements of the above double orbit by z^{-1} and so must yield be another copy of $\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(-1))$ inside $\mathcal{I}' \backslash GL(2, \mathcal{K})$. The action of the fixed points $u_a^- := [z^{-e_a}]$ are given by

$$u_1^- |A, 1\rangle = (A_1 - A_2)\varepsilon\big((A_1 + \tfrac{1}{2})\varepsilon - m\big) |A + (-1, 0), 1\rangle \,, \tag{2.4.60}$$

$$u_1^- |A, w\rangle = \big(2m_{\mathbb{C}} - (A_1 - A_2)\varepsilon\big) A_2 \varepsilon \, |A + (0, -1), w\rangle \,, \tag{2.4.61}$$

$$u_2^- |A, 1\rangle = \big(2m_{\mathbb{C}} - (A_1 - A_2)\varepsilon\big) A_2 \varepsilon \, |A + (0, -1), 1\rangle \,, \tag{2.4.62}$$

$$u_2^- |A, w\rangle = (A_1 - A_2)\varepsilon\big((A_1 + \tfrac{1}{2})\varepsilon - m\big) |A + (-1, 0), w\rangle \,, \tag{2.4.63}$$

$$\tag{2.4.64}$$

and they satisfy relations similar to 2.4.59:

$$su_a^- = u_{w(a)}^- s \qquad u_a^- \varphi_b = (\varphi_b + \delta_{ab}\varepsilon)u_a^-. \tag{2.4.65}$$

And the following relation with the positive abelianized operators, *cf.* Eq. (3.43) of [67] and Eq. (1.2.14) above,

$$u_a^+ u_a^- = P_a^+ \qquad u_a^- u_a^+ = P_a^-. \tag{2.4.66}$$

where

$$P_a^\pm = (\varphi_a \mp \tfrac{1}{2}\varepsilon) \prod_{b \neq a} (\varphi_a - \varphi_b + m_{\mathbb{C}} \mp \tfrac{1}{2}\varepsilon)(\varphi_b - \varphi_a + m_{\mathbb{C}} \pm \tfrac{1}{2}\varepsilon). \tag{2.4.67}$$

Junctions between $\mathbb{V}_{\mathcal{I}}$ and $\mathbb{1}$

Now consider the junctions $\mathrm{Hom}^{\varepsilon}_{A}(\mathbb{V}_{\mathcal{I}},\mathbb{1})$ and $\mathrm{Hom}^{\varepsilon}_{A}(\mathbb{1},\mathbb{V}_{\mathcal{I}})$, which are naturally bi-modules for the algebras $\mathrm{End}^{\varepsilon}_{A}(\mathbb{V}_{\mathcal{I}})$ and $\mathrm{End}^{\varepsilon}_{A}(\mathbb{1})$. Let us start with the space $\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{V}_{\mathcal{I}},\mathbb{1})$. We consider $\overline{GL(2,\mathcal{O})'\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\mathcal{I}}\subset GL(2,\mathcal{K})$. This yields a single point in $GL(2,\mathcal{O})'\backslash GL(2,\mathcal{K})$ and we then have a single operator, call it B_+, that acts as

$$B_+\left|A,\sigma\right\rangle=\left|A\right\rangle. \tag{2.4.68}$$

We expect B_+ to generate the bimodule at the junction. Another (expected) generator is given by

$$B_-:=\tfrac{1}{2}B_+\varphi\Rightarrow B_-\left|A,\sigma\right\rangle=(-1)^{|\sigma|}\big(m_{\mathbb{C}}-(A_1-A_2+\tfrac{1}{2})\varepsilon\big)\left|A\right\rangle, \tag{2.4.69}$$

which is related to B_+ via $B_-\theta=B_+$ and thus $B_+\theta=0$.

Similarly, for $\mathcal{M}_{\mathrm{rav}}(\mathcal{B}_{X_\partial,I_\partial};\mathbb{1},\mathbb{V}_{\mathcal{I}})$ we consider $\overline{\mathcal{I}'\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)GL(2,\mathcal{O})}\subset GL(2,\mathcal{K})$, which yields a $\mathbb{CP}^1$ in $\mathcal{I}'\backslash GL(2,\mathcal{K})$. We find two operators, call them b_- and b_+, with actions

$$b_-\left|A\right\rangle=\frac{\left|A;1\right\rangle-\left|A;w\right\rangle}{m_{\mathbb{C}}-(A_1-A_2+\frac{1}{2})\varepsilon}\qquad b_+\left|A\right\rangle=\tfrac{1}{2}\big(\left|A;1\right\rangle+\left|A;w\right\rangle\big). \tag{2.4.70}$$

Again, we expect that the entire bimodule is generated by either b_- or b_+. In particular one may pass from one to the other via

$$\tfrac{1}{2}\varphi b_-=b_+,\qquad \theta b_+=b_-. \tag{2.4.71}$$

The operators B_+ and b_+ are Weyl symmetric, in the sense that

$$B_+s=B_+,\qquad sb_+=b_+, \tag{2.4.72}$$

while the operators B_- and b_- are Weyl antisymmetric

$$B_- s = -B_-, \qquad s b_- = -b_-. \tag{2.4.73}$$

They furthermore satisfy

$$B_+ b_+ = 1 \qquad B_- b_+ = 0 \qquad B_+ b_- = 0 \qquad B_- b_- = 1. \tag{2.4.74}$$

$\mathrm{End}_A^{\varepsilon}(\mathbb{V}_{\mathcal{I}})$ as a matrix algebra over $\mathrm{End}_A^{\varepsilon}(\mathbb{1})$

There were many hints in the previous subsections that the algebra $\mathrm{End}_A^{\varepsilon}(\mathbb{V}_{\mathcal{I}})$ factors into a product $\mathrm{End}_A^{\varepsilon}(\mathbb{1}) \otimes \mathrm{End}(H_{\bullet}^{\mathbb{C}^*}(\mathbb{CP}^1))$. In particular, we found that the positive and negative monopole operators could all be expressed in terms of products of abelianized monopoles, which only acted on A, and the nil-Hecke algebra, which only acted on σ. Furthermore, we found two "inclusion" maps $b_\pm$ and two "projection" maps $B_\pm$ that relate $\mathrm{End}_A^{\varepsilon}(\mathbb{1})$ and $\mathrm{End}_A^{\varepsilon}(\mathbb{V}_{\mathcal{I}})$. We will write $b_\pm$ as vectors

$$b_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad b_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.4.75}$$

In light of (2.4.74), the operators $B_\pm$ have a natural representation as dual covectors

$$B_+ = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad B_- = \begin{pmatrix} 0 & 1 \end{pmatrix}. \tag{2.4.76}$$

The matrix elements of the various $\mathrm{End}_A^{\varepsilon}(\mathbb{V}_{\mathcal{I}})$ operators found above can be determined by sandwiching between $B_\pm$ and $b_\pm$. For example, the monopole number 0 operators can be represented by matrices over $\mathrm{End}^{\varepsilon}_{A}(\mathbb{1})$ as[15]

$$s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \varphi_1 + \varphi_2 = \begin{pmatrix} -H & 0 \\ 0 & -H \end{pmatrix} \qquad \theta = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \qquad \varphi_2 - \varphi_1 = \begin{pmatrix} 0 & (\varphi_2 - \varphi_1)^2 \\ 1 & 0 \end{pmatrix}. \tag{2.4.77}$$

The operators of nonzero monopole charge can be expressed similarly.

2.4.2 The rational Cherednik algebra

In the previous section we found that the simplest non-abelian A-type line operators — those that break the gauge group but introduce no singularity in the hypermultiplets — are a little *too* simple. In the category of A-type line operators, they are equivalent to direct sums of the trivial line. In this section we study a particular important example involving singularities in the hypermultiplets as well.

We again consider $U(2)$ gauge theory with a single fundamental hypermultiplet and a single adjoint hypermultiplet, *i.e.* $R = \mathbb{C}^2 \oplus \mathfrak{gl}(2, \mathbb{C})$. An algebraic characterization of the desired vortex line is

$$\mathbb{V}_{\mathrm{RCA}} : \mathcal{G}_0 = \mathcal{I}\,, \quad \mathcal{L}_0 = \left\{ X \in \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ z\mathcal{O} & \mathcal{O} \end{pmatrix}, \quad I \in \begin{pmatrix} \mathcal{O} \\ \mathcal{O} \end{pmatrix}, Y \in \begin{pmatrix} \mathcal{O} & z^{-1}\mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}, \quad J^T \in \begin{pmatrix} \mathcal{O} \\ \mathcal{O} \end{pmatrix} \right\}. \tag{2.4.78}$$

Alternatively, we say that we break the gauge symmetry to the Iwahori subgroup $\mathcal{I}$, and require that (X, I) lie in $R_0 = \mathcal{O}^2 \oplus \mathrm{Lie}\mathcal{I}$ and (Y, J) belong to their co-normal fibers within $T^*R(\mathcal{K})$. We will see that, although the two line operators $\mathbb{V}_{\mathcal{I}}$ and $\mathbb{V}_{\mathrm{RCA}}$ have endomorphism algebras with the same labels, *i.e.* by $\mathcal{I}$-equivariant homology classes of $\mathcal{I}'\backslash GL(2, \mathcal{K})$, the corresponding algebras will be dramatically different.

As proven in [249], the algebra of local operators bound to the line operator $\mathbb{V}_{\mathrm{RCA}}$ should reproduce the rational Cherednik algebra for $\mathfrak{gl}(2, \mathbb{C})$. Concretely, this algebra is generated

[15] The operator $(\varphi_2 - \varphi_1)^2$ can be expressed in terms of E, F, H and complex masses, but simply represents the Coulomb branch operator $\mathrm{Tr}\varphi^2$.

by x_i, y_i for $i = 1, 2$ and Σ subject to the following relations:

$$\Sigma^2 = 1 \qquad \Sigma x_1 \Sigma^{-1} = x_2 \qquad \Sigma y_1 \Sigma^{-1} = y_2$$
$$[y_i, x_j] = \begin{cases} -\varepsilon + (m_{\mathbb{C}} - \frac{1}{2}\varepsilon)\Sigma & \text{if } i = j \\ -(m_{\mathbb{C}} - \frac{1}{2}\varepsilon)\Sigma & \text{if } i \neq j \end{cases} \tag{2.4.79}$$

where $\varepsilon, m_{\mathbb{C}}$ are complex parameters. The generator Σ should be thought of as an element of the group algebra of S_2 extending the Weyl algebras generated by the x_i, y_i. In particular, the element $e = \frac{1}{2}(1 + \Sigma)$ can be used to project onto symmetric elements. Correspondingly, the spherical subalgebra is simply the subalgebra generated by elements of the form $e...e$. The generators of the spherical subalgebra described in Section 1.4.2 can be identified as

$$\begin{gathered} Q^1 = e(y_1 + y_2)e \qquad Q^2 = e(x_1 + x_2)e \\ E = \tfrac{1}{2}e(y_1^2 + y_2^2)e \qquad H = \tfrac{1}{2}e(x_1 y_1 + y_1 x_1 + x_2 y_2 + y_2 x_2)e \quad F = -\tfrac{1}{2}e(x_1^2 + x_2^2)e \end{gathered}\,. \tag{2.4.80}$$

A particularly useful, non-redundant set of generators for the rational Cherednik algebra are Σ and the operators $\lambda = \Sigma x_1, \tau = \Sigma y_1$. Although this choice makes the relations somewhat less transparent (see *e.g.* [253, Section 3]), we will see that the actions of λ and τ are much more simple than those of the x_i and y_i.

Boundary local operators

We work with the same conventions as in Section 2.4.1. Indeed, the only difference between this analysis and that of the Iwahori line is the additional zero in R_0. In the presence of the line operator $\mathbb{V}_{\text{RCA}}$, we have

$$\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial, I_\partial}; \mathbb{V}_{\text{RCA}}) \simeq \mathcal{I}\backslash\big(R_0 \cap G(\mathcal{K}) \cdot (X_\partial, I_\partial)\big)\,. \tag{2.4.81}$$

This breaks up into components $\mathcal{M}^{\mathfrak{n}}_{\mathbb{D}}(\mathcal{B}_{X_\partial, I_\partial}; \mathbb{V}_{\text{RCA}})$ and admits a cell decomposition labeled by A, σ with $A_1 + A_2 = \mathfrak{n}$, which take the same form as those found in Section 2.4.1; the only difference between the two situations is the modes that are allowed to be non-zero (and thus the allowed values of A).

We find that $\sigma = 1$ requires $A_1 \geq A_2 \geq 0$ and $\sigma = w$ requires $A_1 - 1 \geq A_2 \geq 0$. The coordinates transform under $T_{\mathbb{C}} \times \mathbb{C}^*_F \times \mathbb{C}^*_\varepsilon$ just as before, and there is once again a (unique) fixed point at the origin of each cell. Putting everything together, we find that the space of local operators at the junction of $\mathbb{V}_{\mathrm{RCA}}$ and the boundary condition $\mathcal{B}_{X_\partial, I_\partial}$ is given by

$$H_\bullet^{\mathbb{C}^*_F \times \mathbb{C}^*_\varepsilon}(\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{X_\partial, I_\partial}; \mathbb{V}_{\mathrm{RCA}})) = \bigoplus_{\sigma \in \{1, w\}} \bigoplus_{A_1 - \delta_{\sigma w} \geq A_2 \geq 0} \mathbb{C} \left| A, \sigma \right\rangle , \tag{2.4.82}$$

where $|A, \sigma\rangle$ is the fundamental class of the fixed point at the cell labeled by A, σ.

Local operators on $\mathbb{V}_{\mathrm{RCA}}$

Finally, we move to the operator algebra itself. As was the case with operators on the Iwahori line, we once again have operators labeled by $\mathcal{I}$-equivariant homology classes of $\mathcal{I}' \backslash G(\mathcal{K})$. We will identify operators that generate the rational Cherednik algebra for $\mathfrak{gl}(2, \mathbb{C})$, namely Σ, λ, and τ.

We begin with the $\mathbb{CP}^1$ cycle coming from $\overline{\mathcal{I}' \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mathcal{I}} \subset G(\mathcal{K})$. Just as before, there are two fixed points and we again denote the pulled-back class by $[1], [w]$. However, these classes do not fit into correspondences over the entire space of X's and I's, *i.e.* the fixed point classes $[1]$ and $[w]$ will have non-trivial excess intersection factors. One finds that

$$[w] \left| A, 1 \right\rangle = (A_1 - A_2)\varepsilon \left| A, w \right\rangle \qquad [w] \left| A, w \right\rangle = \big(2m_{\mathbb{C}} - (1 + A_1 - A_2)\varepsilon\big) \left| A, 1 \right\rangle \tag{2.4.83}$$

and, somewhat more delicately[16]

$$[1] \left| A, \sigma \right\rangle = \big(m_{\mathbb{C}} - \tfrac{1}{2}\varepsilon\big) \left| A, \sigma \right\rangle . \tag{2.4.84}$$

[16]The fixed point 1 does not require any tangent vectors to vanish. Instead, the first order deformation away from the fixed point *does* require the z^0 term in the ${}^1{}_1$ component of the adjoint tangent vectors to vanish.

Putting this together, we arrive at an operator from the fundamental class of this $\mathbb{CP}^1$

$$\Sigma = \frac{[1]-[w]}{\varphi_2 - \varphi_1} \tag{2.4.85}$$

which acts as

$$\begin{aligned}
\Sigma\,|A,1\rangle &= \frac{\big(m_{\mathbb{C}} - \frac{1}{2}\varepsilon\big)\,|A,1\rangle - (A_1 - A_2)\varepsilon\,|A,w\rangle}{m_{\mathbb{C}} - (A_1 - A_2 + \frac{1}{2})\varepsilon} \\
\Sigma\,|A,w\rangle &= \frac{\big(m_{\mathbb{C}} - \frac{1}{2}\varepsilon\big)\,|A,w\rangle - \big(2m_{\mathbb{C}} - (A_1 - A_2 + 1)\varepsilon\big)\,|A,1\rangle}{(A_1 - A_2 + \frac{1}{2})\varepsilon - m_{\mathbb{C}}}\,,
\end{aligned} \tag{2.4.86}$$

from which it follows that $\Sigma^2 = 1$. Note that this is a dramatically different action from the *same* orbit in the affine flag variety $\mathcal{I}'\backslash G(\mathcal{K})$ found in the previous section (where it lead to the operator θ with $\theta^2 = 0$).

