Title
Data-driven Stabilization of Unknown Feedback-Linearizable and Partially Feedback-Linearizable Systems

Author
Fraile Vazquez, Lucas Martin

Publication Date
2022

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA

Los Angeles

Data-driven Stabilization of Unknown Feedback-Linearizable and Partially Feedback-Linearizable Systems

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Electrical and Computer Engineering

by

Lucas Martin Fraile Vazquez

2022

ABSTRACT OF THE DISSERTATION

Data-driven Stabilization of Unknown Feedback-Linearizable and Partially Feedback-Linearizable Systems

by

Lucas Martin Fraile Vazquez
Doctor of Philosophy in Electrical and Computer Engineering
University of California, Los Angeles, 2022
Professor Paulo Tabuada, Chair

This thesis reports on my research in data-driven control, addressing the problem of data-driven stabilization, i.e., stabilization of unknown nonlinear systems without making explicit use of a model. Broadly speaking, it's aim is to answer the question: can we design simple plug-and-play controllers to streamline a control engineer's work? While the trivialization of a controls engineering is probably out of anyone's reach, and most certainly out of mine, I focus on a special class of systems. These are general enough to be practically useful, yet well-behaved to the point of making the problem at hand tractable. My hope is to simplify the design of low-level controllers thus reducing the need for, among other things, hardware-specific controller design, allowing the engineer to focus primarily on the high-level control task. To that end, this work focuses primarily on single-input single-output (SISO) feedback linearizable and partially-feedback linearizable systems with stable or input-to-state stable (ISS) zero-dynamics. We start by considering the stabilization of the former, something accomplished by requiring only a minimal amount of real-time output data and without the need for persistency of excitation. This is achieved through a novel understanding of

how to guarantee the asymptotic stability of unknown continuous-time systems through the use of families of approximate discrete-time models and Lyapunov-based techniques. It follows by expanding onto the multiple-input multiple-output extension and introducing dirty derivatives as a derivative estimation technique with the aim of ameliorating the method's sensitivity to measurement noise. Finally, it concludes by presenting a strikingly simple continuous-time controller capable of stabilizing SISO feedback linearizable and partially-feedback linearizable systems with ISS zero-dynamics. This controller is constructed resorting to well-known techniques, a linear observer and a linear dynamic controller. Based on the simplicity of the methods here developed, the stability guarantees provided and their need of minimal knowledge of the underlying system, I believe they will find future use in practical applications.

The dissertation of Lucas Martin Fraile Vazquez is approved.

Ramanarayan Vasudevan

Bahman Gharesifard

Lieven Vandenberghe

Paulo Tabuada, Committee Chair

University of California, Los Angeles

2022

To my parents . . .

who put our education above all else,

to their morality and dedication

a steadfast compass that guides our lives.

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

The work herein presented would not have come to fruition without the guidance, advice and collaboration of Prof. Paulo Tabuada, who motivated my interest in data-driven control and built the foundations on which this research stands. The remarkable depth and astonishing breath of his knowledge in the fields of control theory, mathematics and physics provide an invaluable source of creativity at the time of problem solving that I can only strive to emulate. Much of the contents of this thesis were developed in direct collaboration with my co-author Matteo Marchi, his insight and superlative attention to detail were instrumental in the development of our methods. I must also acknowledge Luigi Pannocchi and Marcus Lucas, whose experimental contributions and expertise on embedded systems proved crucial in bringing to life the controllers here developed.

I want to thank my Ph.D. dissertation committee members, Prof. Ramanarayan Vasudevan, Prof. Bahman Gharesifard and Prof. Lieven Vandenberghe, for their insightful advice and remarks, which helped shape my research and this manuscript. I am particularly grateful to Prof. Vasudevan for believing in me and encouraging me to pursue a Ph.D. degree. As a Latin American undergraduate exchange student I was not even aware that this was something I could aspire to. He provided me with the tools, training and support required to successfully apply to graduate school, and completely changed my life for the better.

I wish to express my gratitude to the CONIX Research Center, the National Science Foundation, the Army Research Laboratory and the UCLA Electrical and Computer Engineering Department for supporting my research, the results of which are presented in this manuscript.

It would be remiss of me not to mention the rest of my co-workers, friends and members of the Cyber-Physical Systems Laboratory, Tzanis Anevlavis, Jonathan Bunton, Alimzhan Sultangazin, Yanwen Mao and Yskandar Gas, who along with Matteo Marchi, Marcus Lucas and Luigi Pannocchi, offered me their invaluable support and friendship through my Ph.D.

program. The same should be said about Sehajroop Singh, Eden Haney, Josh Hannan, Jared Winer, Bailey Kobelin, Salvador Vazquez, Ken Zhao, Jaume De Dios Pont, Evelina Gromilina and especially my partner Gül Nisa Gürbüz, who have helped me build a home away from home without which I would not have been able to endure the vicissitudes and challenges of the last five years.

Finally I would like to thank those supporting me from half a world away: my parents Marcelo Fraile and Silvia Vazquez, and siblings Matias, Estefania and Martina Fraile, for their unyielding love and care; and my friends Iair Levy and Franco Testori, sources of unending laughter and camaraderie.

VITA

2014-2017	Mechanical Engineering Intern, Computational Mechanics Center, Instituto Tecnológico de Buenos Aires, Argentina.
2015-2017	Remote Undergraduate Student Researcher, Robotics and Optimization for the Analysis of Human Motion Laboratory, University of Michigan, USA.
2016-2017	Research Scholar, Computational Mechanics Center, Instituto Tecnológico de Buenos Aires, Argentina.
2017	B.Eng. (Mechanical Engineering), Instituto Tecnológico de Buenos Aires, Argentina.
2017-2022	Graduate Student Researcher, Cyber-Physical Systems Laboratory, University of California, Los Angeles, USA.
2019	M.Sc. (Electrical and Computer Engineering), University of California, Los Angeles, USA.
2019	Professional Mechanical Engineer, Instituto Tecnológico de Buenos Aires, Argentina.
2021	Software Engineering Intern, Parallel Systems, California, USA.

PUBLICATIONS

P. Tabuada and **L. Fraile**,*"Data-driven control for SISO feedback linearizable systems with unknown control gain"*, IEEE 58th Conference on Decision and Control(CDC), 2019.

P. Tabuada and **L. Fraile**,*"A Note on Data-Driven Control for SISO Feedback Linearizable Systems Without Persistency of Excitation"*, Arxiv: https://arxiv.org/abs/1909.01959, 2019.

L. Fraile, M. Marchi and P. Tabuada,*"Data-driven Stabilization of SISO Feedback Linearizable Systems"*, Arxiv: https://arxiv.org/abs/2003.14240, 2020.

A. Sultangazin, **L. Fraile** and P. Tabuada,*"Exploiting the experts: Learning to control unknown SISO feedback linearizable systems from expert demonstrations "*, IEEE 60th Conference on Decision and Control(CDC), 2021.

A. Sultangazin, L. Pannochi, **L. Fraile** and P. Tabuada,*"Watch and Learn: Learning to control feedback linearizable systems from expert demonstrations"*, IEEE International Conference on Robotics and Automation (ICRA), 2022.

L. Fraile, M. Marchi and P. Tabuada,*"Data-driven control via dirty derivatives and output-feedback stabilization"*, (submitted to) Automatica, 2022.

CHAPTER 1

Introduction

This work aims to address the problem of data-driven stabilization, i.e., stabilization of unknown nonlinear systems without making explicit use of a model. The underlying motivation arises from the proliferation of highly-dynamic cyber-physical systems usually controlled through hierarchical structures. By this I refer to systems in which a high-level controller produces set-points or trajectories that are then taken as references for low-level controllers to follow. A specific example of this are common quad-copters for which a high-level algorithm plans a viable trajectory satisfying some given requirements,e.g., navigating a cluttered environment while avoiding dynamic obstacles, which is then fed to one or more low-level controllers in charge of each individual rotor. Low-level controllers tend to be hardware specific, making hardware replacements problematic and time-consuming, sometimes even requiring changes to the high-level controller. Ameliorate this issue constitutes the motivation to design data-driven controllers capable of "learning" the low-level hardware dynamics, providing stability guarantees despite their design being agnostic of the hardware parameters. The research here presented was performed in collaboration with Matteo Marchi, and under my advisor Prof. Paulo Tabuada, who provided the motivation for this work in his manuscript [1].

Data-driven control has been a field of fast expansion over the recent past, with several hundred related articles published in the top control journals, such as Automatica and Transactions on Automatic Control, in the last five years. The key question in this research area is how to control a system solely based on input and output data, i.e., without explicitly using a model. There is a clear overlap between data-driven control and adaptive control, making their distinction fuzzy at times. There exists a vast variety of methods for data-driven control, such as: PID [2] and intelligent PID control [3, 4]; optimization based methods [5] including iterative and virtual reference feedback tunning; Model-free adaptive control, and other dynamic-linearization-based schemes [6]; the behavioural approach [7, 8]; koopman-operator-based methods [9, 10]; extremum seeking [11, 12]; and reinforcement and machine-learning-based techniques [13, 14]. Each of these methods has its own advantages, yet they tend to suffer from well-documented challenges, many of which are faced by traditional adaptive control methods [15]. These include sensitivity to measurement noise and the need for large amounts of offline or online collected data, for previous on-line training, for the data to be sufficiently informative [16], usually requiring persistency of excitation, or for knowledge of bounds on the unknown systems' parameters. The aim of this thesis is to design a data-driven control methodology that addresses some of these issues.

Chapter 1 presents an overview of the contents of the thesis and defines specific notation to be used throughout the rest of the work.

Chapter 2 reports on the developments of [17], where I presented a methodology for stabilizing single-input single-output (SISO) feedback-linearizable systems by output-feedback when no system model is known and no prior data is available to identify a model. This is accomplished without the need for persistency of excitation or knowledge of bounds on the systems' parameters. I provide therein a specific discrete-time data-driven controller composed of a linear dynamic controller paired with least-squares-based state estimation. Conceptually, this methodology has been greatly inspired by the work of Fliess and Join on intelligent PID controllers (e.g., [3, 4]), and the results in this chapter provide sufficient conditions under which a modified version of their approach is guaranteed to result in asymptotically stable behavior. One of the key advantages of these results is that, contrary to other model-free (or partially model-free) approaches to controlling dynamical systems, such as reinforcement learning, there is no need for extensive training nor large amounts of data. Technically, the results draw heavily from the work of Nesic and co-workers on observer and controller design based on approximate models [18, 19]. Along the way, we also make connections with other well established results such as high-gain observers and adaptive control. Although this chapter focuses only on the simple setting of single-input single-output feedback-linearizable systems, I believe the results therein are already theoretically insightful and practically useful. Despite demonstrating robustness towards external disturbances in experimental settings, such controller is relative sensitive to measurement noise due to relying on unweighted least-squares for state estimation purposes, see [20].

Chapter 3 extends the results to unknown multiple-input multiple-output feedback-linearizable systems and introduces the dirty-derivative estimation approach as a promising substitute for least-squares-based estimation. Regarding the former, I focus on systems that satisfy the condition of sign definiteness in the input gain matrix and discuss the need for this restriction. In terms of the latter, I provide motivation for the use of these dirty derivatives by portraying the robustness to noise and improvement in performance they can offer to the methods presented in Chapter 2.

Chapter 4 leverages the linear dynamic controller proposed in Chapter 2, together with the dirty-derivative-based extended-state high-gain observer introduced in Chapter 3 to show how this simple techniques may be combined to perform data-driven control in continuous time. This is particularly interesting as it provides evidence that the approximate model techniques and the assumption of "sufficiently fast sampling rate" utilized in Chapter 2 are not necessary to recover the results therein presented. The controller introduced in Chapter 2 allows us to avoid the need for bounds on the control gain, usually required by extended-state-observer-based controllers as is the case in active disturbance rejection control [21, 22]. The dirty-derivative-based observer enjoys the properties of traditional high-gain observers without requiring any information of the underlying system's dynamics, making it especially useful when working with unknown partially-feedback linearizable systems and providing more robust estimates against measurement noise when compared to least-squares-based estimation. We note that this controller may be implemented in a discrete fashion through a forward Euler approximation, doing so and following the methodology of Chapter 2 one recovers the results therein presented.

1.1 Notation

1.1.1 Miscellanea

The natural numbers, including zero, are denoted by $\mathbb{N}$, the real numbers by $\mathbb{R}$, the non-negative real numbers by $\mathbb{R}_0^+$, and the positive real numbers by $\mathbb{R}^+$. Given a function $f : \mathcal{X} \to \mathcal{Y}$, the domain of f is $\mathcal{X}$ and its codomain is $\mathcal{Y}$. The image of f is the set of values attained by the function when its argument ranges in its domain. If $c : \mathbb{R} \to \mathbb{R}^n$ is a function of time, we denote its first time derivative by $\dot{c}$. When higher time derivatives are required, we use the notation $c^{(k)}$ defined by the recursion $c^{(1)} = \dot{c}$ and $c^{(k+1)} = \left(c^{(k)}\right)^{(1)}$. The Lie derivative of a function $h : \mathbb{R}^n \to \mathbb{R}$ along a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, given by $\frac{\partial h}{\partial x} f$, is denoted by $L_f h$, the n^{th} Lie derivative of h along the vector field f is denoted by $L_f^n h$.

We denote the identity matrix on $\mathbb{R}^n \times \mathbb{R}^n$ by I_n and the zero matrix on $\mathbb{R}^n \times \mathbb{R}^m$ by $0_{n\times m}$. Positive definite, positive semi-definite, negative semi-definite and negative definite matrices are symmetric square matrices with positive, non-negative, non-positive and negative eigenvalues, respectively. Given a matrix A, we write $A > 0$, $A \geq 0$, $A \leq 0$ or $A < 0$ if A is positive definite, positive semi-definite, negative semi-definite or negative definite, respectively. Consequently, given two matrices A and B, we write $A > B$, $A \geq B$, $A \leq B$ or $A < B$ if $A - B$ is positive definite, positive semi-definite, negative semi-definite or negative definite, respectively. Given a symmetric matrix Q we denote by $\lambda_{\min}(Q)$ its smallest eigenvalue and by $\lambda_{\max}(Q)$ its largest eigenvalue. A square matrix is said to be Hurwitz if all its eigenvalues have negative real part.

Given two compact sets $\mathcal{A}$ and $\mathcal{B}$ we denote that $\mathcal{B}$ is a subset or proper subset of $\mathcal{A}$ by $\mathcal{B} \subseteq \mathcal{A}$ or $\mathcal{B} \subset \mathcal{A}$, respectively. If $\mathcal{B} \subset \mathcal{A}$, we denote the complement of $\mathcal{B}$ in $\mathcal{A}$ by $\mathcal{A} - \mathcal{B}$.

The absolute value of a real number is denoted by $|\cdot|$. The standard Euclidean norm, or the induced matrix 2-norm, is denoted by $\|\cdot\|$. Let $f : \mathcal{S} \to \mathbb{R}$ be a real valued function defined on a set $\mathcal{S} \in \mathbb{R}^n$, then $a \in \mathbb{R}_0^+$ is an upper bound of $f(x)$ in $\mathcal{S}$ if $f(x) < a$ for all $x \in \mathcal{S}$ and the supremum of $f(x)$ in $\mathcal{S}$, $\sup_{x\in\mathcal{S}} |f(x)|$, is the least upper bound of $f(x)$ in $\mathcal{S}$. The L_∞-norm of $f(x)$ is denoted by $\|f\|_\infty$ and defined as $\|f\|_\infty = \sup_{x\in\mathcal{S}} |f(x)|$. In a measure space, a is said to be an essential upper bound of $f(x)$ in $\mathcal{S}$ if it holds that $|f(x)| < a$ for all $x \in \mathcal{S}$ except for a set of measure zero. The essential supremum of f, $\operatorname{ess\,sup}_{x\in\mathcal{S}} |f|$, is then given by the least essential upper bound. In the case of a vector valued function $f : \mathcal{S} \to \mathbb{R}^n$ by $f(x) = (f_1(x), \cdots, f_n(x))$ where $f_i(x)$ are real valued functions, we define the essential supremum as $\operatorname{ess\,sup}_{x\in\mathcal{S}} \|f\| = \operatorname{ess\,sup}_{i\in\{1,\cdots,n\}} (\operatorname{ess\,sup}_{x\in\mathcal{S}} |f_i|)$, we define the supremum and infinity norm of f similarly.

A function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is of class $\mathcal{K}$ if α is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, it is of class $\mathcal{K}_\infty$. A function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is of class $\mathcal{KL}$ if, for fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$ and $\beta(r, \cdot)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

A diffeomorphism is a continuously differentiable map with a continuously differentiable inverse. A function $f : \mathcal{D} \to \mathbb{R}^n$ defined in $\mathcal{D} \subseteq R^n$ is said to be Lipschitz continuous if there exists some constant $L \in \mathbb{R}^+$, named Lipschitz constant, such that for all $x_1, x_2 \in \mathcal{D}$ the bound $\|f(x_1) - f(x_2\| \leq L \|x_1 - x_2\|$ holds. Said function f is called locally Lipschitz continuous if for every $x \in \mathcal{D}$ there exists a neighborhood $\mathcal{D}_0 \in \mathcal{D}$ of x such that f restricted to $\mathcal{D}_0$ is Lipschitz continuous.

1.1.2 Stability

We use the usual definitions of practical, asymptotic, and exponential stability of equilibrium points of dynamical systems, and of input-to-state (ISS) stability of systems as given in Chapter 4 of [23]. Specifically, consider a continuous-time dynamical system of the form:

$$\dot{x} = f(x), \tag{1.1.1}$$

where $x \in \mathbb{R}^n$ is the state and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function. The origin of (1.1.1) is:

- *asymptotically stable* if there exists $\beta \in \mathcal{KL}$ such that:

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall t \geq t_0 \geq 0.$$

- *practically stable* if there exists $\beta \in \mathcal{KL}$ and $c \in \mathbb{R}^+$ such that:

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + c, \quad \forall t \geq t_0 \geq 0.$$

- *exponentially stable* if there exists c, k and $\gamma \in \mathbb{R}^+$ such that:

$$\|x(t)\| \leq k \, \|x(t_0)\| \exp^{-\gamma(t-t_0)}.$$

Consider a continuous-time dynamical system of the form:

$$\dot{x} = f(x, u), \tag{1.1.2}$$

where $x \in \mathbb{R}^n$ is the state, $u(t)$ is a piecewise continuous function and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a piecewise continuous function. System (1.1.2) is said to be input-to-state stable (ISS) if there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all time $t \geq t_0$ and satisfies:

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma(\operatorname*{ess\,sup}_{t_0 \leq \tau \leq t} \|u(\tau)\|).$$

Consider the system:

$$\dot{x} = f(x, u), \tag{1.1.3}$$

where $x \in \mathbb{R}^n$ is the state, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a smooth function and $u \in \mathbb{R}^m$ is an input signal produced by some feedback law satisfying $\dot{v} = \phi(x, v, k),\ u = \chi(v, x, k)$. The system (1.1.3) is said to be rendered semi-globally asymptotically stable if for any given compact set $\mathcal{S}$, controller parameters k can be chosen so that the origin is rendered asymptotically stable for all the solutions starting on $\mathcal{S}$. The same applies, mutatis mutandis, to semi-global practical and exponential stability.

1.1.3 Feedback linearization

We follow the definitions of feedback linearization provided in Chapter 13 of [23], reproduced below for completeness.

A nonlinear system:

$$\dot{x} = f(x) + g(x)u \tag{1.1.4}$$

where $x \in \mathcal{D}$ and $u \in \mathbb{R}$ are the state and input, and $f : \mathcal{D} \to \mathbb{R}^n$ and $g : \mathcal{D} \to \mathbb{R}^n$ are smooth in a domain $\mathcal{D} \subseteq \mathbb{R}^n$, is said to be feedback linearizable if there exists a diffeomorphism $T : \mathcal{D} \to \mathbb{R}^n$ such that $\mathcal{D}_z = T(\mathcal{D})$ contains the origin and the change of variables $z = T(x)$ transforms the system (1.1.4) into the form:

$$\dot{z} = Az + B\left(\alpha(x) + \beta(x)u\right), \tag{1.1.5}$$

with (A, B) controllable, and $\alpha(x)$ and $\beta(x)$ continuous functions with $\beta(x)$ non-zero for all x in $\mathcal{D}$.

Given that one does not always have access to the full state of a system, of particular interest is the input-output case of feedback linearization. Before presenting that case we introduce the concept of relative degree. Consider the single-input single-output system:

$$\dot{x} = f(x) + g(x)u, \quad y = h(x), \tag{1.1.6}$$

where f, g, and h are smooth in a domain $\mathcal{D} \subseteq \mathbb{R}^n$. The system is said to have relative degree ρ, $1 \leq \rho \leq n$, in a region $\mathcal{D}_0 \subseteq \mathcal{D}$ if the Lie derivatives of h along f and g satisfy:

$$L_g L_f^{i-1} h(x) = 0, \ i = 1, 2, \ldots, \rho - 1; \quad L_g L_f^{\rho-1} h(x) \neq 0,$$

for all $x \in D_0$.

With this definition at hand, we introduce Theorem 13.1 from [23], that provides a practical way of verifying if a system of the form (1.1.6) is feedback linearizable. Consider the system (1.1.6) and assume it has relative degree $\rho \leq n$. If $\rho = n$, then for every $x_0 \in \mathcal{D}$, a neighbourhood $\mathcal{N}$ of x_0 exists such that the map:

$$T(x) = (h(x), L_f h(x), \cdots, L_f^{n-1} h(x)),$$

restricted to $\mathcal{N}$, is a diffeomorphism on $\mathcal{N}$.

This implies that a system of the form (1.1.4) is feedback linearizable if a function $h(x)$ exists such that under the output $y = h(x)$ the system has relative degree n. It is shown in Chapter 13.3 of [23] that this is also a necessary condition. Note that in this case the functions $\alpha(x)$ and $\beta(x)$ in (1.1.5) are given by $\alpha(x) = L_f^n h(x)$ and $\beta(x) = L_g L_f^{n-1} h(x)$.

If $\rho < n$, then we recover the partial feedback linearization case. Specifically, if $\rho < n$, then for every x_0 in $\mathcal{D}$, a neighbourhood $\mathcal{N}$ of x_0 and smooth functions $\phi_1(x), \cdots, \phi_{n-\rho}(x)$ exist such that:

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \text{ for } 1 \leq i \leq n - \rho, \forall x \in \mathcal{N},$$

and the map:

$$T(x) = (h(x), L_f h(x), \cdots, L_f^{\rho-1} h(x), \phi_1(x), \cdots, \phi_{n-\rho}(x)),$$

restricted to $\mathcal{N}$, is a diffeomorphism on $\mathcal{N}$. In this case the dynamics of the first ρ elements are called the output dynamics while the dynamics of the last $n - \rho$ elements are named the zero dynamics.

The existence of a function h such that system (1.1.6) has relative degree n can be characterized by necessary and sufficient conditions on the vector fields f and g, first presented by Krener in [24]. Before introducing these conditions we need to define the notions of Lie brackets and invariant distributions. Given two vector fields f and g on $\mathcal{D} \subseteq \mathbb{R}^n$, the Lie bracket $[f, g]$ is a third vector field defined by:

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x),$$

where $\frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial x}$ are Jacobian matrices. One may repeat the bracketing of g with f, we define:

$$ad_f^0 g(x) = g(x), \quad ad_f^1 g(x) = [f, g](x), \quad ad_f^i g(x) = [f, ad_f^{i-1} g(x)](x).$$

For vector fields $f_1,\ f_2,\ \cdots, f_k$ on $\mathcal{D} \subseteq \mathbb{R}^n$, let $\Delta(x) = \text{span}\{f_1(x),\ f_2(x),\ \cdots, f_k(x)\}$ be the subspace of $\mathbb{R}^n$ spanned by the vectors $f_1(x),\ f_2(x),\ \cdots, f_k(x)$ at any fixed $x \in \mathcal{D}$. The collection of all vector spaces $\Delta(x)$ for $x \in \mathcal{D}$ is called a distribution and referred to by:

$$\Delta = \text{span}\{f_1,\ f_2,\ \cdots, f_k\}.$$

The dimension of $\Delta(x)$, defined by $\dim(\Delta(x)) = \text{rank}[f_1(x),\ f_2(x),\ \cdots, f_k(x)]$, may vary with x. If the vector fields $\{f_1(x),\ f_2(x),\ \cdots, f_k(x)\}$ are linearly independent for all $x \in \mathcal{D}$, then $\dim(\Delta(x)) = k$ for all $x \in \mathcal{D}$ and the distribution Δ is said to be a nonsingular distribution in $\mathcal{D}$, generated by $f_1,\ f_2,\ \cdots, f_k$. A distribution is involutive if $f_1 \in \Delta$ and $f_2 \in \Delta$ implies $[f_1, f_2] \in \Delta$. If Δ is a nonsingular distribution generated by $f_1,\ f_2,\ \cdots, f_k$, then Δ is involutive if and only if $[f_i, f_j] \in \Delta$ for all $i, j \in \{1, 2, \cdots, k\}$.

1. the matrix $\mathcal{G}(x) = [g(x), ad_f^1 g(x), \cdots, ad_f^{n-1} g(x)]$ has rank n for all $x \in \mathcal{D}_0$;

2. the distribution $\Delta = \text{span}\{g(x), ad_f^1 g(x), \cdots, ad_f^{n-2} g(x)\}$ is involutive in $\mathcal{D}_0$.

