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Essays on Uncertainty and Stabilization

by

Seung Joo Lee

A dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy in Economics in the Graduate Division of the University of California, Berkeley

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Summer 2022
Essays on Uncertainty and Stabilization

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Seung Joo Lee
Abstract

Essays on Uncertainty and Stabilization
by
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Doctor of Philosophy in Economics
University of California, Berkeley
Professor Yuriy Gorodnichenko, Co-Chair
Professor Nicolae Gârleanu, Co-Chair

How should monetary policy deal with endogenous stock and bond market fluctuations? This dissertation focuses on the interaction among uncertainty arising from financial markets, risk-premium (and term-premium), and the business cycle. The main objective is to study various effective monetary policy responses for stabilization purposes.

The first chapter offers a non-linear version of the standard New-Keynesian framework, in which I provide an illustration of how the consideration of the first-order effects of endogenous and time-varying aggregate risks changes the business cycle dynamics. With conventional monetary policy rules, my non-linear characterization of the solution features interesting potentials for the sunspot equilibria arising from the aggregate business cycle volatility. I provide a new monetary policy rule that restores model determinacy and achieves the economy’s full stabilization again. The entire results rely on the interaction between aggregate consumption demand and the economy’s aggregate risk through the famous precautionary savings channel. This result is novel.

In the second chapter, I develop a more full-fledged New-Keynesian framework with active stock markets that features a potential for self-fulfilling financial uncertainty arising from its interaction with risk-premium, wealth, and aggregate demand. The model remains tractable, providing closed-form expressions for higher-order moments tied to the financial uncertainty and their relations to the rest of the economy. I re-examine the optimality of conventional monetary policy rules and show that the ‘Taylor principle’ no longer guarantees determinacy, with sunspots in aggregate financial volatility not precluded by aggressive targeting of inflation and output gap alone. I then characterize the joint dynamic evolution of financial volatility, risk-premium, asset prices, and the business cycle in a rational expectations equilibrium with sunspots, and uncover that variations in financial uncertainty generate reasonable crises and
booms along the business cycle that are consistent with my empirical estimates based on the US data. As this pitfall of the traditional policy rules lies in their inability to target the expected return on aggregate wealth, the relevant rate in stochastic environments, I then propose a ‘generalized’ Taylor rule that targets risk-premium and asset price, and describe the necessary conditions that restore determinacy and achieve the ultra-divine coincidence: the joint stabilization of inflation, output gap, and risk-premium. Finally, I revisit the zero lower bound (ZLB) and show it amplifies the duration, severity, and welfare costs of fluctuations in financial volatility. Alternative policies such as forward guidance reduce these welfare costs on average, but risk worsening economic situations with a non-zero probability, raising interesting trade-offs for policymakers.

The failure of conventional monetary policy to stabilize the economy at the zero-lower bound (ZLB) has made unconventional interventions more prevalent in recent times, which calls for a new macroeconomic framework for properly analyzing these policies. In the third chapter, I develop a New-Keynesian framework that incorporates the term-structure of financial markets and an active role for government and central bank’s balance sheet size and composition. I show that financial market segmentation and the household’s endogenous portfolio reallocation are crucial features to properly understand the effects of Large-Scale Asset Purchase (LSAP) programs. I propose a new micro-foundation based on imperfect information about expected future asset returns that easily accommodates distinct degrees of market segmentation across asset classes and maturities, while providing intuitive and tractable expressions for the household’s portfolio shares. My analysis reveals that government’s issuance of risk-less bonds stimulates the economy when conventional monetary policy is constrained at the ZLB, which is consistent with the literature on the so-called “safe-asset shortage problems”. I also find that central bank’s bond purchases across different maturities act as a major determinant of the level and slope of the term-structure, and yield-curve-control (YCC) policies that actively manipulate long-term yields are powerful in terms of stabilization both during normal times and at the ZLB. As a drawback, YCC policies increase the likelihood of ZLB episodes and their durations, thereby locking the central bank in a position in which the short-term rate is less useful as a policy tool.
To my family and loved ones.
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Chapter 1

Monetary Policy as a Financial Stabilizer: A Standard New-Keynesian Framework

1.1 Standard Model

We consider a standard non-linear New-Keynesian economy where the stock market is closed. The representative household simply owns the entire firms and get the profit stream in a lump-sum way. For simplicity, we assume a perfectly rigid price: $p_t = \bar{p}$ for $\forall t$ so there is no inflation in the economy. It is not crucial but allows us to focus on the key mechanism we want to illustrate.

The representative household chooses her usual intertemporal consumption-saving decisions, solving the following optimization problem:

$$\max_{\{B_t, C_t, L_t\}_{t \geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \left[ \log C_t - V(L_t) \right] dt \text{ s.t. } \dot{B}_t = i_t B_t - \bar{p} C_t + w_t L_t + D_t,$$

where $C_t$ and $L_t$ are her consumption and labor supply, where $V(L_t)$ is the disutility of labor supply $L_t$, $B_t$ is her nominal bond holding, and $D_t$ is the entire firms' profit and fiscal transfers from the government. $w_t$ is an equilibrium wage, and $i_t$ is the policy rate set by the central bank. We assume that there is no government spending, therefore, the aggregate consumption determines output in this demand-determined environment: $C_t = Y_t$ where $Y_t$ is an aggregate output.

The following equation is the optimality condition for the representative household' in-

\footnote{See Woodford (2003) for the standard treatment of a textbook New-Keynesian model.}
tertemporal consumption-saving decisions:

\[-i_t \, dt = \mathbb{E}_t \left( \frac{d \xi^N_t}{\xi^N_t} \right), \text{ where } \xi^N_t = e^{-\rho t} \frac{1}{C_t} p_t, \tag{1.1.2}\]

where \( \frac{d \xi^N_t}{\xi^N_t} \) is the instantaneous (nominal) stochastic discount factor (SDF) and its expectation yields the nominal risk-free rate \( i_t \). A similar condition holds for the real interest rate \( r_t \) as follows:

\[-r_t \, dt = \mathbb{E}_t \left( \frac{d \xi^r_t}{\xi^r_t} \right), \text{ where } \xi^r_t = e^{-\rho t} \frac{1}{C_t}. \tag{1.1.3}\]

With our rigid price assumption \( p_t = \bar{p} \) for \( \forall t \), equation (C.1.2) and equation (C.1.3) are equivalent with \( i_t = r_t \). We can rewrite equation (C.1.2) can be exactly written as

\[\mathbb{E}_t \left( \frac{d C_t}{C_t} \right) = (i_t - \rho) \, dt + \text{Var}_t \left( \frac{d C_t}{C_t} \right), \tag{1.1.4}\]

where the last term \( \text{Var}_t \left( \frac{d C_t}{C_t} \right) \) is absent in conditions based on log-linearization, and arises here from the effect of ‘endogenous’ volatility of an aggregate consumption process. In contrast to canonical linearized models, our non-linear characterization allows the aggregate risk to be priced, affecting the drift of the aggregate consumption process, where both aggregate risk and drift are endogenous objects. This additional term reflects the precautionary savings channel in which the more volatile a business cycle fluctuation becomes, households engage more in a precautionary saving, reducing their consumption and raising the consumption process’ expected growth.

Let us assume that the ‘natural’ (benchmark) economy’s output \( Y^n_t \) follows the following stochastic process:

\[dY^n_t = \left( r^n_t - \rho + (\sigma_t)^2 \right) \, dt + \sigma_t \, dZ_t, \tag{1.1.5}\]

where \( r^n_t \) is the natural rate of interest. We regard equation (1.1.5) as the exogenous process\(^2\) that our monetary policy cannot affect or control. This benchmark economy can be any target economy the central bank hopes to attain through its monetary policy, including a usual flexible-price economy. Here, we observe that: given \( \{Y^n_t\} \) process, an increase in the ‘natural’ volatility \( \sigma_t \) brings down the natural rate \( r^n_t \) as agents’ demand for the precautionary

\(^2\)Given the \( \{\sigma_t\} \) process, equation (1.1.5) is derived from equation (C.1.3) with \( r_t = r^n_t \) with \( Y_t = Y^n_t \). Therefore, we regard \( dZ_t \) as an aggregate shock that drives the natural output \( Y^n_t \). In most models, it can be a technology shock.
saving rises in response.

Then, let us think about the ‘current’ economy that the central bank manages through its monetary policy. We let $\sigma_s^t$ be an ‘excess’ volatility the current output process $\{Y_t\}$ features compared with the benchmark economy (equation (1.1.5)), therefore:

$$\text{Var}_t \left( \frac{dY_t}{Y_t} \right) = (\sigma_t + \sigma_s^t)^2 dt$$  \hspace{1cm} (1.1.6)

holds. Note that $\sigma_s^t$ can be regarded an ‘endogenous’ volatility to be determined in equilibrium by the monetary policy. By plugging equation (1.1.6) into equation (C.1.2), we obtain

$$\frac{dY_t}{Y_t} = \left( i_t - \rho + (\sigma_t + \sigma_s^t)^2 \right) dt + (\sigma_t + \sigma_s^t) dZ_t.$$  \hspace{1cm} (1.1.7)

With the usual definition of output gap $\hat{Y}_t = \ln \left( \frac{Y_t}{Y_n} \right)$, we can get the following dynamic IS equation written in $\hat{Y}_t$:

$$d\hat{Y}_t = \left( i_t - \left( r_n^0 - \frac{1}{2}(\sigma_t + \sigma_s^t)^2 + \frac{1}{2}(\sigma_t)^2 \right) \right) dt + \sigma_s^t dZ_t.$$  \hspace{1cm} (1.1.8)

The equation (1.1.8) features an interesting feedback effect that is abstracted away in log-linearized equations: given the policy rate $i_t$, an increase in the endogenous volatility $\sigma_s^t$ pushes up the drift of equation (1.1.8), bringing down the current level of output gap $\hat{Y}_t$. It is because a more volatile business cycle induces the household to save more in a precautionary manner (first-order effect) and reduce consumption, thereby inducing a recession.

For the illustration purpose, we can compare the above equation (1.1.8) with the usual IS equation based on the linearization technique, which is given by:

$$d\hat{Y}_t = (i_t - r_n^0) dt + \sigma_s^t dZ_t.$$  \hspace{1cm} (1.1.9)

where an endogenous aggregate volatility $\sigma_s^t$ has no first-order effect on the current level of output. In contrast, our fully non-linear characterization of the solution keeps a proper first-order price of risk, which changes the business cycle dynamics.

We define a risk-adjusted natural rate

$$r_T^t = r_n^0 - \frac{1}{2}(\sigma_t + \sigma_s^t)^2 + \frac{1}{2}(\sigma_t)^2$$  \hspace{1cm} (1.1.10)

\(^3\text{Basically we subtract equation (1.1.5) from equation (1.1.7) based on the continuous-time mathematics to get equation (1.1.8).}\)
that is a function of an endogenous volatility $\sigma_t^s$. In particular, we see $r_t^T$ depends negatively on $\sigma_t^s$. With this new non-linear structure and the feedback effect from the business cycle’s volatility to the drift in mind, we ask a very important question: does a conventional monetary policy, following the Taylor rule, still achieve a model determinacy as it does in the linearized model?

1.2 Taylor rules and Indeterminacy

In this section, we answer whether the conventional Taylor rule guarantees model determinacy and can fully stabilize the economy.\(^4\) We assume that the central bank uses the following conventional monetary policy:

$$i_t = r_t^n + \phi_y \tilde{Y}_t, \text{ where } \phi_y > 0.$$ (1.2.1)

Here, $\phi_y > 0$ is a condition called ‘Taylor principle’ that guarantees no sunspot in the log-linearized model without the first-order effects of volatilities. Here, we ask whether it still can guarantee no sunspot in this non-linear economy with the presence of feedback effects from volatility to drift.

Plugging equation (1.2.1) into equation (1.1.8), we get the following $\tilde{Y}_t$ dynamics.

$$d\tilde{Y}_t = \left( \phi_y \tilde{Y}_t - \frac{(\sigma_t)^2}{2} + \frac{(\sigma_t + \sigma_t^s)^2}{2} \right) dt + \sigma_t^s dZ_t.$$ (1.2.2)

**Multiple equilibria** Instead of equation (1.2.2), if $\tilde{Y}_t$ dynamics is represented by

$$d\tilde{Y}_t = (\phi_y \tilde{Y}_t) dt + \sigma_t^s dZ_t,$$ (1.2.3)

then Blanchard and Kahn (1980) ensures we obtain a unique rational expectation equilibrium: $\tilde{Y}_t = 0$, which is a fully stabilized path.

Now that the endogenous volatility $\sigma_t^s$ affects the drift of equation (1.2.2), we have multiple equilibria and sunspots in $\sigma_t^s$ can appear. We provide one rational expectation equilibrium that supports an initial sunspot $\sigma_0^s > 0$ in aggregate excess volatility, by constructing an equilibrium path where $\{\tilde{Y}_t\}$ process follows martingale. The case for negative sunspot $\sigma_0^s < 0$ can

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\(^4\)As our dynamic IS equation (equation (1.1.8)) has the same mathematical structure as the one in Lee and Carreras (2021a), the same logic applies here. For a more detailed treatment of the issue, see Lee and Carreras (2021a).
be similarly constructed. This equilibrium path should (i) support an initial sunspot $\sigma_0^s > 0$, and (ii) on expectation not diverge in the long-run, following Blanchard and Kahn (1980).

**Martingale equilibrium** Here we provide the explicit equilibrium in which $\sigma_0^s > 0$ appears, the dynamics follows equation (1.2.2), and $\hat{Y}_t$ is martingale. As the drift of the $\{\hat{Y}_t\}$ process (equation (1.2.2)) must be 0, we get the following formula for $\hat{Y}_t$:

$$\hat{Y}_t = -\frac{(\sigma_t + \sigma_t^s)^2}{2\phi_y} + \frac{(\sigma_t)^2}{2\phi_y}.$$  \hspace{1cm} (1.2.4)

The martingale equilibrium guarantees a rationality of the equilibrium, as on average the paths of $\{\hat{Y}_t\}$ stay at the same levels, satisfying $E_0(\hat{Y}_t) = \hat{Y}_0$. The last step to show that there is a stochastic path of $\{\sigma^s_t\}$ starting from $\sigma_0^s$ that supports this equilibrium. This equilibrium then both (i) supports an initial sunspot $\sigma_0^s > 0$ and (ii) does not diverge in the long-run.

Using equation (1.2.2) and equation (1.2.4), we obtain the stochastic process of $\sigma^s_t$ as

$$d\sigma^s_t = -(\phi_y)^2 \frac{(\sigma^s_t)^2}{2(\sigma_t + \sigma_t^s)^3}dt - \phi_y \frac{\sigma^s_t}{\sigma_t + \sigma_t^s}dZ_t.$$ \hspace{1cm} (1.2.5)

Therefore, equation (1.2.4) and equation (1.2.5) constitute this particular rational equilibrium dynamics supporting $\sigma_0^s > 0$. What does this martingale equilibrium look like? The following Proposition 1 sheds lights on behaviors of $\{\hat{Y}_t, \sigma^s_t\}$ paths and argues: business cycle almost surely converge to the perfectly stabilized path in the long run. Those few paths which do not converge can blow up asymptotically and sustain the initial sunspot $\sigma_0^s > 0$, as the economy is forward-looking.

**Proposition 1** (Taylor Rules and Indeterminacy). The rational expectation equilibrium that supports an initial sunspot $\sigma_0^s > 0$ ($\hat{Y}_t$ dynamics in equation (1.2.4), and stochastic process for $\sigma^s_t$ in equation (1.2.5)) features $\sigma_t^s \xrightarrow{a.s.} \sigma_t^\infty = 0$ and $\hat{Y}_t \xrightarrow{a.s.} 0$. And $E_0(\max_t(\sigma_t^s)^2) = \infty$ holds.

The conditions $\sigma_t^s \xrightarrow{a.s.} \sigma_t^\infty = 0$ and $\hat{Y}_t \xrightarrow{a.s.} 0$ imply that equilibrium paths that start from the initial sunspot $\sigma_0^s > 0$ are almost surely stabilized in the long run. Then, how is it possible for a sunspot $\sigma_0^s > 0$ to appear at first? The condition $E_0(\max_t(\sigma_t^s)^2) = \infty$ implies: an initial

\[ d\sigma_t^s = -\frac{\phi^2}{2\sigma_t^2}dt - \phi dZ_t. \]

which stops when $\sigma_t^s$ hits the $\sigma^{a,n} = 0$. For general properties of Bessel process, see Lawler (2019).
spike in $\sigma^s_0$ and the ensuing crisis is sustained by a tiny probability of a very gigantic volatility in the future.

**Intuitions** We explain in a detailed manner (i) how an initial sunspot $\sigma^s_0$ in the aggregate volatility can appear, and (ii) results in Proposition 1. For that purpose, we simplify the economic environment and make the following assumptions:

**A.1** A shock $dZ_t$ at each period takes one of two: $\{+1, -1\}$ with equal probability $\frac{1}{2}$

**A.2** Aggregate demand $\hat{Y}_t$ equals a conditional expected value of the next-period aggregate demand $\hat{Y}_{t+1}$: therefore, if $\hat{Y}_{t+1}$ takes either $\hat{Y}_{t+1}^{(1)}$ or $\hat{Y}_{t+1}^{(2)}$, then $\hat{Y}_t = \frac{1}{2}(\hat{Y}_{t+1}^{(1)} + \hat{Y}_{t+1}^{(2)})$

**A.3** Aggregate demand $\hat{Y}_t$ falls, as a conditional variance of the next-period’s $\hat{Y}_{t+1}$ rises (precautionary saving). Both $\{\hat{Y}_t\}$ and $\{\sigma^s_t\}$ are set to be 0 on the stabilized path

Since we have only two possible realizations of the shock at each period, we can draw a tree diagram as follows.

![Figure 1.1: A sunspot in $\sigma^s_0$ as a rational expectation equilibrium](image)

In Figure 1.1, a thick vertical line represents the stabilized path, with its left and right representing recessions and booms, respectively. The key to build a rational expectation equilibrium supporting a sunspot $\sigma^s_0 > 0$ is to construct a path-dependent consumption strategy of intertemporal agents. First, let us imagine that the current period agents (Agents$_0$) believe
suddenly that the future agents will choose the path-dependent consumption demands so that the next-period’s $\hat{y}_1$ becomes $\hat{y}_1^{(1)}$ after $dZ_0 = +1$ is realized and $\hat{y}_1^{(2)}$ after $dZ_0 = -1$ is realized, with $\hat{y}_1^{(1)} > \hat{y}_1^{(2)}$. Then the current output $\hat{y}_0$ becomes $\hat{y}_0 = \frac{1}{2}(\hat{y}_1^{(1)} + \hat{y}_1^{(2)})$ with $\hat{y}_0$ below the stabilized path, as Agents$_0$ believe there is a dispersion in the next-period business cycle, which is given as $\sigma_1^{s,(1)} = \hat{y}_1^{(1)} - \hat{y}_1^{(2)}$.

Imagine that $dZ_0 = -1$ is realized. For Agents$_0$’s belief that $\hat{y}_1 = \hat{y}_1^{(2)}$ to be correct, Agents$_1$ now must believe the future agents will choose consumption paths in a way that the next period’s $\hat{y}_2$ becomes $\hat{y}_2^{(3)}$ when $dZ_1 = +1$ is realized and $\hat{y}_2^{(4)}$ when $dZ_1 = -1$ is realized, with the conditional volatility $\sigma_2^{s,(2)} = \hat{y}_2^{(3)} - \hat{y}_2^{(4)}$ higher than $\sigma_1^{s,(1)}$, since $\hat{y}_1^{(2)}$ is lower than the initial output $\hat{y}_0$.

After $dZ_1$ is realized, Agents$_1$’s belief about $\hat{y}_2$ can be made consistent by future agents’ coordination and it keeps going on for future agents $\{\text{Agents}_{n \geq 2}\}$. We observe: all the nodes in Figure 1.1 satisfy the assumptions A.2 and A.3, with distance between adjacent nodes getting narrower as the current output gets closer to the stabilized path and wider as the output deviates more from the stabilized level. Since the output $\{\hat{y}_t\}$ is a martingale here, we ensure that the economy does not diverge in the long run in expectation.

In sum, Agents$_0$’s initial doubt (sunspot) that the next-period business cycle would be volatile can be made consistent by coordinations between intertemporal agents (the representative household) at each node.

Note that (i) we have a stochastic aggregate volatility in this equilibrium: i.e., $\sigma_t^s$ is dependent on the path of shocks, as output $\{\hat{y}_t\}$ is stochastic and depends negatively on the conditional volatility of its next-period level. Actually, equation (1.2.5) specifies the exact stochastic process of $\{\sigma_t^s\}$ starting from $\sigma_0^s > 0$, (ii) since volatility $\sigma_t^s$ becomes smaller as the output $\hat{y}_t$ approaches the stabilized path, the economy is likely to stick around the stabilized path if it somehow gets there (therefore, the stabilized path attracting sample paths), justifying the result of Proposition 1 that $\sigma_t^s$ almost surely converges to 0 over time. As volatility $\sigma_t^s$ rises whenever output $\hat{y}_t$ deviates more from the stabilized level, it aligns with the result of Proposition 1 that a maximal $\sigma_t^s$ diverges: $\mathbb{E}_0(\max_t(\sigma_t^s)^2) = \infty$.

While the monetary policy is stabilizing the disruption caused by $\sigma_0^s > 0$ sunspot, the economy features the crisis phase with low aggregate demand and a higher business cycle volatility.

**Escape clause** If the central bank and/or government credibly commit to prevent $\hat{y}_t$ from going below a predetermined threshold through interventions, these sunspot equilibria arising

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6 Their consumption demand determines output in this demand-determined environment.

7 It is possible since all future agents share a common knowledge of their consumption strategies and there is no behavioral friction that blocks communications between intertemporal agents. This sunspot equilibrium is closely related to a notion of ‘self-confirming equilibrium’. For this issue, see Fudenberg and Levine (1993).

8 For example, government might commit to incur huge fiscal deficits whenever the economy undergoes a severe recession. This prescription entails the similar implication about what government can do to restore
from the aggregate financial volatility $\sigma_0^q$ supported by the paths in Figure 1.1 (martingale equilibrium) are not sustained anymore as a possible rational expectations equilibrium (REE). This escape clause illustrates how the credible commitment of the government entity to intervene whenever the economy (probabilistically) enters a big recession actually precludes a possibility of the crisis phase initiated by the positive sunspot shock $\sigma_0^s > 0$.

Whether this type of commitment from government and central bank is credible is important, as here we need a 100% credibility to kill the sunspot equilibrium supporting $\sigma_0^s > 0$.

**Negative sunspot** We can similarly construct a rational expectation equilibrium that supports the initial downward sunspot $\sigma_0^s < 0$. The sunspot equilibrium features the boom phase with buoyant aggregate demand and a lower business cycle volatility. Therefore, our non-linear characterization of the model actually generates a reasonable prediction of (i) why we have boom-crisis phases with a sunspot appearance, and (ii) time-varying behaviors of the first (level) and the second (volatility) moments together during either crisis or boom.9

Now, we study a possible monetary policy rule that restores model determinacy.

### 1.3 A New Monetary Policy

Imagine that instead of equation (1.2.1), the central bank uses the following monetary policy:

$$i_t = r_t^n + \phi_y \hat{Y}_t - \frac{1}{2} \left( (\sigma_t + \sigma_s^t)^2 - (\sigma_t)^2 \right) \text{ (Aggregate volatility targeting)},$$  

where $\phi_y > 0$, (1.3.1)

which targets an aggregate volatility of the business cycle, with targeting coefficient $\frac{1}{2}$, in addition to output gap $\hat{Y}_t$. By plugging the above monetary policy (equation (1.3.1)) into our dynamic IS equation (equation (1.1.8)), we return to equation (1.2.3), which guarantees model determinacy and ensures $\hat{Y}_t = 0$ for $\forall t$ as a unique equilibrium.

**Interpretation** How do we interpret equation (1.3.1)? The additional targeting of aggregate volatility is necessary as it offsets the feedback channel from (endogenous) volatility to (endogenous) drift of the business cycle and allows no sunspot in $\sigma_0^s$.

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9Our sunspot equilibria work in a similar way to how we think about animal spirits and their effects in driving the business cycle. For the neoclassical treatment of this issue, see Angeletos and La’O (2013).
We acknowledge that it sounds very difficult for central banks to target the business cycle volatility directly. In the next chapter 2 (or Lee and Carreras (2021a)), we offer an alternative theoretical framework with explicit stock markets, where agents decide their portfolio choices. It turns out that those volatility targeting can be interpreted as the central bank’s risk-premium targeting.
Chapter 2

Monetary Policy as a Financial Stabilizer

This chapter is coauthored with my classmate and also one of my best friends, Marc Dordal i Carreras. I appreciate him for allowing me to use our joint work as part of this dissertation. All errors are mine.

2.1 Introduction

How should monetary policy respond to stock market fluctuations? The current narrative posits that central banks (-governments) need two separate sets of instruments: macroprudential policies and regulations to ensure the stability of financial markets, and monetary (-fiscal) policies to fulfill the traditional objective of macroeconomic stabilization. However, the debate on this issue is far from being settled for many reasons. For example, the stock market plays a dual role: it is a source of business cycle fluctuations (e.g., the Great Depression) and it is a propagation channel itself (e.g., stock prices merely reflect the collective wisdom on expected future business cycle conditions). Relatedly, resolving this debate has proven difficult because mainstream macroeconomic frameworks lack meaningful stock market fluctuations (if there is a stock market in such models) or rely on approximation techniques and numerical methods which can cloud the economic intuition.

In this paper, we shed some lights on this longstanding debate by proposing a New-Keynesian framework with stock markets and optimal portfolio decisions. We incorporate endogenous and time-varying second-order moments such as stock market volatility and risk-premium. Furthermore, our continuous-time framework allows intuitive analytic expressions.

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1For example, at the press conference held on September 16, 2020, Federal Reserve chair Powell explicitly mentioned “Monetary policy should not be the first line of defense - is not the first line of defense on financial stability. We look to more appropriate tools in the first instance, as a first line of defense. And those would be regulation, supervision, high-capital, high-liquidity stress testing, all of those things, macroprudential tools.”
which highlight the underlying mechanisms behind our results. The model features an important role of financial volatility and risk-premium for business cycle fluctuations: a more volatile financial market (with higher risk-premia) brings down aggregate financial wealth (through individual investor’s portfolio decisions), thereby affecting aggregate demand and output. Because endogenous second-order terms (financial volatility) feed back into the first-order moments (financial wealth and aggregate demand), we explore how monetary policy should be connected to financial stability issues (i.e., financial volatility). We claim that the current monetary policy framework based on two *macroeconomic* mandates (e.g., stable inflation and stable output gap) is not sufficient for macroeconomic stabilization. In addition to these two mandates, we call for targeting time-varying risk premium as a separate policy objective.

Our model solution uncovers that there exists a sunspot equilibrium that arises from aggregate volatility and risk-premium of financial markets: fear of a financial crisis possibly stemming from a rise in risk-premium and stock market volatility, for example, induces investors to reduce their demand for the stock market investment, bringing down the current asset price and wealth and thus generating self-fulfilling increases in the expected stock market return and risk-premium. In particular, we characterize rational expectations equilibria that follow self-fulfilling shocks to the financial volatility and risk-premium, where we derive a tractable expression for the joint dynamic evolution of financial volatility, risk-premium, and business cycle variables after those sunspots appear as a function of fundamentals and policy interventions. We prove that under these sunspot equilibria, the financial volatility gets almost surely stabilized in the long run, but a probability-zero event in which this volatility diverges in the long run leading to a severe recession makes the sunspot’s initial appearance possible. As it takes time for initial volatility sunspots to be eliminated by monetary policy response, our equilibrium features *crisis* periods (with spikes in stock market volatility and risk-premium and drops in wealth and output) and *boom* phases (with low financial volatility and buoyant wealth and production), depending on the directions of initial sunspots.

We then study conventional monetary policy rules in regard to model determinacy and financial stability. Our analysis shows that traditional Taylor rules that focus on macroe-

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2 Even in the 'textbook' New-Keynesian model without explicit stock markets and portfolio decisions, the economy’s *time-varying aggregate risk* can have a first-order impact on the aggregate consumption demand due to the *precautionary savings* channel. That is the reason, in chapter 1, why we provide an alternative standard New-Keynesian model without stock markets and characterize non-linear equilibrium conditions to (i) illustrate the first-order (feedback) effects of endogenous and time-varying aggregate risks on business cycle levels, and (ii) show that sunspot equilibria arise with conventional monetary policy rules. Therefore, most of main results continue to hold in chapter 1 including determinacy issues.

3 This result aligns with Basu et al. (2021), where they emphasize roles of fluctuations in risk-premia as a main driver of the business cycle driving movements and comovements among aggregate variables. In Appendix B.1, we estimate a simple vector autoregression (VAR) with real and financial uncertainty indexes developed by Ludvigson et al. (2015) and uncover that a 1-3% (5-10%) drop in industrial production (S&P-500 index) follows after a one standard deviation shock to financial uncertainty, which our calibrated model replicates.
onomic aggregates (i.e., inflation and output gap) cannot fully prevent the appearance of sunspots in aggregate financial volatility, but a stronger targeting of macroeconomic mandates shortens the time it takes for initial volatility sunspot to get stabilized in our rational expectations equilibrium. This stronger responsiveness of monetary policy comes with a side effect, however: a more aggressive targeting of inflation and output gap amplifies the financial market volatility following sunspot shocks, which generates stronger but short-lived boom and bust financial cycles. We argue that the failure of conventional policy rules to restore determinacy lies in their inability to adequately target the expected risky return of financial markets, which governs the agents’ intertemporal decision-making.

We then propose a generalized policy reaction function that restores determinacy in our stochastic environment. Specifically, we argue that optimal policy rules should target the risk-premium of financial markets in addition to their usual mandates. Intuitively, agents in our model optimally allocate their wealth between risky and riskless assets, and the return on aggregate financial wealth becomes the relevant rate for their intertemporal consumption smoothing decisions. Therefore, the optimal monetary rule aims to control the return on agents’ aggregate wealth, but in order to succeed, it must take into account the risky component of the portfolio return, which is summarized by risk-premium. Thus, our analysis suggests that aggregate wealth should be an intermediate target of the central bank for the purpose of macroeconomic stabilization. This new policy rule that targets risk-premium in a specific way achieves what we describe as ‘ultra-divine’ coincidence: the joint stabilization of inflation, output gap and risk-premium (equivalently, financial volatility).

Following this rule poses its own challenges though, as the central bank is required to target risk premium with just the right amount of responsiveness. If the policy response is too accommodating or strong, monetary policy is again unable to prevent the appearance of sunspots. Nonetheless, even when the central bank is unable to restore the equilibrium determinacy, targeting financial variables remains an optimal strategy as it enables a faster convergence back to the steady state following a sunspot shock.

We then analyze the effects of the zero lower bound (ZLB) on macroeconomic stabilization. The ZLB in our framework causes stock prices to fall, leading to drops in business cycle variables, as in Caballero and Simsek (2020b). We ask whether we should expect heightened financial instability once the policy tool of central banks is constrained at zero, and find that a credible commitment to economic stabilization upon ZLB-exit is enough to ensure financial stability during ZLB episodes. However in cases where post ZLB or forward guidance exit stability is not guaranteed, ZLB is likely to amplify the duration, severity, and welfare costs of fluctuations in financial market volatility after its sunspots appear.\footnote{Even if central bank’s post-ZLB stabilization prevents the additional financial instability at the ZLB,}

\footnote{In Section 2.4.4, we explore two macroprudential policies at the ZLB that induce investors to bear more risks, thereby raising asset prices and business cycle levels: (i) a tax cut on capital gain taxes and (ii) redistribution across agents.}
Related Literature  Our paper is related to a broad literature on the intersection between macroeconomics and finance. Our model builds on the idea that changes in financial wealth levels (usually housing and stock) affect aggregate economic outcomes, documented by Mian et al. (2013), Mian and Sufi (2014), Guerrieri and Iacoviello (2017), Berger et al. (2018), Caballero and Simsek (2020b), Caballero and Simsek (2020a), Di Maggio et al. (2020), Caramp and Silva (2020) and Chodorow-Reich et al. (2021), among others. In line with this literature, an endogenous stock price level shifts aggregate demand in our framework through its effect on aggregate financial wealth. In addition, our framework features endogenous risk-premium and financial volatility as key factors that drive fluctuations in financial markets and the business cycle, in line with arguments made by Gilchrist and Zakrajšek (2012), Brunnermeier and Sannikov (2014), Chodorow-Reich (2014), Stein (2014), Cúrdia and Woodford (2016), Guerrieri and Lorenzoni (2017), Di Tella and Hall (2020), and Basu et al. (2021) among others, that financial (and in particular, credit) disruptions have large impacts on aggregate demand, especially when monetary policy is constrained. Campbell et al. (2020) points out that New-Keynesian channels, through which a higher inflation pushes down bond returns while propping up aggregate output, dividends, and stock returns, can explain the correlation reversal between bond and stock returns which turned negative in recent years. Our framework shares the same intuitions and sheds lights on how stock market fluctuation can be embedded in conventional New-Keynesian models.

Our result that monetary policy must be systematically concerned with financial markets business cycle still can feature high levels of volatility and risk-premium due to fundamental risks (e.g., TFP volatility). In Section 2.4.3, we show that by credibly committing to sacrifice financial stabilization in the future, central bank can attain welfare-enhancing commitment equilibria, in which it boosts asset prices and output, and reduces risk-premium and financial volatility today at the ZLB.

In their works, consumers with a high marginal propensity to consume (MPC) who experience large drops in their housing prices, reduce the consumption amounts due to both wealth effects and a binding credit constraint, the latter of which we do not consider in this paper.

Caramp and Silva (2020) introduced rare-disasters and positive private debt and characterized the roles of time-varying risk-premia and financial wealth in a linearized setting.

Basu et al. (2021) emphasize roles of fluctuations in risk-premia as a business cycle driver, showing that the shock that explains fluctuations in risk-premia can explain a large fraction of business cycle movements and co-movements. They rely on the third-order perturbation to solve their model. In addition, Kekre and Lenel (2021) provide an elegant framework which illustrates the transmission of monetary policy through its impacts on the equilibrium risk-premium level in the environment that features heterogeneity in households’ marginal propensity to take risk (MPR). While their dynamic model relies on global solution methods, their analytic counterpart relies on the third-order approximation.

The previous literature usually focus on channels through which financial wealth and financial market disruptions affect business cycle fluctuations. The other direction, an asset pricing implication of the New-Keynesian model, is also addressed by De Paoli et al. (2010), Weber (2015) and Gorodnichenko and Weber (2016).
stability is related to prior literature including Bernanke and Gertler (2000), Stein (2012), Woodford (2012), Cúrdia and Woodford (2016)\textsuperscript{10}, Caballero and Simsek (2020a), Cieslak and Vissing-Jorgensen (2021), Kekre and Lenel (2021), and Galí (2021)\textsuperscript{11}. In contrast to Bernanke and Gertler (2000)'s findings that monetary policy should not target stock prices, which they concluded based on a model with ad-hoc bubbles, bubble components are omitted in our model and thus only the fundamental stock price level serves as the key factor that determines aggregate demand. Therefore, our specification with the stock price as an aggregate demand shifter leads to the equivalence of targeting of stock price 'level' and more conventional mandates such as output gap, and allows us to connect our work with Cieslak and Vissing-Jorgensen (2021) which conclude that stock market performance is a powerful predictor of the policy rate. In particular, Kekre and Lenel (2021) provide a beautiful theoretical framework in which an accommodation shock in monetary policy redistributes toward those with a higher marginal propensity to take risk (MPR), thereby reducing risk-premium levels and amplifying the monetary transmission. While their focus is on how monetary policy following the conventional Taylor rule affects the economy through its impacts on economy-wide risk-premia in the heterogenous agents New Keynesian (HANK) environment, our analytic approach allows us to spot new indeterminacy around the second-order financial variable (aggregate financial volatility) with conventional Taylor rules,\textsuperscript{12} thereby allowing us to provide a more generalized Taylor rule that targets risk-premium as a way to facilitate stabilization and (possibly) restore model determinacy. Our approach still aligns with their view in that aggregate wealth is to be managed through monetary policy, and our generalized Taylor rule illustrates that an internal rate of return on aggregate wealth, instead of just the risk-free policy rate, must be responding to fluctuations in business cycle variables for the model to restore determinacy and achieve perfect stabilization.