The remaining operators we are interested in have nonzero monopole number. First consider the $\overline{\mathcal{I}'\left(\begin{smallmatrix} 0 & 1 \\ z & 0 \end{smallmatrix}\right)\mathcal{I}} \subset G(\mathcal{K})$; this leads to a single point in $\mathcal{I}'\backslash G(\mathcal{K})$ and it leads to the operator τ. We find that its action on the $|A,\sigma\rangle$ is given by

$$\begin{aligned}
\tau\,|A,1\rangle &= |A+(1,0),w\rangle\,, \\
\tau\,|A,w\rangle &= |A+(0,1),1\rangle\,.
\end{aligned} \tag{2.4.87}$$

Similarly, the $\overline{\mathcal{I}'\left(\begin{smallmatrix} 0 & z^{-1} \\ 1 & 0 \end{smallmatrix}\right)\mathcal{I}} \subset G(\mathcal{K})$ leads to a single point in $\mathcal{I}'\backslash G(\mathcal{K})$ and the operator λ whose action is

$$\begin{aligned}
\lambda\,|A,1\rangle &= A_2\varepsilon\,|A+(0,-1),w\rangle\,, \\
\lambda\,|A,w\rangle &= \big((A_1 + \tfrac{1}{2})\varepsilon - m_{\mathbb{C}}\big)\,|A+(-1,0),1\rangle\,.
\end{aligned} \tag{2.4.88}$$

It is relatively straightforward to check that these operators satisfy the relations spelled out in [253, Section 3]. Moreover, the action provided above exactly agrees with that of the rational Cherednik algebra on its polynomial representation, *cf.* [253, Theorem 4.15].

Chapter 3

HOMFLY-PT from 3d $\mathcal{N} = 4$ Gauge Theory

This chapter aims to describe a new physical framework for constructing HOMFLY-PT link homology. The existence of this triply-graded homology theory was first predicted by Gukov-Schwarz-Vafa and Dunfield-Gukov-Rasmussen [253,254] using M-theory on the resolved conifold, and given its first mathematical definition by Khovanov-Rozansky [251]. Several other conjectural mathematical constructions of HOMFLY-PT homology have since appeared, including

1) In the special case of algebraic knots, Oblomkov-Rasmussen-Shende [99], proposed that HOMFLY-PT homology may be realized as the homology of a related geometric space.

1a) In the case of positive torus knots, the construction 1) can be enhanced (quantized) further to include the representation theory of a rational Cherednik algebra [100].

2) Recent work of Oblomkov-Rozansky [94, 95], involving a representation of the braid group in a monoidal category of matrix factorizations. A closely related (presumed equivalent) construction appears in Gorsky-Neguţ-Rasmussen [255], based on Soergel bimodules.

The construction 2) has already been formulated in terms of 3d TQFT [96,97]; specifically, HOMFLY-PT homology appears as a Hilbert space in B-twisted 3d $\mathcal{N}=4$ gauge theory. We review the physics of construction in Section 3.1.2. Roughly, for a link K realized as the closure of an n-strand braid $K=\overline{\beta}$, Oblomkov-Rozansky consider the rank n ADHM quiver gauge theory given in Figure 1, construct a 2d $\mathcal{N}=(2,2)$ boundary condition $\mathcal{B}_{\overline{\beta}}$ encoding the braid closure, and then realize the "k-th row" of HOMFLY-PT homology of K as the B-twisted Hilbert space $\mathcal{H}_B(\mathcal{B}_{\overline{\beta}},\mathcal{L}_k)$ on a disk D with $\mathcal{B}_{\overline{\beta}}$ on ∂D and a line operator $\mathcal{L}_k$ at $0\in D$. This Hilbert space has many alternative descriptions, *cf.* Section 2.1.3, and a particularly useful one will be as the vector space of local operators at the junction of $\mathcal{L}_k$ and $\mathcal{B}_{\overline{\beta}}$ in the half-space setup.

The goal of this chapter is to show that constructions 1) and 1a) also arise from 3d TQFT; in fact, these constructions are related to construction 2) via 3d mirror symmetry [74–76], as will be described in detail the upcoming [91]. In Section 3.2, we translate each of the ingredients of construction 2) through 3d mirror symmetry and recombine them to obtain another physical construction of HOMFLY-PT homology in terms of an A-twisted Hilbert space $\mathcal{H}_A(\mathcal{B}^!_{\overline{\beta}},\mathcal{L}^!_k)$ of the *same* theory. Again, there are several possible realizations of this A-twisted Hilbert space. Upon specializing to positive algebraic knots, the corresponding computation in the half-space setup [51] exactly reproduces the construction 1) of [100]. A related realization of the A-twisted Hilbert space involves a (conjectural) representation of the braid group in the full category of line operators $\mathcal{C}_A\simeq D\text{−mod}_{GL(n,\mathcal{K})}(\mathfrak{gl}(n,\mathcal{K})\oplus\mathcal{K}^n)$, which we only schematically describe in Section 3.2.3, leaving further investigation to future work.

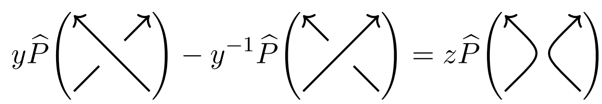
The organization of this chapter is as follows. In Section 3.1 we review various aspects of the HOMFLY-PT invariants as well as the mathematical constructions 1) and 1a) and the physical realization of construction 2). The subsequent section (Section 3.2) translates the various ingredients in construction 2) through 3d mirror symmetry and shows that the half-space setup realizes construction 1) for positive algebraic links. Finally, in Section 3.3 we consider the torus knot $T_{(n,m)}$ as an explicit example and apply the generalized affine Springer theory discussion of Section 2.3.4 to the A-twist realization of construction 1). We show that this physically realizes the construction 1a) for the lowest row of HOMFLY-PT homology and sketch how to understand the representation-theoretic description for the higher rows of HOMFLY-PT homology.

Section 3.1, Section 3.2.1, Section 3.2.3, and Section 3.3.2 are adapted from the upcoming work [91]. Section 3.2.2 and Section 3.3.1 are adapted from [51].

3.1 HOMFLY-PT

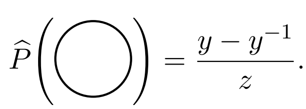
We start with some basic facts about the HOMFLY-PT polynomial and its corresponding triply-graded homology theory.

The HOMFLY-PT polynomial [92, 93] of an oriented link K is a two-variable Laurent polynomial function $\widehat{P}(K)(y,z)$ with integer coefficients, which may be defined by the skein relation



(3.1.1)

and the normalization



(3.1.2)

Under disjoint union it satisfies[1] $\widehat{P}(K \sqcup K') = \widehat{P}(K)\widehat{P}(K')$, with K and K' assumed to be contained in disjoint 3-balls. The HOMFLY-PT polynomial is related to the $SU(N)$ quantum invariant of K, colored by the fundamental representation, upon specializing

$$y = q^{\frac{N}{2}}, \quad z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}\,. \tag{3.1.3}$$

From a physical perspective, this is an expectation value of a fundamental Wilson line in $SU(N)$ Chern-Simons theory at level κ, where $q = e^{\frac{2\pi i}{\kappa+N}}$ [41]. In what follows, we will be interested in the partial specialization $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, and we will denote $y = a^{\frac{1}{2}}$.

[1]A more standard normalization in the mathematics literature is $\widehat{P}(\text{unknot}) = 1$. This slightly modifies the disjoint-union formula. However, the natural normalization for Chern-Simons theory [41], topological strings, and the connections in this chapter is as above.

The rational function $\widehat{P}(K)(a^{\frac{1}{2}}, q^{\frac{1}{2}} - q^{-\frac{1}{2}})$ has a well-defined Laurent-series expansion around $q = 0$, with integer coefficients and integer powers of q and a aside from (possibly) an overall factor of $q^{\frac{1}{2}}$ and/or $a^{\frac{1}{2}}$, which we denote

$$P(K)(a,q) := \widehat{P}(K)(a^{\frac{1}{2}}, q^{\frac{1}{2}} - q^{-\frac{1}{2}})\Big|_{\text{near } q\,=\,0}. \tag{3.1.4}$$

If we have a diagram for K, this property follows noting that $\widetilde{P}(K) = q^{c(K)} a^{\mathrm{wr}(K)} \widetilde{P}(K)$, for $c(K)$ the number of components of K and $\mathrm{wr}(K)$ is the writhe of the diagram,[2] satisfies a skein relation that only depends on $q^{\pm 1}$ and $a^{\pm 1}$. For example,

$$\begin{aligned} P\left(\bigcirc\right) &= \frac{a^{\frac{1}{2}} - a^{-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\Bigg|_{\text{near } q\,=\,0} = a^{-\frac{1}{2}} q^{\frac{1}{2}} \frac{1-a}{1-q}\Bigg|_{\text{near } q\,=\,0} \\ &= a^{-\frac{1}{2}} q^{\frac{1}{2}} (1-a)(1 + q + q^2 + q^3 + \ldots) \end{aligned} \tag{3.1.5}$$

The various versions of HOMFLY-PT homology that we consider will categorify the series $P(K)$.

To an oriented link K, HOMFLY-PT homology associates a $\mathbb{Z}^3$-graded vector space (over $\mathbb{C}$) equipped with a differential Q of degree $(1,0,0)$ (up to quasi-isomorphism), whose cohomology we denote $H^\bullet(K)$. The cohomology has finite graded dimensions. Denoting the three degrees by (R,k,d), one recovers the HOMFLY-PT polynomial as a graded Euler character,

$$a^{-\frac{\#}{2}} q^{-\frac{\#}{2}} P(K) = \mathrm{Tr}_{H^\bullet(K)} a^k q^d (-1)^R = \sum_{(R,k,d)\in\mathbb{Z}^3} \dim_{R,k,d} H^\bullet(K) a^k q^d (-1)^R. \tag{3.1.6}$$

Different constructions of HOMFLY-PT homology produce different models underlying $H^\bullet(K)$. Any such model has a differential Q with degrees $R(Q) = 1$, $k(Q) = 0$, $d(Q) = 0$. (Thus, the R-grading is cohomological, while k and d are auxiliary gradings, preserved by the

[2] The writhe of an an oriented link diagram is the difference in the number of positive and negative crossings. Importantly, the 1st Reidemeister move changes the writhe of a knot diagram; in particular, the writhe of a given diagram can be changed to any integer without changing the underlying link.

differential.) The particular geometric models that we discuss in this chapter will split into a direct sum over finitely many $k \in \mathbb{Z}$, with corresponding cohomology $H^\bullet_k(K)$,

$$H^\bullet(K) = \bigoplus_{k\in\mathbb{Z}} H^\bullet_k(K)\,, \qquad a^{-\frac{\#}{2}} q^{-\frac{\#}{2}} P(K) = \sum_{k\in\mathbb{Z}} a^k \operatorname{Tr}_{H^\bullet_k(K)} q^d (-1)^R\,. \tag{3.1.7}$$

The summands $H^\bullet_k$ are sometimes called "rows" of HOMFLY-PT homology.

3.1.1 HOMFLY-PT from singular curves and Cherednik algebras

The first two (conjectural) constructions of HOMFLY-PT homology we consider in this thesis apply to successively more restrictive classes of knots and links. The first construction, due to Oblomkov-Rasmussen-Shende [99], relates the lowest row of HOMFLY-PT homology for a (positive) algebraic link K to the Borel-Moore homology of a geometric space called the Hilbert scheme of points on an algebraic curve $Z_K \subset \mathbb{C}^2$ encoding the link K. The higher rows of HOMFLY-PT homology are obtained by a generalization of the Hilbert scheme of points, called "incidence varieties" in [99].

The second construction, due to Gorsky-Oblomkov-Rasmussen-Shende [100], specializes yet further to the (n, m) torus *knot* and relates their HOMFLY-PT homology to the representation theory of the rational Cherednik algebra for $\mathfrak{sl}(n, \mathbb{C})$ (at the parameter m/n). These two constructions are related to one another through the results of Oblomkov-Yun [247] using affine Springer theory.

The ORS conjecture and Hilbert schemes of plane curve singularities

We start with the (conjectural) realization for HOMFLY-PT homology due to Oblomkov-Rasmussen-Shende (ORS) [99]. Their construction applies to a special class of links, which will be the main focus of Section 3.2, called positive algebraic links.

A positive algebraic link K is homotopic to the intersection of an infinitesimally small S^3, given by $\{|z|^2 + |w|^2 = r\} \in \mathbb{C}^2$, and a (co)dimension 1 hypersurface $Z_K \subset \mathbb{C}^2$,

$$K \simeq \{|z|^2 + |w|^2 = r\} \cap Z_K\,, \tag{3.1.8}$$

where $Z_K = \{p_K = 0\}$ is the vanishing set of $p_K \in \mathbb{C}[z,w]$. We will implicitly assume that $(0,0) \in Z_K$, *i.e.* p_K has no constant term. For sufficiently small r, the topology of K obtained by this intersection only depends on the topological type of singularity of p_K at the origin. For example, the trefoil knot $\mathbf{3_1}$ may be realized this way by taking $p_{\mathbf{3_1}} = w^2 - z^3$.

More generally, *algebraic* links are associated in the same way to a Laurent polynomial p_K. A particularly special class of these links called the (n,m) torus links, which are links with $\gcd(n,m)$ components, come from $p_K = w^n - z^m$. All algebraic links are cabled torus links [256] — they constitute a relatively small but nonetheless interesting class of links; for example, they are never hyperbolic.

The work [99] conjectures that the HOMFLY-PT homology of a positive algebraic link K, realized as the link of a singular plane curve Z_K, can be recovered from the algebraic geometry of Z_K near the singularity. In essence, the conjecture of ORS says that the algebraic geometry of Z_K near the singularity contains enough information to extract the knot's HOMFLY-PT homology. Their explicit formulae for torus *knots* agree with the known HOMFLY-PT homologies computed by Hogancamp-Mellit [257], but the conjecture is still unproven for more general algebraic links.

Since we will mostly be interested in the behavior of a single singularity, it suffices in most circumstances to consider formal series $p_K \in \mathbb{C}[\![z,w]\!]$; the vanishing locus viewed in formal series, denoted $\widehat{Z}_K$, is called the "germ of the singularity." Somewhat more precisely, let $R_K = \mathbb{C}[\![z,w]\!]/(p_K)$ be the ring of functions of the germ of the singularity $\widehat{Z}_K$. The Hilbert scheme of d points on $\widehat{Z}_K$ is then the moduli space of ideals $I \subseteq R_K$ such that $\dim_{\mathbb{C}} R_K/I = d$

$$\widehat{Z}_K^{[d]} = \mathrm{Hilb}^d(\widehat{Z}_K) = \{I \subset R_K | I \text{ is a codimension } d \text{ ideal}\}. \tag{3.1.9}$$

The Hilbert scheme of d points is a resolution of the singular space $\widehat{Z}_K^d/S_d$ and is therefore an algebraic incarnation of the moduli space of d points on $\widehat{Z}_K$, smoothed at coincident points.

Let M denote the ideal corresponding to the point $z = w = 0$, *i.e.*, $M = (z,w)$. With these ingredients, the work [99] further considers the moduli space of nested ideals, called “incidence varieties” in [99],

$$\widehat{Z}_K^{[d\leq d+k]}\{(I,J)\in \widehat{Z}_K^{[d]}\times \widehat{Z}_K^{[d+k]}|M\cdot J\subset I\subset J\}. \tag{3.1.10}$$

The case of $k=0$ agrees with the Hilbert scheme of d points at the singularity. The ORS conjecture then implies that the HOMFLY-PT polynomial (up to an overall factor) can be obtained from (an algebraic analog of) the graded Euler character of these incidence varieties:

$$a^{-\#/2}q^{-\#/2}P(K)=\sum_{k,d\geq 0} a^k q^d \chi\big(\widehat{Z}_K^{[d\leq d+k]}\big). \tag{3.1.11}$$

The GORS conjecture and rational Cherednik algebras

The third and final realization of the HOMFLY-PT homology that will be of interest to this thesis is due to Gorsky-Oblomkov-Rasmussen-Shende [100]. Their model only applies to the special case of $K=K_{(n,m)}$ is a positive (n,m) torus *knot*, *i.e.* $\gcd(n,m)=1$ with $n,m\geq 1$. In particular, they recover the (reduced) HOMFLY-PT polynomial from characters of the rational Cherednik algebra for $\mathfrak{sl}(n,\mathbb{C})$ (at parameter m/n), and conjecture that the (reduced) homology can be recovered from the representations themselves.