The definitions in this subsection can be extended to the multiple-input multiple-output case, for which we define the concept of vector relative degree. Consider the multiple-input multiple-output system:

$$\dot{x} = f(x) + g(x)u, \quad y = h(x), \tag{1.1.7}$$

where $x \in \mathcal{D}$, $u(t) \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$ are the state, inputs and outputs of the system, $f : \mathcal{D} \to \mathbb{R}^n$, $g : \mathcal{D} \to \mathbb{R}^n \times \mathbb{R}^m$ and $h : \mathcal{D} \to \mathbb{R}^m$ are smooth in a domain $\mathcal{D} \subseteq \mathbb{R}^n$. Let g_i and h_i for $i = \{1, 2, \cdots, m\}$ be the i^{th} column of $g(x)$ and entry of $h(x)$, respectively, then the multivariate nonlinear system (1.1.7) has a vector relative degree $\{\rho_1, \rho_2, \cdots, \rho_m\}$ in a region $\mathcal{D}_0 \subseteq \mathcal{D}$ if:

1. the Lie derivatives of h along f and g satisfy $L_{g_j} L_f^k h_i(x) = 0$, for all $1 \leq j, i \leq m$, for all $k \leq \rho_i - 1$ and for all $x \in \mathcal{D}_0$.

2. the matrix $A \in \mathbb{R}^{m \times m}$:

$$A(x) \begin{bmatrix} L_{g_1} L_f^{\rho_1 - 1} h_1(x) & \cdots & L_{g_m} L_f^{\rho_1 - 1} h_1(x) \\ L_{g_1} L_f^{\rho_2 - 1} h_2(x) & \cdots & L_{g_m} L_f^{\rho_2 - 1} h_2(x) \\ \cdots & \cdots & \cdots \\ L_{g_1} L_f^{\rho_m - 1} h_m(x) & \cdots & L_{g_m} L_f^{\rho_m - 1} h_m(x) \end{bmatrix},$$

is nonsingular for all $x \in \mathcal{D}_0$.

We refer the reader to in Lemma 5.2.1 in [25] for the following result. Consider the system:

$$\dot{x} = f(x) + g(x)u \tag{1.1.8}$$

where $x \in \mathcal{D}$ and $u(t) \in \mathbb{R}^m$ are the state and inputs of the system, $f : \mathcal{D} \to \mathbb{R}^n$ and $g : \mathcal{D} \to \mathbb{R}^n \times \mathbb{R}^m$ are smooth in a domain $\mathcal{D} \subseteq \mathbb{R}^n$. Assume the matrix $g(x)$ has rank m, then the system is feedback linearizable in a region $\mathcal{D}_0 \subseteq \mathcal{D}$ if and only if there exists m real-valued functions $h_1(x), \cdots, h_m(x)$, defined on $\mathcal{D}_0$ such that the resulting system of the form (1.1.7) has some vector relative degree $\{\rho_1, \cdots, \rho_m\}$ in $\mathcal{D}_0$ and $\sum_{i=1}^m r_i = m$.

The textbook [25] presents an in-depth analysis of SISO and MIMO feedback linearization that the interested reader might find appealing.

1.1.4 Big O notation

Consider a function $f : \mathbb{R}_0^+ \times \mathcal{Q} \to \mathbb{R}^n$ with $\mathcal{Q} \subseteq \mathbb{R}^n$. We use the notation $f(t,x) = O_x(T)$ to denote the existence of constants $M, T \in \mathbb{R}^+$ so that for all $t \in [0,T]$ and $x \in \mathcal{Q}$ we have $\|f(t,x)\| \leq MT\|x\|$ with $\|x\|$ denoting the 2-norm of x. Going forward we only consider $T \leq 1$, thus the following rules apply to this notation where the equalities below are to be used to replace the left-hand side with the right-hand side:

$$O_x(T^2) = O_x(T), \quad (O_x(T))^2 = O_{x^2}(T^2), \quad TO_x(T) = O_x(T^2), \quad g(x)O_x(T) = O_x(T).$$

The subscript x^2 in $O_{x^2}(T^2)$ indicates we are squaring the norm, i.e., $O_{x^2}(T^2)$ denotes the upper bound $MT^2\|x\|^2$. Moreover, the function g is assumed to have bounded norm, i.e., there exists $b \in \mathbb{R}^+$ so that $\|g(x)\| \leq b$ for all $x \in \mathcal{Q}$. To illustrate the use of these equalities, consider the equality $f(t,x) = O_x(T^2)$ which is defined by $\|f(t,x)\| \leq MT^2\|x\|$. Given that we chose $T \leq 1$, we have the bound $T^2 \leq T$ that enables us to conclude $\|f(t,x)\| \leq MT\|x\|$, i.e., $f(t,x) = O_x(T)$. Using the above rules we can directly replace $f(t,x) = O_x(T^2)$ with $f(t,x) = O_x(T)$.

1.1.5 Persistency of excitation

There are several equivalent definitions of persistency of excitation, see [26–28], we refer to those provided in the latter, with the discrete version provided in [29], specifically:

1. The piecewise-continuous uniformly bounded function $x : \mathbb{R} \to \mathbb{R}^{1\times n}$ is said to be persistently exciting (of period T) if there exist t_0, T and $a \in \mathbb{R}^+$ such that the following inequality holds uniformly in t_0:

$$\int_{t_0}^{t_0+T} x(\tau)^T x(\tau) d\tau \geq aI_n.$$

2. A sequence $x(t) \in \mathbb{R}^{1\times n}$ is said to be persistently exciting (in N steps), if there exist t_0, N and $a \in \mathbb{R}^+$ such that for all $t \geq t_0$ the following inequality holds:

$$\sum_{t=t_0+1}^{t_0+N} x(t)^T x(t) \geq aI_n.$$

Persistency of excitation is concept that has been present in the adaptive control literature since the 1960s, generally appearing as a requirement on the input signal when performing parameter estimation. It was noted in [30] that "When persistency of excitation is lost in adaptive systems, the system variables from time to time exhibit bursts of oscillatory behaviour". The signal or sequence that must satisfy the persistency of excitation requirement, and in how many steps it must do so to be sufficiently exciting, depends on the adaptive system in question. Despite this, it is generally the case that one can translate the requirement onto the input sequence or onto a sequence of vectors composed by consecutive inputs as is the case in the behavioural setting, see [31]. Informally, one can interpret this as the requirement that the input signal is rich enough such that all the modes of the plant it is applied to are excited. For an in depth analysis of the need for persistency of excitation in adaptive control see Chapters 6 and 8 in [28].

CHAPTER 2

Data-driven stabilization of unknown single-input single-output feedback-linearizable Systems

2.1 Introduction

2.1.1 Motivation

The work presented in this chapter was motivated by two initially independent lines of inquiry: the thought-provoking work of Fliess and Join on intelligent PID controllers [3, 4], and the growing impact of machine learning, in particular deep learning, on a wide variety of engineering problems [14, 32]. Curiously, the techniques of Fliess and Join can be seen as a method to transform sensor measurement data into control inputs with minimal reliance on plant models. Therefore, we can interpret intelligent PID controllers as data-driven[1] controllers and this is the view espoused in this work.

---

[1] The term *model-free* is sometimes used in lieu of *data-driven*. However, we know from behavioral systems theory that data generated by interacting with a system, i.e., its behavior, is essentially a model for such system. Therefore, we find the term data-driven more adequate as it only suggests that state-space models are not explicitly used.

2.1.2 Contribution

The main contribution of this chapter is the identification of a class[2] of nonlinear systems for which a modified version of intelligent PID controllers can guarantee asymptotic stability. This is by no means the largest class of such systems, but a large enough class to make the technical contribution of this chapter relevant to applications, as illustrated by the experimental results presented in Section 2.8. Moreover, the techniques used to prove the results are also of interest as they rely on an apparently unrelated line of work by Nesic and co-workers [18,19] on state estimation and control based on approximate models. In particular, this chapter shows how the results in [18, 19] can be used to provide a formal justification for the working assumption upon which the analysis of Fliess and Join [3,4] relies: *the sampling rate can be made high enough so that the relevant signals can be considered constant in between sampling instants.*

Although the use of learning techniques has been surging[3] within the control community, learning has always been an integral part of the scientific discipline of control. Classical bodies of work within control, such as a system identification [33,34] and adaptive control [28, 35], are essentially learning techniques tailored to the needs of control. The results in this chapter make connections with, and sometimes have been inspired by, such classical results. Several of these connections will be exposed throughout the chapter although readers with a different background may see other connections that have eluded the author. Yet, it matters to highlight the advantages of the results in this chapter over other learning techniques for control. First, the proposed data-driven controllers require neither large amounts of data nor lengthy offline or online training. In this sense, they are much closer to adaptive control than to techniques based on reinforcement learning [36] or deep learning [37]. However, contrary

[2]Essentially single-input single-output feedback-linearizable systems, see Section 2.7 for a formal statement of the main results. Note, however, that the results conceptually extend to multiple-input, multiple-output systems and even to slowly time varying systems.

[3]As revealed, e.g., by a search using the keywords "data-driven" and "control".

to most work on adaptive control that relies on linearly parameterized models (for the plant or controller), the proposed data-driven controllers do not attempt to learn parameters and, instead, directly learn the input to be fed to the plant. Hence, one always works on small finite-dimensional spaces and, for this reason, only needs small amounts of data. A further advantage of the proposed data-driven controllers is that its users only need, yet are not restricted, to employ linear control techniques, an observation that justifies the well crafted title of [3]. Finally, the results in this chapter should be regarded as a design methodology since its key steps can be performed by resorting to different techniques. To show feasibility of the approach, and ease of use, I propose a specific technique for each step although it should be clear these are by no means unique or even the best. We shall return to this point in more detail in Section 2.4 where I provide an outline of the proposed data-driven control methodology. It is worth mentioning that the presented methodology results in asymptotically stable behavior without resorting to persistency of excitation assumptions. This is a key contribution, setting us apart from most adaptive control techniques, since it is often hard to justify or validate persistency of excitation in practical applications.

We would be remiss if we did not give due importance to the limitations of the proposed data-driven control methodology: it can be quite sensitive to measurement noise. This is a consequence of the need to estimate derivatives of sensed signals. While we leave a detailed study of how to best handle noise for future work[4], the experimental results in Section 2.8 already offer evidence that the proposed data-driven methodology can be practically useful despite the aforementioned limitation.

2.1.3 Related work

As previously stated, the results in this chapter were directly inspired by the work of Fliess and Join on intelligent PID controllers. I regard the papers [3, 4] as entry points into this

[4]See Chapter 3 for an incipient step in that direction.

literature since the number of papers on this topic has been growing over the last ten years. The main contributions with respect to this line of work are: 1) to rigorously formalize the idea that signals can be treated as constant in between sampling times provided the sampling rate is high enough; 2) to identify a class of nonlinear systems for which this type of data-driven controllers is guaranteed to result in asymptotically stable behavior. This was accomplished by: 1) proposing several modifications to intelligent PID controllers; 2) a feedback linearizability assumption; and 3) leveraging the work of Nesic and co-workers on estimation and control based on approximate models. Moreover, I also address the case where the control gain is unknown whereas it is assumed to be known in the intelligent PID literature. Although the focus is on the simple case of single-input single-output systems, the attentive reader will notice the results can be generalized to multiple-input multiple-output, and partially feedback-linearizable systems. We discuss such extensions in Section 2.7 and later in Chapter 3.

Two recent papers [38, 39], inspired by behavioral techniques, have also proposed data-driven control techniques. It is shown, in both cases, that the proposed controllers can be used with nonlinear systems even though they were developed for linear systems. The key requirement is that the mismatch between the linear and nonlinear models is small. A similar idea is used in this chapter: by choosing a suitably high sampling rate, a point-wise linear approximation suffices for control. For this reason the author suspects it may be possible to combine these different perspectives to obtain even stronger results. The use of behavioral techniques for the development of data-driven control techniques is not recent and had been advocated before, see [8, 40]. However the algorithms proposed in this earlier work are better suited for offline computation as they require several complex matrix operations. All the aforementioned papers, as well as [41], rely on acquiring enough sufficiently informative data to produce control inputs (see [42] for a discussion on how much informative data is required for different control tasks). This requires that enough experiments are conducted using persistently exciting inputs. In contrast, no prior data or persistency of excitation is required for the results herein presented.

The previous observation sets the current chapter apart from much work on data-driven control as well as other work that, although was not developed under the recent data-driven perspective, can be interpreted as such. One such example is the use of extremum seeking ideas, originally developed for optimization purposes, for stabilization, see [12]. Extremum seeking relies on persistent high-frequency perturbations to estimate gradients and for this reason it is only possible to establish practical stability with this technique, even in the absence of noise. In this line of work, persistency of excitation is typically not stated as an assumption since it is enforced by incorporating high-frequency signals into the input.

Another example is the control of nonlinear systems using Euler approximations that are learned in real-time, see [43]. This line of work bears some similarities with the approach described in this chapter. A key difference is that while in [43] an approximate plant model is learned, in this chapter we directly learn the input to be applied to the plant, which provides the benefit of not requiring knowledge of upper and lower bounds on the control gain. Furthermore, the results in [43] rely again on persistency of excitation which is enforced by design and, for this reason, cannot guarantee asymptotic stability but rather practical stability.

The attentive reader might also find some similarities between the approach presented in this chapter and Khalil's work on extended high-gain observers and feedback control via disturbance compensation [44]. On the one hand, both of these approaches seek to guarantee the observer's and controller's dynamics are sufficiently fast relative to the plant's dynamics. On the other hand, this objective is achieved in very different ways. While in [44] the key technical idea is the use of high gains to "speed up" the controller's dynamics with respect to the plant's, the proposed data-driven controllers "slow down" the plant's dynamics through high-frequency sampling. By dispensing with the need for high gains, our data-driven approach becomes exempt from the peaking phenomenon, thereby not requiring saturation of the input or state estimates.

Preliminary versions of the results in this chapter appeared in the conference publications [1, 45]. While in [1] the control gain is assumed to be known this assumption was dropped in [45]. However, the results in [45] rely on a persistency of excitation assumption that, as previously mentioned, is difficult to verify in practice. In this chapter we assume neither the control gain to be known (although we assume knowledge of its sign) nor persistency of excitation.

2.2 Models

We consider an unknown single-input single-output nonlinear system described by:

$$\dot{x} = f(x) + g(x)u \tag{2.2.1}$$

$$y = h(x) + d, \tag{2.2.2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$, and $h : \mathbb{R}^n \to \mathbb{R}$ are smooth functions and we denote by $y \in \mathbb{R}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $d \in \mathbb{R}$, the output, state, input, and measurement noise, respectively. We make the assumption that this system has relative degree n, i.e., this system is feedback-linearizable. This implies that $L_g L_f^i h(x) = 0$ for $i = 0, \ldots, n-2$ and $L_g L_f^{n-1} h(x) \neq 0$ for all $x \in \mathbb{R}^n$. Since the function $L_g L_f^{n-1} h$ is continuous and never zero, its sign is constant. We assume the sign of $L_g L_f^{n-1} h$ to be known and, without loss of generality, take it to be positive. Knowledge of the sign of $L_g L_f^{n-1} h$ is not a strong assumption beyond $L_g L_f^{n-1} h \neq 0$. A simple input/output experiment can be performed to infer the sign of $L_g L_f^{n-1} h$. While requiring some knowledge of the control gain in the form of bounds is standard practice when handling unknown systems, such is the case in, e.g., [12] and [43], this approach is free of such assumption.

With the objective of presenting the results in its most understandable form, we assume $n = 2$ throughout this chapter, although all the results hold for arbitrary $n \in \mathbb{N}$. This enables us to perform all the necessary computations explicitly and without the need for distracting bookkeeping. To further reduce bookkeeping, we perform most of the analysis under the assumption of noise free measurements (i.e., $d = 0$), which leads to our main result, Theorem 2.7.1. Given that this assumption does not usually hold when working with physical systems, we also provide Theorem 2.7.3 which establishes stability guarantees under essentially bounded measurement noise.

Invoking the feedback linearizability assumption, we can rewrite the unknown dynamics in the coordinates $(z_1, z_2) = \Psi(x) = (h(x), L_f h(x))$:

$$\dot{z}_1 = z_2 \tag{2.2.3}$$

$$\dot{z}_2 = \alpha(z) + \beta(z)u \tag{2.2.4}$$

$$y = z_1, \tag{2.2.5}$$

where $\alpha = L_f^2 h \circ \Psi^{-1}$ and $\beta = L_g L_f h \circ \Psi^{-1}$. We note that f, g, and h are unknown and thus so are α and β. This form of the dynamics has the advantage of using the two scalar valued functions α and β to describe the full dynamics, independently of the value of n. This is a key observation that underlies the claim that the results below hold for arbitrary $n \in \mathbb{N}$.

System (2.2.3)-(2.2.5) will be controlled using piece-wise constant inputs for a sampling time $T \in \mathbb{R}^+$. This means that inputs $u : \mathbb{R}_0^+ \to \mathbb{R}$ satisfy the following equality for all $k \in \mathbb{N}$:

$$u(kT + \tau) = u(kT), \qquad \forall \tau \in [0, T[.$$

It will be convenient to use u to denote an input only defined on $[0, T[$. Since the curve u is constant on the interval $[0, T[$, we identify it with the corresponding element of $\mathbb{R}$.

The solution of (2.2.3)-(2.2.4) is denoted by $F_t^e(z, u) = (F_{t,1}^e(z, u), F_{t,2}^e(z, u))$, for $t \in [0, T[$, and satisfies $F_0^e(z, u) = z$. The superscript "e" reminds us that this is an exact solution. In the next section we discuss approximate solutions.

2.3 Approximate models

In this section we develop an approximate solution of (2.2.3)-(2.2.4) based on the well known Taylor's theorem that we now recall.

Theorem 2.3.1 (See [46]). *Let $c : \mathcal{I} \to \mathbb{R}^n$ be an n times differentiable function where $\mathcal{I} \subseteq \mathbb{R}$ is an open and connected set. For any $t, \tau \in \mathcal{I}$ such that $\tau + t \in \mathcal{I}$ we have:*

$$c(\tau + t) = c(\tau) + c^{(1)}(\tau)t + c^{(2)}(\tau)\frac{t^2}{2} + \ldots + c^{(n-1)}(\tau)\frac{t^{n-1}}{(n-1)!} + c^{(n)}(\tau')\frac{t^n}{n!}, \tag{2.3.1}$$

for some $\tau' \in [\tau, \tau + t]$.

Applying this result to $F^e_{\tau+t,1}$ we obtain:

$$F^e_{\tau+t,1}(z,u) = F^e_{\tau,1}(z,u) + \left(F^e_{\tau,1}\right)^{(1)}(z,u)t + \left(F^e_{\tau,1}\right)^{(2)}(z,u)\frac{t^2}{2} + \left(F^e_{\tau',1}\right)^{(3)}(z,u)\frac{t^3}{3!}.$$

If we only retain the first three terms we obtain an approximate solution with an approximation error given by the magnitude of the (neglected) fourth term. The following result provides a bound for the approximation error in a form useful for the results derived in this chapter.

Proposition 2.3.2. *Let $\mathcal{D} \subset \mathbb{R}^3$ be a compact set. Then, there exist $T \in \mathbb{R}^+$ and $M \in \mathbb{R}^+$ such that:*

$$\left\| \left(F^e_{\tau',1}\right)^{(3)}(z,u)\frac{t^3}{3!} \right\| \leq MT^3 \|(z, u - u_0)\|, \tag{2.3.2}$$

for all $(z,u) \in \mathcal{D}$, all $t, \tau' \in [0,T]$, and where $u_0 = -\beta^{-1}(0)\alpha(0)$.

Using the O notation, this result states that:

$$\left(F^e_{\tau',1}\right)^{(3)}(z,u)\frac{t^3}{3!} = O_{(z,u-u_0)}(T^3).$$

Proof. Since (2.2.3)-(2.2.4) is a smooth differential equation (recall that inputs are constant), solutions exist for all $\tau \in [0, T_{z,u}[$ where $[0, T_{z,u}[$ is the maximal interval for which the solution $F^e_{\tau,1}(z,u)$ exists. The function $(z,u) \mapsto T_{z,u}$ is lower semi-continuous and, given that (z,u)

belongs to the compact set $\mathcal{D}$, it achieves its minimum on $\mathcal{D}$. Let $T \in \mathbb{R}^+$ be smaller than $\min_{(z,u)\in\mathcal{D}} T_{z,u}$. By definition of T, for any $(z,u) \in \mathcal{D}$ solutions exist on the interval $[0,T]$. Consider now the function $\left(F^e_{\tau',1}\right)^{(3)}$ and note it is continuously differentiable, by assumption, and thus Lipschitz continuous on $\mathcal{D} \times \{\tau'\}$ for each fixed $\tau' \in [0,T]$. Hence, by definition of Lipschitz continuity we have:

$$\left\| \left(F^e_{\tau',1}\right)^{(3)}(z,u) - \left(F^e_{\tau',1}\right)^{(3)}(z',u') \right\| \leq L(\tau')\|(z,u) - (z',u')\| \tag{2.3.3}$$

for all $(z,u),(z',u') \in \mathcal{D}$ and all $\tau' \in [0,T]$. Noting that, according to (2.2.3)-(2.2.4), $F^e_{\tau'}(0,u_0) = 0$ for $u_0 = -\beta^{-1}(0)\alpha(0)$ and all $\tau' \in [0,T]$, we conclude that $\left(F^e_{\tau',1}\right)^{(3)}(0,u_0) = 0$. Using this equality in (2.3.3) we obtain:

$$\left\| \left(F^e_{\tau',1}\right)^{(3)}(z,u) \right\| \leq L(\tau')\|(z,u) - (0,u_0)\| = L(\tau')\|(z,u-u_0)\|,$$

by setting $z' = 0$ and $u' = -u_0$. If we now take $M = \frac{1}{3!}\max_{\tau'\in[0,T]} L(\tau')$ we obtain the desired inequality. Note that M is well defined since L is continuous and $[0,T]$ compact. □

Based on Proposition 2.3.2 we can write the exact solution F^e_t of (2.2.3)-(2.2.4) valid for all $t \in [0,T[$, as:

$$F^e_{t,1}(z,u) = z_1 + z_2 t + (\alpha(z) + \beta(z)u)\frac{t^2}{2} + O_{(z,u-u_0)}(T^3) \tag{2.3.4}$$

$$F^e_{t,2}(z,u) = z_2 + (\alpha(z) + \beta(z)u)t + O_{(z,u-u_0)}(T^2). \tag{2.3.5}$$

By setting[5] t equal to T, the previous model provides a *family* of discrete-time *approximate* models indexed by T:

$$z_1(k+1) = z_1(k) + z_2(k)T + (\alpha(k) + \beta(k)u(k))\frac{T^2}{2} \tag{2.3.6}$$

$$z_2(k+1) = z_2(k) + (\alpha(k) + \beta(k)u(k))T, \tag{2.3.7}$$

where $z(k), \alpha(k)$, and $\beta(k)$ denote the value of z, $\alpha(z)$, and $\beta(z)$ at time kT, $k \in \mathbb{N}$, respectively. For later use we introduce the notation:

$$F^a_{T,1}(z,u) \overset{def.}{=} z_1(k) + z_2(k)T + (\alpha(k) + \beta(k)u(k))\frac{T^2}{2}$$

[5] Although $t \in [0,T[$, solutions are not altered by changing the input on a zero measure set.

$$F^a_{T,2}(z,u) \quad \stackrel{def.}{=} \quad z_2(k) + (\alpha(k) + \beta(k)u(k))T,$$

where the superscript "a" emphasizes the fact that z is the solution of an approximate model.

2.4 A data-driven control design methodology

In this section we summarize the proposed data-driven control design methodology that is presented in detail in Sections 2.5 and Section 2.6. The design will be based on different approximate models, all of which are based on the discrete-time approximate model (2.3.6)-(2.3.7). We start by observing that the model (2.3.6)-(2.3.7) is affine and thus all the design techniques described in this chapter only require knowledge of linear systems theory.

The affine nature of the model (2.3.6)-(2.3.7) suggests that we could use the preliminary controller:

$$\overline{u}(k) = \beta^{-1}(z(k))(-\alpha(z(k)) + v(z(k)), \tag{2.4.1}$$

where $v(z)$ is a new input, to cancel the effect of the nonlinear functions α and β provided that $z(k)$ and the values of α and β at the current state $z(k)$ were known. After this preliminary controller, it would be easy to design a virtual controller stabilizing the resulting linear system with input v. As an example design technique, we show in Section 2.6 how to design linear controllers that perform this task.

By considering[6] α and β to be constant functions in (2.3.6)-(2.3.7) we obtain an observable linear system by formally treating $\alpha + \beta u$ as a new state z_3 and using the measurement equation $y = z_1$. Hence, any technique to reconstruct the state of an observable linear system can be employed provided the reconstruction error is of order T, as specified by equation (2.5.6) in Section 2.5. As an example design technique, in Section 2.5 we propose to reconstruct the state by directly solving the equation $Y = \mathcal{O}z$ where Y is a sequence

[6]Formally justifying this design assumption is one of the purposes of the results in Section 2.7.

of measurements and $\mathcal{O}$ is the observability matrix of the aforementioned observable linear system.

Once an estimate of z_3 is obtained, we formally treat z_3 as an observation. It is well known that reconstructing α and β from the measurement equation $z_3 = \alpha + \beta u$ is not possible unless a persistency of excitation assumption is placed on the input u. Rather than assuming persistency of excitation, we note this type of problem has been extensively studied in adaptive control [28, 35] and it is known that any choice of parameters α and β that satisfies the measurement equation $z_3 = \alpha + \beta u$ suffices for control purposes. Inspired by this, we directly utilize the observation z_3 in a dynamic controller generating inputs which asymptotically converge to those generated by our preliminary static controller (2.4.1).

Once the two aforementioned components – state estimator and static controller – have been designed to satisfy the relations (2.5.6) and (2.6.1), it follows from our main result, Theorem 2.7.1, that their concurrent execution, combined with the dynamic controller we provide, results in asymptotically stable behavior.