While Giavazzi and Giovannini (2010), Stein (2012), and Caballero and Simsek (2020a) focus on the preemptive role of monetary policy in avoiding 'future' financial crises, our model features a monetary policy rule targeting the risk-premium of financial markets for the 'current' stabilization purposes, in addition to its traditional inflation and output gap targets. Our result that monetary accommodation props up the business cycle through its effect on the stock market level is in line with evidence provided by Rigobon and Sack (2003), Azali et al. (2013), and Kekre and Lenel (2021).

\textsuperscript{10}Woodford (2012) and Cúrdia and Woodford (2016), in particular, incorporate a friction in financial inter-mediation between agents with different marginal propensities to consume (MPC) and study how the optimal monetary policy rule must be adjusted.

\textsuperscript{11}Galí (2021) introduced rational bubbles in a New-Keynesian model with overlapping generations. He argued that 'leaning against the bubble' policy, if properly specified, insulates the economy from aggregate bubble fluctuations.

\textsuperscript{12}Also, we contribute to the literature by providing an exact stochastic process for each business cycle variable after those sunspots appear in the financial market.
This paper is also related with literature on New-Keynesian environment and monetary policy at the zero lower bound (ZLB). Due to nominal pricing rigidities à la Calvo (1983), our economy is demand-driven and stock market performance drives the aggregate demand. Thus, in order to characterize monetary policy’s stabilization role, endogenous fluctuations in stock markets must be properly taken into account, a topic that has often been overlooked by the previous literature. While several authors focus on demand recessions brought by deleveraging borrowers at the ZLB and aggregate demand externality issues (i.e., Akerlof and Yellen (1985), Blanchard and Kiyotaki (1987), Eggertsson and Krugman (2012), Farhi and Werning (2012), Farhi and Werning (2016), Korinek and Simsek (2016), Schmitt-Grohé and Uribe (2016), and Farhi and Werning (2017)), we turn our attention towards declines in the aggregate demand for risky assets as the key driver behind financial recessions, a channel that has been documented by Caballero and Farhi (2017) and Caballero and Simsek (2020b).

Our paper is similar to Caballero and Simsek (2020b) in terms of how an endogenous asset market is interwoven with business cycle fluctuations. However, while their framework focuses on how behavioral biases can generate interesting crisis dynamics in light with the feedback loop between asset markets and the business cycle\(^\text{13}\), our focus is on the traditional monetary policy rule under rational expectations, and the central bank’s capacity to intervene in financial markets during crisis caused by the ZLB. Our model’s equilibrium determinacy results are similar to Acharya and Dogra (2020) in terms of how countercyclical risks can lead to indeterminacy. While Acharya and Dogra (2020) focus on how determinacy conditions change in the presence of exogenous idiosyncratic risks that are functions of aggregate output, we investigate the existence of sunspots stemming from aggregate financial risk, which is countercyclical in nature and affects both financial markets and business cycle fluctuations, and study the monetary policy mechanisms that restore determinacy and/or improve economic and financial stability.

**Layout** In Section 2.2, we present the model with explicit stock markets and characterize the equilibrium conditions. Section 2.3 focuses on the proper monetary policy rules in lights with our framework’s new features. In Section 2.4, we analyze zero lower bound (ZLB) crises and possible unconventional fiscal and monetary measures that mitigate recessions, focusing on the derivation of key trade-offs that central banks must take into account. Section 3.5 concludes.

In Appendix B.1, we provide evidence on the importance of financial volatility as a driver of business cycle fluctuations, based on a structural Vector Autoregression (VAR) approach. Appendix B.2 contains additional figures and tables. Appendix B.3 contains derivations and

\(^{13}\text{Caballero and Simsek (2020b) features optimists and pessimists who have different beliefs about the probability of an upcoming recession. During ZLB episodes, an endogenous decline in the risky asset valuation generates a demand recession due to a drop in optimists’ wealth.}\)
proofs. Appendix B.4 derives the quadratic welfare loss function in this framework.

2.2 The Model with Stock Markets

In this Section 2.2, we consider a slightly different theoretical framework, which enables us to analyze the effects of higher-order moments tied to the aggregate financial volatility on aggregate demand, and provides us the practical implications about monetary policy rules.

2.2.1 Setting

Time is continuous, and a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, P)\) is given. The economy consists of a measure one of capitalists, who we regard as neoclassical agents, and the same measure of hand-to-mouth workers, who we regard as Keynesian agents. There is a single source of exogenous variation in the aggregate production technology \(A_t\), which is adapted to the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}}\) and evolves according to a geometric process with a possibly time-varying volatility \(\sigma_t\):

\[
\frac{dA_t}{A_t} = g \, dt + \sigma_t \, dZ_t.
\] (2.2.1)

We regard the aggregate TFP’s volatility \(\sigma_t\) as the economy’s ‘fundamental’ risk. We assume it to be constant in most scenarios, but later, as in Caballero and Simsek (2020b), we will allow \(\sigma_t\) to jump and analyze how it affects the equilibrium dynamics. For convenience, we also assume the average growth rate \(g\) to be constant over time.

Finally, there is a standard set of intermediate good producers that face nominal price rigidities, thus making the economy New-Keynesian in nature. Next, we describe roles of each type of agents (capitalists and workers) and firms.

Firms and Workers

There are a measure one of monopolistically competitive firms, each producing a differentiated intermediate good \(y_t(i), i \in [0, 1]\). There also exists a competitive representative firm which transforms intermediates into a final consumption good \(y_t\) according to a Dixit-Stiglitz aggregator with an elasticity of substitution \(\epsilon > 0\) in the following way.

\[
y_t = \left( \int_0^1 y_t(i)^{\epsilon-1} di \right)^{\frac{1}{\epsilon-1}}.
\] (2.2.2)
Each intermediate good firm $i$ has the same production function $y_t(i) = A_t(N_{W,t})^\alpha n_t(i)^{1-\alpha}$, where $N_{W,t}$ is the economy’s aggregate labor and $n_t(i)$ is the labor demand of an individual firm $i$ at time $t$. The reason that we introduce a production externality à la Baxter and King (1991) is that it helps us match empirical regularities on asset price and wage co-movements, and it does not affect other qualitative implications of our framework.\footnote{In our framework, rising asset prices tend to be correlated with the decreasing wage compensation to workers since firm value (stock price) usually rises if firms can pay less to workers. It violates empirical regularities documented by Chodorow-Reich et al. (2021) in which a rise in stock price tends to push up local aggregate demand variables such as employment and wage. Our production function with externality à la Baxter and King (1991) provides us a reasonable calibration that matches these empirical regularities because higher asset prices and aggregate demand raise the firms’ marginal product of labor, thus increasing labor demand and wages. Basically, our externality plays similar roles to the capital in the production function, with a higher degree of tractability.} Each firm $i$ faces the downward-sloping demand curve $y_t(p_t(i)||p_t, y_t)$, where $p_t(i)$ is the price of its own intermediate good and $p_t, y_t$ are the aggregate price index and output, respectively:

$$y_t(p_t(i)||p_t, y_t) = y_t\left(\frac{p_t(i)}{p_t}\right)^{-\epsilon}.$$  \hspace{1cm} (2.2.3)

The set of prices charged by intermediate good firms, $\{p_t(i)\}$, is aggregated into the price index $p_t$ as

$$p_t = \left(\int_0^1 p_t(i)^{1-\epsilon}di\right)^{\frac{1}{1-\epsilon}}.$$  \hspace{1cm} (2.2.4)

We also impose a nominal price rigidity à la Calvo (1983), and firms can change prices of their own intermediate goods with $\delta dt$ probability in a given time interval $dt$. In the cross-section, this implies that a total $\delta dt$ portion of firms reset their prices during a given $dt$ time interval.

A representative hand-to-mouth worker supplies labor to intermediate good producers, gets an equilibrium wage income, and spends every dollar he earns on final good consumption. We assume that each worker solves the following optimization at every moment $t$, where $C_{W,t}$, $N_{W,t}$ and $w_t$ are his consumption, labor supply and wage at time $t$, respectively.

$$\max_{C_{W,t}, N_{W,t}} \frac{C_{W,t}}{A_t} \left(\frac{C_{W,t}}{A_t}\right)^{1-\psi} - \frac{(N_{W,t})^{1+\chi_0}}{1+\chi_0} \text{ s.t. } p_tC_{W,t} = w_tN_{W,t}.$$  \hspace{1cm} (2.2.5)

where $\chi_0$ is the inverse Frisch elasticity of labor supply. Note that we normalize consumption $C_{W,t}$ by technology $A_t$, which governs the economy’s size.\footnote{The qualitative results of the model are not affected by the consumption normalization, which we introduce to simplify the analytic expressions of the model.} As wage $w_t$ is homogeneous across firms, labor demanded by each firm $i$, $\{n_t(i)\}$, are simply combined into aggregate
labor $N_{W,t}$ in a linear manner as

$$N_{W,t} = \int_0^1 n_t(i) di. \quad (2.2.6)$$

Final good output $y_t$ can be written as a function of total labor $N_{W,t}$ by the following aggregate production function with price dispersion $\Delta_t$ defined below.\(^\text{16}\) Due to the Baxter and King (1991) externality, the aggregate production function becomes linear in $N_{W,t}$ as

$$y_t = \frac{A_t N_{W,t}}{\Delta_t}, \text{ where } \Delta_t \equiv \left( \int_0^1 \left( \frac{p_t(i)}{\rho_t} \right)^{-\frac{\alpha}{1-\alpha}} di \right)^{1-\alpha}. \quad (2.2.7)$$

**Financial Market and Capitalists**

Unlike conventional New-Keynesian models where a representative household owns the intermediate goods sector and receives rebated profits in a lump sum way,\(^\text{17}\) we assume that firm profits are capitalized in the financial market as a representative stock fund. Capitalist then face an optimal portfolio decision problem involving the allocation of their wealth between a risk-free bond and the risky stock at every instant $t$.

The total nominal financial wealth of the economy is $p_t A_t Q_t$, where $Q_t$ is the normalized (or TFP detrended) real asset price. $Q_t$ is an endogenous variable adapted to filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$ and assumed to evolve according to the process in equation (2.2.8), with both endogenous drift $\mu^q_t$ and volatility $\sigma^q_t$ terms. In particular, we regard $\sigma^q_t$ as a measure of financial uncertainty or disruption, as we usually observe spikes in asset price volatility during financial crises. Like $Q_t$, we assume that the price aggregator $p_t$ follows the general stochastic process in equation (2.2.9), in which drift $\pi_t$ and volatility $\sigma^p_t$ are endogenous. Thus, it follows that total financial market wealth $p_t A_t Q_t$ evolves as a geometric Brownian motion with volatility $(\sigma_t + \sigma^q_t + \sigma^p_t)$. Intuitively, if some capitalist invests in the stock market, they have to bear all three risks: inflation risk, technology (fundamental) risk, and (detrended) real asset price risk.

$$\frac{dQ_t}{Q_t} = \mu^q_t dt + \sigma^q_t dZ_t, \quad \text{(Financial volatility)} \quad (2.2.8)$$
$$\frac{dp_t}{p_t} = \pi_t dt + \sigma^p_t dZ_t. \quad \text{(Inflation risk)} \quad (2.2.9)$$

\(^{16}\)See Woodford (2003), Yun (2005), Kaplan et al. (2010) among others for the role of relative price dispersion $\Delta_t$ in business cycle fluctuations and economic stabilization issues.

\(^{17}\)We already studied non-linear implications in the context of standard New-Keynesian models in Section 1.1.
Here, $\sigma^q_t$ is determined in equilibrium and can be either positive or negative. $\sigma^q_t < 0$ corresponds to the case where total real wealth $A_t Q_t$ is less volatile than the TFP process $\{A_t\}$. The nominal price process has inflation rate $\pi_t$ as its drift, and in general has a volatility part $\sigma^p_t$, which we call an inflation risk. In most cases other than the flexible price benchmark, we show that $\sigma^p_t = 0$ holds and we do not need to concern ourselves with this term.

In addition to the stock market, we assume that there is a risk-free bond with an associated nominal rate $i_t$ that is controlled by the central bank. Bonds are in zero net supply in equilibrium because all capitalists are equal. A measure one of identical capitalists chooses the portfolio allocation between a risk-free bond and a risky stock, where in the latter case, they earn the profits of the intermediate goods sector as dividends, as well as the nominal price revaluation of the stock due to changes in $p_t$, $A_t$ and $Q_t$. Financial markets are competitive, thus each capitalist takes the nominal risk-free rate $i_t$, expected stochastic stock market return $i^m_t$, and the risk level $\sigma_t + \sigma^q_t + \sigma^p_t$ as given when choosing her portfolio decision. If a capitalist invests a share $\theta_t$ of her wealth $a_t$ in the stock market, she bears a total risk $\theta_t a_t (\sigma_t + \sigma^q_t + \sigma^p_t)$ between $t$ and $t + dt$. Therefore, the riskiness of her portfolio increases proportionally to the investment share $\theta_t$ in the stock. Capitalists are risk-averse, and ask for a risk-premium compensation $i^m_t - i_t$ when they invest in the risky stock, which must also be determined in equilibrium.

Each capitalist with nominal wealth $a_t$ has log-utility and solves the following optimization:

$$\max_{c_t, \omega_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t dt \text{ s.t. } da_t = (a_t i_t + \theta_t (i^m_t - i_t)) - p_t C_t) dt + \theta_t a_t (\sigma_t + \sigma^q_t + \sigma^p_t) dZ_t,$$

(2.2.10)

where $\rho$ is her time discount rate and $C_t$ is final good consumption. At every instant, she earns returns out of both the risk-free bond and the risky stock investments, and spends on final good consumption. From Merton (1971), we know that the solution of the problem features an optimal consumption expenditure rate which is exactly a $\rho$ portion of her wealth $a_t$, thus satisfying

$$\rho_t C_t = \rho a_t.$$  

(2.2.11)

Note that a less patient capitalist (higher $\rho$) increase her instantaneous consumption rate in a proportional manner.

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18 This competitive market assumption is related to the reason we initially assume a measure one of identical capitalists. This assumption turns out to be an important aspect of the framework for explaining inefficiencies caused by the aggregate demand externality that individual capitalist’s financial investment decision imposes on the aggregate economy.
2.2.2 Equilibrium and Asset Pricing

In equilibrium, every agent with the same type (either worker or capitalist) is identical and chooses the same decisions. Because in equilibrium bonds are in zero net supply, each capitalist’s wealth share \( \theta_t \) in the stock market must satisfy \( \theta_t = 1 \), which pins down the equilibrium risk-premium value demanded by capitalists. Due to the log-preference of capitalists, risk-premium is given by \( (\sigma_t + \sigma_q^t + \sigma_r^t)^2 \), as in equation (2.2.12). In equilibrium, capitalists hold a wealth amount that equals the total financial market wealth. These equilibrium conditions can be summarized as follows.

\[
r_	ext{p} t \equiv i^m_t - i_t = (\sigma_t + \sigma_q^t + \sigma_r^t)^2 \quad \text{and} \quad a_t = \frac{p_t A_t Q_t}{\Delta_t},
\]

(2.2.12)

where the risk-premium \( r_	ext{p} t \) demanded by capitalists increases with either of the three volatilities \( \{\sigma_t, \sigma_q^t, \sigma_r^t\} \). As the financial volatility \( \sigma_q^t \) is endogenous, the risk-premium \( r_	ext{p} t \) term is endogenous as well and needs to be determined in equilibrium. Note also that by the previous expression, the wealth gain/loss of the capitalist is equal to the nominal revaluation of the stock.

We can characterize the good’s market equilibrium and the equilibrium asset pricing condition of the expected stock return \( i^m_t \) as follows: Since capitalists spends \( \rho \) portion of their wealth \( a_t \) on consumption expenditure and they hold the entire wealth, \( C_t = \rho A_t Q_t \) holds in equilibrium. Thus we can write the equilibrium condition for the final good market as follows.\(^{19}\)

\[
\rho A_t Q_t + \frac{w_t}{p_t} N_{W,t} = \frac{A_t N_{W,t}}{\Delta_t}.
\]

(2.2.13)

Due to the log-utility of capitalists, their nominal state-price density \( \xi_N^t \) is given in the following way, where the stochastic discount factor between time \( t \) (now) and \( s \) (future) is by definition given as \( \xi_N^s / \xi_N^t \).

\[
\xi_N^t = e^{-\rho t} \frac{1}{C_t} \frac{1}{p_t}.
\]

(2.2.14)

Total stock market wealth \( (p_t A_t Q_t) \) is by definition the sum of discounted profit streams from the intermediate goods sector, which are priced by the above \( \xi_N^t \) because capitalists are natural stock market investors in equilibrium. Thus we can price the entire stock market value as in the following relation, where we discount future profits with the stochastic discount factor generated by the state-price density \( \{\xi_N^t\} \). We know that the entire profit of the intermediate

\(^{19}\)Here \( N_{W,t} \) is the solution of the worker’s optimization problem in equation (2.2.5).

\(^{20}\)A superscript \( N \) means it is a nominal state-price density, where a superscript \( r \) means a real state-price density.
goods sector is given as:

\[
D_t \equiv \int (p_t(i)y_t(i) - w_t n_t(i)) di = \int \underbrace{p_t(i)y_t(i) di}_{= p_t y_t} - \underbrace{w_t N_{W,t}}_{= p_t C_{W,t}} = p_t (y_t - C_{W,t}) = p_t C_t, \tag{2.2.15}
\]

where we use the Dixit-Stiglitz aggregator properties (total expenditure equals the sum of expenditures on each good) and linear aggregation of labor (equation (2.2.6)). Regardless of price dispersion across firms, the aggregate dividend \(D_t\) is equal to the consumption expenditure of capitalists, who are the natural stock investors in equilibrium as hand-to-mouth workers spend all their income on consumption.

Plugging the above expressions into the fundamental asset pricing equation yields the following condition.

\[
p_t A_t Q_t = \mathbb{E}_t \frac{1}{\xi_t^N} \int_t^{\infty} \xi_s^N \left( D_s \right) ds = \frac{p_t C_t}{\rho}, \tag{2.2.16}
\]

which becomes \(C_t = \rho A_t Q_t\), the same expression as capitalist’ optimal consumption (equation (2.2.11)) when \(a_t\) is given by equation (2.2.12). Thus, in order to determine the asset price and close the model, we need an additional condition. In general, we can obtain the exact \(Q_t\) levels when we have the information about equilibrium levels of labor \(N_{W,t}\) from equation (2.2.13). As we know that \(N_{W,t}\) only depends on time \(t\) real wage \(\frac{w_t}{p_t}\), it ultimately requires information about the real wage level to pin down an expression for \(Q_t\).

The nominal expected return on the risky stock \(i_t^m\) in equilibrium consists of the dividend yield from the intermediate goods sector profits and the nominal stock price re-valuation (capital gain) due to fluctuations in \(\{p_t, A_t, Q_t\}\). Within our specifications, the dividend yield always equals \(\rho\), the discount rate of capitalists. Therefore, when \(i_t^m\) changes, only nominal stock prices can adjust endogenously, as the dividend yield is fixed.

With \(\{I_t^m\}\) as the cumulative stock market return process, the following equation (2.2.17)

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21 Usually, a monetary policy rule takes this role in the New-Keynesian literature.
shows the decomposition of $i_t^m$ into dividend yield and stock revaluation:

\[
\begin{align*}
\Delta \ln m_t &= \rho t \left( y_t - \frac{w_t}{p_t} N_{W,t} \right) \\
&= \frac{d(p_t A_t Q_t)}{p_t A_t Q_t} dt + \frac{d(p_t A_t Q_t)}{p_t A_t Q_t} dt = \rho dt + d\left( \frac{p_t A_t Q_t}{p_t A_t Q_t} \right)
\end{align*}
\]  

(2.2.17)

The equilibrium conditions we have obtained consist of the worker’s optimization (solution of equation (2.2.5)), labor aggregation (equation (2.2.6)), total output (equation (2.2.7)), capitalist’s optimization (equation (2.2.12)), the good market equilibrium (equation (2.2.13)), and determination of the risky stock return (equation (2.2.17)). To close the model, we also have to derive the supply block of the economy (pricing decisions of intermediate good firms à la Calvo (1983)) and define the monetary policy rule, which is the most important topic of our interest.

Before we characterize the benchmark case without nominal rigidities, the following Lemma 1 adapts the Fisher equation when there is a correlation between the (aggregate) price process and the wealth process. The Lemma 1 shows that the inflation premium should be added to the original Fisher relation.

**Lemma 1 (Inflation Premium).** Real interest rate is given by the following variant of the Fisher identity.

\[
r_t = i_t - \pi_t + \sigma_t^p \left( \sigma_t^p + \sigma_t^q \right)
\]

(2.2.18)

Lemma 1 is useful when we characterize the flexible price equilibrium of the model where the nominal price process is arbitrary and does not affect the real economy.

### 2.2.3 Flexible Price Equilibrium

As a benchmark case, we study the flexible price equilibrium. When firms can freely reset their prices ($\delta \to \infty$ case), the real wage becomes proportional to aggregate technology $A_t$. The following proposition summarizes the real wage, asset price process, natural rate of interest $r_t^p$ (the real, risk-free rate that prevails in this benchmark economy), and consumption
process of the capitalist in the flexible price equilibrium. Before we proceed, we define the following parameter, which is the effective labor supply elasticity of workers taking their optimal consumption decision into account.

**Definition 1.** Effective labor supply elasticity of workers $\chi^{-1} \equiv \frac{1 - \varphi}{\chi_0 + \varphi}$

**Proposition 2** (Flexible Price Equilibrium). \footnote{We assign a superscript $n$ to denote variables in the flexible price (natural) equilibrium of the economy.} In the flexible price equilibrium, the following conditions for real wage $\frac{w_n^p}{p_t^0}$, asset price $Q_t^n$, natural rate of interest $r_t^n$, and consumption of capitalists $C_t^n$, hold.

(i) Every firm charges the same price ($\Delta_t = 1$, $\forall t$), and the real wage is proportional to aggregate technology $A_t$.

\[ p_t(i) = p_t, \forall i \in [0, 1] \quad \text{and} \quad \frac{w_n^p}{p_t^0} = \frac{(\epsilon - 1)(1 - \alpha)}{\epsilon} A_t \quad (2.2.19) \]

(ii) Equilibrium (detrended) asset price $Q_t^n$ is constant and given as follows.

\[ Q_t^n = \frac{1}{\rho} \left( \frac{(\epsilon - 1)(1 - \alpha)}{\epsilon} \right)^{\frac{1}{2}} \left( 1 - \frac{(\epsilon - 1)(1 - \alpha)}{\epsilon} \right) \quad \text{and} \quad \mu_t^{q,n} = \sigma_t^{q,n} = 0 \quad (2.2.20) \]

(iii) Natural interest rate $r_t^n$ depends on parameters $\rho, g, \sigma_t$ in the following way.

\[ r_t^n = \rho + g - \sigma_t^2 \quad (2.2.21) \]

(iv) Consumption of capitalists evolves with the following stochastic process, which depends on $r^n, \rho, \sigma_t, \chi$.

\[ \frac{dC_t^n}{C_t^n} = (r_t^n - \rho + \sigma_t^2) dt + \sigma_t \cdot dZ_t \equiv \mu_t^{c,n} \equiv \sigma_t^{c,n} \quad (2.2.22) \]

In flexible price equilibrium, proposition 11 shows that we can characterize closed-form expressions of the real wage $\frac{w_n^p}{p_t^0}$, (detrended) stock price $Q_t^n$ and natural rate $r_t^n$. A few points are worth mentioning. In the flexible price economy, $\sigma_t^{q,n} = 0$ holds, which implies that there is no additional financial risk running in the economy, in addition to the TFP risk, $\sigma_t$. This feature arises because our economy features no explicit frictions (other than nominal rigidity, which is absent for now) and thus every variable other than the labor supply $N^{w,n}(t)$ becomes proportional to $A_t$. This means that real wealth $A_t Q_t^n$ has the exact same volatility as $A_t$ itself, and the financial market imposes no additional risk on the economy.

A higher $\epsilon$ increases competition among firms, raising the real wage $\frac{w_n^p}{p_t^0}$. It also has two competing effects on the asset price $Q_t^n$. A higher real wage pushes down the profit.
of the intermediate sector and reduces the stock price $Q_n^t$. On the other hand, a higher wage induces workers to supply more labor to firms, raising output and stock price $Q_n^t$. The effective labor supply elasticity $\chi^{-1}$ matters in this second effect, thus equation (2.2.20) features $\chi^{-1}$ exponent on the term that increases with $\epsilon$. As $Q_n^t$ is constant, its drift $\mu_t^{q,n}$ also satisfies $\mu_t^{q,n} = 0$ for all $t$.

The natural real interest rate $r_n^t$ consists of two parts with countervailing forces. A higher growth rate $g$ induces capitalists to engage in more intertemporal substitution (into both bonds and stocks) and raises the value of $r_n^t$. A higher $\sigma_t$ pushes down the natural rate $r_n^t$ in two ways: with higher $\sigma_t$, capitalists engage more in precautionary savings, bringing down the natural rate $r_n^t$. This effect is well documented in the literature.\footnote{For example, see Acharya and Dogra (2020) for the recent treatment of precautionary saving in the New-Keynesian environment.} Another channel in which a higher $\sigma_t$ pushes down $r_n^t$ works through the risk-premium. A higher $\sigma_t$ raises the equilibrium risk-premium level, inducing capitalists to pull their wealth out of the stock market, forcing $r_n^t$ to go down in order to prevent a fall in the financial wealth. The second channel is present in our framework as we explicitly model the portfolio decision of each capitalist, which collectively pins down the equilibrium wealth and thus the aggregate demand level.

With the flexible price equilibrium as a benchmark, we move on to the sticky price equilibrium and show how our framework differs from the usual New-Keynesian models.

### 2.2.4 Sticky Price Equilibrium

When price resetting is sticky à la Calvo (1983), we obtain the Phillips curve that describes inflation dynamics. Since a fixed portion $\delta dt$ of firms changes their prices on a given infinitesimal interval $dt$, we have no stochastic fluctuation in the price process in equation (2.2.8), thus $\sigma^2_p = 0$ holds. Now, we just need a monetary policy rule to close the model. Before analyzing the proper monetary rule in this framework, we first describe the ‘gap’ economy, which is defined as the economy where every variable is a log-deviation from the corresponding level in the flexible price economy. That is, we define any business cycle variable $x_t$’s gap, $\hat{x}_t$, to be the log-deviation of $x_t$ from its natural level $x_t^n$, which is the level of the variable in the flexible price equilibrium.

$$\hat{x}_t \equiv \ln \frac{x_t}{x_t^n}. \quad (2.2.23)$$

Because the asset price acts as an endogenous aggregate demand shifter, we first write every other variable’s gap in terms of the asset price gap. The following Assumption 1 is the first step.\footnote{Assumption 1 ensures our framework matches the empirical regularities observed in the data, and holds}
Assumption 1 (Labor Supply Elasticity). \( \chi^{-1} > \frac{(\epsilon - 1)(1 - \alpha)}{\epsilon} \frac{\epsilon}{1 - (\epsilon - 1)(1 - \alpha)} \). 

Assumption 1 is needed to guarantee the positive co-movement between the asset price and business cycle variables (e.g., real wage and consumptions of both capitalists and workers) observed in the data. With a large \( \epsilon \), firms’ mark-ups decrease as competition between them intensifies, and real wage level rises as a result. This has a negative impact on the stock price as firm profits decrease, making it harder to satisfy a positive co-movement between the asset price and real wage gaps.\(^{25}\) A larger \( \alpha \) amplifies the effect of the Baxter and King (1991) externality, and an increase in asset price gap can result in higher labor demand and real wage. Without Assumption 1, a positive gap in the asset price depresses wages, labor, and consumption of workers, which might explain a portion of the observed long-run trend towards increased wealth inequality and income stagnation.\(^{26}\)

The following Lemma 2 argues that given Assumption 1, gaps in consumptions of capitalists and workers, asset price, employment, and real wage are all linearly dependent and co-move with one another up to a first-order. Therefore, for stabilization purposes, the central bank only needs to deal with the asset price gap \( \bar{Q}_t \).\(^{27}\) From \( C_t = \rho A_t Q_t \), we infer that \( \bar{Q}_t = \bar{C}_t \) holds. Thus from now on we can interchangeably use \( \bar{Q}_t \) or \( \bar{C}_t \) to denote gaps of asset price \( Q_t \) and consumption of capitalists \( C_t \).

Lemma 2 (Co-movement). Given assumption 1, gaps in consumption of capitalists \( C_t \) and workers \( (C_{W,t}) \), employment \( (N_{W,t}) \), and real wage \( (\frac{w_t}{p_t}) \) co-move with a positive correlation.

under a standard calibration of the model (see Table B.2). Even without Assumption 1, the main qualitative features of our model remain unchanged.

\(^{25}\)When the demand elasticity \( \epsilon \) is larger, profits of firms per unit revenue decrease, as firms face a fiercer competition. In those cases, a drop in profits can lead to decreases in both the asset price and capitalists’ consumption, while hand-to-mouth workers enjoy a rise in wage income, and hence consumption. A higher \( \chi^{-1} \) means a higher output elasticity with respect to aggregate technology, which tends to generate a positive correlation between consumption of capitalists and workers.

\(^{26}\)For example, see Saez and Zucman (2020) for the trend on rising wealth and income inequality in the US. Also, see Autor et al. (2020) for evidence on a decreasing labor share and effects from the rise of market concentration. Especially, growth in pre-tax income for bottom 50% has been only 0.2% on average per year since 1980s, while S&P-500 index has risen almost by 8% per year.

\(^{27}\)In this demand-determined environment, a positive asset price gap induces stronger economic activities in general, resulting in positive gaps in real wage, employment, and consumption.
Up to a first-order, the following approximation holds.

\[
\hat{Q}_t = \hat{C}_t = \left( \chi^{-1} - \frac{(\epsilon - 1)(1 - \alpha)}{1 - (\epsilon - 1)(1 - \alpha)} \right) \frac{\hat{w}_t}{\rho_t} = \frac{\chi^{-1} - \frac{(\epsilon - 1)(1 - \alpha)}{1 - (\epsilon - 1)(1 - \alpha)}}{1 + \chi^{-1}} \hat{C}_{W,t}. \tag{2.2.24}
\]

Using Lemma 2, we can actually get the following relation between \(\hat{Q}_t\) and \(\hat{y}_t\).

\[
\hat{y}_t = \zeta \hat{Q}_t, \text{ where } \zeta \equiv \frac{\chi^{-1}}{(\epsilon - 1)(1 - \alpha)} > 0, \tag{2.2.25}
\]

where Assumption 1 implies \(\varphi > 0\).\(^{28}\)

**Demand block**  Now we formulate one of the key building blocks of this paper, a dynamic \(\{\hat{Q}_t\}\) process. This \(\{\hat{Q}_t\}\) process serves as the demand block of the model, while the Phillips curve will serve as a supply block.

The dynamic IS equation in our model features some important modifications from the canonical New-Keynesian model. Before we characterize it, we define the risk-premium level \(r_{p_t} \equiv (\sigma_t + \sigma^2_t)^2\) and its natural level in the flexible price economy \(r_{p_t}^n \equiv (\sigma_t)^2\) with \(\sigma_t^{2,n} = 0\), as we characterized in equation (2.2.20). By subtracting \(r_{p_t}^n\) from the current risk-premium level \(r_{p_t}\), we define risk-premium gap \(r_{p_t} \equiv r_{p_t} - r_{p_t}^n\). Basically, as the risk-premium gap rises, capitalists ask for a higher compensation to bear financial risks, which causes asset prices to fall below its natural level. We also define the risk-adjusted natural rate \(r_t^T\) as we defined similarly in the standard non-linear New-Keynesian setting (equation (1.1.10)), which is related to its natural correspondent as follows.

\[
r_t^T \equiv r_t^n - \frac{1}{2} r_{p_t}. \tag{2.2.26}
\]

\(r_t^T\) serves as a real rate anchor for monetary policy. A positive risk-premium gap \((r_{p_t} > 0)\), for example, lowers the demand of capitalists for the risky stock compared with the benchmark.

\(^{28}\)Since aggregate production is linear in aggregate labor up to a first-order, aggregate mark up gap becomes negation of the real wage gap.
economy, and thus decreases the risk-free rate \( r_T^t \) that supports the equilibrium dynamics.

In the following proposition, we characterize an asset price gap \( \hat{Q}_t \) process, which is similar to the usual dynamic IS equation in textbook New-Keynesian models but different in a very important aspect: the natural rate \( r_n^t \) is replaced with the risk-adjusted natural rate \( r_T^t \).

**Proposition 3 (Asset Price Gap Process (Dynamic IS Equation)).** With inflation \( \{ \pi_t \} \), we have the following \( \hat{Q}_t \) process, where \( r_T^t \) takes the role of \( r_n^t \) in the conventional IS equation.

\[
d\hat{Q}_t = (i_t - \pi_t - r_T^t)dt + \sigma^q_t dZ_t.
\]  
(2.2.27)

Thus, endogenous financial volatility \( \sigma^q_t \) directly affects the drift of the \( \{ \hat{Q}_t \} \) process, which governs how all other gap variables fluctuate over time.

With \( \sigma^p_t = 0 \) due to the nature of staggered pricing à la Calvo (1983), when capitalists invest in the stock market they bear \( (\sigma_t + \sigma^q_t) \) amount of risk. We know that the log-preference of capitalists determines the risk-premium level to be \( (\sigma_t + \sigma^q_t)^2 \). In flexible price equilibrium, the natural rate is given as \( r_n^t \) and \( \sigma^q_t \) equals \( \sigma_t^q,n = 0 \). Thus, the level of expected (instantaneous) real return in stock market investment becomes \( r_n^t + (\sigma_t)^2 - \frac{1}{2}(\sigma_t)^2 \), where the factor \( \frac{1}{2}(\sigma_t)^2 \) is from the quadratic variation factor that arises from the second-order Taylor expansion. In a sticky price equilibrium with asset price volatility \( \sigma^q_t \), risk premium changes from \( (\sigma_t)^2 \) to \( (\sigma_t + \sigma^q_t)^2 \). Therefore, with monetary policy rate \( i_t \) and inflation \( \pi_t \), the real expected stock market return becomes \( i_t - \pi_t + \frac{1}{2}(\sigma_t + \sigma^q_t)^2 \). If this value differs from \( r_n^t + \frac{1}{2}(\sigma_t)^2 \), then asset price gap \( \hat{Q}_t \) endogenously adjusts, and this adjustment creates a real distortion from its effect on aggregate demand.

Equation (2.2.27) has the same mathematical structure as equation (1.1.8) in the standard New-Keynesian model. In Section 1.1, the endogenous business cycle volatility has a first-order impact on aggregate demand through precautionary savings channel, whereas in the current model with stock markets, an aggregate financial market volatility affects risk-premium and financial wealth, thereby affecting stock prices and aggregate demand. Due to this isomorphic structure between two frameworks, we will show that novel findings in Section 1.1 continue to hold here, with important implications about monetary policy.

