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UNIVERSITY OF CALIFORNIA, SAN DIEGO

**An Analysis of the Multiplicity Spaces in Classical Symplectic Branching**

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Oded Yacobi

Committee in charge:

Professor Nolan Wallach, Chair
Professor Thomas Enright
Professor George Fuller
Professor Wee Teck Gan
Professor Aneesh Manohar

2009

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University of California, San Diego

2009

DEDICATION

לסבא יצחק ולזכרו של סבא עמנואל

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Chapter 7, in full, is a reprint of the material as it will appear in the paper

*A multiplicity formula for tensor products of $SL_2$ modules and an explicit $Sp_{2n}$ to $Sp_{2n-2} \times Sp_2$ branching formula* in Contemp. Math., American Mathematical Society, Providence, R.I., 2009; co-authored with Nolan R. Wallach. I was the principal author of this paper, and made substantial contributions to the research as did my co-author.

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ABSTRACT OF THE DISSERTATION

**An Analysis of the Multiplicity Spaces in**

**Classical Symplectic Branching**

by

Oded Yacobi

Doctor of Philosophy in Mathematics

University of California San Diego, 2009

Professor Nolan Wallach, Chair

The purpose of this dissertation is to develop a new approach to Gelfand-Zeitlin theory for the rank $n$ symplectic group $Sp(n,\mathbb{C})$. Classical Gelfand-Zeitlin theory rests on the fact that branching from $Gl(n,\mathbb{C})$ to $Gl(n-1,\mathbb{C})$ is multiplicity-free. Since branching from $Sp(n,\mathbb{C})$ to $Sp(n-1,\mathbb{C})$ is not multiplicity-free, the theory cannot be directly applied to this case.

Let $L$ be the $n$-fold product of $SL(2,\mathbb{C})$. Our main theorem asserts that each multiplicity space that arises in the restriction of an irreducible representation of $Sp(n,\mathbb{C})$ to $Sp(n-1,\mathbb{C})$, has a unique irreducible $L$-action satisfying certain naturality conditions. We also give an explicit description of the $L$-module structure of each multiplicity space. As an application we obtain a Gelfand-Zeitlin type basis for all irreducible finite dimensional representations of $Sp(n,\mathbb{C})$.

# Chapter 1

# Introduction

## 1.1 The problem

The goal of this dissertation is to develop a new approach to Gelfand-Zeitlin theory for the complex symplectic group. Our main theorem realizes the multiplicity spaces that arise in classical symplectic branching as irreducible modules for a certain product of $SL(2,\mathbb{C})$. The primary application of this theorem is the construction of a Gelfand-Zeitlin type basis for all irreducible finite dimensional representations of the symplectic group.

Before describing our results in more detail, we review the classical Gelfand-Zeitlin theory for the general linear group. Fix $n$ a positive integer, and let $\Lambda_n^+$ be the set of weakly decreasing sequences of length $n$ consisting of non-negative integers. To an element $\lambda \in \Lambda_n^+$ we associate the irreducible (polynomial) representation $V_\lambda$ of $GL(n,\mathbb{C})$ with highest weight $\lambda$.

The general linear groups naturally form a chain

$$GL(n,\mathbb{C}) \supset GL(n-1,\mathbb{C}) \supset \cdots \supset GL(1,\mathbb{C}).$$

Now suppose $1 \leq m < n$ are given and $\lambda \in \Lambda_n^+$. We can view $V_\lambda$ as a $GL(m,\mathbb{C})$ by restriction, and the decomposition of $V_\lambda$ into $GL(m,\mathbb{C})$-modules can be expressed as

$$V_\lambda \cong \bigoplus_{\mu \in \Lambda_m^+} V_\mu \otimes Hom_{GL(m,\mathbb{C})}(V_\mu, V_\lambda).$$

The factor $M_\mu^\lambda = Hom_{GL(m,\mathbb{C})}(V_\mu, V_\lambda)$ is called the **multiplicity space** of $V_\mu$ in $V_\lambda$, since the dimension of $M_\mu^\lambda$ is equal to the multiplicity of $V_\mu$ in $V_\lambda$. Determining the dimension of $M_\mu^\lambda$ for all $\mu \in \Lambda_m^+$ and $\lambda \in \Lambda_n^+$ is known as the **branching multiplicity problem** from $GL(n,\mathbb{C})$ to $GL(m,\mathbb{C})$.

The branching multiplicity problem from $GL(n,\mathbb{C})$ to $GL(n-1,\mathbb{C})$ is well-known. It states that for all $\mu \in \Lambda_{n-1}^+$ and $\lambda \in \Lambda_n^+$, $\dim M_\mu^\lambda \leq 1$, and $M_\mu^\lambda \neq \{0\}$ if, and only if, $\mu$ "interlaces" $\lambda$. In particular, the branching is **multiplicity-free**, which means that an irreducible representation of $GL(n,\mathbb{C})$ decomposes uniquely into irreducible representations of $GL(n-1,\mathbb{C})$. The condition that $\mu$ interlaces $\lambda$, written $\mu < \lambda$, means that the parameters of $\mu$ are bounded above and below by the parameters of $\lambda$ (cf. Definition 3.1.1).

Using this result, Gelfand and Zeitlin constructed a basis for irreducible representations of $GL(n,\mathbb{C})$ [8]. Their construction proceeds as follows. We

restrict the representation $V_\lambda$ ($\lambda \in \Lambda_n^+$) to $GL(n-1,\mathbb{C})$:

$$V_\lambda \cong \bigoplus_{\substack{\mu \in \Lambda_{n-1}^+ \\ \mu < \lambda}} V_\mu \otimes M_\mu^\lambda.$$

Now restrict the resulting representation to $GL(n-2,\mathbb{C})$:

$$V_\lambda \cong \bigoplus_{\substack{\mu \in \Lambda_{n-1}^+ \\ \mu < \lambda}} \bigoplus_{\substack{\eta \in \Lambda_{n-2}^+ \\ \eta < \mu}} V_\eta \otimes M_\eta^\mu \otimes M_\mu^\lambda.$$

We continue restricting in this fashion until we obtain a decomposition into $GL(1,\mathbb{C})$-modules:

$$V_\lambda \cong \bigoplus_{\substack{\lambda^{(i)} \in \Lambda_i^+ \\ \lambda^{(i)} < \lambda^{(i+1)} \\ \lambda^{(n-1)} < \lambda}} V_{\lambda^{(1)}} \otimes M_{\lambda^{(1)}}^{\lambda^{(2)}} \otimes \cdots \otimes M_{\lambda^{(n-1)}}^{\lambda}. \tag{1.1}$$

Notice that $\dim V_{\lambda^{(1)}} = 1$, being an irreducible representation of $GL(1,\mathbb{C})$. Moreover, the multiplicity spaces $M_{\lambda^{(i-1)}}^{\lambda^{(i)}}$ are all one-dimensional by the branching results stated above. Therefore we've canonically decomposed the irreducible representation $V_\lambda$ into one-dimensional spaces. Choosing a nonzero vector in each summand, we obtain a basis of $V_\lambda$ indexed by **Gelfand-Zeitlin patterns**:

$$\{(\lambda^{(1)}, ..., \lambda^{(n)}) : \lambda^{(i)} \in \Lambda_i^+,\ \lambda^{(i)} < \lambda^{(i+1)}, \text{ and } \lambda^{(n)} = \lambda\}.$$

This basis is the **Gelfand-Zeitlin basis** of $V_\lambda$. The basis vectors are unique up to scalar, they are weight vectors for the diagonal torus, and they are orthogonal with respect to the natural inner product on representations of $GL(n,\mathbb{C})$.

The multiplicity-free branching of the chain of general linear groups is useful in many contexts. For example, Guillemin and Sternberg used it to construct a

completely integrable system on the flag manifold [10]. More recently, Kostant and Wallach used a classical mechanical analogue of the Gelfand-Zeitlin algebra to construct a polarization of regular adjoint orbits of $M_n(\mathbb{C})$ [17]. Other applications of classical Gelfand-Zeitlin theory can be found in the literature (e.g. to Poisson Lie groups in [2], and to Schubert varieties in [15]).

It is natural to ask if these results can be carried over to the other classical groups? It turns out that there is a version of Gelfand-Zeitlin theory for the special orthogonal groups, since relative to the standard inclusion

$$SO(n, \mathbb{C}) \supset SO(n-1, \mathbb{C})$$

the branching is also multiplicity-free. Thus there is an analogue of the canonical weight basis for the special orthogonal groups [9]. Many of the applications of Gelfand-Zeitlin theory can be generalized to this setting (e.g. [3]).

This leaves the remaining family of classical groups, the symplectic groups. The symplectic groups also naturally form a chain

$$Sp(n, \mathbb{C}) \supset Sp(n-1, \mathbb{C}) \supset \cdots \supset Sp(1, \mathbb{C})$$

and we can ask if there is a canonical basis for the irreducible representations of $Sp(n, \mathbb{C})$ constructed by restricting down this chain. Unfortunately, this restriction is not multiplicity-free. Therefore the Gelfand-Zeitlin theory is not easily adapted to this setting.

This problem has been addressed by many authors. In this dissertation, we describe a new solution to this problem. Our solution is an application of a

more general theorem, which describes a remarkable irreducible action of an $n$-fold product of $SL(2,\mathbb{C})$ on the multiplicity spaces that arise in branching from $Sp(n,\mathbb{C})$ to $Sp(n-1,\mathbb{C})$. Before describing our results in more detail, we discuss some of the history of this problem; a more comprehensive treatment is found in [21].

## 1.2 Historical background

Let $\mathfrak{sp}(n,\mathbb{C}) = Lie(Sp(n,\mathbb{C}))$. One approach to construct a basis for irreducible representations of the symplectic group is to introduce an intermediate algebra, the "odd symplectic algebra" $\mathfrak{sp}(n-1/2,\mathbb{C})$, between $\mathfrak{sp}(n-1,\mathbb{C})$ and $\mathfrak{sp}(n,\mathbb{C})$. One can restrict an irreducible representation of $\mathfrak{sp}(n,\mathbb{C})$ first to $\mathfrak{sp}(n-1/2,\mathbb{C})$, and then to $\mathfrak{sp}(n-1,\mathbb{C})$. The difficulty is that $\mathfrak{sp}(n-1/2,\mathbb{C})$ is not reductive; indeed it is isomorphic $\mathfrak{sp}(n-1,\mathbb{C}) \oplus \mathfrak{H}$, where $\mathfrak{H}$ is the $2n-1$ dimensional Heisenberg Lie algebra. Nevertheless, this approach was used to resolve multiplicities in the following sense: there is a composition series on any irreducible $\mathfrak{sp}(n,\mathbb{C})$-module by $\mathfrak{sp}(n-1/2,\mathbb{C})$-submodules, such that the corresponding factors are inequivalent and have simple spectra [23].

Another approach uses the restriction of an irreducible representation of $\mathfrak{gl}(2n,\mathbb{C})$ to $\mathfrak{sp}(n,\mathbb{C})$, where $\mathfrak{gl}(2n,\mathbb{C}) = Lie(GL(2n,\mathbb{C}))$. This idea was first developed by Gould and Kalnins [7]. They construct a basis for irreducible representations of $\mathfrak{sp}(n,\mathbb{C})$ parameterized by a certain subset of Gelfand-Zeitlin patterns

for $GL(2n, \mathbb{C})$. This approach was also recently used by Kim to develop a standard monomial theory for the symplectic group [14]. The symplectic flag algebra is a quotient of a suitable flag algebra for $GL(2n, \mathbb{C})$, and Kim constructs a Grobner basis for the associated ideal. From this he obtains a standard monomial bases for irreducible representations of the symplectic group.

Finally, an approach using the theory of quantum groups was used by Molev to obtain a canonical weight basis for irreducible representations of the symplectic group [20]. We review Molev's approach in some detail since it is most closely related to ours.

Let $U(\mathfrak{sp}(n, \mathbb{C}))$ be the universal enveloping algebra of $\mathfrak{sp}(n, \mathbb{C})$. The centralizer algebra $\mathcal{C} = U(\mathfrak{sp}(n, \mathbb{C}))^{\mathfrak{sp}(n-1, \mathbb{C})}$ acts irreducibly on each multiplicity space that occurs in branching from $Sp(n, \mathbb{C})$ to $Sp(n-1, \mathbb{C})$ [5]. By understanding this irreducible action, one might hope to extract a canonical basis for each multiplicity space and thus obtain a basis for the whole representation by induction. The difficulty is that $\mathcal{C}$ is a complicated algebra, and understanding its representation theory seems to be an intractable problem.

Molev gets around this by using a class of quantized enveloping algebras, called twisted Yangians [19]. The twisted Yangian, $Y^-(2)$, is an infinite-dimensional Hopf algebra, which is a flat deformation of the universal enveloping algebra of a suitable polynomial current Lie algebra. The Yangian has been actively studied in recent decades. Drinfeld showed that there is a highest weight theory classifying the irreducible finite-dimensional representations of $Y^-(2)$ [6]. Moreover, Molev

constructed a Gelfand-Zeitlin type basis for these representations [19].

The Yangian is related to branching of the symplectic groups by an algebra map, first realized by Olshanski ([22]), from $Y^-(2)$ to the centralizer algebra $\mathcal{C}$:

$$Y^-(2) \to \mathcal{C}.$$

Using this map, one can pull-back the action of $\mathcal{C}$ to obtain an action of the $Y^-(2)$ on the multiplicity spaces. Molev identifies each multiplicity space as an explicit irreducible representation of $Y^-(2)$. He thus obtains a basis for each multiplicity space, which is then natural from the point of view of Yangian theory.

## 1.3 Overview of main results

Similar to Molev's results discussed above, we realize the multiplicity spaces that arise in branching from $Sp(n, \mathbb{C})$ to $Sp(n-1, \mathbb{C})$ as irreducible modules for a group whose representation theory is well understood. In our case, the group is the $n$-fold product of $SL(2, \mathbb{C})$:

$$L = SL(2, \mathbb{C}) \times \cdots \times SL(2, \mathbb{C}).$$

We show that each multiplicity space has a unique irreducible $L$-action satisfying certain naturality conditions, and we give an explicit description of this $L$-module. Moreover, this $L$-action extends the obvious $Sp(1, \mathbb{C})$ $(= SL(2, \mathbb{C}))$ action on the multiplicity spaces. (That is, the restriction of the action of $L$ to its diagonal subgroup is the action of this $Sp(1, \mathbb{C})$.) Using these results and

elementary representation theory of $SL(2,\mathbb{C})$, we then obtain a canonical weight basis for all irreducible representations of $Sp(n,\mathbb{C})$.

To set up the proof (and statement) of this result we need to undertake a thorough investigation of the multiplicity spaces that arise in branching of the symplectic groups. These investigations proceed in the following manner. For every statement about branching from $Sp(n,\mathbb{C})$ to $Sp(n-1,\mathbb{C})$, we articulate an analogous statement about branching from $GL(n+1,\mathbb{C})$ to $GL(n-1,\mathbb{C})$. These latter statements are manageable, because of the presence of $GL(n,\mathbb{C})$. We then transfer these results back to the symplectic group using Theorem 4.1.2 (stated below), which is an isomorphism of the corresponding "branching algebras", in the sense of [13].

In order to explain our results in more detail we need to introduce some more notation; this notation will be developed more thoroughly in Chapter 2. An element $\lambda \in \Lambda_n^+$ will be thought of interchangeably as a dominant weight for $GL(n,\mathbb{C})$ or $Sp(n,\mathbb{C})$. Let $W_\lambda$ be the irreducible representation of $Sp(n,\mathbb{C})$ of highest weight $\lambda$. Fix a Borel subgroup $B_n \subset GL(n,\mathbb{C})$ (resp. $B_{C_n} \subset Sp(n,\mathbb{C})$) with a decomposition

$$B_n = T_n U_n \text{ (resp. } B_{C_n} = T_{C_n} U_{C_n})$$

into a maximal torus and unipotent radical.

Let $1 \leq m < n$ and let $\mu \in \Lambda_m^+$ and $\lambda \in \Lambda_n^+$. We identify the multiplicity space $M_\mu^\lambda$ with the space $V_\lambda^{U_m}(\mu)$, of $GL(m,\mathbb{C})$ highest weight vectors in $V_\lambda$ of

weight $\mu$. Similarly, set $\widehat{M}_\mu^\lambda = W_\lambda^{U_{C_m}}(\mu)$.

In Chapter 3 we study what we call double interlacing patterns. Let $\mu \in \Lambda_{n-1}^+$ and $\lambda \in \Lambda_{n+1}^+$. Then $M_\mu^\lambda \neq \{0\}$ if, and only if, $\mu$ **double interlaces** $\lambda$ (cf. Definition 3.1.2). We write $\mu \ll \lambda$ to express this relation, and call the tuple $(\mu, \lambda)$ a **double interlacing pattern**. Our object of study is the set

$$\mathfrak{D} = \{(\mu, \lambda) \in \Lambda_{n-1}^+ \times \Lambda_{n+1}^+ : \mu \ll \lambda\}.$$

$\mathfrak{D}$ is a semigroup which is exhausted by a family of sub-semigroups $\{\mathfrak{D}_\sigma\}_{\sigma \in \Sigma}$ indexed by the set $\Sigma$ of **order types** (cf. Section 3.1). One of the main results in this chapter is that these sub-semigroups are remarkably simple; they are each isomorphic to $\Lambda_{2n}^+$ (cf. Lemma 3.1.5).

In this chapter we also prove the fundamental results about branching from $GL(n+1, \mathbb{C})$ to $GL(n-1, \mathbb{C})$, that we will later transfer to the symplectic group. Let $F_k$ be the irreducible representation of $SL(2, \mathbb{C})$ of dimension $k+1$.

**Proposition 3.2.2** *Let $\mu \in \Lambda_{n-1}^+$, $\lambda \in \Lambda_{n+1}^+$. Then $M_\mu^\lambda \neq \{0\}$ if, and only if, $\mu \ll \lambda$. Moreover, if $\mu \ll \lambda$ then as $SL(2, C)$-modules*

$$M_\mu^\lambda \cong \bigotimes_{i=1}^{n} F_{r_i(\mu,\lambda)}$$

*where $SL(2, C)$ acts by the tensor product representation on the right hand side, and*

$$r_i(\mu, \lambda) = \min(\mu_{i-1}, \lambda_i) - \max(\mu_i, \lambda_{i+1}).$$

*In particular,*

$$\dim M_\mu^\lambda = \prod_{i=1}^{n} (r_i(\mu, \lambda) + 1).$$

Our second result in this chapter has to do with Cartan product of multiplicity spaces. For an affine algebraic set $X$, let $\mathcal{O}(X)$ denote its ring of regular functions. Consider the ring

$$\mathcal{M} = \mathcal{O}(GL(n+1, \mathbb{C}))^{\overline{U}_{n+1} \times U_{n-1}}$$

of functions on $GL(n+1, \mathbb{C})$ invariant under right translation by $U_{n-1}$ and under left translation by $\overline{U}_{n+1}$ (the unipotent radical of the Borel subgroup opposite to $B_{n+1}$). This ring plays a special role in studying the restriction of irreducible representations of $GL(n+1, \mathbb{C})$ to $GL(n-1, \mathbb{C})$, since it can be canonically identified with the sum of all multiplicity spaces:

$$\mathcal{M} \cong \bigoplus_{(\mu,\lambda) \in \mathfrak{D}} M_\mu^\lambda.$$

Moreover, $\mathcal{M}$ is $\mathfrak{D}$-graded: $M_\mu^\lambda M_{\mu'}^{\lambda'} \subset M_{\mu+\mu'}^{\lambda+\lambda'}$. From this multiplication we obtain a map

$$M_\mu^\lambda \otimes M_{\mu'}^{\lambda'} \overset{\pi_{\lambda,\lambda'}}{\to} M_{\mu+\mu'}^{\lambda+\lambda'}$$

which we call the **Cartan product of multiplicity spaces**.

Using Proposition 3.2.2 it's easy to see that this map is in general not surjective (cf. Remark 3.3.3). A critical result is that restricted to a fixed order type it is:

**Proposition 3.3.2** *Let $\sigma \in \Sigma$ and let $(\mu, \lambda), (\mu', \lambda') \in \mathfrak{D}_\sigma$. Then the map*

$$M_\mu^\lambda \otimes M_{\mu'}^{\lambda'} \stackrel{\pi_{\lambda,\lambda'}}{\to} M_{\mu+\mu'}^{\lambda+\lambda'}$$

*is surjective.*

In Chapter 4 our goal is to prove analogues of Propositions 3.2.2 and 3.3.2 for the symplectic group. We achieve this by a remarkable isomorphism of rings, which shows that branching from $Sp(n, \mathbb{C})$ to $Sp(n-1, \mathbb{C})$ is intimately connected to branching from $GL(n+1, \mathbb{C})$ to $GL(n-1, \mathbb{C})$. Set

$$\begin{aligned} \widehat{\mathcal{M}} &= \mathcal{O}(Sp(n, \mathbb{C}))^{\overline{U}_{C_n} \times U_{C_{n-1}}} \\ \overline{\mathcal{M}} &= \mathcal{O}(SL(n+1, \mathbb{C}))^{\overline{U}_{n+1} \times U_{n-1}} \end{aligned}$$

defined in analogy with $\mathcal{M}$ above.

These are both rings with an $SL(2, \mathbb{C})$-action by right translation, which we refer to as the **natural** $SL(2, \mathbb{C})$ -action. $\widehat{\mathcal{M}}$ is the sum over all multiplicity spaces that occur in branching from $Sp(n, \mathbb{C})$ to $Sp(n-1, \mathbb{C})$. A priori $\overline{\mathcal{M}}$ is the sum over all multiplicity spaces that occur in branching from $SL(n+1, \mathbb{C})$ to $SL(n-1, \mathbb{C})$. In Lemma 4.1.1, we show that $\overline{\mathcal{M}}$ can be identified with the sum over all the multiplicity spaces that occur in branching from $GL(n+1, \mathbb{C})$ to $GL(n-1, \mathbb{C})$ of the form $M_\mu^{\lambda^+}$. (Here $\lambda^+ \in \Lambda_{n+1}^+$ is obtained from $\lambda \in \Lambda_n^+$ by adding a zero to the end of $\lambda$.) Therefore these rings are both graded by $\mathfrak{D}$. Using results of Zhelobenko, we prove the following theorem, which is termed the "Transfer Theorem".

**Theorem 4.1.2** *There is an isomorphism* $\Psi : \widehat{\mathcal{M}} \to \overline{\mathcal{M}}$ *of* $\mathfrak{D}$*-graded rings. Moreover,* $\Psi$ *intertwines the natural* $SL(2,C)$*-actions.*

The usefulness of this theorem is that it allows us to reduce problems about branching from $Sp(n,\mathbb{C})$ to $Sp(n-1,\mathbb{C})$ to problems about branching from $GL(n+1,\mathbb{C})$ to $GL(n-1,\mathbb{C})$. The latter are more manageable because of the presence of $GL(n,\mathbb{C})$ in between. The first example of this is the next result, which follows by using Theorem 4.1.2 to transfer Proposition 3.2.2 to the symplectic group:

**Corollary 4.1.3** *Let* $\mu \in \Lambda_{n-1}^+$ *and* $\lambda \in \Lambda_n^+$. *Then* $\widehat{M}_\mu^\lambda \neq \{0\}$ *if, and only if,* $\mu \ll \lambda^+$. *Moreover, if* $\mu \ll \lambda^+$ *then as* $SL(2,C)$*-modules,*

$$\widehat{M}_\mu^\lambda \cong \bigotimes_{i=1}^{n} F_{r_i(\mu,\lambda^+)}$$

*where* $SL(2,C)$ *acts diagonally on the right hand side. In particular,* $\dim \widehat{M}_\mu^\lambda = \prod_{i=1}^n (r_i(\mu,\lambda^+)+1)$.

The above result appears in [25], where we give a different proof using the combinatorics of partition functions (cf. Chapter 7).

Another application of Theorem 4.1.2 is the following result about the Cartan product of multiplicity spaces. Here we use Theorem 4.1.2 to transfer Proposition 3.3.2 to the symplectic group:

**Corollary 4.1.4** *Let* $\mu, \mu' \in \Lambda_{n-1}^+$ *and* $\lambda, \lambda' \in \Lambda_n^+$. *Suppose there exists* $\sigma \in \Sigma$ *such that* $(\mu, \lambda^+), (\mu', (\lambda')^+) \in \mathfrak{D}_\sigma$. *Then the map*

$$\widehat{M}_\mu^\lambda \otimes \widehat{M}_{\mu'}^{\lambda'} \overset{\pi_{\lambda,\lambda'}}{\to} \widehat{M}_{\mu+\mu'}^{\lambda+\lambda'}$$

*is surjective.*

In Chapter 5 we use these corollaries to prove our main theorem. Set $\widehat{\mathfrak{D}} = \{(\mu, \lambda) \in \Lambda_{n-1}^+ \times \Lambda_n^+ : \mu \ll \lambda^+\}$, and for $p = (\mu, \lambda) \in \widehat{\mathfrak{D}}$ let $\widehat{M}_p = \widehat{M}_\mu^\lambda$. By Corollary 4.1.3, there is an isomorphism of $SL(2, \mathbb{C})$-modules

$$\widehat{\mathcal{M}} \cong \bigoplus_{p \in \widehat{\mathfrak{D}}} \bigotimes_{i=1}^{n} F_{r_i(p)}. \tag{1.2}$$

In this chapter we will show that there is a natural map exhibiting this isomorphism. We then use this map to obtain an action of $L$ on $\widehat{\mathcal{M}}$, under which $\widehat{M}_p \cong \bigotimes_{i=1}^n F_{r_i(p)}$ as irreducible $L$-modules. We also show that our action is unique, subject to a certain naturality condition, and that it extends the natural action of $SL(2, \mathbb{C})$ on $\widehat{\mathcal{M}}$. These results are summarized as follows:

**Theorem 5.1.2** *The ring $\widehat{\mathcal{M}}$ carries a unique representation of $L$, denoted $\Phi$, satisfying the following two properties:*

1. *Under $\Phi$, $\widehat{M}_p$ is an irreducible $L$-submodule of $\widehat{\mathcal{M}}$ isomorphic to $\bigotimes_{i=1}^n F_{r_i(p)}$.*

2. *$L$ acts as ring automorphisms on $\widehat{\mathcal{M}}_\sigma$ for every $\sigma \in \Sigma$.*

*Moreover, the restriction of $\Phi$ to the diagonal copy of $SL(2, \mathbb{C})$ in $L$ is the natural action of $SL(2, C)$ on $\widehat{\mathcal{M}}$.*

As a scholieum of this theorem, $\widehat{\mathcal{M}}$ contains a family of subrings, $\left\{\widehat{\mathcal{M}}_\sigma\right\}_{\sigma \in \Sigma}$, each of which is isomorphic to the polynomial ring $\mathcal{O}(V)$ for a suitable vector space $V$ (cf. Proposition 5.2.2 and Section 5.5). $L$ acts naturally on $V$, inducing an action of $L$ on $\mathcal{O}(V)$, and $\widehat{\mathcal{M}}_\sigma$ is isomorphic to $\mathcal{O}(V)$ as $L$-rings. Consequently, we see that the product of $\widehat{M}_p$ and $\widehat{M}_{p'}$ in $\widehat{\mathcal{M}}_\sigma$ can be re-formulated as the Cartan product of irreducible $L$-modules in $\mathcal{O}(V)$.