2.5 State estimation

For state estimation purposes it is convenient to formally treat $\alpha(k)+\beta(k)u(k)$, in the family of approximate models (2.3.6)-(2.3.7), as the state z_3 to obtain:

$$z_1(k+1) = z_1(k) + z_2(k)T + z_3(k)\frac{T^2}{2} \tag{2.5.1}$$

$$z_2(k+1) = z_2(k) + z_3(k)T \tag{2.5.2}$$

$$z_3(k+1) = z_3(k). \tag{2.5.3}$$

Note that this *approximate* model states that z_3 is constant although $\left(F^e_{t,1}\right)^{(2)}$ will, in general, not be so. Equality (2.5.3) follows from applying Proposition 2.3.2 to $\left(F^e_{t,1}\right)^{(2)}$ and dropping the error term $O_{(z,u-u_0)}(T)$. Since (2.5.1)-(2.5.3) is a linear model, it can be written in the form:

$$z(k+1) = Az(k), \qquad y(k) \stackrel{def.}{=} z_1(k) = Cz(k).$$

Moreover, it can be easily checked that A is invertible and we thus denote by $\mathcal{O}$ the observability matrix for the pair (A^{-1}, C) which allows us to write:

$$Y(k) \stackrel{def.}{=} \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-\rho+1) \end{bmatrix} = \mathcal{O}z(k), \tag{2.5.4}$$

where $\rho \in \mathbb{N}$, $\rho \geq n+1$, is the number of measurements that will be used for state estimation. The estimate $\widehat{z}(k)$ of the state vector $z(k)$ can then be obtained by solving this equation via least-squares:

$$\widehat{z}(k) = (\mathcal{O}^T\mathcal{O})^{-1}\mathcal{O}^T Y(k). \tag{2.5.5}$$

Given that equalities (2.5.1), (2.5.2), and (2.5.3) only hold up to $O_{(z,u-u_0)}(T^3)$, $O_{(z,u-u_0)}(T^2)$, and $O_{(z,u-u_0)}(T)$, respectively, we can easily establish the equality $z = \widehat{z} + O_{(z,u-u_0)}(T)$. If we introduce the estimation error e_z, defined by $e_z = z - \widehat{z}$, it follows that:

$$e_z = O_{(z,u-u_0)}(T). \tag{2.5.6}$$

It is straightforward to show that in the presence of essentially bounded noise on the measurements (2.5.4) the state estimation error is given by:

$$e_z = O_{(z,u-u_0)}(T) + O_{\overline{d}}(T^{-n}), \quad \overline{d} \stackrel{def.}{=} \operatorname*{ess\,sup}_{t \in \mathbb{R}_0^+} \|d(t)\|, \tag{2.5.7}$$

where n is the relative degree of the system.

As previously stated, the control scheme proposed in Section 2.6 only depends on the preceding equality. Hence, we can replace least-squares estimation with any other estimation technique leading to (2.5.6). In particular, the parameter ρ is not relevant to the theoretical analysis although it will play an important role in mitigating the effect of sensor noise: larger values of ρ "average out" the effect of noise.

Remark 2.5.1. *In [47] it is shown that the algebraic techniques proposed in [48], and used in [3, 4] to estimate derivatives of a measured signal, can be interpreted as estimating the state of the state-space linear model governing the signals* y *satisfying* $y^{(3)} = 0$. *If we denote the constructability Gramian of this linear model by* W_{cn} *and its state-transition matrix by* Φ, *the estimate is given by the well known expression (see (3.9), page 250, [49]):*

$$W_{cn}^{-1} \int_{t_0}^{t_1} \Phi^T(\tau, t_1) C^T y(\tau) d\tau.$$

Equality (2.5.5) *can be seen as the discrete-time analogue of this finite-time estimation technique.*

Remark 2.5.2. *The matrix* $(\mathcal{O}^T\mathcal{O})^{-1}\mathcal{O}^T$ *contains terms of the form* T^{-1} *on its second row and terms of the form* T^{-2} *on its third row. Hence, it can be conceptually understood as a linear high-gain observer with finite-time convergence and where* T *plays the role of the parameter* ε *used in [44]. Similarly to high-gain observers, the estimate provided by* (2.5.5) *can be very sensitive to measurement noise. This can be mitigated by using more samples for estimation so as to "average out" noise, i.e., by increasing* ρ. *Contrary to high-gain observers, however, we do not need to explicitly worry about the peaking phenomenon when computing the estimate since it is not computed recursively. As mentioned before,* (2.5.5) *could be replaced with a high-gain observer or even the more recent low-power high-gain observers [50]. Which specific estimation technique works better in practice, and in the context of the results in this chapter, is an important problem that we revisit in Chapter 3 where I motivate the use of dirty-derivative-based estimation.*

2.6 Controller design

If we assume the parameters α and β to be known, we can design a family of controllers (parameterized by T) for the family of approximate models (2.3.6)-(2.3.7) with the objective of asymptotically stabilizing the origin in the following specific sense: there exists a symmetric and positive definite matrix P_z and constants $\lambda_z, T_0 \in \mathbb{R}^+$ so that $V_z(z) = z^T P_z z$ satisfies:

$$V_z(F_T^a(z,\overline{u})) - V_z(z) \leq \; -\lambda_z T\|z(k)\|^2 + O_{(z,u-u_0)^2}(T^2), \tag{2.6.1}$$

for all T in the interval $[0, T_0]$. Strikingly, we can achieve this inequality with the very simple family of virtual controllers which is independent of T:

$$\begin{aligned} \overline{u} &= \beta^{-1}(-\alpha + v(z)), & (2.6.2)\\ v(z) &= Kz, & (2.6.3) \end{aligned}$$

where K is a suitable matrix. We note that the approximate model (2.3.6)-(2.3.7) can be written as:

$$F_T^a(z(k),\overline{u}) = Az(k) + B\alpha(k) + B\beta(k)\overline{u}(k) = Az(k) + Bv(z(k)), \tag{2.6.4}$$

where the matrices A and B are of the form:

$$A = I + A_1 T, \qquad B = B_1 T + B_2 T^2.$$

Since (A_1, B_1) is a controllable pair, there exists a controller $v(z) = Kz$ and a symmetric and positive definite matrix P_z so that:

$$(A_1 + B_1 K)^T P_z + P_z(A_1 + B_1 K) = -Q, \tag{2.6.5}$$

for some symmetric and positive definite matrix Q. Any choice of K such that the matrix $(A + BK)$ is Hurwitz ensures (2.6.5), and thus (2.6.1), are satisfied, as shown below. Using this controller we have:

$$F_T^a(z(k),\overline{u}) = (A + BK)z = (I + (A_1 + B_1 K)T + B_2 K T^2)z.$$

Computing $V_z(F_T^a(z(k),\overline{u})) - V_z(z)$ provides:

$$\begin{aligned} V_z(F_T^a(z(k),\overline{u})) - V_z(z) &= z^T(A+BK)^T P_z (A+BK)z - z^T P_z z\\ &= z^T((A_1+B_1K)T)^T P_z z + z^T P_z((A_1+B_1K)T)z \end{aligned}$$

$$
\begin{aligned}
&+ O_{z^2}(T^2) + O_{z^2}(T^3) + O_{z^2}(T^4) \\
&= -Tz^TQz + O_{z^2}(T^2) \\
&\leq -\lambda_{\min}(Q)T\|z\|^2 + O_{(z,u-u_0)^2}(T^2),
\end{aligned}
$$

which is the desired inequality (2.6.1).

The dynamics in (2.6.4) are stated for the preliminary control law $\overline{u} = \beta^{-1}(-\alpha + v(z))$, yet since neither α nor β are known, this controller cannot be directly implemented. Instead, we note that this controller enforces $\alpha + \beta u = v(z)$ and design a dynamic controller that asymptotically enforces this equality by guaranteeing convergence to the origin of the error:

$$
e_u(k) = v(z(k)) - (\alpha(k) + \beta(k)u(k)) \,. \tag{2.6.6}
$$

To achieve this we propose a dynamic control law of the following form:

$$
u(k+1) = u(k) + \gamma\left(v(\widehat{z}(k)) - \widehat{z}_3(k)\right), \tag{2.6.7}
$$

where $\gamma \in \mathbb{R}^+$ is sufficiently small[7]. Note that this controller does not enforce persistency of excitation, nor does it require estimating or parameterizing the unknown functions α and β as is often the case in adaptive control, see [51].

In order to fully specify this controller, we need to describe its operation during the initial transient of $\rho - 1$ steps during which enough measurements are collected to produce the first state estimate according to (2.5.5). We simply choose a fixed sequence of inputs $u_0^*, u_1^*, \ldots, u_{\rho-2}^*$ to be used during this transient. Although different sequences lead to different transients, the results in Section 2.7 are independent of this choice.

The main results in the next section explain why such a dynamic controller works despite being designed for an approximate model while assuming knowledge of the exact values of the parameters and states in its design.

[7]If an upper bound $\overline{\beta}$ for β is known, $\gamma < \overline{\beta}^{-1}$ suffices. This is shown in the proof of Theorem 2.7.1, specifically in equation (2.9.11) and the discussion proceeding it.

2.7 Main results

2.7.1 The noise-free scenario

It is pedagogically convenient to start with the noise-free scenario, i.e., $d = 0$ in (2.2.2), as it allows us to expose the key ideas in a simpler manner. Notwithstanding the absence of noise, the proofs of the main results in this section are quite long and for this reason can be found in the Appendix. The author hopes its length does not hide the simple idea upon which it rests: we can formally justify the use of approximate models for observer and controller design by using the frameworks developed by Arcak and Nesic in [18] for the former, and by Nesic and Teel in [19] for the latter. This combination of ingredients shows that for any compact set of initial conditions there exists a sufficiently small sampling time ensuring the proposed controller keeps all the signals bounded and drives the state to the origin.

Theorem 2.7.1. *Consider an unknown nonlinear system of the form* (2.2.1)-(2.2.2) *where the output function h has relative degree n. In the absence of measurement noise, i.e., $d = 0$, for any compact set $\mathcal{S} \subset \mathbb{R}^n$ of initial conditions containing the origin in its interior there exists a time $T^* \in \mathbb{R}^+$ and a constant $b \in \mathbb{R}^+$ (both depending on $\mathcal{S}$) so that for any sampling time $T \in [0, T^*]$, the dynamic controller* (2.6.7)*, where the virtual input v is provided by* (2.6.3)*, using the state estimates provided by an estimation technique satisfying* (2.5.6)*, renders the closed-loop trajectories bounded, i.e., $\|\widehat{z}(k)\| \leq b$ and $\|e_u\| \leq b$ for all $k \in \mathbb{N}$, and $\|x(t)\| \leq b$ for all $t \in \mathbb{R}_0^+$. Moreover:*

$$\lim_{t \to \infty} x(t) = 0.$$

Although the previous result only claims that trajectories converge to the origin, it can be readily applied to trajectory tracking problems by considering convergence to zero of the error between the real trajectory and the trajectory to be tracked.

Extending these results to MIMO control systems is conceptually simple, with the caveat that β and γ (now matrices) must be chosen so that the eigenvalues of $I - \beta\gamma$ reside in the unit circle, ensuring convergence of the error e_u. An extension to partially feedback-linearizable systems is also possible by assuming a well behaved zero dynamics, a continuous-time version is presented in Chapter 4.

We now introduce the following lemma which provides a sufficient condition for the results of Theorem 2.7.1 to hold under a virtual controller v different from the one provided in (2.6.3):

Lemma 2.7.2. *Let the virtual input $v : \mathbb{R}^n \to \mathbb{R}$ be such that the following conditions hold:*

$$V_z\left(F^a(z(k), \beta^{-1}(k)\left(\alpha(k) + v(z(k))\right))\right) - V_z(z(k)) \leq -\lambda T\|z\|^2 + O_{(z,u-u_0)^2}(T^2) \quad (2.7.1)$$

$$v(z(k) + O_{(z,u-u_0)}(T)) = v(z(k)) + O_{(z,u-u_0)}(T), \quad (2.7.2)$$

where V_z is defined in section 2.6, then the results of Theorem 2.7.1 remain unchanged when using such virtual input in place of the one provided by equation (2.6.3).

2.7.2 The noisy scenario

As previously mentioned, in the presence of essentially bounded measurement noise, the state estimation error under the state estimation technique described in Section 2.5 is now given by:

$$e_z = O_{(z,u-u_0)}(T) + O_{\overline{d}}(T^{-n}), \quad \overline{d} \overset{def.}{=} \operatorname*{ess\,sup}_{t\in\mathbb{R}_0^+} \|d(t)\|,$$

where $\overline{d}$ is the noise bound and n is the relative degree of the system. This expression shines light on the trade-off between choosing a small sampling time to render the approximate models adequate and choosing a large sampling time to reduce the amplification effect on noise. As with the noise-free case, the proof of the following result can be found in the Appendix.

Theorem 2.7.3. *Consider an unknown nonlinear system of the form* (2.2.1)-(2.2.2) *where the output function h has relative degree n and assume the noise d to be essentially bounded, i.e., there exists a constant $\overline{d} \in \mathbb{R}_0^+$ satisfying $\overline{d} = \operatorname{ess\,sup}_{t\in\mathbb{R}_0^+} \|d(t)\|$. For any compact* *set* $\mathcal{S} \subset \mathbb{R}^n$ *of initial conditions containing the origin in its interior there exists a time* $T^* \in \mathbb{R}^+$ *(depending on* $\mathcal{S}$*), and constants* $b_1, b_2, b_3 \in \mathbb{R}^+$ *(depending on* $\mathcal{S}$ *and* T^**) so that for any sampling time* $T \in [0, T^*]$*, if* $\overline{d} \leq b_1$*, the dynamic controller* (2.6.7)*, where the virtual input* v *is provided by* (2.6.3)*, using the state estimates provided by an estimation technique satisfying* (2.5.6)*, renders the closed-loop trajectories bounded, i.e.,* $\|\widehat{z}(k)\| \leq b_2$ *and* $\|e_u\| \leq b_2$ *for all* $k \in \mathbb{N}$*, and* $\|x(t)\| \leq b_2$ *for all* $t \in \mathbb{R}_0^+$*. Moreover:*

$$\limsup_{t\to\infty} \|x(t)\| \leq b_3 \, \overline{d} \, T^{-n}$$

2.8 Experimental evaluation

In this section we report on an experimental evaluation of the proposed data-driven controller to regulate the altitude of a quad-copter. The experiments were performed on a Bitcraze Crazyflie 2.1 and an Optitrack Prime 17W motion capture system was used to measure the quad-copter's altitude during the experiments. An experimental demonstration of the robustness of the proposed data-driven controller is available in the video: `https://www.youtube.com/watch?v=9EVcRvLOGVo`.

2.8.1 Experimental setup

The Crazyflie 2.1 is a small open source modular quad-copter designed by Bitcraze AB [52] equipped with an IMU based on a 3-axis accelerometer and gyroscope. The baseline firmware for the Crazyflie includes a PID based flight controller. We partitioned this controller into attitude and altitude controllers, keeping the former and replacing the latter with a data-driven controller.

To provide the data-driven controller with altitude measurements we used eight Optitrack Prime 17W cameras [53] distributed on three sides along the top of a roughly cubic area. The cameras have a refresh rate of up to 360Hz and provide position and pose measurements

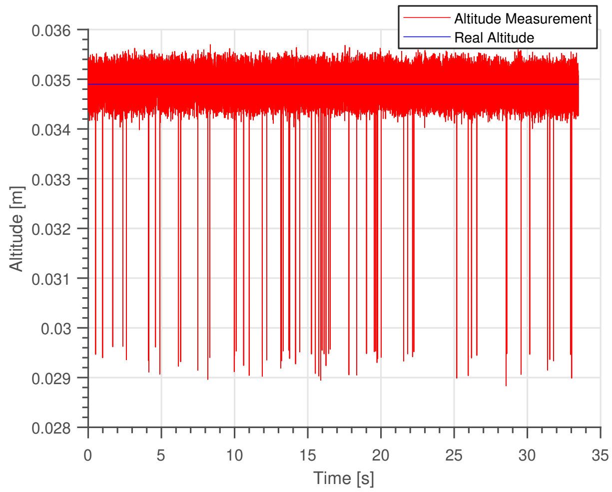

Figure 2.1: Measurement noise while the quad-copter is stationary on the ground.

by triangulating a set of markers placed on the quad-copter. Whereas the PID controller regulating attitude receives measurements from the IMU and the motion capture system, the data-driven controller only receives altitude measurements from the motion capture system.

A qualitative view of the measurement noise, when the quad-copter is static on the floor, is presented in Figure 2.1. The real altitude corresponds to the location of the markers on top of the quad-copter. We observe the noise typically has a magnitude of 1 mm, i.e., $\overline{d} = 0.001$, although there are occasional troughs in the noise signal corresponding to instants where the motion capture system loses track of some of the markers.

2.8.2 Model

To obtain a single-input single-output system we kept the PID controller regulating attitude and restricted the quad-copter's motion to a vertical line. Therefore, assuming perfect attitude regulation, the quad-copter's motion can be described by:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{m} + \frac{1}{m} u_{tr},\end{aligned}$$

$$y = x_1,$$

where x_1 denotes altitude, g is gravity's constant, and the input u_{tr} represents the thrust created by the propellers rotation. The thrust is commanded by a PWM signal[8] and the relation between the commanded PWM signal u and the exerted thrust u_{tr} is well described by the affine map $u_{tr}(u) = \sigma_0 + \sigma_1 u$. The input u in this expression represents the fraction of the maximum allowed thrust, e.g., $u = 0.6$ represents 60% of the maximum thrust. This results in the dynamics:

$$\ddot{x}_1 = \frac{\sigma_0 - g}{m} + \frac{\sigma_1}{m} u, \tag{2.8.1}$$

from which we can infer the relative degree of y to be 2 with $\alpha(x) = \frac{\sigma_0 - g}{m}$ and $\beta(x) = \frac{\sigma_1}{m}$. Since our results apply to the case where α and β are functions, rather than constants, we emulate in software the functions:

$$\beta(x) = \frac{\sigma_1}{m} - \frac{x_1^4}{2}, \qquad \alpha(x) = \frac{\sigma_0 - g}{m} + 2\sin(x_1^2), \tag{2.8.2}$$

i.e., when the data-driven controller requests the input u, we create the input signal $u - \frac{m}{\sigma_1}(x_1^4 u + 4\sin(x_1^2))$. This effectively turns the control gain into a nonlinear state-dependent function. Given that $\frac{\sigma_1}{m} \approx 18$, our assumption that β is greater than zero is satisfied as long as the drone does not reach altitudes higher than 2.4 meters.

In conclusion, the drone dynamics take the form:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{\sigma_0 - g}{m} + 2\sin(x_1^2) + \left(\frac{\sigma_1}{m} - \frac{x_1^4}{2}\right) u, \\
y &= x_1.
\end{aligned} \tag{2.8.3}$$

2.8.3 Data-driven controller and its implementation

The quad-copter receives altitude measurements from the motion capture system and uses them for state estimation using (2.5.5) with $\rho = 4$. This choice of ρ mitigates the effects of

[8]We only use PWM values up to 90% so as to leave some control authority for the attitude controller.

the measurement noise that can be appreciated in Figure 2.1. The resulting state estimate is then fed to the controller (2.6.7) where $K = [-9 \ \ -6]$ so as to place both eigenvalues of $A_1 + B_1 K$ at -3 and $\gamma = 0.002$. For the initial transient we use the sequence of inputs $1.0, 1.0, 1.0, 1.0$. The experiments were executed with a sample time of $T = 0.0028s$ which corresponds to the maximal rate at which the motion capture system provides data.

2.8.4 Experiments

The data-driven controller successfully regulates altitude for the non-linear system (2.8.3) as shown in Figure 2.2: in the top horizontal panel we can observe the desired set-points displayed in red and the quad-copter's trajectory in blue; the second panel from the top shows that this framework achieves altitude steady-state errors consistently below 5 millimeters; the bottom two panels portray the values taken by the state-dependent non-linear functions $\alpha(z)$ and $\beta(z)$. A comparison between the input requested by the static controller (2.6.2)-(2.6.3), assuming knowledge of α and β, and the input generated by the dynamic controller (2.6.7) is presented in Figure 2.3. As expected, we can see the latter converging to the former, made evident in the magnified detail.

The experimental results show that, in spite of measurement errors, the proposed data-driven controller can successfully regulate altitude. In terms of selecting the gains K and γ, we can intuitively understand reductions in γ as leading to both an increase in noise attenuation, through "averaging" of the estimation errors in $\widehat{z}_3$ arising from measurement noise, and a reduction to control responsiveness. Given that large gains in K coupled with a small enough parameter γ will give rise to oscillations in the system's trajectory, as often seen in systems with input delays, we recommend the following heuristic: start the tuning process with low control gains K and a parameter γ of the order of T^{-1}, judiciously increasing the gain K thereafter until either a satisfactory performance is observed or oscillations arise, the latter meaning an increase in γ might be required before continuing to increase K.

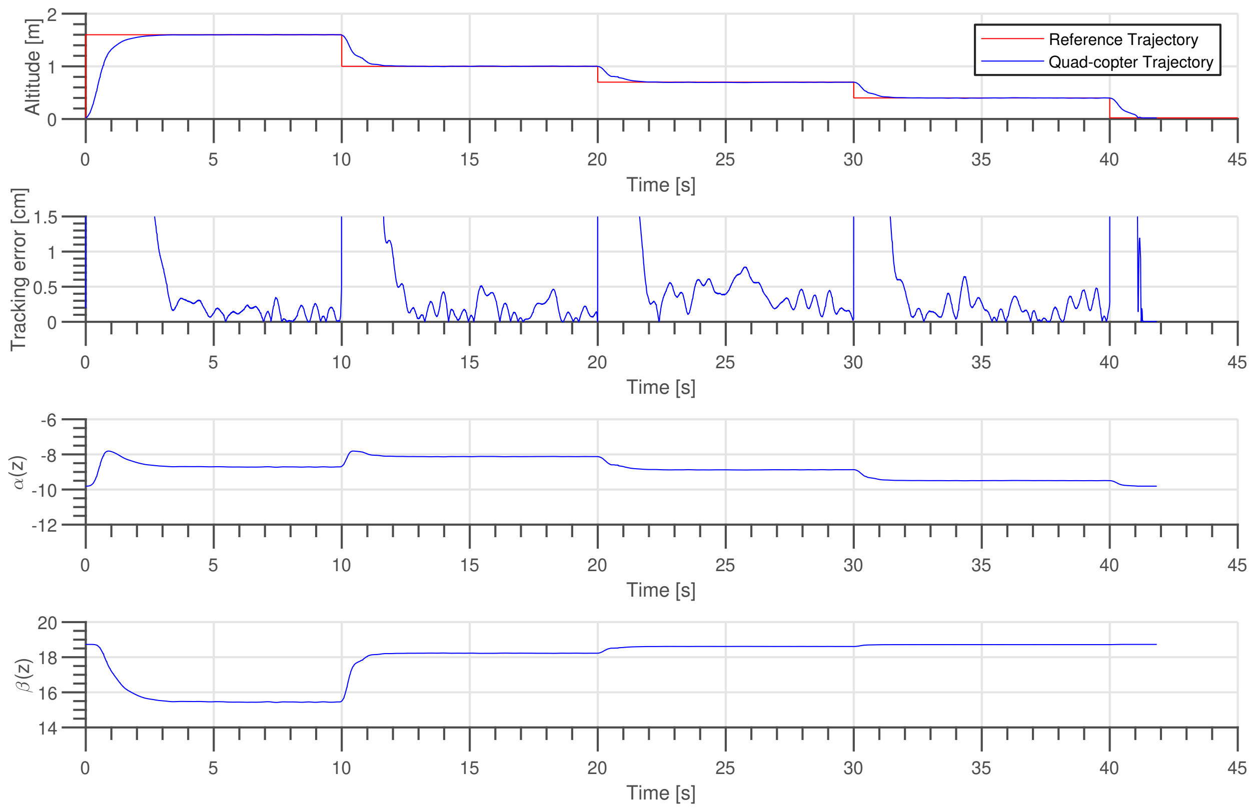

Figure 2.2: Experimental results portraying the quad-copter's trajectory and reference trajectory, tracking error, and values of the state-dependent non-linear functions $\alpha(z)$ and $\beta(z)$.

2.9 Conclusions

In this chapter we have set the foundations in the development of a novel data-driven approach to control that requires neither a model nor previously stored system data. We have proven this framework to stabilize unknown SISO feedback-linearizable systems given sufficiently fast sampling rates and illustrated its practical usefulness and robustness through the experimental stabilization of a quad-rotor.