We start by reviewing properties of the necessary rational Cherednik algebras, for a more thorough review see [184] and references therein. The rational Cherednik algebra for $\mathfrak{gl}(n,\mathbb{C})$, denoted $\overline{\mathcal{H}}_n$, is generated by x_i, y_i and Σ for $i=1,...,n$ and $\Sigma\in S_n$ subject to the following relations:

$$\Sigma x_i\Sigma^{-1}=x_{\Sigma(i)}\,,\quad \Sigma y_i\Sigma^{-1}=y_{\Sigma(i)}\,,\quad [y_i,x_j]=\begin{cases}-\varepsilon+(m_{\mathbb{C}}-\frac{1}{2}\varepsilon)\sum_{k\neq i}(i\,k) & \text{if } i=j\\ -(m_{\mathbb{C}}-\frac{1}{2}\varepsilon)\,(i\,j) & \text{if } i\neq j\end{cases}, \tag{3.1.12}$$

where $(i\,j)\in S_n$ is the transposition that exchanges i and j, and $\varepsilon, m_{\mathbb{C}}$ are complex parameters. The algebra has a symmetry rotating x_i with charge -1, Σ with charge 0, and y_i with charge 1, *i.e.* the algebra is graded by polynomial degree in y minus polynomial degree in x.

In fact, this is an inner grading:

$$H = \tfrac{1}{2}\sum_{i=1}^{n}(x_i y_i + y_i x_i) \quad \rightsquigarrow \quad [H, x_i] = -\varepsilon x_i\,, \quad [H, \Sigma] = 0\,, \quad [H, y_i] = \varepsilon y_i\,. \tag{3.1.13}$$

In addition to being graded by H eigenspaces, the algebra has a filtration $\overline{\mathcal{F}}^{\text{poly}}$ by total polynomial degree:

$$\overline{\mathcal{F}}_i^{\text{poly}} = \left\{ \begin{array}{c} \text{elements of } \overline{\mathcal{H}}_n \text{ that can be written} \\ \text{as a sum of terms of polynomial degree at most } i \end{array} \right\}. \tag{3.1.14}$$

As a vector space, the algebra $\overline{\mathcal{H}}_n$ admits a PBW-type decomposition as

$$\overline{\mathcal{H}}_n \simeq \mathbb{C}[x_1, ..., x_n] \otimes \mathbb{C}[S_n] \otimes \mathbb{C}[y_1, ..., y_n]\,, \tag{3.1.15}$$

from which many analogies between the theory of semisimple Lie algebras arises [184]. The subalgebra $\mathbb{C}[y_1, ..., y_n] \rtimes \mathbb{C}[S_n]$ is analogous to a Borel subalgebra and we can use it to build representations of $\overline{\mathcal{H}}_n$ from representations of S_n, *cf.* building representations of a semisimple Lie algebra by applying raising operators (the x's) to lowest weight vectors (the S_n representation). We choose a representation ρ of S_n and declare that $\mathbb{C}[y_1, ..., y_n]$ acts trivially on the representation. We then define the Verma module

$$\overline{M}(\rho) = \overline{\mathcal{H}}_n \otimes_{\mathbb{C}[y_1,...,y_n] \rtimes \mathbb{C}[S_n]} \rho \simeq \mathbb{C}[x_1, ..., x_n] \otimes \rho\,, \tag{3.1.16}$$

where the last isomorphism is as vector spaces and uses the PBW-type decomposition of $\overline{\mathcal{H}}_n$. For generic $\varepsilon, m_{\mathbb{C}}$ the Verma module $\overline{M}(\rho)$ has no non-trivial submodules but such submodules can appear for non-generic values. We will denote the irreducible quotient of $\overline{M}(\rho)$ by $\overline{L}(\rho)$.

Within a representation V of $\overline{\mathcal{H}}_n$, the vectors v such that $x_i.v = 0$ for all i are called "singular vectors," and are analogs of the lowest weight vectors in the representation theory of semisimple Lie algebras. Importantly, the subspace of singular vectors admits a natural S_n action and if $\rho \subset V$ is a subspace of singular vectors that transform in the S_n representation ρ then V admits a unique $\overline{\mathcal{H}}_n$ homomorphism $\overline{M}(\rho) \to V$.

The rational Cherednik algebra for $\mathfrak{sl}(n, \mathbb{C})$, denoted $\mathcal{H}_n$, differs from $\overline{\mathcal{H}}_n$ only by removing a pair of operators $X = x_1 + ... + x_n$ and $Y = y_1 + ... + y_n$ that generate a (rescaled) copy of the Weyl algebra, or ε-differential operators on $\mathbb{C}$:

$$\overline{\mathcal{H}}_n \simeq \mathcal{H}_n \otimes D_\varepsilon(\mathbb{C}) \,, \qquad D_\varepsilon(\mathbb{C}) = \mathbb{C}\langle X, Y \rangle / ([X, Y] = n\varepsilon). \tag{3.1.17}$$

The subalgebra $\mathcal{H}_n$ is generated by the elements $x_i' = x_i - \frac{1}{n}X$ and $y_i' = y_i - \frac{1}{n}Y$ together with $\mathbb{C}[S_n]$.

Representations of $\mathcal{H}_n$ can be obtained by an induction procedure identical to the above. We denote the Verma modules by $M(\rho)$ and their irreducible quotients by $L(\rho)$. The isomorphism in Eq. (3.1.17) implies that the Verma modules and their irreducible quotients are related as follows:

$$\overline{M}(\rho) \simeq M(\rho) \otimes \mathbb{C}[X] \,, \qquad \overline{L}(\rho) \simeq L(\rho) \otimes \mathbb{C}[X]. \tag{3.1.18}$$

When the parameters are specialized to $m_{\mathbb{C}} - \frac{1}{2}\varepsilon = \frac{m}{n}\varepsilon$, where m a positive integer with $\gcd(n, m) = 1$, the algebra $\mathcal{H}_n$ has a unique finite-dimensional irreducible representation: the irreducible quotient module $L_{m/n} := L(\mathbb{C})$, where $\mathbb{C}$ is the trivial representation of S_n [258]. Singular vectors are defined in the same way as above and satisfy the same properties with respect to homomorphisms from the Verma modules $M(\rho)$.

Within the rational Cherednik algebras $\overline{\mathcal{H}}_n$ and $\mathcal{H}_n$, there is a subalgebra isomorphic to the group algebra $\mathbb{C}[S_n]$. Within this subalgebra is the symmetrizing idempotent e, from which we define the "spherical subalgebras"

$$\overline{\mathcal{H}}_n^{\text{sph}} := e\overline{\mathcal{H}}_n e \subset \overline{\mathcal{H}}_n \qquad \mathcal{H}_n^{\text{sph}} := e\mathcal{H}_n e \subset \mathcal{H}_n \,. \tag{3.1.19}$$

In the limit $\varepsilon \to 0$, the spherical subalgebra $\overline{\mathcal{H}}_n^{\mathrm{sph}}$ is actually a commutative ring that can be identified with the ring of holomorphic functions on the Hilbert scheme of n points on $\mathbb{C}^2$

$$\overline{\mathcal{H}}_n^{\mathrm{sph}} \overset{\varepsilon\to 0}{\longrightarrow} \mathbb{C}[\mathrm{Hilb}^n(\mathbb{C}^2)]\,. \tag{3.1.20}$$

In fact, $\mathrm{Hilb}^n(\mathbb{C}^2)$ is holomorphic symplectic and $\overline{\mathcal{H}}_n^{\mathrm{sph}}$ is a deformation quantization thereof.

Modules for the spherical subalgebra arise by symmetrizing modules of the full algebra; in particular, $eL_{m/n}$ is the unique finite-dimensional module of the spherical subalgebra $\mathcal{H}_n^{\mathrm{sph}}$ when the parameters are specialized to $m_{\mathbb{C}} - \frac{1}{2}\varepsilon = \frac{m}{n}\varepsilon$. There is a similar notion of singular vectors for $\mathcal{H}_n^{\mathrm{sph}}$ with the expected properties as above.

Note that $\mathrm{Hom}_{S_n}(\wedge^k\Box, R)$, for R any S_n representation, is a functor that extracts the S_n submodules of R that transform as k-th antisymmetric tensor representation $\wedge^k\Box$, these are sometimes called the "isotypic components" of R. The work [100] shows that the HOMFLY-PT polynomial of the torus knot $K_{(n,m)}$ can be recovered as a graded character

$$\overline{P}(K_{(n,m)}) = \frac{a^{\frac{(n-1)(m-1)}{2}}(a^{\frac{1}{2}} - a^{-\frac{1}{2}})}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \sum_{k=0}^{n-1} a^{2k}\mathrm{Tr}_{\mathrm{Hom}_{S_n}(\wedge^k\Box, \overline{L}_{m/n})} q^H. \tag{3.1.21}$$

The GORS conjecture[3] states that $\overline{L}_{m/n}$ admits a filtration $\overline{\mathcal{F}}$ (compatible with the filtration $\overline{\mathcal{F}}^{\mathrm{poly}}$ on $\overline{\mathcal{H}}_n$) such that

$$H_k^{\bullet}(K_{(n,m)}) \simeq \mathrm{Hom}_{S_n}(\wedge^k\mathbb{C}^n, \mathrm{gr}^{\overline{\mathcal{F}}}\overline{L}_{m/n}), \tag{3.1.22}$$

where $\mathrm{gr}^{\overline{\mathcal{F}}}\overline{L}_{m/n}$ is the associated graded vector space with respect to the filtration $\overline{\mathcal{F}}$, *i.e.* the ith graded component is the quotient $\overline{\mathcal{F}}_i/\overline{\mathcal{F}}_{i-1}$. The role of the filtration $\overline{\mathcal{F}}$ is to identify the appropriate homological t-grading on impose on the representation $\overline{L}_{m/n}$.

Moreover, GORS show that this result is compatible with the conjecture of ORS through

[3] The actual conjecture of GORS [100, Conjecture 1.2] is formulated in terms of the reduced HOMFLY-PT homology and the module $L_{m/n}$ of the rational Cherednik algebra for $\mathfrak{sl}(n,\mathbb{C})$. This is a restatement in terms of the unreduced homology and the module $\overline{L}_{m/n}$ of the rational Cherednik algebra for $\mathfrak{gl}(n,\mathbb{C})$, *cf.* [100, Conjecture 5.6].

ideas in Springer theory proven by Oblomkov-Yun [247]. The first step is to note that, at least for the lowest a-degree, the (reduced) HOMFLY-PT polynomial can be recovered from the compactified Jacobian $\mathcal{J}_{K_{(n,m)}}$ for the curve $\widehat{Z}_{K_{(n,m)}}$. The compactified Jacobian $\mathcal{J}_{K_{(n,m)}}$ is the moduli space of rank 1, degree 0, torsion-free sheaves on $\widehat{Z}_{K_{(n,m)}}$, or equivalently, of degree 0 line bundles.

The compactified Jacobian $\mathcal{J}_{K_{(n,m)}}$ is known to arise from a certain affine Springer fiber for $SL(n,\mathbb{C})$ by viewing $\widehat{Z}_{K_{(n,m)}}$ as the spectral curve of an element $W_{K_{(n,m)}} \in \mathfrak{sl}(n,\mathcal{O})$, and the work of Yun shows that equivariant homology of a generalized affine Springer fiber (using the Iwahori subgroup) over the same element $W_{K_{(n,m)}}$, denoted $\widetilde{M}_{(n,m)}$, admits an action of the trigonometric Cherednik algebra for $\mathfrak{sl}(n,\mathbb{C})$ [259]. The analysis of [247] then shows that the homology of this affine Springer fiber admits a filtration P, called the perverse filtration, such that the corresponding associated graded vector space $\mathrm{gr}^P H^\bullet(\widetilde{M}_{(n,m)})$ realizes the module $L_{m/n}$ of the rational Cherednik algebra for $\mathfrak{sl}(n,\mathbb{C})$.

The Hilbert scheme of points $\widehat{Z}^d_{(n,m)}$ admits a natural map, the Abel-Jacobi map, to the compactified Jacobian $\mathcal{J}_{K_{(n,m)}}$ which is an isomorphism for sufficiently high d. Moreover, the above perverse filtration comes from these Abel-Jacobi maps [260, 261] and thus the work of [247] implies that the GORS conjecture (with a filtration $\overline{\mathcal{F}}$ determined by the perverse filtration P, see [100, Section 9.2]) is a corollary of the ORS conjecture.

The main take-away for us is the following. The case of $k = 0$ is special and corresponds to the S_n-symmetric part of $\overline{L}_{m/n}$, which is isomorphic to $e\overline{L}_{m/n}$. If we ignore the homological t-grading and only consider q-graded vector spaces, the above says that the $\overline{\mathcal{H}}^{\mathrm{sph}}_n$ module $e\overline{L}_{m/n}$ and the lowest row of HOMFLY-PT homology of $K_{(n,m)}$ are the same.

3.1.2 HOMFLY-PT from the 3d B-model

The final realization of HOMFLY-PT homology that will be of interest to this thesis is due to Oblomkov-Rozansky [94,95], and can be applied to more generally than those in Section 3.1.1. The basic conceptual idea behind the construction of Oblomkov-Rozansky is the following: for any n-strand braid β, they construct a complex of sheaves $\mathcal{S}_\beta$ on $\mathrm{Hilb}^n(\mathbb{C}^2)$ such that the space of global sections of this sheaf exactly reproduces the HOMFLY-PT homology of the closure of β. The explicit construction of the sheaf $\mathcal{S}_\beta$ is quite involved, but nonetheless has a natural interpretation in B-twisted 3d $\mathcal{N} = 4$ gauge theory [96, 97]. In particular, we will consider the rank n ADHM quiver gauge theory given by the quiver in Figure 1; *i.e.* $U(n)$ gauge theory with a single fundamental hypermultiplet and a single adjoint hypermultiplet.

The boundary condition $\mathcal{B}_n$ and boundary line operators

The main object of interest in the Oblomkov-Rozansky construction is certain boundary condition $\mathcal{B}_n$ for the rank n ADHM quiver gauge theory that preserves 2d $\mathcal{N} = (2, 2)$ supersymmetry. The boundary condition $\mathcal{B}_n$ is described as follows. First consider the 2d $\mathcal{N} = (2, 2)$ sigma model $\mathcal{T}_n^{2\mathrm{d}}$ with target space the cotangent bundle to the flag variety $T^*\mathrm{Fl}_n$, where $\mathrm{Fl}_n \simeq GL(n, \mathbb{C})/B$ for B a Borel subgroup. (This theory actually has $\mathcal{N} = (4, 4)$ supersymmetry as $T^*\mathrm{Fl}_n$ is hyperkähler.) This target space admits a $PGL(n, \mathbb{C}) = GL(n, \mathbb{C})/\mathbb{C}^*$ flavor symmetry implemented by holomorphic moment maps $\mu_{2\mathrm{d}}$. The boundary condition $\mathcal{B}_n$ is obtained by gauging the boundary $PGL(n, \mathbb{C})$ flavor symmetry with the 3d gauge fields and additionally introducing the bulk-boundary superpotential $W_\partial = \mathrm{Tr} X \mu_{2\mathrm{d}}$.