In Chapter 6 we apply Theorem 5.1.2 to construct the Gelfand-Zeitlin type basis for irreducible representations of the symplectic group. In the classical setting one defines an action of $(\mathbb{C}^\times)^{\binom{n}{2}}$ on an irreducible representations of $GL(n,\mathbb{C})$, and obtains the Gelfand-Zeitlin basis as the weight basis of this action. We define an action of $(\mathbb{C}^\times)^{n^2}$ on the irreducible representations of $Sp(n,\mathbb{C})$. The corresponding weight basis provides a canonical basis for such representations. The dimensions of these tori agree with the (complex) dimension of a generic coadjoint orbit in each case.

A **symplectic Gelfand-Zeitlin pattern** is a sequence

$$(\gamma^{(1)}, \lambda^{(1)}, ..., \gamma^{(n)}, \lambda^{(n)})$$

where $\gamma^{(i)}, \lambda^{(i)} \in \Lambda_i^+$ satisfy certain interlacing conditions (cf. Section 6.2). We associate to each symplectic Gelfand-Zeitlin pattern a basis vector which is unique up to scalar. These patterns have a very natural interpretation in our set-up via the representation theory of $L$: the $\lambda^{(i)}$ parameters $(i = 1, ..., n)$ determine the isotypic components that the associated vector lies in relative to the chain

$$Sp(n,\mathbb{C}) \supset Sp(n-1,\mathbb{C}) \supset \cdots \supset Sp(1,\mathbb{C}).$$

(This is exactly the role played by the classical Gelfand-Zeitlin patterns which parameterize bases for irreducible representations of $GL(n,\mathbb{C})$.) The "new" parameters, $\gamma^{(i)}$ $(i = 1, ..., n)$, correspond to basis vectors of irreducible $L$-modules. Here, the $\gamma^{(i)}$ parameter corresponds to a canonical basis vector in the multiplicity space $\widehat{M}^{\lambda^{(i)}}_{\lambda^{(i-1)}}$.

# Chapter 2

# Preliminaries

In this chapter we catalog our notation and some basic results that we will be using. Most of what we introduce here is standard, and the reader is encouraged to skim this chapter and refer back to it as the need arises.

## 2.1 The general linear group

Given an affine algebraic variety $X$ defined over $\mathbb{C}$, let $\mathcal{O}(X)$ denote the ring of regular functions on $X$. In particular, if $X$ is a vector space over $\mathbb{C}$, then $\mathcal{O}(X)$ is the ring of polynomial functions on $X$.

Fix a positive integer $n$. Let

$$\Lambda_n^+ = \{(k_1 \geq k_2 \geq \cdots \geq k_n) : k_i \geq 0 \text{ for } i = 1, ..., n\}$$

be the set of weakly decreasing sequences of non-negative integers of length $n$. We refer to elements of $\Lambda_n^+$ as **dominant weights**. The set of dominant weights, $\Lambda_n^+$,

is a semigroup under componentwise addition. It has a set of free generators the **fundamental weights**, $\{\varpi_i : i = 1, ..., n\}$, where

$$\varpi_i = (1, ..., 1, 0, ..., 0)$$

has the first $i$ components are equal to one.

Let $GL(n, \mathbb{C})$ denote the general linear group, i.e. the complex Lie group of invertible $n \times n$ matrices. Let $T_n$ be the subgroup of diagonal matrices in $GL(n, \mathbb{C})$, $U_n$ the subgroup of upper-triangular unipotent matrices, and $\overline{U}_n$ be the subgroup of lower-triangular unipotent matrices.. The product of the two groups, $B_n = T_n U_n$, is a Borel subgroup of $GL(n, \mathbb{C})$, consisting of all upper-triangular matrices.

The subset $\overline{U}_n T_n U_n \subset GL(n, \mathbb{C})$ consists of the **regular elements** of $GL(n, \mathbb{C})$, and is denoted by $GL(n, \mathbb{C})_{reg}$. (Note, this is not the standard usage of the term "regular".) Every element $y \in GL(n, \mathbb{C})_{reg}$ can be expressed uniquely as a product $y = \overline{u}tu$, with $\overline{u} \in \overline{U}_n$, $t \in T_n$, and $u \in U_n$ (cf. Lemma B.2.8 in [11]). This is known as the **Gauss decomposition** of $y$. The set of regular elements is Zariski-dense in $GL(n, \mathbb{C})$. It is given precisely as the non-vanishing of the principal minors:

$$GL(n, \mathbb{C})_{reg} = \{x \in GL(n, \mathbb{C}) : \Delta_i(x) \neq 0 \text{ for } i = 1, ..., n\}$$

where $\Delta_i(x)$ is the determinant of the upper-left-hand $i \times i$ sub-matrix of $x$ (cf. *ibid*).

When convenient, we work in the setting of Lie algebras. In general, we denote the complex Lie algebra of a complex Lie group by the corresponding lower-case fraktur letter. For instance, $\mathfrak{gl}(n,\mathbb{C}) = Lie(GL(n,\mathbb{C}))$, $\mathfrak{t}_n = Lie(T_n)$, etc...

Let $\Phi_n^+ = \Phi(\mathfrak{b}_n, \mathfrak{t}_n)$ be the roots corresponding to the action of $\mathfrak{t}_n$ on $\mathfrak{b}_n$. This is a set of positive roots for the pair $(\mathfrak{gl}(n,\mathbb{C}), \mathfrak{t}_n)$. Let $\varepsilon_i \in \mathfrak{t}_n^*$ be the functional mapping a diagonal matrix to its $i^{th}$ entry. Relative to the choice $\Phi_n^+$ of positive roots, the set of dominant weights for irreducible polynomial representations of $\mathfrak{gl}(n,\mathbb{C})$ is

$$\{k_1\varepsilon_1 + \cdots + k_n\varepsilon_n : k_1 \geq k_2 \geq \cdots \geq k_n \text{ and } k_i \geq 0 \text{ for } i = 1, ..., n\}.$$

We make the obvious identification of this set of dominant weights with $\Lambda_n^+$. For $\lambda \in \Lambda_n^+$, let $V_\lambda$ be an irreducible (polynomial) representation of $GL(n,\mathbb{C})$ with highest weight $\lambda$. In the next section we will fix a particular model for $V_\lambda$.

Let $1 \leq m < n$. We embed $GL(m,\mathbb{C}) \times GL(n-m,\mathbb{C})$ in $GL(n,\mathbb{C})$ as the block diagonal subgroup:

$$\left\{ \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix} : g \in GL(m,\mathbb{C}) \text{ and } h \in GL(n-m,\mathbb{C}) \right\}.$$

Often we want to consider the embedding $GL(m,\mathbb{C}) \subset GL(n,\mathbb{C})$, forgetting about the factor $GL(n-m,\mathbb{C})$. In this case, we are referring to the subgroup:

$$\left\{ \begin{bmatrix} g & \\ & I_{n-m} \end{bmatrix} : g \in GL(m,\mathbb{C}) \right\}.$$

From these embeddings we obtain the chain of subgroups

$$GL(n,\mathbb{C}) \supset GL(n-1,\mathbb{C}) \supset GL(n-2,\mathbb{C}) \supset \cdots. \tag{2.1}$$

Now, suppose $\mu \in \Lambda_m^+$ and $\lambda \in \Lambda_n^+$. The $GL(n,\mathbb{C})$-module $V_\lambda$ is a $GL(m,\mathbb{C})$ module by restriction. We consider the $U_m$-invariants of $V_\lambda$, $V_\lambda^{U_m}$. The torus $T_m$ acts on $V_\lambda^{U_m}$ since $T_m$ normalizes $U_m$. We are interested in the $\mu$ weight space of this action:

$$M_\mu^\lambda = V_\lambda^{U_m}(\mu).$$

$M_\mu^\lambda$ is the space of $GL(m,\mathbb{C})$ highest weight vectors in $V_\lambda$ of weight $\mu$, and, as such, can be identified with the **multiplicity space** $Hom_{GL(m,\mathbb{C})}(V_\mu, V_\lambda)$. Notice that $M_\mu^\lambda$ is naturally a $GL(n-m,\mathbb{C})$-module. Indeed, there is a copy of $GL(n-m,\mathbb{C})$ in $GL(n,\mathbb{C})$ that commutes with $GL(m,\mathbb{C})$:

$$\left\{ \begin{bmatrix} I_m & \\ & g \end{bmatrix} : g \in GL(n-m,\mathbb{C}) \right\}$$

This copy of $GL(n-m,\mathbb{C})$ acts on $M_\mu^\lambda$.

## 2.2 $(GL(n,\mathbb{C}), GL(n,\mathbb{C}))$ duality

In this section we fix models for $V_\lambda$ with $\lambda \in \Lambda_n^+$, and discuss Cartan multiplication among these representations. We begin with a special case of $(GL(n,\mathbb{C}), GL(m,\mathbb{C}))$ duality.

Let $M_n(\mathbb{C})$ be the vector space of $n \times n$ complex matrices. Then $\mathcal{O}(M_n(\mathbb{C}))$ is a $GL(n,\mathbb{C})\times GL(n,\mathbb{C})$-module by left and right translation: if $(g,h) \in GL(n,\mathbb{C})\times GL(n,\mathbb{C})$, $f \in \mathcal{O}(M_n(\mathbb{C}))$, and $A \in M_n(\mathbb{C})$, then

$$((g,h).f)(A) = f(g^{-1}Ah).$$

**Theorem 2.2.1 (cf. Theorem 5.2.7 in [11])** *The space of polynomials on $M_n(\mathbb{C})$ decomposes under the representation of $GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$ as*

$$\mathcal{O}(M_n(\mathbb{C})) \cong \bigoplus_{\lambda \in \Lambda_n^+} V_\lambda^* \otimes V_\lambda.$$

*Furthermore,*

$$\mathcal{O}(M_n(\mathbb{C}))^{\overline{U}_n \times U_n} = \mathbb{C}[\Delta_1, ..., \Delta_n]$$

*is a polynomial ring on $n$ algebraically independent generators.*

The first part of this theorem shows that the left $\overline{U}_n$-invariant polynomials on $M_n(\mathbb{C})$ decompose under the right action of $GL(n,\mathbb{C})$ as

$$\mathcal{O}(M_n(\mathbb{C}))^{\overline{U}_n} \cong \bigoplus_{\lambda \in \Lambda_n^+} V_\lambda.$$

Set $\mathcal{R}_n = \mathcal{O}(M_n(\mathbb{C}))^{\overline{U}_n}$. We henceforth make this identification, and thus regard $V_\lambda \subset \mathcal{R}_n$. By the second part of the theorem, the monomials in $\Delta_1, ..., \Delta_n$ are a basis for the space of highest weight vectors in $\mathcal{R}_n$. More precisely, if $\lambda \in \Lambda_n^+$ and $\lambda = m_1\varpi_1 + \cdots + m_n\varpi_n$ then

$$v_\lambda = \Delta_1^{m_1} \cdots \Delta_n^{m_n} \tag{2.2}$$

is a highest weight vector of $V_\lambda$.

**Proposition 2.2.2** *Let $\lambda, \lambda' \in \Lambda_n^+$. Under the usual multiplication of functions in $\mathcal{R}_n$, $V_\lambda V_{\lambda'} = V_{\lambda+\lambda'}$. In particular, $v_\lambda v_{\lambda'} = v_{\lambda+\lambda'}$ (here $v_\lambda v_{\lambda'}$ denotes the multiplication of $v_\lambda$ and $v_{\lambda'}$ in $\mathcal{R}_n$).*

The above proposition allows us to define maps

$$\pi_{\lambda,\lambda'} : V_\lambda \otimes V_{\lambda'} \to V_{\lambda+\lambda'} \tag{2.3}$$

by $\pi_{\lambda,\lambda'}(v \otimes v') = vv'$, where $vv'$ denotes the multiplication of $v$ and $v'$ in $\mathcal{R}_n$. This map is usually referred to as the **Cartan product** of $V_\lambda$ and $V_{\lambda'}$. Notice that considering $V_\lambda \otimes V_{\lambda'}$ as a $GL(n,\mathbb{C})$-module via the tensor product action, $\pi_{\lambda,\lambda'}$ is a $G$-module morphism. We also define maps

$$j_{\lambda,\lambda'} : V_{\lambda+\lambda'} \to V_\lambda \otimes V_{\lambda'} \tag{2.4}$$

by $j_{\lambda,\lambda'}(v_{\lambda+\lambda'}) = v_\lambda \otimes v_{\lambda'}$ and extending by $G$-linearity. By definition, $j_{\lambda,\lambda'}$ is a $G$-module morphism, which we refer to as the **Cartan embedding** of $V_{\lambda+\lambda'}$ into $V_\lambda \otimes V_{\lambda'}$.

## 2.3 The special linear group

Consider now the special linear group $SL(n+1,\mathbb{C})$ of $(n+1) \times (n+1)$ complex matrices of determinant one. We define the following maximal torus, unipotent subgroup, and Borel subgroup of $SL(n+1,\mathbb{C})$:

$$\begin{aligned} T_{A_n} &= T_{n+1} \cap SL(n+1,\mathbb{C}) \\ U_{A_n} &= U_{n+1} \\ B_{A_n} &= B_{n+1} \cap SL(n+1,\mathbb{C}). \end{aligned}$$

Let $\Phi^+_{A_n} = \Phi(\mathfrak{b}_{A_n}, \mathfrak{t}_{A_n})$ be the roots corresponding to the action of $\mathfrak{t}_{A_n}$ on $\mathfrak{b}_{A_n}$. This is a set of positive roots for the pair $(\mathfrak{sl}(n+1,\mathbb{C}), \mathfrak{t}_{A_n})$. Relative to the choice

$\Phi^+_{A_n}$ of positive roots, a dominant weight of $\mathfrak{sl}(n+1,\mathbb{C})$ is the restriction to $\mathfrak{t}_{A_n}$ of the weight $k_1\varepsilon_1+\cdots+k_n\varepsilon_n$ with $k_1 \geq k_2 \geq \cdots \geq k_n$ and $k_i \geq 0$ for $i = 1, ..., n$ (here $\varepsilon_i \in \mathfrak{t}^*_{n+1}$ is defined as in the previous section). We make the obvious identification of this set of dominant weights with $\Lambda^+_n$. For $\lambda \in \Lambda^+_n$, let $\overline{V}_\lambda$ be an irreducible representation of $SL(n+1,\mathbb{C})$ with highest weight $\lambda$.

For $\lambda = (\lambda_1, ..., \lambda_n) \in \Lambda^+_n$ let $\lambda^+ = (\lambda_1, ..., \lambda_n, 0) \in \Lambda^+_{n+1}$. We realize $\overline{V}_\lambda$ as the restriction of the irreducible representation of $GL(n+1,\mathbb{C})$, $V_{\lambda^+}$, to $SL(n+1,\mathbb{C})$:

$$\overline{V}_\lambda = V_{\lambda^+}|_{SL(n+1,\mathbb{C})}.$$

Mimicking the set-up above for the general linear group, let $1 \leq m < n$. We embed $SL(m,\mathbb{C})$ in $SL(n,\mathbb{C})$ as the block diagonal subgroup:

$$\left\{ \begin{bmatrix} g & 0 \\ 0 & I_{n-m} \end{bmatrix} : g \in SL(m,\mathbb{C}) \right\}.$$

Given $\mu \in \Lambda^+_m$ and $\lambda \in \Lambda^+_n$ set

$$\overline{M}^\lambda_\mu = \overline{V}^{U_m}_\lambda(\mu).$$

The multiplicity space $\overline{M}^\lambda_\mu$ is an $SL(n-m,\mathbb{C})$-module via the action of

$$\left\{ \begin{bmatrix} I_m & \\ & g \end{bmatrix} : g \in SL(n-m,\mathbb{C}) \right\}.$$

The special linear group of rank one plays a special role in this work. Accordingly we introduce a distinguished notation for its representations. We denote the realization of the $(k+1)^{th}$-dimensional irreducible representation of $SL(2,\mathbb{C})$

as degree $k$ homogeneous polynomials in two variables by $F_k$:

$$F_k = \mathcal{O}^k(\mathbb{C}^2) \subset \mathcal{O}(\mathbb{C}^2).$$

Here $SL(2,\mathbb{C})$ acts by right multiplication. We define Cartan maps also for this group: for $k, k' \geq 0$ define

$$\pi_{k,k'} : F_k \otimes F_{k'} \to F_{k+k'}$$

as the usual multiplication of functions, and define the embedding of $SL(2,\mathbb{C})$-modules

$$j_{k,k'} : F_{k+k'} \to F_k \otimes F_{k'}$$

by $j_{k,k'}(e_1^{k+k'}) = e_1^k \otimes e_1^{k'}$, extending by $SL(2,\mathbb{C})$-linearity.

## 2.4 The symplectic group

Label a basis for $\mathbb{C}^{2n}$ as $e_{\pm 1}, ..., e_{\pm n}$ where $e_{-i} = e_{2n+1-i}$. Here we view $\mathbb{C}^{2n}$ as column vectors. Denote by $s_n$ the $n \times n$ matrix with ones on the anti-diagonal and zeros everywhere else. Set

$$J_n = \begin{bmatrix} 0 & s_n \\ -s_n & 0 \end{bmatrix}$$

and define the skew-symmetric bilinear form $\Omega_n(x,y) = x^t J_n y$ on $\mathbb{C}^{2n}$. Define the symplectic group relative to this choice of skew form:

$$Sp(n,\mathbb{C}) = \{g \in GL(2n,\mathbb{C}) : \Omega_n(gx, gy) = \Omega_n(x,y) \text{ for all } x, y \in \mathbb{C}^{2n}\}.$$

Now set

$$\begin{aligned} T_{C_n} &= T_{2n} \cap Sp(n, \mathbb{C}) \\ U_{C_n} &= U_{2n} \cap Sp(n, \mathbb{C}) \\ \overline{U}_{C_n} &= \overline{U}_{2n} \cap Sp(n, \mathbb{C}). \end{aligned}$$

Then $T_{C_n}$ is a maximal torus, and $B_{C_n} = T_{C_n} U_{C_n}$ is a Borel subgroup of $Sp(n, \mathbb{C})$ with unipotent radical $U_{C_n}$.

Let $\Phi^+_{C_n} = \Phi(\mathfrak{b}_{C_n}, \mathfrak{t}_{C_n})$ be the set of roots corresponding to the action of $\mathfrak{t}_{C_n}$ on $\mathfrak{b}_{C_n}$. This is a set of positive roots for the pair $(\mathfrak{sp}(n, \mathbb{C}), \mathfrak{t}_{C_n})$. Let $\varepsilon_i \in \mathfrak{t}^*_{C_n}$ be the functional mapping a diagonal matrix to its $i^{th}$ entry. Relative to the choice $\Phi^+_{C_n}$ of positive roots, the set of dominant weights for $\mathfrak{sp}(n, \mathbb{C})$ is

$$\{k_1\varepsilon_1 + \cdots + k_n\varepsilon_n : k_1 \geq k_2 \geq \cdots \geq k_n \text{ and } k_i \geq 0 \text{ for } i = 1, ..., n\}.$$

We make the obvious identification of this set of dominant weights with $\Lambda^+_n$. For $\lambda \in \Lambda^+_n$, let $W_\lambda$ be an irreducible representation of $Sp(n, \mathbb{C})$ with highest weight $\lambda$. In the next section we will fix a particular model for $W_\lambda$.

We embed $Sp(n-1, \mathbb{C}) \times Sp(1, \mathbb{C})$ in $Sp(n, \mathbb{C})$ as the subgroup leaving the space $Y_n = span\{e_n, e_{-n}\}$ invariant:

$$Sp(n-1, \mathbb{C}) \times Sp(1, \mathbb{C}) \cong \{g \in Sp(n, \mathbb{C}) : gY_n \subset Y_n\}.$$

The subgroup $Sp(1, \mathbb{C}) \cong SL(2, \mathbb{C})$ appears here as the factor

$$\{g \in Sp(n-1, \mathbb{C}) \times Sp(1, \mathbb{C}) : ge_i = e_i \text{ for } i = \pm 1, ..., \pm(n-1)\}, \tag{2.5}$$

while the subgroup $Sp(n-1,\mathbb{C})$ appears as the factor

$$\{g \in Sp(n-1,\mathbb{C}) \times Sp(1,\mathbb{C}) : ge_{\pm n} = e_{\pm n}\}.$$

From these embeddings we obtain the chain of subgroups

$$Sp(n,\mathbb{C}) \supset Sp(n-1,\mathbb{C}) \times Sp(1,\mathbb{C}) \supset Sp(n-2,\mathbb{C}) \times Sp(1,\mathbb{C}) \times Sp(1,\mathbb{C}) \supset \cdots.$$

The last subgroup in this chain is the group:

$$\left\{ \begin{pmatrix} a_1 & & & & & b_1 \\ & \ddots & & & \iddots & \\ & & a_n & b_n & & \\ & & c_n & d_n & & \\ & \iddots & & & \ddots & \\ c_1 & & & & & d_1 \end{pmatrix} : \det \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} = 1 \text{ for } i = 1, ..., n \right\} \tag{2.6}$$

(The nonzero entries occur only along the two main diagonals.) Note that this group is isomorphic to $Sp(1,\mathbb{C}) \times \cdots \times Sp(1,\mathbb{C})$ ($n$ copies), and it contains $T_{C_n}$.

Now, suppose $\mu \in \Lambda_{n-1}^+$ and $\lambda \in \Lambda_n^+$. The $Sp(n,\mathbb{C})$-module $W_\lambda$ is an $Sp(n-1,\mathbb{C})$ module by restriction. Define the multiplicity space

$$\widehat{M}_\mu^\lambda = W_\lambda^{U_{C_{n-1}}}(\mu).$$

$\widehat{M}_\mu^\lambda$ is naturally an $Sp(1,\mathbb{C}) \cong SL(2,\mathbb{C})$-module for the copy of $Sp(1,\mathbb{C})$ given in (2.5).

## 2.5 Algebraic Peter-Weyl Theorem

We work now in slightly greater generality. Let $G$ be a connected classical group, with identity $e \in G$. Fix a maximal torus $T$, Borel subgroup $B$, and

unipotent radical $U \subset B$ so that $TU = B$. Let $\overline{U}$ be the unipotent group opposite $U$: if we take a matrix form of $G$ so that $T$ consists of diagonal matrices and $U$ is upper triangular (this is always possible), then $\overline{U} = U^t$. Fix the choice of positive roots giving $U$: $\Phi^+ = \Phi(B, T)$. Let $\Lambda^+$ to be the corresponding set of dominant integral weights. Let $F_\lambda$ denote the finite-dimensional irreducible representation of $G$ of highest weight $\lambda \in \Lambda^+$.

$G$ has the structure of an affine algebraic variety. Let $G_{reg} = \overline{U}TU$ be the set of **regular elements** in $G$. (This is not the standard usage of the term "regular".) $G_{reg}$ is Zariski dense in $G$. As in the case for the general linear group, an element $g \in G_{reg}$ has a unique decomposition into $g = \overline{u}tu$, where $\overline{u} \in \overline{U}$, $t \in T$, and $u \in U$. This is also known as the **Gauss decomposition** of $g$ (cf. Lemma 11.5.1 in [11]).

The ring of functions $\mathcal{O}(G)$ is a $G \times G$-module under left and right translation. The following theorem is an algebraic analogue of the Peter-Weyl Theorem:

**Theorem 2.5.1 (cf. Corollary 12.1.7 in [11])** *As a $G \times G$-module the ring of functions decomposes as:*

$$\mathcal{O}(G) \cong \bigoplus_{\lambda \in \Lambda^+} F_\lambda^* \otimes F_\lambda.$$

Therefore the left $\overline{U}$-invariant polynomials decompose under the right action of $G$ as:

$$\mathcal{O}(G)^{\overline{U}} \cong \bigoplus_{\lambda \in \Lambda^+} (F_\lambda^*)^{\overline{U}} \otimes F_\lambda \cong \bigoplus_{\lambda \in \Lambda^+} F_\lambda.$$

The space of lowest-weight vectors $(F_\lambda^*)^{\overline{U}}$ is one-dimensional and spanned by a func-

tional of weight $-\lambda$, say $v_\lambda^*$. Then under the above isomorphism, the irreducible representation of highest weight $\lambda$ is realized in $\mathcal{O}(G)^{\overline{U}}$ as the matrix coefficients $f_{v_\lambda^*,v} : g \mapsto v_\lambda^*(g.v)$ for $v \in F_\lambda$. Choose the highest weight vector $v_\lambda \in F_\lambda$ such that $v_\lambda^*(v_\lambda) = 1$. Then $f_{v_\lambda^*,v_\lambda}$ is a highest weight vector in $\mathcal{O}(G)^{\overline{U}}$ of weight, which is independent of the choice of $v_\lambda^*$, and is therefore canonical. Set $f_\lambda = f_{v_\lambda^*,v_\lambda}$. Notice that $f_\lambda f_{\lambda'}$ is $U$-invariant of weight $\lambda+\lambda'$. Since also $f_\lambda f_{\lambda'}(e) = 1$, it follows that $f_\lambda f_{\lambda'} = f_{\lambda+\lambda'}$. Henceforth identify $F_\lambda$ with its image in $\mathcal{O}(G)^{\overline{U}}$. As above we have:

**Proposition 2.5.2** *Let $\lambda, \lambda' \in \Lambda^+$. Under the usual multiplication of functions in $\mathcal{O}(G)^{\overline{U}_n}$, $F_\lambda F_{\lambda'} = F_{\lambda+\lambda'}$.*

Again, we define the corresponding **Cartan product** maps

$$\pi_{\lambda,\lambda'} : F_\lambda \otimes F_{\lambda'} \to F_{\lambda+\lambda'}$$

by $\pi_{\lambda,\lambda'}(v \otimes v') = vv'$, and the **Cartan embeddings**

$$j_{\lambda,\lambda'} : F_{\lambda+\lambda'} \to F_\lambda \otimes F_{\lambda'}$$

by $j_{\lambda,\lambda'}(f_{\lambda+\lambda'}) = f_\lambda \otimes f_{\lambda'}$ and extending by $G$-linearity.

Now we specialize these results to $Sp(n,\mathbb{C})$. In other words, we use the above results to realize the irreducible representations $W_\lambda$ as a submodule of

$$\mathcal{R}_{C_n} = \mathcal{O}(Sp(n,\mathbb{C}))^{\overline{U}_{C_n}}. \tag{2.7}$$

Moreover, we fix highest weight vectors $w_\lambda \in W_\lambda$ such that $w_\lambda w_{\lambda'} = w_{\lambda+\lambda'}$, and

we have the corresponding maps $\pi_{\lambda,\lambda'} : W_\lambda \otimes W_{\lambda'} \to W_{\lambda+\lambda'}$ and $j_{\lambda,\lambda'} : W_{\lambda+\lambda'} \to W_\lambda \otimes W_{\lambda'}$ defined as above.

**Remark 2.5.3** Although we use the same symbol, $\pi_{\lambda,\lambda'}$, to denote the Cartan products of representations of various groups, it will be clear from context which group we have in mind.

# Chapter 3

# Double Interlacing Patterns

In this chapter we study the semigroup of double interlacing patterns, denoted $\mathfrak{D}$. This semigroup arises naturally in branching from $GL(n+1,\mathbb{C})$ to $GL(n-1,\mathbb{C})$, and also in branching from $Sp(n,\mathbb{C})$ to $Sp(n-1,\mathbb{C})$. $\mathfrak{D}$ naturally contains a family of sub-semigroups, each of which is isomorphic to $\Lambda_{2n}^{+}$. Using this basic result we develop the combinatorial properties of $\mathfrak{D}$.

As applications we obtain, in the second section, the $SL(2,\mathbb{C})$ module structure of the multiplicity spaces that arise in the restriction of an irreducible representation of $GL(n+1,\mathbb{C})$ to $GL(n-1,\mathbb{C})$. In the third section we prove a surjectivity result concerning the Cartan product of these multiplicity spaces.