There are several important questions that were left unaddressed here. Feedback linearizability was convenient to construct the technical arguments but one can easily see extensions to partially feedback-linearizable systems with well behaved zero dynamics. The extension

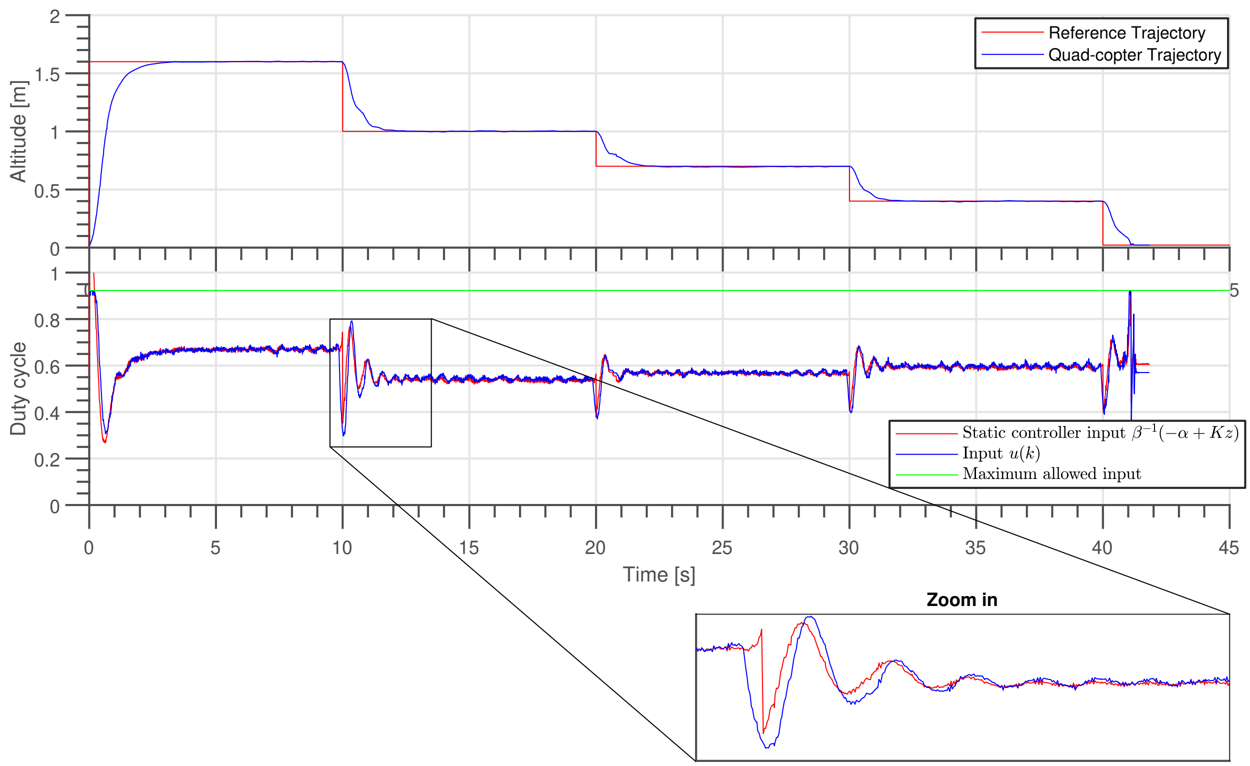

Figure 2.3: Comparison between the input requested by the static controller (2.6.2)-(2.6.3), assuming knowledge of α and β, and the input generated by the dynamic controller (2.6.7), in terms of the PWM's duty cycle. Note that the input is computed on-board the drone and reported, along with the states and parameters, to an external server at a rate of 100 Hz to avoid draining the microprocessors resources.

to the multi-input multi-output case is conceptually easy, as discussed in Section 2.7 and is part of the focus of Chapter 3. Identifying the largest class of systems to which the results in this chapter (or suitable generalizations thereof) apply is also a worthwhile endeavor.

Equally worthwhile is investigating which state estimation and controller design techniques result in better performance in the context of the proposed data-driven methodology since it would make the results more useful in practical applications. In particular, investigating how to best mitigate the effect of measurement noise would be especially important. I motivate so-called dirty derivatives as possible answer to this issue in the second part of Chapter 3 and present a controller based on them, and this chapter's developments, in Chapter 4.

Proofs

Proof Theorem 2.7.1. The proof is be based on the feedback linearized form (2.2.3)-(2.2.5) of the dynamics rather than the original nonlinear form (2.2.1)-(2.2.2). This results in no loss of generality since both systems are related by the diffeomorphism Ψ that satisfies $\Psi(0) = 0$. For simplicity, we denote the set $\Psi(\mathcal{S})$ simply by $\mathcal{S}$. Since Ψ is continuous, $\Psi(\mathcal{S})$ is still a compact set.

The initial transient: the state estimate $\widehat{z}$ requires ρ samples to be collected. To simplify the argument we consider the case where $\rho = 3$ which leads to an initial (fixed) sequence of $\rho - 1 = 2$ inputs u_0^*, u_1^* used at time $k = 0$ and $k = 1$. This corresponds to an initial transient that must be analyzed separately.

By applying Proposition 2.3.2 to the compact set $\mathcal{D} = \mathcal{S} \times \{u_0^*\}$ we conclude the existence of a time T_0 so that trajectories are well defined for all $T \in [0, T_0]$ and for all initial conditions in $\mathcal{S}$. We regard T_0 as the time elapsed during the first time step under input u_0^*. The set of points reached under all these trajectories and for all $T \in [0, T_0]$ is denoted by Z_0. We can repeat this argument, using Z_0 as the set of initial conditions (and assuming the initial time to be zero) and the input u_1^* to conclude the existence of a time T_1 so that trajectories are well defined for all $T \in [0, T_1]$ and for all initial conditions in Z_0. By taking $T_2 = \min\{T_0, T_1\}$ we conclude that solutions are well defined for the sequence of inputs u_0^*, u_1^* where each input is applied for T_2 units of time. Let now Z be the set of points reached under all the trajectories with initial conditions in $\mathcal{S}$ and that result by applying u_0^* for T units of time, followed by applying u_1^* for T units of time with T ranging through all the values in the set $[0, T_2]$. This set is used several times in the remainder of the proof.

At time step $k = 2$ the state estimate is readily available. Let us denote by E the set of possible values taken by the dynamic controller's state e_u at time $k = 2$ depending on the different initial conditions and the chosen constants u_0^*, u_1^*. Since the input error e_u is a continuous function of the initial condition $z(0)$ that belongs to the compact set $\mathcal{S}$ and the constants u_0^*, u_1^*, E is a bounded set. Consider also the set R defined as the smallest sub-level set of $W = V_z + V_{e_u}$ that contains $Z \times E$ where $V_{e_u} : \mathbb{R} \to \mathbb{R}$ is the Lyapunov function defined by $V_{e_u}(s) = s^2$ and V_z is the Lyapunov function satisfying (2.6.1). The objective is to show that R is an invariant set.

Existence of solutions one step beyond the transient: we first show that it is possible to continue the solutions from R by employing again Proposition 2.3.2. For future use, we define the projections $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$, $\pi_Z : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, and $\pi_E : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ defined by $\pi_1(z_1, z_2) = z_1$, $\pi_Z(z, e_u) = z$, and $\pi_E(z, e_u) = e_u$. The dynamic controller is a function of $\widehat{z}$, however, since $\widehat{z}$ is a function of z, we can regard the controller as a smooth function of z. We can thus consider the set of inputs $U \subset \mathbb{R}$ defined by all the inputs obtained via the proposed dynamic controller when $u = u_1^*$, z ranges in $\pi_Z(R)$ and $\widehat{z}$ is given by (2.5.5) with $Y(2)$, defined in (2.5.4), ranging in $(\pi_1 \circ \pi_Z(R))^\rho$, the ρ-fold Cartesian product of $\pi_1 \circ \pi_Z(R)$. By taking its closure, if needed, we can assume the set $\pi_Z(R) \times U$ to be compact and apply Proposition 2.3.2 to obtain a time T_3 ensuring that solutions starting at $\pi_Z(R)$ exist for all $T \in [0, T_3]$. Moreover, Proposition 2.3.2 ensures the existence of a constant M for which the bound (2.3.2) holds and, as a consequence, the approximate model (2.3.6)-(2.3.7) is valid for any solution with initial condition in R and any input in U. If $T_2 < T_3$ we proceed by only considering sampling times in $[0, T_2]$ and note that none of the conclusions reached so far change. If $T_2 > T_3$, we can use sampling times in $[0, T_3]$ while noting that all the reached conclusions remain valid by redefining Z to be the set of points reached for any time in $[0, T_3]$ (if the conclusions hold for the (non-strictly) larger Z set they also hold for the (non-strictly) smaller Z obtained by reducing T_2 to T_3).

Invariance of the set R: we can now establish invariance of R by computing, in several steps, $W(F_T^e(z, u), G_T^e(e_u)) - W(z, e_u)$ for all $(z, e_u) \in R$ with u given by (2.6.7) and G_T^e denoting the exact dynamics of the input error e_u.

In the first step we establish that the evolution of V_z under F_T^e equals the evolution of V_z

under F_T^a up to $O(T^2)$ terms. In order to do so, we recall that $F_T^e(z,u)$ can be expressed as:

$$\begin{aligned} F_T^e(z,u) &= F_T^a(z,u) + O_{(z,u-u_0)}(T^2) \\ &= Az + B(\alpha + \beta u) + O_{(z,u-u_0)}(T^2) \end{aligned} \tag{2.9.1}$$

We then have:

$$\begin{aligned} V_z(F_T^e(z,u)) - V_z(z) &= (F_T^a(z,u) + O_{(z,u-u_0)}(T^2))^T P_z (F_T^a(z,u) + O_{(z,u-u_0)}(T^2)) - z^T P_z z \\ &= V_z(F_T^a(z,u)) - V_z(z) + 2O_{(z,u-u_0)}^T(T^2) P_z F_T^a(z,u) \\ &\quad + O_{(z,u-u_0)}^T(T^2) P_z O_{(z,u-u_0)}(T^2) \\ &\leq V_z(F_T^a(z,u)) - V_z(z) + 2O_{(z,u-u_0)}^T(T^2) P_z F_T^a(z,u) + O_{(z,u-u_0)^2}(T^4) \\ &= V_z(F_T^a(z,u)) - V_z(z) + 2O_{(z,u-u_0)}^T(T^2) P_z Az \\ &\quad + 2O_{(z,u-u_0)}^T(T^2) P_z B(\alpha + \beta u) + O_{(z,u-u_0)^2}(T^4) \\ &= V_z(F_T^a(z,u)) - V_z(z) + O_{(z,u-u_0)^2}(T^2) + O_{(z,u-u_0)^2}(T^3) + O_{(z,u-u_0)^2}(T^4) \\ &= V_z(F_T^a(z,u)) - V_z(z) + O_{(z,u-u_0)^2}(T^2), \end{aligned} \tag{2.9.2}$$

where we used the relationship $\|z\| \leq \|(z, u-u_0)\|$ and boundedness of α, β, and u in virtue of (z,u) belonging to the compact set $\pi_Z(R) \times U$, to obtain the fourth equality.

Noting that from the definition of e_u, equation (2.6.6), one can reach the expression:

$$u = \beta^{-1}(-\alpha + v(z)) - \beta^{-1} e_u, \tag{2.9.3}$$

we consider the term $\|(z, u-u_0)\|$ in more detail,

$$\begin{aligned} \|(z, u-u_0)\| &\leq \|z\| + \left\|\beta^{-1}(-\alpha + v(z)) - \beta^{-1} e_u - u_0\right\| \\ &\leq \|z\| + \left\|\beta^{-1}(-\alpha + v(z)) - u_0\right\| + \left\|\beta^{-1} e_u\right\|. \end{aligned} \tag{2.9.4}$$

As the function $\beta^{-1}(z)(-\alpha(z) + v(z))$ is Lipschitz continuous (with Lipschitz constant L) on $\pi_Z(R)$, and it produces the value u_0 at $z = 0$, we conclude that:

$$\left\|\beta^{-1}\left(-\alpha + v(z)\right) - u_0\right\| \leq L\|z\|. \tag{2.9.5}$$

The preceding sequence of inequalities, and boundedness of β^{-1} on $\pi_Z(R)$, lead to the useful expression:

$$O_{(z,u-u_0)}(T) = O_z(T) + O_{e_u}(T). \tag{2.9.6}$$

Combining the previous bounds (2.9.2) and (2.9.6) we obtain:

$$V_z(F^e_T(z,u)) - V_z(z) = V_z(F^a_T(z,u)) - V_z(z) + O_{z^2}(T^2) + O_{e_u^2}(T^2), \tag{2.9.7}$$

establishing that the decrease of V_z imposed by F^e_T equals the decrease imposed by F^a_T up to $O(T^2)$ terms.

In the second step we use the definition of e_u in the form given by (2.9.3) to show that $V_z(F^a_T(z,u)) - V_z(z)$ is negative definite up to $O_{z^2}(T^2)$ and $O_{e_u^2}(T)$ terms. Using expression (2.9.3), the approximate dynamics are given by:

$$F^a_T(z,u) = Az + Bv(z) - Be_u.$$

We can now compute $V_z(F^a_T(z,u)) - V_z(z)$ as:

$$\begin{aligned}
V_z(F^a_T(z,u)) &- V_z(z) \\
&= (Az + Bv(z))^T P_z(Az + Bv(z)) - V_z(z) \\
&\quad - e_u B^T P_z(Az + Bv(z)) - (Az + Bv(z))^T P_z B e_u + B^T P_z B e_u^2 \\
&= V_z(F^a_T(z,\overline{u})) - V_z(z) \\
&\quad - e_u B^T P_z(Az + Bv(z)) - (Az + Bv(z))^T P_z B e_u + B^T P_z B e_u^2 \\
&\leq -\lambda_{\min}(Q) T\|z\|^2 + O_{(z,u-u_0)^2}(T^2) + 2T\|(A+BK)P_z(B_1 + B_2 T)\|\|z\|\|e_u\| \\
&\quad + T^2(B_1 + TB_2)^T P_z(B_1 + TB_2)e_u^2 \\
&\leq -\frac{\lambda_{\min}(Q)}{2} T\|z\|^2 + cTe_u^2 + O_{e_u^2}(T^2) + O_{(z,u-u_0)^2}(T^2) \\
&\leq -\lambda_z T\|z\|^2 + O_{z^2}(T^2) + O_{e_u^2}(T),
\end{aligned} \tag{2.9.8}$$

where we reach: the second equality due to (2.6.4); the first inequality due to (2.6.1); the second inequality, which holds for any $c \in \mathbb{R}$ satisfying $c > \frac{2}{\lambda_{\min}(Q)}\|(A+BK)P(B_1 + B_2 T)\|^2$, by completing squares; and the last inequality by using equality (2.9.6) and selecting $\lambda_z \in \mathbb{R}^+$ satisfying $\lambda_z \leq \frac{\lambda_{\min}(Q)}{2}$.

In the third step we analyze the effect of using the estimates $\widehat{z}$ and $\widehat{z}_3$ when implementing the control law (2.6.7) by substituting $\widehat{z} = z + e_{z_{(1,2)}}$ and $\widehat{z}_3 = z_3 + e_{z_{(3)}}$, where $e_{z_{(1,2)}}$ represents the vector composed of the first two entries of e_z and $e_{z_{(3)}}$ represents its third entry, and evaluating the dynamics of the error e_u. Based on the relation (2.5.6), the control law (2.6.7) can be expressed as

$$\begin{aligned} u(k+1) &= u + \gamma\left(v(\widehat{z}) - \widehat{z}_3\right) \\ &= u + \gamma\left(v(z) + Ke_{z_{(1,2)}} - (\alpha + \beta u) + e_{z_{(3)}}\right) \\ &= u + \gamma\left(v(z) - (\alpha + \beta u)\right) + Ke_{z_{(1,2)}} + e_{z_{(3)}} \\ &= u + \gamma e_u + O_{(z,u-u_0)}(T), \end{aligned} \tag{2.9.9}$$

Before going forward, we apply Proposition 2.3.2 to $(\alpha, \beta) \circ F_T^e(z, u)$ to obtain

$$\alpha(T) = \alpha(0) + O_{(z,u-u_0)}(T), \quad \beta(T) = \beta(0) + O_{(z,u-u_0)}(T). \tag{2.9.10}$$

With these equalities at hand, we compute $G_T^e(e_u) = e_u(k+1)$:

$$\begin{aligned} G_T^e(e_u(k)) &= v(z(k+1)) - z_3(k+1) \\ &= v\left(z + O_{(z,u-u_0)}(T)\right) - (\alpha(k+1) + \beta(k+1)u(k+1)) \\ &= v(z(k)) + KO_{(z,u-u_0)}(T) - \left(\alpha(k) + \left(\beta(k) + O_{(z,u-u_0)}(T)\right)u(k+1) + O_{(z,u-u_0)}(T)\right) \\ &= v(z(k)) + O_{(z,u-u_0)}(T) - (\alpha(k) + \beta(k)u(k+1)) + u(k+1)O_{(z,u-u_0)}(T) \\ &= v(z(k)) - (\alpha(k) + \beta(k)u(k)) - \beta(k)\gamma e_u(k) - \beta(k)O_{(z,u-u_0)}(T) \\ &\quad + \left(u(k) + \gamma e_u(k) + O_{(z,u-u_0)}(T)\right)O_{(z,u-u_0)}(T) + O_{(z,u-u_0)}(T) \\ &= (1 - \beta(k)\gamma)\, e_u(k) + O_{(z,u-u_0)}(T) + O_{(z,u-u_0)^2}(T^2) \\ &= (1 - \beta(k)\gamma)\, e_u(k) + O_{(z,u-u_0)}(T), \end{aligned} \tag{2.9.11}$$

where we reach: the second equality by using the exact model (2.3.4)-(2.3.5) with $t = T$ aggregating all terms with a T coefficient inside $O_{(z,u-u_0)}(T)$, and the definition of z_3 from

Section 2.5; the third equality by using (2.9.10); the fifth equality by substituting $u(k+1)$ with (2.6.7) and using the definition of $e_u(k)$ from (2.6.6); the sixth equality by absorbing u and e_u into the $O_{(z,u-u_0)}(T)$ term on account of e_u and u being bounded in R; and the last equality by noting that $O_{(z,u-u_0)^2}(T^2) = O_{(z,u-u_0)}(T)$ on account of z and u belonging in the compact sets R and U.

Coming back to the Lyapunov function $V_{e_u}(e_u) = e_u^2$, we can now compute $V_{e_u}(G_T^e(e_u)) - V_{e_u}(e_u)$:

$$\begin{aligned} V_{e_u}(G_T^e(e_u)) - V_{e_u}(e_u) &= \left((1-\beta\gamma)\, e_u + O_{(z,u-u_0)}(T)\right)^2 - e_u^2 \\ &= \left(-2\beta\gamma + \beta^2\gamma^2\right) e_u^2 + 2\,(1-\beta\gamma)\, e_u O_{(z,u-u_0)}(T) + O_{(z,u-u_0)^2}(T^2) \\ &\leq \left(-\beta\gamma + \beta^2\gamma^2\right) e_u^2 + cO_{(z,u-u_0)^2}(T^2) \\ &\leq -\lambda_u e_u^2 + O_{(z,u-u_0)^2}(T^2) \\ &\leq -\lambda_u e_u^2 + O_{z^2}(T^2) + O_{e_u^2}(T^2), \end{aligned} \tag{2.9.12}$$

where: we reach the first inequality, which holds for any $c \in \mathbb{R}$ satisfying $c > 1 + (1-\beta\gamma)^2 / (\beta\gamma)$, by completing squares; the second inequality holds for sufficiently small[9] $\gamma, \lambda_u \in \mathbb{R}^+$; we reach the last inequality by using equality (2.9.6). We now put the three intermediate steps, (2.9.7) and (2.9.8), and (2.9.12) together:

$$\begin{aligned} W(F_T^e(z,u), G_T^e(e_u)) - W(z,e_u) &\leq -\lambda_z T\|z\|^2 - \lambda_u\|e_u\|^2 + O_{z^2}(T^2) + O_{e_u^2}(T) \\ &\leq -\lambda_z T\|z\|^2 - \lambda_u\|e_u\|^2 + MT^2\|z\|^2 + MT\|e_u\|^2. \end{aligned}$$

where $M \in \mathbb{R}^+$ is the largest constant stemming from the definition of the O terms. If we choose $\lambda \in \mathbb{R}^+$ and $T_4 \in \mathbb{R}^+$ satisfying:

$$\begin{aligned} &\lambda < \min\{\lambda_z, \lambda_u\}, \\ &T_4 < \min\left\{\frac{(\lambda_z - \lambda)}{M}, \frac{(\lambda_u - \lambda)}{M}\right\}, \end{aligned}$$

[9]In particular, this inequality holds for any γ and λ_u satisfying $\gamma \leq \overline{\beta}^{-1}$ and $\lambda_u \leq \left(1 - \underline{\beta}\gamma\right)\underline{\beta}\gamma$, where $\overline{\beta} = \max_{z\in R}\beta(z)$ and $\underline{\beta} = \min_{z\in R}\beta(z)$.

it follows that for all $T \in [0, T_4]$ we have:

$$W(F_T^e(z,u), G_T^e(e_u)) - W(z, e_u) \leq -\lambda T \|z\|^2 - \lambda \|e_u\|^2. \tag{2.9.13}$$

Therefore, for any $T \in [0, T_5]$, $T_5 = \min\{T_1, \ldots, T_4\}$, we have that R remains invariant. By noting that trajectories remain in R for any time in $[0, T_5]$ we conclude that we can apply the same argument to establish that trajectories remain in R for any number of time steps since we only assumed that inputs were generated based on output measurements that remained in $\pi_1 \circ \pi_Z(R)$. Compactness of R establishes that trajectories are bounded and thus there exists a constant $b_1 \in \mathbb{R}^+$ so that $\|e_u(k)\| \leq b_1$ and $\|z(k)\| \leq b_1$ for all $k \in \mathbb{N}$. Moreover, (2.9.13) informs us that both z and e_u will converge to the origin. Invoking Theorem 1 in [54], combined with invariance of R and smoothness of the dynamics, we conclude that the solutions of (2.2.1), when using the dynamic controller (2.6.7) , where the virtual input v is provided by (2.6.3), using the state estimates provided by an estimation technique satisfying (2.5.6), are bounded, i.e., there exists a constant $b_2 \in \mathbb{R}^+$ so that $\|x(t)\| \leq b_2$ and, moreover, $\lim_{t\to\infty} x(t) = 0$. Hence, by taking $b = \max\{b_1, b_2\}$ we conclude the proof. □

Proof of Theorem 2.7.2. It is sufficient to verify that the sections of the proof of Theorem 2.7.1 that depend on $v(z)$ hold under any virtual input satisfying the conditions 2.7.1-2.7.2, the rest remains unchanged. An attentive reader will notice that only equations (2.9.8), (2.9.9) and (2.9.11) could be affected by a change in $v(z)$. That being said, it is straightforward to see that if $v(z)$ is such that if condition (2.7.1) is satisfied, then (2.9.8) remains unchanged, and similarly, that if (2.7.2) holds then equation (2.9.9) and inequality (2.9.11) hold. Thus, we conclude that Theorem 2.7.1 holds when replacing $v(z)$ as provided in (2.6.3) with any $v(z)$ satisfying conditions (2.7.1) and (2.7.2). □

Proof of Theorem 2.7.3. As stated in Section 2.5, in the presence of essentially bounded noise the estimation error is given by $e_z = O_{(z,u-u_0)}(T) + O_{\overline{d}}(T^{-n})$. This proof follows the same arguments of the proof of Theorem 2.7.1 while accounting for the effect of measurement noise in e_z. Therefore, we shall describe only the required modifications.

Due to measurement noise, we redefine E as the set of possible values taken by the dynamic controller's state e_u at time $k = 2$ depending on the different initial conditions, and the chosen constants u_0^*, u_1^*. Since the input error e_u is a continuous function of the initial condition $z(0)$ that belongs to the compact set $\mathcal{S}$ and the constants u_0^*, u_1^*, E is a bounded set. Consider also the set R defined as the smallest sub-level set of $W = V_z + V_{e_u}$ that contains $Z \times E$ where $V_{e_u} : \mathbb{R} \to \mathbb{R}$ is the Lyapunov function defined by $V_{e_u}(s) = s^2$ and V_z is the Lyapunov function satisfying (2.6.1). The objective is to show that R is an invariant set.

Existence of solutions one step beyond the transient: we first show that it is possible to continue the solutions from R by employing again Proposition 2.3.2. For future use, we define the projections $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$, $\pi_Z : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, and $\pi_E : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ defined by $\pi_1(z_1, z_2) = z_1$, $\pi_Z(z, e_u) = z$, and $\pi_E(z, e_u) = e_u$. The dynamic controller is a function of $\widehat{z}$, however, since $\widehat{z}$ is a function of z, we can regard the controller as a smooth function of z. We can thus consider the set of inputs $U \subset \mathbb{R}$ defined by all the inputs obtained via the proposed dynamic controller when $u = u_1^*$, z ranges in $\pi_Z(R)$ and $\widehat{z}$ is given by (2.5.5) with $Y(2)$, defined in (2.5.4), ranging in $(\pi_1 \circ \pi_Z(R))^\rho$, the ρ-fold Cartesian product of $\pi_1 \circ \pi_Z(R)$. By taking its closure, if needed, we can assume the set $\pi_Z(R) \times U$ to be compact and apply Proposition 2.3.2 to obtain a time T_3 ensuring that solutions starting at $\pi_Z(R)$ exist for all $T \in [0, T_3]$. Moreover, Proposition 2.3.2 ensures the existence of a constant M for which the bound (2.3.2) holds and, as a consequence, the approximate model (2.3.6)-(2.3.7) is valid for any solution with initial condition in R and any input in U. If $T_2 < T_3$ we proceed by only considering sampling times in $[0, T_2]$ and note that none of the conclusions reached so far change. If $T_2 > T_3$, we can use sampling times in $[0, T_3]$ while noting that all the reached conclusions remain valid by redefining Z to be the set of points reached for any time in $[0, T_3]$ (if the conclusions hold for the (non-strictly) larger Z set they also hold for the (non-strictly) smaller Z obtained by reducing T_2 to T_3).

Due to measurement noise, the subset $U \subset \mathbb{R}$ is now defined by all the inputs obtained via the proposed dynamic controller when $u = u_1^*$, z ranges in $\pi_Z(R)$, $d \in [-\overline{d}, \overline{d}]$ and $\widehat{z}$ is given by (2.5.5) with $Y(2)$, defined in (2.5.4), ranging in $(\pi_1 \circ \pi_Z(R))^\rho$, the ρ-fold Cartesian product of $\pi_1 \circ \pi_Z(R)$. Given that U is still compact we can use the same arguments as in Theorem 2.7.1 to guarantee existence of solutions one step beyond the transient.