The boundary condition $\mathcal{B}_n$ depends on n twisted chiral parameters $\vec{\tau}$ corresponding to the $(n-1)$ Kähler parameters of $T^*\mathrm{Fl}_n$ and the 3d gauge coupling (complexified by a boundary θ-angle). To encode a link as a braid closure, we can wrap the boundary condition a cylinder and allow the parameters $\vec{\tau}$ to depend on the angular coordinate ϑ of the cylinder. For fixed ϑ, the vector $\vec{\tau}(\vartheta)$ describes the positions of n strands of a braid β and, since ϑ is an angular parameter, the functions $\vec{\tau}(\vartheta)$ naturally encode a braid closure $\overline{\beta}$. Since 2d B-model is insensitive to small variations in twisted chiral parameters (so long as they stay from singular configurations), this boundary condition should only depend on the topology of the closure. We denote the corresponding boundary condition $\mathcal{B}_{\overline{\beta}}$.

In the limit where the functions $\vec{\tau}(\vartheta)$ are constant outside a neighborhood of a single angle where the braid element β is concentrated, the boundary condition $\mathcal{B}_{\overline{\beta}}$ can be interpreted in a slightly more digestible fashion. In particular, the boundary condition $\mathcal{B}_{\overline{\beta}}$ can be thought of as the boundary condition $\mathcal{B}_n$ dressed by a boundary line operator $\mathcal{L}_\beta$ encoding the braid element. Alternatively, we could decompose the braid β into elementary braid moves b_i, b_i^{-1} and realize the boundary condition $\mathcal{B}_{\overline{\beta}}$ as being built from a system of boundary line operators $\mathcal{L}_{b_i}, \mathcal{L}_{b_i^{-1}}$ with $\mathcal{L}_\beta$ obtained by colliding the basic line operators. See Figure 3.1.

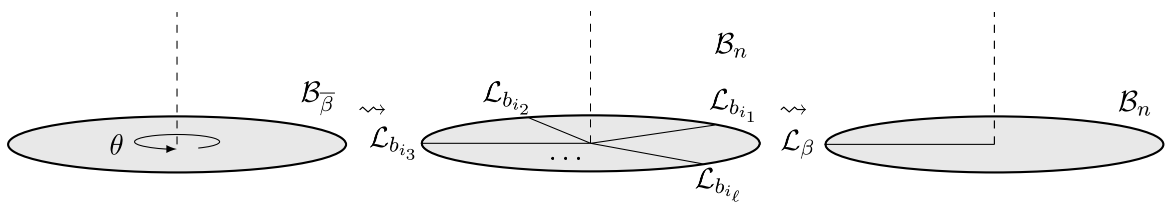


Figure 3.1: Different interpretations of the boundary condition $\mathcal{B}_{\overline{\beta}}$. Left: the boundary condition $\mathcal{B}_{\overline{\beta}}$ obtained from the boundary condition $\mathcal{B}_n$ by allowing the parameters $\vec{\tau}$ depend on the angular parameter θ and trace out the n-strand braid closure $\overline{\beta}$. Middle: a limit where all $\vec{\tau}$ is constant everywhere except finitely many angles $\theta_1, \theta_2, ..., \theta_\ell$ where the strands are permuted by the elementary braid moves $b_{i_1}^{\pm}, b_{i_2}^{\pm}, ..., b_{i_\ell}^{\pm}$, corresponding to boundary line operators $\mathcal{L}_{b_{i_1}}, \mathcal{L}_{b_{i_2}}, ..., \mathcal{L}_{b_{i_\ell}}$. Right: a limit where $\vec{\tau}$ is constant except at a single angle where the parameters rapidly undergo the braid β, corresponding to the boundary line operator $\mathcal{L}_\beta = \mathcal{L}_{b_{i_1}} \otimes \mathcal{L}_{b_{i_2}} \otimes \mathcal{L}_{b_{i_\ell}}$.

From a categorical description of the underlying 3d TQFT, the boundary condition $\mathcal{B}_n$ determines an object in the 2-category of boundary conditions, and line operators bound to the boundary condition $\mathcal{B}_n$ realize a monoidal 1-category of endomorphisms of the object $\mathcal{B}_n$; the monoidal structure comes from colliding boundary line operators. The above analysis suggests that every n-strand braid β should yield a element $\mathcal{L}_\beta$ in the 1-category of line operators on $\mathcal{B}_n$ and, moreover, the collision of line operators $\mathcal{L}_\beta$ and $\mathcal{L}_{\beta'}$ is equivalent to $\mathcal{L}_{\beta\beta'}$.

Oblomkov-Rozansky construction

The work of Oblomkov-Rozansky can be interpreted as providing an explicit model for the monoidal 1-category $\mathcal{C}_{\mathcal{B}_n}$ of boundary line operators on $\mathcal{B}_n$, together with the specific objects $\mathcal{L}_\beta$ realizing the categorical representation $\rho_n : \mathrm{Br}_n \to \mathcal{C}_{\mathcal{B}_n}$. The category $\mathcal{C}_{\mathcal{B}_n}$ is identified with a category of GL_n-equivariant matrix factorizations

$$\mathcal{C}_{\mathcal{B}_n} = \mathrm{MF}_{GL_n}((\mathfrak{gl}(n,\mathbb{C}) \times \mathbb{C}^n \times T^*\mathrm{Fl}_n \times T^*\mathrm{Fl}_n)^{\mathrm{st}}, W), \tag{3.1.23}$$

where the superpotential $W = \mu_1 - \mu_2$ is the difference of the GL_n moment maps acting on the separate $T^*\mathrm{Fl}_n$, and the superscript $^{\mathrm{st}}$ denotes a stability condition. In order to make contact with sheaves on $\mathrm{Hilb}^n(\mathbb{C}^2)$, Oblomkov-Rozansky use a categorical Chern-character map $\mathrm{ch}_{\mathcal{B}_n} : \mathcal{C}_{\mathcal{B}_n} \to D^{\mathrm{per}}(\mathrm{Hilb}^n(\mathbb{C}^2))$ described in [97]. (The category $D^{\mathrm{per}}(\mathrm{Hilb}^n(\mathbb{C}^2))$ is the derived category of (2-periodic) coherent sheaves on $\mathrm{Hilb}^n(\mathbb{C}^2)$.) Moreover, they are able to show that $\mathrm{ch}_{\mathcal{B}_n}(\mathcal{L}_\beta)$ only depends on the closure $\overline{\beta}$.

The final ingredient in recovering HOMFLY-PT homology is the tautological bundle $\mathcal{L}_{\mathrm{taut}}$ on $\mathrm{Hilb}^n(\mathbb{C}^2)$. The Oblomkov-Rozansky construction considers the (derived) tensor product $\mathcal{S}^k_{\overline{\beta}} = \mathrm{ch}_{\mathcal{B}_n}(\mathcal{L}_\beta) \otimes \wedge^k \mathcal{L}_{\mathrm{taut}}$ and then realizes the HOMFLY-PT homology of $\overline{\beta}$ as the total cohomology of this complex of sheaves, *i.e.* the space of extensions between $\mathcal{S}_\beta$ the structure sheaf $\mathrm{Ext}^\bullet(\mathcal{S}_\beta, \mathcal{O}_{\mathrm{Hilb}^n(\mathbb{C}^2)})$. Equivalently, by the Hom-tensor adjunction, one could consider the space of extensions between $\mathrm{ch}_{\mathcal{B}_n}(\mathcal{L}_\beta)$ and $\wedge^k \mathcal{L}^\vee_{\mathrm{taut}}$:

$$H^\bullet_k(\overline{\beta}) = \mathrm{Ext}^\bullet(\mathcal{S}^k_{\overline{\beta}}, \mathcal{O}_{\mathrm{Hilb}^n(\mathbb{C}^2)}) = \mathrm{Ext}^\bullet(\mathrm{ch}_{\mathcal{B}_n}(\mathcal{L}_\beta), \wedge^k \mathcal{L}^\vee_{\mathrm{taut}}). \tag{3.1.24}$$

Physical interpretation

The expressions in Eq. (3.1.24) have a natural physical interpretation. As discussed in [68], the category of line operators in the B-twist of an $\mathcal{N} = 4$ theory (with smooth Higgs branch) corresponds to the derived category of (2-periodic[4]) coherent sheaves on the Higgs branch, which in the present case is exactly $\mathrm{Hilb}^n(\mathbb{C}^2)$. Thus, the categorical Chern-character $\mathrm{ch}_{\mathcal{B}_n}$ should be interpreted as a functor from the category of boundary line operators to the category

[4]The B-twist of a general $\mathcal{N} = 4$ σ-model requires the target to be a hyperkähler manifold [116], and is naturally $\mathbb{Z}/2\mathbb{Z}$ grading by fermion number. When the theory has realizes $U(1)_H \subset SU(2)_H$ R-symmetry, this $\mathbb{Z}/2\mathbb{Z}$ can be enhanced to a full $\mathbb{Z}$-grading; this arises from a choice of $U(1)$ isometry of the hyperkähler target that scales the holomorphic symplectic form Ω (for a complex structure that is invariant under the $U(1)$ isometry) has weight 2.

of bulk line operators.

The bulk line operator $\mathrm{ch}_{\mathcal{B}_n}(\mathbb{1}_{\mathcal{B}_n})$, where $\mathbb{1}_{\mathcal{B}_n}$ is the trivial boundary line operator, is simply the "wrapped boundary condition" in Section 2.1:

$$\mathrm{ch}_{\mathcal{B}_n}(\mathbb{1}_{\mathcal{B}_n}) = \mathrm{ch}(\mathcal{B}_n)\,. \tag{3.1.25}$$

Similarly, the bulk line operator $\mathrm{ch}_{\mathcal{B}_n}(\mathcal{L}_\beta)$ corresponds to the wrapping the boundary condition $\mathcal{B}_n$ dressed by the boundary line operator $\mathcal{L}_\beta$. Alternatively, $\mathrm{ch}_{\mathcal{B}_n}(\mathcal{L}_\beta)$ can be interpreted as wrapping the boundary condition $\mathcal{B}_{\overline{\beta}}$:

$$\mathrm{ch}_{\mathcal{B}_n}(\mathcal{L}_\beta) = \mathrm{ch}(\mathcal{B}_{\overline{\beta}})\,. \tag{3.1.26}$$

The tautological bundle $\mathcal{L}_{\mathrm{taut}}$ is identified with a $\frac{1}{2}$-BPS Wilson line in the fundamental representation of $U(n)$; the algebra of local operators bound to this Wilson line is identified with bundle endomorphisms and, conversely, the space of global sections of this bundle is identified with the local operators on which this fundamental Wilson line can end. The structure sheaf $\mathcal{O}_{\mathrm{Hilb}^n(\mathbb{C}^2)}$ is similarly identified with the trivial bulk line operator $\mathbb{1}$.

We then interpret Eq. (3.1.24) as follows. The line operator $\mathcal{S}^k_{\overline{\beta}}$ is the sheaf-theoretic realization of the collision of the wrapped boundary condition $\mathrm{ch}(\mathcal{B}_{\overline{\beta}})$ and a Wilson line transforming in the k-th exterior power of the fundamental representation, realized as the (derived) tensor product of the corresponding sheaves. Therefore, $\mathrm{Ext}^\bullet(\mathcal{S}^k_{\overline{\beta}}, \mathcal{O}_{\mathrm{Hilb}^n(\mathbb{C}^2)})$ corresponds to the vector space of local operators at the end of the composite line operator $\mathcal{S}^k_{\overline{\beta}}$.

The Hom-tensor adjunction used above physically corresponds to rotating the Wilson line upwards and instead considering the junction of the (conjugate) Wilson line, *cf.* Section 2.1.2, and the wrapped boundary condition $\mathrm{ch}(\mathcal{B}_{\overline{\beta}})$. See Figure 3.2. The expression $\mathrm{Ext}^\bullet(\mathrm{ch}(\mathcal{B}_{\overline{\beta}}), \wedge^k \mathcal{L}^\vee_{\mathrm{taut}})$ corresponds to the vector space of local operators at the junction of the wrapped boundary condition $\mathcal{L}_{\mathcal{B}_{\overline{\beta}}}$ and a Wilson line for the representation k-th exterior power of the anti-fundamental representation $\wedge^k \mathcal{L}^\vee_{\mathrm{taut}}$).

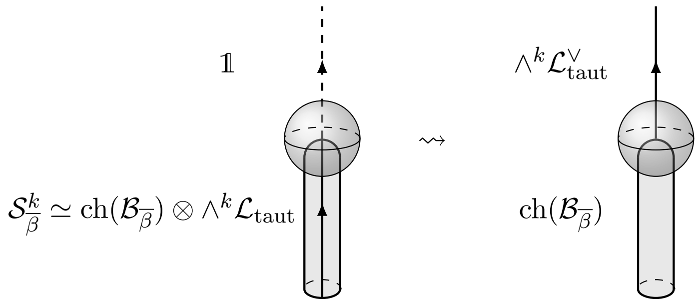


Figure 3.2: Physical interpretation of the Hom-tensor adjunction used in Eq (3.1.24). Left: $\text{Ext}^\bullet(\mathcal{S}_{\overline{\beta}}^k, \mathcal{O}_{\text{Hilb}^n(\mathbb{C}^2)})$ as local operators at the end of the composite line operator $\mathcal{S}_{\overline{\beta}}^k \simeq \text{ch}(\mathcal{B}_{\overline{\beta}}) \otimes \wedge^k \mathcal{L}_{\text{taut}}$ obtaining by colliding the wrapped boundary condition $\text{ch}(\mathcal{B}_{\overline{\beta}})$ and the Wilson line $\wedge^k \mathcal{L}_{\text{taut}}$. Right: $\text{Ext}^\bullet(\text{ch}(\mathcal{B}_{\overline{\beta}}), \wedge^k \mathcal{L}_{\text{taut}}^\vee)$ as local operators at the junction of the wrapped boundary condition $\text{ch}(\mathcal{B}_{\overline{\beta}})$ and the conjugate Wilson line $\wedge^k \mathcal{L}_{\text{taut}}^\vee$.

If we flatten the wrapped boundary condition $\text{ch}(\mathcal{B}_{\overline{\beta}})$, we can obtain a third expression that bares a much closer resemblance to the physical setup described in Section 2.3.4. Upon flattening the wrapped boundary condition, the vector space of local operators at the junction of the bulk Wilson line $\wedge^k \mathcal{L}_{\text{taut}}^\vee$ and the wrapped boundary condition $\text{ch}(\mathcal{B}_{\overline{\beta}})$ becomes identified with the vector space of local operators at the junction of the bulk Wilson line $\wedge^k \mathcal{L}_{\text{taut}}^\vee$ and the boundary condition $\mathcal{B}_{\overline{\beta}}$ defining the braid closure. We could continue deforming the metric to alternatively realize this vector space as the Hilbert space of the theory on a disk D with $\mathcal{B}_{\overline{\beta}}$ wrapped on ∂D and pierced by the Wilson line $\wedge^k \mathcal{L}_{\text{taut}}^\vee$. See Figure 3.3.