## 3.1 Combinatorics of double interlacing patterns

Fix $n$ a positive integer. We begin with the notion of interlacing weights, which naturally arises from the classical branching rules for the restriction of irreducible representations of $GL(n+1,\mathbb{C})$ to $GL(n,\mathbb{C})$ (cf. Proposition 3.2.1).

**Definition 3.1.1** *Let $\gamma = (\gamma_1, ..., \gamma_n) \in \Lambda_n^+$ and $\lambda = (\lambda_1, ..., \lambda_{n+1}) \in \Lambda_{n+1}^+$. We say $\gamma$ **interlaces** $\lambda$, written $\gamma < \lambda$, if $\lambda_i \geq \gamma_i \geq \lambda_{i+1}$ for $i = 1, ..., n$.*

Of present concern is the restriction of representations of $GL(n+1,\mathbb{C})$ to $GL(n-1,\mathbb{C})$. Therefore it is natural to consider the following generalization.

**Definition 3.1.2** *Let $\mu = (\mu_1, ..., \mu_{n-1}) \in \Lambda_{n-1}^+$ and $\lambda = (\lambda_1, ..., \lambda_{n+1}) \in \Lambda_{n+1}^+$. We say $\mu$ **double interlaces** $\lambda$, written $\mu \ll \lambda$, if $\lambda_i \geq \mu_i \geq \lambda_{i+2}$ for $i = 1, ..., n-1$. If $\mu \ll \lambda$, then we say $(\mu, \lambda)$ is a **double interlacing pattern**.*

Note that if there exists $\gamma \in \Lambda_n^+$ such that $\mu < \gamma < \lambda$, then $\mu \ll \lambda$. In fact, the converse is also true, as we shall see below (Lemma 3.1.6). Let $\mathfrak{D} \subset \Lambda_{n-1}^+ \times \Lambda_{n+1}^+$ be the set of double interlacing patterns. An element $p \in \mathfrak{D}$ is a pair $(\mu, \lambda)$. The rest of this section is devoted to the study of $\mathfrak{D}$.

Notice first that $\mathfrak{D}$ is a semigroup: if $\mu \ll \lambda$ and $\mu' \ll \lambda'$ then $\mu + \mu' \ll \lambda + \lambda'$. The next observation is that there are many ways $\mu$ can double interlace $\lambda$. Indeed, the inequalities $\lambda_i \geq \mu_i \geq \lambda_{i+2}$ do not constrain the relationship between $\mu_i$ and $\lambda_{i+1}$. More precisely, if $\mu \ll \lambda$ then for every $i = 1, ..., n-1$, either $\mu_i \geq \lambda_{i+1}$ or $\mu_i \leq \lambda_{i+1}$. A choice of one of these inequalities for each $i = 1, ..., n-1$ is an

**order type**. One can think of an order type as a word, $\sigma$, of length $n-1$ in the alphabet $\{\geq, \leq\}$. A " $\geq$ " in the $i^{th}$ entry means $\mu_i \geq \lambda_{i+1}$, and a " $\leq$ " means $\mu_i \leq \lambda_{i+1}$. An element $p = (\mu, \lambda) \in \mathfrak{D}$ is said to be **of order type** $\sigma = \sigma_1 \cdots \sigma_{n-1}$ if for $i = 1, ..., n-1$,

$$(\sigma_i \text{ is } \geq \ ) \Longrightarrow \mu_i \geq \lambda_{i+1} \tag{3.1}$$

$$(\sigma_i \text{ is } \leq \ ) \Longrightarrow \mu_i \leq \lambda_{i+1}. \tag{3.2}$$

**Example 3.1.3** Let $n = 3$. The order type $(\geq\leq)$ corresponds to double interlacing patterns $(\mu, \lambda)$ that satisfy the additional inequalities $\mu_1 \geq \lambda_2$ and $\mu_2 \leq \lambda_3$.

Let $\Sigma$ be the set of order types,

$$\Sigma = \{(\sigma_1 \cdots \sigma_{n-1}) : \sigma_i \in \{\leq, \geq\} \text{ for } i = 1, ..., n-1\}.$$

$\Sigma$ is a set of cardinality $2^{n-1}$. For $\sigma \in \Sigma$, we write $\mu \ll_\sigma \lambda$ if $\mu$ double interlaces $\lambda$ and satisfies the additional constraints prescribed by $\sigma$. We define the following subset of $\mathfrak{D}$:

$$\mathfrak{D}_\sigma = \{(\mu, \lambda) \in \mathfrak{D} : \mu \ll_\sigma \lambda\}.$$

The set $\mathfrak{D}_\sigma$ is a sub-semigroup of $\mathfrak{D}$. Notice also that $\mathfrak{D} = \bigcup_{\sigma \in \Sigma} \mathfrak{D}_\sigma$, but this is not a partitioning of $\mathfrak{D}$ because we are dealing with weak inequalities.

**Example 3.1.4** Let $n = 3$ and suppose $\sigma = (\geq, \leq)$ and $\sigma' = (\leq, \leq)$. Then $\mathfrak{D}_\sigma \cap \mathfrak{D}_{\sigma'} = \{(\mu, \lambda) \in \mathfrak{D} : \mu_1 = \lambda_2 \text{ and } \mu_2 \leq \lambda_3\}$.

In general, $\mathfrak{D}_\sigma \cap \mathfrak{D}_{\sigma'}$ is the set of double interlacing patterns where there is an equality $\mu_i = \lambda_{i+1}$ for every entry that $\sigma$ and $\sigma'$ disagree. In particular,

$\bigcap_{\sigma\in\Sigma}\mathfrak{D}_\sigma = \{(\mu,\lambda)\in\mathfrak{D} : \mu_i = \lambda_{i+1} \text{ for } i = 1,...,n-1\}$.

Next we introduce the **rearrangement function** on $\mathfrak{D}$. Define $f : \mathfrak{D} \to \Lambda^+_{2n}$ by:

$$(\mu,\lambda) \overset{f}{\mapsto} (x_1, y_1, ..., x_n, y_n)$$

where $\{x_1 \geq y_1 \geq \cdots \geq x_n \geq y_n\}$ is the non-increasing rearrangement of $\{\mu_1, ..., \mu_{n-1}, \lambda_1, ..., \lambda_{n+1}\}$. Notice that $f(\mu,\lambda)$ equals

$$(\lambda_1 \geq \max(\mu_1,\lambda_2) \geq \min(\mu_1,\lambda_2) \geq \cdots \geq \max(\mu_{n-1},\lambda_n) \geq \min(\mu_{n-1},\lambda_n) \geq \lambda_{n+1}).$$

We denote the restriction of the rearrangement function to $\mathfrak{D}_\sigma$ by $f_\sigma$. We now prove a series of lemmas that will be useful in the sequel. The most important one is the following lemma.

**Lemma 3.1.5** *Let $\sigma \in \Sigma$. The rearrangement function $f_\sigma : \mathfrak{D}_\sigma \to \Lambda^+_{2n}$ is a semigroup isomorphism.*

**Proof.** For $(\mu,\lambda) \in \mathfrak{D}_\sigma$ let $f_\sigma(\mu,\lambda) = (f_\sigma(\mu,\lambda)_1, ..., f_\sigma(\mu,\lambda)_{2n})$. Define functions $a : \{1,...,n-1\} \to \{1,...,2n\}$ and $b : \{1,...,n+1\} \to \{1,...,2n\}$ by

$$(\sigma_i \text{ is } \geq\ ) \Longrightarrow a(i) = 2i \text{ and } b(i+1) = 2i+1$$

$$(\sigma_i \text{ is } \leq\ ) \Longrightarrow a(i) = 2i+1 \text{ and } b(i+1) = 2i$$

for $i = 1,...,n-1$. Moreover, let $b(1) = 1$ and $b(n+1) = 2n$. Then for all $(\mu,\lambda) \in \mathfrak{D}_\sigma$

$$\mu_i = f_\sigma(\mu,\lambda)_{a(i)} \text{ for } i = 1,...,n-1$$

$$\lambda_j = f_\sigma(\mu,\lambda)_{b(j)} \text{ for } j = 1,...,n+1.$$

This implies that $f_\sigma$ is an injective semigroup homomorphism. To see that $f_\sigma$ is surjective, suppose $(z_1, ..., z_{2n}) \in \Lambda^+_{2n}$ is given. Define $\mu$ and $\lambda$ by the formulas

$$\begin{aligned} \mu_i &= z_{a(i)} \text{ for } i = 1, ..., n-1 \\ \lambda_j &= z_{b(j)} \text{ for } j = 1, ..., n+1. \end{aligned}$$

Since $a(1) < a(2) < \cdots < a(n-1)$, $\mu \in \Lambda^+_{n-1}$. Similarly $\lambda \in \Lambda^+_{n+1}$. Since $b(i) < a(i) < b(i+2)$, we get that $\mu \ll \lambda$. Finally, suppose $\sigma_i$ is $\geq$. Then $a(i) < b(i+1)$, and so $\mu_i \geq \lambda_{i+1}$. Similarly, if $\sigma_i$ is $\leq$ then $\mu_i \leq \lambda_{i+1}$. Therefore $(\mu, \lambda) \in \mathfrak{D}_\sigma$. Since $f_\sigma(\mu, \lambda) = (z_1, ..., z_{2n})$ we conclude that $f_\sigma$ is surjective. ■

**Lemma 3.1.6** *Let $(\mu, \lambda) \in \mathfrak{D}$. Suppose $f(\mu, \lambda) = (x_1, y_1, ..., x_n, y_n)$ and $\gamma \in \Lambda^+_n$. Then $\mu < \gamma < \lambda$ if, and only if, $y_i \leq \gamma_i \leq x_i$ for $i = 1, ..., n$.*

**Proof.** The condition $\mu < \gamma < \lambda$ is satisfied if, and only if, for every $i = 1, ..., n$, $\lambda_i \geq \gamma_i \geq \lambda_{i+1}$ and, for every $j = 1, ..., n-1$, $\gamma_j \geq \mu_j \geq \gamma_{j+1}$. This translates to precisely the conditions $y_i \leq \gamma_i \leq x_i$ for $i = 1, ..., n$. ■

**Lemma 3.1.7** *Let $a_1, a_2, b, c_1, c_2 \in \mathbb{Z}$ satisfy $a_1 \leq c_1$, $a_2 \leq c_2$, and $a_1 + a_2 \leq b \leq c_1 + c_2$. Then there exist $b_1, b_2 \in \mathbb{Z}$ such that $b = b_1 + b_2$, $a_1 \leq b_1 \leq c_1$, and $a_2 \leq b_2 \leq c_2$.*

**Proof.** Consider the sequence of points in $\mathbb{Z}^2$:

$$(a_1, a_2), (a_1 + 1, a_2), ..., (c_1, a_2), (c_1, a_2 + 1), ..., (c_1, c_2).$$

If we sum the coordinates of each point we get the sequence of consecutive integers

$$a_1 + a_2, a_1 + a_2 + 1, ..., c_1 + c_2.$$

Since $a_1 + a_2 \leq b \leq c_1 + c_2$, somewhere along this latter sequence will occur the number $b$. This corresponds to a point $(b_1, b_2)$ that satisfies the desired properties.

■

**Lemma 3.1.8** *Let $\sigma \in \Sigma$ and let $(\mu, \lambda), (\mu', \lambda') \in \mathfrak{D}_\sigma$. Suppose that $\gamma \in \Lambda_n^+$ satisfies*

$$\mu + \mu' < \gamma < \lambda + \lambda'.$$

*Then there exist $\nu$,$\nu' \in \Lambda_n^+$ such that $\gamma = \nu + \nu'$, $\mu < \nu < \lambda$, and $\mu' < \nu' < \lambda'$.*

**Proof.** Set $f_\sigma(\mu, \lambda) = (x_1, y_1, ..., x_n, y_n)$ and $f_\sigma(\mu', \lambda') = (x'_1, y'_1, ..., x'_n, y'_n)$. By Lemma 3.1.5, since $(\mu, \lambda), (\mu', \lambda')$ both satisfy a common order type $\sigma$,

$$f_\sigma(\mu + \mu', \lambda + \lambda') = (x_1 + x'_1, y_1 + y'_1, ..., x_n + x'_n, y_n + y'_n).$$

Therefore by Lemma 3.1.6, $y_i + y'_i \leq \gamma_i \leq x_i + x'_i$. Now we can apply Lemma 3.1.7 to obtain $\nu_i, \nu'_i$ such that $\gamma_i = \nu_i + \nu'_i$, $y_i \leq \nu_i \leq x_i$, and $y'_i \leq \nu'_i \leq x'_i$. Set $\nu = (\nu_1, ..., \nu_n)$ and $\nu' = (\nu'_1, ..., \nu'_n)$. Clearly $\nu, \nu' \in \Lambda_n^+$ and $\gamma = \nu + \nu'$. Moreover, by Lemma 3.1.6, $\mu < \nu < \lambda$, and $\mu' < \nu' < \lambda'$. ■

## 3.2 Branching from $GL(n+1, \mathbb{C})$ to $GL(n-1, \mathbb{C})$

In this section we prove a result concerning the restriction of an irreducible representation of $GL(n+1, \mathbb{C})$ to $GL(n-1, \mathbb{C})$. The result we prove here is crucial for our construction of a weight bases for irreducible representations of the symplectic group, but also may be of interest in its own right.

First we record the well-known facts about branching from $GL(n+1,\mathbb{C})$ to $GL(n,\mathbb{C})$. These can be found in many texts, e.g. [11], but we also give a self-contained proof in the next chapter (cf. Example 4.2.3).

**Proposition 3.2.1** *Let $\gamma \in \Lambda_n^+$ and $\lambda \in \Lambda_{n+1}^+$. Then*

1. $\dim M_\gamma^\lambda \leq 1$.
2. *$M_\gamma^\lambda \neq \{0\}$ if, and only if, $\gamma < \lambda$.*
3. *Suppose $\gamma < \lambda$. Then as a $GL(1,\mathbb{C})$-module, $M_\gamma^\lambda \cong \mathbb{C}_{|\lambda|-|\gamma|}$.*

We turn our attention now to problem of restricting representations from $GL(n+1,\mathbb{C})$ to $GL(n-1,\mathbb{C})$. By the above proposition, if $\mu \in \Lambda_{n-1}^+$ and $\lambda \in \Lambda_{n+1}^+$, then $M_\mu^\lambda \neq \{0\}$ if, and only if, $\mu \ll \lambda$. Moreover,

$$\dim M_\mu^\lambda = \#\{\gamma \in \Lambda_n^+ : \mu < \gamma < \lambda\}.$$

(See Proposition 3.2.2 for an explicit formula for $\dim M_\mu^\lambda$.)

What we would like to now prove is an analogue of part three of the above proposition for the multiplicity spaces $M_\mu^\lambda$. In particular, $M_\mu^\lambda$ is naturally a $GL(2,\mathbb{C})$-module (cf. Section 2.1), and we would like an explicit description of this module. In fact, $M_\mu^\lambda$ is an $SL(2,\mathbb{C})$-module by restriction, and a more relevant result for our purposes is a description of this action.

We recall first the usual notion of a formal character. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, with Cartan subalgebra $\mathfrak{h}$ and weight lattice $\Lambda$. A finite dimensional $\mathfrak{g}$-module $V$ decomposes as a direct sum of $\mathfrak{h}$ weight spaces $V(\beta)$, with

$\beta \in \Lambda$. Then the **character** of $V$ is the formal sum

$$ch(V) = \sum_{\beta \in \Lambda} \dim V(\beta) e^{\beta}$$

viewed as an element in the integral group algebra of $\Lambda$. This notion is useful since if $V$ and $W$ are finite dimensional $\mathfrak{g}$-modules, then $V \cong W$ if, and only if, $ch(V) = ch(W)$.

Consider now the case of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Let $\mathfrak{h}$ be the Cartan subalgebra of traceless, diagonal, $2 \times 2$ matrices. Set $q = e^{\alpha/2}$, where $\alpha = \varepsilon_1 - \varepsilon_2$ is the usual choice of simple root. Given $(\mu, \lambda) \in \mathfrak{D}$ define $r_i(\mu, \lambda) = x_i - y_i$, where $f(\mu, \lambda) = (x_1, y_1, ..., x_n, y_n)$.

**Proposition 3.2.2** *Let $\mu \in \Lambda^+_{n-1}$, $\lambda \in \Lambda^+_{n+1}$. Then $M^\lambda_\mu \neq \{0\}$ if, and only if, $\mu \ll \lambda$. Moreover, if $\mu \ll \lambda$ then as $SL(2, \mathbb{C})$-modules*

$$M^\lambda_\mu \cong \bigotimes_{i=1}^{n} F_{r_i(\mu,\lambda)}$$

*where $SL(2, \mathbb{C})$ acts by the tensor product action on the right hand side. In particular,* $\dim M^\lambda_\mu = \prod_{i=1}^{n} (r_i(\mu, \lambda) + 1)$.

**Proof.** It remains only to show the second part of the theorem. Let $\lambda \in \Lambda^+_{n+1}$. On the one hand $V_\lambda \cong \bigoplus_{\substack{\mu \in \Lambda^+_{n-1} \\ \mu \ll \lambda}} V_\mu \otimes M^\lambda_\mu$, as $GL(n-1, \mathbb{C}) \times GL(2, \mathbb{C})$-

modules. On the other hand,

$$\begin{aligned} V_\lambda &\cong \bigoplus_{\substack{\gamma\in\Lambda_n^+\\ \gamma<\lambda}} V_\gamma \otimes M_\gamma^\lambda \\ &\cong \bigoplus_{\substack{\gamma\in\Lambda_n^+\\ \gamma<\lambda}} \left( \bigoplus_{\substack{\mu\in\Lambda_{n-1}^+\\ \mu<\gamma}} V_\mu \otimes M_\mu^\lambda \right) \otimes M_\gamma^\lambda \\ &\cong \bigoplus_{\substack{\mu\in\Lambda_{n-1}^+\\ \mu\ll\lambda}} V_\mu \otimes \left( \bigoplus_{\substack{\gamma\in\Lambda_n^+\\ \mu<\gamma<\lambda}} M_\mu^\gamma \otimes M_\gamma^\lambda \right). \end{aligned}$$

Therefore, as $GL(1,\mathbb{C}) \times GL(1,\mathbb{C})$-modules,

$$M_\mu^\lambda \cong \bigoplus_{\substack{\gamma\in\Lambda_n^+\\ \mu<\gamma<\lambda}} M_\mu^\gamma \otimes M_\gamma^\lambda.$$

By Proposition 3.2.1(3), $d(x)$ acts on $M_\mu^\gamma \otimes M_\gamma^\lambda$ by the scalar $x^{2|\gamma|-|\lambda|-|\mu|}$. Moreover, $\dim M_\mu^\gamma \otimes M_\gamma^\lambda = 1$ for $\gamma$ such that $\mu < \gamma < \lambda$. Therefore

$$ch(M_\mu^\lambda) = \sum_{\substack{\gamma\in\Lambda_n^+\\ \mu<\gamma<\lambda}} q^{2|\gamma|-|\lambda|-|\mu|}.$$

Now set $f(\mu,\lambda) = (x_1, y_1, ..., x_n, y_n)$ and $r_i = r_i(\mu,\lambda)$. By Lemma 3.1.6,

$$\begin{aligned} \sum_{\substack{\gamma\in\Lambda_n^+\\ \mu<\gamma<\lambda}} q^{2|\gamma|-|\lambda|-|\mu|} &= \sum_{y_i\le\gamma_i\le x_i} q^{2(\gamma_1+\cdots+\gamma_{n-1})-|\lambda|-|\mu|} \\ &= \sum_{0\le j_i\le r_i} q^{2(y_1+\cdots+y_n+j_1+\cdots+j_n)-(x_1+\cdots+x_n+y_1+\cdots+y_n)} \\ &= \sum_{0\le j_i\le r_i} q^{(-r_1+2j_1)+\cdots+(-r_n+2j_n)} \\ &= \prod_{i=1}^{n}\sum_{j=0}^{r_i} q^{-r_i+2j} \\ &= \prod_{i=1}^{n} ch(F_{r_i}) \end{aligned}$$

Therefore $ch(M_\mu^\lambda) = ch\left(\bigotimes_{i=1}^{n-1} F_{r_i(\mu,\lambda)}\right)$. ■

## 3.3 Cartan product of the multiplicity spaces $M_\mu^\lambda$

We now prove a result about the Cartan product of multiplicity spaces that occur in branching from $GL(n+1,\mathbb{C})$ to $GL(n-1,\mathbb{C})$. We will show that under certain conditions, this product surjects onto another multiplicity space. This significance of this result will become clear in Chapter 5, where we use this result to interpret the Cartan product of multiplicity spaces as a Cartan product of irreducible modules for $SL(2,\mathbb{C}) \times \cdots \times SL(2,\mathbb{C})$ ($n$ copies).

In this section it will be convenient to denote the general linear groups and their Borel subgroups, which we express as the product of the diagonal torus and upper triangular unipotent groups as follows:

$$\begin{array}{ccccc} \overbrace{GL(n-1,\mathbb{C})}^{G'} & \subset & \overbrace{GL(n,\mathbb{C})}^{G''} & \subset & \overbrace{GL(n+1,\mathbb{C})}^{G} \\ \cup & & \cup & & \cup \\ \underbrace{T_{n-1}U_{n-1}}_{T'U'} & \subset & \underbrace{T_nU_n}_{T''U''} & \subset & \underbrace{T_{n+1}U_{n+1}}_{TU} \end{array}$$

Set $\mathfrak{D}$ to be the semigroup of double interlacing patterns defined above (cf. Section 3.1).

**Lemma 3.3.1** *Let* $(\mu,\lambda),(\mu',\lambda') \in \mathfrak{D}$. *Then* $\pi_{\lambda,\lambda'}(M_\mu^\lambda \otimes M_{\mu'}^{\lambda'}) \subset M_{\mu+\mu'}^{\lambda+\lambda'}$.

**Proof.** Since $\pi_{\lambda,\lambda'}$ is a $G$-module morphism, in particular it is an $U'$-module morphism. Therefore, $\pi_{\lambda,\lambda'}(V_\lambda \otimes V_{\lambda'})^{U'}) \subset V_{\lambda+\lambda'}^{U'}$. Since $V_\lambda^{U'} \otimes V_{\lambda'}^{U'} \subset (V_\lambda \otimes V_{\lambda'})^{U'}$ we get that $\pi_{\lambda,\lambda'}(V_\lambda^{U'} \otimes V_{\lambda'}^{U'}) \subset V_{\lambda+\lambda'}^{U'}$. Since $T'$ normalizes $U'$ there is an action of

$T'$ on $V_\lambda^{U'} \otimes V_{\lambda'}^{U'}$ and $V_{\lambda+\lambda'}^{U'}$, and, moreover, $\pi_{\lambda,\lambda'}$ intertwines this action. Therefore $\pi_{\lambda,\lambda'}$ maps the weight space $(V_\lambda^{U'} \otimes V_{\lambda'}^{U'})(\mu+\mu')$ to the weight space $V_{\lambda+\lambda'}^{U'}(\mu+\mu')$. Since $M_\mu^\lambda \otimes M_{\mu'}^{\lambda'} \subset (V_\lambda^{U'} \otimes V_{\lambda'}^{U'})(\mu+\mu')$, and $M_{\mu+\mu'}^{\lambda+\lambda'} = V_{\lambda+\lambda'}^{U'}(\mu+\mu')$ the proof is complete. ■

By the above lemma the restriction of $\pi_{\lambda,\lambda'}$ to $M_\mu^\lambda \otimes M_{\mu'}^{\lambda'}$ is a map whose range lies in $M_{\mu+\mu'}^{\lambda+\lambda'}$. Abusing notation a bit, we denote the restriction of $\pi_{\lambda,\lambda'}$ to $M_\mu^\lambda \otimes M_{\mu'}^{\lambda'}$ also by $\pi_{\lambda,\lambda'}$. This will cause no confusion since we will explicitly write $M_\mu^\lambda \otimes M_{\mu'}^{\lambda'} \overset{\pi_{\lambda,\lambda'}}{\to} M_{\mu+\mu'}^{\lambda+\lambda'}$, when referring to this map. We call $M_\mu^\lambda \otimes M_{\mu'}^{\lambda'} \overset{\pi_{\lambda,\lambda'}}{\to} M_{\mu+\mu'}^{\lambda+\lambda'}$ the **Cartan product of multiplicity spaces**. The main result of this section is the following proposition.

**Proposition 3.3.2** *Let $\sigma \in \Sigma$ and let $(\mu,\lambda),(\mu',\lambda') \in \mathfrak{D}_\sigma$. Then the map*

$$M_\mu^\lambda \otimes M_{\mu'}^{\lambda'} \overset{\pi_{\lambda,\lambda'}}{\to} M_{\mu+\mu'}^{\lambda+\lambda'}$$

*is surjective.*

We now develop the necessary preliminary results for our proof of this lemma. First a remark:

**Remark 3.3.3** The hypothesis that $(\mu,\lambda)$ and $(\mu',\lambda')$ have the same order type is necessary. Indeed, consider the example when $n=2$, and $\lambda = \lambda' = (2,1,0)$, $\mu = (2)$, and $\mu' = (0)$. Note that $(\mu,\lambda)$ and $(\mu',\lambda')$ do not satisfy a common order type. By Proposition 3.2.2, $\dim M_\mu^\lambda = 2$, $\dim M_{\mu'}^{\lambda'} = 2$, $\dim M_{\mu+\mu'}^{\lambda+\lambda'} = 9$. Therefore the multiplication map $\pi_{\lambda,\lambda'}$ cannot be surjective.

Suppose $\gamma \in \Lambda_n^+$ and $\lambda \in \Lambda_{n+1}^+$. We may view $V_\lambda$ as a $G''$-module by restriction, and, as such, define $V_\lambda[\gamma]$ to be the $\gamma$-isotypic component of $V_\lambda$. Let $p_\gamma^\lambda : V_\lambda \to V_\lambda[\gamma]$ be the corresponding projection.

**Lemma 3.3.4** *Let $\nu, \nu' \in \Lambda_n^+$ and $\lambda, \lambda' \in \Lambda_{n+1}^+$. Suppose that $0 \neq x \in V_\lambda[\nu]$ and $0 \neq x' \in V_{\lambda'}[\nu']$. Then $p_{\nu+\nu'}^{\lambda+\lambda'}(\pi_{\lambda,\lambda'}(x \otimes x')) \neq 0$.*

Before beginning the proof of this lemma we make a simple observation.

**Observation**: Suppose $\mathfrak{S}$ is a semigroup and $\mathcal{V}$,$\mathcal{W}$ are $\mathfrak{S}$-graded vector spaces:

$$\begin{aligned} \mathcal{V} &= \bigoplus_{i \in \mathfrak{S}} V_i \\ \mathcal{W} &= \bigoplus_{i \in \mathfrak{S}} W_i. \end{aligned}$$

Suppose there are linear maps $\pi_{i,j} : V_i \otimes V_j \to V_{i+j}$ and $\tau_{i,j} : W_i \otimes W_j \to W_{i+j}$ for every $i, j \in \mathfrak{S}$. We refer to these maps as **products** on the vector spaces. Finally, suppose also there is an $\mathfrak{S}$-graded isomorphism $T : \mathcal{V} \to \mathcal{W}$ that preserves the products on $\mathcal{V}$ and $\mathcal{W}$ in the following sense: for all $i, j \in \mathfrak{S}$ the following diagram commutes:

$$\begin{array}{ccc} V_i \otimes V_j & \xrightarrow{\pi_{i,j}} & V_{i+j} \\ \downarrow T & & \downarrow T \\ W_i \otimes W_j & \xrightarrow{\tau_{i,j}} & W_{i+j} \end{array}$$

Then if $x \in V_i$ and $y \in V_j$ and $\tau_{i,j}(T(x) \otimes T(y)) \neq 0$, then $\pi_{i,j}(x \otimes y) \neq 0$.