We now note that to establish boundedness of all the signals it is sufficient to establish the existence of a sub-level set R of W that is forward invariant and satisfies $\mathcal{S} \subseteq \pi_Z(R)$. As in the previous proof we define R to be the smallest sub-level set of $W = V_z + V_{e_u}$ that contains $Z \times E$ at the end of the initial transient. Given that R is a compact set we can define $c_1 = \max_{(z,e_u)\in R} \|(z, u - u_0)\|$ and $c_2 = \min_{(z,e_u)\in \delta R} \sqrt{T\|z\|^2 + \|e_u\|^2}$, where δR is the boundary of the set R. Note that c_2 is greater than zero as the origin is assumed to be contained in the interior of $\mathcal{S}$ which is itself contained in the interior of R.

Under measurement noise d, equality (2.9.9) becomes:

$$u(k+1) = u(k) + \gamma e_u + O_{(z,u-u_0)}(T) + O_{\overline{d}^m}(T^{-n}). \tag{2.9.14}$$

This in turn results in $G_T^e(e_u) = e_u(k+1)$ becoming:

$$G_T^e(e_u(k)) = (1 - \beta(k)\gamma)\, e_u(k) + O_{(z,u-u_0)}(T) + O_{\overline{d}}(T^{-n}) \tag{2.9.15}$$

Based on this equality, it can be shown that:

$$V_{e_u}(G_T^e(e_u)) - V_{e_u}(e_u) \leq -\lambda_u e_u^2 + O_{z^2}(T^2) + O_{e_u^2}(T^2) + O_{\overline{d}^2}(T^{-2n}) \tag{2.9.16}$$

Inequality (2.9.16) allows us to conclude that:

$$\begin{aligned} W(F_T^e(z,u), G_T^e(e_u)) - W(z, e_u) &\leq -\lambda T\|z\|^2 - \lambda\|e_u\|^2 + M\overline{d}^2 T^{-2n}, \\ &\leq -\lambda c_2^2 + M\overline{d}^2 T^{-2n}. \end{aligned}$$

Thus,

$$W(F_T^e(z,u), G_T^e(e_u)) - W(z, e_u) < 0$$

holds for all $(z, e_u) \in \delta R$ and $\overline{d} < b_1 = \left(\sqrt{\frac{\lambda}{M}} c_2\right)^{\frac{1}{2}} T^n$, showing that R is invariant. This guarantees that all signals remain bounded, i.e., if we define b_2' as the radius of the smallest ball containing R we conclude that $\|\widehat{z}(k)\| \leq b_2'$ and $\|e_u\| \leq b_2'$ for all $k \in \mathbb{N}$. By using arguments similar to those employed in the proof of Theorem 2.7.1, there exists a constant b_2'' so that $\|x(t)\| \leq b_2''$ for all $t \in \mathbb{R}$ and we can define b_2 to be $\max\{b_2', b_2''\}$.

Moreover, trajectories will converge to the smallest sub-level set of W containing the ball of radius r centered at zero where r is the smallest real number satisfying $-\lambda r^2 + M\overline{d}^2 T^{-2n} \leq 0$, i.e., $r = \overline{d}\, T^{-n} \left(\frac{\lambda}{M}\right)^{-\frac{1}{2}}$. Since said sub-level set is contained in the ball centered at the origin and of radius $r\lambda_{\max}(P_w)/\lambda_{\min}(P_w)$ where P_w is the matrix defining the quadratic Lyapunov function $W(z, e_u) = w^T P_w w$, $w = (z, e_u)$, the result is proved by taking:

$$b_3 = \left(\frac{\lambda}{M}\right)^{-\frac{1}{2}} \frac{\lambda_{\max}(P_w)}{\lambda_{\min}(P_w)}.$$

□

CHAPTER 3

Extensions to data-driven control

3.1 Motivation

The work presented in this chapter involves two crucial extensions in the path to turning the theory developed in Chapter 2 into a practically useful method at large. While not directly connected, both the extension to multiple-input multiple-output feedback-linearizable systems and the introduction of dirty-derivative estimation provides data-driven control with a larger range of applicability, enabling its application to benchmark control systems such as quad-copters and increasing robustness to measurement noise injected by real sensors. With this in mind, this chapter is divided in two sections.

3.2 Extension to multiple-input multiple-output systems

3.2.1 Model

Extending the previous chapter's findings to multiple-input multiple-output systems is conceptually straightforward, thus I only present here the changes that would be required in Chapter 3 for this extension to hold. In place of system (2.2.1)-(2.2.2), consider the following system:

$$\dot{x} = f(x) + g(x)u \tag{3.2.1}$$

$$y = h(x) + d, \tag{3.2.2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $h : \mathbb{R}^n \to \mathbb{R}^m$ are smooth functions and we denote by $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^m$, $d \in \mathbb{R}^m$, the state, output, input, and measurement noise, respectively. We make the assumption that each element h_i of the output function h has a respective relative degree s_i, such that $\sum_1^m s_i = n$, i.e., this system is feedback-linearizable. This implies that $L_g L_f^j h_i(x) = 0$ for $j = 0, \ldots, s_i - 2$ for each $i = 0, \ldots, m$, and that the matrix

$$\begin{bmatrix} L_g L_f^{s_1-1} h_1(x) \\ L_g L_f^{s_2-1} h_2(x) \\ \vdots \\ L_g L_f^{s_p-1} h_l(x) \end{bmatrix}$$

is non-singular for all $x \in \mathbb{R}^n$, i.e., it has a non-zero determinant.

We consider the dimensions $n = 5$, $m = 2$ and $s_1 = 2, s_2 = 3$ in an attempt to preserve simplicity yet still demonstrate the richness of the method. Invoking the feedback linearizability assumption, we can rewrite the unknown dynamics in the coordinates[1] $(z_1, z_2, z_4, z_5, z_6) = \Psi(x) = (h_1(x), L_f h_1(x), h_2(x), L_f h_2(x), L_f^2 h_2(x))$:

$$y_1 = z_1 \tag{3.2.3}$$

$$\dot{z}_1 = z_2 \tag{3.2.4}$$

$$\dot{z}_2 = \alpha_1(z) + \beta_{1,1}(z)u_1 + \beta_{1,2}(z)u_2 \tag{3.2.5}$$

$$y_2 = z_4 \tag{3.2.6}$$

$$\dot{z}_4 = z_5 \tag{3.2.7}$$

$$\dot{z}_5 = z_6 \tag{3.2.8}$$

$$\dot{z}_6 = \alpha_2(z) + \beta_{2,1}(z)u_1 + \beta_{2,2}(z)u_2 \tag{3.2.9}$$

where $\alpha_1 = L_f^2 h_1 \circ \Psi^{-1}$, $\alpha_2 = L_f^3 h_2 \circ \Psi^{-1}$ and $\beta_{1,1} = L_{g_1} L_f h_1 \circ \Psi^{-1}$, $\beta_{1,2} = L_{g_2} L_f h_1 \circ \Psi^{-1}$, $\beta_{2,1} = L_{g_1} L_f^2 h_2 \circ \Psi^{-1}$ and $\beta_{2,2} = L_{g_2} L_f^2 h_2 \circ \Psi^{-1}$. We note that f, g, and h are unknown and thus so are α_1, α_2 and $\beta_{1,1}$, $\beta_{1,2}$, $\beta_{2,1}$ and $\beta_{2,2}$. As seen before, this form of the dynamics has

[1] Note that we skip z_3 here to ease with notation as we later use a dynamic extension.

the advantage of using the scalar valued functions α_i and $\beta_{i,j}$ to describe the full dynamics, independently of the values of s_i. This is a key observation that underlies the claim that the results below hold for arbitrary $s_i \in \mathbb{N}$.

System (3.2.3)-(3.2.9) will be controlled using piece-wise constant inputs for a sampling time $T \in \mathbb{R}^+$. This means that inputs $u_i : \mathbb{R}_0^+ \to \mathbb{R}$ satisfy the following equality for all $k \in \mathbb{N}$:

$$u_i(kT + \tau) = u_i(kT), \qquad \forall \tau \in [0, T[.$$

It will be convenient to use u_i to denote an input only defined on $[0, T[$. Since the curve u_i is constant on the interval $[0, T[$, we identify it with the corresponding element of $\mathbb{R}$.

The solution of (2.2.3)-(2.2.4) is denoted by $F_t^e(z, u) = (F_{t,1}^e(z, u), F_{t,2}^e(z, u))$, for $t \in [0, T[$, and satisfies $F_0^e(z, u) = z$. The superscript "e" reminds us that this is an exact solution. In the next section we discuss approximate solutions.

3.2.2 Aproximate Models

Note that proposition 2.3.2 can be readily modified to hold for $u = [u_1, u_2]$, and applied separately to each output, by redefining:

$$\alpha(k) = \begin{bmatrix} \alpha_1(k) \\ \alpha_2(k) \end{bmatrix} \tag{3.2.10}$$

$$\beta(k) = \begin{bmatrix} \beta_{1,1}(k) & \beta_{1,2}(k) \\ \beta_{2,1}(k) & \beta_{2,2}(k) \end{bmatrix} \tag{3.2.11}$$

$$u_0 = -\beta(0)^{-1}\alpha(0)$$

Doing so leads to the expressions:

$$\left\| \left(F_{\tau',1}^e\right)^{(3)} (z, u) \frac{t^3}{3!} \right\| \leq M_1 T^3 \|(z, u - u_0)\|, \tag{3.2.12}$$

$$\left\| \left(F_{\tau',3}^e\right)^{(4)} (z, u) \frac{t^4}{4!} \right\| \leq M_2 T^4 \|(z, u - u_0)\|. \tag{3.2.13}$$

These allow us to write the exact models:

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} (k+1) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} (k) + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} (\alpha + \beta u)_1 + \begin{bmatrix} O_{(z,u-u_0)}(T^3) \\ O_{(z,u-u_0)}(T^2) \end{bmatrix} \tag{3.2.14}$$

$$\begin{bmatrix} z_4 \\ z_5 \\ z_6 \end{bmatrix} (k+1) = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_4 \\ z_5 \\ z_6 \end{bmatrix} (k) + \begin{bmatrix} \frac{T^3}{6} \\ \frac{T^2}{2} \\ T \end{bmatrix} (\alpha + \beta u)_2 + \begin{bmatrix} O_{(z,u-u_0)}(T^4) \\ O_{(z,u-u_0)}(T^3) \\ O_{(z,u-u_0)}(T^2) \end{bmatrix} \tag{3.2.15}$$

where $(\alpha + \beta u)_i$ represents the ith entry of the vector $(\alpha + \beta u)$. Mirroring Chapter 2, we drop the O terms to build the approximate models:

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} (\alpha(k) + \beta(k)u(k))_1 \tag{3.2.16}$$

$$\begin{bmatrix} z_4(k+1) \\ z_5(k+1) \\ z_6(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_4(k) \\ z_5(k) \\ z_6(k) \end{bmatrix} + \begin{bmatrix} \frac{T^3}{6} \\ \frac{T^2}{2} \\ T \end{bmatrix} (\alpha(k) + \beta(k)u(k))_2, \tag{3.2.17}$$

which can be written as:

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = A_{s_1} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + B_{s_1}(\alpha(k) + \beta(k)u(k))_1 \tag{3.2.18}$$

$$\begin{bmatrix} z_4(k+1) \\ z_5(k+1) \\ z_6(k+1) \end{bmatrix} = A_{s_2} \begin{bmatrix} z_4(k) \\ z_5(k) \\ z_6(k) \end{bmatrix} + B_{s_2}(\alpha(k) + \beta(k)u(k))_2, \tag{3.2.19}$$

3.2.3 State Estimation

State estimation does not differ much from what was developed in Section 2.5, as we can treat each output independently as if working with individual single-input single-output systems. We are able to do this due to the s_i derivative of output i being considered constant for the purposes of state estimation. Following this idea we estimate the extended state $(z_1,\ z_2,\ z_3,\ z_4,\ z_5,\ z_6,\ z_7)$, only requiring measurements of the outputs z_1 and z_4 to obtain the sets of estimates $(\widehat{z_1},\ \widehat{z_2},\ \widehat{z_3})$, and $(\widehat{z_4},\ \widehat{z_5},\ \widehat{z_6},\ \widehat{z_7})$, respectively.

Each of these state observers may use a different number of consecutive measurements ρ_i where $\rho_i \geq s_i + 1$. The results in Section 2.5 extend directly, from which we conclude that each of these observers achieves an error $e_{\bar{z}_i} = O_{(z_1,z_2),u-u_0)}(T)$ where $\bar{z}_i$ represents the states estimated by observer i. One can easily show that running both observers synchronously results in $e_z = O_{(z,u-u_0)}(T)$, according to the equation:

$$
\begin{aligned}
e_{\bar{z}_1} &= O_{((z_1,z_2,z_3),u-u_0)}(T) = M_1 T \|((z_1, z_2), u - u_0)\| && (3.2.20)\\
e_{\bar{z}_2} &= O_{((z_4,z_5,z_6,z_7),u-u_0)}(T) = M_2 T \|((z_4, z_5, z_6), u - u_0)\| && (3.2.21)\\
e_z &\leq (M_1 + M_2) T \|(z, u - u_0)\| = O_{(z,u-u_0)}(T). && (3.2.22)
\end{aligned}
$$

Two observers provide the possibility of having different sampling periods, T_1 and T_2. Let $\overline{T} = \max\{T_1, T_2\}$, $\underline{T} = \min\{T_1, T_2\}$ and $\overline{T} = q\underline{T}$ for some finite $q \in \mathbb{Z}^+$. Without loss of generality assume that $\overline{T} = T_1$ and $\underline{T} = T_2$ and let $\rho_2 \geq q(s_2 + 1)$. As before, $e_{\bar{z}_1} = O_{(z,u-u_0)}(T_1)$ and $e_{\bar{z}_2} = O_{(z,u-u_0)}(T_2)$, furthermore, since $T_1 > T_2$ it follows that $M_2 T_2 \|((z_4, z_5, z_6), u - u_0)\| < M_2 T_1 \|((z_4, z_5, z_6), u - u_0)\|$, implying $e_{\bar{z}_2} = O_{(z,u-u_0)}(T_1)$. Setting $T = T_1$ gives the state estimation error $e_z = O_{(z,u-u_0)}(T)$ under a similar argument as the previous paragraph. Due to the condition $T = qT_2$, the observers for the states, parameters and the controller can be synchronized.

By using two sampling periods the user has access to a design tool that introduces a trade-off in each output between increased noise effect due to smaller sampling periods and decreased noise effect due to a higher degree of averaging by using more samples for state estimation.

3.2.4 Controller design

The family of controllers we use mirror that one presented in Chapter 2 and the development is identical with the exception that now α and β are respectively given by (3.2.10) and (3.2.11). Given systems (3.2.16)-(3.2.19), it follows that:

$$A_{s_1} = I_2 + A_{1_{s_1}} T, \quad B_{s_1} = B_{1_{s_1}} T + B_{2_{s_1}} T^2 A$$

$$A_{s_2} = I_3 + A_{1_{s_2}}T + A_{2_{s_2}}T^2, \quad B_{s_2} = B_{1_{s_2}}T + B_{2_{s_2}}T^2 + B_{3_{s_2}}T^3.$$

Let $\overline{A}$ and $\overline{B}$ be defined as:

$$\overline{A} = \begin{bmatrix} A_{1_{s_1}} & 0_{2\times 3} \\ 0_{3\times 2} & A_{1_{s_2}} \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} B_{1_{s_1}} & 0_{2\times 1} \\ 0_{3\times 1} & B_{1_{s_2}} \end{bmatrix},$$

as the pair $(\overline{A}, \overline{B})$ is controllable, there exists a controller $v(z) = Kz$ and symmetric and positive definite matrices P_z and Q such that:

$$(\overline{A} + \overline{B}K)^T P_z + P_z(\overline{A} + \overline{B}K) = -Q.$$

As in chapter 2, the control law satisfies the dynamics:

$$u(k+1) = u(k) + \gamma \left(K\widehat{z}(k) - \begin{bmatrix} \widehat{z}_3(k) \\ \widehat{z}_7(k) \end{bmatrix} \right), \tag{3.2.23}$$

where $\gamma \in \mathbb{R}^{n\times n}$ is chosen such that the eigenvalues, η_i, of $(I_2 - \beta\gamma)$ lie inside the unit circle. In order to fully specify this controller, we need to describe its operation during the initial transient of $\rho = \max\{\rho_1, \ldots, \rho_p\}$ steps during which enough measurements are collected to produce the first state estimate for all elements of (3.2.23). We simply choose a fixed sequence of inputs $u_0^*, u_1^*, \ldots, u_{\rho-1}^*$ to be used during this transient. Although different sequences lead to different transients, the results are independent of this choice.

3.2.5 Changes to the main proof

The main proof follows with basically no changes, except for the following required alteration to inequality (2.9.12): Under a choice of γ guaranteeing $max_{i=1\ldots n}\|\eta_i\| < 1$, where $\{\eta_1, \ldots, \eta_n\}$ are the eigenvalues of $(I_2 - \beta\gamma)$, there exists some positive definite matrices P and Q such that

$$(I_2 - \beta\gamma)^T P(I_2 - \beta\gamma) - P = -Q.$$

With this in mind, we define the Lyapunov function $V_{e_u}(e_u) = e_u^T P e_u$ and compute $V_{e_u}(G_T^e(e_u)) - V_{e_u}(e_u)$:

$$V_{e_u}(G_T^e(e_u)) - V_{e_u}(e_u)$$

$$
\begin{aligned}
&= \left((I_2 - \beta\gamma)\, e_u + O_{(z,u-u_0)}(T)\right)^T P \left((I_2 - \beta\gamma)\, e_u + O_{(z,u-u_0)}(T)\right) - e_u^T P e_u \\
&= e_u^T \left((I_2 - \beta\gamma)^T P (I_2 - \beta\gamma) - P\right) e_u + O_{(z,u-u_0)^2}(T^2) \\
&\qquad + e_u^T \left(I_2 - \beta\gamma\right)^T P O_{(z,u-u_0)}(T) + O_{(z,u-u_0)}(T)^T P \left(I_2 - \beta\gamma\right) e_u \\
&\leq -e_u^T Q e_u + O_{(z,u-u_0)^2}(T^2) + 2\|P\left(I_2 - \beta\gamma\right)\|\|e_u\|\|O_{(z,u-u_0)}(T)\| \\
&\leq -\lambda_u \|e_u\|^2 + O_{(z,u-u_0)^2}(T^2) + 2\|P\left(I_2 - \beta\gamma\right)\|\|e_u\|\|O_{(z,u-u_0)}(T)\| \\
&\leq -\frac{\lambda_u}{2}\|e_u\|^2 + cO_{(z,u-u_0)^2}(T^2) \\
&\leq -\frac{\lambda_u}{2}\|e_u\|^2 + O_{z^2}(T^2) + O_{e_u^2}(T^2),
\end{aligned} \tag{3.2.24}
$$

where we reach the second inequality by taking $\lambda_u \in \mathbb{R}^+,\ \lambda_u \leq \lambda_{min}(Q)$, and the third by completing squares, where $c \in \mathbb{R},\ c > \frac{2}{\lambda_u}\|P(I_2 - \beta\gamma)\|^2$.

Remark 3.2.1. *The same changes as those applied to the noise-free case are required in the proof of Theorem* (2.7.3)*.*

With this we conclude the extension of the theory to the multiple-input multiple-output case. Despite this extension being conceptually simple, applying the resulting controller does present a major challenge. Designing γ without knowledge of β is not an easy task. In the following subsection I analyse the limitations posed by this issue, specifically what level of knowledge of β is required to successfully design γ.

3.2.6 Designing γ

We can reach an equivalent formulation of the problem by recasting the controller design in Section 3.2.4. We consider the design of a control law such that the quantity e_u converges to zero. This quantity evolves according to the equations:

$$
\begin{aligned}
e_u(k) &= v(k) - (\alpha(k) + \beta(k)u(k)) \\
e_u(k+1) &= v(k) - \left(\alpha(k) + \beta(k)u(k+1)\right).
\end{aligned}
$$

Without loss of generality we note that $u(k+1) = u(k) + \delta u(k)$, which allows us to write the dynamical system for e_u:

$$e_u(k+1) = e_u(k) - \beta(k)\delta u(k).$$

If we now choose the Lyapunov function $V_{e_u}(s) = s^T s$ we have that:

$$\begin{aligned} V_{e_u}(e_u(k+1)) - V_{e_u}(e_u(k)) &= e_u(k+1)^T e_u(k+1) - e_u(k)^T e_u(k) \\ &= (e_u(k) - \beta(k)\delta u(k))^T (e_u(k) - \beta(k)\delta u(k)) - e_u(k)^T e_u(k) \\ &= -e_u(k)^T \beta(k)\delta u(k) - \delta u(k)^T \beta(k)^T e_u(k) \\ &\quad +\delta u(k)^T \beta(k)^T \beta(k)\delta u(k). \end{aligned}$$

Selecting $\delta u(k) = \gamma e_u(k)$ where $\gamma \in \mathbb{R}^{m\times m}$ results in:

$$V_{e_u}(e_u(k+1)) - V_{e_u}(e_u(k)) = -e_u(k)^T \left(\beta(k)\gamma + \gamma^T \beta(k)^T - \gamma^T \beta(k)^T \beta(k)\gamma\right) e_u(k).$$

To show the system is stable we need to solve the Riccati-like equation:

$$\beta(k)\gamma + \gamma^T \beta(k)^T - \gamma^T \beta(k)^T \beta(k)\gamma = Q, \tag{3.2.25}$$

where Q is positive definite, β unknown and γ to be designed.

Claim 3.2.2. *Designing γ such that $\beta\gamma$ is positive definite with eigenvalues in $]0, 2[$ is required for equation (3.2.25) to hold.*

Proof. We begin by rewriting equation (3.2.25) as:

$$I_2 - (I_2 - \beta(k)\gamma)^T (I_2 - \beta(k)\gamma) = Q. \tag{3.2.26}$$

Given that the term $(I_2 - \beta(k)\gamma)^T (I_2 - \beta(k)\gamma)$ is a quadratic form of the matrix $(I_2 - \beta(k)\gamma)$, we can conclude that it represents a positive semi-definite matrix with eigenvalues in the ray $[0, \infty[$ depending on the eigenvalues of the matrix $\beta\gamma$. Let a_i, b_i and q_i with $i \in \{1, 2, \ldots, n\}$

represent, in descending order, the eigenvalues of matrices I_2, $(I_2-\beta(k)\gamma)^T(I_2-\beta(k)\gamma)$ and Q. By inequalities (2) and (11) in [55], it follows that:

$$q_1 \leq a_1 + b_1 \qquad q_n \geq a_n + b_n,$$

we conclude that for Q to be positive definite, considering that $a_i = 1\ \forall i \in \{1, 2, \ldots, n\}$, the eigenvalues of $(I_2-\beta(k)\gamma)^T(I_2-\beta(k)\gamma)$ must reside in the interval $[0, 1[$.

The eigenvalues b_i are squares of the eigenvalues of $(I_2-\beta(k)\gamma)$, due to the latter being a square matrix. This implies that for the eigenvalues b_i to reside in the interval $[0, 1[$, γ must be designed such that the eigenvalues of $(I_2-\beta(k)\gamma)$ reside in the interval $]-1, 1[$. Moreover, the eigenvalues of $I_2-\beta\gamma$ satisfy $(I_2-\beta\gamma)v = \lambda v$, from where it is straightforward to see that the eigenvalues of $\beta\gamma$ must satisfy $(\beta\gamma)v = (1-\lambda)v$. Thus we conclude that the eigenvalues of $\beta\gamma$ must reside in the interval $]0, 2[$ for a positive definite matrix Q to exist such that equation (3.2.25) to holds. □

Rendering $\beta\gamma$ positive definite through a suitable choice of γ can easily be done if the definiteness of β is known, i.e., if β is known to be either positive or negative definite. In this case the problem of selecting γ reduces to selecting a sufficiently small scalar number of the same sign as the eigenvalues of β. If β satisfies neither of those properties, then more knowledge of β is required. This requirement is a recurring issue across the literature, [8, 39, 40, 42] and aligns with the observations by Nussbaum in [56] regarding the impossibly of solving this problem with a rational dynamic controller when the sign of the controller gain is unknown, in that case a specific class of controllers is needed. This type of controllers have the property that as the controller output's magnitude increases so does the frequency with which it changes sign. These have been widely studied, and have come to be known as Nussbaum gain controllers. Despite their capacity to stabilize unknown systems in which not even the control gain is known, their aggressive transient behaviour and lack of robustness makes them somewhat impractical for implementation [57].