Thus we get the lesson that the monetary policy \( i_t \) should take deviation in risk-premium from its natural level into account as well as the natural rate of interest \( r_n^t \), since otherwise asset price \( Q_t \) will deviate from its natural level and generate business cycle fluctuation. \( r_T^t \) can be interpreted as the real risk-free rate that ensures that the real return on stock market investment is equal to its level in the benchmark economy, as shown in the following equation (2.2.28).

\[
r_n^t + \frac{1}{2} (\sigma_t)^2 = r_T^t = \frac{1}{2} (\sigma_t + \sigma^q_t)^2.
\]  
(2.2.28)
When \( \sigma_t^q = \sigma_{t,n}^q = 0 \) holds, the risk-adjusted rate \( r_t^T \) equals the natural rate \( r_t^n \) and equation (2.2.27) becomes the canonical New-Keynesian IS equation in equation (2.2.29).

\[
d\hat{C}_t = (i_t - \pi_t - r_t^n) dt.
\]  

(2.2.29)

The crux of the problem is that \( \sigma_t^q \) is itself an endogenous variable to be determined in equilibrium, with no guarantee that it will equate its natural level \( \sigma_{t,n}^q = 0 \).

The endogenous financial volatility \( \sigma_t^q \) can be interpreted a measure of financial disruption, as its rise, given monetary policy rate \( i_t \), reduces stock prices and thus aggregate demand, dragging the economy into recession. This channel has been pointed out by many authors including Gilchrist and Zakrajšek (2012), Stein (2014), Chodorow-Reich (2014), Guerrieri and Lorenzoni (2017), Di Tella and Hall (2020) among others, with different aspects of financial disruption affecting economic activity. Woodford (2012) and Cúrdia and Woodford (2016) especially introduced a friction in credit intermediation between borrowers and savers to the New-Keynesian framework and derived similar dynamics for output gap, but their friction is exogenous and relies on ad-hoc assumptions.

The existence of this new stock market volatility channel invites us to re-think the traditional monetary policy framework, to which we devote Section 2.3. Before we jump on to the next topic, if we plug equation (2.2.21) into equation (2.2.26), we get the following expression for \( r_t^T \).

\[
r_t^T = \rho + g - \frac{\sigma_t^2}{2} - \frac{(\sigma_t + \sigma_t^q)^2}{2}.
\]  

(2.2.30)

Figure 2.1a represents \( r_t^T \) as a function of \( \sigma_t^q \) given \( \sigma_t \) level. Intuitively, when \( \sigma_t^q \) jumps up, a rise in risk-premium \( r_p \) ensues and the rate \( r_t^T \) falls. We see \( r_t^T \) aligns with the natural rate \( r_t^n \) when \( \sigma_t^q \) equals \( \sigma_{t,n}^q = 0 \). Figure 2.1b illustrates the effect of a spike in \( \sigma_t \). When \( \sigma_t \) rises, the curve in Figure 2.1a uniformly shifts down. The formula \( \sigma_{t,n}^q = 0 \) in equation (2.2.20) implies that \( \sigma_{t,n}^q \) remains unchanged, but the natural rate of interest \( r_t^n \) still falls due to equation (2.2.21).

Supply block  We follow the standard literature on pricing à la Calvo (1983) to determine inflation dynamics. The above Lemma 2 allows us to express the firms’ aggregate marginal cost gap in terms of the asset price gap up to a first order, as asset price determines aggregate demand, which in turn determines such variables as the aggregate marginal cost.

The following Phillips curve in Proposition 4 describes \( \pi_t \) dynamics, and is of the same form as in many New-Keynesian models.

**Proposition 4 (Phillips Curve).** Inflation \( \pi_t \) evolves according to the following stochastic pro-
cess with $\hat{Q}_t$ entering in the position of output gap in conventional New-Keynesian models.\footnote{The coefficient $\chi\delta(\delta + \rho)\Theta$ is attached to the output gap $\hat{y}_t$ in equation (2.2.31). In standard New-Keynesian models with a representative agent whose utility is of the same form as our workers’, the coefficient becomes $(\chi_0 + \varphi)\delta(\delta + \rho)\Theta$, which is different from $\chi\delta(\delta + \rho)\Theta$ as $\chi \neq \chi_0 + \varphi$.}

$$\mathbb{E}_t\pi_t = (\rho\pi_t - \frac{\kappa}{\zeta}\hat{y}_t)dt \text{ where, } \kappa = \frac{\delta(\delta + \rho)\Theta}{(\epsilon - 1)(1 - \alpha)}, \quad \Theta = \frac{1 - \alpha}{1 - \alpha + \alpha\epsilon}. \quad (2.2.31)$$

Plugging equation (2.2.25) into the Phillips curve, we get $\mathbb{E}_t\pi_t = (\rho\pi_t - \kappa\hat{Q}_t)dt$, which is expressed in terms of $\hat{Q}_t$. Under Assumption 1, a higher asset price gap $\hat{Q}_t$ means the economy is over-heated, and thus inflation rates would jump up. Note that: as price resetting probability increases ($\delta \to \infty$), then we have $\kappa \to \infty$ and $\hat{Q}_t = 0$ in equilibrium. Thus, we achieve the flexible price equilibrium when $\delta \to \infty$.

Now that we characterize the model’s demand block (the IS equation for $\hat{Q}_t$ (equation (2.2.27))) and supply block (Phillips curve in equation (2.2.31)), we need to specify the policy reaction function $i_t$ to close the model. Before we move on to the analysis of policy rules, we briefly discuss the traditional approach to the problem of financial and macroeconomic stabilization in the literature.
Macroprudential policies and regulations  There are in general two goals in short (and/or medium)-run macroeconomics: macro-stabilization and financial stability. Many policymakers (including central bankers) and academic economists believe that financial stability should be dealt with by regulations and macroprudential policies imposed on banks and financial institutions, with business cycle stabilization being the sole focus of monetary policy. Because our model is parsimonious and does not include any complex financial market participants, those macroprudential regulations that tackle potential financial instabilities can be regarded as a policy avenue to prevent $\sigma^q_t$ from deviating from $\sigma^{q,n}_t = 0$. If $\sigma^q_t = \sigma^{q,n}_t = 0$, then as in equation (2.2.29), our model features exactly the same dynamics as conventional New Keynesian models. Therefore, in that case a conventional monetary policy rule can solely focus on business cycle stabilization.

One interesting aspect built in our model is that financial stability (volatility and risk-premium) issues are intertwined with macro-stabilization. The more volatile financial markets features higher risk-premium levels, thereby driving down aggregate financial wealth and aggregate demand. Our view is that even without perfect macroprudential policies to guarantee $\sigma^q_t = \sigma^{q,n}_t = 0$, monetary policy might be able to tackle both concerns simultaneously, as stabilization in one dimension might help stabilize the other.

Now we move onto the analysis of distinct monetary policy rules and revisit the classical question on the role of monetary policy as a financial stabilizer.

## 2.3 Monetary Policy

In this Section 2.3, we study the monetary policy’s roles of macroeconomic stabilization in the context of our model. First, we analyze conventional Taylor rules with inflation and output gap as policy targets. After showing limitations of such policies and how sunspot equilibria can arise, we propose a generalized version of the Taylor rule for stochastic environments that successfully achieve twin objectives of financial and economic stability.

For simplicity, we assume throughout Section 2.3 the constant TFP volatility $\sigma_t = \sigma$ for all $t$ such that the real natural rate $r^*_t = \rho + g - \sigma^2 > 0$ and the natural risk-premium $r^{p^*_t} = \sigma^2$ are constants.
2.3.1 Old Monetary Rule

Conventional Taylor rule and Bernanke and Gertler (2000) rule

We start with a conventional Taylor rule with a constant intercept equal to the natural rate \( r^n \), and standard inflation and output gap targets.

\[
i_t = r^n + \phi_\pi \pi_t + \phi_y \hat{y}_t.
\]  

(2.3.1)

where \( \hat{y}_t \) is the output gap, \( \pi_t \) inflation and note we implicitly assume a zero trend inflation target, \( \bar{\pi} = 0 \). As output gap \( \hat{y}_t \) is positively correlated with the asset price gap \( \hat{Q}_t \) as in equation (2.2.25), we can express equation (2.3.1) as the monetary policy rule that targets asset price \( \hat{Q}_t \) as well as inflation:

\[
i_t = r^n + \phi_\pi \pi_t + \phi_q \hat{Q}_t
\]  

≡ \( \phi_q > 0 \)

(2.3.2)

Bernanke and Gertler (2000), by adding stochastic ad-hoc bubbles to the fundamental asset price in a model based on Bernanke et al. (1999), conducted an analysis on whether monetary rules that directly target asset price as in equation (2.3.2) can effectively stabilize the economy. They conclude that such rules are undesirable as they deter real economic activity when the ‘bubble’ appears and bursts.\(^{30}\) In contrast, our framework features no irrational asset price bubble: here, fluctuations in \( \hat{Q}_t \) reflect rational expectation about future business cycle fluctuations, and thus from central bank’s perspective, targeting the asset price gap \( \hat{Q}_t \) is equivalent to targeting the output gap \( \hat{y}_t \), as the two gaps are perfectly correlated up to a first-order. Therefore in our model, a conventional monetary policy rule is equivalent to the rule of Bernanke and Gertler (2000).

Now we study whether equation (2.3.2) achieves divine coincidence as in textbook New-Keynesian models. Our objective now is to show that this rule cannot guarantee equilibrium determinacy even if it satisfies the so-called Taylor principle. Let us assume the monetary authority relies on Bernanke and Gertler (2000) rule in equation (2.3.2) that targets two factors, \( \pi_t \) and \( \hat{Q}_t \). We define the coefficient \( \phi \equiv \phi_q + \kappa(\phi_{\pi} - 1) > 0 \), which is the total responsiveness of monetary policy to inflation and asset price gap. \( \phi > 0 \) corresponds to the conventional Taylor principle that excludes the possibility of sunspot in inflation. Thus,

\(^{30}\)Galí (2021) introduces rational bubbles in a New-Keynesian model with overlapping generations. He argues that ‘leaning against the bubble’ monetary policy, if properly specified, can insulate the economy from the aggregate bubble fluctuations, as only rational bubbles shift the aggregate output in his framework.
\(i_t\) follows
\[
i_t = r^n + \phi_r \pi_t + \phi_q \hat{Q}_t, \quad \text{where} \quad \phi \equiv \phi_q + \frac{\kappa(\phi_r - 1)}{\rho} > 0. \tag{2.3.3}
\]

Plugging equation (2.3.3) into equation (2.2.27), we get the following \(\hat{Q}_t\) dynamics.
\[
d\hat{Q}_t = \left( (\phi_r - 1) \pi_t + \phi_q \hat{Q}_t - \frac{\sigma_q^2}{2} + \frac{(\sigma + \sigma_q^2)^2}{2} \right) dt + \sigma_q^2 dZ_t. \tag{2.3.4}
\]

**Multiple equilibria** Instead of equation (2.3.4), if \(\hat{Q}_t\) dynamics is represented by
\[
d\hat{Q}_t = \left( (\phi_r - 1) \pi_t + \phi_q \hat{Q}_t \right) dt + \sigma_q^2 dZ_t, \tag{2.3.5}
\]
then, with the Taylor principle \(\phi > 0\) satisfied we achieve divine coincidence: \(\hat{Q}_t = \pi_t = 0\) is the unique possible rational expectations equilibrium from the Blanchard and Kahn (1980).

In contrast, now that the financial volatility \(\sigma_q^2\) affects the drift of equation (2.3.4), we have multiple equilibria and sunspots in \(\sigma_q^2\) can possibly appear. The reason is similar to the reason why we might have sunspots in aggregate business cycle volatility in the standard New-Keynesian model in Section 1.1. Here, the dynamic IS equation in (2.3.4) features a countercyclical financial volatility \(\sigma_q^2\). Since an increase in \(\sigma_q^2\) raises the risk-premium, it brings down financial wealth and aggregate demand (thus, raising the drift of equation (2.3.4)).

For example, imagine that capitalists fear of a possible financial crisis arising from higher levels of risk-premium and financial volatility: they respond by reducing the demand for the risky stock, which leads to the collapse of the asset price, and self-justifies a higher expected return in the stock market investment and a rise in risk-premium. This result is related to Acharya and Dogra (2020)'s findings about equilibrium determinacy issues in models with countercyclical income risks, even though their paper focuses on idiosyncratic risks and effects from precautionary savings, while ours centers on the sunspot equilibria stemming from aggregate endogenous risk.

We now formalize the multiple equilibrium intuition presented above by constructing a rational expectations equilibrium that supports an initial sunspot \(\sigma_0^2\). For simplicity, we focus on the case in which \(\sigma_0^2\) jumps off from \(\sigma_{n}^2 = 0\) (thus, \(\sigma_0^2 > 0\)), and study how the sunspot \(\sigma_0^2\) can be rationally sustained in equilibrium. For that purpose, a rational expectations equilibrium must: (i) support an initial hike \(\sigma_0^2 > 0\), and (ii) not diverge (on expectation) in the long-run, following Blanchard and Kahn (1980).

\footnote{Monetary policy in equation (2.3.2) responds when its mandates \(\hat{Q}_t\) and \(\pi_t\) are affected by a sunspot in \(\sigma_q^2\), but does not directly target the sunspot or volatility \(\sigma_q^2\).}
Martingale equilibrium In particular, we study one rational expectations equilibrium that supports an initial sunspot \( \sigma_0^q > 0 \): the equilibrium in which asset price gap \( \hat{Q}_t \) follows a martingale after the initial sunspot \( \sigma_0^q \) happens. As \( \hat{Q}_t \) is martingale, we get the following formula for \( \pi_t \) by iterating equation (2.2.31) over time.

\[
\pi_t = \kappa \int_{\tau}^{\infty} e^{\rho(s-t)} \mathbb{E}_s(\hat{Q}_s) \, ds = \frac{\kappa}{\rho} \hat{Q}_t, \tag{2.3.6}
\]

which implies inflation closely follows the trajectory of \( \hat{Q}_t \). Plugging equation (2.3.6) into equation (2.3.4) and imposing a martingale condition, we obtain

\[
\hat{Q}_t = -\left( \frac{\sigma + \sigma_t^q}{2\phi} \right)^2 + \frac{\sigma^2}{2\phi} \text{ and } \pi_t = \frac{\kappa}{\rho} \left( -\frac{(\sigma + \sigma_t^q)^2}{2\phi} + \frac{\sigma^2}{2\phi} \right). \tag{2.3.7}
\]

Our martingale equilibrium does not diverge (on expectation) in the long-run, as the paths of \( \{\hat{Q}_t, \pi_t\} \) stay, on expectation, at the initial values of the variables, thus satisfying \( \mathbb{E}_0(\pi_t) = \pi_0 \) and \( \mathbb{E}_0(\hat{Q}_t) = \hat{Q}_0, \forall t \geq 0 \). The last step is to show that there exists a stochastic path of \( \{\sigma_t^q\} \) starting from \( \sigma_0^q \) that supports this equilibrium. This equilibrium then both (i) supports an initial sunspot \( \sigma_0^q > 0 \) and (ii) does not diverge in the long-run. Using equation (2.3.4) and equation (2.3.7), \( \sigma_t^q \) we obtain the stochastic process of \( \sigma_t^q \) as

\[
\frac{d\sigma_t^q}{\sigma_t^q} = -\frac{\phi^2(\sigma_t^q)^2}{2(\sigma + \sigma_t^q)^2} \, dt - \phi \frac{\sigma_t^q}{\sigma + \sigma_t^q} \, dZ_t. \tag{2.3.9}
\]

Both equation (2.3.7) and equation (2.3.9) constitute the dynamics of this particular rational equilibrium supporting \( \sigma_t^q > 0 \). What does this equilibrium look like? The next Proposition 5 sheds light on the behavior of \( \hat{Q}_t \) and \( \pi_t \) paths and argues that business cycles almost surely converge to a perfectly stabilized path in the long run. The very few paths that do not converge can blow up asymptotically and, together with the forward-looking nature of

\[\text{Footnotes:}\]

32 Under some regularity conditions dictating how the expected risk-premium evolves in the long run, our martingale equilibrium becomes a ‘unique’ rational expectations equilibrium that supports an initial sunspot \( \sigma_0^q > 0 \). A martingale process for \( \hat{Q}_t \) is consistent with the previous findings of the literature on the ‘Efficient Market Hypothesis (EMH)’ (For example, see Fama (1970)).

33 Thus in this particular equilibrium, \( \sigma_t^q > \sigma^{q,n} = 0 \) causes \( \hat{Q}_t \) to drop below zero, causing a recession.

34 Since \( \hat{Q}_t \) process is a martingale, the drift part in equation (2.3.4) must be 0.

35 When \( \sigma = 0 \), this process becomes the following Bessel process:

\[
\frac{d\sigma_t^q}{\sigma_t^q} = -\frac{\phi^2}{2\sigma_t^q} \, dt - \phi \, dZ_t. \tag{2.3.8}
\]

which stops when \( \sigma_t^q \) reaches \( \sigma^{q,n} = 0 \). For general properties of Bessel processes, see Lawler (2019).
the economy, help sustain the initial crisis.

**Proposition 5** (Bernanke and Gertler (2000) Rule and Indeterminacy). For any value of Taylor responsiveness $\phi > 0$:

1. Indeterminacy: there is always a rational expectations equilibrium (REE) that supports initial sunspot $\sigma_0^q > 0$ and is represented by $\hat{Q}_t$ and $\pi_t$ dynamics in equation (2.3.7), and $\sigma_t^q$ process in equation (2.3.9)

2. Properties: the rational expectations equilibrium that supports an initial sunspot $\sigma_0^q > 0$ satisfies:

   $(i)$ $\sigma_t^q \xrightarrow{a.s.} \sigma_\infty^q = \sigma^{q,n} = 0$, $(ii)$ $\hat{Q}_t \xrightarrow{a.s.} 0$ and $\pi_t \xrightarrow{a.s.} 0$, and $(iii)$ $\mathbb{E}_0 (\max_t (\sigma_t^q)^2) = \infty$

The conditions $\sigma_t^q \xrightarrow{a.s.} \sigma_\infty^q = \sigma^{q,n}$, $\hat{Q}_t \xrightarrow{a.s.} 0$, and $\pi_t \xrightarrow{a.s.} 0$ imply that equilibrium paths supporting an initial sunspot $\sigma_0^q > 0$ are almost surely stabilized in the long run. Then, how is it possible for a sunspot $\sigma_0^q > 0$ to appear at first? The finding $\mathbb{E}_0 (\max_t (\sigma_t^q)^2) = \infty$ implies that an initial spike in $\sigma_0^q$ and the ensuing crisis is sustained by the tiny probability of an $\infty$-severe financial disruption in the future. This result has similar implications to Martin (2012) in a sense that our framework does not assume the existence of specific disasters but disaster risk is always present even if monetary authority satisfies the Taylor principle and actively stabilizes the business cycle. Martin (2012) applied a similar logic to pure asset pricing contexts and showed that the pricing of a broad class of long-dated assets is driven by the possibility of extraordinarily bad news in the future. The intuitions we derived here continue to hold in our simple discrete-time framework in Lee and Carreras (2021b).

**Calibration and Simulation** For the rest of the paper, we calibrate the parameters of our model to values commonly found in the literature: see Table B.2 in Appendix B.2 for further details. A few points are worth mentioning. For worker’s risk-aversion parameter $\varphi$, we use $\varphi = 0.2$ following Gandelman and Hernández-Murillo (2014). For individual firm’s labor share in production, we use $1 - \alpha = 0.6$ following Alvarez-Cuadrado et al. (2018), as we regard the aggregate labor in the production function as a proxy for the capital in conventional macroeconomic models. With this calibration, our co-movement condition (Assumption 1) is satisfied.

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36Their estimates of $\varphi$ range between 0.2 and 10. In our environment, a higher risk-aversion of workers makes their labor supply (and therefore, output) less responsive to business cycle fluctuations. In that scenario, a higher asset price tends to translate into less wage income distributed to workers, making it harder to satisfy the co-movement condition (Assumption 1). Thus, we pick a value on the lower end of the acceptable range and set $\varphi = 0.2$. 
35

(a) With $\phi_{\pi} = 1.5$  
(b) With $\phi_{\pi} = 2.5$.

Figure 2.2: $\{\sigma^q_t, \hat{Q}_t\}$ dynamics when $\sigma^{q,n}_0 = 0$ and $\sigma^q_0 = 0.9$, with calibration in Table B.2

Figure 2.2 illustrates the martingale equilibrium’s dynamic paths of $\{\sigma^q_t, \hat{Q}_t\}$ supporting $\sigma^q_0 = 0.9 > \sigma^{q,n}_0 = 0$. Normalization shows that as $\sigma^q_0$ jumps off by $\sigma$, stock price falls by $2-10\%$, which is consistent with our empirical findings in Appendix B.1 (Figure B.1b). Figure 2.2 also explores the effects on the martingale equilibrium of a change in policy responsiveness to inflation $\phi_{\pi}$. The right panel 2.2b uses the default calibration value $\phi_{\pi} = 2.5$, while the left panel 2.2a assumes a more accommodating stance $\phi_{\pi} = 1.5$. As we raise $\phi_{\pi}$, the average sample path converges faster towards full stabilization, but at the expense of an increased likelihood of a more severe crisis path in a given period of time. We obtain similar results when looking at changes in policy responsiveness to the asset price gap $\phi_q$ (alternatively, output gap $\phi_y$), and find that a change in $\phi \equiv \phi_q + (\phi_{\pi} - 1)\kappa\rho$, the measure of combined responsiveness of monetary policy $i_t$, brought by any combination $\{\phi_{\pi}, \phi_q\}$ follows the same patterns depicted in Figure 2.2.

**Booms** In an analogous way, we can construct a rational expectations equilibrium that supports an initial downward sunspot $\sigma^q_0 < \sigma^{q,n}_0 \equiv 0$. The equilibrium paths feature a boom phase with buoyant production and consumption with lower levels of financial volatility and risk-premium. A higher $\phi$ value speeds up the stabilization process, but increases the likelihood of an equilibrium path with an overheated economy.37

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37We have two singular points in the $\{\sigma^q_t\}$ process (equation (2.3.9)): as $\sigma^q_t$ hits $-\sigma$, the drift and volatility of the process diverge, and $\{\sigma^q_t\}$ process features a jump. When $\sigma^q_t$ hits 0, it stays there forever. Thus, when $\sigma^q_0$ is below $-\sigma$, we might end up in paths where we have a jump in $\sigma^q_t$ to a positive value, which eventually converges to 0.
2.3.2 Modified Monetary Rule

A modified monetary policy rule includes risk-premium as a separate factor in the following way:

\[ i_t = r^n + \phi_{\pi} \pi_t + \phi_q \hat{Q}_t - \frac{1}{2} \hat{r} p_t, \text{ where } \hat{r} p_t \equiv r_p - r_p^n. \]  

(2.3.10)

Thus, the above monetary policy rule contains a ‘risk-premium gap term’ as a factor in addition to inflation and asset price gap. It also can be written in terms of the risk-adjusted natural rate \( r_T \) as

\[ i_t = r_T + \phi_{\pi} \pi_t + \phi_q \hat{Q}_t, \]  

(2.3.11)

where a higher \( \hat{r} p_t \) brings down \( r_T \), forcing \( i_t \) to fall. The next Proposition 6 establishes that a monetary policy rule consistent with equation (2.3.10) and that satisfies the Taylor principle (corresponding to \( \phi > 0 \)) restores equilibrium determinacy and fully stabilizes the economy.

**Proposition 6** (Ultra-Divine Coincidence with Risk-Premium Targeting). The monetary policy rule

\[ i_t = r^n + \phi_{\pi} \pi_t + \phi_q \hat{Q}_t - \frac{1}{2} \hat{r} p_t, \text{ where } \phi \equiv \phi_q + \frac{\kappa(\phi_{\pi} - 1)}{\rho} > 0, \]  

(2.3.12)

achieves \( \hat{Q}_t = \pi_t = \hat{r} p_t = 0 \). Therefore, the monetary policy rule in equation (2.3.12) attains (i) output (asset price) stabilization, (ii) price level (inflation) stabilization, and (iii) financial market (financial volatility and risk-premium) stabilization. We call it a ultra-divine coincidence.

This result is a direct consequence of Blanchard and Kahn (1980) and Buiter (1984). The reason central banks must target risk-premium as a separate factor is that this term directly appears in the drift part of our dynamic IS equation (equation (2.2.27)). According to the rule in equation (2.3.12), a central bank lowers the policy rate \( i_t \) when \( r_p > r_p^n \) to boost \( \hat{Q}_t \) and \( \hat{C}_t \), since a higher risk-premium drags down asset price and business cycle levels. If monetary policy kills an initial excess volatility (or excess risk-premium) with this additional target in its rule, it precludes the possibility of sunspots in financial volatility that we discussed. Since the Taylor principle (\( \phi > 0 \)) guarantees there is no sunspot inflation,

---

\(^{38}\)Even with Bernanke and Gertler (2000) rule, monetary policy responds to a rise in risk-premium since it negatively affects the asset price gap \( \hat{Q}_t \) and inflation \( \pi_t \). Our point here is that the policy rate must systematically respond to deviations of \( r_p \) from its natural level \( r_p^n \) given \( \hat{Q}_t \) and \( \pi_t \) levels.
the policy rule in equation (2.3.12) restores equilibrium determinacy and achieves both macro stability (with $\hat{Q}_t = \pi_t = 0$) and financial stability (with $r p_t = 0$, which implies $r p_t = r p^n$ and $\sigma^q_t = \sigma^{q,n}_t = 0$). The equilibrium interest rate then becomes $i_t = r^n$, which is the same level as in the equilibrium path of a canonical New-Keynesian model. Therefore, the ultra-divine coincidence result implies: one policy tool ($i_t$ rule) achieves an additional objective (financial stability) in addition to the two usual mandates (output gap and inflation stability). This is possible in our framework because financial markets and the business cycle are tightly interwoven and real and financial instabilities are equivalent to each other.

A common view in the literature holds that monetary policy must respond to financial market disruptions only when they affect (or to the degree that they affect) the original central bank mandates of inflation stability and full employment (or full output). This premise is at odds with the results of our paper: the failure to target the risk-premium of financial markets subjects the economy to the apparition of sunspot shocks and the corresponding recessions and overheating episodes that ensue. Only by targeting risk-premium in the particular way characterized in equation (2.3.10), the monetary authority can re-establish equilibrium determinacy and achieve the ultra-divine coincidence outlined in the previous paragraphs.

**Interpretation** We can rewrite our modified Taylor rule (equation (2.3.12)) as

\[
\begin{align*}
  i_t + r p_t - \frac{1}{2} r p_t &= r^n + r p^n - \frac{1}{2} r p^n + \phi \pi_t + \phi q \hat{Q}_t, \\
  \text{or equivalently as} \\
  \rho + \mathbb{E}_t \left( \frac{d \log a_t}{dt} \right) &= \rho + \mathbb{E}_t \left( \frac{d \log a^n_t}{dt} \right) + \phi \pi_t + \phi q \hat{Q}_t,
\end{align*}
\]

where $a_t$ is the economy’s aggregate financial wealth and $a^n_t$ is the aggregate wealth of the natural (flexible price) economy. Our modified monetary policy that targets a risk-premium as prescribed in equation (2.3.12) thus can be interpreted as the rule on the rate of change of log-aggregate wealth as a function of traditional inflation and output gap (asset price) targets.

In the standard linearized New-Keynesian model (or alternatively, a model under perfect foresight), the economy’s risk-free rate (i.e., policy rate) equals the rate of change in log-wealth, whereas the expected stock market return takes that role in our model with risk. Therefore, equation (2.3.14) restores determinacy and attains divine coincidence both
in the standard linearized model and in our framework where the endogenous volatility of stocks (equivalently, the risk premium) affects expected asset returns. We interpret equation (2.3.14) as the generalized Taylor rule that holds in both linearized and risk-centric environments. With this rule, the central bank uses the aggregate wealth and its rate of return as intermediate targets towards achieving business cycle stabilization, as wealth itself affects aggregate demand, and its internal rate of return changes how a demand-driven economy evolves along the cycle.

**Practicality** Some issues exist about the feasibility to implement this new policy rule. First, the risk-premium gap $\hat{r}_p$ in equation (2.3.10) depends on the natural risk-premium level, which is a counterfactual variable by definition, and therefore its observation is subject to some form of measurement error. Second, there are multiple kinds of risk-premia in financial markets that can be possibly targeted through monetary policy, and the construction of an aggregate risk-premium index as featured in our model might be subject to error as well.\(^{39,40}\)

More importantly, and related to the previous two points, the coefficient attached to risk-premium in equation (2.3.10) is exactly $\frac{1}{2}$. Given the potential for measurement error in $\hat{r}_p$, it might be impossible for the central bank to target the risk-premium with the exact strength demanded by equation (2.3.10).\(^{41}\) To understand the consequences of deviating from the $\frac{1}{2}$ risk-premium target, we consider the following alternative rule:

$$i_t = r^n + \phi_r \pi_t + \phi_q \hat{Q}_t - \phi_{rp} \hat{r}_p,$$

(2.3.15)

where $\phi_{rp}$ is a constant term potentially different from $\frac{1}{2}$. We have the following $\{\hat{Q}_t\}$ process with the policy rule in equation (2.3.15):

$$d\hat{Q}_t = \left( (\phi_r - 1)\pi_t + \phi_q \hat{Q}_t + \left( \frac{1}{2} - \phi_{rp} \right) \hat{r}_p \right) dt + \sigma_q^2 dZ_t.$$

(2.3.16)

With $\phi_{rp} = \frac{1}{2}$, we return to determinacy (Proposition 6). With $\phi_{rp} \neq \frac{1}{2}$, the martingale

---

\(^{39}\)Our framework features only an 'index' of the stock market as a feasible vehicle to invest in, but there are multiple risk-premia (including term-premia) covering stocks and bonds in the real world.

\(^{40}\)There have been long-standing debates about whether monetary authorities should adjust policy rates in response to fluctuations in risk-premia of financial markets. For example, \textit{Doh et al.} (2015) argued "adjusting short-term interest rates in response to various estimated risk premium levels could be appropriate, especially if the risk premiums are low for a sustained period. In contrast, if policymakers are predominantly concerned about the most likely macroeconomic outcome, monitoring estimated risk premiums and adjusting the monetary policy stance accordingly may be of little benefit." This argument is based on the fact that information about possible tail risks is summarized by the risk-premia levels in financial markets.

\(^{41}\)As an example, consider a multiplicative measurement error $\epsilon_t$ such that $\hat{r}_p^{obs} = \epsilon_t \cdot \hat{r}_p$, where $\hat{r}_p^{obs}$ stands for the observed risk-premium. It is easy to see that the central bank following the policy rule in equation (2.3.10) will target the 'true' risk-premium with a coefficient $\neq \frac{1}{2}$.
equilibrium reappears and is characterized by

$$
\dot{Q}_t = -\frac{(\sigma + \sigma_t^q)^2}{2\phi_{\phi_{rp}}} + \frac{\sigma_t^2}{2\phi_{\phi_{rp}}} \quad \text{and} \quad \pi_t = \frac{\kappa}{\rho} \left( -\frac{(\sigma + \sigma_t^q)^2}{2\phi_{\phi_{rp}}} + \frac{\sigma^2}{2\phi_{\phi_{rp}}} \right) \quad \text{with} \quad \phi_{\phi_{rn}} \equiv \frac{\phi}{1 - 2\phi_{rp}},
$$

(2.3.17)

where \( \{\sigma_t^q\} \)’s stochastic process after an initial sunspot \( \sigma_0^q \) appears is given as

$$
d\sigma_t^q = -\frac{\phi_{\phi_{rn}}^2 (\sigma_t^q)^2}{2(\sigma + \sigma_t^q)^3} dt - \phi_{\phi_{rn}} \frac{\sigma_t^q}{\sigma + \sigma_t^q} dZ_t.
$$

(2.3.18)

When \( \phi_{rp} < \frac{1}{2} \) (including Proposition 5, the case of \( \phi_{rp} = 0 \)), a rise in \( \phi_{rp} \) leads to an increase in \( \phi_{\phi_{rn}} \) in equation (2.3.17). From equation (2.3.18) we observe that a higher \( \phi_{\phi_{rn}} \) accelerates the convergence of sample paths while creating more amplified paths after the initial sunspot \( \sigma_0^q \) appears. As far as \( \phi_{rp} < \frac{1}{2} \), a higher \( \phi_{rp} \) means monetary policy responds more strongly to fluctuations in \( r\rho_t \), which allows faster stabilization. As \( \phi_{rp} \) goes up from 0 to \( \frac{1}{2} \), fluctuations in \( r\rho_t \) have less direct effects on dynamics (equation (2.3.16)). Thus, the volatility of the \( \{\sigma_t^q\} \) process (in equation (2.3.18)) must rise to ensure that \( \{\dot{Q}_t\} \) eventually is stabilized\(^{43}\), which results, on average, on shorter but more amplified sample paths.

\( \phi_{rp} < 0 \) case is interesting since it implies central bank raises the policy rate when risk-premia rise in financial markets. It is consistent with the “Real Bills Doctrine” which was a popular idea during the first half of the 20th century. Basically, the doctrine advocated for the Fed discount rate to track the average interest rate of the financial markets, as a means of stabilization.\(^{44}\) In our framework, \( \phi_{rp} < 0 \) pushes down \( \phi_{\phi_{rn}} \) from \( \phi \), which effectively slows down the pace of stabilization after sunspots hit the stock market. Therefore, we see that “Real Bills Doctrine” with \( \phi_{rp} < 0 \) is not suitable for stabilization purposes, as empirically documented by Richardson and Troost (2009).

With \( \phi_{rp} > \frac{1}{2} \), monetary policy responds too strongly to fluctuations in risk-premium, thus with an initial positive sunspot \( \sigma_0^q > 0 \), policy rate drops excessively and creates an artificial boom instead of a crisis.\(^{45}\) A higher \( \phi_{rp} \) reduces \( |\phi_{\phi_{rn}}| \) and slows down stabilization since a

\(^{42}\)The equations (equation (2.3.15) and equation (2.3.17)) are easily derived in a similar way to the proof of Proposition 5 in Appendix B.3.

\(^{43}\)Here with the monetary policy in equation (2.3.15), \( (\frac{1}{2} - \phi_{\phi_{rn}}) r\rho_t \) appears in the drift of the \( \{\dot{Q}_t\} \) process (equation (2.3.16)). When \( \phi_{rp} < \frac{1}{2} \), a higher \( \phi_{\phi_{rn}} \) implies that \( \{\sigma_t^q\} \) path, on average, features more volatility (of \( \{\sigma_t^q\} \) path itself) to raise \( r\rho_t \) given the levels of \( \dot{Q}_t \) and \( \pi_t \), as \( r\rho_t \) is a convex function of \( \sigma_t^q \). Eventually, \( \dot{Q}_t \) and \( \pi_t \) adjust as they are jump variables.

\(^{44}\)Richardson and Troost (2009) studied the effects of such policy during the Great Depression era, exploiting the fact that the state of Mississippi is divided by the Federal Reserve act between the 6th (Atlanta) and 8th (St. Louis) districts which had different approaches to the economy-wide banking panics and depressions.

\(^{45}\)With \( \phi_{rp} > \frac{1}{2} \), we have \( \phi_{\phi_{rn}} < 0 \) from equation (2.3.17), thus \( \sigma_t^q > 0 \) is equivalent to the boom phase with \( \pi_t > 0 \) and \( \dot{Q}_t > 0 \).
higher $\phi_{rp}$ means monetary policy deviates more from determinacy (the case of $\phi_{rp} = \frac{1}{2}$), and therefore becomes less efficient at stabilization. Figure 2.3 illustrates that with $\phi_{rp} > \frac{1}{2}$, a spike in financial volatility, $\sigma_t^q > 0$, actually acts as a boon to the economy, as we have $\hat{Q}_t > 0$ and $\pi_t > 0$ along sample paths. Moreover, with $\phi_{\pi} = 2.5$ fixed, as we raise $\phi_{rp}$ (from 1 to 1.5), stabilization slows down\(^{46}\) as we further deviate from the determinacy case ($\phi_{rp} = \frac{1}{2}$).

These results are summarized in Table 2.1.