**Proof (of Lemma 3.3.4).** For our purposes we use the semigroup $\mathfrak{S} = \{(\gamma, \lambda) \in \Lambda_n^+ \times \Lambda_{n+1}^+ : \gamma < \lambda\}$. We now define three $\mathfrak{S}$-graded vector spaces, each of which is equipped with product maps. We will also show that there exist

graded isomorphisms between these vector spaces that respect products in the sense defined above. We then use the above observation to complete the proof.

- $\mathcal{V}_1 = \bigoplus_{(\gamma,\lambda)\in\mathfrak{S}} V_\lambda[\gamma]$, with product maps defined as follows: if $x \in V_\lambda[\gamma]$ and $x' \in V_{\lambda'}[\gamma']$, then

$$xx' = p^{\lambda+\lambda'}_{\nu+\nu'}(\pi_{\lambda,\lambda'}(x \otimes x')).$$

- $\mathcal{V}_2 = \bigoplus_{(\gamma,\lambda)\in\mathfrak{S}} V_\gamma \otimes Hom_{G''}(V_\gamma, V_\lambda)$, with product maps defined as follows: if $v \otimes f \in V_\gamma \otimes Hom_{G''}(V_\gamma, V_\lambda)$ and $v' \otimes f' \in V_{\gamma'} \otimes Hom_{G''}(V_{\gamma'}, V_{\lambda'})$, then

$$(v \otimes f)(v' \otimes f') = \pi_{\gamma,\gamma'}(v \otimes v') \otimes (\pi_{\lambda,\lambda'} \circ (f \otimes f') \circ j_{\gamma,\gamma'}).$$

  (See (2.4) for the definition of the maps $j_{\lambda,\lambda'} : V_{\lambda+\lambda'} \hookrightarrow V_\lambda \otimes V_{\lambda'}$.)

- $\mathcal{V}_3 = \bigoplus_{(\gamma,\lambda)\in\mathfrak{S}} V_\gamma \otimes V^{U''}_\lambda(\gamma)$, with product maps defined as follows: if $v \otimes w \in V_\gamma \otimes V^{U''}_\lambda(\gamma)$ and $v' \otimes w' \in V_{\gamma'} \otimes V^{U''}_{\lambda'}(\gamma')$, then

$$(v \otimes w)(v' \otimes w') = \pi_{\gamma,\gamma'}(v \otimes v') \otimes \pi_{\lambda,\lambda'}(w \otimes w').$$

Now define $T : \mathcal{V}_2 \to \mathcal{V}_1$ by $T(v \otimes f) = f(v)$. This is clearly a linear isomorphism. Let $v \otimes f$,$v' \otimes f'$ be chosen as in the definition of $\mathcal{V}_2$. Then

$$T((v \otimes f)(v' \otimes f')) = (\pi_{\lambda,\lambda'} \circ (f \otimes f') \circ j_{\gamma,\gamma'})(\pi_{\gamma,\gamma'}(v \otimes v'))$$

while

$$T(v \otimes f)T(v' \otimes f') = p^{\lambda+\lambda'}_{\gamma+\gamma'}(\pi_{\lambda,\lambda'}(f(v) \otimes f(v'))).$$

We must show that these are equal. Let $z = v \otimes v'$. Since $z \in V_\gamma \otimes V_{\gamma'}$ we can write $z = z_0 + z_1$, where $z_0 \in (V_\gamma \otimes V_{\gamma'})[\gamma + \gamma']$ and $z_1 \in \sum_{\tau\neq\gamma+\gamma'}(V_\gamma \otimes V_{\gamma'})[\tau]$.

By the definition of $\pi_{\gamma,\gamma'}$ and $j_{\gamma,\gamma'}$, the composition $j_{\gamma,\gamma'} \circ \pi_{\gamma,\gamma'}$ is the projection of $V_\gamma \otimes V_{\gamma'}$ onto its isotypic component $(V_\gamma \otimes V_{\gamma'})[\gamma + \gamma']$. Therefore,

$$(\pi_{\lambda,\lambda'} \circ (f \otimes f') \circ j_{\gamma,\gamma'})(\pi_{\gamma,\gamma'}(z)) = (\pi_{\lambda,\lambda'} \circ (f \otimes f'))(z_0).$$

On the other hand,

$$\begin{aligned} p^{\lambda+\lambda'}_{\gamma+\gamma'}(\pi_{\lambda,\lambda'}(f(v) \otimes f(v'))) &= p^{\lambda+\lambda'}_{\gamma+\gamma'}(\pi_{\lambda,\lambda'}(f \otimes f'))(z) \\ &= p^{\lambda+\lambda'}_{\gamma+\gamma'}(\pi_{\lambda,\lambda'}((f \otimes f')(z_0))) + p^{\lambda+\lambda'}_{\gamma+\gamma'}(\pi_{\lambda,\lambda'}((f \otimes f')(z_1))) \\ &= \pi_{\lambda,\lambda'}((f \otimes f')(z_0)). \end{aligned}$$

Therefore $T$ preserves the products on $\mathcal{V}_1$ and $\mathcal{V}_2$.

Now we show that there exists a product-preserving graded isomorphism between $\mathcal{V}_2$ and $\mathcal{V}_3$. Define $S : \mathcal{V}_2 \to \mathcal{V}_3$ by $S(v \otimes f) = v \otimes f(v_\gamma)$, where recall that $v_\gamma$ is a fixed highest weight vector in $V_\gamma$ that we used to define the Cartan embeddings $j_{\gamma,\gamma'}$ (cf. (2.2)). This is clearly a linear isomorphism. We must show $S$ respects the product maps. Let $v \otimes f$,$v' \otimes f'$ be chosen as in the definition of $\mathcal{V}_2$. Then

$$\begin{aligned} S((v \otimes f)(v' \otimes f')) &= S(\pi_{\gamma,\gamma'}(v \otimes v') \otimes (\pi_{\lambda,\lambda'} \circ (f \otimes f') \circ j_{\gamma,\gamma'})) \\ &= \pi_{\gamma,\gamma'}(v \otimes v') \otimes \pi_{\lambda,\lambda'}(f(v_\gamma) \otimes f(v_{\gamma'})) \end{aligned}$$

whereas

$$S(v \otimes f)S(v' \otimes f') = (v \otimes f(v_\gamma))(v' \otimes f(v_{\gamma'})) = \pi_{\gamma,\gamma'}(v \otimes v') \otimes \pi_{\lambda,\lambda'}(f(v_\gamma) \otimes f(v_{\gamma'})).$$

Therefore $S$ preserves the products on $\mathcal{V}_2$ and $\mathcal{V}_3$.

Now $S \circ T^{-1}$ is a graded isomorphism of $\mathcal{V}_1$ and $\mathcal{V}_3$ that respects products. Consider $0 \neq x \in V_\lambda[\nu]$. Under the isomorphism $S \circ T^{-1}$, $x$ is mapped to a simple tensor $v \otimes w$. Indeed, by Proposition 3.2.1 $\dim V_\lambda^{U''}(\nu) = 1$, and $x$ is mapped to the summand $V_\nu \otimes V_\lambda^{U''}(\nu)$. Similarly, $0 \neq x' \in V_{\lambda'}[\nu']$ is mapped to a simple tensor $v' \otimes w'$. By the definition of multiplication in $V_3$, we have that $(v \otimes w)(v' \otimes w') = \pi_{\nu,\nu'}(v \otimes v') \otimes \pi_{\lambda,\lambda'}(w \otimes w')$. By definition, $\pi_{\nu,\nu'}(v \otimes v')$ (resp. $\pi_{\lambda,\lambda'}(w \otimes w')$) is simply the product of $v$ and $v'$ (resp. $w$ and $w'$) in $\mathcal{R}_{n+1}$ (resp. $\mathcal{R}_n$) (cf. Section 2.2). Since $\mathcal{R}_{n+1}$ (resp. $\mathcal{R}_n$) does not have zero divisors, it follows that $(v \otimes w)(v' \otimes w') \neq 0$. By the observation above we conclude that $xx' \neq 0$ in $\mathcal{V}_1$, i.e. $p_{\gamma+\gamma'}^{\lambda+\lambda'}(\pi_{\lambda,\lambda'}(x \otimes x')) \neq 0$. ■

The space $\mathcal{V}_1$ defined in the above proof is a special case of a construction of Vinberg's known as the asymptotic cone [24]. Similar constructions (in much greater generality) are studied from the perspective of toric deformations in [4], [1].

We note that the product defined on $\mathcal{V}_1$ actually makes it into an associative ring (which is isomorphic to the rings defined by the products on $\mathcal{V}_2$ and $\mathcal{V}_3$). One way to prove this is to show that $\mathcal{V}_1$ is a flat deformation of $\mathcal{R}_{n+1}$, but we do not need this so we omit the proof.

We are now ready to prove Proposition 3.3.2. Let $(\mathfrak{t}_n^*)_\mathbb{R}$ be the real form of $\mathfrak{t}_n^*$ spanned by the functionals $\{\varepsilon_i : i = 1, ..., n\}$ (cf. Section 2.1). Let $(\cdot, \cdot)$ be the inner product on $(\mathfrak{t}_n^*)_\mathbb{R}$ defined by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, and let $\|\gamma\|^2 = (\gamma, \gamma)$ define the associated norm. Denote by $\preceq$ the positive root ordering on $\mathfrak{t}_n^*$, defined relative to

the set of positive roots: $\{\varepsilon_i - \varepsilon_j : i < j\}$. In other words, $\alpha \preceq \beta$ means $\beta - \alpha$ is a nonnegative integer combination of positive roots. Recall the following standard result (for a proof see e.g. Proposition 5.1.12 in [11]):

**Proposition 3.3.5** *Let $\nu, \nu', \gamma \in \Lambda_n^+$. If $Hom_{G''}(V_\gamma, V_\nu \otimes V_{\nu'}) \neq \{0\}$ then $\gamma \preceq \nu + \nu'$.*

**Proof (of Proposition 3.3.2).** To ease notation let $X = M_\mu^\lambda$, $X' = M_{\mu'}^{\lambda'}$, $Y = M_{\mu+\mu'}^{\lambda+\lambda'}$, and $\pi = \pi_{\lambda,\lambda'}$. For $\gamma \in \Lambda_n^+$ set $Y[\gamma] = p_\gamma^{\lambda+\lambda'}(Y)$, $X[\gamma] = p_\gamma^\lambda(X)$, and $X'[\gamma] = p_\gamma^{\lambda'}(X')$.

Note that $Y = \bigoplus_{\gamma \in \Lambda_n^+} Y[\gamma]$, and, by Proposition 3.2.1, $\dim Y[\gamma]$ is zero or one. Moreover, $Y[\gamma] \neq \{0\}$ if, and only if, $\mu + \mu' < \gamma < \lambda + \lambda'$. We will prove by induction on $\|\gamma\|$ that $Y[\gamma]$ is in the image of $\pi$.

Let $\gamma \in \Lambda_n^+$ be of minimal norm such that $Y[\gamma] \neq \{0\}$. Our base case is to show that $Y[\gamma]$ is in the image of $\pi$. Since $(\mu, \lambda), (\mu', \lambda') \in \mathfrak{D}_\sigma$ we can apply Lemma 3.1.8 to obtain $\nu$,$\nu' \in \Lambda_n^+$ such that $\gamma = \nu + \nu'$, $\mu < \nu < \lambda$, and $\mu' < \nu' < \lambda'$. Choose $0 \neq x \in X[\nu]$ and $0 \neq x' \in X'[\nu']$, and let $z = x \otimes x'$.

Now, $\pi(z) = \sum_{\tau \in \Lambda_n^+} p_\tau^{\lambda+\lambda'}(\pi(z))$ is a decomposition of $\pi(z)$ in $Y = \bigoplus_{\gamma \in \Lambda_n^+} Y[\gamma]$. By Proposition 3.3.5, $p_\tau^{\lambda+\lambda'}(\pi(z)) = 0$ for $\tau \succ \gamma$. Therefore $\pi(z) = \sum_{\tau \preceq \gamma} p_\tau^{\lambda+\lambda'}(\pi(z))$. Since $\tau \prec \gamma$ implies $\|\tau\| < \|\gamma\|$, and by hypothesis $\gamma$ is of minimal norm such that $Y[\gamma] \neq \{0\}$, $p_\tau^{\lambda+\lambda'}(\pi(z)) = 0$ for $\tau \prec \gamma$. Therefore $\pi(z) = p_\gamma^{\lambda+\lambda'}(\pi(z)) \in Y[\gamma]$. By definition, $\pi(z)$ is the product of $x$ and $x'$ in $\mathcal{R}_{n+1}$. Therefore, $\pi(z) \neq 0$ since $\mathcal{R}_{n+1}$ has no zero divisors. Since $\dim Y[\gamma] = 1$, we conclude that $Y[\gamma]$ is in the

image of $\pi$. This completes the base case.

Now fix $\gamma \in \Lambda_n^+$ such that $Y[\gamma] \neq \{0\}$, and suppose $Y[\tau]$ is in the image of $\pi$ for all $\tau$ such that $\|\tau\| < \|\gamma\|$. Using Lemma 3.1.8 again, we choose $\nu$,$\nu' \in \Lambda_{n-1}^+$ such that $\gamma = \nu + \nu'$, $\mu < \nu < \lambda$, and $\mu' < \nu' < \lambda'$. Also choose $0 \neq y \in Y[\gamma]$, $0 \neq x \in X[\nu]$, and $0 \neq x' \in X'[\nu']$. By Lemma 3.3.4, $p_\gamma^{\lambda+\lambda'}(\pi(x \otimes x')) \neq 0$. Therefore, by Proposition 3.3.5,

$$\pi(x \otimes x') \in \mathbb{C}^\times y + \sum_{\tau \prec \gamma} Y[\tau].$$

Since $\tau \prec \gamma$ implies $\|\tau\| < \|\gamma\|$, by the inductive hypothesis we obtain an element $\xi \in X \otimes X'$ such that $\pi(\xi) = y$. Since $\dim Y[\gamma] = 1$, this shows that $Y[\gamma]$ is in the image of $\pi$. This completes the induction. ■

# Chapter 4

# The Transfer Theorem

This purpose of this chapter is to prove results about branching from $Sp(n,\mathbb{C})$ to $Sp(n-1,\mathbb{C})$. The main theorem is the so-called Transfer Theorem, which shows that the algebra associated to branching from $Sp(n,\mathbb{C})$ to $Sp(n-1,\mathbb{C})$ is isomorphic to the algebra associated to branching from $SL(n+1,\mathbb{C})$ to $SL(n-1,\mathbb{C})$. In the language of geometric invariant theory this translates to

$$\overline{U}_{C_n}\backslash\backslash Sp(n,\mathbb{C})//U_{C_{n-1}} \cong \overline{U}_{n+1}\backslash\backslash SL(n+1,\mathbb{C})//U_{n-1}.$$

Since this is an isomorphism also of $\mathfrak{D}$-graded $SL(2,\mathbb{C})$-rings, we can transfer Propositions 3.2.2 and 3.3.2 from the previous chapter to the symplectic group.

In section 1 we state the Transfer Theorem and its corollaries. Section 2 is devoted our main tool in proving the Transfer Theorem: Zhelobenko's "indicator system". This is a realization of irreducible representations of a complex semisimple Lie group in the space of functions on the unipotent radical of a Borel

subgroup. In section 3 we write down the indicator system for the symplectic group, and in section 4 prove the lemma that relates branching from $GL(n+1,\mathbb{C})$ to $GL(n-1,\mathbb{C})$, to branching from $Sp(n,\mathbb{C})$ to $Sp(n-1,\mathbb{C})$. The methods of our proof are all due to Zhelobenko. Finally, in section 5 we put together all the ingredients to prove the Transfer Theorem.

## 4.1 Statement of the Transfer Theorem

In this section we state the theorem that allows us to transfer the results that we obtained about the general linear group in the previous chapter to the symplectic group. In particular, as corollaries of the theorem we obtain the $SL(2,\mathbb{C})$-module structure of the multiplicity spaces that arise in branching from $Sp(n,\mathbb{C})$ to $Sp(n-1,\mathbb{C})$, and an analogue of Proposition 3.3.2 for the symplectic group about the Cartan product of multiplicity spaces. The proof of the theorem will be deferred to the end of the chapter.

We begin with the space of left $\overline{U}_{C_n}$-invariant and right $U_{C_{n-1}}$-invariant functions on $Sp(n,\mathbb{C})$:

$$\widehat{\mathcal{M}} = \mathcal{O}(Sp(n,\mathbb{C}))^{\overline{U}_{C_n}\times U_{C_{n-1}}}. \tag{4.1}$$

Via the embedding of $Sp(1,\mathbb{C}) \cong SL(2,\mathbb{C})$ in $Sp(n,\mathbb{C})$ described in Section 2.4, $\widehat{\mathcal{M}}$ is an $SL(2,\mathbb{C})$-ring by right translation.

As in Lemma 3.3.1, $\widehat{M}^{\lambda}_{\mu}\widehat{M}^{\lambda'}_{\mu'} \subset \widehat{M}^{\lambda+\lambda'}_{\mu+\mu'}$. Therefore $\widehat{\mathcal{M}}$ is graded by the

semigroup $\Lambda_{n-1}^+ \times \Lambda_n^+$:

$$\widehat{\mathcal{M}} = \bigoplus_{(\mu,\lambda)\in\Lambda_{n-1}^+\times\Lambda_n^+} \widehat{M}_\mu^\lambda. \tag{4.2}$$

Now consider

$$\overline{\mathcal{M}} = \mathcal{O}(SL(n+1,\mathbb{C}))^{\overline{U}_{n+1}\times U_{n-1}}.$$

$\overline{\mathcal{M}}$ is again an $SL(2,\mathbb{C})$-ring, and by similar considerations, $\overline{\mathcal{M}}$ is graded by the semigroup $\Lambda_{n-2}^+ \times \Lambda_n^+$:

$$\overline{\mathcal{M}} = \bigoplus_{(\kappa,\lambda)\in\Lambda_{n-2}^+\times\Lambda_n^+} \overline{M}_\kappa^\lambda.$$

Our first order of business is to show that the grade on $\overline{\mathcal{M}}$ can be refined to $\Lambda_{n-1}^+ \times \Lambda_n^+$.

For $\lambda = (\lambda_1, ..., \lambda_n) \in \Lambda_n^+$ let $\lambda^- = (\lambda_1 - \lambda_n, ..., \lambda_{n-1} - \lambda_n) \in \Lambda_{n-1}^+$. This is a left-inverse to the $(\cdot)^+$ operation defined in Section 2.3: $(\lambda^+)^- = \lambda$. On the other hand, in general $(\lambda^-)^+ \neq \lambda$. Note that for $\lambda \in \Lambda_n^+$, the restriction $V_\lambda|_{SL(n,\mathbb{C})}$ is isomorphic to $\overline{V}_{\lambda^-}$.

**Lemma 4.1.1** *As $SL(2,\mathbb{C})$-rings,*

$$\overline{\mathcal{M}} \cong \bigoplus_{(\mu,\lambda)\in\Lambda_{n-1}^+\times\Lambda_n^+} M_\mu^{\lambda^+}$$

*where the right hand side is a subring of*

$$\mathcal{M} = \mathcal{O}(M_{n+1}(\mathbb{C}))^{\overline{U}_{n+1}\times U_{n-1}}.$$

**Proof.** For ease of notation let

$$\mathcal{M}' = \bigoplus_{\mu\in\Lambda_{n-1}^+,\lambda\in\Lambda_n^+} M_\mu^{\lambda^+} \subset \mathcal{M}.$$

We want to show that as $SL(2,\mathbb{C})$-rings, $\overline{\mathcal{M}} \cong \mathcal{M}'$. Consider the diamond of groups:

$$\begin{array}{ccccc} & & GL(n+1,\mathbb{C}) & & \\ & \swarrow & & \searrow & \\ SL(n+1,\mathbb{C}) & & & & GL(n-1,\mathbb{C}) \times SL(2,\mathbb{C}) \\ & \searrow & & \swarrow & \\ & & SL(n-1,\mathbb{C}) \times SL(2,\mathbb{C}) & & \end{array}$$

Fix $\lambda \in \Lambda_n^+$. We restrict the representation $V_{\lambda^+}$ of $GL(n+1,\mathbb{C})$ down both sides of the diamond:

$$\begin{array}{cc} V_{\lambda^+} & V_{\lambda^+} \\ | & | \\ \overline{V}_\lambda & \bigoplus_{\mu \in \Lambda_{n-1}^+} V_\mu \otimes M_\mu^{\lambda^+} \\ | & | \\ \bigoplus_{\beta \in \Lambda_{n-2}^+} \overline{V}_\beta \otimes \overline{M}_\beta^\lambda & \bigoplus_{\mu \in \Lambda_{n-1}^+} \overline{V}_{\mu^-} \otimes M_\mu^{\lambda^+} \end{array}$$

Hence

$$\bigoplus_{\beta \in \Lambda_{n-2}^+} \overline{V}_\beta \otimes \overline{M}_\beta^\lambda \cong \bigoplus_{\mu \in \Lambda_{n-1}^+} \overline{V}_{\mu^-} \otimes M_\mu^{\lambda^+}$$

as $SL(n-1,\mathbb{C}) \times SL(2,\mathbb{C})$-modules. Therefore

$$\overline{M}_\beta^\lambda \cong \bigoplus_{\substack{\mu \in \Lambda_{n-1}^+ \\ \mu^- = \beta}} M_\mu^{\lambda^+}.$$

Summing over all multiplicity spaces we get

$$\overline{\mathcal{M}} = \bigoplus_{\substack{\lambda \in \Lambda_n^+ \\ \beta \in \Lambda_{n-2}^+}} \overline{M}_\beta^\lambda \cong \bigoplus_{\substack{\lambda \in \Lambda_n^+ \\ \beta \in \Lambda_{n-2}^+}} \bigoplus_{\substack{\mu \in \Lambda_{n-1}^+ \\ \mu^- = \beta}} M_\mu^{\lambda^+} \cong \mathcal{M}'.$$

This show that as $SL(2,\mathbb{C})$-modules $\overline{\mathcal{M}} \cong \mathcal{M}'$.

$\mathcal{M}'$ inherits a product from $\mathcal{M}$. To conclude the proof we need to show that this product agrees with the product on $\overline{\mathcal{M}}$. We have a ring homomorphism given by restriction of functions:

$$\mathcal{M} \to \overline{\mathcal{M}}.$$

What we've just done shows that under this map the subring $\mathcal{M}'$ maps bijectively onto $\overline{\mathcal{M}}$. Therefore the restriction maps $\mathcal{M}'$ isomorphically onto $\overline{\mathcal{M}}$. ■

Lemma 4.1.1 shows that, like $\widehat{\mathcal{M}}$, $\overline{\mathcal{M}}$ is also graded by $\Lambda_{n-1}^+ \times \Lambda_n^+$. We are now ready to state the "Transfer Theorem".

**Theorem 4.1.2** *There is an isomorphism $\Psi : \widehat{\mathcal{M}} \to \overline{\mathcal{M}}$ of $\Lambda_{n-1}^+ \times \Lambda_n^+$-graded rings. Moreover, $\Psi$ intertwines the $SL(2,\mathbb{C})$-actions.*

The remainder of the chapter is devoted to a proof of this theorem. We conclude this section with two corollaries of Theorem 4.1.2. Both corollaries use the above theorem to transfer results about branching from $GL(n+1,\mathbb{C})$ to $GL(n-1,\mathbb{C})$, to results about branching from $Sp(n,\mathbb{C})$ to $Sp(n-1,\mathbb{C})$. The first follows by applying Theorem 3.2.2, and the second one by Proposition 3.3.2.

**Corollary 4.1.3** *Let $\mu \in \Lambda_{n-1}^+$ and $\lambda \in \Lambda_n^+$. Then $\widehat{M}_\mu^\lambda \neq \{0\}$ if, and only if, $\mu \ll \lambda^+$. Moreover, if $\mu \ll \lambda^+$ then as $SL(2,\mathbb{C})$-modules,*

$$\widehat{M}_\mu^\lambda \cong \bigotimes_{i=1}^{n} F_{r_i(\mu,\lambda^+)} \tag{4.3}$$

*where $SL(2,\mathbb{C})$ acts by the tensor product action on the right hand side. In particular, $\dim \widehat{M}_\mu^\lambda = \prod_{i=1}^n (r_i(\mu,\lambda^+)+1)$.*

**Corollary 4.1.4** *Let $\mu, \mu' \in \Lambda_{n-1}^+$ and $\lambda, \lambda' \in \Lambda_n^+$. Suppose there exists $\sigma \in \Sigma$ such that $(\mu, \lambda^+), (\mu', (\lambda')^+) \in \mathfrak{D}_\sigma$. Then the map*

$$\widehat{M}_\mu^\lambda \otimes \widehat{M}_{\mu'}^{\lambda'} \stackrel{\pi_{\lambda,\lambda'}}{\to} \widehat{M}_{\mu+\mu'}^{\lambda+\lambda'}$$

*is surjective.*

Another proof of Corollary 4.1.3 using the combinatorics of partition functions appears in Appendix A. On the other hand, we are not aware of a proof of Corollary 4.1.4 which does not reduce the problem to a question about the general linear groups.

## 4.2 Zhelobenko's indicator system

In order to prove the Transfer Theorem (Theorem 4.1.2) we utilize a realization of irreducible representations of connected classical groups due to D.P. Zhelobenko. In this section we explain the basic idea behind Zhelobenko's construction. For detail and proofs consult [26].

Let $G$ be a connected classical group. We use freely the notation from Section 2.5. By Theorem 2.5.1, the irreducible representation of highest weight $\lambda \in \Lambda^+$ can be realized in $\mathcal{O}(G)^{\overline{U}}$ as precisely the matrix coefficients $f_v : g \mapsto v_\lambda^*(g.v)$ for $v \in F_\lambda$. Recall here that $v_\lambda^*$ is a choice of nonzero functional in $(F_\lambda^*)^{\overline{U}}$, necessarily of weight $-\lambda$. $G$ acts on these functions by right translation, and they manifestly satisfy the equation $f(\overline{u}tu) = t^\lambda f(u)$, where $\overline{u} \in \overline{U}$, $t \in T$, and $u \in U$.

In particular, since $G_{reg}$ is dense, $f$ is completely determined by its values on $U$. Hence there is a an embedding (as vector spaces) of $F_\lambda$ in $\mathcal{O}(U)$, given by

$$v \overset{j}{\mapsto} f_v|_U.$$

Set $Z_\lambda(U) = j(F_\lambda)$. If there is no cause for confusion, we write simply $Z_\lambda = Z_\lambda(U)$.

The action of $G$ on $Z_\lambda$ that makes $j$ an isomorphism of $G$-modules is described as follows. Let $e^\lambda : T \longrightarrow \mathbb{C}$ be the character of $T$ given by $t \mapsto t^\lambda$. We extend this character to $G_{reg}$ by defining $e^\lambda(\overline{u}tu) = t^\lambda$. Then by continuity $e^\lambda$ is defined on all of $G$. Now let $u \in U$, $g \in u^{-1}G_{reg}$, and $f \in Z_\lambda$. Write $ug = \overline{u}_1 t_1 u_1 \in G_{reg}$. Then

$$g.f(u) = e^\lambda(t_1) f(u_1).$$

Again, since $u^{-1}G_{reg}$ is dense, we extend this action to all of $G$. Note that this action is easy to describe in two cases: if $u, u' \in U$ then $u.f(u') = f(u'u)$, and, moreover, if $t \in T$ then $t.f(u) = e^\lambda(t) f(t^{-1}ut)$. In particular, the constant function $z_\lambda : u \mapsto 1$ is a canonical highest weight vector in $Z_\lambda$.