3.3 Estimation through dirty derivatives

In this section we motivate the use of the so-called dirty derivatives as a state estimation technique for data-driven control. Let $x : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function, we call $v : \mathbb{R}^+ \to \mathbb{R}$ a dirty derivative of $x(t)$ if their Laplace transforms, respectively $V(s)$ and $X(s)$, satisfy the relationship:

$$V(s) = \frac{\sigma s}{s+\sigma} X(s),$$

for some positive $\sigma \in \mathbb{R}^+$. If one considers $sX(s)$ as the transform of the derivative of $x(t)$, it is easy to see that the dirty derivative is nothing more than the output of a low-pass filter with input $\dot{x}(t)$ and a pole set at σ. While this implies that they can be quite useful to filter a noisy derivative signal, the property that makes dirty derivatives particularly interesting is that we can obtain $v(t)$ solely based on $x(t)$ by means of the dynamical system:

$$\dot{q}(t) = -\sigma(q(t) + \sigma x(t)) \tag{3.3.1}$$

$$v(t) = q(t) + \sigma x(t), \tag{3.3.2}$$

providing a low-gain approach for "estimating"[2] derivatives through integration. This itself results in discrete approximations which do not required dividing by the sampling time, which in on itself greatly reduces the effect of measurement noise in the estimates. This can be seen when using a simple forward Euler approximation method:

$$q(k+1) = q(k) - \sigma T\left(q(k) + \sigma x(k)\right) \tag{3.3.3}$$

$$v(k+1) = q(k+1) + \sigma x(k). \tag{3.3.4}$$

Dirty-derivative estimates can be recursively obtained for derivatives of higher order. Let $x_1(t)$ be a smooth signal of which we wish to estimate its $n-1$ first derivatives, and denote by $\widehat{x}_i$ the estimate of $x^{(i-1)}(t)$ for $i \in \{2, \cdots, n\}$. Then the dirty-derivative estimates are

[2] I use quotation marks here as dirty derivatives are not exactly estimates of the derivatives of a signal. In a slight abuse of notation we omit the quotation marks going forward.

defined as:

$$\begin{aligned}\dot{q}_1(t) &= -\sigma(q_1(t) + \sigma x_1(t)),\\ \widehat{x}_2(t) &= q_1(t) + \sigma x_1(t),\\ \dot{q}_i(t) &= -\sigma(q_i(t) + \sigma \widehat{x}_i(t)),\\ \widehat{x}_{i+1}(t) &= q_i(t) + \sigma \widehat{x}_i(t),\end{aligned} \tag{3.3.5}$$

for all $i \in \{2, \cdots, n-1\}$. Similarly to (3.3.3), we can obtain a discrete approximation of this system through the forward Euler method that contains no divisions by the sampling time T. Note that (3.3.5) implies that the dirty derivative's own derivatives become:

$$\dot{\widehat{x}}_{i+1} = \dot{q}_i + \sigma \dot{\widehat{x}}_i = -\sigma(q_i + \sigma \widehat{x}_i) + \sigma \dot{\widehat{x}}_i = -\sigma(\widehat{x}_{i+1} - \dot{\widehat{x}}_i).$$

The analysis in chapter 4 is based on the latter expression.

Khalil introduced cascaded high-gain observers in [58], of which dirty-derivative-based observers can be shown to be a special case, with particularly well-behaved autonomous dynamics due to all eigenvalues being placed at $-\sigma$. This type of observers have been widely used in practical [2, 59] and theoretical [60–63] applications for decades but, to the best of my knowledge, no stability proof existed prior to Khalil's work. To portray the potential benefits of using dirty derivatives as a proxy for derivative estimation, I provide two simulation examples: the first one presenting the dirty derivatives as an estimation method and comparing its performance with a state-of-the-art observer, a low-power peaking-free high-gain observer; and the second one presents the improvement in close-loop performance when replacing the estimation method introduced in Section 2.5 with the forward Euler approximation of system (3.3.5).

3.3.1 Dirty derivatives as an observer

We begin this subsection by reminding the reader that a dirty derivative is not exactly an estimate of the derivative of a signal. This is important to notice as one would expect the performance of any observer to outperform a dirty-derivative-based estimation. While this is true in the absence of noise, the low-pass filtering effect, low implementation gain and estimation through integration aspect of dirty derivatives provide such robustness against noise that we found dirty derivatives to routinely match or outperform classical observers when measurement noise is present in the simulation.

For the examples that follows we consider the recently developed low-power peaking-free high-gain observer [50], which has been shown to outperform traditional low-power high-gain observers. We reproduce the example presented therein, including observer gains and system's initial conditions. The system under consideration is given by:

$$\dot{x}_1 = x_2 \tag{3.3.6}$$

$$\dot{x}_2 = x_3 \tag{3.3.7}$$

$$\dot{x}_3 = x_4 \tag{3.3.8}$$

$$\dot{x}_4 = x_5 \tag{3.3.9}$$

$$\dot{x}_5 = 0.2(x_1^2 - 1) - x_2 - x_3 - 4x_4 - x_5, \tag{3.3.10}$$

and the low-power peaking-free high-gain observer presented in equations (11a) in [50] has full knowledge of the dynamics. We obtain our dirty-derivative estimates from system (3.3.5) with $n = 5$ and $\sigma = 1$. While in the presence of noise the dirty-derivative-based observer matches or outperforms the high-gain obeserver in all the estimates, we present only the first two derivatives to simplify the exposition.

Figure 3.1 portrays the original signal $x(t)$ overlaid onto the noisy signal fed to both observers obtained by adding measurement noise to $x(t)$ generated as white noise with frequencies in the band 0-200Hz. Figures 3.2 and 3.3 respectively compare the estimates and absolute estimation errors for the first and second derivatives of $x(t)$ as estimated under measurement noise by both observers.

I believe these figures underscore the potential robustness against measurement noise that dirty derivatives can offer. The next subsection emphasizes this point in a close-loop example.

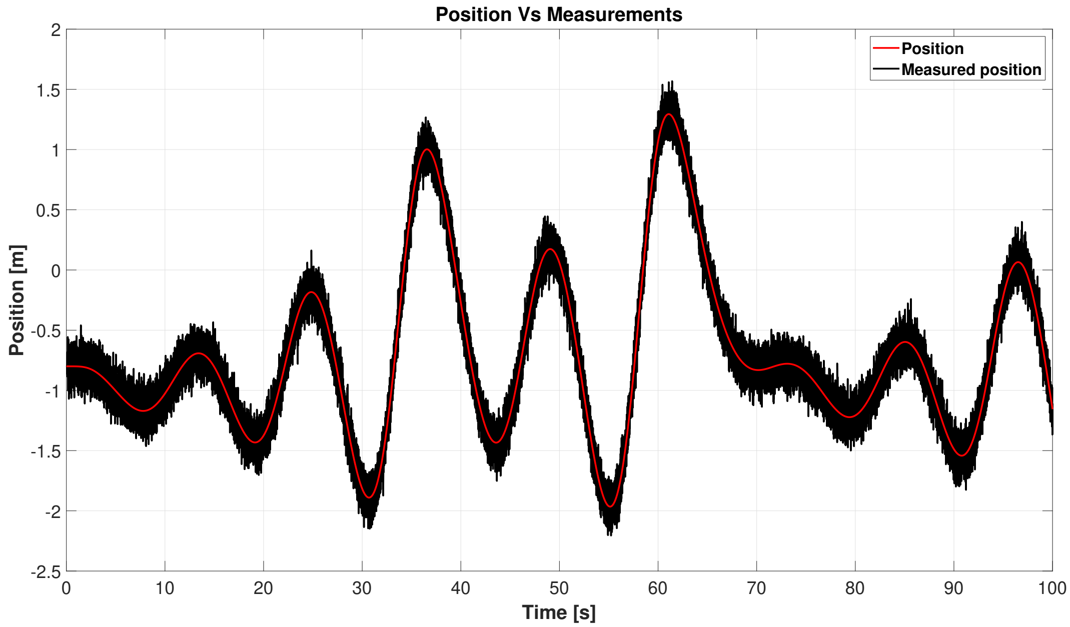

Figure 3.1: Original signal overlaid onto the noisy signal created by adding measurement noise generated as white noise in the band 0-200 Hz.

3.3.2 Dirty derivatives in data-driven control

For this section we simulate the model presented in Section 2.8, specifically the system described by (2.8.1), comparing the performance of the data-driven controller designed in Section 2.6 in close loop with the least-squares estimation method developed in Section 2.5 and the dirty-derivative estimation method. As mentioned before, the least-squares estimation is highly sensitive to noise, so it is no surprise that the data-driven controller performs noticeably better when in close-loop with the dirty-derivative estimation.

Figures 3.4 and 3.5 respectively compare the trajectories and absolute trajectory tracking errors for the data-driven controller designed in Section 2.6 in close-loop with the least-squares estimation method and the dirty derivative estimation method under the measurement noise modeled as white noise with an amplitude of $0.05m$.

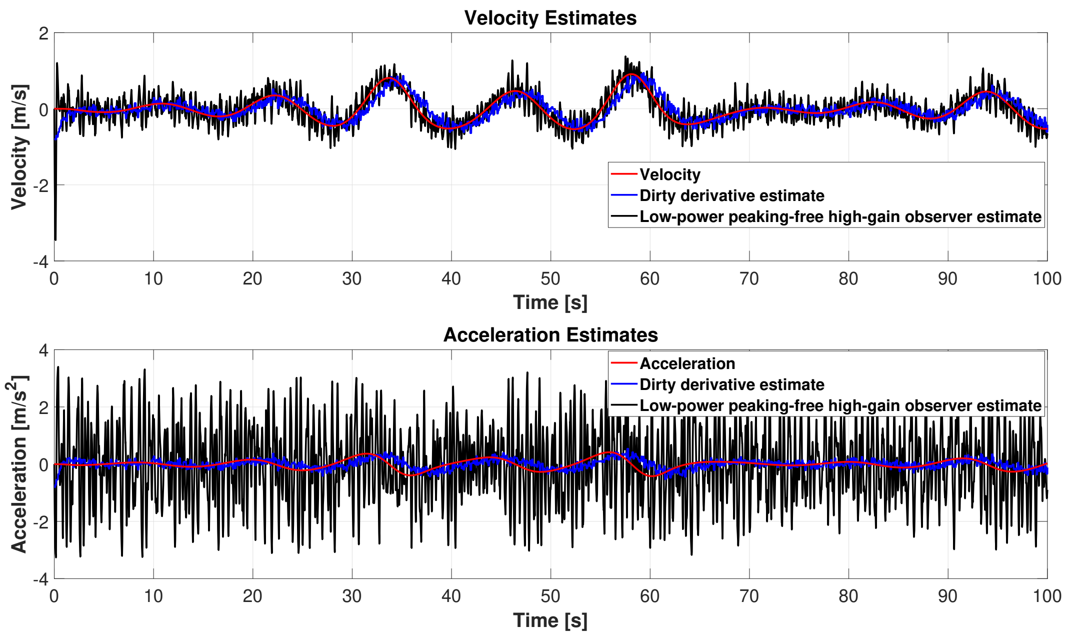

Figure 3.2: Comparison between estimates for the first and second derivatives of the signal in Figure 3.1 provided by a peaking-free low-power high-gain observer and the dirty-derivative method.

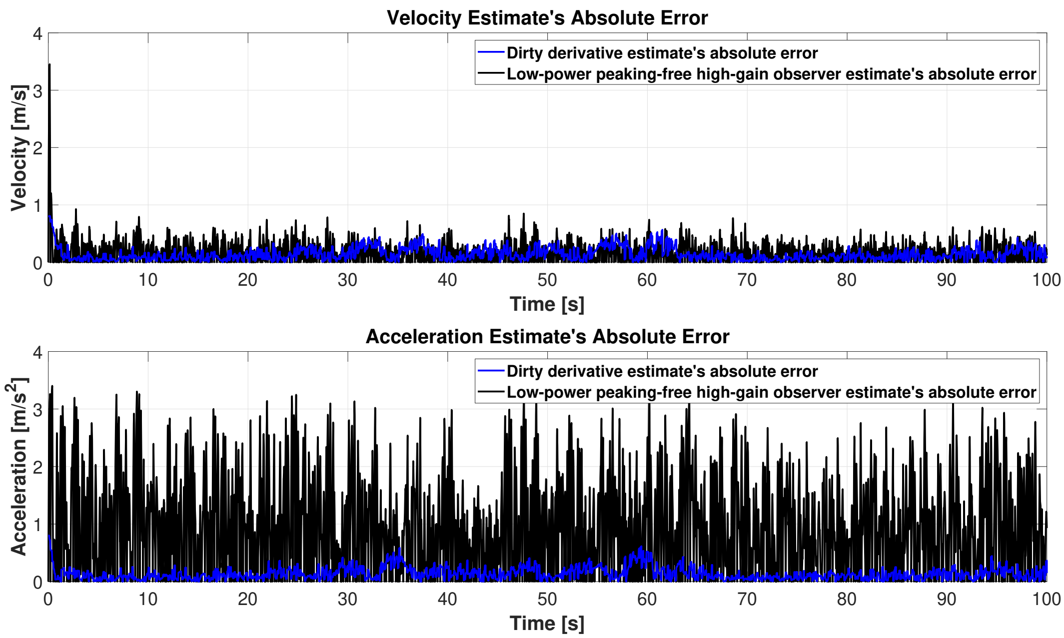

Figure 3.3: Comparison between absolute estimation error for the first and second derivatives of the signal in Figure 3.1 provided by a peaking-free low-power high-gain observer and the dirty-derivative method.

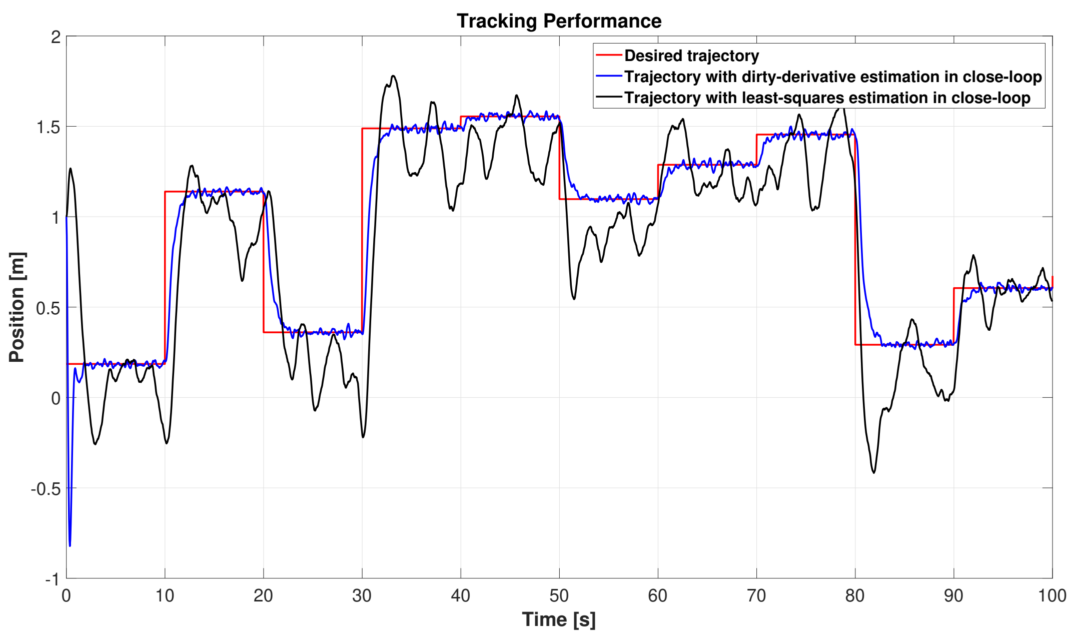

Figure 3.4: Comparison between the desired trajectory and the achieved trajectories between the data-driven controller designed in Section 2.6 in close-loop with the least-squares estimation method and the dirty derivative estimation method under measurement noise modeled as white noise with an amplitude of $0.05m$.

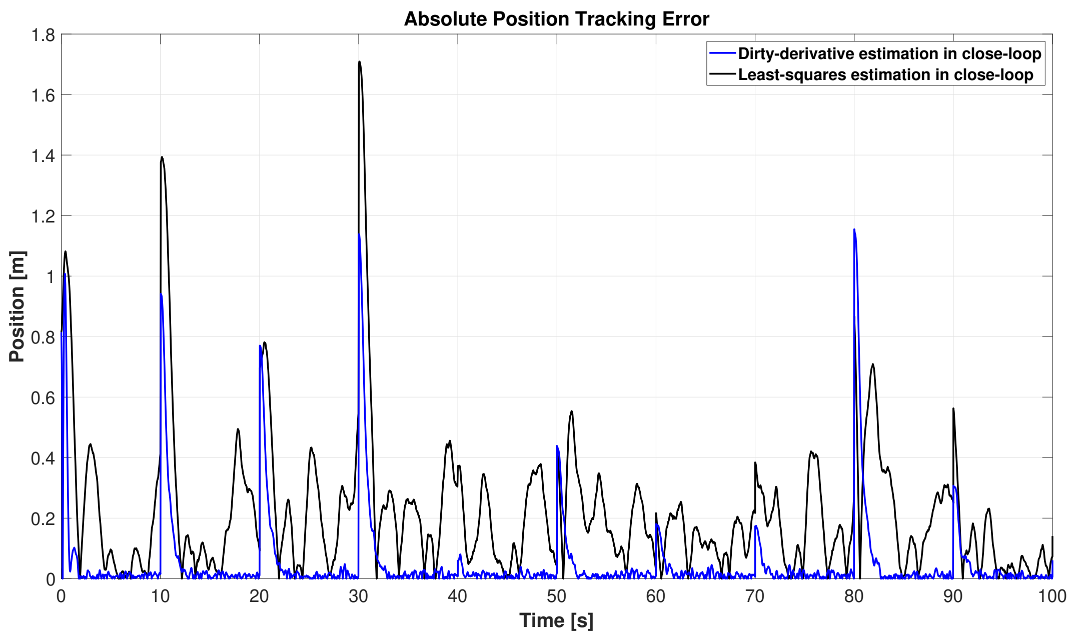

Figure 3.5: Comparison between absolute tracking error between the data-driven controller designed in Section 2.6 in close-loop with the least-squares estimation method and the dirty derivative estimation method under measurement noise modeled as white noise with an amplitude of $0.05m$.

CHAPTER 4

Data-driven control via dirty derivatives and output-feedback stabilization

4.1 Introduction

As stated in Chapter 1, the main contribution of this chapter is to provide a concrete example on how simple and widely known control techniques may be combined to avoid most of the challenges currently faced by data-driven control. To that end I leverage the linear dynamic controller developed in Chapter 2 together with the dirty-derivative-based extended-state high-gain observer, introduced in Chapter 3. This linear dynamic controller allows us to avoid the need for bounds on the control gain, usually required by extended-state-observer-based controllers as is the case in active disturbance rejection control [21, 22]. The dirty-derivative-based observer enjoys the properties of traditional high-gain observers without requiring any information of the underlying system's dynamics, making it especially useful when working with unknown partially feedback linearizable systems and providing more robust estimates against measurement noise when compared to least-squares-based estimation. The resulting scheme may be compared to the one recently presented in [64], where semi-global practical stability of unknown partially-feedback linearizable SISO systems under unmodeled dynamics is achieved. The controller herein proposed offers better stability guarantees as long as the unmodeled dynamics do not affect the relative degree of the system, recovering the stability guarantees provided in [64] otherwise. In particular, I demonstrate semi-global exponential stability in two cases: when the system is feedback linearizable; and when it is partially feedback linearizable and the unforced zero dynamics are exponentially stable. These results are further extended to global exponential stability when the unknown functions in the output dynamics and their Jacobians are globally bounded and the zero dynamics are globally Lipschitz. One may apply the techniques developed in Chapter 3 to the forward Euler discretization of this controller to reach similar stability results as therein.

4.2 Problem Statement

We consider an unknown single-input single-output nonlinear system of the form:

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x),$$

where now $f : \mathbb{R}^{\overline{n}} \to \mathbb{R}^{\overline{n}}$, $g : \mathbb{R}^{\overline{n}} \to \mathbb{R}^{\overline{n}}$, and $h : \mathbb{R}^{\overline{n}} \to \mathbb{R}$ are smooth functions and we denote by $y \in \mathbb{R}$, $x \in \mathbb{R}^{\overline{n}}$ and $u \in \mathbb{R}$, the output, state and input respectively, with $\overline{n} \in \mathbb{N}$. We make the assumption that the output function h has relative degree[1] $n-1$, where $n - 1 + \rho = \overline{n}$ for some $n,\ \rho \in \mathbb{N}$, i.e., this system is partially feedback linearizable. This implies that $L_g L_f^i h(x) = 0$ for $i = 0, \dots, n-3$ and $L_g L_f^{n-2} h(x) \neq 0$ for all $x \in \mathbb{R}^n$. Since the function $L_g L_f^{n-2} h$ is continuous and never zero, its sign is constant. We assume the sign of $L_g L_f^{n-2} h$ to be known and, without loss of generality, take it to be positive. Invoking the partial feedback linearizability assumption, we can rewrite the unknown dynamics in the coordinates:

$$(\eta, z) = \Psi(x) = (h(x), L_f h(x), \dots, L_f^{n-2} h(x)), \tag{4.2.1}$$

[1] The choice of $n-1$ instead of n simplifies notation in the main theorem's proof.

with dynamics:

$$\begin{aligned}
\dot{\eta}_1 &= \eta_2 \\
&\vdots \\
\dot{\eta}_{n-2} &= \eta_{n-1} \\
\dot{\eta}_{n-1} &= \alpha(\eta, z) + \beta(\eta, z)u \\
\dot{z} &= \omega(\eta, z) \\
y &= \eta_1,
\end{aligned} \tag{4.2.2}$$

where $\alpha = L_f^{n-1}h \circ \Psi^{-1}$ and $\beta = L_g L_f^{n-2} h \circ \Psi^{-1}$. We note that f, g, and h are unknown and so are α, β and ω. This description of the system has the advantage of using the two scalar valued functions α and β to describe the full output dynamics, independently of the value of m or n.

4.3 Controller design

We propose a surprisingly simple dynamic controller in order to stabilize the system despite the function $\alpha(\eta, z)$ and, more importantly, the controller gain $\beta(\eta, z)$ being unknown. To do so, we first introduce the following notation:

- Given a column vector $x \in \mathbb{R}^n$, and some $i, j \in \{1, 2, \dots, n\}$, $i \leq j$, we denote by $x_{i:j} = (x_i, x_{i+1}, \dots, x_j)$ the column vector composed of the entries i through j of x.
- We define $A \in \mathbb{R}^{(n-1)\times(n-1)}$ and $B \in \mathbb{R}^{(n-1)\times 1}$ as:

$$A = \begin{bmatrix} 0_{(n-2)\times 1} & I_{n-2} \\ 0 & 0_{1\times(n-2)} \end{bmatrix}, \quad B = \begin{bmatrix} 0_{(n-2)\times 1} \\ 1 \end{bmatrix}.$$

Drawing inspiration from the controller presented in Chapter 2, we propose the use of a dynamic controller of the form:

$$\dot{u} = -\gamma \left(\dot{\eta}_{n-1} - K\eta \right), \tag{4.3.1}$$

where $\gamma \in \mathbb{R}_{>0}$ and $K \in \mathbb{R}^{1\times(n-1)}$ is such that the equality:

$$(A + BK)^T P + P(A + BK) = -Q, \tag{4.3.2}$$

is satisfied for some symmetric positive definite matrices P and Q, existence of which is guaranteed due to (A, B) being a controllable pair. The motivation for the use of this controller is that it attempts to asymptotically enforce the equality $\dot{\eta}_{n-1} = K\eta$ without the need of explicitly estimating $\beta(\eta, z)$. If one could enforce that equality, then the output dynamics would become:

$$\dot{\eta} = (A + BK)\eta.$$

In the following, we use η_n to denote $\dot{\eta}_{n-1}$ and η to denote $(\eta_1, \ldots, \eta_n)$, which allows us to rewrite (4.3.1) as:

$$\dot{u} = -\gamma\left(\eta_n - K\eta_{1:n-1}\right). \tag{4.3.3}$$

Closing the loop with controller (4.3.3) in state-feedback fashion results in the closed-loop output-dynamics system:

$$\begin{aligned}
\dot{\eta}_1 =& \eta_2 \\
\vdots& \\
\dot{\eta}_{n-1} =& \eta_n \\
\dot{\eta}_n =& \varphi_z(\eta_{1:n}, z)\omega + \varphi_\eta(\eta_{1:n}, z)\eta_{2:n} - \beta(\eta_{1:n-1}, z)\gamma\left(\eta_n - K\eta_{1:n-1}\right) \\
y =& \eta_1
\end{aligned} \tag{4.3.4}$$

where

$$\begin{aligned}
\varphi_\eta(\eta, z) =& \frac{\partial\alpha(\eta_{1:n-1}, z)}{\partial\eta_{1:n-1}} + \frac{\partial\beta(\eta_{1:n-1}, z)}{\partial\eta_{1:n-1}} u \\
=& \frac{\partial\alpha}{\partial\eta_{1:n-1}} + \frac{\partial\beta}{\partial\eta_{1:n-1}}\beta^{-1}(\eta_n - \alpha),
\end{aligned}$$

and where we obtained the final expression by using the identity:

$$\eta_n = \alpha(\eta_{1:n-1}, z) + \beta(\eta_{1:n-1}, z)u.$$

We define $\varphi_z(\eta, z)$ in a similar manner as $\varphi_\eta(\eta, z)$, where the partial derivatives of α and β are taken with respect to z instead of $\eta_{1:n-1}$.

As we do not have access to the system's state, the controller (4.3.3) is implemented by using state estimates provided by a dirty-derivative-based observer.

4.4 State estimation

The motivation behind the use of dirty-derivative estimation in place of traditional estimation methods is bipartite. Dirty derivatives have long been used in industry as the preferred derivative estimation method due to their innate noise filtering properties [2,59,65,66] when compared to algebraic estimation. Moreover dirty derivatives do not require knowledge of the control input, or control gain, to be implemented. This last trait is particularly useful in our setting, allowing us to use this observers in conjunction with the data-driven controller developed in Chapter 2.