3.2 HOMFLY-PT from the 3d A-model

The 3d theories with $\mathcal{N} = 4$ supersymmetry that we consider in this thesis enjoy an IR duality known as 3d mirror symmetry [74–76]. At the level of the 3d $\mathcal{N} = 4$ supersymmetry algebra, 3d mirror symmetry is an involution that exchanges the roles of $SU(2)_H$ and $SU(2)_C$ R-symmetries. (This is directly analogous to the classic description of mirror symmetry in 2d $\mathcal{N} = (2, 2)$ theories [78], as exchanging the role of axial and vector R-symmetries.) In sufficiently nice cases, 3d mirror symmetry also exchanges one gauge theory with linear matter for another. In such a case, many of the structures discussed throughout this thesis

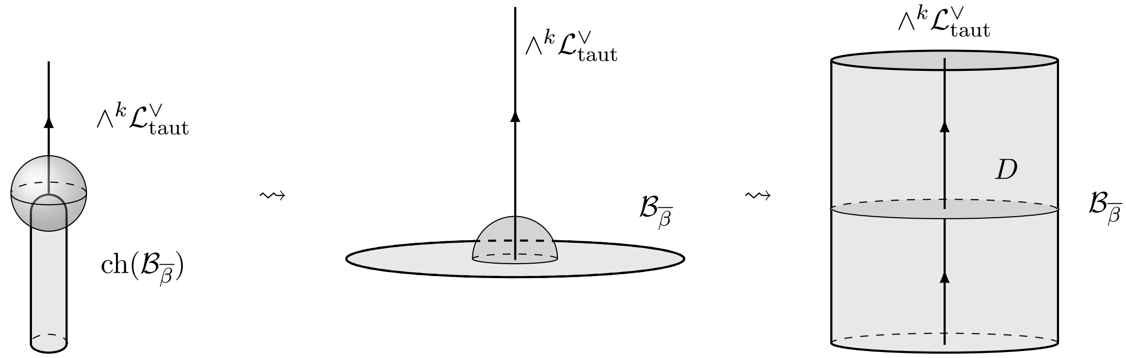


Figure 3.3: Different interpretations of the Oblomkov-Rozansky construction of the (k-th row of) HOMFLY-PT homology for the braid closure $\overline{\beta}$. Left: $\text{Ext}^\bullet(\text{ch}(\mathcal{B}_{\overline{\beta}}), \wedge^k\mathcal{L}^\vee_{\text{taut}})$ as local operators at the junction of the wrapped boundary condition $\text{ch}(\mathcal{B}_{\overline{\beta}})$ and the conjugate Wilson line $\wedge^k\mathcal{L}^\vee_{\text{taut}}$. Center: $\rho_{\mathcal{B}_{\overline{\beta}}}(\wedge^k\mathcal{L}^\vee_{\text{taut}})$ as local operators at the junction of the boundary condition $\text{ch}(\mathcal{B}_{\overline{\beta}})$ and the conjugate Wilson line $\wedge^k\mathcal{L}^\vee_{\text{taut}}$. Right: $\mathcal{H}(\mathcal{B}_{\overline{\beta}}; \wedge^k\mathcal{L}^\vee_{\text{taut}})$ as the Hilbert space on a disk D with boundary condition $\mathcal{B}_{\overline{\beta}}$ on ∂D and pierced by the Wilson line $\wedge^k\mathcal{L}^\vee_{\text{taut}}$.

are swapped:

$$\begin{array}{ccc} \text{SQM}_A & \leftrightarrow & \text{SQM}_B \\ Q_A & \leftrightarrow & Q_B \\ \text{vortex lines} & \leftrightarrow & \text{Wilson lines} \\ \text{Hom}_A(\mathcal{L}, \mathcal{L}') & \leftrightarrow & \text{Hom}_B(\mathcal{L}^!, \mathcal{L}'^!) \\ \mathcal{M}_C,\ \mathbb{C}[\mathcal{M}_C] & \leftrightarrow & \mathcal{M}_H,\ \mathbb{C}[\mathcal{M}_H] \end{array} \tag{3.2.1}$$

In particular, $\frac{1}{2}$-BPS Wilson lines (and the BPS local operators bound to them) will be mapped to $\frac{1}{2}$-BPS vortex lines (and the BPS local operators bound to them), and vice versa.

We saw in Section 3.1.2 that the construction of HOMFLY-PT homology due to Oblomkov-Rozansky can be viewed as a construction in the B-twist of the rank n ADHM quiver gauge theory, which is famously self-mirror [74, 75]. It is therefore natural to ask for the A-twist construction that is mirror to the construction in Section 3.1.2. This is the main focus of the upcoming paper [91].

In Section 3.2.1, we explain how each of the ingredients, *i.e.* the Wilson lines $\wedge^k\mathcal{L}_{\text{taut}}$ and the boundary condition $\mathcal{B}_n$, map across 3d mirror symmetry, and the resulting mirror construction. Then, in Section 3.2.2 we show that the proposed mirror cleanly reproduces the construction of [99]. In particular, when the link can be described in terms of a plane curve singularity, we show that the mirror construction can be realized algebraically as computing the equivariant homology of generalized affine Springer fibers isomorphic to the incidence varieties of [99]. Finally, in Section 3.2.3 we discuss steps towards to generalizing the setup of Section 3.2.2 to more general links.

3.2.1 3d mirror map

The main ingredients in the construction of HOMFLY-PT homology due to Oblomkov-Rozansky were a boundary condition $\mathcal{B}_n$, dependent on n complex parameters, and Wilson lines transforming in exterior powers of the fundamental representation of $U(n)$. The upcoming work [91] explicitly maps each of these ingredients across 3d mirror symmetry to realize a construction in the A-twist of the same theory and we summarize the results below. In Section 3.2.2 we provide a rather robust check of the proposed 3d mirror construction by showing that it realizes the (conjectural) constructions of HOMFLY-PT homology due to Oblomkov-Rasmussen-Shende [99].

Line operator

The mapping of the Wilson line across mirror symmetry is realized by applying the analysis of Assel-Gomis [80]. The mirror line operator $\mathcal{L}^!_k$ is realized by coupling to a 1d $\mathcal{N} = 4$ super quantum mechanical gauge theory, *cf.* Section 2.2.4, and additionally introducing a bulk-line superpotential. Explicitly, the 1d quantum theory is described by the quiver given in Figure 3.4 and the superpotential is given by $W_{\mathrm{1d}} = \mathrm{Tr}(Xps)$, where p, s are scalars in the chiral multiplets.

This line operator admits an algebraic description: the superpotential W_{1d} allows Y to have a simple pole with residue ps and forces X to have dual zeros; the singular part of Y has rank at most k and, dually, the constant part of X has rank at most $(n-k)$. We can use the bulk $GL(n, \mathcal{O})$ gauge symmetry to require the poles of Y to happen in the first k rows

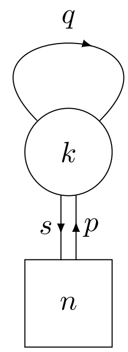


Figure 3.4: The quiver describing the 1d $\mathcal{N}=4$ super quantum mechanical gauge theory used in describing the vortex line operator $\mathcal{L}^!_k$; this uses $\mathcal{N}=2$ notation for the multiplets. The 3d bulk theory couples to this 1d theory by gauging the $PSU(n)$ flavor symmetry and the superpotential $W_{1\mathrm{d}} = \mathrm{Tr}(Xps)$.

and, dually, force the first k columns of X to have a zero

$$Y \in \mathfrak{g}_k^\perp = \begin{pmatrix} z^{-1}\mathcal{O}^{k\times k} & z^{-1}\mathcal{O}^{(n-k)\times k} \\ \mathcal{O}^{k\times(n-k)} & \mathcal{O}^{(n-k)\times(n-k)} \end{pmatrix}, \quad X \in \mathfrak{g}_k = \begin{pmatrix} z\mathcal{O}^{k\times k} & \mathcal{O}^{(n-k)\times k} \\ z\mathcal{O}^{k\times(n-k)} & \mathcal{O}^{(n-k)\times(n-k)} \end{pmatrix}. \tag{3.2.2}$$

This choice breaks $GL(n,\mathcal{O})$ to the parahoric subgroup $P_k(\mathcal{O})$ given by

$$P_k(\mathcal{O}) = \left\{ g \in GL(n,\mathcal{O}) \middle| g(0) = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \begin{pmatrix} \mathbb{C}^{k\times k} & \mathbb{C}^{(n-k)\times k} \\ \mathbb{C}^{k\times(n-k)} & \mathbb{C}^{(n-k)\times(n-k)} \end{pmatrix} \right\}. \tag{3.2.3}$$

All together, we find that the mirror vortex line operator $\mathcal{L}^!_k$ can be described by the geometric data of $\mathcal{L}_0 = N^*(\mathfrak{g}_k \oplus \mathcal{O}^n) = (\mathfrak{g}_k \oplus \mathcal{O}^n) \oplus (\mathfrak{g}_k^\perp \oplus (\mathcal{O}^n)^*)$ and $\mathcal{G}_0 = P_k(\mathcal{O})$.

Boundary condition

The boundary condition $\mathcal{B}_n$ is somewhat more delicate. The analysis of the upcoming [91] approaches this boundary condition by first realizing the boundary $T^*\mathrm{Fl}_n$ σ-model as the (2d) Higgs branch of the $\mathcal{N}=(2,2)$ gauged linear σ-model $\widetilde{\mathcal{T}}^{2\mathrm{d}}_n$ with field content given by the quiver in Figure 3.5 together with a purely 2d superpotential W_{2d} given by

$$W_{2d} = \sum_{i=1}^{n-1} \text{Tr}[X_i(w_i z_i - z_{i-1} w_{i-1})], \tag{3.2.4}$$

where $z_0 = w_0 = 0$. In fact, $\widetilde{\mathcal{T}}_n^{2d}$ has $\mathcal{N} = (4,4)$ supersymmetry and corresponds to the the dimensional reduction of the $T[SU(n)]$ theory of Gaiotto-Witten [55, 56]. In terms of the gauged linear σ-model, the Kähler parameters are realized by complexified FI parameters and the bulk-boundary superpotential W_∂ is given by

$$W_\partial = \text{Tr}[X(z_{n-1} w_{n-1})]. \tag{3.2.5}$$

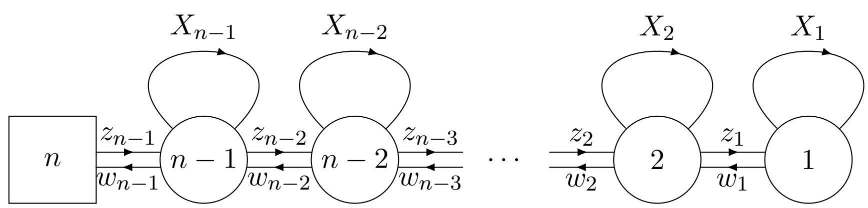


Figure 3.5: The quiver describing the $\mathcal{N} = (2,2)$ gauged linear σ-model $\widetilde{\mathcal{T}}_n^{2d}$ that flows to the non-linear σ-model $\mathcal{T}_n^{2d}$ in the IR. The theory $\widetilde{\mathcal{T}}_n^{2d}$ additionally has a superpotential W_{2d} given in Eq. (3.2.4).

To determine the 3d mirror of the boundary condition $\mathcal{B}_n$, we propose a Hanany-Witten-type brane construction [79] of the rank n ADHM quiver gauge theory with the boundary condition $\mathcal{B}_n$. We wrap the branes as in Figure 3.6, with D3 branes on $\mu = 0,1,2,3$, the NS5 branes on $\mu = 0,1,2,4,5,6$, the NS5' branes on $\mu = 0,1,2,3,4,5,7$, and the D5 branes on $\mu = 0,1,2,7,8,9$.

In the limit that the x_3 circle goes to zero size, the worldvolume theory of the n D3 branes yields the rank n ADHM quiver theory, but the circle reduction of the n NS5' branes is more subtle. Instead, we can consider the flat limit of the T-dual configuration in Figure 3.7. Under the $x_3 \to \widetilde{x}_3$ T-duality, the D3 branes become D2 branes wrapping $\mu = 0,1,2$, and the D5 brane becomes a D6 brane wrapping $\mu = 0,1,2,\widetilde{3},4,5,6$. Similarly, since the NS5' brane wraps x_3, the NS5' brane in the T-dual description wraps $\mu = 0,1,\widetilde{3},4,5,7$. On

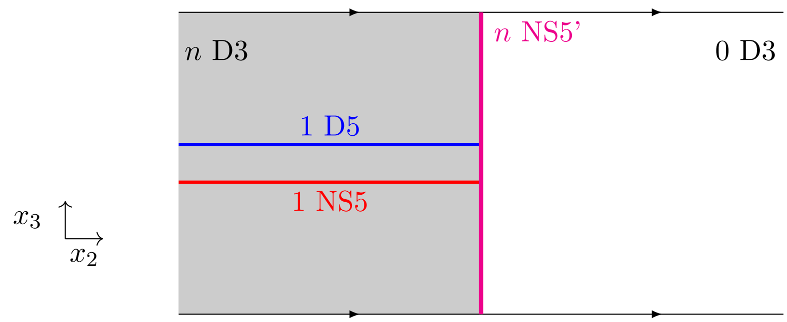


Figure 3.6: Proposed type IIB brane construction realizing the 3d $\mathcal{N} = 4$ rank n ADHM quiver gauge theory with boundary condition $\mathcal{B}_n$ used in the Oblomkov-Rozansky construction of HOMFLY-PT homology.

the other hand, the NS5 brane is transverse to x_3 and hence disappears from the T-dual description, instead resulting in a Taub-NUT spacetime wrapping $\mu = \widetilde{3}, 4, 5, 6$. In the flat limit, the corresponding Taub-NUT geometry limits to $\mathbb{C}^2$.

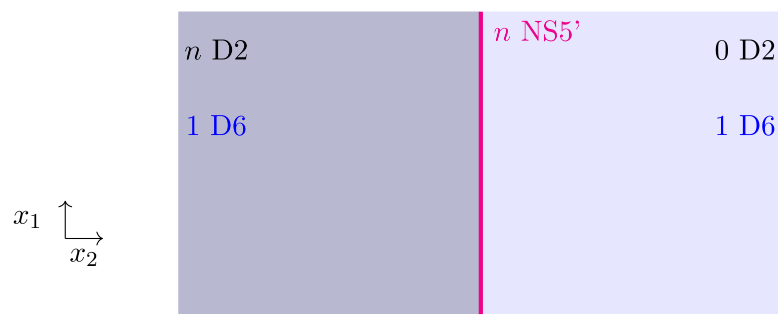


Figure 3.7: The IIA brane construction that is T-dual to Figure 3.6.

The worldvolume theory of the n D2s "dissolved" in the D6 is again the rank n ADHM quiver gauge theory, and we see that the boundary condition at the x_2 position of the NS5' brane again yields a boundary condition for this 3d gauge theory. It is argued in [91] that the boundary condition exactly reproduces $\mathcal{B}_n$. Roughly speaking, the n NS5' branes can be resolved by separating them in the x_2 direction. The single interface becomes n separate interfaces that sequentially terminate a single D2, until there are none left. In the IR, the 3d theories trapped between the NS5 interfaces become effective 2d degrees of freedom realizing the theory $\widetilde{T}_n^{\text{2d}}$.[5] Finally, the (complexified) boundary FI parameters of this boundary

[5]More precisely, the effective 3d degrees of freedom in the x_2 region between the $(n-k)$th and $(n-k+1)$th

condition are determined by the 89 positions of the NS5' branes.

We now apply the S-duality of [79] to the IIB brane system in Figure 3.6 and repeat the T-duality to realize the desired mirror boundary condition. The S-duality transformation is realized by exchanging NS5 branes and D5 branes as well as the rotation sending $x^\mu \to x^{\mu+3}$ and $x^{\mu+3} \to -x^\mu$ for $\mu = 4, 5, 6$. The corresponding brane configuration is given in Figure 3.8. The D3, D5, and NS5 branes are wrapped on the same directions as before; the D5' branes are wrapped on $\mu = 0, 1, 2, 4, 7, 8$.

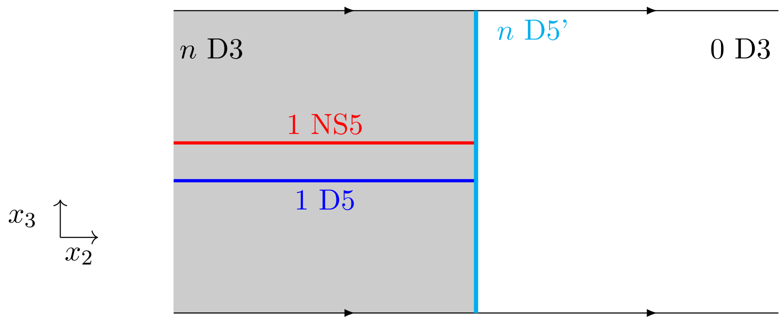


Figure 3.8: Type IIB brane configuration S-dual to Figure 3.6.

Applying T-duality to this setup is equally straight-forward, and the resulting type IIA brane setup is given in Figure 3.9. The only change from the previous setup are D4' branes wrapping $\mu = 0, 1, 4, 7, 8$ in place of the NS5' branes. We can again understand the boundary condition induced by the n D4' branes by separating them along x_2 so that 1 D2 brane ends on each. The resulting boundary condition does not have additional boundary degrees of freedom but instead breaks gauge symmetry and specifies the boundary values of X and I.