The model $Z_\lambda$ has two advantages: firstly, the representations are realized in a space of polynomials with the minimal number of variables, and, secondly, and most importantly for us, the space $Z_\lambda$ is the vanishing set of a system of differential operators. We now go on to describe this system of operators, the so-called "indicator system".

Let $\{\alpha_1, ..., \alpha_n\}$ be a set of simple roots relative to the positive roots $\Phi^+$. For each $\alpha_i$ choose a nonzero root vector $X_i \in \mathfrak{g}_{\alpha_i}$. (As usual $\mathfrak{g} = Lie(G)$.)

Notice that $X_i \in Lie(U)$ as well. Now let $D_i$ be the differential operator on $\mathcal{O}(U)$ corresponding to action of the simple root vector $X_i$ acting on $\mathcal{O}(U)$ by left translation. Finally, let $\{\varpi_1, ..., \varpi_n\}$ be the fundamental weights and fix $\lambda = m_1\varpi_1 + \cdots + m_n\varpi_n \in \Lambda^+$.

**Proposition 4.2.1 ([26], p. 324)** *$Z_\lambda = \{f \in \mathcal{O}(U) : D_i^{m_i+1} f = 0$ for $i = 1, ..., n\}$.*

In [26] the system of differential equations $\{D_i^{m_i+1} f = 0 : i = 1, ..., n\}$ is termed the "indicator system".

**Example 4.2.2 ($G = SL(2, \mathbb{C})$)** We make the identification $U = \mathbb{C}$, so then $\mathcal{O}(U) = \mathbb{C}[x]$ and $D = \frac{d}{dx}$ is the operator corresponding to the simple root vector. The above proposition realizes the representation $F_k$ as the space spanned by $1, x, ..., x^k$. The action is given by "linear fractional transformations":

$$g.f(x) = (g_{11} + xg_{21})^k f(\frac{g_{12} + xg_{22}}{g_{11} + xg_{21}}).$$

**Example 4.2.3 ($G = GL(n, \mathbb{C})$)** Let $U = U_n$. It is readily seen that the operator $D_i$ corresponding to the simple root $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ is

$$\frac{\partial}{\partial u_{i,i+1}} + \sum_{k=i+2}^{n} u_{i+1,k} \frac{\partial}{\partial u_{i,k}}$$

where the $u_{ij}$ are standard coordinates on $U$ (recall that $U$ is isomorphic to affine space). We will now use this expression to prove parts one and two of Proposition 3.2.1.

Let $U' = U_{n-1}$ and $T' = T_{n-1}$. To describe the decomposition of $Z_\lambda$ into irreducible representations of $GL(n-1,\mathbb{C})$, we will find a $T'$-weight basis for $Z_\lambda^{U'}$. This well tell us, of course, the highest weights of all the $GL(n-1,\mathbb{C})$-modules that appear in $Z_\lambda$.

Set $u_{in} = x_i$ for $i = 1, ..., n-1$. Then $\mathcal{O}(U)^{U'} = \mathbb{C}[x_1, ..., x_{n-1}]$. If $f \in \mathbb{C}[x_1, ..., x_{n-1}]$, then $D_i(f) = x_{i+1}\frac{\partial}{\partial x_i}(f)$ (where $x_n = 1$). Since $[x_{i+1}, \frac{\partial}{\partial x_i}] = 0$, $D_i^{m_i+1} f = 0$ for $i = 1, ..., n-1$ if, and only if, $f$ is a linear combination of monomials $x_1^{j_1} \cdots x_{n-1}^{j_{n-1}}$ where $0 \leq j_i \leq m_i$.

We now have that $Z_\lambda^{U'}$ has a basis $\{x_1^{j_1} \cdots x_{n-1}^{j_{n-1}} : 0 \leq j_i \leq n_i\}$. The $GL(n-1,\mathbb{C})$-weight of $x_1^{j_1} \cdots x_{n-1}^{j_{n-1}}$ is $(\lambda_1 - j_1, ..., \lambda_{n-1} - j_{n-1})$. These weights are all distinct, proving part one of Proposition 3.2.1. Recall that $m_i = \lambda_i - \lambda_{i+1}$; substituting this into the constraint $0 \leq j_i \leq m_i$ we obtain the interlacing condition $\lambda_i \geq \lambda_i - j_i \geq \lambda_{i+1}$. This proves part two of Proposition 3.2.1.

We conclude with a general observation about products of the representations $Z_\lambda$. Notice that $Z_\lambda Z'_\lambda = Z_{\lambda+\lambda'}$, where here we are just taking the usual product of functions. Indeed, $Z_\lambda Z'_\lambda \subset Z_{\lambda+\lambda'}$ by the Leibniz rule, and, moreover, $Z_\lambda Z'_\lambda$ is a nonzero invariant subspace. By irreducibility equality holds.

Consider the ring $\mathcal{O}(U \times \mathbb{C}^n)$. Let $t_1, ..., t_n$ be the standard coordinates on $\mathbb{C}^n$. Then $\mathcal{O}(U \times \mathbb{C}^n) = \bigoplus_{\overline{m} \in \mathbb{N}^n} \mathcal{O}(U) \otimes t_1^{m_1} \cdots t_n^{m_n}$. Set $Z_{\overline{m}} = Z_\lambda$ and $z_{\overline{m}} = z_\lambda$ where $\lambda = m_1\varpi_1 + \cdots + m_n\varpi_n$. We form the subring of $\mathcal{O}(U \times \mathbb{C}^n)$:

$$\bigoplus_{\overline{m} \in \mathbb{N}^n} Z_{\overline{m}} \otimes t^{\overline{m}}.$$

This subring is a $G$-ring, with $G$ acting on the left factor.

Recall from Section 2.5 that we've realized all the irreducible representations of $G$ in $\mathcal{O}(G)^{\overline{U}}$, and, moreover, we've defined canonical highest weight vectors $f_\lambda \in F_\lambda$ for every $\lambda \in \Lambda^+$. These vectors satisfy the property that $f_\lambda f_{\lambda'} = f_{\lambda+\lambda'}$. Let $f_{\overline{m}} = f_\lambda$.

Define a map

$$\bigoplus_{\overline{m}\in\mathbb{N}^n} Z_{\overline{m}} \otimes t^{\overline{m}} \to \mathcal{O}(G)^{\overline{U}} \tag{4.4}$$

by $z_{\overline{m}} \otimes t^{\overline{m}} \mapsto f_{\overline{m}}$ and extending by $G$-linearity. We have proved:

**Proposition 4.2.4** *The map (4.4) is an isomorphism of $G$-rings.*

The point of introducing these auxiliary coordinates $t_1, ..., t_n$ is to differentiate between the different summands. For instance, the constant function $\mathbf{1} : u \mapsto 1$ appears in $Z_\lambda$ for any $\lambda$ as the highest weight vector. But $\mathbf{1} \otimes t^{\overline{m}}$ is unambiguously the highest weight vector for $Z_{\overline{m}}$.

## 4.3 The indicator system for the symplectic group

In this section we analyze the indicator system of the symplectic group. Let $\{\alpha_1, ..., \alpha_n\}$ be the set of simple roots in $\Phi^+_{C_n}$ given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, ..., n-1$ and $\alpha_n = 2\varepsilon_n$.

Now we find coordinates for the affine space $U = U_{C_n}$. Consider the

subgroup

$$G' = \{k \in Sp(n, \mathbb{C}) : ke_{\pm 1} = e_{\pm 1}\}.$$

$G'$ is isomorphic to $Sp(n-1, \mathbb{C})$, but notice that this is not the copy of $Sp(n-1, \mathbb{C})$ in $Sp(n, \mathbb{C})$ that we chose in Section 2.4. Let $U' = U \cap G'$. Let $J$ be the subgroup consisting of matrices

$$\begin{bmatrix} 1 & v & z \\ & I_{2(n-1)} & \widetilde{v} \\ & & 1 \end{bmatrix}$$

where $v \in (\mathbb{C}^{n-2})^t$, $z \in \mathbb{C}$, and $\widetilde{v}$ depends linearly on $v$. It's straight-forward to verify that $U = U'J$ and $U' \cap J$ is the identity matrix. Therefore every element $u \in U$ can be expressed uniquely as a product $u = u'j$, with $u' \in U'$ and $j \in J$.

Applying the same decomposition to the element $u' = u''j' \in U'$, and so on, we see that the following can be taken as coordinates on $U$:

$$\begin{array}{ccccccccc} 1 & u_{12} & & & \cdots & & u_{1n-1} & u_{1n} \\ & 1 & u_{23} & & \cdots & & u_{2n-1} & \\ & & \ddots & \ddots & & \iddots & & \\ & & & 1 & u_{n,n+1} & & & \end{array} \tag{4.5}$$

(The one's are retained here in order to preserve the symmetry of the entries.) The other entries of $U$ are polynomials in these coordinates.

**Example 4.3.1** For $n = 2$ the coordinates of the element

$$\begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} \\ & 1 & u_{23} & u_{24} \\ & & 1 & u_{34} \\ & & & 1 \end{bmatrix}$$

are the entries

$$\begin{pmatrix} 1 & u_{12} & u_{13} & u_{14} \\ & 1 & u_{23} & \end{pmatrix}$$

**Lemma 4.3.2 ([26], p. 328)** *Let $D_i$ be the differential operator corresponding to the simple root $\alpha_i$. Then*

$$D_i = \frac{\partial}{\partial u_{i,i+1}} + \sum_{j=i}^{n-i-1} u_{i+1,j+2} \frac{\partial}{\partial u_{i,j+2}}.$$

Notice that in the formula for $D_i$, the terms in the expression are written in the coordinate system on $U$, except for one term: when $j = n - i - 1$, the variable $u_{i+1,n-i+1}$ is not among the coordinates in (4.5). That means it can be expressed as a polynomial in the coordinates (4.5). While it is possible to write this explicitly (cf. [26]), this will not be necessary for our purposes.

## 4.4 Branching for the symplectic group

In this section we revert to the usual embedding of $Sp(n-1,\mathbb{C}) \subset Sp(n,\mathbb{C})$ as defined in Section 2.4:

$$Sp(n-1,\mathbb{C}) = \{k \in Sp(n,\mathbb{C}) : ke_{\pm n} = e_{\pm n}\}.$$

The ring $\mathcal{O}(U_{C_n})$ is a polynomial algebra in precisely the variables (4.5) described above. A straight-forward calculation shows that $f \in \mathcal{O}(U_{C_n})^{U_{C_{n-1}}}$ if, and only if,

it's a polynomial in the variables

$$\begin{matrix} u_{1,n} & u_{1,n+1} \\ u_{2,n} & u_{2,n+1} \\ \vdots & \vdots \\ u_{n-1,n} & u_{n-1,n+1} \\ 1 & u_{n,n+1} \end{matrix} \tag{4.6}$$

In other words, $\mathcal{O}(U_{C_n})^{U_{C_{n-1}}}$ is a polynomial ring in $2n-1$ variables.

Compare this now to the general linear group. Recall that $U_{n-1} \subset GL(n+1, \mathbb{C})$ consists of matrices of the form

$$\begin{bmatrix} 1 & x_{12} & \cdots & x_{1,n-1} & 0 & 0 \\ & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & \ddots & x_{n-2,n-1} & & \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{bmatrix}$$

Therefore $\mathcal{O}(U_{n+1})^{U_{n-1}}$ is the polynomial ring in the variables

$$\begin{matrix} x_{1,n} & x_{1,n+1} \\ x_{2,n} & x_{2,n+1} \\ \vdots & \vdots \\ x_{n-1,n} & x_{n-1,n+1} \\ 1 & x_{n,n+1} \end{matrix} \tag{4.7}$$

We see that $\mathcal{O}(U_{n+1})^{U_{n-1}}$ is also a polynomial ring in $2n-1$ variables.

To make this explicit, we define an isomorphism $\psi : \mathcal{O}(U_{C_n})^{U_{C_{n-1}}} \to \mathcal{O}(U_{n+1})^{U_{n-1}}$ as follows. For $g \in GL(2n, \mathbb{C})$, let $g_p$ be its $(n+1) \times (n+1)$

principal sub-matrix, or "cut-off". Given $f \in \mathcal{O}(U_{C_n})^{U_{C_{n-1}}}$ and $x \in U_{n+1}$, define $\psi(f)(x) = f(g)$, where $g \in U_{C_n}$ and $g_p = x$. By our computation above (cf. (4.5)), there always exists such a $g$. Moreover, if $g, h \in U$ and $g_p = h_p$, then $f(g) = f(h)$ since $f \in \mathcal{O}(U_{C_n})^{U_{C_{n-1}}}$. Therefore $\psi$ is well-defined. By our descriptions of the rings $\mathcal{O}(U_{C_n})^{U_{C_{n-1}}}$ and $\mathcal{O}(U_{n+1})^{U_{n-1}}$, $\psi$ is a ring isomorphism. Intuitively, $\psi(f)$ simply replaces the variables $u_{ij}$ with $x_{ij}$ in the expression for the polynomial $f$.

Recall from the previous section that, for $\lambda \in \Lambda_n^+$, $Z_\lambda(U_{C_n})$ is an irreducible representation of $Sp(n, \mathbb{C})$ of highest weight $\lambda$. Similarly, $Z_{\lambda^+}(U_{n+1})$ is an irreducible representation of $GL(n+1, \mathbb{C})$ of highest weight $\lambda^+$.

**Lemma 4.4.1** *Let $\lambda \in \Lambda_n^+$. The map $\psi$ restricts to a linear isomorphism $\psi : Z_\lambda(U_{C_n})^{U_{C_{n-1}}} \xrightarrow{\simeq} Z_{\lambda^+}(U_{n+1})^{U_{n-1}}$.*

**Proof.** Let $f \in \mathcal{O}(U_{C_n})^{U_{C_{n-1}}}$. Then $f$ is a polynomial in the variables (4.6). By Lemma 4.3.2,

$$D_i(f) = (u_{i+1,n}\frac{\partial}{\partial u_{i,n}} + u_{i+1,n+1}\frac{\partial}{\partial u_{i,n+1}})(f)$$

where here $D_i$ is the differential operator on $\mathcal{O}(U_{C_n})$ corresponding to the simple root $\alpha_i$ of $Sp(n, \mathbb{C})$. Let $\lambda = m_1\varpi_1 + \cdots + m_n\varpi_n$. This gives us the following description of the space of $Sp(n-1, \mathbb{C})$ highest weight vectors in $Z_\lambda(U_{C_n})$. $Z_\lambda(U_{C_n})^{U_{C_{n-1}}}$ equals

$$\left\{ f \in \mathcal{O}(U_{C_n})^{U_{C_{n-1}}} : (u_{i+1,n}\frac{\partial}{\partial u_{i,n}} + u_{i+1,n+1}\frac{\partial}{\partial u_{i,n+1}})^{m_i+1}(f) = 0 \text{ for } i = 1, ..., n \right\}. \tag{4.8}$$

Now consider $f \in \mathcal{O}(U_{n+1})^{U_{n-1}}$. Then $f$ is a polynomial in the variables (4.7). By Example 4.2.3,

$$D_i(f) = (x_{i+1,n}\frac{\partial}{\partial x_{i,n}} + x_{i+1,n+1}\frac{\partial}{\partial x_{i,n+1}})(f)$$

where here $D_i$ is the differential operator on $\mathcal{O}(U_{n+1})$ corresponding to the simple root $\alpha_i$ of $GL(n+1,\mathbb{C})$. Therefore, $Z_{\lambda^+}(U_{n+1})^{U_{n-1}}$ equals

$$\left\{ f \in \mathcal{O}(U_{n+1})^{U_{n-1}} : (x_{i+1,n}\frac{\partial}{\partial x_{i,n}} + x_{i+1,n+1}\frac{\partial}{\partial x_{i,n+1}})^{m_i+1}(f) = 0 \text{ for } i = 1, ..., n \right\}. \tag{4.9}$$

With these descriptions in hand it follows that $\psi(Z_\lambda(U_{C_n})^{U_{C_{n-1}}}) = Z_{\lambda^+}(U_{n+1})^{U_{n-1}}$.

■

Our next task is to show that $\psi$ intertwines certain actions. We need the following observation, the proof of which follows from the obvious computation using block matrices.

**Lemma 4.4.2** *Let $g \in GL(2n,\mathbb{C})_{reg}$ and let $g = ytx$ be its Gauss decomposition. Then $g_p \in GL(n+1,\mathbb{C})_{reg}$ and, moreover, $g_p = y_p t_p x_p$ is its Gauss decomposition.*

Now $T_{C_{n-1}}$ acts on $Z_\lambda(U_{C_n})^{U_{C_{n-1}}}$, while $T_{n-1}$ acts on $Z_{\lambda^+}(U_{n+1})^{U_{n-1}}$. In the next lemma, these tori are both identified with $(\mathbb{C}^\times)^{n-1}$.

**Lemma 4.4.3** *Let $\lambda \in \Lambda_n^+$. Then $\psi : Z_\lambda(U_{C_n})^{U_{C_{n-1}}} \to Z_{\lambda^+}(U_{n+1})^{U_{n-1}}$ is a $(\mathbb{C}^\times)^{n-1}$-isomorphism.*

**Proof.** By Lemma 4.4.1, it remains to show only that $\psi$ intertwines the $(\mathbb{C}^\times)^{n-1}$-action. Fix $f \in Z_\lambda(U_{C_n})^{U_{C_{n-1}}}$. Let $t \in (\mathbb{C}^\times)^{n-1}$ and $x \in U_{n+1}$. Choose

$g \in U_{C_n}$ such that $g_p = x$. Regarding $t$ as an element of $T_{C_{n-1}} \subset Sp(n, \mathbb{C})$, we write $gt = tu$ ($u = t^{-1}gt$) is the Gauss decomposition of $gt$ in $Sp(n, \mathbb{C})$ (and in $GL(2n, \mathbb{C})$). Therefore

$$\psi(t.f)(x) = (t.f)(g) = t^{\lambda} f(u).$$

Now regarding $t$ as an element of $T_{n-1} \subset GL(n+1, \mathbb{C})$, by Lemma 4.4.2, $xt = tu_p$ is the Gauss decomposition of $xt$ in $GL(n-1, \mathbb{C})$. Therefore

$$(t.\psi(f))(x) = t^{\lambda}\psi(f)(u_p) = t^{\lambda} f(u).$$

Hence $\psi(t.f) = t.\psi(f)$. ■

If $\mu = (\mu_1, ..., \mu_{n-1}) \in \Lambda^+_{n-1}$ is weight of $(\mathbb{C}^\times)^{n-1}$, then Lemma 4.4.3 implies that $\psi : Z_\lambda(U_{C_n})^{U_{C_{n-1}}}(\mu) \xrightarrow{\simeq} Z_{\lambda^+}(U_{n+1})^{U_{n-1}}(\mu)$ is a linear isomorphism of $(\mathbb{C}^\times)^{n-1}$ weight spaces. These spaces are none other than the multiplicity spaces $M^\lambda_\mu$ and $\widehat{M^\lambda_\mu}$.

Recall that $SL(2, \mathbb{C})$ acts on the spaces $Z_\lambda(U_{C_n})^{U_{C_{n-1}}}(\mu)$ and $Z_{\lambda^+}(U_{n+1})^{U_{n-1}}(\mu)$. The same argument as in Lemma 4.4.3 shows that $\psi$ intertwines this action. Therefore we have:

**Lemma 4.4.4** *Let $\lambda \in \Lambda^+_n$ and $\mu \in \Lambda^+_{n-1}$. Then $\psi : Z_\lambda(U_{C_n})^{U_{C_{n-1}}}(\mu) \to Z_{\lambda^+}(U_{n+1})^{U_{n-1}}(\mu)$ is an $SL(2, \mathbb{C})$-isomorphism.*

## 4.5 Proof of the Transfer Theorem

We've completed the groundwork necessary to prove Theorem 4.1.2. As observed above, $\widehat{M}_\mu^\lambda \cong Z_\lambda(U_{C_n})^{U_{C_{n-1}}}(\mu)$ and $M_\mu^{\lambda^+} \cong Z_{\lambda^+}(U_{n+1})^{U_{n-1}}(\mu)$ as $SL(2,\mathbb{C})$-modules. Recall from Section 4.2 that we allow for the following notational convention: if $\lambda = m_1\varpi_1 + \cdots + m_n\varpi_n$ then $Z_{\overline{m}}(U_{C_n}) = Z_\lambda(U_{C_n})$, where $\overline{m} = (m_1, ..., m_n) \in \mathbb{N}^n$. In addition, we let $Z_{\overline{m}}(U_{n+1})^{U_{n-1}}(\mu) = Z_{\lambda^+}(U_{n+1})^{U_{n-1}}(\mu)$. Now, by Proposition 4.2.4,

$$\begin{aligned}
\widehat{\mathcal{M}} &\cong \bigoplus_{\substack{\mu\in\Lambda_{n-1}^+,\\ \overline{m}\in\mathbb{N}^n}} Z_{\overline{m}}(U_{C_n})^{U_{C_{n-1}}}(\mu)\otimes t^{\overline{m}} \\
\overline{\mathcal{M}} &\cong \bigoplus_{\substack{\mu\in\Lambda_{n-1}^+,\\ \overline{m}\in\mathbb{N}^n}} Z_{\overline{m}}(U_{n+1})^{U_{n-1}}(\mu)\otimes t^{\overline{m}}
\end{aligned}$$

as graded $SL(2,\mathbb{C})$-rings. We define $\Psi : \widehat{\mathcal{M}} \to \overline{\mathcal{M}}$ by

$$\Psi(x \otimes t^{\overline{m}}) = \psi(x) \otimes t^{\overline{m}}.$$

By Lemma 4.4.4, $\Psi$ is a grade-preserving isomorphism of $SL(2,\mathbb{C})$-modules. Since $\psi : \mathcal{O}(U_{C_n})^{U_{C_{n-1}}} \to \mathcal{O}(U_{n+1})^{U_{n-1}}$ is a ring homomorphism, it follows that $\Psi$ is also. This completes the proof of Theorem 4.1.2.

# Chapter 5

# The Extension Theorem

Let $L = \prod_{i=1}^{n} SL(2, \mathbb{C})$. In this chapter we prove the main theorem of the dissertation, showing that the natural $SL(2, \mathbb{C})$-action on $\widehat{\mathcal{M}}$ (cf. (4.1)) can be canonically extended to an $L$-action.

In section 1 we state the Extension Theorem. In section 2 we introduce an auxiliary family of $L$-rings, which we will show are isomorphic to certain subrings of $\widehat{\mathcal{M}}$. In section 3 we prove some lemmas that are necessary to prove Proposition 5.4.2. This proposition essentially says that the product of multiplicity spaces in $\widehat{\mathcal{M}}$ is the Cartan product of irreducible $L$-modules. In section 5 we use this to prove the Extension Theorem.

## 5.1 Statement of the Extension Theorem

Fix $n$ a positive integer. Set

$$L = SL(2,\mathbb{C}) \times \cdots \times SL(2,\mathbb{C}) \ (n \text{ copies}).$$

Our object of study is the ring of functions $\widehat{\mathcal{M}}$, as defined in (4.1). Recall that $\widehat{\mathcal{M}}$ has an $SL(2,\mathbb{C})$-action by right translation (cf. Section 4.1). We refer to this action of $SL(2,\mathbb{C})$ on $\widehat{\mathcal{M}}$ as the **natural action**. The Extension Theorem says that this action can be naturally extended to an $L$-action. In this section we state the Extension Theorem; its proof will be deferred to subsequent sections.

We begin by adapting some definitions from Section 3.1 to the setting of the symplectic group. First recall that for $\lambda = (\lambda_1, ..., \lambda_n) \in \Lambda_n^+$ we set $\lambda^+ = (\lambda_1, ..., \lambda_n, 0) \in \Lambda_{n+1}^+$. Let $\widehat{\mathfrak{D}} \subset \Lambda_{n-1}^+ \times \Lambda_n^+$ be the following semigroup of double interlacing patterns:

$$\widehat{\mathfrak{D}} = \{(\mu, \lambda) \in \Lambda_{n-1}^+ \times \Lambda_n^+ : \mu \ll \lambda^+\}.$$

Next we define the **rearrangement function** on $\widehat{\mathfrak{D}}$. Let $f : \widehat{\mathfrak{D}} \to \Lambda_{2n}^+$ be given by $(\mu, \lambda) \mapsto f(\mu, \lambda^+)$, where $f(\mu, \lambda^+)$ is defined in Section 3.1. The image of $f$ is thus all sequences in $\Lambda_{2n}^+$ ending in zero. As before, we define the functions $r_i : \widehat{\mathfrak{D}} \to \mathbb{Z}$ by $r_i(p) = x_i - y_i$, where $f(p) = (x_1, y_1, ..., x_n, y_n)$ and $p = (\mu, \lambda) \in \widehat{\mathfrak{D}}$.

To each $p = (\mu, \lambda) \in \widehat{\mathfrak{D}}$ we attach the multiplicity space $\widehat{M}_p = \widehat{M}_\mu^\lambda$ defined in Section 2.4. By Corollary 4.1.3, $\widehat{M}_\mu^\lambda \neq \{0\}$ if, and only if, $\mu \ll \lambda^+$. Thus the spaces $\widehat{M}_p$ (as $p$ ranges over $\widehat{\mathfrak{D}}$) are all the non-trivial multiplicity spaces. As

noted in (4.2), the ring $\widehat{\mathcal{M}}$ is naturally $\widehat{\mathfrak{D}}$-graded:

$$\widehat{\mathcal{M}} = \bigoplus_{p \in \widehat{\mathfrak{D}}} \widehat{M}_p.$$

Let $\Sigma$ be the set of order types defined in Section 3.1. Given $\sigma \in \Sigma$, let $\widehat{\mathfrak{D}}_\sigma = \{(\mu, \lambda) \in \widehat{\mathfrak{D}} : \mu \ll_\sigma \lambda^+\}$. The argument in Lemma 3.1.5 shows that $\widehat{\mathfrak{D}}_\sigma$ is semigroup isomorphic to $\Lambda^+_{2n-1}$. Set $f_\sigma = f|_{\widehat{\mathfrak{D}}_\sigma}$. Then $f_\sigma$ is a semigroup embedding, with image the sequences in $\Lambda^+_{2n}$ ending in zero. In particular, $f_\sigma^{-1}$ is defined on the set of such sequences. In this chapter we will only deal with sequences ending in zero, so $f_\sigma^{-1}$ will always be well-defined.

To each $\sigma \in \Sigma$ we attach the subspace

$$\widehat{\mathcal{M}}_\sigma = \bigoplus_{p \in \widehat{\mathfrak{D}}_\sigma} \widehat{M}_p,$$

which clearly is an $SL(2, \mathbb{C})$-submodule of $\widehat{\mathcal{M}}$.

**Lemma 5.1.1** *Let $\sigma \in \Sigma$. Then $\widehat{\mathcal{M}}_\sigma$ is a unital subring of $\widehat{\mathcal{M}}$.*

**Proof.** Suppose that $\mu \ll_\sigma \lambda$ and $\mu' \ll_\sigma \lambda'$. As in Lemma 3.3.1, $\widehat{M}^\lambda_\mu \widehat{M}^{\lambda'}_{\mu'} \subset \widehat{M}^{\lambda+\lambda'}_{\mu+\mu'}$. Since $\mu + \mu' \ll_\sigma \lambda + \lambda'$, it follows that $\widehat{M}^\lambda_\mu \widehat{M}^{\lambda'}_{\mu'} \subset \widehat{\mathcal{M}}_\sigma$. The claim that $\widehat{\mathcal{M}}_\sigma$ is unital just amounts to the observation that the trivial interlacing pair $(0, 0) \in \widehat{\mathfrak{D}}_\sigma$. Then $1 \in \widehat{M}^0_0$ is the unit of $\widehat{\mathcal{M}}_\sigma$ (and of $\widehat{\mathcal{M}}$). ■

Define the **diagonal embedding** $\delta : SL(2, \mathbb{C}) \hookrightarrow L$ by

$$\delta(x) = (x, ..., x).$$

We identify $SL(2, \mathbb{C})$ with its image under $\delta$, and thus view it as a subgroup of $L$. The following theorem is what we call the "Extension Theorem".