As most high-gain observers, dirty derivatives suffer from peaking during an initial transient period. To ameliorate this issue, saturation is applied to the estimates as proposed in [58]. To that end we define the function $\mathrm{sat}(y, M) = M \min\{\frac{|y|}{M}, 1\}\mathrm{sign}(y)$. Let $\widehat{\eta}$ denote the vector $\left[\widehat{\eta}_1, \dots, \widehat{\eta}_n\right]^T$ where $\widehat{\eta}_i$ is an estimate of η_i, for every $i \in \{1, 2, \dots, n\}$, and let

$\xi \in \mathbb{R}^n$ denote the internal state of our observer. Then $\widehat{\eta}$ satisfies the dynamics:

$$\begin{aligned}
\widehat{\eta}_1 &= \eta_1 \\
\dot{\xi}_1 &= -\sigma(\xi_1 - \eta_1) \\
\widehat{\eta}_2 &= \text{sat}(-\sigma(\xi_1 - \eta_1), M_1) \\
\dot{\xi}_2 &= -\sigma(\xi_2 - \widehat{\eta}_2) \\
\widehat{\eta}_3 &= \text{sat}(-\sigma(\xi_2 - \widehat{\eta}_2), M_2) \\
&\vdots \\
\dot{\xi}_n &= -\sigma(\xi_n - \widehat{\eta}_n) \\
\widehat{\eta}_n &= \text{sat}(-\sigma(\xi_n - \widehat{\eta}_n), M_n),
\end{aligned} \tag{4.4.1}$$

where $M_1, \ldots, M_n$ are positive design constants. The selection of constants M_i is quite straightforward and requires no knowledge of the system's dynamics. Whether the user is performing set-point regulation or trajectory tracking, these constants should be selected such that the set in which the output y and its derivatives are desired to be contained is a proper subset of $M = [-M_1, M_1] \times [-M_2, M_2] \times \cdots \times [-M_n, M_n]$. As shown in the main result of the paper, Theorem 4.5.1, as long as the initial conditions of the system belong to said set, sufficiently large choices of γ and σ guarantee said set to be forward invariant.

While the dirty derivatives as presented in (4.4.1) might seem different that those defined in Chapter 3 even when saturation doesn't take place, both systems produce the same estimates under the assumption of equal initial conditions. We use this representation here to more closely resemble the system analysed in [58].

4.5 Main result

Consider the following set of assumptions:

(i) The function $\beta(\eta_{1:n-1}, z)$ satisfies:

$$\beta(\eta_{1:n-1}, z) > 0 \quad \forall \eta_{1:n-1} \in \mathbb{R}^{n-1}, \forall z \in \mathbb{R}^{\rho}.$$

(ii) The zero dynamics are ISS with respect to $\eta_{1:n-1}$.

(iii) The origin is an exponentially stable equilibrium of the zero dynamics when $\eta_{1:n-1} = 0$.

Theorem 4.5.1. *Let assumptions (i)-(ii) hold. Consider system* (2.2.1)*, for any compact set* $\mathcal{S} \subset \mathbb{R}^{\overline{n}}$ *there exists* $\underline{\gamma} \in \mathbb{R}_{>0}$*, such that for any* $\gamma \in (\underline{\gamma}, \infty)$ *there exists* $\underline{\sigma} \in \mathbb{R}_{>0}$ *such that for any* $\sigma \in (\underline{\sigma}, \infty)$ *the dynamic controller defined by* (4.3.3)*, implemented using the estimates provided by the dirty-derivative-based observer defined in* (4.4.1)*, renders the origin into:*

a) a semi-globally practically stable equilibrium for the closed-loop system.

b) a semi-globally exponentially stable equilibrium for the closed-loop system if system (2.2.1) *is feedback-linearizable, i.e., there are no zero dynamics.*

c) a semi-globally exponentially stable equilibrium for the closed-loop system if assumption (iii) holds.

Furthermore, if the functions $\alpha(\eta_{1:n-1}, z)$*,* $\beta(\eta_{1:n-1}, z)$ *and their Jacobians are uniformly globally bounded then one recovers the global versions of cases a) and b), and if* $w(\eta_{1:n-1}, z)$ *is also globally Lipschitz, the global version of case c).*

The proof of this theorem can be found in the appendix.

Although the previous result only claims that trajectories converge to the origin, it can be readily applied to trajectory tracking problems by considering convergence to zero of the error between the system's trajectory and the desired trajectory to be tracked. Extending this result to multiple-input multiple-output systems is conceptually simple and can be achieved by applying the observer (4.4.1) to each output individually and selecting the matrix γ so that the matrix product $\beta\gamma$, is positive definite, much like what was done in Chapter 3.

4.6 Conclusion

In this chapter I have provided a strikingly simple data-driven linear dynamic controller to semi-globally stabilize unknown partially-feedback linearizable SISO nonlinear systems. The required knowledge of the system is only the relative degree of the output and the sign of the control gain, dispensing with the need of upper bounds for the latter, as is often the case in adaptive and data-driven control methods. Moreover, the controller does not require persistency of excitation to be enforced or previously recorded data to be provided. This was achieved using well-known control techniques, a linear dynamic controller and a dirty-derivative-based observer. Through a Lyapunov-based proof I demonstrate several different degrees of stability are attainable depending on the unknown system's properties. This positions the proposed controller as an easy-to-use, straight forward control method in the absence of system models.

Appendix

Proof of Theorem 4.5.1

Proof. Let $\Psi(\mathcal{S})$ denote the image of $\mathcal{S}$ under the diffeomorphism Ψ, given by (4.2.1), which satisfies $\Psi(0) = 0$. Then $\Psi(\mathcal{S})$ is the set of initial conditions for the state of system (4.2.2) given the set of initial conditions $\mathcal{S}$ for system (2.2.1). Due to Ψ being continuous, $\Psi(\mathcal{S})$ is a compact set. Consider the function $\phi : \mathbb{R}^{n-1} \times \mathbb{R}^{\rho} \times \mathbb{R} \to \mathbb{R}^{m+1}$ by:

$$\phi(\eta_{1:n-1}, z, u) = (\eta_{1:n-1}, \alpha(\eta_{1:n-1}, z) + \beta(\eta_{1:n-1}, z)u - K\eta_{1:n-1}, z). \tag{4.6.1}$$

Given the set $\Psi(\mathcal{S})$ and any compact set $U_0 \subset \mathbb{R}$ of initial inputs u_0^*, let the set $\overline{\mathcal{S}} \subset \mathbb{R}^{m+1}$ denote the image of $\phi(\eta_{1:n-1}, z, u)$ when $(\eta_{1:n-1}, z, u)$ range in $\Psi(\mathcal{S}) \times U_0$. Due to ϕ being a bijection, it follows that the sets $\Psi(\mathcal{S}) \times U_0$ and $\overline{\mathcal{S}}$ are isomorphic to each other. Thus, without loss of generality, we base the following proof on system (4.2.2) and the set of initial conditions $\overline{\mathcal{S}}$.

For brevity, we may use $\alpha,\ \beta,\ \varphi_\eta, \varphi_z$ and ω to denote $\alpha(\eta_{1:n-1}, z),\ \beta(\eta_{1:n-1}, z), \varphi_\eta(\eta_{1:n}, z)$, $\varphi_z(\eta_{1:n}, z)$ and $\omega(\eta_{1:n-1}, z)$ respectively. Let e_u and $e_{i,0}$ be defined as:

$$e_u = \eta_n - K\eta_{1:n-1}, \qquad e_{i,0} = \widehat{\eta}_i - \eta_i, \tag{4.6.2}$$

for all $i \in \{2, \ldots, n\}$. We denote by e the vector $(e_{1,0}, \ldots, e_{n,0})$, where[2] $e_{1,0} = 0$. We use (4.6.2)-(4.6.12) to rewrite $\dot{\eta}_n$ as:

$$\dot{\eta}_n = \varphi_z(\eta_{1:n}, z)\omega + \varphi_\eta(\eta_{1:n}, z)\dot{\eta}_{1:n-1} - \beta(\eta_{1:n-1})\gamma(e_u + \overline{K}e),$$

and the derivatives of $\eta_1, \eta_2, \ldots, \eta_{n-1}$ as:

$$\dot{\eta}_{1:n-1} = A\eta_{1:n-1} + B\eta_n = (A + BK)\eta_{1:n-1} + Be_u, \tag{4.6.3}$$

where we used (4.6.2) to obtain the second equality. This enables us to compute the derivative of e_u:

$$\begin{aligned} \dot{e}_u &= \dot{\eta}_n - K\dot{\eta}_{1:n-1} \\ &= -\left(\beta\gamma + \varphi' B\right) e_u + \varphi_z\omega - \varphi'\overline{A}\eta_{1:n-1} - \beta\gamma\overline{K}e, \end{aligned} \tag{4.6.4}$$

where we define $\overline{A} = A + BK$ and $\varphi' = K - \varphi_\eta$, and $\overline{K} = \begin{bmatrix} -K & 1 \end{bmatrix}$.

To finalize this preamble we present the following useful lemma:

Lemma 4.6.1. *All triples a, b, ε with $a, b \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}_{>0}$ satisfy the inequality:*

$$\pm\, 2a^T b \leq \varepsilon \left\|a\right\|^2 + \frac{1}{\varepsilon}\left\|b\right\|^2.$$

First step: we show that system (4.3.4) is ISS with respect to z and e. Consider the Lyapunov function $V_1 : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ given by $V_1(\eta_{1:n-1}, e_u) = \eta_{1:n-1}^T P\eta_{1:n-1} + e_u^2$, where P

[2] This is done to simplify bookkeeping throughout the proof.

satisfies (4.3.2) by assumption. We compute the time-derivative of $V_1(\eta_{1:n-1}, e_u)$ as:

$$\begin{aligned}
\dot{V}_1(\eta_{1:n-1}, e_u) =& \eta_{1:n-1}^T P\overline{A}\eta_{1:n-1} + \eta_{1:n-1}^T \overline{A}^T P\eta_{1:n-1} + 2\eta_{1:n-1}^T PBe_u \\
& - 2e_u \left((\beta\gamma + \varphi' B)\, e_u + \varphi_z\omega + \varphi'\overline{A}\eta_{1:n-1} + \beta\gamma\overline{K}e \right) \\
=& - \eta_{1:n-1}^T Q\eta_{1:n-1} + 2\eta_{1:n-1}^T PBe_u - 2(\beta\gamma + \varphi' B)e_u^2 - 2e_u\varphi'\overline{A}\eta_{1:n-1} \\
& - 2e_u\beta\gamma\overline{K}e - 2e_u\varphi_z\omega \\
\leq& - \lambda \left\|\eta_{1:n-1}\right\|^2 - 2(\beta\gamma - \|\varphi' B\|) \left\|e_u\right\|^2 + 2\left(\left\|\varphi'\overline{A}\right\| + \|PB\| \right) \|\eta_{1:n-1}\| \, \|e_u\| \\
& + 2\beta\gamma \left\|\overline{K}\right\| \|e_u\| \, \|e\| + 2\|e_u\| \, \|\varphi_z\omega\|
\end{aligned}$$

where we reach: the first equality by using (4.6.3) and (4.6.4); the second equality by way of (4.3.2); and the last inequality by taking norms and noting that $\eta_{1:n-1}^T Q\eta_{1:n-1} \leq -\lambda\eta_{1:n-1}^T\eta_{1:n-1}$, where $\lambda = \lambda_{\min}(Q)$. We now bound the terms of the last inequality in an individual fashion using lemma (4.6.1) to reach the inequalities:

$$\begin{aligned}
2\left\|\varphi'\overline{A}\right\| \|\eta_{1:n-1}\| \, \|e_u\| \leq& \frac{\lambda}{8} \left\|\eta_{1:n-1}\right\|^2 + \frac{8}{\lambda}\left\|\varphi'\overline{A}\right\|^2 e_u^2, \\
2\|PB\| \, \|\eta_{1:n-1}\| \, \|e_u\| \leq& \frac{\lambda}{8} \left\|\eta_{1:n-1}\right\|^2 + \frac{8}{\lambda}\|PB\|^2 e_u^2, \\
2\beta\gamma\left\|\overline{K}\right\| \|e\| \, \|e_u\| \leq& \beta\gamma e_u^2 + \beta\gamma \left\|\overline{K}\right\| \|e\|^2, \\
2\|\varphi_z\| \, \|e_u\| \leq& \frac{\beta\gamma}{2} e_u^2 + \frac{2}{\beta\gamma}\left\|\varphi_z\omega\right\|^2.
\end{aligned}$$

Assumption (ii) implies the function $\omega(\eta_{1:n-1}, z)$ is locally Lipschitz and satisfies $\omega(0, 0) = 0$. Let $\mathcal{V} \subset \mathbb{R}^{m+1}$ be a compact set, to be defined later in the proof, and L_ω denote the Lipschitz constant of ω when the state $(\eta_{1:n-1}, e_u, z)$ ranges in $\mathcal{V}$, it follows that:

$$\|\omega(\eta_{1:n-1}, z)\| \leq L_\omega \|(\eta_{1:n-1}, z)\|.$$

Consequently, one obtains the bound:

$$\|\varphi_z\omega(\eta_{1:n-1}, z)\|^2 \leq \|\varphi_z\|^2 L_\omega^2 \|\eta_{1:n-1}\|^2 + \|\varphi_z\|^2 L_\omega^2 \|z\|^2. \tag{4.6.5}$$

Substituting the previous bounds in (4.6.5) we reach:

$$\begin{aligned}\dot{V}_1(\eta_{1:n-1}, e_u) \leq & -\left(\frac{3\lambda}{4} - \frac{2}{\beta\gamma}\left\|\varphi_z\right\|^2 L_\omega^2\right)\left\|\eta_{1:n-1}\right\|^2 \\ & -\left(\frac{\beta\gamma}{2} - 2\left\|\varphi' B\right\| - \frac{8}{\lambda}\left\|\varphi'\overline{A}\right\|^2 - \frac{8}{\lambda}\left\|PB\right\|^2\right)\left\|e_u\right\|^2 \\ & + \beta\gamma\left\|\overline{K}\right\|\left\|e\right\|^2 + \frac{2}{\beta\gamma}\left\|\varphi_z\right\|^2 L_\omega^2\left\|z\right\|^2 .\end{aligned} \tag{4.6.6}$$

Inequality (4.6.6) establishes ISS of the system (4.6.3)-(4.6.4) with respect to z and e, provided[3] β, φ_z, φ_η and L_ω are bounded. Due to these functions being smooth, bounds are guaranteed to exist as long as their domains are contained in some compact set. We show that for any such compact domain, sufficiently large gains may be selected to guarantee stability thereby rendering our results semi-global. Note that β is a continuous function of $\eta_{1:n-1}$ and z, and φ_η, φ_z, and φ' are continuous functions of $\eta_{1:n}$ and z. Given that $\eta_n = e_u + K\eta_{1:n-1}$, the functions $\varphi_z(\eta_{1:n}, z)$, $\varphi_\eta(\eta_{1:n}, z)$ and $\varphi'(\eta_{1:n}, z)$ can be expressed as a functions of $\eta_{1:n-1}$, z and e_u: $\varphi_\eta(\eta_{1:n-1}, z, e_u)$, $\varphi_z(\eta_{1:n-1}, z, e_u)$ and $\varphi'(\eta_{1:n-1}, z, e_u)$. Then, if we consider states $(\eta_{1:n-1}, e_u, z)$ in a compact set[4] $\mathcal{V}$, we can define the following bounds:

$$\begin{aligned}\left\|\varphi_\eta(\eta_{1:n}, z)\right\| &\leq \max_{(\eta_{1:n-1}, e_u, z)\in\mathcal{V}} \left\|\varphi_\eta(\eta_{1:n-1}, z, e_u)\right\| = \overline{\varphi}_\eta, \\ \left\|\varphi_z(\eta_{1:n}, z)\right\| &\leq \max_{(\eta_{1:n-1}, e_u, z)\in\mathcal{V}} \left\|\varphi_z(\eta_{1:n-1}, z, e_u)\right\| = \overline{\varphi}_z, \\ \left\|\varphi'(\eta_{1:n})\right\| &\leq \max_{(\eta_{1:n-1}, e_u, z)\in\mathcal{V}} \left\|\varphi'(\eta_{1:n-1}, z, e_u)\right\| = \overline{\varphi}', \\ \beta(\eta_{1:n-1}, z) &\leq \max_{(\eta_{1:n-1}, e_u, z)\in\mathcal{V}} \beta(\eta_{1:n-1}, z) = \overline{\beta}, \\ \beta(\eta_{1:n-1}, z) &\geq \min_{(\eta_{1:n-1}, e_u, z)\in\mathcal{V}} \beta(\eta_{1:n-1}, z) = \underline{\beta},\end{aligned} \tag{4.6.7}$$

where $\underline{\beta} > 0$ due to assumption (i). Let $\overline{\mathcal{S}}_\eta$ and $\overline{\mathcal{S}}_z$ denote the projections of $\overline{\mathcal{S}}$ on its first n and last ρ dimensions, respectively. Note that $\overline{\mathcal{S}}_\eta$ and $\overline{\mathcal{S}}_z$ contain all the initial conditions

[3]Note that the existence of a bound for the latter implies a bound for φ'.

[4]If α, β and their Jacobians are globally bounded, and ω is globally Lipschitz, the need to define the compact set $\mathcal{V}$ disappears and one recovers the global results corresponding to the semi-global cases provided by this theorem.

for $(\eta_{1:n-1}, e_u)$ and z, respectively. Let $\mathcal{V}_{\mathcal{S}_\eta}$ be the smallest sub-level set of V_1 containing $\overline{\mathcal{S}}_\eta$, and $\mathcal{V}_1$ be any sub-level set of V_1 strictly larger[5] than $\mathcal{V}_{\mathcal{S}_\eta}$. Under assumption (ii) the zero dynamics are ISS with respect to $\eta_{1:n-1}$, thus there exist some Lyapunov function $V_2 : \mathbb{R}^\rho \to \mathbb{R}$ such that $\dot{V}_2(z) = -\kappa_1(\|z\|) + \kappa_2(\|\eta_{1:n-1}\|)$ where κ_1 and κ_2 are class $\mathcal{K}_\infty$ functions. Let $\mathcal{V}_2 = \{z \ : \ V_2(z) < c_2\}$ be any sub-level set of V_2 that is forward invariant when $\eta_{1:n-1}$ ranges in $\mathcal{V}_1$ and that contains the set of initial conditions $\overline{\mathcal{S}}_z$. We define $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$.

With the bounds (4.6.7) at hand we conclude the inequality:

$$\dot{V}_1(\eta_{1:n-1}, e_u) \leq -\frac{\lambda}{2}\left\|(\eta_{1:n-1}, e_u)\right\|^2 + \frac{\lambda\gamma}{4\underline{\gamma}}\|z\|^2 + \beta\gamma\left\|\overline{K}\right\|\|e\|^2, \tag{4.6.8}$$

holds for $(\eta_{1:n-1}, e_u, z) \in \mathcal{V}$ for any $\gamma \in \mathbb{R}_{>0}$ satisfying:

$$\begin{aligned} &\gamma > \max\{\underline{\gamma}, \underline{\gamma}\frac{c_1}{c_0}\}, \\ &\underline{\gamma} \geq \max\left\{\frac{2}{\underline{\beta}}\left(\frac{\lambda}{2} + 2\left\|\overline{\varphi}' B\right\| + \frac{8}{\lambda}\left\|\overline{\varphi}'\overline{A}\right\|^2 + \frac{8}{\lambda}\|PB\|^2\right), \frac{8}{\underline{\beta}\lambda}\overline{\varphi}_z^2 L_\omega^2\right\}, \end{aligned} \tag{4.6.9}$$

where $c_0 = \max_{(\eta_{1:n-1}, e_u)\in\mathcal{V}_{\mathcal{S}_\eta}}\left\|(\eta_i, e_u)\right\|^2$ and $c_1 = \max_{z\in\mathcal{V}_2}\|z\|^2$. The inequality (4.6.8) allows us to conclude that the output dynamics' system described by (4.6.3)-(4.6.4) is ISS with respect to z and e. Notice that z is guaranteed to remain in $\mathcal{V}$ as long as $(\eta_{1:n-1}, e_u)$ remains in $\mathcal{V}_1$. In the following we show that $\mathcal{V}_1$ is a forward invariant set.

Second step: We now consider the dirty-derivative estimates. Due to the peaking phenomena we partition the analysis in two segments: the first one dealing with the transient period involving peaking and during which saturation of the variables takes place; and the second one considering the steady-state phase where no saturation takes place and the estimation errors follow smooth dynamics.

Based on $\mathcal{V}$ we define M_i, the saturation limits in (4.4.1), as $M_i > \max_{(\eta_{1:n-1}, e_u, z)\in\mathcal{V}}\|\eta_i\|$ for all i in the set $\{2, 3, \ldots, n\}$. Consider (4.6.8), (4.6.9) and let $\overline{M} = \max_{i\in\{1,\ldots,n\}} M_i$, given

[5]This requirement is needed for the analysis of the transient period.

that the set of initial conditions for $\eta_{1:n-1}$ are contained in $\mathcal{V}_{\mathcal{S}_\eta} \subset \mathcal{V}_1$, the chosen values for M_i guarantee the bound:

$$\dot{V}_1(\eta_{1:n-1}, e_u) \leq -\frac{\lambda}{4}c_0 + 2n\beta\gamma \left\|\overline{K}\right\| \overline{M}, \tag{4.6.10}$$

holds as long as $(\eta_{1:n-1}, e_u) \in \mathcal{V}_1 - \mathcal{V}_{\mathcal{S}_\eta}$. Noting that the right hand side of (4.6.10) is positive for sufficiently large values of γ and that $\mathcal{S}_\eta \subseteq \mathcal{V}_{\mathcal{S}_\eta} \subset \mathcal{V}_1$, there exists a time $t_1 \in \mathbb{R}_{>0}$, independent on the value of σ, such that $(\eta_{1:n-1}, e_u) \in \mathcal{V}_1$ for all $t \in [0; t_1[$. Thus the state $(\eta_{1:n-1}, e_u, z)$ remains in $\mathcal{V}$, and as such bounded, for all $t \in [0; t_1[$. We may then apply Theorem 1 in [58]. This implies that there exists $\sigma^* \in \mathbb{R}^+$ such that for all $\sigma > \sigma^*$ the estimation error e remains bounded and there exists time $T(\sigma)$ satisfying $\lim_{\sigma\to\infty} T(\sigma) = 0$, such that $\|e\| < a\sigma^{-1}$ for all $t > T(\sigma)$, for some positive constant a. As a consequence, the peaking period may be rendered arbitrarily short, and the norm of the estimation error arbitrarily small, by choosing sufficiently large values for the gain σ. This allows us to guarantee forward invariance of $\mathcal{V}$ for all time $t > 0$. Specifically, one may select σ such that the bound $\|e\| < \sqrt{\lambda(4\beta\gamma\left\|\overline{K}\right\|)^{-1}c_0}$ holds for all $t > t^*$ for an arbitrarily small t^*. Given that t_1 is independent of σ, we require:

$$\sigma > \sigma^*, \text{where } T(\sigma^*) = t^* < t_1. \tag{4.6.11}$$

Consequently, it follows that $\dot{V}_1(\eta_{1:n-1}, e_u) < 0$ for all $t > t^*$ and $(\eta_{1:n-1}, e_u) \in \mathcal{V}_1 - \mathcal{V}_{\mathcal{S}_\eta}$. Given that $(\eta_{1:n-1}, e_u) \in \mathcal{V}_1$ for all $t \in [0, t_1]$, this implies trajectories starting in $\mathcal{V}_{\mathcal{S}_\eta}$ will remain in $\mathcal{V}$ for all time t.

To analyse the steady-state behaviour of the estimation error dynamics we introduce the auxiliary error states:

$$e_{i,j} = \widehat{\eta}_i^{(j)} - \eta_{i+j}, \quad \forall i \in \{2, \ldots, n\}, \forall j \in \{1, \ldots, n-i\} \tag{4.6.12}$$

where $\widehat{\eta}_i^{(j)}$ denotes the j^{th} derivative of $\widehat{\eta}_i$ with respect to time. Note that the maximum value of j is $n-i$, so that $i + j \leq n$ and the error term $e_{i,j}$ is always well-defined. This leaves us with a set of $\frac{n(n-1)}{2}$ error variables for the state estimates. We define $\overline{e}$ to be the vector containing all $e_{i,j}$ for $i \in \{2, \dots, n\}$ and $j \in \{0, \dots, n-i\}$. Given (4.6.12), the time derivatives of the error variables when no saturation is active are given by:

$$\begin{aligned} \dot{e}_{i,j} = \dot{\widehat{\eta}}_i^{(j)} - \dot{\eta}_{i+j} &= -\sigma \left(\widehat{\eta}_i - \dot{\widehat{\eta}}_{i-1} \right)^{(j)} - \dot{\eta}_{i+j} \\ &= -\sigma(\widehat{\eta}_i^{(j)} - \widehat{\eta}_{i-1}^{(j+1)}) - \dot{\eta}_{i+j}, \end{aligned} \tag{4.6.13}$$

where the second equality is obtained by substituting $\dot{\widehat{\eta}}_i$ with the corresponding right-hand side in the equations of the dynamical system (4.3.4). This results in the following two cases:

$$\begin{aligned} e_{2,j} =& -\sigma(\widehat{\eta}_2^{(j)} - \eta_2^{(j)}) - \dot{\eta}_{2+j} = -\sigma e_{1,j} - \dot{\eta}_{2+j}, \\ e_{i,j} =& -\sigma(\widehat{\eta}_i^{(j)} - e_{i-1,j+1} - \eta_{i+j}) - \dot{\eta}_{2+j} \\ =& -\sigma e_{i,j} + \sigma e_{i-1,j+1} - \dot{\eta}_{2+j}, \end{aligned}$$

where we used the identity $\widehat{\eta}_{i-1}^{(j+1)} = -e_{i-1,j+1} - \eta_{i+j}$ in the second case. With these at hand, consider the Lyapunov function $V_2 : \mathbb{R}^{\frac{n(n+1)}{2}} \to \mathbb{R}_{\geq 0}$ given by:

$$V(\overline{e}) = \overline{e}^T \overline{e} = \sum_{i=1}^{n} \sum_{j=0}^{n-i} e_{i,j}^2. \tag{4.6.14}$$

Computing the time derivative of (4.6.14) results in:

$$\begin{aligned} \dot{V}_2(\overline{e}) =& 2 \sum_{i=2}^{n} \sum_{j=0}^{n-i} e_{i,j} \dot{e}_{i,j} \\ =& -2 \left(\sigma \left[\sum_{i=2}^{n} \sum_{j=0}^{n-i} e_{i,j}^2 - \sum_{i=3}^{n} \sum_{j=0}^{n-i} e_{i,j} e_{i-1,j+1} \right] + (\varphi_\eta B + \beta\gamma) e_u \sum_{i=2}^{n} e_{i,n-i} \right. \\ & \left. -2\varphi_z \omega \sum_{i=2}^{n} e_{i,n-i} + \beta\gamma \overline{K} e \sum_{i=2}^{n} e_{i,n-i} + \varphi_\eta \overline{A} \eta_{1:n-1} \sum_{i=2}^{n} e_{i,n-i} + \sum_{i=2}^{n-1} \sum_{j=0}^{n-i-1} e_{i,j} \eta_{i+j+1} \right). \end{aligned} \tag{4.6.15}$$

Just as in the first step, we proceed by deriving upper bounds for all of the terms in (4.6.15). The first term can be written as:

$$-2\sigma \left[\sum_{i=2}^{n} \sum_{j=0}^{n-i} e_{i,j}^2 - \sum_{i=3}^{n} \sum_{j=0}^{n-i} e_{i,j} e_{i-1,j+1} \right] = 2\sigma \sum_{k=2}^{n} \left(-\sum_{\ell=0}^{k-1} e_{k-\ell,\ell}^2 + \sum_{\ell=0}^{k-2} e_{k-\ell,\ell} e_{k-\ell-1,\ell+1} \right),$$

where, through a slight abuse of notation, we define the sum $\sum_i^j 1$ to be zero whenever $j < i$. We claim that for any k there exists a constant $d_k \in \mathbb{R}_{>0}$ such that:

$$-\sum_{\ell=0}^{k-1} e_{k-\ell,\ell}^2 + \sum_{\ell=0}^{k-2} e_{k-\ell,\ell} e_{k-\ell-1,\ell+1} \leq -d_k \sum_{\ell=0}^{k-1} e_{k-\ell,\ell}^2. \tag{4.6.16}$$

This can be shown by first defining $\eta_\ell = e_{k-\ell,\ell}$ and applying the following lemma[6].