<table>
<thead>
<tr>
<th>$\phi_{rp} &lt; 0$ (Real Bills Doctrine)</th>
<th>$0 \leq \phi_{rp} &lt; \frac{1}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) With $\phi_{rp} \downarrow$, convergence speed $\downarrow$ and less amplified paths</td>
<td>(i) With $\phi_{rp} \uparrow$, convergence speed $\uparrow$ and more amplified paths</td>
</tr>
<tr>
<td>(ii) $\sigma_t^q &gt; \sigma_t^{q,n} = 0$ means a crisis ($\hat{Q}_t &lt; 0$ and $\pi_t &lt; 0$)</td>
<td>(ii) $\sigma_t^q &gt; \sigma_t^{q,n} = 0$ means a crisis ($\hat{Q}_t &lt; 0$ and $\pi_t &lt; 0$)</td>
</tr>
<tr>
<td>$\phi_{rp} = \frac{1}{2}$</td>
<td>$\phi_{rp} &gt; \frac{1}{2}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>No sunspot (ultra-divine coincidence)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>As $\phi \uparrow$, convergence speed $\uparrow$ and $\exists$ more amplified paths</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.1: Effects of different parameters $\{\phi_{rp}, \phi\}$ on stabilization

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\(^{46}\)Also, a higher $\phi_{rp}$ causes less amplification from the initial sunspot $\sigma_0^q > 0$. 

---

Figure 2.3: $\{\sigma_t^q, \hat{Q}_t\}$ dynamics when $\sigma^{q,n} = 0$ and $\sigma_0^q = 0.9$, with varying $\phi_{rp} > \frac{1}{2}$.

(a) With $\phi_{rp} = 1$  
(b) With $\phi_{rp} = 1.5$. 

---

40
In Figure 2.2, we observe that an initial sunspot $\sigma_0^q > 0$ can be amplified endogenously through monetary policy’s responses to the business cycle fluctuation, which might drag the economy into zero lower bound (ZLB) episodes when the $\{\sigma_t^q\}$ path hits some threshold from below. When monetary policy is constrained at those episodes, both asset market and business cycle would collapse, which we observed in the 2007-2009 Global Financial Crisis (GFC). In the next Sections, we formally study ZLB issues following the prior literature and discuss possible fiscal-monetary policies that mitigate recessionary pressures and stabilize financial markets and the real economy.

2.4 Zero Lower Bound (ZLB) and Forward Guidance

The ZLB featured prominently during the Great Financial Crisis, and in this section we will show that it has, indeed, very interesting implications for the costs of financial volatility sunspot shocks. In the previous section, and as depicted in Figure 2.2, we have shown that a less responsive monetary policy (to $\pi_t$ and $\dot{Q}_t$) results in persistent sunspot recessions. During a ZLB episode, the policy rate is stuck at zero, and therefore the unresponsiveness of monetary policy can be approximated, to a first pass, as an extreme case of very low monetary responsiveness, in which case ZLB amplifies the duration -and the costs- of positive sunspot shocks. Central banks have developed alternative tools like forward guidance to retain their capacity to intervene in the economy and minimize the costs of ZLB recessions. In this section, we will study the capacity of this tools to stabilize the economy and financial markets, and the potential trade-offs in terms of stabilization that their use entails.

Following Werning (2012), we consider a scenario in which exogenous TFP volatility $\sigma_t$ jumps in a deterministic manner between $t = 0$ and $T$. In particular, we consider the case where $\sigma_t = \bar{\sigma}$ for $0 \leq t \leq T$ and $\sigma_t = \sigma$ for $t \geq T$. We assume that $r \equiv r^n(\bar{\sigma}) < 0$ and $\bar{r} \equiv r^n(\sigma) > 0$, thus monetary policy is constrained by the ZLB until $t = T$.

2.4.1 Perfect Stabilization after ZLB or Forward Guidance

Perfect stabilization after ZLB We assume that after $t = T$, monetary policy follows the modified Taylor rule (equation (2.3.12)) and achieves perfect stabilization, satisfying $\pi_t = \dot{Q}_t = 0$ for $t \geq T$. From equation (2.2.27) and equation (2.2.31), the fact that

\textsuperscript{47}From equation (2.2.21) we can express natural rate $r^n$ as a function of fundamental volatility $\sigma_t$ only. Throughout this Section 2.4.1 we assume $r^n(\bar{\sigma}) < 0$ and $r^n(\sigma) > 0$.

\textsuperscript{48} Basically, we assume that monetary policy returns to the modified Taylor rule that includes a risk-premium factor as one of targets as in equation (2.3.10) after $t = T$. For $t \leq T$, since $\sigma_t = \bar{\sigma}$ satisfies $r^n(\bar{\sigma}) < 0$, monetary authority cannot implement this rule and is constrained by the ZLB.
\( \hat{Q}_T = \pi_T = 0 \) is pinned down implies that there is no volatility for both \( \hat{Q}_t \) and \( \pi_t \) processes before \( T \) thus \( \sigma^q_t = \sigma^{q,n}_t = 0 \) and \( \sigma_{\pi,t} = 0 \) for \( t \leq T \). Therefore, in this case the risk-adjusted natural rate \( r^T_t \) equals the natural rate \( \hat{r} = r^\sigma(\bar{\sigma}) \) for \( t \leq T \) and we get exactly the same dynamics for \( \pi_t \) and \( \hat{Q}_t \) as in Werning (2012) and Cochrane (2017). For a reasonable calibration, \(^4^9\) Figure 2.4 (dashed black for \{\( \hat{Q}_t \)\} and dashed gray for \{\( \pi_t \)\}) illustrates \( \hat{Q}_t \) and \( \pi_t \) dynamics during ZLB crises. Both variables are negative until \( T \) and become stabilized after \( T \) due to our generalized Taylor rule that targets risk-premium.

Notice that even though we have similar dynamics for \{\( \hat{Q}_t, \pi_t \)\} to the ones in Werning (2012) and Cochrane (2017), the forces that drive our results are different. Here, the ZLB constraint causes asset price \( \hat{Q}_t \) to fall, as the ZLB is higher than the risk-free rate that is needed for full stabilization, which reduces capitalists’ demand for stock market investment. This eventually translates into a reduction of aggregate financial wealth and demand.\(^5^0\) In canonical New-Keynesian models, on the other hand, the ZLB induces agents to engage in deleveraging and reduce consumption, which collectively lowers the aggregate output through the aggregate-demand externality. Note also that the dynamics in Figure 2.4 depend on the perfect stabilization after \( T \) due to our modified Taylor rule. We get an important lesson from Figure 2.4: central banks can prevent the appearance of sunspot equilibria at the ZLB by credibly committing to stabilize financial markets at some future date \( T < +\infty \). Therefore, even if the monetary authority is unable to temporarily follow the modified Taylor rule in equation (2.3.12) due to a binding ZLB, the additional financial stability costs of policy inaction can be contained (indeed, eliminated) by a credible commitment to stabilization upon-ZLB exit. A corollary of this result is that the costs of the ZLB might be highly heterogeneous across countries: countries with a monetary authority committed to financial stabilization will ‘only’ experience the demand-driven recession outlined above, while countries without that capacity (or willingness) to stabilize financial markets might suffer additional costs from long-lived sunspot shocks to financial volatility.

Now we turn our eyes to the forward guidance policy, in which central banks commit to keep the policy rate at zero for a longer duration than \( T \). In our framework, forward guidance is a powerful tool as in Werning (2012) and Cochrane (2017), with the premise that after the forward guidance is over, central banks return to our modified monetary policy rule that stabilizes the financial market.

\(^ {4^9}\)We use parameters: \( T = 3, \sigma = 0.0090, \bar{\sigma} = 0.2090, \bar{r} = -1.54\%, \tilde{r} = 2.82\% \) in simulating our equilibrium throughout this Section 2.4. The ZLB can be created by not only a spike in \( \sigma_t \), but also downward jump in \( g \).

\(^ {5^0}\)This is the other side of Caballero and Farhi (2017). While Caballero and Farhi (2017) showed that a high demand for safe assets drags the economy into recession when monetary policy is constrained, we argue that it causes marginal investors to pull their wealth out of the stock market, thereby reducing stock market wealth and aggregate demand.
Perfect stabilization after forward guidance. Now central bank keeps $i_t$ at zero until $\hat{T} > T$. After $\hat{T}$, central bank fully stabilizes the economy with $\hat{Q}_t = \pi_t = 0$ for $t \geq \hat{T}$ with the generalized Taylor rule in equation (2.3.10). Due to the same reason exposed in the analysis of ZLB, we know that $\sigma_q^2 = \sigma_{q,n}^2 = 0$ and $\sigma_{\pi,t} = 0$ for $t \leq \hat{T}$ and therefore, equation (2.2.29) and equation (2.2.31) characterize $\{\hat{Q}_t, \pi_t\}$ dynamics until $\hat{T}$ with $\hat{Q}_\hat{T} = \pi_\hat{T} = 0$. Forward guidance in our framework features similar dynamics to the ones in Werning (2012) and Cochrane (2017), but acts through a different mechanism. In our framework, forward guidance is powerful because it raises asset price $\hat{Q}_t$ from $T$ to $\hat{T}$, leading to a rise in $\hat{Q}_t$ even before $T$, thus increasing the consumption level of capitalists. In traditional New-Keynesian models, forward guidance is useful as it raises household consumption and thus income from $T$ to $\hat{T}$, leading to a rise in consumption before $T$ due to the usual intertemporal substitution channels and general equilibrium effects.

To characterize optimal $\hat{T}$, we minimize the following quadratic loss function with respect to $\hat{T}$, which is derived in Appendix B.4.

$$L(\{\hat{Q}_t, \pi_t\}_{t \geq 0}) = E_0 \int_0^\infty e^{-\rho t}(\hat{Q}_t^2 + \Gamma \pi_t^2)dt.$$ (2.4.1)

Figure 2.4 (thick blue for $\{\hat{Q}_t\}$ and red for $\{\pi_t\}$) illustrates $\{\hat{Q}_t, \pi_t\}$ dynamics with the
optimal forward guidance duration $\hat{T} = 3.602 > T = 3$. $\hat{Q}_t$ and $\pi_t$ drop less than in cases without forward guidance (dashed black for $\{\hat{Q}_t\}$ and dashed gray for $\{\pi_t\}$) and $\hat{Q}_t$ is even positive for some periods during the ZLB episode due to forward guidance. After $\hat{T} = 3.602$, $\hat{Q}_t$ and $\pi_t$ are both stabilized since monetary policy becomes active again and targets risk premium in addition to $\{\hat{Q}_t, \pi_t\}$.

Note that the heterogeneous costs of the ZLB reappear under forward guidance: central banks that can credibly commit to stabilize financial markets after $\hat{T}$ do not face any adverse financial volatility costs from the implementation of such policies, and results are unambiguously positive in terms of welfare and business cycle stabilization. This conclusion changes dramatically when a central bank cannot commit to perfect stabilization after $\hat{T}$, as the economy is subject to the appearance of sunspot shocks (which are specially costly at the ZLB). In that scenario, voluntarily lengthening the time spent at the ZLB is a risky business: by keeping a passive monetary policy until $\hat{T}$, a central bank risks aggravating the costs stemming from the endogenous financial volatility of the economy - precisely at the moment in which those costs are higher-.\footnote{To be clear, even though the scope for forward guidance policies is reduced when the central bank cannot commit to future stabilization, the gains from the policy are still positive, on expectation.}

We believe this constitutes a novel result on the trade-offs involved in forward guidance policy, and think that it might be one of the reasons behind the cautiousness with which central bankers around the world approach the implementation of such policies in practice. In the next Section 2.4.2, we look into this case where central banks cannot credibly commit to attain perfect stabilization after ZLB or forward guidance, and instead uses a conventional Taylor rule out of the ZLB, only targeting usual mandates.

### 2.4.2 Imperfect Stabilization after ZLB or Forward Guidance

For analytic tractability, we assume in this Section 2.4.2 that inflation is fixed at zero $\pi_t = 0$ for $\forall t$, which corresponds to a perfectly rigid-price economy. First, we consider the usual ZLB case without forward guidance.

**Imperfect stabilization after ZLB** In this section, we explore the case in which the central bank cannot achieve full stabilization after $T$ and instead relies on the conventional Taylor rule $i_t = \bar{r} + \phi_0 \hat{Q}_t$ for $t \geq T$. Under imperfect financial stabilization, sunspots in $\sigma^q_t$ can appear, and in this section we provide a rational expectations equilibrium of the economy at the ZLB.

We can tackle the problem based on the standard backward induction logic. Assume that a positive sunspot $\sigma^q_0 > 0$ arises, based on the fact that the central bank cannot fully stabilize the economy after $T$. Then, rational agents expect that upon-ZLB exit for $t \geq T$, the economy follows the martingale equilibrium outlined in Section 2.3.2, with an initial date...
\( t = T \) endogenous volatility \( \sigma^q_t \). Therefore, they expected the asset price gap \( \tilde{Q}_t \) path for \( t \geq T \) to evolve according to equation (2.4.2).

\[
\tilde{Q}_t = -\frac{(\sigma + \sigma^q_t)^2}{2\phi_q} + \left( \frac{\sigma}{2\phi_q} \right) \text{ for } t \geq T,
\]

(2.4.2)

with \( \sigma^q_T \geq 0 \) being stochastic from a \( t = 0 \) point of view. The asset price gap \( \tilde{Q}_t \) process for \( t < T \) follows

\[
d\tilde{Q}_t = \left( -r + \frac{1}{2}(\hat{\sigma} + \sigma^q_t)^2 - \frac{1}{2}(\bar{\sigma})^2 \right) dt + \sigma^q_t d\tilde{Z}_t.
\]

(2.4.3)

We can integrate equation (2.4.3) and obtain \( \tilde{Q}_t \) for \( t \leq T \) as:

\[
\tilde{Q}_t = -\frac{E_t \left( (\sigma + \sigma^q_t)^2 \right)}{2\phi_q} \left( \frac{\sigma}{2\phi_q} \right) + \int_t^T \left( r - \frac{1}{2}E_t \left( (\hat{\sigma} + \sigma^q_s)^2 \right) + \frac{1}{2}(\bar{\sigma})^2 \right) ds.
\]

(2.4.4)

As equation (2.4.4) must satisfy the dynamic IS equation (equation (2.4.3)), we take a total derivative to equation (2.4.4) to obtain:

\[
-\frac{dE_t \left( (\sigma + \sigma^q_t)^2 \right)}{2\phi_q} - \frac{1}{2} \int_t^T dE_t \left( (\hat{\sigma} + \sigma^q_s)^2 \right) ds = \sigma^q_t d\tilde{Z}_t.
\]

(2.4.5)

Equation (2.4.5) is the stochastic differential equation (SDE) that the \( \{\sigma^q_t\} \)-path starting from \( \sigma^q_0 > 0 \) follows until \( T \) during the ZLB. Unfortunately, this SDE has no known analytic solution. Here, we try to understand heuristically how \( \sigma^q_t \) evolves during the ZLB. Note that in equation (2.4.5), without the new ZLB term, equation (2.4.5) becomes exactly the same as the SDE that characterizes the martingale equilibrium.

We already know from Section 2.3.1 that outside ZLB episodes, a lower \( \phi_q \) value slows down the stabilization process of \( \sigma^q_t \) in the martingale equilibrium. Therefore, we guess that ZLB actually is very ineffective in stabilizing \( \sigma^q_0 > 0 \) and \( \sigma^q_T \) can still be large compared to \( \sigma^q_0 \). In other words, under ZLB regimes, a positive sunspot \( \sigma^q_0 \) is unlikely to disappear until the economy exits the ZLB at \( t = T \) and monetary policy becomes active again.

Therefore, in our framework the ZLB raises the welfare costs of business cycle fluctuations in two ways: (i) it brings down asset prices, financial wealth, and aggregate output (level effect), and (ii) it keeps \( \sigma^q_t \) sunspots alive, making business cycle more volatile (volatility effect). Therefore, the inability of conventional monetary policy to intervene at the ZLB and prevent financial disruption (in terms of endogenous volatility \( \sigma^q_t \)) supposes an additional business cycle cost, in addition to the level effect.
In this case, $\hat{Q}_t$-dynamics at the ZLB can be illustrated as in Figure 2.5. The solid blue line corresponds to the equilibrium with perfect stabilization after $T$, while the two dashed lines correspond to the other case where the central bank uses the conventional Taylor rule and $\sigma_0^q > 0$ appears at $t = 0$.

![Figure 2.5: ZLB dynamics (Taylor rule after $T$) with initial sunspot $\sigma_0^q > 0$](image)

We observe: (i) compared with the perfect stabilization case (solid blue), paths under imperfect future stabilization feature lower $\{\hat{Q}_t\}$ levels, as financial volatility $\{\sigma_t^q > 0\}$ starting from $\sigma_0^q > 0$ raises risk-premium and additionally depresses the levels of $\{\hat{Q}_t\}$. (ii) $\{\hat{Q}_t\}$ is stochastic, and $\{\sigma_t^q\}$ converges slowly to zero during the ZLB and faster after $t \geq T$ as monetary policy becomes active again and responds to $\{\hat{Q}_t\}$ following the conventional Taylor rule. The dashed lines in Figure 2.5 correspond to two possible sample paths with stochastic $\hat{Q}_T$ ($\hat{Q}_T^{(1)}$ or $\hat{Q}_T^{(2)}$) from $t = 0$ perspective.

**Mathematical explanation** In the martingale equilibrium out of the ZLB (for $t \geq T$), $\{\sigma_t^q\}$ starting from $\sigma_0^q$ follows:

$$d\sigma_t^q = -\frac{(\phi_q)^2}{2(\sigma + \sigma_t^q)^3}dt - \phi_q \frac{\sigma_t^q}{\sigma + \sigma_t^q}dZ_t, \forall t \geq T. \quad (2.4.6)$$
Here, we provide a very heuristic explanation of why $\sigma_t^q$ does not fall, in general, during ZLB regimes. Our strategy is to start with the $\sigma_t^q$-process in equation (2.4.6) and how it should be modified to satisfy equation (2.4.5) at the ZLB (for $t < T$).

First, for simplicity, we replace equation (2.4.5) with

$$
-d \left( \frac{(\sigma + \sigma_t^q)^2}{2\phi_q} \right) - \frac{1}{2} \int_t^T d \left( \left( \bar{\sigma} + \sigma_t^q \right)^2 \right) ds = \sigma_t^q dZ_t,
$$

(2.4.7)

where we replace $E_t((\sigma + \sigma_t^q)^2)$ by $(\sigma + \sigma_t^q)^2$ (which holds with equation (2.4.6)) and $E_t((\bar{\sigma} + \sigma_t^q)^2)$ by $(\bar{\sigma} + \sigma_t^q)^2$ (which does not hold technically, but allows us to obtain some intuitions as equation (2.4.7) is simpler than equation (2.4.5)). Since

$$
d \left( \left( \bar{\sigma} + \sigma_t^q \right)^2 \right) = \left( \phi_q \frac{\sigma_t^q}{\sigma + \sigma_t^q} \right)^2 \left( -\frac{\bar{\sigma} + \sigma_t^q}{\sigma + \sigma_t^q} + 1 \right) dt - 2\phi_q \left( \frac{\bar{\sigma} + \sigma_t^q}{\sigma + \sigma_t^q} \right) \sigma_t^q dZ_t,
$$

(2.4.8)

therefore, the $(\bar{\sigma} + \sigma_t^q)^2$-process has a negative drift, which by equation (2.4.7) implies that $(\sigma + \sigma_t^q)^2$ must have a positive drift because the $\sigma_t^q dZ_t$-process does not contain a drift term. As the $\{\sigma_t^q\}$ process in equation (2.4.6) implies $(\sigma + \sigma_t^q)^2$ is a martingale without drift, the new process that satisfies equation (2.4.7) must have a higher drift than equation (2.4.6), which implies a slower stabilization under ZLB than in the martingale equilibrium in equation (2.4.6).

**Imperfect stabilization after forward guidance** With the lesson that keeping the policy rate at zero does not help stabilize the financial volatility sunspot in mind, we consider the case in which central banks return to the conventional Taylor rule $i_t = \bar{r} + \phi_q \hat{Q}_t$ after its forward guidance program ends at $\hat{T}$. Forward guidance has two countervailing effects. (i) It raises asset prices, aggregate financial wealth, and aggregate demand until $\hat{T}$ (level effect). (ii) Slows the stabilization of financial volatility $\sigma_t^q$ between $T \leq t \leq \hat{T}$ by prolonging the period of policy inaction, thereby generating a welfare loss (volatility effect). Figure 2.6 illustrates those two opposite forces generated by forward guidance under imperfect stabilization with an initial sunspot $\sigma_0^q > 0$. Here, the solid gray line represents the $\hat{Q}_t$ path with perfect stabilization after $\hat{T}$, while the dashed lines represent stochastic paths under imperfect future stabilization after $\hat{T}$.

In the next Section 2.4.4, we turn our attention to the study of possible macroprudential interventions from the fiscal side to raise asset prices $\hat{Q}_t$ and inflation $\pi_t$ during a ZLB crisis.
2.4.3 Intertemporal Stabilization Trade-off with Commitment

During the Global Financial Crisis and afterwards, central banks around the world purchased large amounts of assets in financial markets, which mitigated collapses in asset prices and brought down levels of risk-premia for a variety of assets.\textsuperscript{53} As our framework’s Ricardian structure does not allow the central bank’s balance sheet quantities to affect the equilibrium, we turn our eyes to a different type of policy that can prop up asset markets and the business cycle: a central bank commitment to passive financial stabilization \textit{in the future} in exchange for lower financial volatility at the ZLB.

To obtain sharper analytic results, in the following Section 2.4.3 we consider the rigid-price case with no inflation (i.e. $\pi_t = 0 \forall t$). In Section 2.4.3 we consider the 3-equations sticky price model (equation (2.2.27), equation (2.2.31), equation (2.3.10)) and extend our

\textsuperscript{53}See Gagnon et al. (2010), Krishnamurthy and Vissing-Jorgensen (2011), and Gorodnichenko and Ray (2018) for empirical evidence on how the Fed’s QE1 and QE2 programs affected asset prices and risk-premia in financial markets. For example, Gagnon et al. (2010) presented evidence that asset purchases led to reductions in interest rates on a range of securities, which reflects reductions in the levels of risk premia. Krishnamurthy and Vissing-Jorgensen (2011) found that QE1, which involved large purchases of agency-backed MBSs, yielded huge reductions in mortgage rates, while QE2, in contrast, involved only Treasury purchases and brought down Treasury bond rates only. Gorodnichenko and Ray (2018) found: the more disrupted financial markets are, asset purchases act more strongly as local demand shocks in financial markets.
logic from Section 2.4.3. In this Section 2.4.3, we assume the same ZLB environment as in Section 2.4.1.

**Rigid Price Case**

We assume: \( \pi_t = 0, \forall t \), which corresponds to the limit case with zero price-resetting probability, instead of the Phillips curve (equation (2.2.31)). This simplification allows us to derive sharper analytic implications about the optimal commitment path.

**General idea**  We discuss a possible central bank’s commitment path at the ZLB which brings the higher expected welfare than the conventional forward guidance studied in the previous sections. For that purpose, we make a strong assumption: *central bank can choose the path of endogenous financial volatility \( \{\sigma_t^q\} \) if that path is consistent with the dynamic IS equation (equation (2.2.27)) and the Phillips curve (equation (2.2.31)).* In the forward guidance path we described in Section 2.4.1, the fact that the central bank achieves perfect stabilization after \( \hat{T} \) (in Figure 2.4) determines financial volatility levels during the ZLB, including the forward guidance period (between \( T \) and \( \hat{T} \)). To be specific, we derived that \( \sigma_t^q = \sigma_t^{q,n} = 0 \) for both \( T \leq t \leq \hat{T} \) (forward guidance period) and \( t < T \) (ZLB period). Therefore, the risk-premium level before \( \hat{T} \) is completely determined to be the same as its natural correspondent \( \rho \) and we have \( \hat{\rho} = 0 \) for \( t \leq \hat{T} \). This logic can be illustrated by the following diagram.

1. Central bank achieves perfect stabilization after \( \hat{T} \) (\( \hat{Q}_t = 0, \forall t \geq \hat{T} \))
2. \( \hat{Q} = 0 \) guarantees \( \sigma_t^q = \sigma_t^{q,n} = 0 \), \( \hat{\rho} = \rho_t^0 \) for \( t \leq \hat{T} \)

*Figure 2.7: Mechanism under the Forward Guidance*

*We ask the following question:* What if central bank can commit to forgo full stabilization after the forward guidance period (after \( \hat{T} \) in Figure 2.4) while engineering an equilibrium path with lower levels of risk-premium \( \rho_t \) both during the ZLB (until \( T \)) and the forward guidance period (from \( T \) to \( \hat{T} \))? Specifically, from the central bank’s point of view, the risk premium level, which is determined by the volatility of \( \{A_t\} \) process, is too high during the ZLB episode and it causes the asset price to plummet, bringing the economy into a harsh recession. Thus, central banks might conclude that pushing down \( \sigma_t^q \) levels (or equivalently, risk-premium levels) from \( \sigma^{q,n} = 0 \) will mitigate the crisis during the ZLB by propping up
asset price levels, which eventually stimulate the aggregate demand. This policy can be related to the central bank’s efforts to supply liquidity to financial markets through LSAP policies, recapitalization and/or government guarantee of financial entities, and many other bail-out policies aimed (in part) at reducing financial uncertainty, whereas in our model, this reduction in uncertainty comes from the central bank’s commitment to deviate from the perfectly stabilized path in the future (after $\hat{T}$).

The possibility that $\sigma^q_t$ differs from $\sigma^{q,n}_t = 0$ causes asset price gap $\hat{Q}_t$ to fluctuate in a stochastic manner until $\hat{T}$ and therefore, the central bank cannot attain $\hat{Q}_\hat{T} = 0$ surely. Thus, the conjectured engineered path in which $\{\sigma^q_t\}$ is pushed down below $\{\sigma^{q,n}_t = 0\}$ until $\hat{T}$ for stabilization purpose would be successful only if the central bank commits ex-ante not to pursue perfect stabilization even after the forward guidance period is over and the economy returns to normal.\footnote{From equation (2.2.27), $\sigma^q_t \neq \sigma^{q,n}_t = 0$ creates a stochastic movement in the asset price gap process $\{\hat{Q}_t\}$ until $\hat{T}$.} \footnote{It can be understood as follows: a $\sigma^q_t$ different from $\sigma^{q,n}_t = 0$ until $\hat{T}$ means that the stock market becomes separated from what is stipulated by the real economy. It will eventually lead to a failure of the monetary authority to stabilize the economy at $\hat{T}$, the moment when forward guidance ends. If we interpret it in a backward way, it implies: when the central bank tries to engineer business cycle paths while bringing down the risk-premium it deems ‘too high’ during ZLB, it must consider how these paths would trouble the economy after forward guidance is over. This feature arises as financial market and the real economy are connected with each other in our model.} \footnote{For example, we assume that after $\hat{T}$ the central bank uses the passive policy rule with just $i_t = r^n(\sigma)$, creating a possibility of multiple equilibria after $\hat{T}$. In this Section 2.4.3, we select one equilibrium in which we have $\sigma^q_t = \sigma^{q,n} = 0$ after forward guidance ends (after $\hat{T}$ in Figure 2.4). Therefore, $\hat{Q}_t$ will remain at $\hat{Q}_\hat{T}$ after $t \geq \hat{T}$ with $i_t = r^n(\sigma)$.} This logic can be interpreted as a contrapositive to the one in Figure 2.7 and is illustrated by Figure 2.8.

In other words, when engineering an equilibrium path, the central bank must juggle between boosting the economy during ZLB by lowering the financial volatility $\{\sigma^q_t\}$ and risk-premium $r_{p_t}$, and the perfect stabilization after the fundamentals return to normal, thus effectively future stability for present stabilization at the ZLB.

\begin{itemize}
\item[-2.] $\sigma^q_t < \sigma^{q,n}_t = 0$, $r_{p_t} < r^p_{n}$ for $t \leq \hat{T}$
\item[-1.] $\hat{Q}_\hat{T} \neq 0$. Central bank commits not to perfectly stabilize the economy after $\hat{T}$
\end{itemize}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.8.png}
\caption{Financial Market Intervention and Stabilization}
\end{figure}

One thing to notice is that the central bank would like to reduce the financial volatility level and the risk-premium even after the TFP volatility $\sigma_t$ returns to $\sigma$ (from $T$ to $\hat{T}$) while...
it still follows the forward guidance rate prescription \((i_t = 0)\). It is because it would push up \(\hat{Q}_t\) between \(T ≤ t ≤ \hat{T}\), which further raises the asset price \(\hat{Q}_t\) during high-TFP volatility periods \((\sigma_t = \bar{\sigma} \text{ for } t ≤ T)\) as \(\hat{Q}_t\) is forward-looking. Since we have introduced another way (manipulating financial market volatility) to stimulate the economy during and after ZLB, forward guidance is not as necessary to prop up \(\hat{Q}_t\) as in Section 2.4.1, thus its duration must decrease with this intervention.

To be more formal, we define \(rp^1_t ≡ (\bar{\sigma})^2\), \(rp^2_t ≡ (\bar{\sigma}^2)^2\), and \(rp^3_t ≡ (\bar{\sigma})^2\), the risk-premium levels (i.e., \((\sigma_t + \sigma_q^2)^2\)) in time-intervals \(t ≤ T\), \(T < t ≤ \hat{T}\), and \(t ≥ \hat{T}\), respectively when the central bank perfectly stabilizes the economy after ZLB. We define \(\hat{T}'\) to be the new forward guidance duration under the newly engineered path, which is possibly different from \(\hat{T}\), the original forward guidance duration. For simplicity, we assume that the central bank maintains the same financial volatility and risk-premium levels in the same regime: specifically, financial volatility \(\sigma^q_t\) is \(\sigma^q_{1,\ell}\) from 0 to \(T\) (High TFP volatility region), \(\sigma^q_{2,\ell}\) from \(T\) to \(\hat{T}'\) (Low TFP volatility region with the forward guidance), and 0 after \(\hat{T}'\) (Low TFP volatility region). The assumption that \(\sigma^q_t = 0\) after \(\hat{T}'\) means that the central bank does not manipulate financial markets after \(\hat{T}'\) when the forward guidance ends.

Therefore, under this newly engineered path, the risk-premium levels \(rp_t = (\sigma_t + \sigma^q_t)^2\) in each time-interval become \(rp^1_t ≡ (\bar{\sigma} + \sigma^q_{1,\ell})^2 < rp^1_t\) for \(t ≤ T\), \(rp^2_t ≡ (\bar{\sigma} + \sigma^q_{2,\ell})^2 < rp^2_t\) for \(T < t ≤ \hat{T}'\), and \(rp^3_t ≡ (\bar{\sigma})^2\) after \(\hat{T}'\). Since the policy intervention lowers the economy’s total risk, risk-premium levels fall and both asset price and business cycle levels rise in response. From equation (2.4.30), we can express the risk-adjusted natural rates \(r^T_1\) and \(r^T_2\) that enter the dynamic IS equation (equation (2.2.27)) as functions of \(\sigma^q_{1,\ell}\) and \(\sigma^q_{2,\ell}\), respectively, as

\[
\begin{align*}
\hat{r}_1^T(\sigma^q_{1,\ell}) &\equiv \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{(\bar{\sigma} + \sigma^q_{1,\ell})^2}{2} > \bar{\ell} \equiv r^\ell(\bar{\sigma}) = r^T_1(0) \text{ when } \sigma^q_{1,\ell} < 0, \\
\hat{r}_2^T(\sigma^q_{2,\ell}) &\equiv \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{(\bar{\sigma} + \sigma^q_{2,\ell})^2}{2} > \bar{\ell} \equiv r^\ell(\bar{\sigma}) = r^T_2(0) \text{ when } \sigma^q_{2,\ell} < 0.
\end{align*}
\]

We observe \(\sigma^q_{1,\ell} < 0\) and \(\sigma^q_{2,\ell} < 0\) imply \(r^T_1 > \bar{\ell}\) and \(r^T_2 > \bar{\ell}\), respectively, which yield higher levels of \(\{\hat{Q}_t\}\) during and after the ZLB on average. That would be the reason the central bank wants to push down \(\sigma^q_{1,\ell}\) and \(\sigma^q_{2,\ell}\) from zero, the natural level of financial volatility, but from equation (2.2.27) we see that it creates a stochastic fluctuation of \(\hat{Q}_t\), which brings additional costs in terms of stabilization. Therefore, central bank faces a trade-off between future and current stability when it engineers the new commitment path.

To pin down equilibrium paths (there are multiple equilibria and we need to select one equilibrium), we assume that, at \(t = 0\), the monetary authority anchors an expected value of

\[57\text{We will eventually prove } \sigma^q_{1,\ell} < \sigma^q_{2,\ell} = 0 \text{ and } \sigma^q_{2,\ell} < \sigma^q_{3,\ell} = 0 \text{ at optimum. For illustration purposes, we assume these conditions are satisfied in the rest of the argument in Section 2.4.3.}
Figure 2.9: Possible Intervention Dynamics of \( \{ \hat{Q}_t \} \) with \( \sigma_1^{q,L} < 0 \), \( \sigma_2^{q,L} < 0 \), and \( \hat{T}' < \hat{T} \)

\( \hat{Q}_{\hat{T}',} \), the asset price gap level when the forward guidance is over, at zero, i.e., \( \mathbb{E}_0 \hat{Q}_{\hat{T}',} = 0 \). In Figure 2.9, the gray line is the original forward guidance path, while the green one is the path of average (or deterministic component of) \( \{ \hat{Q}_t \} \) when the central bank engineers a new path with \( \sigma_1^{q,L} < 0 \) and \( \sigma_2^{q,L} < 0 \). As we now have \( \sigma_1^{q,L} \neq 0 \) and \( \sigma_2^{q,L} \neq 0 \), stochastic fluctuations of \( \{ \hat{Q}_t \} \) around the deterministic path are generated and illustrated by two possible sample paths (dashed lines) in Figure 2.9. These stochastic fluctuations bring the welfare loss. We also observe that: with \( \sigma_2^{q,L} < 0 \) and \( \mathbb{E}_0 \hat{Q}_{\hat{T}',} = 0 \), the average level of \( \hat{Q}_t \) becomes higher from \( T \) to \( \hat{T}' \) than in the gray forward guidance path. The rises in \( \hat{Q}_t \) from \( T \) to \( \hat{T}' \) bring up \( \hat{Q}_t \) levels before \( T \). In addition, \( \sigma_1^{q,L} < 0 \) for \( t \leq T \) further props up the average level of \( \hat{Q}_t \) during the high TFP volatility period (before \( T \)) and thus \( \hat{Q}_0 \) falls less than in the (gray) forward guidance path in Figure 2.9.

In sum, this type of financial market intervention must exploit a trade-off between higher asset price and output levels before \( \hat{T}' \) and the future stabilization after the ZLB. Central banks should balance this trade-off when manipulating \( (\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}') \). Now, the next step is

\(^{58}\)It does not have to be the asset price gap, since the other gap variables are all proportional to \( \hat{Q}_t \).

\(^{59}\)When we draw the deterministic path in Figure 2.9, we ignore the fact that we have stochastic fluctuations of \( \hat{Q}_t \).
to check whether our conjecture-based analysis in Section 2.4.3 indeed holds as the optimal commitment solution.