**Theorem 5.1.2** *The ring $\widehat{\mathcal{M}}$ carries a unique representation of $L$, denoted $\Phi$, satisfying the following two properties:*

1. *Under $\Phi$, $\widehat{M_p}$ is an irreducible $L$-submodule of $\widehat{\mathcal{M}}$ isomorphic to $\bigotimes_{i=1}^{n} F_{r_i(p)}$.*

2. *$L$ acts as algebra automorphisms on $\widehat{\mathcal{M}_\sigma}$ for every $\sigma \in \Sigma$.*

*Moreover, $\Phi$ extends the natural action of $SL(2,\mathbb{C})$ on $\widehat{\mathcal{M}}$. In other words,*

$$Res^{L}_{SL(2,\mathbb{C})}(\Phi)$$

*is the natural action of $SL(2,\mathbb{C})$ on $\widehat{\mathcal{M}}$.*

We note that it if $p, p' \in \widehat{\mathfrak{D}}$ do not satisfy a common order type, then it is not necessarily true that the produt map $\widehat{M_p} \otimes \widehat{M_{p'}} \to \widehat{M_{p+p'}}$ is a homomorphism of $L$-modules. Indeed, the following shows that, in effect, Theorem 5.1.2 is best possible.

**Example 5.1.3** There is no representation of $L$ on $\widehat{\mathcal{M}}$ satisfying the first condition of Theorem 5.1.2, and the following strengthening of the second condition:

2'. L acts as algebra automorphisms on $\widehat{\mathcal{M}}$.

Indeed, suppose $(\Phi, \widehat{\mathcal{M}})$ is a representation of $L$ satisfying conditions 1 and 2'. Set

$$\begin{aligned} p &= (\varpi_{n-1}, \varpi_{n-1}) \in \widehat{\mathfrak{D}} \\ p' &= (\varpi_{n-2}, \varpi_n) \in \widehat{\mathfrak{D}}. \end{aligned}$$

(Recall that if $(\mu, \lambda) \in \widehat{\mathfrak{D}}$ then, by definition, $\mu \in \Lambda_{n-1}^+$ and $\lambda \in \Lambda_n^+$.) Notice that $p$ and $p'$ do not satisfy a common order type. By condition (1),

$$\widehat{M}_p \cong \widehat{M}_{p'} \cong F_0 \otimes \cdots \otimes F_0$$

are both trivial $L$-modules. On the other hand,

$$\widehat{M}_{p+p'} \cong F_0 \otimes \cdots \otimes F_0 \otimes F_1 \otimes F_1$$

is a non-trivial irreducible $L$-module. Now let

$$\pi : \widehat{M}_p \otimes \widehat{M}_{p'} \to \widehat{M}_{p+p'}$$

be the product map of $\widehat{M}_p$ with $\widehat{M}_{p'}$. By condition (2'), $\pi$ is a morphism of $L$-modules. But $\pi$ is nonzero since $\mathcal{R}_{C_n}$ as no zero divisors. Therefore it follows that $\widehat{M}_{p+p'}$ contains a trivial $L$-module, a contradiction.

## 5.2 The rings $\mathcal{A}_\sigma$

In order to prove Theorem 5.1.2 we introduce an auxiliary family of $L$-rings, which we will show in the next sections are canonically isomorphic to the rings $\widehat{\mathcal{M}}_\sigma$.

Define $s_i : \widehat{\mathfrak{D}} \to \mathbb{Z}$ by $s_i(p) = x_i + y_i$, where, as usual, $f(p) = (x_1, y_1, ..., x_n, y_n)$. Let $H = (\mathbb{C}^\times)^{n-1}$. For $p \in \widehat{\mathfrak{D}}$ let $(\theta_p, A_p)$ be the irreducible $L \times H$-module defined as the outer tensor product

$$A_p = \left(\bigotimes_{i=1}^n F_{r_i(p)}\right) \widehat{\otimes} \left(\bigotimes_{i=1}^{n-1} \mathbb{C}_{s_i(p)}\right).$$

Here $\mathbb{C}_k$ denotes the one-dimensional $\mathbb{C}^\times$-module given by the character $z \mapsto z^k$. For $\sigma \in \Sigma$ let $\theta_\sigma = \bigoplus_{p \in \widehat{\mathfrak{D}}_\sigma} \theta_p$. The representation $\theta_\sigma$ is on the space

$$\mathcal{A}_\sigma = \bigoplus_{p \in \widehat{\mathfrak{D}}_\sigma} A_p.$$

We now introduce a product structure on $\mathcal{A}_\sigma$. The crucial observation is the following lemma.

**Lemma 5.2.1** *Let $\sigma \in \Sigma$. Suppose that $p, p' \in \widehat{\mathfrak{D}}_\sigma$. Then*

$$A_{p+p'} = \left(\bigotimes_{i=1}^{n} F_{r_i(p)+r_i(p')}\right) \otimes \left(\bigotimes_{i=1}^{n-1} \mathbb{C}_{s_i(p)+s_i(p')}\right).$$

**Proof.** Let $f_\sigma(p) = (x_1, y_1, ..., x_n, y_n)$ and $f_\sigma(p') = (x'_1, y'_1, ..., x'_n, y'_n)$. By Lemma 3.1.5, $f_\sigma(p + p') = (x_1 + x'_1, y_1 + y'_1, ..., x_n + x'_n, y_n + y'_n)$. Therefore $r_i(p + p') = r_i(p) + r_i(p')$ and $s_i(p + p') = s_i(p) + s_i(p')$. Hence

$$\begin{aligned} A_{p+p'} &= \left(\bigotimes_{i=1}^{n} F_{r_i(p+p')}\right) \otimes \left(\bigotimes_{i=1}^{n-1} \mathbb{C}_{s_i(p+p')}\right) \\ &= \left(\bigotimes_{i=1}^{n} F_{r_i(p)+r_i(p')}\right) \otimes \left(\bigotimes_{i=1}^{n-1} \mathbb{C}_{s_i(p)+s_i(p')}\right). \end{aligned}$$

■

Recall that $\pi_{k,k'} : F_k \otimes F_{k'} \longrightarrow F_{k+k'}$ is the Cartan multiplication of irreducible $SL(2, \mathbb{C})$-modules (cf. Section 2.3). Moreover, let $m_{k,k'} : \mathbb{C}_k \otimes \mathbb{C}_{k'} \to \mathbb{C}_{k+k'}$ to be the usual multiplication map. Let $\sigma \in \Sigma$ and $p, p' \in \widehat{\mathfrak{D}}_\sigma$. By the above lemma, there is a map $\alpha_{p,p'} : A_p \otimes A_{p'} \to A_{p+p'}$ given by

$$\alpha_{p,p'} = \left(\pi_{r_1(p),r_1(p')} \otimes \cdots \otimes \pi_{r_n(p),r_n(p')}\right) \otimes \left(m_{s_1(p),s_1(p')} \otimes \cdots \otimes m_{s_{n-1}(p),s_{n-1}(p')}\right).$$

This endows a product on $\mathcal{A}_\sigma$. We will now show that equipped with this product $\mathcal{A}_\sigma$ is a polynomial ring.

Consider the vector space $V = W \times Z$, where

$$\begin{aligned} W &= \mathbb{C}^2 \times \cdots \times \mathbb{C}^2 \ (n \text{ copies}) \\ Z &= \mathbb{C} \times \cdots \times \mathbb{C} \ (n-1 \text{ copies}). \end{aligned}$$

We define a representation $(\theta, V)$ of $L \times H$ as follows: if $g = (g_1, ..., g_n) \in L$, $h = (h_1, ..., h_{n-1}) \in H$, $w = (w_1, ..., w_n) \in W$, and $z = (z_1, ..., z_{n-1}) \in Z$ then

$$\theta(g,h)(w,z) = (g_1 w_1, ..., g_n w_n, h_1 z_1, ..., h_{n-1} z_{n-1}).$$

In other words, we consider $W$ as the standard module for $L$, and $Z$ as the standard module for $H$. $L \times H$ acts on the polynomial ring $\mathcal{O}(V)$ by left translation.

**Proposition 5.2.2** *Let $\sigma \in \Sigma$. There is a canoncial isomorphism of as $L \times H$ modules $T : \mathcal{A}_\sigma \to \mathcal{O}(V)$, such that for $p, p' \in \widehat{\mathfrak{D}}_\sigma$, $y \in A_p$, and $y' \in A_{p'}$*

$$T(\alpha_{p,p'}(y \otimes y')) = T(y)T(y').$$

*Therefore $\mathcal{A}_\sigma$ is a polynomial ring isomorphic to $\mathcal{O}(V)$.*

**Proof.** Let $t_1, ..., t_{n-1}$ be the standard coordinate functions on $\mathbb{C}^{n-1}$. Decompose $\mathcal{O}(V)$ into graded components as follows:

$$\mathcal{O}(V) \cong \bigoplus_{\substack{r_j \geq 0 \\ j=1,...,n}} \bigoplus_{\substack{s_k \geq 0 \\ k=1,...,n-1}} F_{r_1} \otimes \cdots \otimes F_{r_n} \otimes t_1^{s_1} \cdots t_{n-1}^{s_{n-1}}. \tag{5.1}$$

This is a decomposition of $\mathcal{O}(V)$ into irreducible $L \times H$-modules. We define the obvious map $T$ from $\mathcal{A}_\sigma$ to $\mathcal{O}(V)$: for $p \in \widehat{\mathfrak{D}}_\sigma$ and $v \in A_p$, let $T(v) = v \otimes t_1^{s_1(p)} \cdots t_{n-1}^{s_{n-1}(p)}$. We extend $T$ linearly to $\mathcal{A}_\sigma$.

$T$ is an injective homomorphism of $L \times H$-modules. $T$ is surjective since given the parameters $r_1, ..., r_n, s_1, ..., s_{n-1}$, one can recover $f_\sigma(p)$ by the formulae $r_j = x_j - y_j$, $s_k = x_k + y_k$, and $y_n = 0$. Lemma 3.1.5 allows us to recover $p = f_\sigma^{-1}(f_\sigma(p))$.

Finally, note that, via the isomorphism (5.1), the multiplication in $\mathcal{O}(V)$ is given by tensoring Cartan multiplication of irreducible $L$-modules with the usual multiplication of monomials. This is equivalent to the multiplication defined by the maps $\alpha_{p,p'}$. Therefore $T$ preserves the ring structure. ■

## 5.3 A filtration on $\widehat{\mathfrak{D}}_\sigma$

For the rest of this chapter we view the rings $\mathcal{A}_\sigma$ as $SL(2,\mathbb{C})$-rings by restricting to the diagonal $SL(2,\mathbb{C})$ in $L$. The main step in proving Theorem 5.1.2 is showing that $\mathcal{A}_\sigma$ and $\widehat{\mathcal{M}}_\sigma$ are canonically isomorphic as $SL(2,\mathbb{C})$-rings (see Proposition 5.4.2 below). We will prove this by induction on a certain filtration of $\widehat{\mathfrak{D}}_\sigma$. In this section we introduce this filtration, and prove a series of lemmas that form the technical underpinnings of the proof of Proposition 5.4.2.

For $p \in \widehat{\mathfrak{D}}$ let $p_{\max}$ be the largest entry of $p$ thought of as a collection of $2n-1$ integers. In other words, if $p = (\mu, \lambda)$ and $\lambda = (\lambda_1, ..., \lambda_n)$, then $p_{\max} = \lambda_1$.

For every $\sigma \in \Sigma$ we define the set

$$\widehat{\mathfrak{D}}_{\sigma,m} = \{p \in \widehat{\mathfrak{D}}_\sigma : p_{\max} \leq m\}.$$

Clearly $\widehat{\mathfrak{D}}_{\sigma,m}$ is a finite set. Moreover, $\widehat{\mathfrak{D}}_{\sigma,m-1} \subset \widehat{\mathfrak{D}}_{\sigma,m}$ and

$$\bigcup_{m\geq 0} \widehat{\mathfrak{D}}_{\sigma,m} = \widehat{\mathfrak{D}}. \tag{5.2}$$

**Lemma 5.3.1** *Let $m > 1$, $\sigma \in \Sigma$, and suppose $p \in \widehat{\mathfrak{D}}_{\sigma,m}$ satisfies $p_{\max} = m$. Then there exist $p', p'' \in \widehat{\mathfrak{D}}_{\sigma,m-1}$ such that $p = p' + p''$.*

**Proof.** Let $f_\sigma(p) = (z_1, ..., z_{2n})$. Define

$$z_i' = \begin{cases} 1 \text{ if } z_i \geq 1 \\ 0 \text{ if } z_i = 0 \end{cases}$$

and $z_i'' = z_i - z_i'$. It's trivial to check that $\xi' = (z_1', ..., z_{2n}'), \xi'' = (z_1'', ..., z_{2n}'') \in \Lambda_{2n}^+$. Then let $p' = f_\sigma^{-1}(\xi')$ and $p'' = f_\sigma^{-1}(\xi'')$. This is well-defined by Lemma 3.1.5. Lemma 3.1.5 also shows that $p = p' + p''$. By construction, since $m > 1$, $p', p'' \in \widehat{\mathfrak{D}}_{\sigma,m-1}$. ■

**Corollary 5.3.2** *Let $m > 1$, $\sigma, \tau \in \Sigma$, and suppose $p \in \widehat{\mathfrak{D}}_{\sigma,m} \cap \widehat{\mathfrak{D}}_{\tau,m}$ satisfies $p_{\max} = m$. Then there exist $p', p'' \in \widehat{\mathfrak{D}}_{\sigma,m-1} \cap \widehat{\mathfrak{D}}_{\tau,m-1}$ such that $p = p' + p''$.*

**Proof.** Let $p', p'' \in \widehat{\mathfrak{D}}_{\sigma,m-1}$ be constructed as in Lemma 5.3.1. We must show that $p', p'' \in \widehat{\mathfrak{D}}_\sigma \cap \widehat{\mathfrak{D}}_\tau$. The set $f_\sigma(\widehat{\mathfrak{D}}_\sigma \cap \widehat{\mathfrak{D}}_\tau)$ is characterized by a collection of certain equalities among consecutive entries. More precisely, if $\sigma = (\sigma_1, ..., \sigma_{n-1})$ and $\tau = (\tau_1, ..., \tau_{n-1})$ (cf. Section 3.1), then every $i$ such that $\sigma_i \neq \tau_i$ forces the

equality $\mu_i = \lambda_{i+1}$ among the entries of $p$. Therefore, if $f_\sigma(p) = (z_1, ..., z_{2n})$ and $\sigma_i \neq \tau_i$, then $z_{2i+1} = z_{2i+2}$. Now note that in the definition of $\xi', \xi''$, if $z_{2i+1} = z_{2i+2}$ then $z'_{2i+1} = z'_{2i+2}$ and $z''_{2i+1} = z''_{2i+2}$. Hence the entries of $\xi', \xi''$ satisfy the same equalities that $f_\sigma(p)$ satisfies, which implies that $p', p'' \in \widehat{\mathfrak{D}}_\sigma \cap \widehat{\mathfrak{D}}_\tau$. ■

**Lemma 5.3.3** *Let $m > 1$, $\sigma \in \Sigma$, and suppose $p \in \widehat{\mathfrak{D}}_{\sigma,m}$ satisfies $p_{\max} = m$. Then there exist $q_1, ..., q_n \in \widehat{\mathfrak{D}}_{\sigma,m-1}$ such that*

1. *$p = q_1 + \cdots + q_n$*
2. *$A_{q_i}$ is an irreducible $SL(2,\mathbb{C})$-module.*
3. *Either $A_{q_1} \otimes \cdots \otimes A_{q_n} \cong A_p$ as $SL(2,\mathbb{C})$-modules, or $A_p$ is irreducible as a $SL(2,\mathbb{C})$-module, and the multiplication map $A_{q_1} \otimes \cdots \otimes A_{q_n} \to A_p$ is a projection onto the Cartan component $A_p$ of $A_{q_1} \otimes \cdots \otimes A_{q_n}$.*

**Proof.** Let $f_\sigma(p) = (x_1, y_1, ..., x_n, y_n)$. Define

$$\xi_i = (\underbrace{x_i - x_{i+1}, ..., x_i - x_{i+1}}_{2i-1}, y_i - x_{i+1}, 0, ..., 0)$$

for $i = 1, ..., n-1$, and set

$$\xi_n = (x_n, ..., x_n, 0).$$

The argument breaks up into cases.

Case 1: Suppose $f_\sigma^{-1}(\xi_i) \notin \widehat{\mathfrak{D}}_{\sigma,m-1}$ for some $i < n$. Then $x_i - x_{i+1} = m$ and therefore

$$f_\sigma(p) = (m, ..., m, b, 0, ..., 0)$$

for some $b \leq m$ in the $(2i)^{th}$ entry. Therefore $A_p$ is irreducible as an $SL(2,\mathbb{C})$-module.

Now choose $\xi', \xi''$ as in the proof of Lemma 5.3.1 and consider the associated $p', p''$. By the lemma $p', p'' \in \widehat{\mathfrak{D}}_{\sigma,m-1}$. Moreover, by our construction of $\xi', \xi''$ from $\xi$, $A_{p'}, A_{p''}$ are irreducible $SL(2,\mathbb{C})$-modules. Therefore the map $A_{p'} \otimes A_{p''} \to A_p$ is a projection onto the Cartan component of $A_{p'} \otimes A_{p''}$, and the lemma is satisfied with $q_1 = p', q_2 = p''$, and $q_i = 0$ for $i > 2$.

Case 3: Suppose that $f_\sigma^{-1}(\xi_i) \in \widehat{\mathfrak{D}}_{\sigma,m-1}$ for $i = 1, ..., n$. Then set $q_i = f_\sigma^{-1}(\xi_i)$. Since $\xi = \xi_1 + \cdots + \xi_n$, by Lemma 3.1.5, $p = q_1 + \cdots + q_n$. By the definition of $\xi_i$ we also have that

$$A_{q_i} = F_0 \otimes \cdots \otimes \underbrace{F_{x_i - y_i}}_{i^{th}} \otimes \cdots \otimes F_0.$$

Therefore $A_{q_i}$ is an irreducible $SL(2,\mathbb{C})$-module, and $A_{q_1} \otimes \cdots \otimes A_{q_n} \cong A_p$. ■

**Remark 5.3.4** In the proof of Lemma 5.3.3 all we used was the $SL(2,\mathbb{C})$-module structure of $A_p$. Therefore, by Corollary 4.1.3, the statement holds with $A_p$ replaced by $\widehat{M}_p$ and $A_{q_i}$ replaced by $\widehat{M}_{q_i}$.

**Lemma 5.3.5** *Let $m > 1$, $\sigma \in \Sigma$, and suppose $p \in \widehat{\mathfrak{D}}_{\sigma,m}$ satisfies $p_{\max} = m$. Let $q_1, ..., q_n \in \widehat{\mathfrak{D}}_{\sigma,m-1}$ be given as in Lemma 5.3.3. Suppose also we are given $SL(2,\mathbb{C})$-isomorphisms $\phi_i : \widehat{M}_{q_i} \to A_{q_i}$ for $i = 1, ..., n$. Let*

$$\begin{aligned} K &= \ker(\widehat{M}_{q_1} \otimes \cdots \otimes \widehat{M}_{q_n} \xrightarrow{\tau} \widehat{M}_p) \\ J &= \ker(A_{q_1} \otimes \cdots \otimes A_{q_n} \xrightarrow{\kappa} A_p) \end{aligned}$$

*be the kernels of the multiplication maps coming from the rings* $\widehat{M}_\sigma$ *and* $A_\sigma$*, which we denote here by* $\tau$ *and* $\kappa$*. Set* $\phi = \phi_1 \otimes \cdots \otimes \phi_n$*. Then* $\phi(K) = J$*. Consequently, there is an* $SL(2,\mathbb{C})$*-isomorphism* $\psi : \widehat{M}_p \to A_p$ *making the following diagram commute:*

$$\begin{array}{ccc} \widehat{M}_{q_1} \otimes \cdots \otimes \widehat{M}_{q_n} & \longrightarrow & \widehat{M}_p \\ \downarrow \phi & & \downarrow \psi \\ A_{q_1} \otimes \cdots \otimes A_{q_n} & \longrightarrow & A_p \end{array} \tag{5.3}$$

**Proof.** Clearly $\kappa$ is surjective. By the Surjectivity Lemma for the symplectic group (Corollary 4.1.4), $\tau$ is surjective. Therefore we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \widehat{M}_{q_1} \otimes \cdots \otimes \widehat{M}_{q_n} & \longrightarrow & \widehat{M}_p & \longrightarrow & 0 \\ & & & & \downarrow \phi & & & & \\ 0 & \longrightarrow & J & \longrightarrow & A_{q_1} \otimes \cdots \otimes A_{q_n} & \longrightarrow & A_p & \longrightarrow & 0 \end{array}$$

According to Lemma 5.3.3 there are two possibilities. Either $A_{q_1} \otimes \cdots \otimes A_{q_n} \cong A_p$ as $SL(2,\mathbb{C})$-modules, or $A_p$ is irreducible as a $SL(2,\mathbb{C})$-module, and the multiplication map $A_{q_1} \otimes \cdots \otimes A_{q_n} \to A_p$ is a projection onto the Cartan component $A_p$ of $A_{q_1} \otimes \cdots \otimes A_{q_n}$. If $A_{q_1} \otimes \cdots \otimes A_{q_n} \cong A_p$ then $J = \{0\}$, and by the above remark $\widehat{M}_{q_1} \otimes \cdots \otimes \widehat{M}_{q_n} \cong \widehat{M}_p$. Therefore $K = \{0\}$, and so clearly $\phi(K) = J$.

In the other case, $A_p \cong \widehat{M}_p$ are irreducible $SL(2,\mathbb{C})$-modules. Choose $k$ so that $A_p \cong \widehat{M}_p \cong F_k$. Since the maps $\kappa, \tau$ are both projections onto the Cartan component $F_k$, their kernels are given as sums of $SL(2,\mathbb{C})$-isotypic components

$$\begin{aligned} K &= \sum_{n<k} (\widehat{M}_{q_1} \otimes \cdots \otimes \widehat{M}_{q_n})[j] \\ J &= \sum_{j<k} (A_{q_1} \otimes \cdots \otimes A_{q_n})[j]. \end{aligned}$$

Since $\phi$ intertwines the $SL(2,\mathbb{C})$-action, $\phi(K) \subset J$. Moreover, $\kappa$ and $\tau$ are both Cartan multiplications of the same $SL(2,\mathbb{C})$-modules, and so $\dim K = \dim J$. Therefore $K = J$. ■

**Lemma 5.3.6** *Let $m > 1$, $\sigma \in \Sigma$, and $p \in \widehat{\mathfrak{D}}_{\sigma,m}$. Suppose we are given $p', p'', q', q'' \in \widehat{\mathfrak{D}}_{\sigma,m-1}$ such that $p' + p'' = p = q' + q''$. Then there exist $t', t'', r', r'' \in \widehat{\mathfrak{D}}_{\sigma,m-1}$ such that*

$$\begin{aligned} t' + r' &= p' \\ t'' + r'' &= p'' \\ t' + t'' &= q' \\ r' + r'' &= q'' \end{aligned}$$

**Proof.** Every element of $\Lambda_{2n}^+$ can be uniquely expressed as a nonnegative integer combination of fundamental weights $\varpi_i$ for $i = 1, ..., 2n$. (Here $\varpi_i = (1, ..., 1, 0, ..., 0)$ has $i$ one's.) Since we are only concerned with elements of $\Lambda_{2n}^+$ that end with zero, we don't need $\varpi_{2n}$. Therefore for some nonnegative integers $n_i', n_i'', m_i', m_i''$ we have

$$\begin{aligned} f_\sigma(p') &= \textstyle\sum_{i=1}^{2n-1} n_i' \varpi_i,\ f_\sigma(p'') = \sum_{i=1}^{2n-1} n_i'' \varpi_i \\ f_\sigma(q') &= \textstyle\sum_{i=1}^{2n-1} m_i' \varpi_i,\ f_\sigma(q'') = \sum_{i=1}^{2n-1} m_i'' \varpi_i \end{aligned}$$

Define:

$$\begin{aligned}
\tau' &= \sum_{i=1}^{2n-1}(m_i'' - \min(n_i'', m_i''))\varpi_i \\
\rho' &= \sum_{i=1}^{2n-1}(n_i' - m_i'' + \min(n_i'', m_i''))\varpi_i \\
\tau'' &= \sum_{i=1}^{2n-1}\min(n_i'', m_i'')\varpi_i \\
\rho'' &= \sum_{i=1}^{2n-1}(n_i'' - \min(n_i'', m_i''))\varpi_i.
\end{aligned}$$

Firstly, note that $\tau', \tau'', \rho', \rho'' \in \Lambda_{2n}^+$. This is clear for $\tau', \tau''$, and $\rho''$. To see that $\rho' \in \Lambda_{2n}^+$ we have to show that for all $i$, $n_i' - m_i'' + \min(n_i'', m_i'') \geq 0$. The equality $p' + p'' = q' + q''$ implies that $n_i' - m_i^{'} = m_i'' - n_i''$ for all $i$. Therefore,

$$\min(n_i', m_i') = m_i' \Leftrightarrow \min(n_i'', m_i'') = n_i''$$

from which it follows that $n_i' - m_i'' + \min(n_i'', m_i'') = \min(n_i', m_i')$. So $\rho' = \sum_{i=1}^{2n-1}\min(n_i', m_i')\varpi_i \in \Lambda_{2n}^+$. Secondly, note that

$$\begin{aligned}
f_\sigma(p') &= \tau' + \rho' \\
f_\sigma(p'') &= \tau'' + \rho'' \\
f_\sigma(q') &= \rho' + \rho'' \\
f_\sigma(q'') &= \tau' + \tau''.
\end{aligned}$$

Now set

$$\begin{aligned}
t' &= f_\sigma^{-1}(\tau'),\ \ t'' = f_\sigma^{-1}(\tau'') \\
r' &= f_\sigma^{-1}(\rho'),\ \ \rho'' = f_\sigma^{-1}(\rho'')
\end{aligned}$$

Since $f_\sigma$ is a semigroup isomorphism and $p', p'', q', q'' \in \widehat{\mathfrak{D}}_{\sigma,m-1}$, it follows that $t', t'', r', r'' \in \widehat{\mathfrak{D}}_{\sigma,m-1}$ and they satisfy the desired equations. ■

## 5.4 An induction on $\widehat{\mathfrak{D}}_\sigma$

We are now ready to prove the main result need for the Extension Theorem.