We conclude that mirror boundary condition $\mathcal{B}^!_n$ is a Dirichlet boundary condition that fully breaks gauge symmetry. Moreover, the eigenvalues of the matrix hypermultiplet scalar X (which are determined by the 56 positions of the D4 branes), are mirror to the boundary parameters $\vec{\tau}$.

NS5' brane is exactly the rank k ADHM quiver theory. The NS5'-induced interface before this region of x_2 and carries boundary a pair of 2d $\mathcal{N} = (2, 2)$ chiral multiplets z_k, w_k that couple the hypermultiplets on either side of the interface with a superpotential of the form $W_k = \mathrm{Tr}(z_k X_{k+1} w_k) - \mathrm{Tr}(w_k X_k z_k) + J_{k+1} w_k I_k$. The boundary conditions preserves gauge symmetry on both sides on both sides and force $Y_k = z_k w_k = w_{k-1} z_{k-1}$, $J_k = J_{k+1} w_k$, and $I_{k+1} = w_k I_k$, so they may be removed. The effective 2d degrees of freedom are then parameterized by the z_k, w_k, X_k with the superpotential in Eq. (3.2.4), and are coupled to the 3d bulk by the superpotential in Eq. (3.2.5).

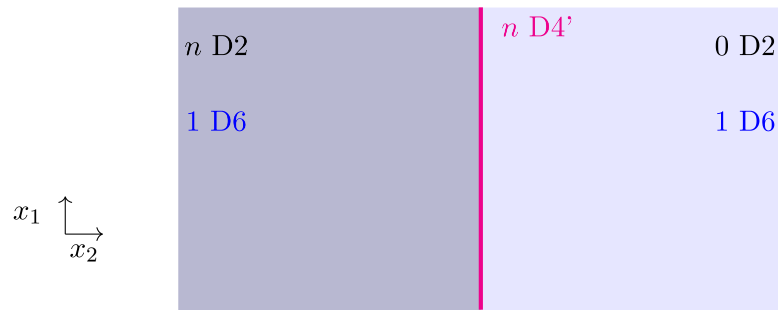


Figure 3.9: The IIA brane construction that is T-dual to Figure 3.8.

Construction in the A-model

Given the 3d mirrors of the ingredients used in the Oblomkov-Rozansky construction of HOMFLY-PT homology, we can piece them together to realize a 3d mirror of the entire construction.

We consider the Dirichlet boundary condition $\mathcal{B}^!_n$ that gives the hypermultiplet scalars X, I boundary values X_∂, I_∂ that fully break the gauge and flavor symmetries. If we wrap this boundary condition on cylinder and allow the values X_∂, I_∂ to vary along the angular coordinate ϑ of this cylinder, the eigenvalues of X_∂ trace out a link realized as the closure of an n-strand braid β. We denote the corresponding boundary condition $\mathcal{B}^!_{\overline{\beta}}$. The k-th row of HOMFLY-PT homology for the braid closure $\overline{\beta}$ should be realizable as the vector space of local operators at the junction of the mirror vortex line operator $\mathcal{L}^!_k$ and the boundary condition $\mathcal{B}^!_{\overline{\beta}}$.

Just as in the discussion in Section 3.1.2, there should be several descriptions of this space of local operators depending on the explicit realization of the boundary condition $\mathcal{B}^!_{\overline{\beta}}$. The categories of boundary line operators for Dirichlet boundary conditions are not well understood in the A-twist, so it is presently infeasible to realize the mirror of the full braid group representation $\rho_n : \mathrm{Br}_n \to \mathcal{C}_{\mathcal{B}_n}$ used by Oblomkov-Rozansky. Nonetheless, if the boundary condition $\mathcal{B}^!_{\overline{\beta}}$ admits an algebraic realization, the vector space of local operators at the junction of the line operator $\mathcal{L}^!_k$ can be described in terms of the homology of a generalized affine Springer fiber, *cf.* Section 2.3.4.

Note that there are many choices of X_∂, I_∂ that result in the same braid closure; since the construction is locally insensitive to deformations of the boundary values X_∂, I_∂, we expect that any two choices of the boundary values that fully break the gauge flavor symmetry and yield the same braid closure should be equivalent in all computations. We will give an explicit example of one such choice in Section 3.2.2.

3.2.2 Hilbert schemes of singular curves as generalized affine Springer fibers

With a proposal for the 3d mirror of the Oblomkov-Rozansky construction of HOMFLY-PT homology described in Section 3.1.2, we now check its validity by showing that the proposed mirror construction exactly reproduces the (conjecturally equivalent) mathematical construction described in Section 3.1.1 and due to Oblomkov-Rasmussen-Shende [99]. We will see that it is further compatible with the results of Gorsky-Oblomkov-Rasmussen-Shende [100] in Section 3.3.

We once again consider to positive algebraic link K realized as the link of a plane curve singularity. We saw in Section 3.1.1 that the HOMFLY-PT homology of this class of links can be (conjecturally) realized in terms of the algebraic geometry of the corresponding singularity. Since we are interested in the behavior of the singularity at $z = w = 0$, it suffices to work over formal series in z, w, *i.e.* with the germ of the singularity $\widehat{Z}_K$. The ring of functions on this germ is precisely $R_K = \mathbb{C}[\widehat{Z}_K] \simeq \mathbb{C}[\![w, z]\!]/(p_K)$.

By Weierstrass preparation, any plane curve singularity is equivalent to one where p_K is a monic polynomial in w and we write $p_K = w^n - a_n w^{n-1} - ... - a_1$ with $a_i \in \mathbb{C}[\![z]\!]$. For fixed z, the solutions to $p_K = 0$ (viewed as an equation for w) are viewed as the positions of the braid strands and, in particular, K corresponds to an n-strand braid closure when p_K is degree n in w.

Given this algebraic description of the n-strand braid closure K, we seek a choice of Dirichlet boundary condition $\mathcal{B}^!_K$ for the rank n ADHM gauge theory such that the boundary values X_∂, I_∂ fully breaks boundary flavor and gauge symmetry and such that the eigenvalues of the matrix X_∂ exactly reproduces the solutions to $p_K = 0$. A natural choice of X_∂ is the companion matrix of p_K

$$W_K = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ a_1 & a_2 & \dots & a_{n-1} & a_n \end{pmatrix}. \tag{3.2.6}$$

For fixed z, the n solutions to $p_K = 0$ realize the n eigenvalues of the matrix W_K. Alternatively, the vanishing locus $Z_K = \{p_K = 0\}$ is simply the spectral curve of the matrix W_K. Given the choice $X_\partial = W_K$, there are many compatible choices of I_∂ that fully breaks and gauge symmetries. One such choice is simply $I_\partial = e_n$, the n-th standard basis vector.

We conclude that for K a positive algebraic knot realized as the link of the singularity of $p_K(z, w) = 0$ with p_K of degree n in w, we can choose the boundary condition $\mathcal{B}^!_K$ to be the generic Dirichlet boundary condition with boundary values $(X_\partial, I_\partial) = (W_K, e_n)$. The machinery of Section 2.3.4 can then be applied to describing the vector space of local operators at the junction of $\mathcal{B}^!_K$ and the vortex line operator $\mathcal{L}^!_k$; we will find that this exactly reproduces the (conjectural) realization of HOMFLY-PT due to Oblomkov-Rasmussen-Shende [99].

The lowest row: $k = 0$

First consider the case of $k = 0$, *i.e.* when $\mathcal{L}^!_k$ is the trivial line operator $\mathbb{1}$. In this case, the vector space of local operators can be described as the homology $H_\bullet(\mathcal{M}_\mathbb{D}(\mathcal{B}_{W_K, e_n}; \mathbb{1}))$, where $\mathcal{M}_\mathbb{D}(\mathcal{B}_{W_K, e_n}; \mathbb{1})$ is the following generalized affine Springer fiber:

$$\mathcal{M}_\mathbb{D}(\mathcal{B}_{W_K, e_n}; \mathbb{1}) = \left\{ [g] \in \mathrm{Gr}_{GL(n,\mathbb{C})} \middle| \begin{array}{c} g^{-1} W_K g \in \mathfrak{gl}(n, \mathcal{O}) \\ g^{-1} e_n \in \mathcal{O}^n \end{array} \right\}. \tag{3.2.7}$$

On the other hand, the ORS conjecture for $k = 0$ is concerned with the Hilbert scheme of points on $\widehat{Z}_K$; the lowest row of HOMFLY-PT homology for the knot K corresponding to the (germ of the) plane curve singularity $\widehat{Z}_K$ is obtained from the Borel-Moore homology of $\widehat{Z}^\bullet_K = \bigcup \widehat{Z}^{[d]}_K$. We now show the above generalized affine Springer fiber can be identified with

$\widehat{Z}^{\bullet}_K$.

To realize this identification, we can use $w^n = a_1 + a_2 w + ... + a_n w^{n-1}$ to write any element $f \in R_K$ as a (row) vector of series in z $f(z,w) = f_1 + ... + f_n w^{n-1} \leftrightarrow (f_1, ..., f_n)$ and thereby identify $R_K \simeq (\mathcal{O}^n)^*$ as an $\mathcal{O}$-module. An ideal of R_K must be closed under multiplication by both z and w; under the above identification, this says that an ideal I should then be an $\mathcal{O}$-submodule of $R_K \simeq (\mathcal{O}^n)^*$ closed under the action of the matrix W_K given in Eq. (3.2.6).

Somewhat more generally, we could have considered $\mathcal{O}$-submodules that live in all of $(\mathcal{K}^n)^*$. These $\mathcal{O}$-submodules are often called "lattices" and the distinguished lattice $R_K = (\mathcal{O}^n)^* = \Lambda_0$ is called the "standard lattice." We see that the moduli space of (nonzero) ideals of R_K, *i.e.* the Hilbert scheme, can be identified with the moduli space of lattices contained in the standard lattice closed under the action of W_K:

$$\widehat{Z}^{\bullet}_K \simeq \left\{ \begin{array}{c} \text{lattices } \Lambda \text{ in } (\mathcal{K}^n)^* \\ \text{with } \Lambda W_K \subset \Lambda \subseteq \Lambda_0 \end{array} \right\} \tag{3.2.8}$$

Any lattice Λ can be realized as $\Lambda = \Lambda_0 g^{-1}$ for some $g \in GL(n, \mathcal{K})$ and $\Lambda = \Lambda_0$ if and only if $g \in GL(n, \mathcal{O})$. Thus, the moduli space of all lattices in $(\mathcal{K}^n)^*$ is isomorphic to the homogeneous space $GL(n,\mathcal{K})/GL(n,\mathcal{O})$, also known as the affine Grassmannian $\mathrm{Gr}_{GL(n,\mathbb{C})}$:

$$\{\text{lattices } \Lambda \text{ in } (\mathcal{K}^n)^*\} \simeq GL(n,\mathcal{K})/GL(n,\mathcal{O}) = \mathrm{Gr}_{GL(n,\mathbb{C})} \tag{3.2.9}$$

The lattices corresponding to ideals of R_K were quite special. If the lattice $\Lambda = \Lambda_0 g^{-1}$ is closed under the action of W_K, *i.e.* under multiplication by w, it follows that $g^{-1} W_K g$ stabilizes Λ_0, thus $g^{-1} W_K g \in \mathfrak{gl}(n, \mathcal{O})$. In other words, the space of lattices $(\mathcal{K}^n)^*$ closed under multiplication by W_K exactly reproduces the "classical" affine Springer fiber over $W_K \in \mathfrak{gl}(n, \mathcal{O})$:

$$\left\{ \begin{array}{c} \text{lattices } \Lambda \text{ in } (\mathcal{K}^n)^* \\ \text{with } \Lambda W_K \subset \Lambda \end{array} \right\} \simeq \{[g] \in \mathrm{Gr}_{GL(n,\mathbb{C})} | g^{-1} W_K g \in \mathfrak{gl}(n, \mathcal{O})\}. \tag{3.2.10}$$

To finally identify the lattice Λ with an ideal, we need to ensure that the lattice Λ is actually contained in the standard lattice $R = \Lambda_0$. This is accomplished if and only if the translating matrix g^{-1} has entries that belong to $\mathcal{O}$ $g^{-1} \in G(\mathcal{K}) \cap \mathfrak{gl}(n, \mathcal{O})$. In particular, it follows that $g^{-1}e_n \in \mathcal{O}^n$.

Conversely, given $g \in GL(n, \mathcal{K})$ with $g^{-1}W_K g \in \mathfrak{gl}(n, \mathcal{O})$, it follows immediately that $\Lambda = \Lambda_0 g^{-1}$ is stable under W_K. Moreover, if $g^{-1}e_n \in \mathcal{O}^n$ it follows that $g^{-1}e_\ell \in \mathcal{O}^n$ for all ℓ since $g^{-1}e_\ell = (g^{-1}\gamma g)g^{-1}e_{\ell+1}$, *i.e.* the entries of g^{-1} belong to $\mathcal{O}$ and therefore $\Lambda \subset \Lambda_0$. Combined with the earlier result, we conclude that the Hilbert scheme of points on (the germ of) $\{p_K = 0\}$ can be identified with the generalized affine Springer fiber over $(W_K, e_n) \in \mathfrak{gl}(n, \mathcal{O}) \oplus \mathcal{O}^n$:

$$\widehat{Z}_K^{\bullet} \simeq \left\{ [g] \in \mathrm{Gr}_{GL(n,\mathbb{C})} \middle| \begin{array}{c} g^{-1}W_K g \in \mathfrak{gl}(n, \mathcal{O}) \\ g^{-1}e_n \in \mathcal{O}^n \end{array} \right\} \simeq \mathcal{M}_{\mathbb{D}}(\mathcal{B}_{W_K, e_n}; \mathbb{1}), \tag{3.2.11}$$

For the ideal to belong to $\widehat{Z}_K^d$, the translating element is of degree $-d$, *i.e.*, $\det g^{-1} = \# z^d + \dots$.

Higher rows: $k > 0$

The above isomorphism has a easy generalization to the higher rows of HOMFLY-PT $k > 0$. The algebraic description of the line operator $\mathcal{L}_k^!$ was given in Section 3.2.1: it breaks gauge symmetry to the parahoric subgroup $P_k(\mathcal{O})$ and constrains the hypermultiplet scalar I to belong to $\mathcal{O}^n$ and the hypermultiplet scalar X to belong to $\mathfrak{gl}(n, \mathcal{O})z^{A_k} =: \mathfrak{g}_k$, where $A_k = (1, 1, ..., 1, 0, 0, ..., 0)$ with k 1's, *i.e.* X is required to have zeros in the first k rows

$$\mathfrak{g}_k = \left\{ X \in \mathfrak{gl}(n, \mathcal{O}) \middle| X(0) = \begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix} \in \begin{pmatrix} \mathbb{C}^{k\times k} & \mathbb{C}^{(n-k)\times k} \\ \mathbb{C}^{k\times(n-k)} & \mathbb{C}^{(n-k)\times(n-k)} \end{pmatrix} \right\}. \tag{3.2.12}$$

The ORS conjectures are interested the incidence varieties $Z_K^{[d \le d+k]}$ in the moduli space of pairs of ideals $(I, J) \in \mathrm{Hilb}^d(Z_K) \times \mathrm{Hilb}^{d+k}(Z_K)$ such that $J \supset I \supset MJ$. We again identify ideals of $\mathbb{C}[\![\lambda, z]\!]/(p_K)$ and lattices $\Lambda \subset (\mathcal{K}^n)^*$ closed under the action of W_K; a chain of ideals $J \supset I \supset MJ$ is then identified with a pair of lattices (Λ_1, Λ_2) $\Lambda_0 \supset \Lambda_1 \supset \Lambda_2$, each closed under the action of W_K, such that (1) $\Lambda_2 \supset z\Lambda_1$ and (2) $\Lambda_2 \supset \Lambda_1 W_K$.