Central bank’s optimal commitment path  The central bank chooses optimal \( \sigma_{1}^{q,L}, \sigma_{2}^{q,L} \), and \( \hat{T}' \) to minimize the loss function in equation (2.4.1), with functions \( r_{1}^{T}(\cdot) \) and \( r_{2}^{T}(\cdot) \) defined in equation (2.4.9). We assume that the central bank keeps \( i_{t} = 0 \) until \( \hat{T}' \). Therefore, \( \hat{T}' \) is our new forward guidance period as we explained above. After \( \hat{T}' \), it implements a passive monetary policy (interest rate anchoring) with \( i_{t} = r_{2}^{T}(0) = r^{n}(\sigma) \), not seeking to stabilize the economy. In sum, central bank solves the following optimization problem:

\[
\min_{\sigma_{1}^{q,L},\sigma_{2}^{q,L},\hat{T}} \mathbb{E}_{0} \int_{0}^{\infty} e^{-\rho t} (\hat{Q}_{t})^{2} dt, \text{ s.t. } \begin{cases} 
\frac{d\hat{Q}_{t}}{dt} = -(r_{1}^{T}(\sigma_{1}^{q,L}))dt + (\sigma_{1}^{q,L})dZ_{t}, & \text{for } t < T, \\
\frac{d\hat{Q}_{t}}{dt} = -(r_{2}^{T}(\sigma_{2}^{q,L}))dt + (\sigma_{2}^{q,L})dZ_{t}, & \text{for } T \leq t < \hat{T}', \\
\frac{d\hat{Q}_{t}}{dt} = 0, & \text{for } t \geq \hat{T}',
\end{cases}
\]

(2.4.10)

with \( \hat{Q}_{0} = r_{1}^{T}(\sigma_{1}^{q,L})T + r_{2}^{T}(\sigma_{2}^{q,L})(\hat{T}' - T) \).

The forward guidance path (Section 2.4.1 with \( \pi_{t} \equiv 0 \)) corresponds to \( (\sigma_{1}^{q,L},\sigma_{2}^{q,L},\hat{T}') = (0,0,\hat{T}) \) case, and thus optimal \( (\sigma_{1}^{q,L},\sigma_{2}^{q,L},\hat{T}') \) combination yields the lower level of (quadratic) loss function. It turns out our conjecture in Section 2.4.3 that we have \( \sigma_{1}^{q,L} < 0, \sigma_{2}^{q,L} < 0, \) and \( \hat{T}' < \hat{T} \) at optimum is correct, as summarized by the next Proposition 7.

**Proposition 7 (Optimal Commitment Path).** The solution of the central bank’s optimization program in equation (2.4.10) features \( \sigma_{1}^{q,L} < 0, \sigma_{2}^{q,L} < 0, \) and \( \hat{T}' < \hat{T} \) hold.

We solve equation (2.4.10) with \( T = 4.5 \) and parameters in Table B.2, and simulate optimal commitment paths. We calculate a sample estimate of the loss function with:

\[
\int_{0}^{\infty} e^{-\rho t} \mathbb{E}_{0}(\hat{Q}_{t}^{2}) dt \approx \int_{0}^{\infty} e^{-\rho t} \frac{1}{s} \sum_{i=1}^{s} (\hat{Q}_{t}^{(i)})^{2} dt.
\]

(2.4.11)

where \( \hat{Q}_{t}^{(i)} \) is the \( i \)th realized sample path.\(^{60} \) Our result reveals when \( \sigma_{1}^{q,L}, \sigma_{2}^{q,L} \) and \( \hat{T} \) are chosen optimally, the loss value is reduced by 0.4239%, which constitutes a moderate gain. In our simulation, we observe \( \sigma_{1}^{q,L} < 0, \sigma_{2}^{q,L} < 0, \) and \( \hat{T}' < \hat{T} \) all hold at optimum,\(^{61} \) which

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\(^{60}\)We use \( s = 1000 \) number of sample paths in our simulation.

\(^{61}\)Our simulation yields \( \hat{T}' = 5.612 < \hat{T} = 5.614, \sigma_{1}^{q,L} = -1.4325 \times 10^{-4} < 0, \) and \( \sigma_{2}^{q,L} = -1.0467 \times 10^{-6} < 0. \)
aligns with Proposition 7, but with optimal volatilities of very small magnitudes. The reason that the optimal commitment path features very small degrees of financial volatility (and risk-premium) manipulation is because after $T$, there is no ZLB possibility at all, which raises the cost of destabilization when central bank’s monetary policy becomes passive after $\hat{T}'$.

In a more realistic setting, there exists the possibility that the economy gets trapped at the ZLB in a stochastic way, which can raise the degree to which the central bank manipulates financial market volatilities and risk-premia. This exercise clearly illustrates concerns that a central bank must consider when it tries to change the business cycle path by manipulating financial market variables at the ZLB.

**Sticky Price Case**

In cases where inflation $\pi_t$ evolves according to the Phillips curve (equation (2.2.31)), we still preserve the logic of intervention from Section 2.4.3. Since we assume that central bank does not pursue full stabilization after $\hat{T}'$ based on interest rate anchoring, there arise multiple equilibria in $\{\hat{Q}_t, \pi_t\}$ and we focus on the particular equilibrium where we have $\sigma_{\pi,t} = 0, \forall t$ in equation (2.2.31). As in Section 2.4.3, we assume that the central bank manipulates the financial volatility levels, and $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ are those levels at the ZLB ($t \leq T$) and in the forward guidance ($T \leq t < \hat{T}'$) region, respectively, that central bank targets while it tries to engineer a new commitment path.

Before we state the central bank’s optimization problem, we first consider the dynamics of $\{\hat{Q}_t\}$ without its stochastic component ($\sigma_q dZ_t$ in equation (2.2.27)), where we assign superscript $c$ to denote it is a counterfactual. We do so in order to identify the pure benefits that central bank’s manipulation of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ brings in terms of welfare. In this particular counterfactual path, we assume that after $\hat{T}'$, the economy is fully stabilized so $\hat{Q}_t^c = \pi_t^c = 0$ for $\forall t \geq \hat{T}'$. Then we have the following IS equations, which together with our Phillips curve (equation (2.2.31) with $\sigma_{\pi,t} = 0$ for $\forall t$), characterize the complete $\{\hat{Q}_t^c, \pi_t^c\}$ dynamics.

\[
d\hat{Q}_t^c = \begin{cases} 
-(r^T_1(\sigma_1^{q,L}) + \pi_t^c)dt & \text{for } t < T, \\
-(r^T_2(\sigma_2^{q,L}) + \pi_t^c)dt & \text{for } T \leq t < \hat{T}', \\
\hat{Q}_t^c = \pi_t^c = 0, & \text{for } t \geq \hat{T}'.
\end{cases}
\]

(2.4.12)

Observe $\sigma_1^{q,L} < 0$ and $\sigma_2^{q,L} < 0$ imply $r^T_1(\sigma_1^{q,L}) > r^T_1(0) = r^n(\bar{\sigma})$ and $r^T_2(\sigma_2^{q,L}) > r^T_2(0) = r^n(\bar{\sigma})$. Therefore, the above counterfactual path in general yields a higher welfare than the forward guidance path in Section 2.4.1, as falls in financial volatilities and risk-premia mitigate the severity of ZLB. This welfare gain constitutes the benefit that the central bank’s manipulation of asset price volatilities ($\sigma_1^{q,L}, \sigma_2^{q,L}$) brings to the economy. We express the path $\{\hat{Q}_t^c(\sigma_1^{q,L}, \sigma_2^{q,L}), \pi_t^c(\sigma_1^{q,L}, \sigma_2^{q,L})\}$ satisfying equation (2.4.12) as a function of $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$, as equation (2.4.12) is deterministic. The more $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ are reduced from their natural
levels, the more the levels of $\hat{Q}_t^c(\sigma_1^{q,L}, \sigma_2^{q,L})$ and $\pi_t^c(\sigma_1^{q,L}, \sigma_2^{q,L})$ rise, in a similar manner to Cochrane (2017).

Now, we consider the case where there is a stochastic fluctuation term ($\sigma_d^q dZ_t$ in equation (2.2.27)). We assume that the path starts at $\{Q_0^c(\sigma_1^{q,L}, \sigma_2^{q,L}), \pi_0^c(\sigma_1^{q,L}, \sigma_2^{q,L})\}$ given the financial volatility levels $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ that central bank manipulates. And we assume that after $\hat{T}'$ central bank follows a passive monetary policy rule (interest rate anchoring) $i_t = r^n(\sigma) = r^0_T(0)$, committing not to seek to stabilize the economy, as in Section 2.4.3.

Therefore, we can write the central bank’s optimization problem in the following way.

$$\begin{align*}
\min_{\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}'} \mathbb{E}_0 \int_0^\infty e^{-\rho t} (\hat{Q}_t^2 + \Gamma \pi_t^2) dt,
\end{align*}$$

with $\hat{Q}_0 = \hat{Q}_0^c(\sigma_1^{q,L}, \sigma_2^{q,L})$ and $\pi_0 = \pi_0^c(\sigma_1^{q,L}, \sigma_2^{q,L})$.

As it is not possible to analytically characterize optimal $\sigma_1^{q,L}, \sigma_2^{q,L}$, and $\hat{T}'$, we rely on numerical simulation to check whether our intuitions in Section 2.4.3 still hold in the sticky price case. The simulation result confirms our intuition that at the optimal commitment equilibrium central bank is better off by pushing down $\sigma_1^{q,L}$ and $\sigma_2^{q,L}$ from zero, which is the natural level of financial volatility, and the forward guidance duration $\hat{T}'$ is shortened from $\hat{T}$, the original duration. Basically, when the central bank tries to engineer a new commitment path by controlling the financial volatility level and the endogenous risk-premium, it must

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62It is similar to our assumption in Section 2.4.3 that central bank anchors $\mathbb{E}_0 \hat{Q}_t = 0$. Here, initial levels of $\hat{Q}_0, \pi_0$ are those that arise when there is no stochastic fluctuation in both variables. This way of selecting an equilibrium helps us to separate benefits and costs that the optimal commitment path brings, and deal with each part carefully.

63In particular, we use the following approximation similar to equation (2.4.11):

$$\int_0^\infty e^{-\rho t} \mathbb{E}_0 (\hat{Q}_t^2 + \Gamma \pi_t^2) dt \approx \int_0^\infty e^{-\rho t} \frac{1}{s} \sum_{i=1}^s ((\hat{Q}_t^{(i)})^2 + (\pi_t^{(i)})^2) dt,$$

where $\hat{Q}_t^{(i)}, \pi_t^{(i)}$ are $i$'th sample paths of $\{\hat{Q}_t, \pi_t\}$. We use $s = 1000$ for the number of sample paths.

64With parameters in Section 2.4.1, the optimal forward guidance duration becomes $\hat{T}' = 3.601 < \hat{T} = 3.602$, and we have $\sigma_1^{q,L} = -1.0491 \times 10^{-5} < 0$ and $\sigma_2^{q,L} = -9.7365 \times 10^{-6} < 0$. The loss function value drops by 0.03%.

65As we emphasize, the whole point of this exercise is not to accurately quantify the actual benefits of this type of stock market intervention as is done in quantitative DSGE models, but to convey the key intuitions that must be taken into account in conjunction with our monetary policy framework.
trade-off some future stability (after the ZLB) to stabilize the present economy (during the ZLB), and accept the future destabilizing effects that the policy entails.

### 2.4.4 Macroprudential Policies

In Section 2.4.4, we analyze two types of stimulative macroprudential policies at the ZLB. First, we introduce a fiscal subsidy that incentivizes capitalists to bear higher levels of risks, effectively raising asset price and other real activities. Second, we study direct fiscal redistribution from capitalists to hand-to-mouth workers whose marginal propensity to consume (MPC) is much higher than former. We show that this policy raises total dividends of the stock market and eventually, asset price $\hat{Q}_t$ and consumption. To isolate the effects of macroprudential policies on the business cycle, forward guidance policy is not considered in this Section 2.4.4.

#### Fiscal Subsidy on Stock Market Investment

For $t \in [0, T]$, monetary policy is constrained by the ZLB, and the risk-premium level demanded by capitalists puts a downward pressure on $\hat{Q}_t$ and $\pi_t$ through the aggregate demand externality that financial decisions of individual capitalist exert.\(^{66}\) In this section, we develop a subsidy policy that induces capitalists to increase their demand for risky stocks, which raises the aggregate asset price level $Q_t$ and corrects the aggregate demand externalities affecting the economy.

We start by considering a government subsidy for the purchase of risky stock market assets\(^{67}\), and which is funded by imposing a lump-sum tax on capitalists. In specific, instead of the original expected return $i^{m}_t$, capitalists get $(1 + \tau_t) i^{m}_t$ in expectation out of a 1$ stock market investment. If the government imposes a $T_t$ lump-sum tax on capitalists to finance this subsidy, then each capitalist with nominal wealth $a_t$ solves the following optimization problem.

$$
\max_{C_t, \theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t \, dt \quad \text{s.t.} \quad da_t = \left( a_t((1+\tau_t)(i^{m}_t-i_t)) - p_tC_t - T_t \right) dt + \theta_t a_t(\sigma_t + \sigma^q_t) dZ_t.
$$

\(^{(2.4.15)}\)

\(^{66}\) A number of papers have been written over relations between externalities (either pecuniary or aggregate-demand) and macroprudential policies. See Caballero and Krishnamurthy (2001), Lorenzoni (2008), Farhi et al. (2009), Bianchi and Mendoza (2010), Jeanne and Korinek (2010), Stein (2012), Farhi and Werning (2012, 2016, 2017), Korinek and Simsek (2016), Davila and Korinek (2018) among others.

\(^{67}\) In our model, a stock market subsidy is equivalent to a tax break on capital income, which is most often the policy implemented in practice by governments. We model this policy using the ‘subsidy’ version in order to economize on notation.
In equilibrium, capitalists end up paying a tax amount \( T_t = \tau_t a_t \theta_t i_t^m \) to finance the subsidy on their own stock market investment. Imposing \( \theta_t = 1 \) as the equilibrium condition, the equilibrium stock market’s expected return can be expressed as

\[
  i_t^m = i_t + \frac{(\sigma_t + \sigma_t^q)^2}{1 + \tau_t} = \underbrace{\rho}_{\text{Dividend yield}} + g + \mu_t^q + \sigma_t \sigma_t^q. \tag{2.4.16}
\]

Note that given \( i_t = 0 \) (ZLB), the presence of subsidy \( \tau_t > 0 \) pushes down the equilibrium levels of \( i_t^m \) and \( \mu_t \), boosting the current \( \hat{Q}_t \) until \( T \). Therefore, this policy mitigates the severity of recessions, as summarized by the following Proposition 8. Notice that when \( \tau_t \to \infty \), we immediately escape from the ZLB crisis and return to the fully stabilized economy.\(^{68}\)

**Proposition 8** (Fiscal Subsidy on Stock Market (Expected) Return). Under the ZLB environment of 2.4.1 with the fiscal subsidy \( \tau_t > 0 \) on the expected stock market return, a dynamic IS equation for \( \hat{Q}_t \) can be written as

\[
  d\hat{Q}_t = -\left( r + \frac{\tau_t}{1+\tau_t} (\bar{\sigma})^2 + \pi_t \right) dt. \tag{2.4.17}
\]

Since the central bank fully stabilizes the economy after \( T \), we have \( \sigma_t^q = \sigma_t^{\text{a,n}} = 0 \) for all \( t \).

In equation (2.4.17), a positive \( \tau_t > 0 \) raises the effective natural rate from \( r \) to \( r + \frac{\tau_t}{1+\tau_t} (\bar{\sigma})^2 \), reducing the gap between the ZLB and the effective natural rate. The following Figure 2.10 confirms recessionary pressures at the ZLB are alleviated and both \( \hat{Q}_t \) and \( \pi_t \) drop by less upon entering the ZLB, as we raise the subsidy rate \( \tau_t \).

A subsidy on risky asset holdings (or equivalently, a tax cut on capital gains) effectively raises capitalists’ stock market demand, stimulating both financial markets and real activity at the ZLB.

**Tax on whom?** What if government instead imposes a lump-sum tax \( T_t \) on hand-to-mouth workers? In this case, each worker’s budget constraint changes as

\[
  \frac{w_t}{p_t} N_{W,t} = C_{W,t} + \frac{T_t}{p_t}. \tag{2.4.18}
\]

As workers are hand-to-mouth, this taxation reduces their consumption one-by-one and negatively impacts the stock market’s total dividend amount and stock price \( \hat{Q}_t \). Therefore, with

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\(^{68}\)Since we always have \( r + (\bar{\sigma})^2 > 0 \), with \( \tau_t \to \infty \), the effective natural rate becomes positive, and therefore, central bank can attain full stabilization.
Figure 2.10: Zero lower bound (ZLB) crisis with varying $\tau_t$ rates: $\{\hat{Q}_t, \pi_t\}$ dynamics

A lump-sum tax on workers the stock market’s expected return $i_t^m$ is represented as

$$i_t^m = \frac{y_t - \frac{w_t}{\rho} N_{W,t}}{A_t Q_t} + \mathbb{E}_t \left( \frac{d(p_t A_t Q_t)}{p_t A_t Q_t} \right) = \rho - \tau_t i_t^m + \mathbb{E}_t \left( \frac{d(p_t A_t Q_t)}{p_t A_t Q_t} \right).$$ (2.4.19)

where we plug $T_t = \tau_t i_t^m p_t A_t Q_t$ into the worker’s budget constraint (equation (2.4.18)). As it will turn out, the negative effect on $\hat{Q}_t$ from drops in workers’ consumption (from the tax) exactly cancels out with the positive effect of the stock subsidies on capitalists. Therefore, there is no net effect (beyond pure redistribution from workers to capitalists) of the subsidy policy on $\{\hat{Q}_t, \pi_t\}$ dynamics during a ZLB episode when it is financed by a tax on workers. The following Proposition 9 summarizes this point.

**Proposition 9** (Fiscal Subsidy and Tax on Workers). The policy of Section 2.4.4 that subsidizes the expected return on risky stocks, if financed through a lump-sum taxation on workers, would have no effect on $\{\hat{Q}_t, \pi_t\}$ dynamics during the ZLB. Such policy features the same dynamics as in Figure 2.4.

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$^{69}$Therefore in equation (2.4.19), the dividend yield jumps down from $\rho$ to $\rho - \tau_t i_t^m$. 
Fiscal Redistribution

A direct fiscal transfer $T_t > 0$ from capitalists to hand-to-mouth workers can raise the total amount of dividends in the financial market, thereby pushing up the current $\hat{Q}_t$ at the ZLB. The formula for the expected stock market return $i_t^m$ in this case can be written as

$$i_t^m = \frac{y_t - \frac{w_t N_{W,t}}{A_t Q_t}}{A_t Q_t} + \mathbb{E}_t \left( \frac{d(p_t A_t Q_t)}{p_t A_t Q_t} \right) = \rho + \frac{T_t}{p_t A_t Q_t} + \mathbb{E}_t \left( \frac{d(p_t A_t Q_t)}{p_t A_t Q_t} \right). \quad (2.4.20)$$

If we assume capitalists pay $\varphi_t$ portion of their wealth to finance this transfer, we will have $T_t = \varphi_t p_t A_t Q_t$ and dividend yield becomes $\rho + \varphi_t$ instead of just $\rho$. This raises the effective natural rate of interest from $r$ to $r + \varphi_t$ during the ZLB, thereby increasing $\hat{Q}_t$ and $\pi_t$. The following Proposition 10 summarizes this result.

**Proposition 10 (Direct Redistribution).** Under the ZLB environment of Section 2.4.1 and a direct (instantaneous) transfer of $\varphi_t$ portion of capitalists’ aggregate financial wealth towards hand-to-mouth workers, the dynamic IS equation for $\hat{Q}_t$ can be expressed as

$$d\hat{Q}_t = -(r \equiv r(\theta) < 0 \quad + \varphi_t \quad + \pi_t) dt. \quad (2.4.21)$$

It is worth mentioning that from the perspective of capitalists with a nominal wealth $a_t$, paying $T_t = \varphi_t a_t$ effectively lowers their equilibrium return by $\varphi_t$. The equilibrium return jumps down by $\varphi_t$ because the asset price value $\hat{Q}_t$ rises so that the capital gain is reduced also by $\varphi_t$. Thus, a transfer to workers with a higher marginal propensity to consume (MPC) brings an additional stabilizing effect other than just pushing up the current consumption and aggregate demand. This policy also raises the dividend yield, boosting $\hat{Q}_t$ and therefore increasing aggregate demand through a wealth effect.\(^{71}\)

\(^{70}\)If the dividend yield increases, the required capital gain for given policy rate falls and the current asset price $\hat{Q}_t$ jumps up at the ZLB.

\(^{71}\)The policy that subsidizes firms’ payroll based on tax imposed on capitalists achieves exactly the same dynamic IS equation as in equation (2.4.21). To be precise, if firms pay $w_t N_{W,t} - T_t$ instead of $w_t N_{W,t}$ as the total payroll to workers, where $T_t$ is financed by capitalists in a lump-sum way, this policy props up the profit amounts that are to be distributed as dividends to capitalists, raising stock prices and business cycle. One difference is that this policy lowers the effective marginal cost for firms, which affects firms’ pricing decisions and distorts the Phillips curve as

$$d\pi_t = (\rho \pi_t + \delta(\delta + \rho)\Theta \tau_t - \kappa \hat{Q}_t) dt, \quad (2.4.22)$$
In sum, macroprudential policies help to mitigate recessionary forces during ZLB crises through their impact on financial markets and asset price $Q_t$. However, there exist other types of financial market interventions that are able to substitute monetary policy’s lack of ammunition power during the ZLB and stimulate the economy through their impact on asset markets. In the next Section 2.4.3, we analyze possible financial market interventions and look into subtle issues that arise when the central bank distorts the stock market and manipulates its risk in order to prop up the economy.

2.5 Conclusion

In this paper, we illustrate that properly accounting for higher-order moments related to the business cycle and stock markets changes the business cycle dynamics of the New-Keynesian framework and provides new implications about monetary policy. To that end, we develop a model with stock markets that features higher-order stock market variables (time-varying aggregate financial volatility and risk-premium). This setup allows a tractable analytical characterization of the equilibrium conditions and uncovers interesting dynamics stemming from the role of aggregate financial volatility: a rise in aggregate financial volatility raises the risk-premium, reducing wealth and aggregate demand. This feedback structure from higher-order terms (aggregate volatility and risk-premium) to first-order ones (wealth and aggregate demand) opens up the possibility of second-order sunspot equilibria, which require a different set of monetary policy rules for stabilization purposes.

Our analysis reveals that conventional monetary policy rules, even with aggressive targeting of traditional macroeconomic measures, cannot guarantee determinacy. This failure of conventional rules in ensuring determinacy lies in their inability to adequately target the ‘expected risky return’ of financial markets, the relevant rate in a stochastic environment. We then propose a generalized Taylor rule that restores determinacy, with which the central bank targets not only conventional mandates (inflation and output gap), but also the risk-premium in a specific way, thus effectively managing the expected rate of return on aggregate financial wealth. This new policy rule achieves what we describe as the *ultra-divine coincidence*: joint stabilization of inflation, output gap and risk-premium. Finally, we study various policy options when the policy rate is constrained by the zero lower bound (ZLB).

Our framework opens new avenues for future research focused on understanding the interaction of the real economy and the higher-order variables of financial markets. For example, in our model we largely abstract from wealth inequality and potentially heterogenous sensitivity of economic players to financial volatility. We view future work aiming to incorporate

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where this additional term would negatively affect $\pi_t$ and $\hat{Q}_t$ during a ZLB crisis.
these realistic features as a particularly fruitful direction to pursue.
Chapter 3

A Unified Theory of the Term-Structure and Monetary Stabilization

This chapter is coauthored with my classmate and also one of my best friends, Marc Dordal i Carreras. I appreciate him for allowing me to use our joint work as part of this dissertation. All errors are mine.

3.1 Introduction

Central banks' various “unconventional” monetary policies\(^1\) have never been more “conventional” than in recent decades, especially after the 2007-2008 Global Financial Crisis (GFC) and the subsequent Great Recession. In environments where the short-term policy rate is constrained by its lower bound (ZLB),\(^2\) policymakers devised ways to reduce long-maturity rates, hoping that falling long-term rates would boost aggregate demand and mitigate the recessionary pressure on the economy, and to that end, central banks tremendously increased sizes of their balance sheets. Simultaneously, governments financed their spending increases by raising their debt issuance. Unfortunately, this unprecedented economic environment intensified in the wake of the Covid-19 pandemic recently and the Federal Reserve, as its policy rate hit its effective lower bound again, has undertook another round of those measures for stabilization purposes.\(^3\)

\(^1\)For example, the Quantitative Easing (QE) programs, and large scale asset purchases (LSAPs) programs in general, and Operation Twist (OT) are possible forms of the unconventional monetary policy.

\(^2\)For the long-term downward trend of neutral interest rates, see Rachel and Smith (2017). This trend amplifies the important stabilizing roles of unconventional monetary policies.

\(^3\)The Federal Reserve lowered its short-term interest rates to a range of 0% to 0.25% in March 2020. While increasing its securities holdings and the size of its balance sheet tremendously, the Fed pledged not to raise interest rates until the economy reaches full employment and consistently maintain 2% inflation. The
The textbook New-Keynesian framework features a single policy rate, abstracting from the term-structure of interest rates and the presence of multiple assets. This omission is not a simplification that can be easily incorporated, as expected returns across assets and maturities are usually equalized in equilibrium in these models, rendering any additional assets as a fully-dependent function of the policy rate, and therefore redundant for the study of monetary policy.

In this paper, we build a tractable New-Keynesian framework featuring the endogenous term-structure of interest rates of bond markets and private capital markets, with which we study effects of alternative monetary (conventional and unconventional) and fiscal policies. Following prior theoretical and empirical works that point out ‘market segmentation’ across bonds of different maturities as a critical feature in explaining the effectiveness of quantitative easing programs, we provide an alternative microfoundation that enables us to incorporate (i) bonds market segmentation, (ii) the household’s endogenous portfolio choices across different asset classes and maturities, (iii) real effects of the government and central bank’s balance sheet size and composition: all building blocks necessary for understanding the transmission channel of unconventional monetary policies.

Under market segmentation, our framework predicts that the total amount and maturity structure of the government’s bond issuance affects the equilibrium interest rate levels and the slope of the yield curve, while the central bank’s relative bond purchases across bonds of different maturities are negatively related with yields. These results are consistent with the findings of Krishnamurthy and Vissing-Jorgensen (2012) and Greenwood and Vayanos (2014), who illustrate the short-run and long-run importance of both relative asset demand

4This result follows from the log-linearization technique and leads to the famous expectation hypothesis, which holds in most log-linearized New-Keynesian models. According to this hypothesis, returns on long-term bonds become just the average of expected future short-term rates.

5For empirical assessments of the market-segmentation hypothesis as a key determinant of the term structure, see D’Amico and King (2013) and Droste et al. (2021). For theory side, Ray (2019) adapts the preferred-habitat framework developed by Vayanos and Vila (2021) and proposes a New-Keynesian model that features bond market segmentation, revealing many interesting relationships between monetary policy and the term-structure. Gourinchas et al. (2020) and Greenwood et al. (2020) study implications of the preferred-habitat setting in joint determination of exchange rates and the term-structure of interest rates.

6For example, (ii) endogenous portfolio decision is important: a relative fall in the short-term rate leads households to reallocate their savings into other assets and/or longer maturities bonds, thus diminishing marginal effects of further changes in the policy rate on the household’s intertemporal consumption decision, and also generates spill-over effects relevant for the determination of other rates.

7Krishnamurthy and Vissing-Jorgensen (2012) found that a higher debt-to-GDP ratio leads to lower corporate credit spreads, and this effect becomes stronger for longer maturities. Likewise, Greenwood and Vayanos (2014) documented that the relative abundance of long term bonds supply (with respect to short term bonds) is positively correlated with the term-spread.
and supply across maturities in determining the yield curve.

We also study cyclical properties of distinct monetary interventions in the form of simple policy rules. By explicitly incorporating the government and central bank’s balance sheets, our framework easily accommodates policies aimed at controlling the yields, bond supplies or a mixture of both at different maturities. We begin by focusing on the implementation of a conventional policy rule on the short-term rate and its effect of the entire yield curve and the economy. Then, we develop a more general yield-curve control (YCC) policy in which the central bank directly controls the entire bond market yield curve. Our framework reveals interesting phenomena and differences across policies, which become especially relevant when the economy enters the ZLB episodes (thus, the short-term rate is constrained by the ZLB). For example, our analysis reveals that when the central bank follows a conventional monetary policy on short-term rates, a reduction in the government’s risk-free bond supply is recessionary at the ZLB, as argued by the literature on safe-asset shortage (SAS) problems (see for example, Caballero and Farhi (2017) and Caballero et al. (2021)). In contrast, under the YCC policy, the central bank swiftly shifts down the entire yield curve and lowers the effective savings rate of households, boosting aggregate demand and preventing the economy’s collapses.\(^8\) We find that YCC, in general, is a more powerful policy in terms of economic stabilization which increases household’s welfare compared with a conventional short-term rate policy.

However, YCC policy has interesting side-effects in forms of more frequent and prolonged ZLB episodes. Active easing of long-maturity yields imposes an additional downward pressure on the returns of short-term bonds, which stems from the household’s endogenous portfolio reallocation: falling long-term rates induce the household to pull its wealth out of long-term bonds and instead invest into (i) short-maturity bonds, imposing additional downward pressures on short-term yields, and (ii) private capital (loan) markets, reducing firms’ borrowing costs, and hence the consumption prices due to lower production costs.\(^9\) When the ZLB binds, YCC policy disproportionately controls the yields of long-term bonds, which places additional downward pressure on short-term rates and delays an exit from the ZLB. Therefore, the household’s endogenous portfolio reallocation results in a feedback loop between ZLB duration and the need for YCC policies: YCC raises ZLB duration and frequency, while the economy relies more on the YCC’s stabilization power during ZLB episodes. Up to our best knowledge, this result is new to the literature.\(^10\)

\(^8\) Even under the conventional policy, a falling short-term rate reduces long-term bond yields due to the endogenous portfolio reallocation of the household, thereby reducing the effective savings rate. However, this channel is insufficient for boosting aggregate demand, especially when the economy hits the ZLB constraint and conventional policy becomes inactive.

\(^9\) Drops in the aggregate price index further impose downward pressures on the policy rate following conventional Taylor rules, which is already constrained at the ZLB.

\(^10\) A similar result, but obtained through the completely different channel, is presented by Karadi and Nakov.
We propose a new theoretical foundation for the financial market segmentation based on imperfect information about asset returns. We assume that the household is subdivided into a continuum of families and family members, each of them having distinct and imperfect information sets about those future asset returns. Then, unable to extract a common signal from the pool of heterogeneous information sets, the household evenly splits aggregate savings across its members and lets them allocate their share on the assets that they deem more profitable. This investment strategy effectively results in market segmentation, with cross-sectional dispersions in each individual’s expectation of asset returns determining degrees of market segmentation associated with each class of assets. We simplify the aggregation problem of individual portfolio choices among members by modeling the differences on expected asset returns as Fréchet-distributed shocks around the respective rationally expected levels of returns.11 This aggregation technique, which we borrow from the international trade literature (e.g., Eaton and Kortum (2002)), allows us to easily incorporate new asset varieties and distinct degrees of market segmentation across different assets and maturities, while providing analytically tractable expressions for the household’s portfolio shares as a function of relative expected asset returns. Our formulation is fairly general and nests the famous expectations hypothesis as a one special case, allowing deviations due to imperfect information and behavioral reasons. A final benefit of this framework is that the demand elasticity of each asset class becomes a sufficient statistic for its particular degree of market segmentation, making the segmented market hypothesis easy to test and calibrate in contexts of our model.

Related Literature This paper contributes to several different strands of the literature in macroeconomics and finance. First, previous works have shown that macroeconomic factors are important in explaining behaviors of the term-structure of interest rates (e.g., Ang and Piazzesi (2003), Rudebusch and Wu (2008), and Bekaert et al. (2010)).12 The frameworks developed in this literature are usually based on an ad-hoc affine term-structure (e.g., Duffie and Kan (1996)) without specific equilibrium micro-foundations.13 We contribute to this

11For general properties of the Fréchet distribution, see Gumbel (1958).
12For example, Ang and Piazzesi (2003) found that models with business cycle factors forecast better than models with only unobservable factors by analyzing the joint dynamics of bond yields and macroeconomic variables in a VAR, with no-arbitrage as an identifying restriction.
13Bekaert et al. (2010) combined the no-arbitrage term-structure with a canonical New-Keynesian framework, maintaining consistency between the household’s IS (intertemporal substitution) equation and the affine pricing kernel. Even though their model delivers strong contemporaneous responses of the entire term structure to various macroeconomic shocks, the household does not invest in the entire yield curve, and it does not feature a full general equilibrium.
literature by pinning down the term-structure of interest rates in the presence of multiple asset classes (e.g., bonds for intertemporal smoothing and private loans for productive investments) and nominal rigidities, with explicit roles of the government and central bank’s balance sheets and the household’s endogenous portfolio choices along the entire yield curve, which allows us to characterize how the business cycle variables and financial markets (including the term-structure) are intertwined.

Another relatively nascent literature focuses on the relationship between central bank’s endogenous balance sheet composition and monetary policy (e.g. Gertler and Karadi (2011), Cúrdia and Woodford (2011), Christensen and Krogstrup (2018), Christensen and Krogstrup (2019), Karadi and Nakov (2021)). This literature provides new insights on how the central bank’s large scale asset purchase programs (LSAPs) help mitigate financial market disruptions, but in many cases abstracts from the study of multiple bond market maturities, and focuses instead on the aggregate expansion of central bank’s balance sheets. We contribute to the literature by providing a unified framework that describes how central banks can manipulate their bond portfolios in order to control targeted rates along the yield curve for stabilization purposes. Especially, our implication that an active endogenous manipulation of central bank’s long-term bond holdings can be welfare-improving aligns with Sims and Wu (2021).

Our analysis of the zero lower bound (ZLB) closely follows the previous literature (e.g., Swanson and Williams (2014), Caballero and Farhi (2017), and Caballero et al. (2021)) and describes additional benefits of an active manipulation of the central bank’s balance sheet (size and composition along the entire yield curve) when the economy enters the ZLB. To the best of our knowledge, our paper is the first to characterize a general equilibrium economy featuring both the term-structure of interest rates and a possibility of the binding ZLB, together with the presence of multiple financial assets.

**Layout** In Section 3.2, we present a New-Keynesian framework incorporating capital markets and the term-structure of interest rates, and derive the main theoretical results on how imperfect information leads to market segmentation. Section 3.3 focuses on the steady state implications of distinct policies and the model calibration. Section 3.4 studies the cyclical (short-run) responses of our model to different shocks under alternative monetary policy rules and economic situations, including the ZLB. Section 3.5 concludes.

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14 For example, Gertler and Karadi (2011) pointed out that (i) central banks are not balance sheet constrained, and (ii) as the balance sheet constraints on private intermediaries tighten during financial crises, a net benefit from the central bank’s intermediation increases.

15 Cúrdia and Woodford (2011), for example, showed that targeted asset purchases by central bank’s credit expansion is effective when financial markets are highly disrupted for some *exogenous* reason.

16 The term-structure of interest rates is abstracted away in Sims and Wu (2021). Instead, Sims and Wu (2021) assumes that a wholesale firm and fiscal authorities issue perpetuities with decaying coupon payments.
3.2 Model

3.2.1 Non-technical Summary

We start by providing a non-technical overview of our theoretical framework and its key components. There is a representative household and a continuum of monopolistically competitive firms producing differentiated goods and subject to price stickiness à la Calvo (1983). Firms use labor and capital in production, with the latter rented to firms by the competitive capital producer. Firms are subject to a cash-in-advance constraint when renting capital, and borrow from households through one-period loan contracts (equivalently, corporate one-period bond) in order to fulfill the capital rental payment ahead of their production. Firms also pay a wage to the household in exchange for its labor, and the household allocates its income between consumption and savings.

The household allocates its savings across a menu of different assets that includes firm loans and zero-coupon risk-free bonds. In contrast to canonical New-Keynesian models, the bond market is comprised of bonds with multiple maturities. The household contains a continuum of individuals, each with a different information set regarding the assets’ profitability. In order to construct its portfolio, we assume that the household evenly splits the savings across its members and lets them freely allocate their share to their preferred asset. We show that this difference in information sets across household members leads to financial market segmentation, which breaks the conventional expectations hypothesis of linearized monetary models and allows us to study the distinct impacts of different monetary policies on distinct assets and maturities.