**Definition 5.4.1** *Let $\mathcal{F} = \{\phi_p : \widehat{M}_p \to A_p\}_{p\in\widehat{\mathfrak{D}}}$ be a family of $SL(2,\mathbb{C})$-isomorphisms indexed by $\widehat{\mathfrak{D}}$. Then $\mathcal{F}$ is a **compatible family** if it satisfies the following condition: for any $\sigma \in \Sigma$ and $p', p'' \in \widehat{\mathfrak{D}}_\sigma$ the following diagram commutes*

$$\begin{array}{ccc} \widehat{M}_{p'} \otimes \widehat{M}_{p''} & \longrightarrow & \widehat{M}_{p'+p''} \\ \downarrow & & \downarrow \\ A_{p'} \otimes A_{p''} & \longrightarrow & A_{p'+p''} \end{array} \tag{5.4}$$

*The vertical maps are given by $\phi_{p'} \otimes \phi_{p''}$ and $\phi_{p'+p''}$, while the horizantel maps are the product maps in the rings $\widehat{\mathcal{M}}_\sigma$ and $\mathcal{A}_\sigma$.*

**Proposition 5.4.2** *There exists a compatible family $\mathcal{F} = \{\phi_p : \widehat{M}_p \to A_p\}_{p\in\widehat{\mathfrak{D}}}$ of $SL(2,\mathbb{C})$-isomorphisms. Moreover, each map $\phi_p \in \mathcal{F}$ is unique up to scalar.*

**Proof.** Let $\widehat{\mathfrak{D}}_m = \{p \in \widehat{\mathfrak{D}} : p_{\max} \leq m\}$. We first prove by induction on $m$ that there is a family of $SL(2,\mathbb{C})$ isomorphisms

$$\mathcal{F}_m = \{\phi_p : \widehat{M}_p \to A_p\}_{p\in\widehat{\mathfrak{D}}_m}$$

such that for any $p \in \widehat{\mathfrak{D}}_m$, $\sigma \in \Sigma$, and $p', p'' \in \widehat{\mathfrak{D}}_\sigma$ such that $p = p' + p''$, diagram (5.4) commutes.

For the base case we construct $\mathcal{F}_1$. If $p_{\max} = 0$ then $p = p_0 = (0,0)$. We define $\phi_{p_0} : \widehat{M}_{p_0} \to A_{p_0}$ by $1 \in \widehat{M}_{p_0} \longmapsto 1 \otimes \cdots \otimes 1 \in A_{p_0}$. Of course, if $p' + p'' = p_0$ then $p' = p'' = p_0$ and (5.4) trivially commutes. Suppose now that $p_{\max} = 1$. Then $p = (\mu, \lambda)$ with $\lambda$ a fundamental weight, and $\mu$ either zero or a fundamental weight. In any case, $A_p$ is an irreducible $SL(2,\mathbb{C})$-module, and by Corollary 4.1.3, $\widehat{M}_p \cong A_p$ as $SL(2,\mathbb{C})$-modules. We choose an $SL(2,\mathbb{C})$-isomorphism $\phi_p : \widehat{M}_p \to A_p$. By Schur's Lemma, $\phi_p$ is unique up to scalar. Now suppose $\sigma \in \Sigma$, $p', p'' \in \widehat{\mathfrak{D}}_\sigma$, and $p' + p'' = p$. Then either $p'_{\max} = 0$ or $p''_{\max} = 0$. Assume, without loss of generality, that $p'_{\max} = 0$. Then $p' = p_0$, and by our construction of $\phi_{p_0}$, diagram (5.4) commutes. Set $\mathcal{F}_1 = \{\phi_p : \widehat{M}_p \to A_p\}_{p \in \widehat{\mathfrak{D}}_1}$; this completes the base case.

Let $m > 1$ and suppose that $\mathcal{F}_{m-1}$ exists and satisfies the desired properties. We must construct $\mathcal{F}_m$ and show that it satisfies the desired properties. For $p \in \widehat{\mathfrak{D}}_m$ such that $p_{\max} < m$, there exists $\phi_p \in \mathcal{F}_{m-1}$ by hypothesis. We take these $\phi_p$ and include them in our set $\mathcal{F}_m$. Notice that we can already conclude the following: if $\sigma \in \Sigma$, $p', p'' \in \widehat{\mathfrak{D}}_\sigma$, and $p = p' + p''$, then diagram (5.4) commutes. Indeed, $p = p' + p''$ implies that $p', p'' \in \widehat{\mathfrak{D}}_{m-1}$. Therefore $\phi_{p'}$ and $\phi_{p''}$ are also obtained from $\mathcal{F}_{m-1}$, and diagram (5.4) commutes by hypothesis. We now construct the maps $\phi_p$ for $p \in \widehat{\mathfrak{D}}_m$ such that $p_{\max} = m$, assuming that the maps $\phi_p$ for $p$ such that $p_{\max} < m$ already exist.

Suppose $p \in \widehat{\mathfrak{D}}_m$ satisfies $p_{\max} = m$. Choose an order type $\sigma \in \Sigma$ such that $p \in \widehat{\mathfrak{D}}_\sigma$. Note that $\sigma$ may not be unique. Choose $q_1, ..., q_n \in \widehat{\mathfrak{D}}_{\sigma,m-1}$ by Lemma 5.3.3. Now apply Lemma 5.3.5 to obtain an $SL(2,\mathbb{C})$-isomorphism $\psi : \widehat{M}_p \to A_p$

such that the following diagram commutes:

$$\begin{array}{ccc} \widehat{M}_{q_1} \otimes \cdots \otimes \widehat{M}_{q_n} & \longrightarrow & \widehat{M}_p \\ \downarrow \phi & & \downarrow \psi \\ A_{q_1} \otimes \cdots \otimes A_{q_n} & \longrightarrow & A_p \end{array} \tag{5.5}$$

where $\phi = \phi_{q_1} \otimes \cdots \otimes \phi_{q_n}$. We now show that (i) if $p', p'' \in \widehat{\mathfrak{D}}_\sigma$ satisfy $p = p' + p''$ then

$$\begin{array}{ccc} \widehat{M}_{p'} \otimes \widehat{M}_{p''} & \longrightarrow & \widehat{M}_p \\ \downarrow & & \downarrow \psi \\ A_{p'} \otimes A_{p''} & \longrightarrow & A_p \end{array} \tag{5.6}$$

commutes, (ii) $\psi$ is independent of the choice of $q_1, ..., q_n$, and (iii) $\psi$ is independent of the choice of $\sigma$.

First note that (i) implies (ii). Indeed, suppose $q_1', ..., q_n' \in \widehat{\mathfrak{D}}_{\sigma,m-1}$ is another set of patterns satisfying the conditions of Lemma 5.3.3, and $\psi' : \widehat{M}_p \to A_p$ is the associated $SL(2,\mathbb{C})$-isomorphism obtained by Lemma 5.3.5. By (i) both $\psi$ and $\psi'$ would make (5.6) commute. But since all the maps in the diagram are surjective, there is a unique map making (5.6) commute. Therefore $\psi = \psi'$.

Now we prove (i). Let $p', p'' \in \widehat{\mathfrak{D}}_\sigma$ satisfy $p = p' + p''$. If $p'_{\max} = m$ (resp. $p''_{\max} = m$) then $p'' = p_0$ (resp. $p' = p_0$), and (5.6) commutes by our choice of $\phi_{p_0}$. Therefore we may assume that $p'_{\max}, p''_{\max} < m$. By renumbering the $q_j$ if necessary, we may assume that $(q_1)_{\max} \neq 0$. Let $q' = q_1$ and $q'' = q_2 + \cdots + q_n$. Then $q', q'' \in \widehat{\mathfrak{D}}_{\sigma,m-1}$ and $q' + q'' = p$. By inductive hypothesis the following diagram commutes:

$$\begin{array}{ccc} \widehat{M}_{q_1} \otimes \cdots \otimes \widehat{M}_{q_n} & \longrightarrow & \widehat{M}_{q'} \otimes \widehat{M}_{q''} \\ \downarrow & & \downarrow \\ A_{q_1} \otimes \cdots \otimes A_{q_n} & \longrightarrow & A_{q'} \otimes A_{q''} \end{array} \tag{5.7}$$

where the vertical map on the left is $\phi = \phi_{q_1} \otimes \cdots \otimes \phi_{q_n}$, and the one on the right is $\phi_{q'} \otimes \phi_{q''}$. Combining (5.5) and (5.7) and the fact that all the maps are surjective

(Corollary 4.1.4), we conclude that

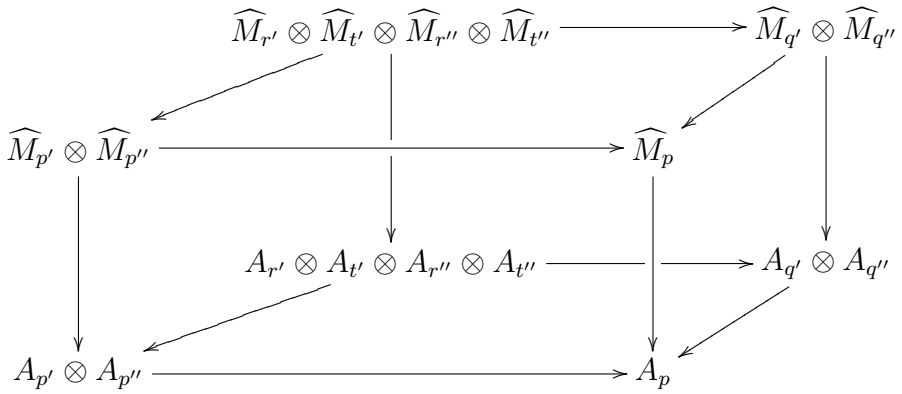
$$\begin{array}{ccc} \widehat{M}_{q'} \otimes \widehat{M}_{q''} & \longrightarrow & \widehat{M}_p \\ \downarrow & & \downarrow \psi \\ A_{q'} \otimes A_{q''} & \longrightarrow & A_p \end{array} \tag{5.8}$$

commutes.

By Lemma 5.3.6, there exist $t', t'', r', r'' \in \widehat{\mathfrak{D}}_{\sigma,m-1}$ such that

$$\begin{aligned} t' + r' &= p' \\ t'' + r'' &= p'' \\ t' + t'' &= q' \\ r' + r'' &= q''. \end{aligned}$$

Therefore we can consider the following diagram:




The top square commutes by associativity of the product in $\widehat{M}_\sigma$. The left and back squares commute by inductive hypothesis. The right square commutes since it is the diagram (5.8). The bottom square commutes by associativity of the product in $A_\sigma$. By chasing this diagram and repeatedly using Corollary 4.1.4, it follows that the front square commutes. This proves (ii).

We now prove (iii), namely that $\psi$ is independent of $\sigma$. Indeed, suppose $\tau \in \Sigma$ is another order type such that $p \in \widehat{\mathfrak{D}}_\tau$. By the above argument we obtain an $SL(2,\mathbb{C})$ isomorphism $\zeta : \widehat{M}_p \to A_p$ such that (5.6) commutes for all $p', p'' \in \widehat{\mathfrak{D}}_\tau$. We must show that $\psi = \zeta$. By Lemma 5.3.2 there exist $p', p'' \in \widehat{\mathfrak{D}}_\sigma \cap \widehat{\mathfrak{D}}_\tau$ such that $p = p' + p''$. Therefore both $\psi$ and $\zeta$ make the following diagram commute:

$$\begin{array}{ccc} \widehat{M}_{p'} \otimes \widehat{M}_{p''} & \longrightarrow & \widehat{M}_p \\ \downarrow & & \downarrow \psi, \zeta \\ A_{p'} \otimes A_{p''} & \longrightarrow & A_p \end{array} \tag{5.9}$$

Hence $\psi = \zeta$. This proves (iii).

At this point we've shown for any $p \in \widehat{\mathfrak{D}}_m$ there is an $SL(2,\mathbb{C})$ isomorphism $\psi : \widehat{M}_p \to A_p$ satisfying the property: for any $\sigma \in \Sigma$ and $p', p'' \in \widehat{\mathfrak{D}}_\sigma$ such that $p = p' + p''$, diagram (5.6) commutes. Set $\phi_p = \psi$ and define $\mathcal{F}_m = \{\phi_p : \widehat{M}_p \to A_p\}_{p \in \widehat{\mathfrak{D}}_m}$. This completes the induction.

Notice that we've constructed a nested sequence $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$. Let

$$\mathcal{F} = \bigcup_{m=1}^{\infty} \mathcal{F}_m.$$

Then $\mathcal{F}$ is a family of $SL(2,\mathbb{C})$ isomorphisms $\{\phi_p : \widehat{M}_p \to A_p\}_{p \in \widehat{\mathfrak{D}}}$. We claim that $\mathcal{F}$ is our desired family of functions. Indeed, suppose $\sigma \in \Sigma$ and $p', p'' \in \widehat{\mathfrak{D}}_\sigma$. Then $p' + p'' \in \widehat{\mathfrak{D}}_m$ for some $m \geq 1$, and so $\phi_{p'+p''}, \phi_{p'}, \phi_{p''} \in \mathcal{F}_m$. Therefore, by the construction of $\mathcal{F}_m$, diagram (5.4) commutes. This completes the proof of the first statement of the proposition.

To complete the proof of the proposition we need to prove the second statement. Suppose $\widetilde{\mathcal{F}} = \{\widetilde{\phi}_p : \widehat{M}_p \to A_p\}_{p \in \widehat{\mathfrak{D}}}$ is another compatible family of $SL(2,\mathbb{C})$ - isomorphisms. We will show by induction on $p_{\max}$ that there exist a set of

nonzero scalars indexed by $\widehat{\mathfrak{D}}$, $\{c_p \in \mathbb{C}^\times : p \in \widehat{\mathfrak{D}}\}$, such that

$$\phi_p = c_p \widetilde{\phi}_p$$

for all $p \in \widehat{\mathfrak{D}}$.

We already noted that by Schur's Lemma each isomorphism $\phi_p$ with $p_{\max} = 1$ is unique up to scalar. Therefore there exist $c_p \in \mathbb{C}^\times$ such that

$$\phi_p = c_p \widetilde{\phi}_p \tag{5.10}$$

for all $p$ with $p_{\max} = 1$. Let $m > 1$. Suppose now that there exist scalars so that (5.10) holds for all $p \in \widehat{\mathfrak{D}}$ such that $p_{\max} < m$. Let $p \in \widehat{\mathfrak{D}}$ with $p_{\max} = m$. Choose some $\sigma \in \Sigma$ such that $p \in \widehat{\mathfrak{D}}_\sigma$. By Lemma 5.3.1, there exist $p', p'' \in \widehat{\mathfrak{D}}_{\sigma,m-1}$ such that $p = p' + p''$. Then by the compatibility of $\mathcal{F}$ the following diagram commutes:

$$\begin{array}{ccc} \widehat{M}_{p'} \otimes \widehat{M}_{p''} & \longrightarrow & \widehat{M}_p \\ \downarrow & & \downarrow \\ A_{p'} \otimes A_{p''} & \longrightarrow & A_p \end{array} \tag{5.11}$$

where the vertical maps are $\phi_{p'} \otimes \phi_{p''}$ and $\phi_p$. By hypothesis, $\phi_{p'} \otimes \phi_{p''} = c_{p'} c_{p''} \widetilde{\phi}_{p'} \otimes \widetilde{\phi}_{p''}$. Therefore (5.11) commutes with the vertical maps replaced by $\widetilde{\phi}_{p'} \otimes \widetilde{\phi}_{p''}$ and $\frac{1}{c_{p'} c_{p''}} \phi_p$. Hence $\frac{1}{c_{p'} c_{p''}} \phi_p = \widetilde{\phi}_p$, or, in other words, $c_p = c_{p'} c_{p''}$ and $\phi_p = c_p \widetilde{\phi}_p$. This completes the induction, and shows that $\phi_p \in \mathcal{F}$ is unique up to scalar. ■

## 5.5 Proof of the Extension Theorem

**Existence**: Let $\mathcal{F} = \{\phi_p : \widehat{M}_p \to A_p\}_{p \in \widehat{\mathfrak{D}}}$ be a compatible family of $SL(2, \mathbb{C})$-isomorphisms (cf. Definition 5.4.1 and Proposition 5.4.2). We will now

construct a representation $(\Phi_{\mathcal{F}}, \widehat{\mathcal{M}})$ of $L$ satisfying conditions one and two of Theorem 5.1.2.

For every $\sigma \in \Sigma$ define a map

$$\phi_\sigma : \widehat{\mathcal{M}}_\sigma \to \mathcal{A}_\sigma$$

by

$$\phi_\sigma|_{\widehat{M}_p} = \phi_p$$

for all $p \in \widehat{\mathfrak{D}}_\sigma$, and extend linearly. Since $\mathcal{F}$ is a compatible family, $\phi_\sigma$ is an isomorphism of $SL(2,\mathbb{C})$ - rings. Indeed, the commutativity of diagram (5.4) means precisely that $\phi_\sigma$ is a ring homomorphism.

Define a representation of $L$ on $\widehat{\mathcal{M}}_\sigma$, denoted $\Phi_\sigma$, by the formula

$$\Phi_\sigma(g) = \phi_\sigma^{-1} \circ \theta_\sigma(g) \circ \phi_\sigma$$

for $g \in L$. The representations $\{\Phi_\sigma\}_{\sigma\in\Sigma}$ satisfy four desirable properties, all of which are almost tautologies.

(i) For any $p \in \widehat{\mathfrak{D}}_\sigma$, $\widehat{M}_p$ is an irreducible $L$-submodule isomorphic to $\bigotimes_{i=1}^n F_{r_i(p)}$. Indeed, by definition of $\Phi_\sigma$, $\phi_p : \widehat{M}_p \to A_p$ is an isomorphism of $L$-modules.

(ii) $L$ acts as algebra automorphisms on $\widehat{\mathcal{M}}_\sigma$. In other words, we claim that for $p, p' \in \widehat{\mathfrak{D}}_\sigma$, the produt map, $\widehat{M}_p \otimes \widehat{M}_{p'} \to \widehat{M}_{p+p'}$, is a homomorphism of $L$-modules. Indeed, by the compatibility of $\mathcal{F}$, the product map factors as follows:

$$\begin{array}{ccc} \widehat{M}_p \otimes \widehat{M}_{p'} & \longrightarrow & \widehat{M}_{p+p'} \\ \downarrow & & \uparrow \phi_{p+p'}^{-1} \\ A_p \otimes A_{p'} & \longrightarrow & A_{p+p'} \end{array}$$

Since the three lower maps are $L$-module morphisms, it follows that the top map is too.

(iii) $Res^L_{SL(2,\mathbb{C})}(\Phi_\sigma)$ is the natural action of $SL(2,\mathbb{C})$ on $\widehat{\mathcal{M}}_\sigma$. In other words, for $x \in SL(2,\mathbb{C})$

$$x|_{\widehat{\mathcal{M}}_\sigma} = \Phi_\sigma(\delta(x))$$

where $x|_{\widehat{\mathcal{M}}_\sigma}$ denotes the natural action of $x$ on $\widehat{\mathcal{M}}_\sigma$. Indeed, $\phi_\sigma$ intertwines the natural action of $SL(2,\mathbb{C})$ on $\widehat{\mathcal{M}}_\sigma$ with the diagonal $SL(2,\mathbb{C})$-action on $\mathcal{A}_\sigma$. This means that

$$\phi_\sigma \circ (x|_{\widehat{\mathcal{M}}_\sigma}) = \theta_\sigma(\delta(x)) \circ \phi_\sigma.$$

Therefore

$$\begin{aligned} \Phi_\sigma(\delta(x)) &= \phi_\sigma^{-1} \circ \theta_\sigma(\delta(x)) \circ \phi_\sigma \\ &= \phi_\sigma^{-1} \circ \phi_\sigma \circ (x|_{\widehat{\mathcal{M}}_\sigma}) \\ &= x|_{\widehat{\mathcal{M}}_\sigma}. \end{aligned}$$

(iv) For any $\sigma_1, \sigma_2 \in \Sigma$ and $g \in L$

$$\Phi_{\sigma_1}(g)|_{\widehat{\mathcal{M}}_{\sigma_1} \cap \widehat{\mathcal{M}}_{\sigma_2}} = \Phi_{\sigma_2}(g)|_{\widehat{\mathcal{M}}_{\sigma_1} \cap \widehat{\mathcal{M}}_{\sigma_2}}. \tag{5.12}$$

Indeed, suppose $p \in \widehat{\mathfrak{D}}_{\sigma_1} \cap \widehat{\mathfrak{D}}_{\sigma_2}$. Then

$$\phi_{\sigma_1}|_{\widehat{M}_p} = \phi_p = \phi_{\sigma_2}|_{\widehat{M}_p}$$

from which (5.12) immediately follows.

We are now ready to construct the representation $(\Phi_{\mathcal{F}}, \widehat{\mathcal{M}})$ of $L$. Let $g \in L$. Define $\Phi_{\mathcal{F}}(g)$ on $\widehat{\mathcal{M}}$ by

$$\Phi_{\mathcal{F}}(g)|_{\widehat{\mathcal{M}}_\sigma} = \Phi_\sigma(g).$$

By (5.12) this is well-defined, and since $\sum_\sigma \widehat{\mathcal{M}}_\sigma = \widehat{\mathcal{M}}$, this gives an action of $L$ on all of $\widehat{\mathcal{M}}$. Moreover, by properties (i) and (ii), $(\Phi_{\mathcal{F}}, \widehat{\mathcal{M}})$ satisfies the conditions of Theorem 5.1.2. By property (iii) $(\Phi_{\mathcal{F}}, \widehat{\mathcal{M}})$ extends the natural action of $SL(2, \mathbb{C})$ on $\widehat{\mathcal{M}}$.

**Uniqueness**: Suppose $(\Phi, \widehat{\mathcal{M}})$ is a representation of $L$ satisfying the conditions of Theorem 5.1.2. We first show that there exists a compatible family $\widetilde{\mathcal{F}} = \{\widetilde{\phi}_p : \widehat{M}_p \to A_p\}_{p \in \widehat{\mathfrak{D}}}$ such that $\Phi = \Phi_{\widetilde{\mathcal{F}}}$. Indeed, by condition (1) of Theorem 5.1.2, $\widehat{M}_p$ is isomorphic to $A_p$ as $L$-modules for every $p \in \widehat{\mathfrak{D}}$. In particular, we can choose a set of $L$-isomorphisms $\{\widetilde{\phi}_p : \widehat{M}_p \to A_p\}_{p_{\max}=1}$. From this set we construct a compatible family $\widetilde{\mathcal{F}} = \{\widetilde{\phi}_p : \widehat{M}_p \to A_p\}_{p \in \widehat{\mathfrak{D}}}$ as in the proof of Proposition 5.4.2.

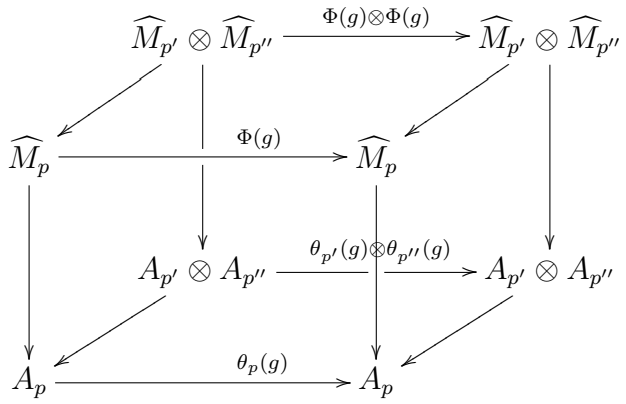
To show that $\Phi = \Phi_{\widetilde{\mathcal{F}}}$, we need to show that for all $g \in L$ and $p \in \widehat{\mathfrak{D}}$, the following diagram commutes:

$$\begin{array}{ccc} \widehat{M}_p & \xrightarrow{\Phi(g)} & \widehat{M}_p \\ \downarrow & & \downarrow \\ A_p & \xrightarrow{\theta_p(g)} & A_p \end{array} \tag{5.13}$$

where the vertical maps are both $\widetilde{\phi}_p$. We prove this by induction on $p_{\max}$.

Let $p \in \widehat{\mathfrak{D}}$. If $p_{\max} = 1$, then (5.13) commutes by our choice of $\{\widetilde{\phi}_p : \widehat{M}_p \to A_p\}_{p_{\max}=1}$ above. Let $m > 1$ and assume (5.13) commutes for all $p$ such that $p_{\max} < m$. Suppose then that $p_{\max} = m$. Choose some $\sigma \in \Sigma$ such that $p \in \widehat{\mathfrak{D}}_\sigma$. By Lemma 5.3.1, there exist $p', p'' \in \widehat{\mathfrak{D}}_{\sigma, m-1}$ such that $p = p' + p''$. Consider the

following cube:




The top square commutes since $\Phi$ satisfies condition (2) of Theorem 5.1.2. The left and right squares commute by the compatibility of $\mathcal{F}$. The bottom square commutes since the product on $\mathcal{A}_\sigma$ intertwines the $L$-action. Finally, the back square commutes by inductive hypothesis. Since all the maps are surjective, we conclude that the front square commutes. This completes the induction, and proves that $\Phi = \Phi_{\widetilde{\mathcal{F}}}$.

It remains to prove that $\Phi_{\mathcal{F}} = \Phi_{\widetilde{\mathcal{F}}}$ for any two compatible families $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ of $SL(2,\mathbb{C})$ isomorphisms. By Proposition 5.4.2 there exists a set of nonzero scalars indexed by $\widehat{\mathfrak{D}}$, $\{c_p \in \mathbb{C}^\times : p \in \widehat{\mathfrak{D}}\}$, such that

$$\phi_p = c_p \widetilde{\phi}_p$$

for all $p \in \widehat{\mathfrak{D}}$. Let $p \in \widehat{\mathfrak{D}}_\sigma$. Then for any $g \in L$,

$$\begin{aligned}
\Phi_{\mathcal{F}}(g)|_{\widehat{M}_p} &= \phi_p^{-1} \circ \theta_p(g) \circ \phi_p \\
&= (c_p\widetilde{\phi}_p)^{-1} \circ \theta_p(g) \circ (c_p\widetilde{\phi}_p) \\
&= \widetilde{\phi}_p^{-1} \circ \theta_p(g) \circ \widetilde{\phi}_p \\
&= \Phi_{\widetilde{\mathcal{F}}}(g)|_{\widehat{M}_p}.
\end{aligned}$$

Therefore $\Phi_{\mathcal{F}} = \Phi_{\widetilde{\mathcal{F}}}$. This completes the proof of Theorem 5.1.2.

**Remark 5.5.1** As a scholium of this proof and Proposition 5.2.2, each subalgebra $\widehat{\mathcal{M}}_\sigma$ is isomorphic to the polynomial algebra $\mathcal{O}(V)$. (Recall that $V$ is a vector space of dimension $3n-1$, with an action of $L \times H$.) Thus, although $\widehat{\mathcal{M}}$ is far from being a polynomial algebra, it has a family of polynomial subalgebras. Moreover, $\widehat{\mathcal{M}}_\sigma$ is isomorphic to $\mathcal{O}(V)$ as $L$-algebras. Therefore the product in these subalgebras, which is a priori just the product inherited from $\widehat{\mathcal{M}}$, has a simple formulation: it is the Cartan product of irreducible $L$-modules. An interesting open question is to understand geometrically the meaning of the subalgebras $\widehat{\mathcal{M}}_\sigma$. In other words, for every $\sigma \in \Sigma$ there is an embedding

$$\widehat{\mathcal{M}}_\sigma \hookrightarrow \widehat{\mathcal{M}},$$

which corresponds to a projection

$$\mathbb{A}^{3n-1} \twoheadleftarrow Spec(\widehat{\mathcal{M}}).$$

Can one show the existence of these projections independently?