The first constraint together with $(I, J) \in \mathrm{Hilb}^d(Z_K) \times \mathrm{Hilb}^{d+k}(Z_K)$ says that (Λ_1, Λ_2) is a $G(\mathcal{K})$ translate of the pair $(\Lambda_0, \Lambda_0 z^{A_k})$. Moreover, the translating element $g \in G(\mathcal{K})$ is well defined up to the action of the parahoric subgroup $P_k(\mathcal{O})$; the moduli space of chains of lattices $\Lambda_1 \supset \Lambda_2$ satisfying (1) is the partial affine flag variety $\mathrm{Fl}_{P_k(\mathcal{O})} \simeq GL(n, \mathcal{K})/P_k(\mathcal{O})$. The second condition implies that $g^{-1} W_K g \in \mathfrak{g}_k$.

Just as above, we get an isomorphism between $Z_K^{[\bullet \le \bullet + k]}$ and this generalized affine Springer fiber:

$$\widehat{Z}_K^{[\bullet \le \bullet + k]} \simeq \left\{ [g] \in \mathrm{Fl}_{P_k(\mathcal{O})} \;\middle|\; \begin{array}{c} g^{-1} W_K g \in \mathfrak{g}_k \\ g^{-1} e_n \in \mathcal{O}^n \end{array} \right\} \simeq \mathcal{M}_{\mathbb{D}}(\mathcal{B}_{W_K, e_n}; \mathcal{L}^!_k). \tag{3.2.13}$$

It is worth noting that this isomorphism can be generalized yet further to higher incidence varieties $\widehat{Z}^{[d_1 \le \dots \le d_\ell]}$ by realizing those chains of ideals within an appropriate partial affine flag variety. The role of these higher incidence varieties in knot theory is unclear.

3.2.3 Towards more general knots

In the previous section, we saw that it is possible to (conjecturally) realize the HOMFLY-PT homology of positive algebraic links K can be obtained from applying the physical construction of Section 2.3.4 for particular choice of $\frac{1}{2}$-BPS vortex line operator $\mathcal{L}^!_k$ and boundary condition $\mathcal{B}^!_K = \mathcal{B}_{W_K, e_n}$ for the 3d $\mathcal{N} = 4$ ADHM quiver gauge theory.

It is natural to ask whether it is possible to realize the HOMFLY-PT homology of other links using a similar construction. If K can be represented by the closure of an n-strand braid, the kth row of HOMFLY-PT homology should be obtained as a vector space of boundary local operators in the rank n ADHM quiver gauge theory in the presence of the $\frac{1}{2}$-BPS vortex line $\mathcal{L}^!_k$ and a boundary condition $\mathcal{B}^!_K$ encoding the link. While the underlying physical setup is be the same as before, it is unreasonable to expect a simple algebraic realization thereof unless K is itself algebraic.

Unlike the case of Rozansky-Witten theory studied in [68], the 2-category of boundary conditions in the A-twist is not well understood and so it is not currently feasible to realize the categorical representation of the braid group within the category of boundary line operators. Nonetheless, there should still be a categorical Chern-character map that takes realizes the more general boundary conditions $\mathcal{B}^!_{\overline{\beta}}$ within the category of bulk line operators.

As described in Section 2.2, the category of bulk line operators in the A-twist of the rank n ADHM gauge theory can be identified within the derived category of $GL(n, \mathcal{K})$-equivariant D-modules on $\mathfrak{gl}(n, \mathcal{K}) \oplus \mathcal{K}^n$:

$$\mathcal{C}_A \simeq \text{D-mod}_{GL(n,\mathcal{K})}(\mathfrak{gl}(n, \mathcal{K}) \oplus \mathcal{K}^n) \tag{3.2.14}$$

In particular, for any n-strand braid β there should be a complex of D-modules $\text{ch}_{\mathcal{B}^!_{\overline{\beta}}}(\mathbb{1}_{2\text{d}})$ and an isomorphism

$$H^\bullet_k(\overline{\beta}) = \text{Ext}^\bullet_{\mathcal{C}_A}(\text{ch}_{\mathcal{B}^!_{\overline{\beta}}}(\mathbb{1}_{2\text{d}}), \mathcal{L}_k). \tag{3.2.15}$$

Explicitly identifying the complex of D-modules $\text{ch}_{\mathcal{B}^!_{\overline{\beta}}}(\mathbb{1}_{2\text{d}})$ will likely be a non-trivial task even for the simplest non-algebraic knots.

3.3 Example: positive torus knots

Let us return once more to the positive torus knot $T_{(n,m)}$. The HOMFLY-PT homology of this knot admits a complementary (conjectural) description, due to Gorsky-Oblomkov-Rasmussen-Shende [100], in terms the representation theory of the rational Cherednik algebra $\overline{\mathcal{H}}_n$, at parameter m/n.

The algebra $\overline{\mathcal{H}}_n$ is graded with respect to the eigenvalues of $h = -\frac{1}{2}\sum_i (x_i y_i + y_i x_i)$ and is filtered by total polynomial degree $\overline{\mathcal{F}}^{\text{poly}}$. Gorsky-Oblomkov-Rasmussen-Shende conjecture that the HOMFLY-PT homology of $T_{(n,m)}$ can be obtained from the representation $\overline{L}_{m/n}$; we only state the part relevant for minimal a-degree: there exists a filtration $\overline{\mathcal{F}}_{(n,m)}$ on $\overline{L}_{m/n}$

compatible with the filtration $\overline{\mathcal{F}}^{\text{poly}}$ on $\overline{\mathcal{H}}_n$ such that

$$H_0^\bullet(T_{(n,m)}) \simeq \mathrm{Hom}_{S_n}(\mathbb{C}, \mathrm{gr}^{\overline{\mathcal{F}}_{(n,m)}} \overline{L}_{m/n}) \equiv \mathrm{gr}^{\overline{\mathcal{F}}_{(n,m)}} e \overline{L}_{m/n}, \tag{3.3.1}$$

as graded vector spaces. The filtration $\overline{\mathcal{F}}_{(n,m)}$ is used to identify the appropriate homological t-grading on the representation. The work of Hogancamp-Mellit [257] shows that the HOMFLY-PT homology of these torus knots is supported in even homological degree, so one can specialized $t \to -1$ by simply forgetting the t-grading and/or the filtrations $\overline{\mathcal{F}}_{(n,m)}$.

The work [100] moreover relates this (conjectural) representation-theoretic realization of HOMFLY-PT homology to the work [99], by showing that the rational Cherednik algebra action can be realized (somewhat circuitously) through (usual) affine Springer theory [247]. In the work [51], it is shown that this action is more cleanly realized in *generalized* affine Springer theory, which we review below.

Based on the physical description in Section 3.2, there should be an action of the algebra of bulk local operators for any choice of link; the work [51] realizes this action for links obtained as the link of a plane curve singularity by showing Hilbert schemes of more general singular curves as a generalized affine Springer fiber. One caveat is that the boundary condition need not be compatible with an Omega-background, *i.e.*, this action of local operators need not extend to the quantization $\overline{\mathcal{H}}_n^{\text{sph}}$. Quasi-homogeneous singularities, in particular the torus links $T_{(n,m)}$, are exceptions to this general phenomenon. See [100] for various other mathematical speculations.

3.3.1 The lowest row: $k = 0$

We consider the generalized affine Springer fiber over $(W_{(n,m)}, e_n) \in \mathfrak{gl}(n, \mathcal{O}) \oplus \mathcal{O}^n$. Let us denote the corresponding generalized affine Springer fiber $\mathcal{M}_{(n,m)}$:

$$\mathcal{M}_{(n,m)} = \mathcal{M}_{\mathbb{D}}(\mathcal{B}_{W_{(n,m)}, e_n}; \mathbb{1}). \tag{3.3.2}$$

We start by showing that the stabilizer subgroup for this choice is simply a copy of $\mathbb{C}^*$ (denoted $\mathbb{C}^*_{(n,m)}$) and, moreover, that the induced action on the generalized affine Springer fiber has isolated fixed points. Using this, we can analyze the action of the quantized Coulomb-branch chiral ring by fixed point localization and identify the module coming from the Dirichlet boundary condition $\mathcal{B}_{X_\partial, I_\partial}$ with $X_\partial = W_{(n,m)}, I_\partial = e_n$.

As discussed in 1.4.2, there is a flavor symmetry that acts by dilating $\mathfrak{gl}(n, \mathcal{O})$ and an extension $\widehat{G} = GL(n, \mathbb{C}) \times \mathbb{C}^*_F$ that is a simple product of groups, and the group $\widehat{G}^{\mathcal{O}}(\mathcal{K})$ is identified with $GL(n, \mathcal{K}) \times \mathbb{C}^*_F(\mathcal{O})$. The action of $(g(z), \mu(z), \lambda) \in \widehat{G}^{\mathcal{O}}(\mathcal{K}) \rtimes \mathbb{C}^*_\varepsilon$ on $(X, I) \in \mathfrak{gl}(n, \mathcal{O})$ is given by

$$(g(z), \mu(z), \lambda) : X(z), I(z) \mapsto \lambda^{-\frac{1}{2}} \mu(\lambda^{-1} z) g(\lambda^{-1} z) X(\lambda^{-1} z) g^{-1}(\lambda^{-1} z), \lambda^{-\frac{1}{2}} g(\lambda^{-1} z) I(\lambda^{-1} z). \tag{3.3.3}$$

The stabilizer of the vector $(W_{(n,m)}, e_n)$ can be found as follows. If $(W_{(n,m)}, e_n)$ is stabilized by $(g(z), \mu(z), \lambda)$, then so too is the determinant of $W_{(n,m)}$. The determinant of $W_{(n,m)}$ is invariant under the $GL(n, \mathcal{K})$ action, so $(g(z), \mu(z), \lambda)$ stabilizes $\det W_{(n,m)} = -z^m$ if and only if

$$\mu^n \lambda^{-(m + \frac{n}{2})} = 1 \Rightarrow \mu = \nu^{(m + \frac{n}{2})}, \lambda = \nu^n \tag{3.3.4}$$

for $\nu \in \mathbb{C}^*$. Now, $(g(z), \mu, \lambda)$ stabilizes $e_n \in \mathcal{O}^n$ if and only if $g(\lambda^{-1} z) e_n = \lambda^{\frac{1}{2}} e_n$, thus the last column of $g(z)$ must be $\lambda^{\frac{1}{2}} e_n$. We can then work column by column in $g(z)$ to show that the stabilizer of $(W_{(n,m)}, e_n)$ is exactly given by $(g(z), \mu, \nu)$ satisfying $\mu^n \lambda^{-(m+\frac{n}{2})} = 1$ and with $g(z)$ a diagonal matrix with entries $(\mu^{-(n-1)} \lambda^{\frac{n}{2}}, ..., \mu^{-1} \lambda, \lambda^{\frac{1}{2}})$:

$$\mathbb{C}^*_{(n,m)} = \left\{ (\mathrm{diag}(\mu^{-(n-1)} \lambda^{\frac{n}{2}}, ..., \mu^{-1} \lambda, \lambda^{-\frac{1}{2}}), \mu, \lambda) \middle| \mu^n = \lambda^{m + \frac{n}{2}} \right\} \simeq \mathbb{C}^*. \tag{3.3.5}$$

It is straightforward to check that the induced action of $\mathbb{C}^*_{(n,m)}$ on the generalized affine Springer fiber $\mathcal{M}_{(n,m)}$ has isolated fixed points if $\gcd(n, m) = 1$. The fixed points of $\mathbb{C}^*_{(n,m)}$ are labeled by cocharacters $[z^{-A}] \in \mathrm{Gr}_{GL(n,\mathbb{C})}$, and we find that $[z^{-A}]$ belongs to the fiber if

$0 \leq A_n \leq ... \leq A_1 \leq A_n + m$.[6] Each such fixed point yields an $\mathbb{C}^*_{(n,m)}$ equivariant homology class after localization, which we simply denote $|A\rangle$, and the (localized) $\mathbb{C}^*_{(n,m)}$ equivariant homology of our generalized affine Springer $\mathcal{M}_{(n,m)}$ is spanned by such classes $|A\rangle$. See [51] for more details. Let $\mathfrak{A}_{(n,m)}$ denote the set of such A:

$$\mathfrak{A}_{(n,m)} = \{A \in \mathbb{Z}^n | 0 \leq A_n \leq ... \leq A_1 \leq A_n + m\}, \tag{3.3.6}$$

then we have

$$H_\bullet^{\mathbb{C}^*_{(n,m)}}(\mathcal{M}_{(n,m)}) \overset{\text{localization}}{\simeq} \bigoplus_{A \in \mathfrak{A}_{(n,m)}} \mathbb{C}\, |A\rangle \,. \tag{3.3.7}$$

We can now proceed as in Section 2.4 to determine the action of the quantized Coulomb-branch chiral ring, *i.e.* the spherical subalgebra $\overline{\mathcal{H}}_n^{\text{sph}}$. Note that, as discussed in Section 2.3.4, since the stabilizer $\mathbb{C}^*_{(n,m)}$ can be identified with a proper subgroup of $\mathbb{C}^*_F \times \mathbb{C}^*_\varepsilon$ the deformation and quantization parameters of $\overline{\mathcal{H}}_n^{\text{sph}}$ must be specialized. Physically, this says that the Omega background is only compatible with a single value of the complex mass: we need $m_{\mathbb{C}} - \frac{1}{2}\varepsilon = \frac{m}{n}\varepsilon$. This exactly says that $\overline{\mathcal{H}}_n^{\text{sph}}$ algebra acts with parameter m/n.

The action of the complex scalars φ_a on the fixed-point class $|A\rangle$ can be found by investing the compensating gauge transformation required to keep $[z^{-A}]$ fixed under the action of $\mathbb{C}^*_{(n,m)}$. We find that they action is on the class $|A\rangle$ is given by

$$\varphi_a\, |A\rangle = \big(A_a + \tfrac{1}{2} - \tfrac{m(n-a)}{n}\big)\varepsilon\, |A\rangle \,. \tag{3.3.8}$$

This says that the distinguished operator $H = -(\varphi_1 + ... + \varphi_n)$ acts as

$$H\, |A\rangle = (\tfrac{1}{2}(mn - m - n) - \mathfrak{n})\varepsilon\, |A\rangle \,, \tag{3.3.9}$$

[6]The corresponding ideal in the Hilbert scheme is generated by the monomials $z^{A_a} w^{n-a}$ for $a = 1, ..., n$. The constraint $A_n \leq ... \leq A_1 \leq A_n + m$ ensures that the subspace generated by the $z^{A_a} w^{n-a}$ over series in z is closed under multiplication by w and compatible with $w^n = z^m$. The constraint $0 \leq A_n$ ensures that this w-closed subspace is actually an ideal, as opposed to a fractional ideal.

where $\mathfrak{n} = A_1 + ... + A_n$. It follows that the character of this representation is given by

$$\chi_q\big(H_\bullet^{L_{(n,m)}}(\mathcal{M}_{(n,m)})\big) = q^{\frac{m+n-mn}{2}} \sum_{A\in\mathfrak{A}_{(n,m)}} q^{\mathfrak{n}} = \frac{q^{\frac{m+n-mn}{2}}}{1-q^n} \begin{bmatrix} n+m-1 \\ n-1 \end{bmatrix}_q , \tag{3.3.10}$$

where

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \prod_{i=1}^{b-1} \frac{1-q^{a-i}}{1-q^{i+1}} \tag{3.3.11}$$

is the q-binomial coefficient.