Our framework also contains the government with an exogenous public consumption demand financed through taxation and bond issuance. By sustaining a positive steady state deficit, which is a plausible assumption for most advanced economies, the government becomes the natural issuer of the entire risk-free bond market. The central bank then implements its preferred monetary policy by controlling the economy's yield curve through open market bond purchases that affect the size and composition of its balance sheet. We consider two distinct policy rules that the central bank follows. First, a conventional policy rule that controls the short(est)-term yield through active manipulation of its short-term bond holdings, together with a passive targeting of the balance sheet volumes of longer bond maturities. A movement of the policy rate leads the household to reallocate their portfolio (i)
across bonds of different maturities, and (ii) between bonds and loans. This reallocation changes the household’s effective savings rate and affects its consumption through the usual intertemporal substitution channel (‘demand block’). In addition, a policy movement alters the share of savings flowing into firms as private loans, which affects the effective loan rate, firms’ capital demand, and the output level (‘supply block’).

Second, we consider a yield-curve control monetary policy rule that targets the entire yield curve by actively trading bonds across all different maturities.\footnote{More precisely, we assume that the central bank sets a Taylor-type rule for each yield along the entire yield curve.} We show that this policy is very powerful in terms of stabilization, as it enables the central bank to lower the effective savings rate of households even at the zero-lower bound (ZLB), during which conventional policy is ineffective. As a drawback, this policy raises the likelihood and durations of ZLB episodes, since falling long-maturity yields increases the demand for short-term bonds and imposes additional downward pressure on their yields. Albeit being optimal from a welfare perspective, the prolonged ZLB episodes that accompany such policy further amplify its usefulness and chronify its application, suggesting that more frequent ZLB spells and unconventional interventions such as Large Scale Asset Purchases (LSAPs) might become the new normal.

The key economic agents and financial markets of our framework are summarized in Figure C.1. Next, we formally present the main components of our model.

### 3.2.2 Representative Household

The representative household maximizes the following objective function:

$$\max_{\{C_{t+j}, N_{t+j}\}} \mathbb{E}_{t} \sum_{j=0}^{\infty} \beta^j \left[ \log (C_{t+j}) - \left( \frac{\eta}{\eta + 1} \right) \left( \frac{N_{t+j}}{\bar{N}_{t+j}} \right)^{1+\frac{1}{\eta}} \right],$$

(3.2.1)

where $N_t = (\int_0^1 N_t(\nu) \frac{d\nu}{\eta} + 1)^{\frac{\eta}{\eta+1}}$ is an aggregate labor index, $N(\nu)$ is labor supplied to intermediate industry $\nu$, $\eta$ is the Frisch labor supply elasticity, and $\bar{N}_t$ is the balanced growth path population, which grows at constant gross rate $GN$. Variable $C_t$ is consumption of the final good.

At each period $t$, the representative household can invest in $f$-period zero-coupon government bonds where $f$ varies from 1 to $F$, and also provide loans to the firms.\footnote{Alternatively, we interpret it as households purchasing one-period corporate bonds.}\footnote{Banks and financial intermediaries are abstracted away in our framework, and the representative household issues direct loans to the firms instead. Without any relevant intermediation frictions, the results of both representations are equivalent.} Therefore,
the representative household’s period $t$ budget constraint is written as

$$C_t + \frac{L_t}{P_t} + \sum_{f=1}^{F} B_{t}^{H,f} = \frac{\sum_{f=0}^{F-1} R_{t}^{K} B_{t-1}^{H,f}}{P_t} + \frac{R_{t}^{K} L_{t-1}}{P_t} + \int_{0}^{1} \frac{W_t(\nu) N_t(\nu)}{P_t} \, d\nu + \frac{\Lambda_t}{P_t},$$

(3.2.2)

where $L_t$ is the amount of one period loans to firms, with associated return $R_{t}^{K}$ determined upon issuance. $W_t(\nu)$ is the wage paid by industry $\nu$, $P_t$ is the price index of the final good, and $\Lambda_t$ are transfers from different sources, including government’s lump sum taxation and profits of the central bank and firms. $B_{t}^{H,f} \equiv Q_{t}^{f} B_{t}^{H,f}$ is the nominal amount of dollars invested in the $f$-maturity government bond paying one dollar at the terminal period $t + f$. $Q_{t}^{f}$ is the price of such bond, with $Q_{t}^{0}$ equal to one. $\tilde{B}_{t}^{H,f}$ is the amount of $f$-maturity bonds held by the household, and we assume that the household is unable to credibly issue risk-free bonds and therefore is constrained to hold a non-negative quantity of them, $\tilde{B}_{t}^{H,f} \geq 0$ for all $f$. Variable $R_{t}^{f}$ is the return earned on an $f$-period bond, which corresponds to the rate of bond price revaluation between two adjacent quarters, $R_{t}^{f} = Q_{t}^{f}/Q_{t-1}^{f+1}$. The gross yield of a zero-coupon bond of maturity $f$ is conventionally defined as $YD_{t}^{f} \equiv (Q_{t}^{f})^{\frac{-f}{f+1}}$, and hence we can alternatively express bond return $R_{t}^{f}$ as

$$R_{t}^{f} = \frac{(YD_{t}^{f})^{\frac{-f}{f+1}}}{(YD_{t-1}^{f+1})^{(f+1)}}.$$

**Individual Savings**

The representative household chooses the optimal level of consumption, employment, and savings $S_t$, with the latter allocated either into government bonds $B_{t}^{H} = \sum_{f=1}^{F} B_{t}^{H,f}$ or firm loans $L_{t}$, thus satisfying $S_t = B_{t}^{H} + L_{t}$. To generate a downward sloping demand curve for each investment vehicle, we introduce the following machinery: After deciding the savings level $S_{t}$, the household is equally split into a $[0, 1]$ continuum of families that differ in their preferred savings vehicle, which can either be loans or bonds. If a family prefers to invest in the bond market instead of issuing loans, then the family is again split into a $[0, 1]$ measure of family members that differ on the preferred bond maturity $f = 1 \sim F$. We use index $m$ to identify a family within the continuum, and index $n$ to refer to one of its family members. Each family $m$ and each member $n$ in the bond family $m$ all have the same amount of savings $S_{t}$ as the household. We solve the allocation problem recursively in the following way.

Assuming that a family $m$ has chosen bonds as their preferred savings vehicle, its member

\footnote{Otherwise, linearization of the model results in the perfect equalization, in equilibrium, of all expected asset returns (including different bond maturities), which is consistent with the standard expectation hypothesis (see Froot (1989)).}
$n$ maximizes the expected savings return, solving the following problem.

$$\max \sum_{f=1}^{F} \mathbb{E}_{m,n,t} \left[ Q_{t,t+1} R_{t+1}^{f-1} B_{m,n,t}^{H,f} \right] \text{ s.t. } B_{m,n,t}^{H} \equiv \sum_{f=1}^{F} B_{m,n,t}^{H,f} = S_{t}, \quad B_{m,n,t}^{H,f} \geq 0,$$

where $\mathbb{E}_{m,n,t}$ is the expectations operator for member $n$ in family $m$ and $Q_{t,t+1}$ is the stochastic discount factor of the household. Due to the linear nature of the problem, we reach a corner solution in which member $n$ allocates her entire share of savings to the bond with the highest expected discounted return.\(^{22}\) Formally,

$$B_{m,n,t}^{H,i} = \begin{cases} S_{t}, & \text{if } i = \arg \max_{1 \leq j \leq F} \{ \mathbb{E}_{m,n,t} \left[ Q_{t,t+1} R_{t+1}^{j-1} \right] \} , \\ 0 , & \text{otherwise.} \end{cases}$$

In the benchmark rational expectations environment, all members in the bond family $m$ choose the same allocation, and expected discounted returns $\mathbb{E}_{t} \left[ Q_{t,t+1} R_{t+1}^{f-1} \right]$ for any maturity $f$ are equalized in equilibrium. This case aligns with the 'expectation hypothesis' in the log-linearized economy, where long term rates are approximated as the average of future expected short term rates.\(^{23}\) Since the short term rate $R_{t+1}^0$ is controlled by the central bank, longer yield maturities are fully determined by conventional monetary policy in this environment. This precludes any meaningful role for other central bank policies such as QE, despite empirical evidence on the contrary.\(^{24}\)

We deviate from the expectations hypothesis and generate a downward sloping demand curve for each bond of maturity $f$ by imposing additional structure on the portfolio allocation problem. We assume that each member $n$ of the family has different expectations about the discounted future returns of bonds. This difference can be attributed to each agent having access to a distinct and imperfect information set (in a similar manner to Angeletos and La’O (2013)) or simply by behavioral assumptions. In addition, we assume that family $m$ doesn’t have the capacity to aggregate the individual information from its members and perform a centralized portfolio allocation based on signal extraction. Therefore, the family decides to equally split the savings among its members and allows them to decide on the allocation of

\(^{22}\)The exception would be if two or more bonds have exactly the same highest expected discounted return, in which case the member $n$ would be indifferent between allocations across these bonds. This, in general, will happen with zero probability as it will become clear in the derivations below.

\(^{23}\)In the linearized economy, the co-variation between $Q_{t,t+1}$ and returns $R_{t+1}^{f-1}$ is omitted together with other higher-order terms. Thus, expected returns for each bond maturity are equalized.

\(^{24}\)For example, Krishnamurthy and Vissing-Jorgensen (2011) show that LSAP interventions lower long-term interest rates.
their individual share. We assume the following functional form for member $n$ expectations:

$$E_{m,n,t} [Q_{t,t+1} R_{t+1}^{f-1}] = z_{f,n,t} \cdot E_{t} [Q_{t,t+1} R_{t+1}^{f-1}], \quad \forall f = 1, \ldots, F,$$

where the expectation operator $E_{m,n,t}$ is a member-specific expectation, whereas $E_{t}$ is the rational expectation. $z_{f,n,t}$ are maturity-$f$ specific shocks to member $n$’s expectations. Note that, ceteris paribus, a high realization of $z_{f,n,t}$ makes member $n$ more willing to save in the $f$-maturity bond.

For tractability, we model $z_{f,n,t}$ as a Fréchet-distributed shock with location parameter zero, scale parameter $z_{f,t}$ and shape parameter $\kappa_B$, and assume it to be i.i.d. across members $n$, maturities $f$ and quarters $t$.\footnote{See Eaton and Kortum (2002) and Carreras et al. (2021) for applications of the Fréchet-distribution and aggregation issues in international trade and macroeconomics literature.} Shape parameter $\kappa_B$ determines the volatility of these expectation shocks, with $\lim_{\kappa_B \to \infty} \text{Var}(z_{f,t,n}) = 0$. Therefore, with $z_{f,t} = \Gamma (1 - 1/\kappa_B)^{-1}$ and $\kappa_B \to \infty$, the model collapses to the standard rational expectations case with $E_{m,n,t}$ aligning with $E_{t}$. Otherwise, individual expectations deviate from the rational expectation.\footnote{If we set the scale parameter to $z_{f,t} = \Gamma (1 - 1/\kappa_B)^{-1}$, then we have $\mathbb{E}(z_{f,t,n}) = 1$, and member-specific expectations fluctuate around the rational expectation.}

We define $\lambda_{t}^{HB,f}$ as the probability that the $f$-period bond provides the highest expected discounted return to a family member $n$. By the properties of the Fréchet distribution, we obtain a nice analytical expression for this probability as

$$\lambda_{t}^{HB,f} = \left( \frac{z_{f,t} E_{t} [Q_{t,t+1} R_{t+1}^{f-1}]}{\Phi_{t}^{B}} \right)^{\kappa_{B}}, \quad (3.2.3)$$

where $\Phi_{t}^{B} \equiv \left[ \sum_{j=1}^{F} \left( z_{f,j} E_{t} [Q_{t,t+1} R_{t+1}^{j-1}] \right)^{\kappa_{B}} \right]^{1/\kappa_{B}}$ is an aggregate index that captures the average expected discounted return across bonds of different maturities. Equation (3.2.3) implies that demand for savings in the $f$-maturity bond increases when its return $R_{t+1}^{f-1}$ is relatively higher to that of the average bond return across all maturities, $\Phi_{t}^{B}$.

Aggregating across families and family members, we obtain an expression for the household’s total holdings of each $f$-maturity bond as

$$B_{t}^{H,f} = \lambda_{t}^{HB,f} \cdot B_{t}^{H}, \quad \forall f = 1, \ldots, F, \quad (3.2.4)$$

where $B_{t}^{H}$ is the household’s aggregate bond holding amounts. Using equation (3.2.4), we

\footnote{Thus, our framework nests the above benchmark case (no-arbitrage term structure) as a limiting case.}
obtain an aggregate expression for the returns to the household’s bond portfolio as

\[ R^{HB}_{t+1} = \sum_{f=0}^{F-1} \lambda^{HB,f}_{t+1} R^f_{t+1}. \]  
(3.2.5)

Now that we have found the allocation of savings across bond maturities, we turn our eyes into the problem of how each family \( m \) decides between depositing its savings either in bonds or loans. Family \( m \) maximizes savings returns out of the set of possible asset classes (bonds and loans in our model) by solving the following problem:

\[
\max \ E_{m,t} \left[ Q_{t,t+1} R^{HB}_{t+1} B^H_{m,t} \right] + E_{m,t} \left[ Q_{t,t+1} R^K_{t+1} L_{m,t} \right] \quad \text{s.t.} \quad B^H_{m,t} + L_{m,t} = S_t, \quad B^H_{m,t} \geq 0, \quad \text{and} \quad L_{m,t} \geq 0. 
\]

Family \( m \) takes as given that if it becomes a bond family, it will follow the investment strategy outlined in equation (3.2.3) and obtain aggregate returns \( R^{HB}_{t+1} \) (equation (3.2.5)) on its bond portfolio. In the benchmark rational expectation environment, all families choose the same allocation, and in equilibrium, expected discounted returns \( E_{m,t} \left[ Q_{t,t+1} R^{HB}_{t+1} \right] \) and \( E_{m,t} \left[ Q_{t,t+1} R^K_{t+1} \right] \) are equalized, making families indifferent in their portfolio allocation.

As before, we generate a downward-sloping demands for bonds and loans in the linearized economy by assuming that each family \( m \)'s expectation operator deviates from the rational expectation as follows:

\[
E_{m,t} \left[ Q_{t,t+1} R^K_{t+1} \right] = z^K_{m,t} \cdot E_t \left[ Q_{t,t+1} R^K_{t+1} \right],
\]

where \( E_t \) is the rational expectation whereas \( E_{m,t} \) is a family \( m \)-specific expectation. We model \( z^K_{m,t} \) as a Fréchet-distributed shock with location parameter zero, scale parameter \( z^K_t \) and shape parameter \( \kappa_S \), and assume it to be i.i.d. across families \( m \) and quarters \( t \). As before, \( \kappa_S \) governs the expectation shock’s volatility, satisfying \( \lim_{\kappa_S \to \infty} \Var (z^K_{m,t}) = 0 \). Thus, when \( z^K_t = \Gamma (1 - 1/\kappa_S)^{-1} \) and \( \kappa_S \to \infty \), the model collapses to the standard rational expectation case with \( E_{m,t} \) aligning with \( E_t \).

Similar to what we found before, we can now aggregate decisions of each family \( m \) and find the share of aggregate savings that will be allocated to loans as

\[
\lambda^K_t = \left( \frac{z^K_t E_t \left[ Q_{t,t+1} R^K_{t+1} \right]}{\Phi^S_t} \right)^{\kappa_S}, \tag{3.2.6}
\]

where \( \Phi^S_t = \left[ (E_t \left[ Q_{t,t+1} R^{HB}_{t+1} \right])^{\kappa_S} + (z^K_t E_t \left[ Q_{t,t+1} R^K_{t+1} \right])^{\kappa_S} \right]^{\frac{1}{\kappa_S}} \) is an aggregate index that
captures the average expected discounted return of bonds and loans. Using equation (3.2.6),
we can now express the aggregate amount of savings flowing into bonds of each maturity as
\[ B_t^{H,f} = (1 - \lambda_t^K) \cdot \lambda_t^{HB,f} \cdot S_t, \quad \forall f = 1, \ldots, F, \]
and the aggregate return on household savings,
\[ R_t^S = (1 - \lambda_{t-1}^K) R_t^{HB} + \lambda_{t-1}^K R_t^K. \quad (3.2.7) \]
Observe that \( R_t^S \) depends on the rates of all available assets including (i) bonds of different maturities and (ii) loans (private one-period bond) with endogenous weights determined by relative returns of these assets. Finally, we can rewrite the budget constraint in equation (3.2.2) as
\[ C_t + S_t = \frac{R_t^S S_{t-1}}{P_t} + \int_0^1 \frac{W_t(\nu) N_t(\nu)}{P_t} d\nu + \Lambda_t P_t. \quad (3.2.8) \]
Note that the representative household problem now resembles that of a conventional New-Keynesian model, which constitutes a remarkably tractable result given the asset variety and market segmentation that we introduce.

**Remarks on aggregation:** The assumption about separate information sets on asset returns (which we model as extreme type Fréchet deviations from the rational equilibrium), effectively creates market segmentations (i) between bond and loan markets, and (ii) among different maturities in the bond market, which is also empirically supported by the literature (see D’Amico and King (2013)). Shape parameters \((\kappa_B, \kappa_S)\) control the degree of market segmentation across maturities and assets, respectively, and the conventional expectations hypothesis framework without market segmentation is nested as an special case of our model when \(\kappa_B, \kappa_S \to \infty\). Most notably, the nested CES structure of our asset markets can be easily extended to accommodate a wide variety of assets and maturity structures. Also, the shape parameters \((\kappa_B, \kappa_S)\) summarize the demand elasticity of financial products to movements in their expected returns (equations (3.2.3) and (3.2.6)). These elasticities can take distinct values across asset classes, and they can be easily estimated from the data in order to capture different degrees of market segmentation across assets and maturities.

\[ \text{Observe that equation (3.2.6) implies that family preference for issuing loans increases when the return on loans } R_{t+1}^K \text{ becomes relatively higher to that of the aggregate bond portfolio } R_{t+1}^{HB}. \]
Optimality Conditions

The solution to the household’s problem in equation (3.2.1) subject to the budget in equation (3.2.8) brings the following equilibrium conditions:

\[
\left( \frac{N_t(\nu)}{\bar{N}_t} \right)^{\frac{1}{\eta}} = \left( \frac{C_t}{\bar{N}_t} \right)^{-1} \frac{W_t(\nu)}{P_t}, \tag{3.2.9a}
\]

\[
1 = \beta E_t \left[ \frac{R_{t+1}^S C_t}{C_{t+1} \Pi_{t+1}} \right], \tag{3.2.9b}
\]

where \( \Pi_{t+1} \equiv \frac{P_{t+1}}{P_t} \) is the gross inflation rate. Note that the Euler equation (3.2.9b) informs us that the effective savings rate \( R_{t+1}^S \) has now become the reference rate for the household’s intertemporal consumption decisions.

3.2.3 Capital Producer

There is a representative firm that produces capital \( K_t \) and rents it to the intermediate good producers at price \( P^K_t \). Capital is produced by using the final good as an investment input, depreciates at rate \( \delta \) and there is one-period lag for investment \( I_t \) to be deployed as new capital. The evolution of capital is thus given by

\[
K_t = (1 - \delta) K_{t-1} + I_{t-1}.
\]

Solving the capital producer’s profit maximization problem with respect to investment \( I_t \), we obtain the following first order condition.

\[
1 = E_t \left[ Q_{t,t+1} \Pi_{t+1} \left( (1 - \delta) + \frac{P^K_{t+1}}{P_{t+1}} \right) \right].
\]

3.2.4 Firms

There is a continuum \( \nu \in [0, 1] \) of intermediate goods, each produced by a monopolist \( \nu \) with the following production function employing capital and labor:

\[
Y_t(\nu) = \left( \frac{K_t(\nu)}{\alpha} \right)^\alpha \left( \frac{A_t N_t(\nu)}{1 - \alpha} \right)^{1-\alpha}, \tag{3.2.10}
\]

where \( A_t \) is the aggregate production technology which grows at the exogenous gross rate \( GA_t \). A representative and perfectly competitive firm aggregates intermediate products into
a final good according to the familiar Dixit-Stiglitz aggregator,

\[ Y_t = \left[ \int_0^1 Y_t(\nu) \frac{\epsilon-1}{\epsilon} \, d\nu \right]^{\frac{\epsilon}{\epsilon-1}}, \]

where \( \epsilon > 1 \) is the elasticity of substitution between varieties. Household’s demand for intermediate good \( \nu \) is given by

\[ Y_t(\nu) = \left( \frac{P_t(\nu)}{P_t} \right)^{-\epsilon} Y_t, \quad (3.2.11) \]

where \( P(\nu) \) is the price of intermediate \( \nu \). The aggregate price index is given by

\[ P_t = \left[ \int_0^1 P_t(\nu)^{1-\epsilon} \, d\nu \right]^{\frac{1}{1-\epsilon}}. \quad (3.2.12) \]

Intermediate producers have sticky prices à la Calvo (1983), and reset prices at the beginning of the quarter with probability \( 1 - \theta \). All price-changing firms reset to the same optimal price within a given period, which we denote by \( P^*_t \). This allows us to recursively express equation (3.2.12) as

\[ P_t^{1-\epsilon} = (1 - \theta) (P^*_t)^{1-\epsilon} + \theta (P_{t-1})^{1-\epsilon}. \quad (3.2.13) \]

Firms are subject to a cash-in-advance constraint on rented capital, which they finance via one-period household loans.\(^{29}\) Formally, firm \( \nu \) constraint is

\[ L_t(\nu) \geq P^K_t K_t(\nu), \]

where \( L_t(\nu) \) and \( K_t(\nu) \) are the loans and capital rented at \( t \) by firm \( \nu \), respectively. Finally, minimizing a firm \( \nu \)'s production costs with respect to labor and capital, we obtain the following demand for inputs.

\[
N_t(\nu) = (1 - \alpha) \frac{Y_t(\nu)}{A_t} \left( \frac{\mathbb{E}_t \left[ Q_{t,t+1} R^K_{t+1} \right] \left( \frac{P^K_t}{P_t} \right) }{W_t(\nu) \frac{P_t}{A_t}} \right)^{\alpha}.
\quad (3.2.14)
\]

\[
\frac{K_t(\nu)}{A_t} = \alpha \frac{Y_t(\nu)}{A_t} \left( \frac{\mathbb{E}_t \left[ Q_{t,t+1} R^K_{t+1} \right] \left( \frac{P^K_t}{P_t} \right) }{W_t(\nu) \frac{P_t}{A_t}} \right)^{-(1-\alpha)}.
\quad (3.2.15)
\]

\(^{29}\)Inada conditions ensure that capital utilization, and therefore loan demand, will stay positive throughout the cycle.
3.2.5 Bond Market

The equilibrium condition in the bond market can be written as

$$B_t^{H,f} + B_t^{G,f} + B_t^{CB,f} = 0, \quad \forall f = 1, \ldots, F,$$

(3.2.16)

where $B_t^{G,f}$ and $B_t^{CB,f}$ are nominal bonds held by the government and central bank, respectively. We assume that the government and the central bank are the only agents capable of issuing risk-less claims, and therefore sustain a negative position in bond markets. For central banks, a negative position can be rationalized, for example, as the outcome of an interest-bearing policy on excess reserves.\(^{31}\) We define $\lambda_t^{G,f}$ and $\lambda_t^{CB,f}$ as the fractions of nominal $f$-maturity bonds held by the government and central bank, respectively. We assume that government’s bond portfolio shares $\{\lambda_t^{G,f}\}$ are stochastic and exogenously given, while those of the central bank $\{\lambda_t^{CB,f}\}$ are determined in equilibrium as a function of the monetary policy implemented.

3.2.6 Government

The budget constraint of the government is given by

$$\frac{B_t^G}{P_t} = \frac{R_t^G B_{t-1}^G}{P_t} - \left[ \zeta_t^G + \zeta_t^F - \zeta_t^T \right] Y_t,$$

(3.2.17)

where $R_t^G = \sum_{f=0}^{F-1} \lambda_{t-1}^{G,f+1} R_t^f$ is the aggregate return on the government’s bond portfolio $B_{t-1}^G$ and $\zeta_t^G$, $\zeta_t^F$ and $\zeta_t^T$ are government’s spending, production subsidy and taxation as a share of GDP, respectively. We assume $\zeta_t^G$ and $\zeta_t^T$ follow exogenous processes given by

$$\zeta_t^G = \frac{1}{1 + a^G \exp (-u_t^G)}, \quad \zeta_t^T = \frac{1}{1 + a^T \exp (-u_t^T)},$$

where $u_t^G$, $u_t^T$ are AR(1) shocks.

3.2.7 Monetary Policy

The term-structure of interest rates, when paired with market segmentation, adds $F$ new degrees of freedom tied to the nominal yield of risk-free bonds. This implies that a deter-

\(^{30}\)Since we assume the government issues bonds across different maturities, $B_t^{G,f} \leq 0$ for all $f = 1 \sim F$.

\(^{31}\)For theoretical and empirical analyses of the excess reserve’s roles in conjunction with the federal fund market and interbank credit market in general, see Frost (1971), Güntner (2015), Mattingly and Abou-Zaid (2015), Primus (2017), and Ennis (2018) among others.
minimize nominal equilibrium requires the central bank’s monetary policy to be expanded with \( F \) new equilibrium conditions pinning down the nominal yields along the maturity curve. For simplicity, we assume that the central bank follows policy rules similar in spirit to the Taylor or money-growth rules of traditional New-Keynesian models. The central bank can state its policy goals as a function of bond yields, bond portfolio holdings, or a combination of both at different maturities. We classify the policy options of the central bank into three main categories:

1. The central bank sets a rule on the bond holding amount of each maturity \( f \), \( \{B_{CB}^{f}\} \). Then, bond prices (and hence, yields) adjust endogenously.\(^{32}\)

2. The central bank sets a Taylor rule on the yield of each \( f \)-maturity bond, \( \{YD_f\} \) (or equivalently, on its price \( Q_f \)). Then, it adjusts its bond portfolio holdings \( \{B_{CB}^{f}\} \) to meet the target yield.

3. A combination of the previous options at different maturities.

Case 1 resembles textbook money supply rules, whereas not money but long-term bond supplies are controlled by the central bank.\(^{33}\) Case 2 captures a policy avenue often called ‘yield-curve control (YCC)’, which Japan employed in 2016.\(^{34}\) Case 3 is a mix of the previous options, and includes widely employed rules like the traditional short-term rate target of conventional monetary policy, as we shall see below. In this paper, we want to study the distinct economic and welfare implications of conventional and unconventional policy interventions. The specific implementation of the latter type of policies can potentially take the form of any of the three cases considered. For simplicity, we assume that the fundamental trait that characterizes unconventional interventions (e.g. QE and LSAP) is its aim to affect the asset returns along the entire yield curve (as opposed to conventional policy focused on short-term rates), and hence we adopt a YCC policy rule as the representative unconventional policy of our framework. Below, we provide a formal characterization of the equations that describe conventional and YCC policy rules.

\(^{32}\)Basically, central bank chooses its bond portfolios across different maturities \( \{\lambda_{CB}^{f}\} \), as well as gross debt position \( B_{CB} \), the former of which (portfolio) is usually omitted in traditional New-Keynesian models without an explicit term-structure.

\(^{33}\)For example, Sims and Wu (2021) study the optimal response of central bank bond holdings (\( B_{CB}^{f} \) in our model) to business cycle fluctuations.

\(^{34}\)For example, on September 21, the Bank of Japan combined a new long-term interest rate target with its existing short-term interest rate target to give the bank ‘yield-curve control’, with which the Bank of Japan set its short-term policy target—a rate paid on bank reserves—at \(-0.1\%\), and capped its long-term target rate—on 10-year government bonds—at approximately zero. For the United States, see Humpage (2016) for the Fed’s yield-curve control policy during WW2, and possible benefits and costs of the policy.
**Conventional policy** We approximate a conventional monetary policy intervention by assuming that the central bank follows a Taylor rule on the short-term rate $YD^1_t$, while keeping a passive position on its portfolio of long-term bonds. Formally:

$$R^0_{t+1} \equiv YD^1_t = \max \{ YD^1_{t^*}, 1 \},$$  \hspace{1cm} (3.2.18a)  

$$YD^1_{t^*} = YD^1 \left( \frac{\Pi_t}{\bar{\Pi}} \right)^{\gamma_x} \left( \frac{Y_t}{\bar{Y}} \right)^{\gamma_y} \cdot \exp \left( \tilde{\varepsilon}^{YD^1}_t \right),$$  \hspace{1cm} (3.2.18b)  

$$\frac{B^C_{CB,t}}{A_t N_t P_t} = \frac{B^C_{CB,t}}{ANP} \quad \forall f = 2, \ldots, F,$$  \hspace{1cm} (3.2.18c)  

where $YD^1_{t^*}$ follows a Taylor rule with inflation and output targets, and $\tilde{\varepsilon}^{YD^1}_t$ is a monetary policy shock. When $YD^1_{t^*}$ is below one, monetary policy is constrained by the zero lower bound (ZLB), as implied by equation (3.2.18a). Equation (3.2.18c) captures the passive stance on real long-term portfolio bond holdings, adjusted by population and technical growth.

**Yield-curve control (YCC) policy** YCC policy is defined by the central bank targeting of bond returns along the entire yield curve. Formally, we model it as the central bank following an individual Taylor rule for each bond maturity as

$$YD^{GP,1}_t = \max \{ YD^{1*}_t, 1 \},$$  \hspace{1cm} (3.2.19a)  

$$YD^{1*}_t = YD^f \left( \frac{\Pi_t}{\bar{\Pi}} \right)^{\gamma_x^f} \left( \frac{Y_t}{\bar{Y}} \right)^{\gamma_y^f} \cdot \exp \left( \tilde{\varepsilon}^{YD^f}_t \right),$$  \hspace{1cm} (3.2.19b)  

$$YD^{GP,f}_t = YD^{SP,f} \gamma_{SP}^f \left[ \left( \frac{\Pi_t}{\bar{\Pi}} \right)^{\gamma_x^f} \left( \frac{Y_t}{\bar{Y}} \right)^{\gamma_y^f} \cdot \exp \left( \tilde{\varepsilon}^{YD^f}_t \right) \right]^{1-\gamma_{SP}^f}, \quad f \geq 2,$$  \hspace{1cm} (3.2.19c)  

where $YD^{GP,f}_t$ is the YCC policy yield of an $f$-maturity bond, and $YD^{SP,f}_t$ is the yield that prevails in a counterfactual economy under the conventional monetary policy described in equation (3.2.18a–c), and $\tilde{\varepsilon}^{YD^f}_t$ is a monetary policy shock to the maturity $f$ yield. Parameters $\gamma_x^f$ and $\gamma_y^f$ capture the responsiveness of monetary policy to inflation and output, respectively, and can take distinct values across maturities.

This policy constitutes a generalization of the traditional Taylor rule, and nests the conventional monetary policy regime as a limiting case when $\gamma_{SP}^f = 1$. As $\gamma_{SP}^f$ approaches zero, the rule converges to a YCC policy that only targets inflation and output, and which proves quite convenient to study the transition from a conventional policy rule towards an YCC regime.\(^{35}\)

\(^{35}\)Conventional monetary policy assumes that the central bank passively targets long-term bond holding amounts (see equation (3.2.18c)), while YCC policy directly controls the yields of long-term bonds, which
3.2.8 Market Clearing

Using the bond market equilibrium (equation (3.2.16)), we can express total transfers to the household from firms, central bank, capital producer, and government as

\[ \Lambda_t \equiv \Lambda^F_t + \Lambda^{CB}_t + \Lambda^K_t - P_t T_t = P_t Y_t - P_t G_t - \int_0^1 W_t(\nu) N_t(\nu) \, d\nu + S_t - R^S_t S_{t-1} - P_t I_t, \]

where \( T_t \) and \( G_t \) are the government lump-sum taxes and spending, respectively. Combining equation (3.2.16) with the household’s budget constraint (equation (3.2.2)), we obtain the following usual aggregate market clearing condition.

\[ C_t + G_t + I_t = Y_t. \]

Next, we present the steady state of the economy and present several comparative statics exercises where we study the long-run equilibrium effects of distinct degrees of market segmentation and maturity composition of the government’s and central bank’s bond portfolio.

3.3 Steady-State (Long-Run) Analysis

3.3.1 Steady-State Relations

Given the exogenous composition of the central bank’s portfolio \( \{\lambda^{CB,f}\}_{f=1} \), the bond market equilibrium (equation (3.2.16)) at the steady state can be expressed as

\[ \lambda^{HB,f}_t = \lambda^{G,f}_t + \lambda^{CB,f}_t \zeta^{CB} \left( 1 + \zeta^{CB} \right). \]

where \( \zeta^{CB} = B^{CB}/B^G \) is the steady state share of government bonds in the central bank’s portfolio.\(^{36}\) Thus, the household’s bond portfolio shares across different maturities are determined by the exogenous parameters \( \{\lambda^{G,f}, \lambda^{CB,f}\}_{f=1} \) and \( \zeta^{CB} \).

Rearranging the government’s budget constraint (equation (3.2.17)), we can obtain an expression for the balanced growth path debt-to-GDP ratio as

\[ \frac{B^G_t}{P_t Y_t} = - \left( 1 - \frac{R^G}{\Pi \cdot GA \cdot GN} \right)^{-1} \left[ \zeta^G + \zeta^F - \zeta^T \right]. \]

makes central bank’s long-term bond purchases endogenously determined by the levels of its controlled yields (see equation (3.2.19c)).

\(^{36}\)Since \( B^G < 0 \) and \( B^{CB} > 0 \) in the steady state, we have \( \zeta^{CB} < 0 \).
Notice that both a higher return $R^G$ on the government’s bond portfolio and a higher deficit $\zeta^G + \zeta^F - \zeta^T$ lead to a larger equilibrium debt-to-GDP ratio.\(^{37}\) In the former case, an increase in $R^G$ leads to higher debt servicing costs, which in equilibrium are financed by additional bond issuance. After some manipulations, we can write the balanced growth path (normalized) output $\frac{Y}{AN}$ as

$$\frac{Y_t}{A_tN_t} = \xi Y \left[ (1 - \zeta^G) - \xi^C \left( 1 - \frac{R^G}{\Pi \cdot GA \cdot GN} \right)^{-1} \left( \frac{\lambda^K}{1 - \lambda^K} \right) \left[ \left( \frac{\lambda^K}{1 - \lambda^K} \right) \right]^{-\frac{n}{1+n}} (R^K)^{-\frac{1}{1+n}} \right], \quad (3.3.1)$$

as a function of the aggregate government’s rate, $R^G$, the share of household’s savings that flow into firms as loans, $\lambda^K$, and their associated interest rate, $R^K$.\(^{38}\) For comparative statics, we assume that $\xi^Y > 0$ and $\xi^C > 0$, which is satisfied under a reasonable model calibration. An increase in $R^K$ affects output $\frac{Y}{A_tN_t}$ through two opposite channels. First, an increase in $R^K$ results in a higher rental (loan) cost of capital from the firms’ perspectives, which reduces aggregate capital and eventually, output.\(^{39}\) On the other hand, a higher $R^K$ raises the share of savings flowing into firms, $\lambda^K$, as households reallocate funds out of the bond market and towards the issuance of more loans to firms. The increased availability loans then leads to higher aggregate capital and output levels.

We describe the full set of steady state equilibrium conditions in Appendix Steady-State Derivations in Section 3.3.1. Next, we explain the model’s calibration and perform several comparative static exercises that show the effects of the parameters on the equilibrium of the economy.

### 3.3.2 Results

#### Calibration and Yield Curve

Using publicly available data on (i) treasury yields, (ii) federal reserve’s holdings of treasury bonds, and (iii) U.S Treasury’s outstanding bonds,\(^ {40}\) we calibrate the parameters of our model to match the average yield curve for 1990 to 2007 period. For that purpose, we develop a version of the model with a hundred and twenty distinct maturities, $F = 120$, which accounts

\(^{37}\)A primary deficit $\zeta^G + \zeta^F - \zeta^T > 0$ and government bond issuance $B^G < 0$ can only be jointly sustained if $R^G < \Pi \cdot GA \cdot GN$. This condition is satisfied under our model calibration.