Lemma 4.6.2. *For any $k \in \mathbb{N}_{>0}$, the quadratic form $-\sum_{i=2}^k \eta_i^2 + \sum_{i=2}^{k-1} \eta_i \eta_{i+1}$ is negative definite.*

Finally, inequality (4.6.16) is reached by noting that the negative definiteness of the left terms implies the existence of $d_k \in \mathbb{R}_{>0}$ such that the inequality holds. Inequality (4.6.16) allows us to write:

$$\begin{aligned} -2\sigma \left[\sum_{i=2}^{n} \sum_{j=0}^{n-i} e_{i,j}^2 - \sum_{i=3}^{n} \sum_{j=0}^{n-i} e_{i,j} e_{i-1,j+1} \right] =& 2\sigma \sum_{k=2}^{n} \left(-\sum_{\ell=0}^{k-1} e_{k-\ell,\ell}^2 + \sum_{\ell=0}^{k-2} e_{k-\ell,\ell} e_{k-\ell-1,\ell+1} \right) \\ \leq& -2\sigma \sum_{k=2}^{n} d_k \sum_{\ell=0}^{k-1} e_{k-\ell,\ell}^2 = -2\sigma d \bar{e}^T \bar{e}, \end{aligned}$$

where $\sum_{i=2}^n \sum_{j=0}^{n-i} e_{i,j}^2 = \sum_{k=2}^n \sum_{\ell=0}^{k-1} e_{k-\ell,\ell}^2$, and $d = \min\{d_1, d_2, \ldots, d_n\}$. All other terms may

[6] This lemma can be easily proved by rewriting the quadratic form as $-\eta^T M \eta$, with $M \in \mathbb{R}^{k\times k}$ a symmetric matrix and noting that M can be reduced to upper triangular form with strictly negative elements on the diagonal by using row operations. This implies that $\det(M) < 0$ and since M is negative semidefinite by Gershgorin's circle theorem, it's enough to prove strict negative definiteness of M.

be bounded by resorting to lemma 4.6.2, resulting in the bounds:

$$\begin{aligned}
-2\left(\beta\gamma+\varphi_\eta B\right)e_u\sum_{i=2}^{n}e_{i,n-i} \leq& \frac{\lambda}{16}e_u^2+\frac{16n}{\lambda}\left(\beta^2\gamma^2+\left\|\varphi_\eta B\right\|^2\right)\overline{e}^T\overline{e},\\
-2\varphi_z\omega\sum_{i=2}^{n}e_{i,n-i} \leq& \frac{2n}{\sigma}\overline{\varphi}_z^2L_\omega^2\left\|\eta_{1:n-1}\right\|^2+\frac{2n}{\sigma}\overline{\varphi}_z^2L_\omega^2\left\|z\right\|^2+\frac{\sigma}{2}\overline{e}^T\overline{e},\\
-2\beta\gamma\overline{K}e\sum_{i=2}^{n}e_{i,n-i} \leq& 2n\beta\gamma\left\|\overline{K}\right\|\overline{e}^T\overline{e},\\
-2\varphi_\eta\overline{A}\eta_{1:n-1}\sum_{i=2}^{n}e_{i,n-i} \leq& \frac{\lambda}{16}\left\|\eta_{1:n-1}\right\|^2+\frac{16n}{\lambda}\left\|\varphi_\eta\overline{A}\right\|^2\overline{e}^T\overline{e},\\
-2\sum_{i=2}^{n-1}\sum_{j=0}^{n-i-1}e_{i,j}\eta_{i+j+1} \leq& c_3\overline{e}^T\overline{e}+\frac{\lambda}{16}\left\|\eta_{1:n-1}\right\|^2+\frac{\lambda}{16}\left\|e_u\right\|^2,
\end{aligned}$$

where $c_3=\left(\frac{16(n^2-5n+6)}{\lambda}+\frac{16(n-2)}{\lambda}+\frac{16(n-2)}{\lambda}KK^T\right)$. Note that all η terms in the right-hand side of the second equality have indices between 1 and $n-1$. Substituting all of these bounds in (4.6.15), we can bound $\dot{V}_2$ by:

$$\begin{aligned}
\dot{V}_2(\overline{e}) \leq& -\left(\frac{3}{2}\sigma d-\frac{16n}{\lambda}\left(\beta^2\gamma^2+\left\|\varphi_\eta B\right\|^2+\left\|\varphi_\eta\overline{A}\right\|^2\right)-2n\beta\gamma\left\|\overline{K}\right\|-c_3\right)\overline{e}^T\overline{e}+\frac{\lambda}{4}e_u^2\\
&+\left(\frac{\lambda}{8}+\frac{2n}{\sigma}\overline{\varphi}_z^2L_\omega^2\right)\left\|\eta_{1:n-1}\right\|^2+\frac{2n}{\sigma}\overline{\varphi}_z^2L_\omega^2\left\|z\right\|^2\\
\leq& -\sigma d\left\|\overline{e}\right\|^2+\frac{\lambda}{4}\left\|(\eta_{1:n-1},e_u)\right\|^2+\frac{\lambda\underline{\sigma}}{8\sigma}\left\|z\right\|^2,
\end{aligned} \tag{4.6.17}$$

where the last inequality holds for all $\sigma\in\mathbb{R}_{>0}$ satisfying:

$$\sigma\geq\underline{\sigma}=\max\left\{\frac{2}{d}\left(\frac{16n}{\lambda}\left(\overline{\beta}^2\gamma^2+\left\|\overline{\varphi}B\right\|^2+\left\|\overline{\varphi}_\eta\overline{A}\right\|^2\right)+2n\overline{\beta}\gamma\left\|\overline{K}\right\|+c_3\right),\frac{16n}{\lambda}\overline{\varphi}_z^2L_\omega^2\right\}. \tag{4.6.18}$$

Due to inequality (4.6.17) we conclude that the estimation error dynamics are ISS with respect to $\eta_{1:n-1}$, e_u and z.

Step three: we now show that we satisfy the assumptions of the small-gain theorem for the closed-loop system, when considering z as an external signal. We refer to Theorem 3.1 in [67], were we equate u_1 and u_2 in [67] to z. Assumption (8) is satisfied due to V_1 and V_2 being quadratic, assumptions (15) and (16) are satisfied with $\theta_1^u(|u_1|) = \theta_2^u(|u_2|) = d_1 = d_2 = 0$, and based on (4.6.6) and (4.6.17) we define:

$$\begin{aligned}
\alpha_1(V_1(\eta_{1:n-1}, e_u)) &= \frac{\lambda}{2\lambda_{\max}(\overline{P})} V_1(\eta_{1:n-1}, e_u), \\
\alpha_2(V_2(\overline{e})) &= \sigma d V_2(\overline{e}), \\
\theta_1^\eta(V_2(\overline{e})) &= \overline{\beta}\gamma \left\|\overline{K}\right\| V_2(\overline{e}), \\
\theta_2^\eta(V_1(\eta_{1:n-1}, e_u)) &= \frac{\lambda}{4\lambda_{\min}(\overline{P})} V_1(\eta_{1:n-1}, e_u),
\end{aligned}$$

where $\overline{P}$ is the block diagonal matrix with blocks P and 1. Following Corollary 3.1 in [67], we define $\kappa_1 = \frac{\overline{\beta}\gamma\left\|\overline{K}\right\|}{\sigma d}$ and $\kappa_2 = \frac{1}{2}\frac{\lambda_{\max}(\overline{P})}{\lambda_{\min}(\overline{P})}$, such that $\theta_1^\eta(V_2(\overline{e})) = \kappa_1 \alpha_2(V_2(\overline{e}))$ and $\theta_2^\eta(V_1(\eta_{1:n-1}, e_u)) = \kappa_2 \alpha_1(V_1(\eta_{1:n-1}, e_u))$, and note that both κ_1 and κ_2 are positive. It is straight-forward to see that by choosing σ to be sufficiently large one can make the interconnection gain to be arbitrarily small, fulfilling the requirement that $\kappa_1\kappa_2 < 1$, thus satisfying all the assumptions of Corollary 3.1, and consequently Theorem 3.1, in [67]. By applying said theorem we conclude that the output-dynamics in closed-loop with the proposed controller and observer are ISS with respect to z. Furthermore, [67] informs us that the composite function $\overline{V}(\eta_{1:n-1}, e_u, \overline{e}) = V_1(\eta_{1:n-1}, e_u) + c_4 V_2(\overline{e})$ is a valid Lyapunov function for said system, where c_4 satisfies $\kappa_1 < c_4 < \kappa_2^{-1}$. Given that $\kappa_2^{-1} \leq 2$ and that κ_1 can be made arbitrarily small by increasing σ, we may select $c_4 < 2$ and it follows that:

$$\dot{\overline{V}}(\eta_{1:n-1}, e_u, \overline{e}) < -\, c_5 \left\|(\eta_{1:n-1}, e_u, \overline{e})\right\|^2 + \left(\frac{c_4 \lambda \sigma}{8\sigma} + \frac{\lambda\gamma}{4\gamma}\right) \left\|z\right\|^2,$$

where[7] $0 \leq c_5 \leq (1 - \frac{c_4}{2})\frac{\lambda}{2}$. Note that γ and σ can always be increased, respecting the bounds (4.6.9), (4.6.11), and (4.6.18), such that the coefficient $\left(\frac{c_4\lambda\sigma}{8\sigma} + \frac{\lambda\gamma}{4\gamma}\right)$ is rendered arbitrarily small.

As a result we conclude:

a) If system (2.2.1) is feedback-linearizable, i.e., there are no zero dynamics, then the

[7] Note that we can increase σ to ensure $k_1 < \left(\sigma d k - \beta\gamma \left\|\overline{K}\right\|\right)$ without affecting the choice of c_4.

closed-loop system is rendered semi-globally exponentially[8]

b) If the zero dynamics are ISS with respect to $\eta_{1:n-1}$ and exponentially stable when $\eta_{1:n-1} = 0$ then the closed-loop system is rendered semi-globally exponentially stable. Exponential stability of the unforced system implies, through Theorem 4.14 in [23], that the unforced zero-dynamics admit a quadratic Lyapunov function V_2. Given than ω is locally Lipschitz in $\mathcal{V}$ it follows that $\dot{V}_2 \leq -c_6 \|z\|^2 + c_7 \|\eta_{1:n-1}\|^2$ for some c_6 and $c_7 \in \mathbb{R}^+$. Applying again the small-gain theorem as we have done in step 3 using $\overline{V}$ and V_2 recovers the result.

c) If the zero dynamics are ISS with respect to $\eta_{1:n-1}$, the closed-loop system is rendered semi-globally practically stable.

d) If α, β and their Jacobians are globally bounded then one recovers global stability results in cases a) and c), and if ω is also globally Lipschitz one recovers global exponential stability in b). This is due to there not being a need for defining $\mathcal{V}$. In this case the proof follows through without needing saturation in the observer, yet this is not advised in practice as it might result in aggressive transients.

□

[8]It can readily be shown that $\bar{e}$ is a linear function of e, $\eta_{3:n-1}$ and e_u, thus $\overline{V} = V_1 + c_4 V_2$ is a positive-definite, quadratic function of (η, e_u, e). It can be similarly shown that the time derivative of $\overline{V}$ is a negative-definite quadratic function of (η, e_u, e).

CHAPTER 5

Conclusion

The previous chapters have addressed the problem of stabilization of unknown nonlinear systems without making explicit use of a model. I have focused on partially and fully feedback single-input single-output linearizable system, utilizing both recent results and well-known techniques to circumvent issues and constraints well-documented in the field of adaptive control. The result is a pair of controllers, a discrete-time and a continuous-time one, characterized by their ease of implementation, minimal system model knowledge requirements and asymptotic stability guarantees. An extension of this results to the multiple-input multiple-output case was presented and its practical limitations analysed, matching what is currently the state-of-the-art.

There are several further avenues to expand the results herein contained, amongst them non-feedback linearizable systems and applications to specific classes of systems, such as Euler-Lagrangian systems. With collaborators and coauthors leading the charge, I am hopeful this research will provide the foundations for simple and safe, plug-and-play data-driven control of a large class of nonlinear systems.

REFERENCES

[1] P. Tabuada, W.-L. Ma, J. Grizzle, and A. D. Ames, "Data-driven control for feedback linearizable single-input systems," in *56th IEEE Conference on Decision and Control (CDC)*. IEEE, 2017.

[2] K. J. Aström and T. Hägglund, *PID Controllers: Theory, Design, and Tuning*. ISA - The Instrumentation, Systems and Automation Society, 1995.

[3] M. Fliess and C. Join, "Model-free control and intelligent pid controllers: towards a possible trivialization of nonlinear control?" *ArXiv*, vol. abs/0904.0322, 2009.

[4] ——, "Model-free control," *International Journal of Control*, vol. 86, no. 12, p. 2228–2252, Dec 2013. [Online]. Available: http://dx.doi.org/10.1080/00207179.2013.810345

[5] A. S. Bazanella, L. Campestrini, and D. Eckhard, *Data-driven controller design: the H2 approach*. Springer Science & Business Media, 2011.

[6] Z. Hou, R. Chi, and H. Gao, "An overview of dynamic-linearization-based data-driven control and applications," *IEEE Transactions on Industrial Electronics*, vol. 64, no. 5, pp. 4076–4090, 2016.

[7] J. C. Willems, P. Rapisarda, I. Markovsky, and B. L. De Moor, "A note on persistency of excitation," *Systems & Control Letters*, vol. 54, no. 4, pp. 325 – 329, 2005.

[8] T. Maupong and P. Rapisarda, "Data-driven control: A behavioral approach," *Systems & Control Letters*, vol. 101, pp. 37–43, 2009.

[9] E. Kaiser, J. N. Kutz, and S. L. Brunton, "Data-driven discovery of Koopman eigenfunctions for control," *arXiv preprint arXiv:1707.01146*, 2017.

[10] A. Mauroy, Y. Susuki, and I. Mezić, *Koopman operator in systems and control*. Springer, 2020.

[11] Y. Tan, W. H. Moase, C. Manzie, D. Nešić, and I. M. Mareels, "Extremum seeking from 1922 to 2010," in *Proceedings of the 29th Chinese control conference*. IEEE, 2010, pp. 14–26.

[12] A. Scheinker and M. Krstić, "Minimum-Seeking for CLFs: Universal Semiglobally Stabilizing Feedback Under Unknown Control Directions," *IEEE Transactions on Automatic Control*, vol. 58, no. 5, pp. 1107–1122, 2013.

[13] R. Kamalapurkar, P. S. Walters, J. A. Rosenfeld, and W. E. Dixon, *Reinforcement Learning for Optimal Feedback Control: A Lyapunov-based Approach*. Springer, 2018.

[14] Y. LeCun, Y. Bengio, and G. Hinton, "Deep Learning," *Nature*, vol. 521, no. 7553, pp. 436–444, 2015. [Online]. Available: https://doi.org/10.1038/nature14539

[15] B. D. Anderson and A. Dehghani, "Challenges of adaptive control–past, permanent and future," *Annual reviews in control*, vol. 32, no. 2, pp. 123–135, 2008.

[16] H. J. van Waarde, J. Eising, H. L. Trentelman, and M. K. Camlibel, "Data Informativity: A New Perspective on Data-Driven Analysis and Control," *IEEE Transactions on Automatic Control*, vol. 65, no. 11, pp. 4753–4768, 2020.

[17] L. Fraile, M. Marchi, and P. Tabuada, "Data-driven Stabilization of SISO Feedback Linearizable Systems," 2021. [Online]. Available: https://arxiv.org/abs/2003.14240v3

[18] M. Arcak and D. Nešić, "A framework for nonlinear sampled-data observer design via approximate discrete-time models and emulation," *Automatica*, vol. 40, no. 11, pp. 1931–1938, 2004. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S000510980400175X

[19] D. Nešić and A. R. Teel, "A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models," *IEEE Transactions on Automatic Control*, vol. 49, no. 7, pp. 1103–1122, 2004.

[20] P. C. Hansen, V. Pereyra, and G. Scherer, *Least squares data fitting with applications*. JHU Press, 2013.

[21] B.-Z. Guo and Z.-L. Zhao, *Active disturbance rejection control for nonlinear systems: An introduction.* John Wiley & Sons, 2016.

[22] H. Feng and B.-Z. Guo, "Active disturbance rejection control: Old and new results," *Annual Reviews in Control*, vol. 44, pp. 238–248, 2017.

[23] H. K. Khalil, *Nonlinear Systems*, ser. Pearson Education. Prentice Hall, 2002.

[24] A. J. Krener, "Approximate linearization by state feedback and coordinate change," *Systems & Control Letters*, vol. 5, no. 3, pp. 181–185, 1984.

[25] A. Isidori, *Nonlinear Control Systems*, ser. Communications and Control Engineering. Springer-Verlag London, 1995.

[26] G. Tao, *Adaptive control design and analysis.* John Wiley & Sons, 2003, vol. 37.

[27] K. J. Aström and B. Wittenmark, *Adaptive control.* Courier Corporation, 2013.

[28] K. S. Narendra and A. M. Annaswamy, *Stable adaptive systems.* Courier Corporation, 2012.

[29] E.-W. Bai and S. S. Sastry, "Persistency of excitation, sufficient richness and parameter convergence in discrete time adaptive control," *Systems & control letters*, vol. 6, no. 3, pp. 153–163, 1985.

[30] B. D. Anderson, "Adaptive systems, lack of persistency of excitation and bursting phenomena," *Automatica*, vol. 21, no. 3, pp. 247–258, 1985.

[31] J. C. Willems, P. Rapisarda, I. Markovsky, and B. L. De Moor, "A note on persistency of excitation," *Systems & Control Letters*, vol. 54, no. 4, pp. 325–329, 2005.

[32] J. Schmidhuber, "Deep learning in neural networks: An overview," *Neural Networks*, vol. 61, pp. 85 – 117, 2015. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0893608014002135

[33] L. Ljung, *System Identification: Theory for the User.* Prentice Hall, 1999.

[34] P. van Overschee and B. L. De Moor, *Subspace Identification for Linear Systems: Theory, Implementation, Applications.* Springer, 2012.

[35] P. Ioannou and J. Sun, *Robust and Adaptive Control.* Dover Publications, 2012.

[36] R. S. Sutton and A. G. Barto, *Reinforcement Learning, An Introduction.* MIT Press, 2018.

[37] I. Goodfellow, Y. Bengio, and A. Courville, *Deep Learning.* MIT Press, 2016.

[38] J. Coulson, J. Lygeros, and F. Dörfler, "Data-enabled predictive control: In the shallows of the DeePC," in *18th European Control Conference (ECC)*, 2019.

[39] C. D. Persis and P. Tesi, "Formulas for Data-Driven Control: Stabilization, Optimality, and Robustness," *IEEE Transactions on Automatic Control*, vol. 65, no. 3, pp. 909–924, 2020.

[40] I. Markovsky and P. Rapisarda, "Data-driven simulation and control," *International Journal of Control*, vol. 81, no. 12, pp. 1946–1959, 2008.

[41] G. Baggio, V. Katewa, and F. Pasqualetti, "Data-driven minimum-energy controls for linear systems," *IEEE Control Systems Letters*, vol. 3, no. 3, pp. 589–594, 2019.

[42] H. J. van Waarde, J. Eising, H. L. Trentelman, and M. K. Camlibel, "Data informativity: a new perspective on data-driven analysis and control," 2019.

[43] I. M. Mareels, H. Penfold, and R. Evans, "Controlling nonlinear time-varying systems via euler approximations," *Automatica*, vol. 28, no. 4, pp. 681 – 696, 1992. [Online]. Available: http://www.sciencedirect.com/science/article/pii/000510989290030J

[44] H. K. Khalil, *High-gain Observers in Nonlinear Feedback Control.* Society for Industrial and Applied Mathematics, 2017.

[45] P. Tabuada and L. Fraile, "Data-driven control for SISO feedback linearizable systems with unknown control gain," in *58th IEEE Conference on Decision and Control (CDC).* IEEE, 2019.

[46] W. Rudin, *Principles of Mathematical Analysis.* McGraw-Hill, 1976.

[47] J. Reger and J. Jouffroy, "On algebraic time-derivative estimation and deadbeat state reconstruction," in *Proceedings of the 48h IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference*, 2009, pp. 1740–1745.

[48] M. Mboup, C. Join, and M. Fliess, "A revised look at numerical differentiation with an application to nonlinear feedback control," in *15th Mediterranean Conference on Control and Automation*, 2007.

[49] P. Antsaklis and A. Michel, *Linear Systems.* Birkhauser, 2005.

[50] L. P. Daniele Astolfi, Lorenzo Marconi and A. R. Teel, "Low-power peaking-free high-gain observers," *Automatica*, vol. 98, pp. 169–179, 2018. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0005109818304345

[51] S. Sastry and P. Kokotovic, "Feedback linearization in the presence of uncertainties," *International Journal of Adaptive Control and Signal Processing*, vol. 2, no. 4, pp. 327–346, 1988.

[52] "Crazyflie specifications." [Online]. Available: https://store.bitcraze.io/products/crazyflie-2-1

[53] "Optitrack prime 17w specifications." [Online]. Available: https://optitrack.com/products/prime-17w/specs.html

[54] D. Nešić, A. Teel, and E. D. Sontag., "Formulas relating KL stability estimates of discrete-time and sampled-data nonlinear systems," *Systems & Control Letters*, vol. 38, pp. 49–60, 1999.

[55] W. Fulton, "Eigenvalues, invariant factors, highest weights, and Schubert calculus," *Bull. Amer. Math. Soc*, pp. 209–249, 2000.

[56] R. D. Nussbaum, "Some remarks on a conjecture in parameter adaptive control," *Systems & control letters*, vol. 3, no. 5, pp. 243–246, 1983.

[57] A. Astolfi, D. Karagiannis, and R. Ortega, *Nonlinear and adaptive control with applications.* Springer, 2008, vol. 187.

[58] H. K. Khalil, "Cascade high-gain observers in output feedback control," *Automatica*, vol. 80, pp. 110–118, 2017.

[59] K. H. Ang, G. Chong, and Y. Li, "PID control system analysis, design, and technology," *IEEE Transactions on Control Systems Technology*, vol. 13, no. 4, pp. 559–576, 2005.

[60] A. Loria, "Observers are Unnecessary for Output-Feedback Control of Lagrangian Systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 4, pp. 905–920, 2016.

[61] H. Sira-Ramirez, E. Zurita-Bustamante, and C. Huang, "Equivalence Among Flat Filters, Dirty Derivative-Based PID Controllers, ADRC, and Integral Reconstructor-Based Sliding Mode Control," *IEEE Transactions on Control Systems Technology*, vol. PP, pp. 1–15, 06 2019.

[62] I. Furtat, A. Fradkov, and Y. Orlov, "State Feedback Finite Time Sliding Mode Stabilization Using Dirty Differentiation," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 9619–9624, 2017, 20th IFAC World Congress. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S240589631732311X

[63] R. H. Jaafar and S. S. Saab, "Sliding Mode Control with Dirty Derivatives Filter for Rigid Robot Manipulators," in *2018 9th IEEE Annual Ubiquitous Computing, Electronics Mobile Communication Conference (UEMCON)*, 2018, pp. 716–720.

[64] M. Ran, J. Li, and L. Xie, "A new extended state observer for uncertain nonlinear systems," *Automatica*, vol. 131, p. 109772, 2021.

[65] E. Nunes and L. Hsu, "Global tracking for robot manipulators using a simple causal PD controller plus feedforward," *Robotica*, vol. 28, pp. 23–34, 01 2010.

[66] K. J. Aström and T. Hägglund, "The future of PID control," *Control Engineering Practice*, vol. 9, no. 11, pp. 1163–1175, 2001, pID Control. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0967066101000624

[67] Z.-P. Jiang, I. M. Mareels, and Y. Wang, "A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems," *Automatica*, vol. 32, no. 8, pp. 1211–1215, 1996. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0005109896000519