The fixed point class $[z^\lambda]$ acts via Eq. (2.4.29):

$$[z^\lambda]\,|A\rangle = \Bigg(\prod_{\lambda_a<0}\prod_{\alpha=0}^{|\lambda_a|-1}(A_a-\alpha+\tfrac{m(n-a)}{n})\varepsilon\Bigg)\Bigg(\prod_{\lambda_a>\lambda_b}\prod_{\beta=0}^{\lambda_a-\lambda_b-1}(A_b-A_a-\beta+\tfrac{m(b-a+1)}{n})\varepsilon\Bigg)\,|A+\lambda\rangle\,. \tag{3.3.12}$$

The first term comes form the tangent vectors in the $\mathcal{O}^n$ directions, and the second term comes from tangent vectors in the $\mathfrak{gl}(n,\mathcal{O})$ directions. We see that $[z^\lambda]\,|A\rangle = 0$ if and only if $A' = A + \lambda$ if one of these two products vanish. For $\gcd(n,m) = 1$, the first term vanishes if and only if $A'_n = A_n + \lambda_n < 0$. The second term vanishes if and only if $A'_{a+1} > A'_a$ for some a or if $A'_1 > A'_n + m$. Namely, $[z^\lambda]\,|A\rangle = 0$ if and only if $[z^{-(A+\lambda)}]$ lies outside the generalized affine Springer fiber over $(W_{(n,m)}, e_n)$. The honest homology classes, *i.e.* those arising from the double orbits $GL(n,\mathcal{O})' z^\lambda GL(n,\mathcal{O})$ for λ minuscule, can be expressed in terms of the fixed point classes via a straightforward generalization of (2.4.28)

$$[\mathrm{Gr}^{\leq\lambda}_{GL(n,\mathbb{C})}] = \sum_{w\in W/W_\lambda} \frac{[z^{w.\lambda}]}{e(T_{w.\lambda}\mathrm{Gr}^{\leq\lambda}_{GL(n,\mathbb{C})})}, \tag{3.3.13}$$

We can show that this representation is irreducible if there is a unique singular vector, *i.e.* the mutual kernel of the homology classes $[\mathrm{Gr}^{\leq\lambda}_{GL(n,\mathbb{C})}]$, and their dressed versions, for all of the minuscule cocharacters $\lambda_\ell = (0,...,0,-1,...,-1)$ for $\ell = 1,...,n$ is 1 dimensional. In [51] it is shown that $|0\rangle$ is the unique singular vector, as expected. The proof works inductively, showing that any fixed-point class $|A\rangle$ in the kernel of the dressed monopoles for $\lambda = (0, ..., 0, -1, ..., -1)$, with i -1's, must have $A_a = 0$ for all $a \geq i$.

In order to identify this irreducible representation, we recall that the spherical subalgebra $\mathcal{H}_n^{\text{sph}}$ has a unique finite-dimensional, irreducible representation $eL_{m/n}$. Moreover, the irreducible modules of $\mathcal{H}_n^{\text{sph}}$ are of the form $\overline{V} = \mathbb{C}[X] \otimes V$ for V an irreducible representation of $\mathcal{H}_n^{\text{sph}}$. Finally, since X has q-degree 1, it follows that the dimension of V (if it is finite) can be expressed as

$$\dim_{\mathbb{C}} V = \lim_{q \to 1} \frac{\chi_q(\overline{V})}{\chi_q(\mathbb{C}[X])} = \lim_{q \to 1} \frac{\chi_q(\overline{V})}{1-q} \,. \tag{3.3.14}$$

For the irreducible representation $H_\bullet^{\mathbb{C}^*_{(n,m)}}(\mathcal{M}_{(n,m)})$, we find

$$\lim_{q \to 1} \frac{\chi_q\big(H_\bullet^{\mathbb{C}^*_{(n,m)}}(\mathcal{M}_{(n,m)})\big)}{(1-q)} = \frac{1}{n}\binom{n+m-1}{n-1} = \frac{(n+m-1)!}{n!\,m!} \tag{3.3.15}$$

is finite and equal to the dimension of $eL_{m/n}$ [100, 258]. Thus, we conclude that

$$H_\bullet^{\mathbb{C}^*_{(n,m)}}(\mathcal{M}_{(n,m)}) \simeq e\overline{L}_{m/n} \tag{3.3.16}$$

as representations for $\overline{\mathcal{H}}_n^{\text{sph}}$.

We will end by mentioning that we saw an incarnation of this result in Section 2.4.1. In particular, there is a natural limit $m \to \infty$ of the torus knots $K_{(n,m)}$, denoted $K_{(n,\infty)}$, which corresponds to the considering the polynomial $p_{K_{(n,\infty)}} = w^n$. Running the above argument leads to the generalize affine Springer fiber $\mathcal{M}_{(n,\infty)}$ over $(W_{(n,\infty)}, e_n)$; we saw the case of $n = 2$ in Section 2.4.1. The stabilizer in this case is actually all of $\mathbb{C}^*_F \times \mathbb{C}^*_\varepsilon$ and it's action has isolated fixed points, whence the action can be defined for any $m_{\mathbb{C}}, \varepsilon$. Again, the class $|0\rangle$ is the unique singular state for generic $m_{\mathbb{C}}$, but there are an non-trivial submodule when the complex mass and Omega background parameter align as $m_{\mathbb{C}} - \frac{1}{2}\varepsilon = \frac{m}{n}\varepsilon$ for integers $m, n \geq 1$; the submodule generated by $|0\rangle$ cannot reach the fixed-point classes with $A_1 > A_n + m$ for these special parameters. This representation is equivalent to a Verma module, which has an irreducible quotient $e\overline{L}_{(n,m)}$ at these special values of the parameters with $\gcd(n, m) = 1$.

3.3.2 The rational Cherednik algebra and parabolic Hilbert schemes

The analysis of the convolution algebra $H_\bullet(\mathcal{M}_{\text{rav}}(\mathcal{B}_R; \mathcal{L}^!_k, \mathcal{L}^!_k))$ is highly non-trivial for $k > 0$. Thankfully, there may be a workaround coming from the theory of rational Cherednik algebras. The work of Braverman-Etingof-Finkelberg [249] considers the line operator $\mathbb{V}_{\text{RCA}}$ given by the geometric data of $\mathcal{G}_0 = \mathcal{I}$, for $\mathcal{I}$ an Iwahori subgroup of $GL(n, \mathcal{K})$, and $R_0 = \text{Lie}\mathcal{I} \oplus \mathcal{O}^n$ and show that the convolution algebra $H^{\mathbb{C}^*_F \times \mathbb{C}^*_\varepsilon}_\bullet(\mathcal{M}_{\text{rav}}(\mathcal{B}_R; \mathbb{V}_{\text{RCA}}, \mathbb{V}_{\text{RCA}}))$ realizes the full rational Cherednik algebra

$$\overline{\mathcal{H}}_n \simeq H^{\mathbb{C}^*_F \times \mathbb{C}^*_\varepsilon}_\bullet(\mathcal{M}_{\text{rav}}(\mathcal{B}_R; \mathbb{V}_{\text{RCA}}, \mathbb{V}_{\text{RCA}}))\,. \tag{3.3.17}$$

Moreover, using [51, 253], the homology of the generalized affine Springer fiber over the same (W_K, e_n) can be identified with the parabolic flag Hilbert scheme for the variety $\widehat{Z}_K$

$$\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{(W_K, e_n)}; \mathbb{V}_{\text{RCA}}) \simeq \text{PHilb}^\bullet(\widehat{Z}_K) = \left\{ \begin{array}{c} \text{flags of ideals of } I^\bullet = I^1 \supset I^2 \supset \ldots \supset I^n \supset zI^1 \\ \text{such that } \dim_{\mathbb{C}} I^i/I^{i+1} = 1 \end{array} \right\}. \tag{3.3.18}$$

For the case where $K = K_{(n,m)}$ is a torus knot, the equivariant homology of $\text{PHilb}^\bullet(\widehat{Z}_{K_{(n,m)}})$ admits an action of the rational Cherednik algebra using the usual generalized affine Springer theory machinery and one finds

$$H^{\mathbb{C}^*_{(n,m)}}_\bullet(\mathcal{M}_{\mathbb{D}}(\mathcal{B}_{(W_{K_{(n,m)}}, e_n)}) \simeq \overline{L}_{m/n}\,. \tag{3.3.19}$$

As suggested by [100], the k-th row of HOMFLY-PT homology should be obtained by projecting to the isotypic component transforming as $\wedge^k \Box$ should then yield the a-degree k part of the HOMFLY-PT homology. The action of (an appropriately sphericized subalgebra of) the rational Cherednik on this component would yield the desired action.

A natural, physical way to obtain such a projection is as follows. Since local operators bound to interfaces between two line operators are bi-modules for the corresponding algebras of local operators bound to each line operator. We can then tensor the bimodule at the junction between $\mathbb{V}_{\mathrm{RCA}}$ and $\mathcal{L}^!_k$ with the the module $\rho_{\mathcal{B}^!_K}(\mathbb{V}_{\mathrm{RCA}})$ for the rational Cherednik algebra to obtain the module $\rho_{\mathcal{B}^!_K}(\mathbb{V}_{\mathrm{RCA}})$ for $\mathrm{End}_{\mathcal{C}_A}(\mathcal{L}^!_k, \mathcal{L}^!_k)$. Physically, this final module corresponds to realizing boundary local operators $\rho_{\mathcal{B}^!_K}(\mathbb{V}_{\mathrm{RCA}})$ as collisions of local operators at the junction between $\mathbb{V}_{\mathrm{RCA}}$ and $\mathcal{L}^!_k$ with the boundary local operators $\rho_{\mathcal{B}^!_K}(\mathbb{V}_{\mathrm{RCA}})$, *cf.* Figure 2.4.

Many operators bound to $\mathcal{L}^!_k$ can be realized by colliding the above interface and its conjugate, called "sandwiching" in [111]. In particular, a local operator bound to $\mathbb{V}_{\mathrm{RCA}}$ surrounded by a local operator at a $\mathcal{L}^!_k \to \mathbb{V}_{\mathrm{RCA}}$ interface below and by an local operator at a $\mathbb{V}_{\mathrm{RCA}} \to \mathcal{L}^!_k$ interface above, upon colliding the two interfaces, gives a local operator bound to the $\mathcal{L}^!_k$ line operator. The action of the resulting local operator then be obtained by successively colliding these three separate local operators with the boundary. See Figure 3.10. This seems to be a promising route for understanding both the algebra of local operators bound to the line operator $\mathcal{L}^!_k$ and their action on the higher a-degrees of HOMFLY-PT homology, at least for algebraic knots and links.

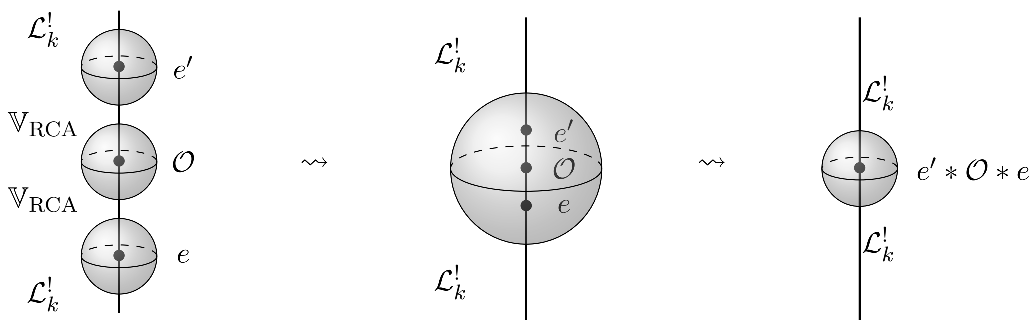


Figure 3.10: An illustration of the "sandwiching" process described in [111], whereby a local operator bound to $\mathcal{L}^!_k$ can be obtained from local operators on $\mathbb{V}_{\mathrm{RCA}}$ by colliding with $\mathcal{L}^!_k \to \mathbb{V}_{\mathrm{RCA}}$ and $\mathbb{V}_{\mathrm{RCA}} \to \mathcal{L}^!_k$ interfaces.

Conclusion

We end with a brief summary of the results of this thesis and mention several directions for future research.

As emphasized in Chapter 1, and used to great effect in Chapter 2, a remarkably powerful perspective on 3d $\mathcal{N}=4$ theories comes from viewing the theory as a supersymmetric quantum mechanical theory with infinite dimensional target space. Modulo subtleties about infinite dimensionality and singular behavior, much of the structure of the topological twists of the 3d theory follow from considerations in the corresponding twisted quantum mechanics problem. When applied to the topological A-twist of a 3d $\mathcal{N}=4$ gauge theory, this naturally leads to the recent construction of the Coulomb branches of these theories due to BFN [72], as described in Section 1.3.3, and the related physical analysis of [67], realized via a fixed-point localization or "abelianization" procedure. Moreover, this perspective is compatible with inserting line operators running along the "time" axis, and thus can be used to describe A-type line operators, as described in Chapter 2.

Using this framework, we propose a mathematically precise, geometric category that models the category of A-type line operators in these gauge theories, and provide means by which one can perform explicit computations in this category. Physically, we introduce a large class of vortex line operators realized by coupling to an auxiliary quantum mechanical system supported on the line or, equivalently, by specifying a breaking of gauge symmetry in the neighborhood of the line and a compatible singularity structure for the hypermultiplets. Moreover, we determine the vector spaces of (possibly dressed) monopole operators that can be used to join two such vortex line operators, as well as how to collide these junctions with one another.

As witnessed by the Chern-Simons/WZW correspondence [41] and its analog in 3d $\mathcal{N} = 4$ theories [53, 61], there is a rich interplay between boundary conditions for a 3d TQFT and the TQFT's line operators. In those contexts, local operators at the junction of a bulk line operator and the boundary are viewed as modules for the algebra of boundary local operators. Alternatively, we can view this same junction as a module for local operators bound to the bulk line operator, or, better, a representation of the category of line operators, *cf.* Section 2.1.3. Of course, these two module structures should be compatible with one another and this should lead to non-trivial statements for both perspectives. For example, if we know the category of bulk line operators is a semisimple category, then so too must be the (suitably defined) category of modules for the algebra of boundary local operators.

When we take this latter perspective for A-type vortex line operators in 3d $\mathcal{N} = 4$ gauge theories, there is a natural connection to the far-reaching mathematical subject known as (generalized affine) Springer theory. Much like the supersymmetric Hilbert spaces in supersymmetric quantum mechanics, we explicitly realize the vector space of boundary local operators for certain Dirichlet boundary conditions as the cohomology (really, Borel-Moore homology) of a certain moduli space of BPS configurations. The moduli spaces that arise admit an algebraic reformulation in term of a generalization of the classical notion of affine Springer fibers, which are fundamental objects in geometric representation theory [243–247]; indeed, the collision of local operators bound to these line operators with the boundary generalizes the actions that appear in (affine) Springer theory. Although the moduli spaces can be quite complex, a particularly simple situation arises when the theory has enough flavor symmetry so that there are isolated vacua in the presence of generic complex masses. In particular, when the Dirichlet boundary condition is compatible with the complex masses, this induces an explicit description of the vector space of boundary local operators at the junction with a given vortex line operator, as well as an action of local operators bound to the corresponding line operator (or joining it to another line operator).

Using the description in terms of generalized affine Springer theory, it would be interesting to approach the more traditional perspective in terms of modules for the algebra of boundary local operators. In particular, generalized affine Springer fibers in the affine Grassmannian should admit an algebra structure and those in other affine flag varieties should admit a module structure for the corresponding algebra. This boundary algebra should have non-trivial monopole operators and should admit a description similar to local operators on Dirichlet boundary conditions for 3d $\mathcal{N} = 2$ gauge theories, *cf.* [152].

Instead of pursuing the algebra structure on the boundary local operators on Dirichlet boundary conditions for 3d $\mathcal{N} = 4$ gauge theories, in Chapter 3 we use the above analysis to test a proposed 3d mirror to a recent construction of HOMFLY-PT knot homology due to Oblomkov-Rozansky [94, 96]. Oblomkov-Rozansky propose a sophisticated mathematical construction of this knot homology that is rooted in the topological B-twist of a 3d $\mathcal{N} = 4$ gauge theory, which we describe in Section 3.1.2. Applying 3d mirror symmetry to the various ingredients of their construction, we arrive at a setup in the 3d A-twist and can apply the above machinery of vortex line operators ending on Dirichlet boundary conditions to show that it reproduces yet another (conjectural) instance of HOMFLY-PT homology known to arise from generalized affine Springer theory [99, 100].

There are still several aspects of this A-twist construction that still need to be worked out. First, it is most easily applied to a class of knots/links known as positive algebraic knots/links; one important direction is to generalize the construction to other classes of knots/links, *cf.* Section 3.2.3. Second, even for positive algebraic knots/links it is easiest to understand the "lowest row" of HOMFLY-PT homology, but the higher rows are somewhat more delicate and deserve a more detailed understanding. Finally, it would be interesting understand how to extend the present construction to colored HOMFLY-PT invariants and what this corresponds to in the mirror B-twist construction of Oblomkov-Rozansky.

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