\(^{38}\)Coefficients $\xi^Y$ and $\xi^C$ in equation (3.3.1) are given in Appendix Steady-State Derivations in Section 3.3.1.

\(^{39}\)Since firms’ elasticity of substitution between capital and labor is 1, which is finite, replacement of capital by labor is not enough to prevent the output level from dropping.

\(^{40}\)https://fiscaldata.treasury.gov/datasets/monthly-statement-public-debt
for long-term bonds of up to thirty years duration, with maturity increments measured at a quarterly frequency.\footnote{In cases where data on a specific maturity is missing, we relied on interpolation methods in order to generate a smooth yield curve with quarterly maturity increments.}

Shape parameters $\kappa_B$ and $\kappa_S$ capture the household’s elasticity of substitution between bond maturities, and across capital loans and bonds, respectively. Pending a proper empirical estimation of these parameters, we temporarily calibrate them at $\kappa_B = 10$ and $\kappa_S = 1$ to reflect our priors on a relatively higher degree of substitution across bond maturities than across asset classes.\footnote{The qualitative results reported in the paper are robust to alternative calibrations of elasticities $\kappa_B, \kappa_S$.} Figure C.3 illustrates the steady state flattening of the yield curve as the elasticity of substitution across bond maturities increases, $\kappa_B \to \infty$, and the model converges towards the expectations hypothesis case.

Scale parameters $\{z^f\}_{f=1}^F$ are then calibrated to match the yield curve’s shape (relative yields across maturities) while scale parameter $z^K$ is calibrated to match the spread between the short-term rate $R^0$ and the loan rate $R^K$. The detailed calibration procedure for $\{z^f\}_{f=1}^F$ and $z^K$ is described in Appendix Calibrating $\{z^f\}$ and $z^K$ in the Steady State. Figure 3.1 shows the distribution of bond portfolio shares across maturities for each agent (household, government, and central bank) and the associated yield curve. The values used to calibrate $z^K$ and $\{z^f\}_{f=1}^F$ are reported in Table C.2 and Figure C.2, respectively. Note that the steady-state preference for the short-term bond, $z^1$, is especially large compared to that for longer-term maturities, $z^f$ for $f \geq 2$. This captures the empirical evidence on the historically low yield of short-term bonds relative to longer-term maturities, and is consistent with theories of the safety and/or liquidity premium.\footnote{See for example, Krishnamurthy and Vissing-Jorgensen (2012) and Caballero and Farhi (2017).}

The remaining parameters of the model are frequently present in the macroeconomics literature, and we calibrate them to commonly accepted values. Table C.1 summarizes the parameter calibration of our model.

Next, we study the effect of different calibrations on the steady state equilibrium of the economy.

**Government’s Bond Supply and Central Bank’s Bond Demand**

Steady state government bond issuance shares $\{\lambda^{G,f}\}_{f=1}^F$ are exogenous parameters in our model, which we calibrate to their observed average value between 1990 and 2007. We begin our study of steady-state outcomes by considering the effects of alternative issuance distributions across maturities. Figure 3.2 plots alternative government bond issuance shares (left panel), and the resulting changes in the yield curve (right panel). The Figure illustrates that our model generates a positive relationship between the yield $Y^{D_f}$ of an $f$-maturity bond and its issuance share $\lambda^{G,f}$, a result which is consistent with previous empirical findings of the
The effect comes through the relative increase in the supply of $f$-maturity bonds, which decreases its equilibrium price and thereby raises its yield. In addition, the change in government’s issuance shares triggers an endogenous household portfolio reallocation that affects the equilibrium returns across all bond maturities and loan markets, affecting the aggregate borrowing costs of the government and the steady-state debt level.

Figure C.4 illustrates a change in the composition of the central bank’s bond portfolio, and where we observe that central bank’s relative purchase of each maturity is negatively related with the bond’s yield. This result follows from the central bank being an additional source of demand in bond markets, and which, under market segmentation, generates asymmetrical pressures on the price of bonds when the central bank’s demand composition shifts.

Note that the strength of the yield curve response to changes in $\{\lambda^{G,f}, \lambda^{CB,f}\}_{f=1}^{F}$ depends on the degree of market segmentation across bond maturities, which is controlled by parame-

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Footnotes:

44 For example, Greenwood and Vayanos (2014) documents that the supply of long-term bonds, relative to that of short-term maturities, is positively correlated with the term spread.

45 For example, Ray (2019) and Droste et al. (2021), based on a preferred-habitat environment developed by Vayanos and Vila (2021), derive similar implications. Krishnamurthy and Vissing-Jorgensen (2011) empirically find that QE policies (both QE1 and QE2) affect particular assets differently. QE2, which primarily focused on treasury bonds, had a disproportionate impact on Treasuries and Agency bonds, relative to mortgage-backed securities and corporate bonds. Also, D’Amico and King (2013) identify QE programs’ stock and flow effects on treasury yields, and find supporting evidence for segmented markets and/or imperfect substitution within the Treasury market.
In the limiting case $\kappa_B \to \infty$, the expectation hypothesis is restored across maturities and the government and central bank portfolio shares have no effect on the determination of the yield curve.

**Other Comparative Statics**

Figures C.5 and C.6 document the effect of variations on the deficit ratio $\zeta^F + \zeta^G - \zeta^T$. A higher steady-state deficit ratio can be sustained by (i) a higher government bond issuance (holding output constant), or (ii) a lower government borrowing rate, $R^G$, or (iii) a lower output level (holding bond issuance constant), or any appropriate combination of the previous. We first consider (i) and observe that it is an impossible option to sustain in the long-run: if the government issues more debt in order to finance a higher deficit (given a fixed output level), it raises the government’s effective bond return $R^G$ (supply effect in Section 3.3.2), which results in the issuance of more bonds to finance the additional interest costs, further raising $R^G$ in an ad infinitum loop. Options (ii) and (iii) work together: a higher deficit ratio brings down output, consumption, and capital, which lowers the deficit size (nominal) and the government’s bond issuance, pushing down its bond return $R^G$. Capital loan rate $R^K$ responds mildly and the credit spread $r^K - r^{HB}$ rises in response. The result that the debt-to-GDP ratio $\frac{B^G}{\Gamma}$ decreases while the entire yield curve shifts down in response to a higher
deficit ratio is in line with prior literature including Laubach (2009).

Figures C.7 and C.8 describe comparative statics of the scale parameter $z^K$, which controls the household’s steady-state preference for capital loans as savings vehicle (see equation (3.2.6)). A higher $z^K$ increases the household’s willingness to issue loans to firms rather than invest in the bond market, which raise steady-state $\lambda^K$ and capital, output, and consumption in response. As households increase their loan issuance, the average marginal propensity to consume (MPC) drops and the loan rate $R^K$ falls due to the increased supply of funds. Then, the household’s endogenous portfolio reallocation towards bonds shifts the yield curve downwards, resulting in a widening of the credit spread. This result is specially interesting and highlights the importance of general equilibrium effects and the endogenous portfolio re-allocation. Given the initial preference shock towards capital loans, we would have otherwise expected a narrowing of the credit spread. As $R^G$ falls, the government debt-to-GDP ratio falls too.

Figures C.9 and C.10 describe comparative statics of the shape parameter $\kappa_S$ that controls the degree of market segmentation between bonds and capital and doubles as the elasticity of loan supply (see equation (3.2.6)). A higher $\kappa_S$ reduces $\lambda^K$, the household’s loan share out of total savings. This in turn, raises the loan rate $R^K$, and reduces capital (as firms face a higher marginal cost), output, and consumption while raising the average marginal propensity to consume (MPC). The credit spreads increase, with a higher $R^K$ bringing up the government’s effective bond return $R^G$ and the entire yield curve, which results in a higher debt-to-GDP ratio.

### 3.4 Short-Run Analysis

#### 3.4.1 Log-linearization

In this section, we present the solution to a first-order log-approximation of our model. We use lower-case letters to denote the logarithm of normalized variables, while hats correspond to deviations from the respective steady-state. We present the most interesting equilibrium

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$^{46}$Laubach (2009) empirically found that a 1% point increase in the projected debt-to-GDP ratio is estimated to raise long-term interest rates by roughly 3-4 basis points.

$^{47}$We normalize non-stationary variables in order to account for trend population and technological growth, for example

$$k_t \equiv \log \left( \frac{K_t}{A_{t-1}N_{t-1}} \right), \quad y_t \equiv \log \left( \frac{Y_t}{A_tN_t} \right), \quad c_t \equiv \log \left( \frac{C_t}{A_tN_t} \right), \quad n_t \equiv \log \left( \frac{N_t}{N_t} \right), \quad \rho^K_t \equiv \log \left( \frac{P^K_t}{P_t} \right).$$

where we divide $K_t$ by $A_{t-1}N_{t-1}$ because it is a variable determined at quarter $t-1$. 

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equations in this section, and relegate the detailed derivation of the model’s solution to Derivation and Proofs for 3.

Linearizing the Euler-equation (equation (3.2.9b)) yields the usual dynamic IS equation

\[
\hat{c}_t = \mathbb{E}_t \left[ \hat{c}_{t+1} - \left( \hat{r}_t^S - \hat{r}_{t+1} \right) \right], \tag{3.4.1}
\]

where \(\hat{r}_t^S\) is the household’s effective savings rate that can be derived from equation (3.2.7) as

\[
\hat{r}_t^S = \lambda^K \left( R^K - R^{HB} \right) + \frac{(1 - \lambda^K) R^{HB}}{R^K} \hat{r}_t^H + \frac{\lambda^K R^K}{R^K} \hat{r}_t^K.
\]

Deviations in \(\hat{r}_t^S\) depend on the household’s aggregate bond portfolio return \(\hat{r}_t^{HB}\) and the loan rate \(\hat{r}_t^K\), both weighted by their relative contribution to the aggregate savings rate. The third term, \(\lambda^K\), captures movements in \(\hat{r}_t^S\) arising from a portfolio reallocation between bonds and loans, and is weighted by the credit spread between both asset classes. A linearization of equation (3.2.5) bring the following expression for the aggregate bond portfolio return,

\[
\hat{r}_t^{HB} = \sum_{f=1}^{F} \frac{\lambda^{HB,f} R^{f-1}}{R^{HB}} \left[ \hat{\lambda}^{HB,f}_{t-1} - (f - 1) \cdot \hat{y}_{t-1}^{f} + f \cdot \hat{y}_{t-1}^{f} \right],
\]

where \(\hat{r}_t^{HB}\) is affected by movements of the current and past yields \(\{\hat{y}_{t-1}^{f}, \hat{y}_{t-1}^{f-1}\}_{f=1}^{F}\), which capture the effects of price revaluation on \(f\)-maturity bond returns. The term \(\hat{\lambda}^{HB,f}_{t-1}\) captures the effect of portfolio rebalancing across maturities on the aggregate bond return.

Portfolio reallocation across asset classes or maturities is an important determinant of the effective savings rate, \(\hat{r}_t^S\), as seen above. Linearizing equation (3.2.6), we obtain an intuitive expression for the share of savings \(\hat{\lambda}_t^K\) allocated to the capital loan market as

\[
\hat{\lambda}_t^K = \kappa^S \left( 1 - \lambda^K \right) \left( \hat{z}_t^K + \mathbb{E}_t \left[ \hat{r}_{t+1}^H - \hat{r}_{t+1}^{HB} \right] \right), \tag{3.4.2}
\]

where \(\hat{\lambda}_t^K\) is positively related to the household’s preference for firm loans, \(\hat{z}_t^K\), and to the credit spread between the loan rate and the aggregate return on the bond portfolio. Note that more segmented markets (lower \(\kappa_S\)) will feature a milder portfolio relocation in response to movements in the credit spread. Equation equation (3.4.2) also highlights a new monetary policy transmission channel which is separate form the traditional intertemporal substitution channel (equation (3.4.1)). By indirectly manipulating the credit spread of the economy through the aggregate bond portfolio return \(\hat{r}_{t+1}^{HB}\), central banks can shift the flow of savings towards capital loans, which relaxes the firm’s cash-in-advance constraint on capital utilization and increases aggregate employment, output and consumption.
We provide a complete list of the linearized equilibrium conditions of our model in Appendix Summary of Conventional Policy Linearized Equations (conventional policy case) and Appendix Summary of YCC Policy Linearized Equations (YCC policy case).

### 3.4.2 Welfare

Following Coibion et al. (2012), in Welfare we provide a detailed derivation of the second-order approximation to the household’s welfare loss around the efficient steady state with positive trend inflation ($\Pi > 1$). We summarize our results in the following Proposition:

**Proposition 11.** A 2\textsuperscript{nd}-order approximation to the expected per-period welfare loss of the household is given by

$$
\mathbb{E} U_t - \bar{U}^F = \Omega_0 + \Omega_n \text{Var}(\hat{h}_t) + \Omega_\pi \text{Var}(\hat{\pi}_t) + t.i.p + h.o.t,
$$

where $\Omega_0$, $\Omega_n$, and $\Omega_\pi$ are coefficients defined in equation (C.3.40), equation (C.3.41), and equation (C.3.42), and $\bar{U}^F$ is the efficient (flexible-price with transfers) steady-state utility of the household.

### 3.4.3 Results

**Impulse-Response without the ZLB**

First, we present impulse-responses to various shocks when the ZLB does not bind. We consider: $z_1^t$ and $z_K^t$ (household preferences for bond and loan investments, respectively), $\epsilon^A_t$ (technology growth shock), $\epsilon^{YD}_t$ (short-term rate policy shock), and $\epsilon^T_t$ (fiscal shock).

**Bond preference shock, $z_1^t$:** Figure 3.3 presents an impulse-response to a $z_1^t$ shock, which drives the household’s portfolio demand for the shortest maturity bond. More broadly, a positive $z_1^t$ shock can be interpreted as a flight-to-safety (or liquidity) shock, as the shortest-maturity bond (federal funds market) usually features the highest degrees of safety and liquidity. The blue and red lines depict the responses under conventional and YCC monetary policy rules, respectively.

With conventional policy, a hike in $z_1^t$ increases the household’s portfolio demand for short maturity bonds and reduces short rates, which eventually leads the returns on other maturities and capital loans, as well wage, to fall as the household re-optimizes its portfolio choices and
Figure 3.3: Impulse response to $z^1_t$ shock
firms substitute between capital and labor. Output falls as labor supply decreases together with wages, which results in a lower inflation rate. Monetary policy reaction is stabilizing and boosts aggregate demand (from consumption and investment), but is not sufficient to prevent a fall in output, which decreases by 4% in response to a one standard-deviation shock to $z_t^1$.

YCC policy is very effective at insulating the economy from a $z_t^1$ shock. The reason is simple: a $z_t^1$ shock modifies the household’s portfolio preferences and asymmetrically affects the returns and household demand for other assets due to market segmentation. Under YCC policy, the central bank can easily accommodate fluctuations in household’s bond demand by modifying the composition of its own portfolio, which leaves the bond yield curve virtually unchanged. The central bank cannot directly affect the supply of loans to firms, and hence a relatively stronger preference for short-term bonds leads to an increase in capital loan rates. Nonetheless, the effect is quantitatively small and output, wages and inflation remain virtually unchanged.

**Fiscal shock, $\varepsilon_T^T$:** Figure 3.4 presents an impulse-response to a temporary shock to government tax revenues, $\varepsilon_T^T$. Notice that in traditional models, such shock has no impact on the economy due to Ricardian equivalence between taxation and bond issuance. With the inclusion of market segmentation, the relative supply of assets has real economic implications in our model, and temporary movements in taxation break Ricardian equivalence by altering the relative government’s bond issuance vis-à-vis existing capital loan supply.

Under conventional policy, a positive $\varepsilon_T^T$ shock results in lower issuance of risk-less government bonds, which places downward pressure on their returns. This leads to a decrease in the return on capital loans, wages and inflation as a result of the household’s endogenous portfolio reshuffling and firms’ substitution across inputs. Monetary policy response to inflation further decreases short-term rates, which boosts aggregate demand and raises output.

A YCC rule achieves similar results to conventional policy, and by lowering rates along the entire yield curve, it reduces the fluctuations in the short-term rate necessary to achieve a reduction of the effective savings rate of households.

**Other shocks:** Figure C.11 depicts an impulse response to a capital loan preference shock, $z^K_t$: a positive jump in $z^K_t$ induces the household to issue more loans to intermediate firms, raising aggregate capital and pushing down its return. Output and inflation also increase.

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48 The representative household’s endogenous portfolio choice (as a function of relative rates) is crucial in generating this phenomenon. For example, if the household’s portfolio weight is fixed as in Ray (2019), the loan rate would rise in response to a positive $z_t^1$ shock that reduces the household’s loan issuance supply.

49 Under this particular calibration, monetary policy is strong enough to actually offset the negative effects of a shortage in bonds supply on output.
Figure 3.4: Impulse response to $\varepsilon^T_t$ shock
and monetary policy tightens in response. YCC policy is more effective in terms of economic stabilization, but displays a similar response pattern to conventional interventions.

Figure C.12 presents the impulse response to a technology growth shock, $\epsilon_A^t$: a positive jump in $GA_t$ generates similar effects to those found by prior literature,\(^{50}\) where output rises while inflation falls. Higher productivity brings up the return on capital, and reduces the capital stock in the short-run due to the firms becoming more efficient in the use of inputs. In the long-run, the capital stock returns to the steady state as firms adjust their prices and consumption demand expands. With YCC policy, output (which is normalized by trend technology) falls on impact: as inflation falls, the bond yield curve shifts downward and capital returns and wage decrease compared to conventional policy, which leads the household to reduce its labor supply. However, actual output (not normalized by technology) increases in response to a $GA_t$ as one would expect.

Figure C.13 presents the impulse response to a monetary policy shock, $\epsilon_Y^t$: a contractionary policy shock brings down consumption demand, which in turn reduces firms’ capital and labor demand. Both output, inflation, capital returns and wages fall in response. On the other hand, YCC policy almost perfectly insulates the economy from the monetary policy shock. Following a shock, the central bank shifts up the entire yield curve, and prevents input prices (capital return and wage), inflation and output from falling.

**Impulse-Response at the ZLB**

Now, we consider the impulse-response to the previous shocks when the short-term rate is constrained at the ZLB. We bring the economy to the ZLB by increasing the size of the shocks appropriately until it binds. For graphical representation purposes, we consider very large shocks, which might be very unlikely to happen otherwise.

**Bond preference shock, $z^t$:** Figure 3.5 presents the impulse-response to a preference shock for the short-maturity bond, $z^t$. Blue lines depict the response under conventional monetary policy, and red lines represent the response under the YCC regime. The impulse-responses are similar to those of Figure 3.3 (the case without ZLB), except for the temporary ZLB constraint observed a few quarters.

YCC regime achieves almost perfect stabilization. However, note that it generates a longer ZLB duration than conventional policy: under when the economy enters a ZLB episode under YCC policy, the central bank increases its purchase of long-term bonds, thus reducing its yields. This action imposes additional downward pressure on short yields and capital

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\(^{50}\) For the effects of technology shock in New-Keynesian models, see Ireland (2004).
loan returns due to the household’s endogenous portfolio reallocation,\textsuperscript{51} which prolongs the duration of the ZLB. Note that the YCC regime also insulates the economy more effectively from adverse shocks, and in theory, it can lift the economy off the ZLB sooner. Nonetheless, under our model calibration, this effect is weaker compared to the endogenous portfolio effect described above.

Note that, while the YCC regime helps to insulate the economy from various shocks (which is good from a welfare perspective, as we shall see in next section), it also generates prolonged ZLB episodes, thus making the economy more reliant on YCC policies as they are the most effective at the ZLB. This generates a reinforcing feedback loop in which central banks become ‘addicted’ to the use of unconventional monetary policy tools.\textsuperscript{52}

**Capital loan preference shock,** $z^K_t$: Figure C.14 represents an impulse-response to a preference shock for capital loans, $z^K_t$. A negative shock to $z^K_t$ induces households to issue less loans to intermediate firms and invest more in the bond markets. Bond rates fall and the short-term rate becomes constrained by the ZLB. Output, capital, inflation, and capital loan returns jump down in response. In contrast, YCC policy is very effective at stabilizing the economy, but at the expense of longer ZLB spell.

**Fiscal shock,** $\varepsilon^T_t$: Figure 3.6 presents an impulse-response to a positive $\varepsilon^T_t$ shock that raises government’s tax revenues. In contrast to Figure 3.4 (non-ZLB case), the economy experiences a recession with conventional policy: with a high enough tax increase shock, the government significantly reduces its bond issuance, dragging the economy into a ZLB recession. Output, capital, inflation, and capital loan returns all drop. This experiment emphasizes stabilizing role of the supply of safe bond at the ZLB, as pointed out by prior literature as Caballero and Farhi (2017) and Caballero et al. (2021) in the global economy context. With YCC policy, the central bank shifts down the entire yield curve and lowers the household’s effective savings rate. This action boosts aggregate demand, and thus output and capital stock. As a result, inflation and capital returns fall less than in the conventional policy case. Note also that in this case, the YCC regime generates a longer ZLB episode than conventional policy due to the same portfolio re-balancing effects previously explained.

\textsuperscript{51}For example, as capital loan return falls, it also reduces wages through the firms’ substitution across inputs, which in turn reduces inflation falls and increases the likelihood of a binding ZLB.

\textsuperscript{52}Karadi and Nakov (2021) model an environment where quantitative easing (QE) policies are effective in fighting against financial disruptions in the banking sector, but at the cost of encouraging banks’ ‘addiction’ to QE. Our framework emphasizes a similar phenomenon, in which the YCC policy generates longer ZLB episodes, which makes the economy more dependent on YCC policy’s ammunition power in insulating the economy from the adverse effects of shocks.
Figure 3.5: Impulse response to $z_t^1$ shock with ZLB
Figure 3.6: Impulse response to $\epsilon_t$ shock with ZLB
Policy Comparison

In this section, we compare alternative monetary policies using the welfare criterion characterized in Proposition 11. We consider three distinct regimes: (i) conventional policy (equations (3.2.18a–c)), (ii) yield-curve control (YCC) policy (equations (3.2.19a–c)), and (iii) mixed policy, where the central bank implements an YCC rule when the economy is trapped at the ZLB, and follows a conventional short-term rate policy otherwise. We simulate the alternative policy regimes following the method described by Carreras et al. (2016) to accommodate an occasionally binding ZLB constraint. Table 3.1 reports summary statistics from this exercise.

Table 3.1 findings can be summarized as: (i) compared with conventional policy, YCC and mixed regimes improve welfare by around 0.4% per-quarter steady state consumption, (ii) YCC and mixed policy prolong ZLB episodes, with higher ZLB frequency and longer ZLB duration than conventional policy, and (iii) YCC and mixed policy regimes yield fairly similar results in terms of welfare and ZLB behavior.

The YCC regime, by easing long-term yields outside the ZLB, imposes additional downward pressure on short-term rates compared to mixed policy, which accounts for the slightly higher ZLB frequency. Nonetheless, the similarity between YCC and mixed regimes suggests that the fundamental difference with respect to conventional policy does not lie on the creation of new ZLB episodes, but rather on the extension of ZLB spells that would occur under either regimes. Previous literature has noted the non-linear increasing welfare costs of ZLB duration under conventional policy (see Carreras et al. (2016)). This result does not extend to unconventional policies that manipulate the entire yield curve as, unlike conventional policy, they retain the capacity to influence the effective savings rate of households and successfully stimulate the economy. Rather, those regimes present an interesting outcome in the form of ‘addiction’ to unconventional policies: by endogenously extending the duration of ZLB episodes, unconventional policies make themselves even more necessary as the only effective monetary policy tools at the ZLB.

Finally, our framework does not include any welfare costs associated to the size or composition of the central bank’s balance sheet, which has been a topic of contentious public debate as a result of the ballooning Fed and ECB balance sheets following the Great Financial Crisis. Incorporating various economic and political costs of the central bank’s direct manipulation of the yield curve might have interesting policy implications, which we leave to future research.

53 For simplicity, we assume that upon ZLB exit, the central bank adjusts its holdings of long-term bonds to their steady-state levels. For a study of optimal exit strategy from QE policies, see Karadi and Nakov (2021).
54 For example, Karadi and Nakov (2021) introduced a small quadratic efficiency cost to QE as a reduced-form proxy for un-modeled distortions and political costs of maintaining a positive central bank balance sheet.
Table 3.1: Policy comparisons. We define mean and median ZLB duration, respectively, as the sample mean and median duration, measured in quarters, of a ZLB episode. A ZLB episode is defined as continuous and uninterrupted period, measured from start to end date, of a binding ZLB constraint. ZLB frequency is measured as the average number of periods within our sample with an actively binding ZLB constraint. Welfare is defined as the second order welfare loss described in Proposition 11, and measured in percents per-quarter of steady state consumption.

3.5 Conclusion

This paper develops a New-Keynesian model that incorporates the term-structure of financial markets and an active role for government and central bank’s balance sheet size and maturity structure. We show that market segmentation across assets and maturities and the household’s endogenous portfolio reallocation are two necessary elements for understanding of the effects of unconventional monetary interventions. For that purpose, we show how standard techniques from the international trade literature (see Eaton and Kortum (2002)) can be employed in the macroeconomics literature to parsimoniously accommodate market segmentation arising from differences in asset return expectations. Our economy, even after log-linearization, features an equilibrium term-structure that deviates from the so-called expectation hypothesis, and which allows unconventional monetary policies such as LSAPs to affect the yield curve and thereby, have some stabilizing powers.

We find that government’s issuance and the central bank’s purchase of different bond maturities act as two major determinants of the yield curve level and slope, and government’s issuance of risk-less bonds stimulates the economy when conventional monetary policy is constrained by the ZLB, as documented by previous works on the so-called ‘safe-asset shortage problems’. We also study different policy regimes, and reveal that yield-curve control (YCC) interventions where the central bank actively manipulates the entire yield curve are more stabilizing than conventional policy both in normal times and during ZLB episodes. However, our YCC policy poses interesting side-effects, as it raises frequency and duration of ZLB episodes. This result comes from the portfolio balancing channel: the central bank’s active easing of long-term rates imposes additional downward pressure on short-term rates by inducing households to endogenously rebalance their portfolios. Therefore, unconventional
policies are addictive: central banks resort to them as the most powerful tools at the ZLB, but in doing so perpetuate the ZLB conditions that render conventional policy ineffective.

Now that the balance sheet of central banks dramatically expanded in most advanced economies as a result of the unconventional policies that were adopted following the 2007 Great Financial crisis, we believe that our framework will be useful to future research looking into the political economy implications and risks to the taxpayer that originate from an expanded central bank’s balance sheet.

In addition, we aim to extend our framework to the international macro setting and revisit global imbalance issues (e.g., Caballero et al. (2008, 2021)) and the global monetary cycles (e.g., Miranda-Agrippino and Rey (2021)) with endogenous fluctuations in the term-structure of interest rates.
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Appendix A

Derivations and Proofs for Chapter 1

Derivation of equation (1.1.4): From the definition of (nominal) state-price density \( \xi^N_t = e^{-\rho t \frac{1}{C_t} \frac{1}{p_t}} \), we get:

\[
\frac{d\xi^N_t}{\xi^N_t} = -\rho dt - \frac{dC_t}{C_t} - \frac{dp_t}{p_t} + \left( \frac{dC_t}{C_t} \right)^2 + \left( \frac{dp_t}{p_t} \right)^2 + \frac{dC_t dp_t}{C_t p_t}.
\] (A.0.1)

Since we have a perfectly rigid price \((p_t = \bar{p} \text{ for } \forall t)\), the above expression becomes:

\[
\frac{d\xi^N_t}{\xi^N_t} = -\rho dt - \frac{dC_t}{C_t} + \left( \frac{dC_t}{C_t} \right)^2
\] (A.0.2)

\[
= -\rho dt - \frac{dC_t}{C_t} + \text{Var}_t \left( \frac{dC_t}{C_t} \right).
\] (A.0.3)

Plugging equation (B.3.2) into equation (C.1.2), we get the following equation (1.1.4).

\[
\mathbb{E}_t \left( \frac{dC_t}{C_t} \right) = (i_t - \rho) dt + \text{Var}_t \left( \frac{dC_t}{C_t} \right).
\] (A.0.4)

Derivation of equation (1.1.8): From equation (1.1.7), we obtain

\[
d\ln Y_t = \left( i_t - \rho + \frac{1}{2}(\sigma_t + \sigma_t^2) \right) dt + (\sigma_t + \sigma_t^2) dZ_t.
\] (A.0.5)

From equation (1.1.5), we obtain

\[
d\ln Y^n_t = \left( r^n_t - \rho + \frac{1}{2}(\sigma_t)^2 \right) dt + \sigma_t dZ_t.
\] (A.0.6)

Therefore, by subtracting equation (B.3.6) from equation (B.3.5), we obtain

\[
d\hat{Y}_t = \left( i_t - \left( r^n_t - \frac{1}{2}(\sigma_t + \sigma_t^2)^2 + \frac{1}{2}(\sigma_t)^2 \right) \right) dt + \sigma_t^2 dZ_t.
\] (A.0.7)
which is equation (1.1.8).

**Proof of Proposition 1.** From equation (1.2.5), \( \{\sigma_t^s\} \) process can be written as

\[
\frac{\sigma_t^s}{\sigma_t + \sigma_t^s} dt - \phi_y \frac{\sigma_t^s}{\sigma_t + \sigma_t^s} dZ_t. \tag{A.0.8}
\]

Using Ito’s lemma, we get the process for \( (\sigma + \sigma_t^s)^2 \) which is a martingale, as

\[
d(\sigma_t + \sigma_t^s)^2 = 2(\sigma_t + \sigma_t^s) d\sigma_t^s + (d\sigma_t^s)^2
\]

\[
= 2(\sigma_t + \sigma_t^s) \left( -\frac{(\phi_y)^2 (\sigma_t^s)^2}{2(\sigma_t + \sigma_t^s)^3} dt - \phi_y \frac{\sigma_t^s}{\sigma_t + \sigma_t^s} dZ_t \right) + (\phi_y)^2 \frac{(\sigma_t^s)^2}{(\sigma_t + \sigma_t^s)^2} dt
\]

\[
= -2\phi_y (\sigma_t^s) dZ_t. \tag{A.0.9}
\]

Therefore, we have \( \mathbb{E}_0((\sigma + \sigma_t^s)^2) = (\sigma + \sigma_0^s)^2 \). By Doob’s martingale convergence theorem (as \( (\sigma + \sigma_t^s)^2 \geq 0, \forall t \)), we know \( \sigma_t^s \xrightarrow{a.s.} \sigma_\infty^s = 0 \) since:

\[
d\sigma_t^s = -\frac{(\phi_y)^2 (\sigma_t^s)^2}{2(\sigma_t + \sigma_t^s)^3} dt - \phi_y \frac{\sigma_t^s}{\sigma_t + \sigma_t^s} dZ_t. \tag{A.0.10}
\]

Thus, equation (B.3.10) proves \( \sigma_t^s \xrightarrow{a.s.} \sigma_\infty^s = 0 \). From equation (1.2.4) \( \sigma_t^s \xrightarrow{a.s.} \sigma_\infty^s = 0 \) leads to \( \hat{Y}_t \xrightarrow{a.s.} 0 \). Finally, we must have \( \mathbb{E}_0(\max_t (\sigma_t^s)^2) = \infty \), since otherwise the uniform integrability implies \( \mathbb{E}_0((\sigma + \sigma_\infty^s)^2) = (\sigma + \sigma_0^s)^2 \), which is a contradiction to our earlier result \( \sigma_t^s \xrightarrow{a.s.} 0 \) since \( \sigma_\infty^s = 0 \) and \( \sigma_0^s > 0 \) by assumption in Proposition 1.

\( \square \)
Appendix B

Appendices to Chapter 2

B.1 Suggestive Evidence

Stock market volatility is commonly viewed in the literature as a proxy of financial and economic uncertainty, which Bloom (2009) and later Gilchrist and Zakrajšek (2012), Bachmann et al. (2013), Jurado et al. (2015), Caldara et al. (2016), Baker et al. (2020), Coibion et al. (2021) further studied as a driving force behind business cycles fluctuations. In this Section, we will evaluate these claims and present interesting empirical results. Figure B.3 provides the first piece of supportive evidence in that direction. Panel B.3a depicts several variables commonly used in the literature to measure financial uncertainty. The correlation between series is remarkably high and they all display positive spikes at the beginning and/or initial months following an NBER-dated recession, which is consistent with the evidence that many of these episodes were financial in nature.\(^1\) Panel B.3b plots Ludvigson et al. (2015) (henceforth, LMN) financial and real (i.e. non-financial) uncertainty series. These variables are positively correlated and display a similar propensity to increase around recessions, though a different type of crisis (e.g. financial or not) is correlated with a different type of uncertainty playing the dominant role. For example, the massive spike in real vis-à-vis financial uncertainty following the recent Covid-19 recession, which initially was a health crisis that spilled into the real economy, can be observed in Panel B.3b.

The patterns displayed in Figure B.3 do not yet constitute a proof of the importance of financial uncertainty as a driver of the business cycle, as we still should worry about the possibility of reverse causation running from unfavorable economic conditions towards uncertainty. We tackle this issue by proposing a simple Vector Autoregression (VAR) with the structural identification strategy based on the timing of macroeconomic shocks similar to Bloom (2009). Equation (B.1.1) presents the variables considered and their ordering, with non-financial series first and financial variables last.\(^2\)

---

\(^1\)See Reinhart and Rogoff (2009) and Romer and Romer (2017) for classification of the past recessions. Their analysis showed many recessions had roots in financial markets.

\(^2\)The ordering is also used by Ludvigson et al. (2015), which, using identification strategy based on event constraints, find that the uncertainty of financial markets tends to be an exogenous source of business cycle fluctuations, while real uncertainty tends to be an endogenous response to the business cycle fluctuations. We also have considered alternative specifications and orderings that produced qualitatively similar results (not
Both LMN real and financial uncertainty measures are included to differentiate the effects of financial volatility shocks from the effects from real uncertainty. For similar reasons, we include the S&P-500 index in our VAR to empirically distinguish between shocks affecting the level of financial markets and shocks affecting their volatility. In order to ameliorate possible concerns about the validity of the structural identification strategy, we estimate our VAR using monthly data, where the identification assumptions are more likely to hold. Figure B.1 presents the impulse responses to the orthogonalized financial uncertainty shock. Panel B.1a plots the response of industrial production, which falls by up to 2.5% and displays moderate persistence following a one standard deviation shock to financial uncertainty. Panel B.1b plots the response of the S&P-500 Index, which drops up to 12% within the first four months before gradually recovering. Together, both pictures imply that an increase of financial uncertainty tends to depress both industrial activity and financial markets.

Figure B.1 also features alternative estimates using common financial uncertainty proxies such as Bloom (2009) stock market volatility index and 10-years premium on Baa-rated corporate bonds. The responses are generally more muted, and take the opposite sign in the case of the S&P Index. These results can be explained by the fact that standard proxies contain information unrelated to financial uncertainty that distorts our estimates (see Jurado et al. (2015) for a discussion), and therefore we choose LMN as our preferred financial uncertainty measure. In Appendix B.2, we report additional impulse response estimates. Especially, the Figure B.5 shows that monetary authorities respond with accommodating interest rate movements to financial uncertainty shocks, while real uncertainty has no statistically significant effect on either interest rates or stock market fluctuations. We will further discuss optimal monetary policy response to financial volatility shocks in Section 2.3.