# Chapter 6

# The Symplectic Gelfand-Zeitlin Basis

In this chapter we apply the Extension Theorem to obtain a symplectic Gelfand-Zeitlin basis. In section 1 we use the Extension Theorem to resolve the multiplicities that arise in branching from $Sp(n, \mathbb{C})$ to $Sp(n-1, \mathbb{C})$. In section 2 we construct the symplectic Gelfand-Zeitlin basis. In particular, we show that the symplectic Gelfand-Zeitlin basis is the weight basis for an $n^2$-dimensional torus acting on a given irreducible representation of $Sp(n, \mathbb{C})$. In contrast, the classical Gelfand-Zeitlin basis for irreducible representations of $GL(n, \mathbb{C})$ can be extracted as the weight basis of an $\binom{n}{2}$ dimensional torus.

## 6.1 Resolution of multiplicities

In this section we complete our program to construct a canonical weight basis for the irreducible finite dimesional representations of $Sp(n,\mathbb{C})$.

Let

$$K = K_1 \times \cdots \times K_n \subset Sp(n,\mathbb{C})$$

so that $K_i \cong Sp(1,\mathbb{C}) \cong SL(2,\mathbb{C})$ and $K_i$ is a subgroup of the centralizer of $Sp(i-1,\mathbb{C})$ in $Sp(i,\mathbb{C})$. (See (2.6) for an explicit description of $K$.) Let $\lambda \in \Lambda_n^+$. By Corollary 4.1.3, when we restrict $W_\lambda$ to $K$ we obtain a sum over double interlacing patterns,

$$W_\lambda \cong \bigoplus_{\substack{\lambda^{(i)} \in \Lambda_i^+ \\ \lambda^{(i)} \ll \lambda^{(i+1)} \\ \lambda^{(n-1)} \ll \lambda}} W_{\lambda^{(1)}} \otimes \widehat{M}^{\lambda^{(2)}}_{\lambda^{(1)}} \otimes \cdots \otimes \widehat{M}^{\lambda}_{\lambda^{(n-1)}}. \tag{6.1}$$

where $K_i$ acts on the $i^{th}$ factor in each summand. Identify $W_\lambda$ with this direct sum.

Set $T = T_{C_n}$ and let $T_{K_i} \subset K_i$ be the maximal torus given by $T_{K_i} = T \cap K_i$. Then

$$T = T_{K_1} \times \cdots \times T_{K_n}.$$

The problem of constructing a canonical $T$ weight basis of $W_\lambda$ reduces to finding a canonical $T_{K_i}$ weight basis of $\widehat{M}^{\lambda^{(i)}}_{\lambda^{(i-1)}}$ for $i = 2, ..., n$ (where $\lambda^{(n)} = \lambda$) and a $T_{K_1}$ weight basis for $W_{\lambda^{(1)}}$. To achieve this we use the representation $(\Phi, \widehat{\mathcal{M}})$ of $L$ afforded by Theorem 5.1.2.

In the previous chapter we identified $K_n$ with $SL(2,\mathbb{C})$, and referred to its action on $\widehat{\mathcal{M}}$ as the "natural" action of $SL(2,\mathbb{C})$ on $\widehat{\mathcal{M}}$. At present, we have several different copies of $SL(2,\mathbb{C})$ and it will be important for us to differentiate between them.

Let $\Delta \subset L$ denote the diagonal copy of $SL(2,\mathbb{C})$ in $L$,

$$\Delta = \{(x, ..., x) : x \in SL(2,\mathbb{C})\}$$

and let $T_\Delta \subset \Delta$ be the torus consisting of diagonal elements,

$$T_\Delta = \{(x, ..., x) \in \Delta : x \text{ is diagonal}\}.$$

Let $\epsilon : SL(2,\mathbb{C}) \to K_n$ be the tautological identification. By Theorem 5.1.2, for all $x \in SL(2,\mathbb{C})$, the actions of $\epsilon(x)$ and $\delta(x)$ on $\widehat{\mathcal{M}}$ agree:

$$\epsilon(x)|_{\widehat{\mathcal{M}}} = \Phi(\delta(x)). \tag{6.2}$$

Therefore finding a canonical $T_{K_n}$ weight basis of $\widehat{M}_p$ is equivalent to finding a canonical $T_\Delta$ weight basis of $\widehat{M}_p$.

By Theorem 5.1.2, $\widehat{M}_p$ is isomorphic as an $L$-module to $\bigotimes_{i=1}^n F_{r_i(p)}$. Suppose we decompose $\widehat{M}_p$ into weight spaces for some maximal torus $T_L \subset L$. Since these weight spaces are one dimensional, we get a basis of $\widehat{M}_p$ which is unique up to scalar. But, remember, we need this basis to be a weight basis for $T_\Delta$. Therefore, we must require that $T_\Delta \subset T_L$. There is only one maximal torus of $L$ that contains $T_\Delta$:

**Lemma 6.1.1** *Let $T_L$ be a maximal torus of $L$ containing $T_\Delta$. Then $T_L$ is the product of the diagonal maximal tori in each factor:*

$$T_L = \{(t_1, ..., t_n) \in L : \ t_i \text{ is diagonal}\}.$$

**Proof.** Write $L = L_1 \times \cdots \times L_n$, with $L_i = SL(2, \mathbb{C})$. Let $pr_i : L \to L_i$ be the corresponding projection. Then $pr_i(T_L)$ is a maximal torus of $L_i$, and by the maximality condition on $T_L$,

$$T_L = pr_1(T_L) \times \cdots \times pr_n(T_L).$$

Since $T_\Delta \subset T_L$, $pr_i(T_L)$ consists of the diagonal matrices in $L_i$. Therefore $T_L$ is the product of diagonal matrices in each factor. ■

Set

$$T_L = \{(t_1, ..., t_n) \in L : t_i \text{ is diagonal}\}. \tag{6.3}$$

To summarize briefly, our choice of torus $T \subset Sp(n, \mathbb{C})$ induces the torus $T_{K_n} = T \cap K_n \subset K_n$. Associated to the torus $T_{K_n}$, in turn, there is a unique torus $T_L \subset L$ such that the weight spaces for $T_L$ on $\widehat{\mathcal{M}}$ are also weight spaces for $T_{K_n}$. Schematically,

$$T \rightsquigarrow T_{K_n} \rightsquigarrow T_L.$$

We now examine the weight spaces of $T_L$ on $\widehat{M}_p$ ($p \in \widehat{\mathfrak{D}}$) more closely.

**Lemma 6.1.2** *Let $p = (\mu, \lambda) \in \widehat{\mathfrak{D}}$. The weight spaces of $T_L$ on $\widehat{M}_p$ are indexed by the dominant weights $\{\gamma \in \Lambda_n^+ : \mu < \gamma < \lambda^+\}$. A dominant weight $\gamma$ corresponds*

*to the weight*

$$(t_1, ..., t_n) \mapsto \prod_{i=1}^{n} t_i^{2\gamma_i - x_i - y_i}.$$

*where* $f(p) = (x_1, y_1, ..., x_n, y_n)$. *In particular,* $T_{K_n}$ *acts on the weight space indexed by* $\gamma$ *by the character*

$$t \mapsto t^{2|\gamma| - |\lambda| - |\mu|}.$$

**Proof.** Suppose $f(p) = (x_1, y_1, ..., x_n, y_n)$. Since $\widehat{M}_p$ is isomorphic as an $L$-module to $\bigotimes_{i=1}^{n} F_{r_i(p)}$, the weight spaces of $T_L$ on $\widehat{M}_p$ are indexed by

$$\{(j_1, ..., j_n) : 0 \leq j_i \leq r_i(p) \text{ for } i = 1, ..., n\}.$$

Indeed, such a sequence $(j_1, ..., j_n)$ corresponds to the weight

$$(t_1, ..., t_n) \mapsto \prod_{i=1}^{n} t_i^{(-r_i(p) + 2j_i)}.$$

Now there is a one-to-one correspondence between

$$\{(j_1, ..., j_n) : 0 \leq j_i \leq r_i(p) \text{ for } i = 1, ..., n\} \tag{6.4}$$

and

$$\{(\gamma_1, ..., \gamma_n) : y_i \leq \gamma_i \leq x_i \text{ for } i = 1, ..., n\}$$

given by $(j_1, ..., j_n) \mapsto (j_1 + y_1, ..., j_n + y_n)$. Moreover, by Lemma 3.1.6 there is a one-to-one correspondence,

$$\{(\gamma_1, ..., \gamma_n) : y_i \leq \gamma_i \leq x_i \text{ for } i = 1, ..., n\} \leftrightarrow \{\gamma \in \Lambda_n^+ : \mu < \gamma < \lambda^+\}.$$

Consequently the weight spaces of $T_L$ on $\widehat{M}_p$ are indexed by $\{\gamma \in \Lambda_n^+ : \mu < \gamma < \lambda^+\}$. Unwinding these identifications, we see that a pattern $\gamma$ corresponds to the

weight

$$(t_1, ..., t_n) \mapsto \prod_{i=1}^{n} t_i^{-(x_i+y_i)+2\gamma_i}.$$

■

By elementary representation theory of $SL(2, \mathbb{C})$, the weight spaces of $T_L$ on $\widehat{M_p} = \widehat{M_\mu^\lambda}$ are one-dimensional. By the above lemma we choose a weight basis

$$\{w_\mu^\lambda(\gamma) : \mu < \gamma < \lambda^+\}$$

of $\widehat{M_\mu^\lambda}$, where $w_\mu^\lambda(\gamma)$ is in the weight space indexed by $\gamma$. This basis is unique up to scalar, and $T_{K_n}$ acts on $w_\mu^\lambda(\gamma)$ by the character

$$t \mapsto t^{2|\gamma|-|\lambda|-|\mu|}.$$

We now remark about the action of $Sp(1, \mathbb{C})$ on an irreducible representation $W_\lambda$, where $\lambda \in \Lambda_1^+ = \mathbb{N}$. Since $Sp(1, \mathbb{C}) \cong SL(2, \mathbb{C})$ acts irreducibly, the weight spaces of $T_{C_1}$ are one-dimensional. Hence we get a canonical $T_{C_1}$ weight basis of $W_\lambda$ for free, which is unique up to scalar. We choose such a basis and label it $\{w^\lambda(i) : 0 \leq i \leq \lambda\}$, where $T_{C_1}$ acts on $w^\lambda(i)$ by the character

$$t \mapsto t^{-\lambda+2i}.$$

## 6.2 The symplectic Gelfand-Zeitlin basis

We now have all the ingredients to construct the advertised basis. We define the **Gelfand-Zeitlin patterns for the symplectic group**:

$$\mathbf{GZ}_{C_n} = \{(\gamma^{(1)}, \lambda^{(1)}, ..., \gamma^{(n)}, \lambda^{(n)})\}$$

where $\gamma^{(i)}, \lambda^{(i)} \in \Lambda_i^+$ and

$$\lambda^{(i-1)} < \gamma^{(i)} < (\lambda^{(i)})^+ \tag{6.5}$$

for $i = 1, ..., n$. To $P = (\gamma^{(1)}, \lambda^{(1)}, ..., \gamma^{(n)}, \lambda^{(n)}) \in \mathbf{GZ}_{C_n}$ we associate the vector

$$\begin{aligned} w_P &= w^{\lambda^{(1)}}(\gamma^{(1)}) \otimes w_{\lambda^{(1)}}^{\lambda^{(2)}}(\gamma^{(2)}) \otimes \cdots \otimes w_{\lambda^{(n-1)}}^{\lambda^{(n)}}(\gamma^{(n)}) && (6.6) \\ &\in W_{\lambda^{(1)}} \otimes \widehat{M}_{\lambda^{(1)}}^{\lambda^{(2)}} \otimes \cdots \otimes \widehat{M}_{\lambda^{(n-1)}}^{\lambda^{(n)}}. && (6.7) \end{aligned}$$

The vector $w_P$ is unique up to scalar and the collection $\{w_P\}_{P \in \mathbf{GZ}_{C_n}(\lambda)}$ is a basis of $W_\lambda$, where $\mathbf{GZ}_{C_n}(\lambda)$ is the set of all $P \in \mathbf{GZ}_{C_n}$ such that $\lambda^{(n)} = \lambda$. We term this basis the **symplectic Gelfand-Zeitlin basis**. Some observations are in order.

First, note that this provides a combinatorial formula to compute the dimension of an irreducible representation $W_\lambda$ ($\lambda \in \Lambda_n^+$):

$$\dim W_\lambda = \#\{(\gamma^{(1)}, \lambda^{(1)}, ..., \gamma^{(n)}, \lambda^{(n)}) \in \mathbf{GZ}_{C_n} : \lambda^{(n)} = \lambda\}.$$

Second, the symplectic Gelfand-Zeitlin basis is a weight basis for the maximal torus $T_{C_n}$. Indeed, by Lemma 6.1.2, $T_{K_i}$ acts on $w_{\lambda^{(i-1)}}^{\lambda^{(i)}}(\gamma^{(i)})$ by the weight $t \mapsto t^{2|\gamma^{(i)}|-|\lambda^{(i)}|-|\lambda^{(i-1)}|}$. A toral element $t = diag(t_1, ..., t_n, t_n^{-1}, ..., t_1^{-1}) \in T_{C_n}$ acts on $w_P$ as follows:

$$t.w_P = t_1^{2|\gamma^{(1)}|-|\lambda^{(1)}|} t_2^{2|\gamma^{(2)}|-|\lambda^{(2)}|-|\lambda^{(1)}|} \cdots t_n^{2|\gamma^{(n)}|-|\lambda^{(n)}|-|\lambda^{(n-1)}|} w_P,$$

which shows that the weight of $w_P$ is

$$(2|\gamma^{(1)}| - |\lambda^{(1)}|, 2|\gamma^{(2)}| - |\lambda^{(2)}| - |\lambda^{(1)}|, ..., 2|\gamma^{(n)}| - |\lambda^{(n)}| - |\lambda^{(n-1)}|).$$

We conclude this section with a comparison of the classical Gelfand-Zeitlin basis with the symplectic Gelfand-Zeitlin using the so-called Thimm torus action [12]. Consider first the classical Gelfand-Zeitlin basis discussed in Section 1.1. Instead of obtaining this basis by restricting down the chain of groups, we define a big torus action on $V_\lambda$ whose weight space decomposition yields the basis.

Let $0 < i \leq n$ and consider $V_\lambda$ as a $GL(i,\mathbb{C})$-module by restriction. Given a $GL(i,\mathbb{C})$-type $\eta \in \Lambda_i^+$, let $V_\lambda[\eta]$ denote the $\eta$-isotypic component of $V_\lambda$. Now, let $(\mathbb{C}^\times)^i$ act on $V_\lambda[\eta]$ by the character $t \mapsto t^\eta$, and extend this action linearly to all of $V_\lambda$. Notice that this action differs from the action of the maximal torus of $GL(i,\mathbb{C})$ on $V_\lambda$.

For $0 < i, j < n$ we have actions of $(\mathbb{C}^\times)^i$ and $(\mathbb{C}^\times)^j$ on $V_\lambda$, and these actions commute. This is easily seen by taking a vector in $V_\lambda$, and decomposing it first with respect to isotypic components of $GL(i,\mathbb{C})$, and then with respect to isotypic components of $GL(j,\mathbb{C})$. Therefore we obtain an action of the **Thimm torus** $\mathbf{H} = (\mathbb{C}^\times)^1 \times (\mathbb{C}^\times)^2 \times \cdots \times (\mathbb{C}^\times)^{n-1}$ on $V_\lambda$.

Notice that the summand $V_{\lambda^{(1)}} \otimes M^{\lambda^{(2)}}_{\lambda^{(1)}} \otimes \cdots \otimes M^{\lambda}_{\lambda^{(n-1)}}$ (cf. 1.1) is in $V_\lambda[\lambda^{(i)}]$ for $i = 1, ..., n-1$. Therefore given $h_i \in (\mathbb{C}^\times)^i$, the element $h = (h_1, ..., h_{n-1}) \in \mathbf{H}$ acts on $V_{\lambda^{(1)}} \otimes M^{\lambda^{(2)}}_{\lambda^{(1)}} \otimes \cdots \otimes M^{\lambda}_{\lambda^{(n-1)}}$ by the scalar

$$h_1^{\lambda^{(1)}} \cdots h_{n-1}^{\lambda^{(n-1)}}.$$

This shows that the Gelfand-Zeitlin basis is the (canonical) set of weight vectors for the action of the $\mathbf{H}$ on $V_\lambda$.

We now show how to extract the symplectic Gelfand-Zeitlin basis as the weight vectors of a torus of dimension $n^2$. Let $L^{(i)} = SL(2,\mathbb{C}) \times \cdots \times SL(2,\mathbb{C})$ ($i$ copies) and set $T_{L^{(i)}} \subset L^{(i)}$ to be the maximal torus as defined in (6.3). Define an action of $\mathbf{H}$ on $W_\lambda$ as above, by letting $(\mathbb{C}^\times)^i$ act on the isotypic components $W_\lambda[\lambda^{(i)}]$ by a scalar. This action of $\mathbf{H}$ on $W_\lambda$ is not sufficient to extract a canonical basis since its weight spaces are not one-dimensional. We remedy this by using the action of

$$L^{(\frac{(n+1)(n)}{2})} = L^{(1)} \times \cdots \times L^{(n)}$$

on $W_\lambda$. This action leaves the weight spaces of $\mathbf{H}$ invariant, and therefore $L^{(\frac{(n+1)(n)}{2})}$ commutes with $\mathbf{H}$. Moreover, the weight spaces of

$$\mathbf{H}' = T_{L^{(\frac{(n+1)(n)}{2})}}$$

are one-dimensional on each weight space of $\mathbf{H}$. The symplectic Gelfand-Zeitlin basis $\{w_P\}_{P \in \mathbf{GZ}_{C_n}(\lambda)}$ is the set of weight vectors for $\mathbf{H} \times \mathbf{H}' = (\mathbb{C}^\times)^{n^2}$ acting on $W_\lambda$.

In conclusion, we see that while the classical Gelfand-Zeitlin basis for irreducible representations of $GL(n,\mathbb{C})$ can be extracted as the weight basis of a torus of dimension $\binom{n}{2}$, the symplectic Gelfand-Zeitlin basis is the weight basis of an $n^2$-dimensional torus. These numbers agree with the (complex) dimension of a generic coadjoint orbit in each case.

# Chapter 7

# Appendix

The purpose of this appendix is to give an elementary proof of formula (4.3) in Corollary 4.1.3. This proof is purely combinatorial, and doesn't rely on Zhelobenko's indicator system. As a byproduct of this proof we derive a formula for the decomposition of arbitrary tensor products of irreducible representations of $SL(2,\mathbb{C})$, generalizing the Clebsch-Gordan formula. Here the multiplicities are given as a difference of two generalized Kostant partition functions. This proof appears in [25].

## 7.1 A combinatorial proof of Corollary 4.1.3

The Clebsch-Gordan formula implies that if $r_1 \geq r_2$ then

$$F_{r_1} \otimes F_{r_2} \cong F_{r_1+r_2} \oplus F_{r_1+r_2-2} \oplus \cdots \oplus F_{r_1-r_2}. \tag{7.1}$$

We now extend the Clebsch-Gordan formula to an arbitrary tensor product of representations of $SL(2,\mathbb{C})$.

We begin by setting up some notation. Let $\{v_1,...,v_n\}$ be the standard basis for $\mathbb{R}^n$ and set $\Sigma_n = \{v_1 \pm v_n, ..., v_{n-1} \pm v_n\}$. We identify $\mathbb{R}^n$ with $(\mathbb{R}^n)^{**}$; thus if $v \in \mathbb{R}^n$, $e^v$ is a function on $(\mathbb{R}^n)^*$. Denote by $\mathcal{P}_n(v)$ the coefficient of $e^v$ in the formal product

$$\prod_{w\in\Sigma_n} \frac{1}{1-e^w} .$$

This says that $\mathcal{P}_n(v)$ is the number of ways of writing

$$v = \sum_{w\in\Sigma_n} c_w w,\ c_w \in \mathbb{N}.$$

The following is a reinterpretation of formula (7.1).

**Lemma 7.1.1** *Let $r_1, r_2, l \in \mathbb{N}$. Then*

$$\dim Hom_{SL(2,\mathbb{C})}(F_l, F_{r_1} \otimes F_{r_2}) = \mathcal{P}_2(r_1v_1 + r_2v_2 - lv_2) - \mathcal{P}_2(r_1v_1 + r_2v_2 + (l+2)v_2).$$

**Proof.** Note that $\mathcal{P}_2(av_1+bv_2) = 1$ if and only if $b \in \{-a, 2-a, ..., a-2, a\}$. The result follows by considering the cases $r_1 \leq r_2$ and $r_1 > r_2$ separately. ■

Before generalizing Lemma 7.1.1 to a tensor product of an arbitrary number of irreducible $SL(2,\mathbb{C})$-modules, we develop some combinatorial properties of $\mathcal{P}_n$.

Let $\Sigma_n^+ = \{v_1+v_n, ..., v_{n-1}+v_n\}$ and $\Sigma_n^- = \{v_1-v_n, ..., v_{n-1}-v_n\}$. Denote by $\mathcal{P}_n^{\pm}(v)$ the coefficient of $e^v$ in

$$\prod_{w\in\Sigma_n^{\pm}} \frac{1}{1-e^w} .$$

It is easy to see that

$$\mathcal{P}_n(v) = \sum_{u+w=v} \mathcal{P}_n^+(u)\mathcal{P}_n^-(w).$$

Since $\Sigma_n^+, \Sigma_n^-$ are linearly independent the corresponding partition functions take only values 0 or 1. Furthermore, one can easily check that

$$\mathcal{P}_n^+(a_1v_1 + \cdots + a_nv_n) = 1 \Leftrightarrow a_1, ..., a_{n-1} \in \mathbb{N} \text{ and } \sum_{j=1}^{n-1} a_j = a_n$$

$$\mathcal{P}_n^-(b_1v_1 + \cdots + b_nv_n) = 1 \Leftrightarrow b_1, ..., b_{n-1} \in \mathbb{N} \text{ and } \sum_{j=1}^{n-1} b_j = -b_n$$

Let $v = c_1v_1 + \cdots + c_nv_n$ and suppose $v = u + w$ with $u = a_1v_1 + \cdots + a_nv_n$ and $w = b_1v_1 + \cdots + b_nv_n$. Then $a_j + b_j = c_j$ for $j = 1, ..., n$. If $\mathcal{P}_n^+(u)\mathcal{P}_n^-(w) = 1$ then

$$c_n = \sum_{j=1}^{n-1} a_j - b_j. \tag{7.2}$$

Define a **bisection** of a natural number $m$ to be a two-part partition of $m$. Then $\mathcal{P}_n(v)$ counts the number of bisections of $c_1, ..., c_{n-1}$ that satisfy (7.2). This description provides a useful recursive formula.

**Lemma 7.1.2**

$$\mathcal{P}_n(c_1v_1 + \cdots + c_nv_n) = \sum_{i=0}^{c_{n-1}} \mathcal{P}_{n-1}(c_1v_1 + \cdots + c_{n-2}v_{n-2} + (c_{n-1} + c_n - 2i)v_{n-1})$$

**Proof.** The $i^{\text{th}}$ summand on the right hand side counts the number of bisections of $c_1, ..., c_{n-2}$ that satisfy $c_{n-1} + c_n - 2i = \sum_{j=1}^{n-2} a_j - b_j$. (Here $c_j = a_j + b_j$ for $j = 1, ..., n-2$.) These bisections correspond to the bisections of $c_1, ..., c_{n-1}$ that satisfy $c_n = \sum_{j=1}^{n-1} a_j - b_j$ with $a_{n-1} = i$ and $b_{n-1} = c_{n-1} - i$. ■

**Theorem 7.1.3** *Let* $r_1, ..., r_n, l \in \mathbb{N}$. *Then*

$$\dim Hom_{SL(2,\mathbb{C})}(F_l, F_{r_1} \otimes \cdots \otimes F_{r_n})$$

*equals*

$$\mathcal{P}_n(r_1v_1 + \cdots + r_nv_n - lv_n) - \mathcal{P}_n(r_1v_1 + \cdots + r_nv_n + (l+2)v_n).$$

**Proof.** We proceed by induction on $n \geq 2$. If $n = 2$ use Lemma 2.1. Now suppose $n > 2$ and the claim holds for $n-1$. Let $r_1, ..., r_n, l \in \mathbb{N}$ and to simplify matters write $S_k = \sum_{j=1}^{k} r_j v_j$ and $Q(t) = \mathcal{P}_{n-1}(S_{n-2} + tv_{n-1})$. By Lemma 2.2 we obtain

$$\mathcal{P}_n(S_n - lv_n) - \mathcal{P}_n(S_n + (l+2)v_n) = \sum_{i=0}^{r_{n-1}} Q(r_{n-1} + r_n - 2i - l) - Q(r_{n-1} + r_n - 2i + l + 2).$$

If $r_{n-1} \leq r_n$ then $r_{n-1} + r_n - 2i \geq 0$ so by the inductive hypothesis

$$Q(r_{n-1} + r_n - 2i - l) - Q(r_{n-1} + r_n - 2i + l + 2) = \boldsymbol{m}_l(r_1, ..., r_{n-2}, r_{n-1} + r_n - 2i).$$

By the Clebsch-Gordan formula

$$\sum_{i=0}^{r_{n-1}} \boldsymbol{m}_l(r_1, ..., r_{n-2}, r_{n-1} + r_n - 2i) = \boldsymbol{m}_l(r_1, ..., r_{n-2}, r_{n-1}, r_n).$$

If $r_{n-1} > r_n$ the situation is not as straightforward. As above we have

$$\mathcal{P}_n(S_n - lv_n) - \mathcal{P}_n(S_n + (l+2)v_n) = \boldsymbol{m}_l(r_1, ..., r_{n-2}, r_{n-1}, r_n) + E$$

where

$$E = \sum_{i=r_n+1}^{r_{n-1}} Q(r_{n-1} + r_n - 2i - l) - Q(r_{n-1} + r_n - 2i + l + 2).$$

Rewrite $E$ as

$$\sum_{i=1}^{r_{n-1}-r_n} Q(r_{n-1}-r_n-2i-l)-Q(r_{n-1}-r_n-2i+l+2)$$

and notice that

$$r_{n-1}-r_n-2i-l=-(r_{n-1}-r_n-2(r_{n-1}-r_n+1-i)+l+2).$$

Therefore if we set $C_i=r_{n-1}-r_n-2i-l$ then by rearranging terms

$$E=\sum_{i=1}^{r_{n-1}-r_n} Q(C_i)-Q(-C_i).$$

But clearly $Q(t)=Q(-t)$ so $E=0$. ■

Now let $\mu\in\Lambda_{n-1}^+$ and $\lambda\in\Lambda_n^+$ and suppose $\mu\ll\lambda^+$. Combining Theorem 7.1.3 with the following theorem, due to J. Lepowsky ([18]), proves formula (4.3) of Corollary 4.1.3.

**Theorem 7.1.4 ([16], Proposition 9.5.9)** *Let $\mu\in\Lambda_{n-1}^+$ and $\lambda\in\Lambda_n^+$ and suppose $\mu\ll\lambda^+$. Set $r_i=r_i(\mu,\lambda)$. Then for $l\in\mathbb{N}$,*

$$\dim Hom_{SL(2,\mathbb{C})}(F_l,\widehat{M}_\mu^\lambda)=\mathcal{P}_n(r_1v_1+\cdots+r_nv_n-lv_n)-\mathcal{P}_n(r_1v_1+\cdots+r_nv_n+(l+2)v_n).$$

Chapter 7, in full, is a reprint of the material as it will appear in the paper *A multiplicity formula for tensor products of $SL_2$ modules and an explicit $Sp_{2n}$ to $Sp_{2n-2}\times Sp_2$ branching formula* in Contemp. Math., American Mathematical Society, Providence, R.I., 2009; co-authored with Nolan R. Wallach. I was the principal author of this paper, and made substantial contributions to the research as did my co-author.

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