Finally, we can further explore the contribution of financial uncertainty to business cycles fluctuations by looking at Table B.1 in Appendix B.2, which reports the Forecast Error Variance Decomposition (FEVD) of Industrial Production and the S&P-500 Index. Financial uncertainty shocks explain close to 5% of the fluctuations in both series, while real uncertainty explains an additional 2-4% of movements in industrial activity in the medium run. Figure B.2 provides a more graphical illustration of these results by plotting the historical decomposition reported, provided upon request).
of the series. We observe the contribution of financial uncertainty rivals that of shocks to the level of financial variables captured by the S&P-500 shock, and is especially important in driving industrial production boom-bust patterns during and in the preceding months of recessionary episodes, as it can be seen during the Global Financial Crisis (2007).

In this Appendix B.1, we have revisited the empirical evidence on financial market volatility and shown that it acts as a major driving force of the business cycle.
B.2 Additional Figures and Tables

Figure B.1: Impulse Response Functions (IRFs), selected series. Figures B.1a and B.1b display the response to a one standard deviation financial uncertainty shock of monthly (log) Industrial Production and (log) S&P-500 Index series, respectively, using a VAR-11 with equation (B.1.1) variable composition and ordering. Shaded area indicates 95% confidence interval around preferred financial uncertainty measure computed using standard bootstrap techniques.

Figure B.2: Historical Decomposition, selected series. Figures B.2a and B.2b display historical decomposition of monthly Industrial Production and S&P-500 Index series, respectively, based on the VAR-11 with equation (B.1.1) variable composition and ordering. Shaded areas indicate NBER dated recessions (peak trough the through). Variables of interest are detrended by subtracting the contribution of initial conditions and constant terms after series decomposition. Columns report a contribution of each shock to the fluctuations around trend of the variable considered.
Figure B.3: Uncertainty series. Figure B.3a displays common measures of financial uncertainty. Figure B.3b displays Ludvigson et al. (2015) (henceforth, LMN) measures of financial and real economic uncertainty. Shaded areas indicate NBER dated recessions (peak trough the through). LMN financial and real economic uncertainty series are constructed as the average volatility of the residuals from predictive regressions on financial and real economic variables, respectively (See Ludvigson et al. (2015) for the series construction). Bloom (2009)’s stock market volatility variable is constructed using VXO data from 1987 onward and the monthly volatility of the S&P 500 index normalized to the same mean and variance in the overlapping interval for the 1960-1987 period (See Bloom (2009) for the series construction). The bond risk-premia series is the Moody’s seasoned Baa corporate bond yield relative to the yield on a 10-year treasury bond at constant maturity. For graphical comparison purposes, the depicted series have a normalized zero mean and one standard deviation.
(i) Industrial Production

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(ii) S&P-500 Index

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(iii) Fed Funds Rate

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</table>

Table B.1: Forecast Error Variance Decomposition (FEVD). The table presents the variance contribution (in percentage) of financial and real uncertainty shocks to selected series at different time horizons (in months). The FEVD is constructed using a VAR-11 with equation (B.1.1) variable composition and ordering. The first two columns report the contribution of LMN financial and real uncertainty shocks, respectively. The last two columns report alternative VAR specifications where the preferred LMN financial uncertainty measure (column one) is replaced by common proxies employed in the literature, either Bloom (2009) stock market volatility measure or the Baa 10-years corporate bond premia, respectively.
Figure B.4: Impulse Response Functions (IRFs), selected series. Figures B.4a and B.4b display the response to one standard deviation real uncertainty shock by monthly (log) Industrial Production and (log) S&P-500 Index series, respectively, using a VAR-11 with equation (B.1.1) variable composition and ordering. Shaded area indicates 95% confidence interval around preferred financial uncertainty measure computed using standard bootstrap techniques.
Figure B.5: Impulse Response Functions (IRFs), Fed Funds Rate. This Figure displays the response to a one standard deviation uncertainty (financial or real) shock by monthly Fed Funds Rate series, using a VAR-11 with equation (B.1.1) variable composition and ordering. Panel B.5a plots the response to a financial uncertainty shock, and Panel B.5b to a real uncertainty shock. Shaded area indicates 95% confidence interval around preferred financial/real uncertainty measure computed using standard bootstrap techniques. Additional lines display alternative impulse responses obtained by substituting preferred LMN financial uncertainty measure with common proxies employed in the literature.
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<th>Parameter</th>
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<th>Description</th>
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</tr>
<tr>
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<td>Inverse Frisch labor supply elasticity</td>
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<tr>
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<td>Subjective time discount factor</td>
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<tr>
<td>$\sigma$</td>
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<td>$g$</td>
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<td>$\epsilon$</td>
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<td>Policy rule output gap response</td>
</tr>
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<td>$\phi_{rp}$</td>
<td>0</td>
<td>Policy rule risk premium response</td>
</tr>
<tr>
<td>$\bar{\pi}$</td>
<td>0</td>
<td>Steady state trend inflation target</td>
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</table>

Table B.2: The table presents the baseline parameter calibration used in Sections 2.3 and 2.4.3 of the paper.
B.3 Derivations and Proofs for Chapter 2

B.3.1 Section 1.1

Derivation of equation (1.1.4): From the definition of (nominal) state-price density \( \xi_t^N = e^{-\rho t \frac{1}{C_t} p_t} \), we get:

\[
\frac{d \xi_t^N}{\xi_t^N} = -\rho dt - \frac{dC_t}{C_t} - \frac{dp_t}{p_t} + \left( \frac{dC_t}{C_t} \right)^2 + \frac{dC_t}{C_t} \frac{dp_t}{p_t}.
\] (B.3.1)

Since we have a perfectly rigid price \((p_t = \bar{p} \text{ for } \forall t)\), the above expression becomes:

\[
\frac{d \xi_t^N}{\xi_t^N} = -\rho dt - \frac{dC_t}{C_t} + \text{Var}_{t} \left( \frac{dC_t}{C_t} \right). 
\] (B.3.2)

\[
\frac{d \xi_t^N}{\xi_t^N} = -\rho dt - \frac{dC_t}{C_t} + \text{Var}_{t} \left( \frac{dC_t}{C_t} \right). 
\] (B.3.3)

Plugging equation (B.3.2) into equation (C.1.2), we get the following equation (1.1.4).

\[
\mathbb{E}_t \left( \frac{dC_t}{C_t} \right) = (i_t - \rho) dt + \text{Var}_{t} \left( \frac{dC_t}{C_t} \right). 
\] (B.3.4)

Derivation of equation (1.1.8): From equation (1.1.7), we obtain

\[
d \ln Y_t = \left( i_t - \rho + \frac{1}{2} \left( \sigma_t + \sigma_t^z \right)^2 \right) dt + (\sigma_t + \sigma_t^z) dZ_t. 
\] (B.3.5)

From equation (1.1.5), we obtain

\[
d \ln Y''_t = \left( r^n_t - \rho + \frac{1}{2} (\sigma_t)^2 \right) dt + \sigma_t dZ_t. 
\] (B.3.6)

Therefore, by subtracting equation (B.3.6) from equation (B.3.5), we obtain

\[
d \hat{Y}_t = \left( i_t - \left( r^n_t - \frac{1}{2} (\sigma_t + \sigma_t^z)^2 + \frac{1}{2} (\sigma_t)^2 \right) \right) dt + \sigma_t^z dZ_t, 
\] (B.3.7)

which is equation (1.1.8).

Proof of Proposition 1. From equation (1.2.5), \( \{\sigma_t^z\} \) process can be written as

\[
d \sigma_t^z = -\left( \phi_y \right)^2 \frac{(\sigma_t^z)^2}{2(\sigma_t + \sigma_t^z)^3} dt - \phi_y \frac{\sigma_t^z}{\sigma_t + \sigma_t^z} dZ_t. 
\] (B.3.8)

Using Ito’s lemma, we get the process for \((\sigma + \sigma_t^z)^2\) which is a martingale, as
\[
d(\sigma_t + \sigma_t^s)^2 = 2(\sigma_t + \sigma_t^s)d\sigma_t^s + (d\sigma_t^s)^2
\]
\[
= 2(\sigma_t + \sigma_t^s)\left(-\frac{(\phi_y)^2(\sigma_t^s)^2}{2(\sigma_t + \sigma_t^s)^3}dt - \phi_y\frac{\sigma_t^s}{\sigma_t + \sigma_t^s}dZ_t\right) + (\phi_y)^2\frac{(\sigma_t^s)^2}{(\sigma_t + \sigma_t^s)^2}dt
\]
\[
= -2\phi_y(\sigma_t^s)dZ_t. \tag{B.3.9}
\]

Therefore, we have \( \mathbb{E}_0((\sigma + \sigma_t^s)^2) = (\sigma + \sigma_0^s)^2 \). By Doob’s martingale convergence theorem (as \( (\sigma + \sigma_t^s)^2 \geq 0, \forall t \)), we know \( \sigma_t^s \xrightarrow{a.s.} \sigma_\infty^s = 0 \) since:

\[
d\sigma_t^s = -\frac{(\phi_y)^2(\sigma_t^s)^2}{2(\sigma_t + \sigma_t^s)^3}dt - \phi_y\frac{\sigma_t^s}{\sigma_t + \sigma_t^s}dZ_t. \tag{B.3.10}
\]

Thus equation (B.3.10) proves \( \sigma_t^s \xrightarrow{a.s.} \sigma_\infty^s = 0 \). From equation (1.2.4) \( \sigma_t^s \xrightarrow{a.s.} \sigma_\infty^s = 0 \) leads to \( \dot{Y}_t \xrightarrow{a.s.} 0 \). Finally, we must have \( \mathbb{E}_0(\max_t(\sigma_t^s)^2) = \infty \), since otherwise the uniform integrability says \( \mathbb{E}_0((\sigma + \sigma_\infty^s)^2) = (\sigma + \sigma_0^s)^2 \), which is a contradiction to our earlier result \( \sigma_t^s \xrightarrow{a.s.} 0 \) since \( \sigma_\infty^s = 0 \) and \( \sigma_0^s > 0 \) by assumption in Proposition 1.

\[
\square
\]

### B.3.2 Section 2.2

#### Section 2.2.1

Here we solve the optimization problems of workers (equation (2.2.5)) and capitalists (equation (2.2.10)).

**Worker’s Optimization** : Workers solve the following optimization problem in equation (2.2.5).

\[
\max_{C_{W,t},N_{W,t}} \left( \frac{C_{W,t}}{A_t} \right)^{1-\varphi} \left( \frac{C_{W,t}}{1-\varphi} \right) - \left( \frac{N_{W,t}}{1+\chi_0} \right)^{1+\chi_0} \quad \text{s.t.} \quad p_tC_{W,t} = w_tN_{W,t}. \tag{B.3.11}
\]

If we let \( \lambda_tA_t^{\varphi-1} \) be the multiplier on the budget constraint, then solution is easy to compute as follows.

\[
C_{W,t}^\varphi = \lambda_t p_t, \quad A_t^{1-\varphi}(N_{W,t})^{\chi_0} = \lambda_t N_{W,t} = \frac{w_t}{p_t}C_{W,t}^\varphi = \left( \frac{w_t}{p_t} \right)^{1-\varphi}N_{W,t}^{-\varphi},
\]

\[
\therefore N_{W,t} = \left( \frac{w_t}{p_t} \right)^{\chi_0+\varphi} \left( \frac{p_t}{w_t} \right)^{-\frac{1}{1-\varphi}} \left( \frac{1}{A_t^\varphi} \right) \text{ with } \chi_0 + \varphi = 1 - \varphi, \quad C_{W,t} = \frac{w_t}{p_t}N_{W,t} = \left( \frac{w_t}{p_t} \right)^{\frac{1}{1-\varphi}} \left( \frac{1}{A_t^\varphi} \right). \tag{B.3.12}
\]

**Capitalist’s Optimization** : Each capitalist with wealth \( a_t \) solves the following optimization
in equation (2.2.10).

\[
\max_{C_t, \omega_t} \mathbb{E}_0 \int_0^\infty e^{-rt} \log C_t \, dt \quad \text{s.t.} \quad da_t = (a_t(i_t + \theta_t(i^m_t - i_t)) - p_t C_t) \, dt + \theta_t a_t(\sigma_t^q + \sigma_t^p) dZ_t.
\]

Putting all the state variables \((i_t, p_t, i^m_t, \sigma_t, \sigma_t^q, \sigma_t^p)\) into the vector \(S_t\), then Hamilton-Jacobi-Bellman (HJB) equation can be written in the following way.

\[
\rho V(a_t, S_t, t) = \max_{C_t, \theta_t} \log C_t + \frac{\partial V}{\partial a_t}(a_t(i_t + \theta_t(i^m_t - i_t)) - p_t C_t) + \frac{1}{2} \theta_t^2 a_t^2(\sigma_t^q + \sigma_t^p)^2 + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S_t} \mathbb{E}_t(\frac{dS_t}{dt}) + \frac{1}{2} \text{Tr} \left( \frac{\partial^2 V}{\partial S_t \partial S_t'} dS_t dS_t' \right).
\]

Following Merton (1971), we know the value function has the following form.

\[
V(a_t, S_t, t) = \frac{1}{\rho} \log a_t + f(S_t, t).
\]

The first-order conditions for \(C_t\) and \(\theta_t\) are easy to compute as follows.

\[
p_t C_t = \rho a_t \quad \text{and} \quad \frac{i^m_t - i_t}{\sigma_t + \sigma_t^q + \sigma_t^p} = \theta_t(\sigma_t^q + \sigma_t^p) \quad \text{Price of risk Sharpe ratio}
\]

If we plug the guessed value function form (equation (B.3.15)) into HJB equation, we get the following partial differential equation (PDE) for the function \(f(S_t, t)\), verifying our form in equation (B.3.15) is a reasonable guess.

\[
\rho f(S_t, t) = \log \frac{\rho}{p_t} + \frac{1}{\rho}(i_t + \theta_t(i^m_t - i_t) - \rho) - \frac{1}{2} \theta_t^2 (\sigma_t^q + \sigma_t^p)^2 + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S_t} \mathbb{E}_t(dS_t) + \frac{1}{2} \text{Tr} \left( \frac{\partial^2 f}{\partial S_t \partial S_t'} dS_t dS_t' \right) \quad \text{with} \quad \theta_t = \frac{i^m_t - i_t}{(\sigma_t + \sigma_t^q + \sigma_t^p)^2}.
\]

Thus solving the partial differential equation in equation (B.3.17) restores the functional form \(f(S_t, t)\).

**Section 2.2.2**

We can easily derive the equilibrium condition in equation (2.2.12) by plugging in \(\theta_t = 1\) to equation (B.3.16). \(a_t = p_t A_t Q_t\) holds since all capitalists are identical both ex-ante and ex-post. Now we prove Lemma 1.

**Proof of Lemma 1.** First we start by stating capitalist’s nominal state-price density \(\xi_t^N\) and real state-price density \(\xi_t^r\). Nominal state-price density will be relevant to the nominal interest
rate, while real state-price density matters when we calculate the real interest rate.

\[ \xi_t^N = e^{-\rho t} \frac{1}{C_t^t} \xi_t^r = e^{-\rho t} \frac{1}{C_t^t} = p_t \xi_t^N. \]  

(B.3.18)

If \( \lambda_t \) is price of risk ((\( \sigma_t + \sigma^q_t + \sigma^p_t \)) in this model), the nominal pricing kernel evolves with the following process.

\[ \frac{d\xi_t^N}{\xi_t^N} = -i_t dt - \lambda_t dZ_t, \quad \xi_t^N = \exp \left( - \int_0^t (i_s + \frac{1}{2} \lambda_s^2) ds - \int_0^t \lambda_s dZ_s \right). \]  

(B.3.19)

If we apply Ito’s lemma to the relation \( \xi_t^r = p_t \xi_t^N \) in equation (B.3.18), we get the following process for real pricing kernel \( \xi_t^r \).

\[ \frac{d\xi_t^r}{\xi_t^r} = (\pi_t - i_t - \sigma^p_t \lambda_t) dt - (\sigma + \sigma^q_t) dZ_t. \]  

(B.3.20)

Thus we get the following Fisher identity with the inflation premium in equation (2.2.18).

\[ r_t = i_t - \pi_t + \sigma^p_t (\sigma_t + \sigma^q_t + \sigma^p_t). \]  

(B.3.21)

B.3.3 Section 2.2.3

Here we prove the Proposition 11 based on the results above.

Proof of Proposition 11. We start from the pricing decision of intermediate good firms. Since we have an externality à la Baxter and King (1991), we need to have additional step to aggregate across each firm. Let firm \( i \) take his demand as given and choose the optimal price \( p_t(i) \) at given moment \( t \). With \( E_t \equiv (N_{W,t})^\alpha \), we can get the following conditions for \( \{n_t(i), y_t(i)\} \).

\[ n_t(i) = \left( \frac{y_t(i)}{A_t E_t} \right)^{\frac{1}{1-\alpha}}, \quad y_t(i) = y_t \left( \frac{p_t(i)}{p_t} \right)^{-\varepsilon}. \]  

(B.3.22)

Each firm \( i \) chooses \( p_t \) that maximizes its profit, solving the following optimization.

\[ \max_{p_t(i)} p_t(i) \left( \frac{p_t(i)}{p_t} \right)^{-\varepsilon} y_t - w_t \left( \frac{y_t}{A_t E_t} \right)^{\frac{1}{1-\alpha}} \left( \frac{p_t(i)}{p_t} \right)^{-\varepsilon}. \]  

(B.3.23)

Here all firms charge the same price \( (p_t(i) = p_t \) holds for \( \forall i \)). The solution of equation (B.3.23) combined with this condition yields the following solution. In equilibrium, we
also know \( n(i) = N_{W,t} \) for \( \forall i \).

\[
\frac{w_t^n}{p_t^n} = \left( \frac{\epsilon - 1}{\epsilon} \right) \left( 1 - \alpha \right) y_t^{\frac{1}{\alpha}} (A_t E_t)^{\frac{1}{1 - \alpha}}
\]

\[
= \frac{\epsilon - 1}{\epsilon} \left( 1 - \alpha \right) y_t^{\frac{1}{\alpha}} (A_t)^{\frac{1}{\alpha}} N_{W,t}^{\frac{1}{\alpha}}
\]

\[
= \frac{\epsilon - 1}{\epsilon} \left( 1 - \alpha \right) y_t^{\frac{1}{\alpha}} (A_t)^{\frac{1}{\alpha}} \left( \frac{w_t^n}{p_t^n} \right)^{\frac{1}{\alpha}} A_t^{\frac{1}{\alpha}} \left( \frac{A_t^{1 - \alpha}}{A_t} \right).
\]

Thus we get the following condition for the real wage.

\[
\frac{w_t^n}{p_t^n} = \left( \frac{\epsilon - 1}{\epsilon} \right) \left( 1 - \alpha \right) y_t^{\frac{1}{\alpha}} (A_t)^{\frac{1}{\alpha}} \left( \frac{A_t^{1 - \alpha}}{A_t} \right).
\]

(B.3.25)

And then we know the aggregate production is linear, thus \( y_t = A_t N_{W,t} \) due to the externality. Thus we have:

\[
y_t = A_t N_{W,t} = A_t \left( \frac{w_t^n}{p_t^n} \right)^{\frac{1}{\alpha}} \frac{1}{A_t^\frac{1}{\alpha}}
\]

\[
= A_t \left( \frac{\epsilon - 1}{\epsilon} \right) \left( 1 - \alpha \right) y_t^{\frac{1}{\alpha}} A_t^{\frac{1}{\alpha}}. \quad (B.3.26)
\]

Thus we get the natural level of output \( y_t^n \) and the natural level of real wage \( w_t^n / p_t^n \):

\[
y_t^n = \left( \frac{\epsilon - 1}{\epsilon} \right) \left( 1 - \alpha \right) A_t, \quad \frac{w_t^n}{p_t^n} = \frac{\epsilon - 1}{\epsilon} (1 - \alpha) A_t, \quad (B.3.27)
\]

from which we get the following consumption and labor for workers.

\[
N_{W,t}^n = \left( \frac{\epsilon - 1}{\epsilon} \right) \left( 1 - \alpha \right)^{\frac{1}{\alpha}} A_t, \quad C_{W,t}^n = \left( \frac{\epsilon - 1}{\epsilon} \right)^{1 + \frac{1}{\alpha}} (1 - \alpha) A_t. \quad (B.3.28)
\]

In equilibrium, consumptions of capitalists and workers add up to the amount of final good output. If we plug the real wage in equation (B.3.24) into workers' consumption and the labor supply decision in equation (B.3.12), we get the following good-market equilibrium condition, where we define \( Q_t^n \) to be the natural level of detrended stock price. Also from equation (B.3.16), we see the consumption of capitalists would be \( C_t = \rho A_t Q_t^n \) in equilibrium.

\[
\rho A_t Q_t^n + \left( \frac{\epsilon - 1}{\epsilon} \right) \left( 1 - \alpha \right)^{1 + \frac{1}{\alpha}} A_t = \left( \frac{\epsilon - 1}{\epsilon} \right) \left( 1 - \alpha \right)^{\frac{1}{\alpha}} A_t. \quad (B.3.29)
\]

Thus we get the following expression for \( Q_t^n \) and \( C_t^n \), a natural asset price level and capitalists' consumption in the flexible price equilibrium.

\[
Q_t^n = \frac{1}{\rho} \left( \frac{\epsilon - 1}{\epsilon} \right)^{\frac{1}{\alpha}} \left( 1 - \frac{(\epsilon - 1)(1 - \alpha)}{\epsilon} \right),
\]

\[
C_t^n = A_t \left( \frac{\epsilon - 1}{\epsilon} \right)^{\frac{1}{\alpha}} \left( 1 - \frac{(\epsilon - 1)(1 - \alpha)}{\epsilon} \right). \quad (B.3.30)
\]
Since $Q^n_t$ is constant, there should be no drift and volatility for its process in the flexible price economy, thus we have $\mu^n_q = \sigma^n_q = 0$. To calculate the natural interest rate $r^n_t$, we start from the capital gain component in equation (2.2.17). If we apply Ito’s lemma, we get the following capital gain formula.

$$E \left( d\left( p_t A_t Q_t \right) \right) = \pi_t + \mu^q_t + g + \sigma^q_t \sigma^p_t + \sigma_t (\sigma^p_t + \sigma^q_t).$$  \hspace{1cm} (B.3.31)

As dividend yield is always $\rho$, imposing expectation on both sides of equation (2.2.17) and combining with the equilibrium condition in equation (2.2.12) yield the following relation.

$$E(i^m_t) = \rho + \pi_t + g + \sigma_t \sigma^p_t = i_t + (\sigma_t + \sigma^p_t)^2.$$  \hspace{1cm} (B.3.32)

Using Lemma 1, we finally express natural rate of interest $r^n_t$ as a function of structural parameters and $\sigma_t$, which proves (iii) of Proposition 11.

$$r^n_t = i_t - \pi_t + \sigma^p_t (\sigma_t + \sigma^n_q + \sigma^q_t) = \rho + g - \sigma^2_t.$$  \hspace{1cm} (B.3.33)

For the capitalist’s consumption process in the flexible price case, Since their consumption $C^n_t$ is directly proportional to TFP $A_t$, we know

$$\frac{dC^n_t}{C^n_t} = g dt + \sigma_t dZ_t = \left( r^n_t - \rho + \sigma^2_t \right) dt + \sigma_t dZ_t.$$  \hspace{1cm} (B.3.34)

where we use $r^n_t - \rho + \sigma^2_t = g$ from equation (B.3.33).

**Section 2.2.4**

**Proof of Lemma 2.** First from $C_t = \rho A_t Q_t$, we get $\hat{C}_t = \hat{Q}_t$. We start from the flexible price case’s good market equilibrium condition, where we use equation (B.3.12). Here $\frac{w^n_t}{p^n_t}$ is the real wage level in the flexible price economy.

$$A_t \left( \frac{w^n_t}{p^n_t} \right)^{\frac{1}{x}} \frac{1}{A_t^{\frac{1}{x}}} = \rho A_t Q^n_t + \left( \frac{w^n_t}{p^n_t} \right)^{1 + \frac{1}{x}} \frac{1}{A_t^{\frac{1}{x}}}.$$  \hspace{1cm} (B.3.35)

We subtract equation (B.3.35) from the same good market equilibrium condition in sticky price economy.

$$A_t \left( \left( \frac{w_t}{p_t} \right)^{\frac{1}{x}} - \left( \frac{w^n_t}{p^n_t} \right)^{\frac{1}{x}} \right) \frac{1}{A_t^{\frac{1}{x}}} = C_t - C^n_t + \left( \left( \frac{w_t}{p_t} \right)^{1 + \frac{1}{x}} - \left( \frac{w^n_t}{p^n_t} \right)^{1 + \frac{1}{x}} \right) \frac{1}{A_t^{\frac{1}{x}}}.$$  \hspace{1cm} (B.3.36)

where we divide both sides of equation (B.3.36) by $A_t^{1 - \frac{1}{x}} (\frac{w^n_t}{p^n_t})^{\frac{1}{x}}$ and obtain
\[
\frac{\left(\frac{W_t}{p_t}\right)^{\frac{1}{\chi}} - \left(\frac{W^n_t}{p^n_t}\right)^{\frac{1}{\chi}}}{\left(\frac{W^n_t}{p^n_t}\right)^{\frac{1}{\chi}}} = \frac{C^n_t}{A_t^{1-\frac{1}{\chi}\left(\frac{W^n_t}{p^n_t}\right)^{\frac{1}{\chi}}} = 1 - \frac{(\epsilon - 1)(1 - \alpha)}{\chi} \frac{\hat{w}_t}{p_t} \left(\frac{W^n_t}{p^n_t}\right)^{\frac{1}{\chi}} = \frac{\hat{C}_t}{C^{\alpha}(t)}. \tag{B.3.37}
\]

Thus equation (B.3.37) can be written in the following way, which proves Lemma 2.

\[
\frac{1}{\chi} \frac{\hat{w}_t}{p_t} = (1 - \frac{(\epsilon - 1)(1 - \alpha)}{\epsilon}) \frac{\hat{C}_t}{\epsilon} + \frac{(\epsilon - 1)(1 - \alpha)}{\epsilon} \left(\frac{1 + \frac{1}{\chi}}{\hat{Q}_t}\right) = \hat{C}^{\alpha}(t). \tag{B.3.38}
\]

We finally obtain

\[
\hat{Q}_t = \hat{C}_t = \left(\chi^{-1} - \frac{(\epsilon - 1)(1 - \alpha)}{\epsilon} \frac{1}{1 - \frac{(\epsilon - 1)(1 - \alpha)}{\epsilon}} \frac{\hat{w}_t}{p_t}\right) = \left(\frac{\epsilon}{1 - (\epsilon - 1)(1 - \alpha)} \frac{\chi^{-1} - \frac{1}{\epsilon} \frac{(\epsilon - 1)(1 - \alpha)}{1 + \chi^{-1}}}{\epsilon} \frac{1}{\epsilon} \frac{\hat{w}_t}{p_t}\right) \hat{C}^{\alpha}(t), \tag{B.3.39}
\]

We see that Assumption 1 guarantees that all gaps (asset price, consumptions for both capitalists and workers, employment, and real wage) co-move with positive correlations. Now we can use \(\hat{Q}_t\) and \(\hat{C}_t\) interchangeably, and if one gap variable becomes 0, then all other gap variables become also stabilized and 0.

\[
\text{Proof of Proposition 3.} \hspace{1cm} \text{In sticky price equilibrium, we have } \sigma^n_t \equiv 0, \text{ as over the small time period } dt \text{ a } \delta dt \text{ portion of firms get to change their prices and there is no stochastic change in aggregate price level } p_t. \text{ Thus capitalist’s consumption } C_t \text{ has the following process, where we use the equilibrium condition } i^m_t = i_t + (\sigma + \sigma^n_t)^2.
\]

\[
\frac{dC_t}{C_t} = (i_t^m - \pi_t - \rho) dt + (\sigma_t + \sigma^q_t) dZ_t
\]

\[
= (i_t + (\sigma_t + \sigma^q_t)^2 - \pi_t - \rho) dt + (\sigma_t + \sigma^q_t) dZ_t. \tag{B.3.40}
\]

Thus we have the following two process for \(\ln C_t\) and \(\ln C^n_t\):

\[
d\ln C_t = \left(i_t - \pi_t + \frac{(\sigma_t + \sigma^q_t)^2}{2} - \rho\right) dt + (\sigma_t + \sigma^q_t) dZ_t, \quad \ln C^n_t = \left(r^n_t - \rho + \frac{\sigma^2_t}{2}\right) dt + \sigma_t dZ_t, \tag{B.3.41}
\]

of which the latter is from equation (B.3.34). We get the following \(\hat{C}_t = \hat{Q}_t\) gap from
equation (B.3.41):
\[
\begin{align*}
&\dot{Q}_t = \dot{C}_t = \left( i_t - \pi_t - \left( r_t^n - \frac{(\sigma_t + \sigma^q_t)^2}{2} + \frac{\sigma_t^2}{2} \right) \right) dt + \sigma_t^q dZ_t \\
&= \left( i_t - \pi_t - r_t^T \right) dt + (\sigma_t^q - \sigma_t^n) dZ_t.
\end{align*}
\] (B.3.42)

As we have risk-premium levels \( r_p_t = (\sigma_t + \sigma^q_t)^2 \) in the sticky price case and \( r_p^n = (\sigma_t)^2 \) in the flexible price economy, we can express \( r_t^T \) as
\[
\begin{align*}
&\dot{r}_t^T = r_t^n - \frac{1}{2}(r_p_t - r_p^n) = r_t^n - \frac{1}{2} r_{\hat{p}} t, \\
&\text{where we know that when } \sigma_t^q = \sigma_t^n = 0 \text{ holds, then we have } \dot{r}_{\hat{p}} t = 0 \text{ and } r_t^T = r_t^n.
\end{align*}
\] (B.3.43)

**Proof of Proposition 4.** We assume that firms change their prices with instantaneous probability \( \delta dt \) à la Calvo (1983). If there is price dispersion \( \Delta_t \), as defined in equation (2.2.7), across intermediate goods firms, then labor market equilibrium condition can be written as follows.

\[
N_{W,t} = \int_0^1 n_t(i) di = \left( \frac{y_t}{A_t(N_{W,t})^\alpha} \right)^{\frac{1}{1-\alpha}} \int_0^1 \left( \frac{\rho_t(i)}{p_t} \right)^{-\frac{\epsilon}{1-\alpha}} di, \quad y(t) = \frac{A_t N_{W,t}}{\Delta_t} = C_t + C_{W,t}.
\] (B.3.44)

Plugging equation (B.3.12) and equation (B.3.16) (optimal consumption decisions of workers and capitalists) into equation (B.3.44), we get the following equilibrium condition.

\[
\rho A_t Q_t + A_t \left( \frac{w_t}{p_t A_t} \right)^{\frac{1+\epsilon}{\chi}} = A_t \left( \frac{w_t}{p_t A_t} \right)^{\frac{1}{\chi}} \frac{1}{\Delta_t}.
\] (B.3.45)

Since a price level (nominal side) does not matter for the allocation of resources in the flexible price economy, we can regard \( \hat{x}_t \) to be the log-deviation of \( x_t \) from the constant price flexible price equilibrium value of itself. From price aggregator in equation (2.2.4), we get the log-linearize version easily.

\[
\hat{p}_t^{1-\epsilon} = \int_0^1 p_t(i)^{1-\epsilon} di \quad \text{thus } \hat{p}_t = \int_0^1 \hat{p}_t(i) di.
\] (B.3.46)

To get a sense of price dispersion \( \Delta_t \), we illustrate Woodford (2003)’s treatment of \( \Delta_t \) up to first-order. Up to the first-order we can regard \( \Delta_t \approx 1 \) because \( \Delta_t \) is in nature the second
order variable, as the following relation shows.

\[
\frac{\Delta_t}{1-\alpha} = \frac{1}{1-\alpha} \ln \frac{\Delta_t}{\Delta^n} = \ln \int_0^1 \exp \left[ -\frac{\epsilon}{1-\alpha} (p_t(i) - \hat{p}_t) \right] di
\]

\[
= \ln \int_0^1 \left[ 1 - \frac{\epsilon}{1-\alpha} (p_t(t) - \hat{p}_t) + \frac{1}{2} \left( \frac{\epsilon}{1-\alpha} \right)^2 (p_t(i) - \hat{p}_t)^2 \right] di
\]

\[
\simeq \frac{1}{2} \left( \frac{\epsilon}{1-\alpha} \right)^2 \text{Var}(p_t(i)).
\]  

(Pricing is standard, except that our model is in continuous time. With \(\delta dt\) probability at time \(t\), individual firm can change the price instantaneously from \(t\) to \(t + dt\). From time-0 perspective, a probability that firm can reset its price for the first time at time \(t\) is \(\delta e^{-\delta t} dt = \delta dt \cdot e^{-\delta t}\).)

At time \(t\), price-changing firm \(i\) solves the following optimization to choose \(p_{it}\):

\[
\max \frac{1}{p_{t(i)}} \int_t^\infty \xi^N \frac{p_s}{\hat{p}_t} \left( \frac{p_t(i)}{p_s} y_i(s|t) - \frac{1}{p_s} C(y_i(s|t)) \right) ds,
\]

where  

\[
y_i(s|t) = \left( \frac{p_t(i)}{p_s} \right)^{-1} y_s,
\]

and 

\[
C(\cdot) \text{ is the nominal cost function for each firm. Let } MC_{s|t} \text{ and } \varphi_{s|t} \text{ be the nominal and real marginal cost at time } s \text{ conditional on price resetting at prior time } t. \text{ The nominal pricing kernel has following simple formula due to log-preference of capitalists.}
\]

\[
\xi^N = e^{-\rho_s} \frac{1}{p_s \hat{C}_s}, \quad \frac{\xi^N}{\xi^N_p} = e^{-\rho(s-t)} \frac{\hat{C}_t}{\hat{C}_s}.
\]

Thus optimal adjusted price \(p^*_t(i)\) is given as the following first-order condition. Here \(\varphi_{s|t}\), a real marginal cost of firms at time \(s\) given time \(t\) price resetting, appears, where \(\varphi\) is flexible-price (natural) equilibrium level of real marginal cost, which is \(\epsilon^{-1}\).

\[
\hat{p}_t^* = \left( \frac{\hat{C}_t}{\hat{C}_s} \right) \frac{\varphi_{s|t}}{\hat{p}_s} ds.
\]

If we log-linearize this equation around the steady state equilibrium with the constant price as in equation (B.3.46), we obtain the following log-linearized \(\hat{p}_t^*\) expressed as

\[
\hat{p}_t^* = (\delta + \rho) \hat{p}_t^* \int_t^\infty e^{-\epsilon(s-t)} (\hat{p}_{s|t} + \hat{p}_s) ds.
\]
We know that the total conditional real cost and real marginal cost can be written as

\[ \frac{1}{\rho_s} C(y_{s|t}) = \frac{w_s}{\rho_s} \left( \frac{y_{s|t}}{A_s(N_{W,s})^\alpha} \right)^{1/\alpha}, \quad \varphi_{s|t} = \frac{w_s}{\rho_s} \left( \frac{y_{s|t}}{A_s(N_{W,s})^\alpha} \right)^{\alpha/\alpha - 1} \frac{1}{A_s(N_{W,s})^\alpha}. \]  

(B.3.52)

A conditional real marginal cost gap at time \( s \) conditional on price resetting at time \( t \) can be written as

\[ \varphi_{s|t} = \frac{\dot{w}_s}{\rho_s} - \frac{\alpha \varepsilon}{1-\alpha} (\hat{\rho}_t - \hat{\rho}_s) = \varphi_s - \frac{\alpha \varepsilon}{1-\alpha} (\hat{\rho}_t - \hat{\rho}_s). \]  

(B.3.53)

Thus \( \varphi_s \) is the aggregate marginal cost index, and since production becomes linear in aggregate level, it equals the real wage gap. We then characterize the change in aggregate price gap \( \hat{\rho}_t \), using equation (B.3.46).

\[ d\hat{\rho}_t = \delta dt (\hat{\rho}_t - \hat{\rho}_s) = \delta dt (\delta + \rho) \mathbb{E}_t \int_t^\infty e^{-(\delta + \rho)(s-t)} (\Theta \varphi_s + \hat{\rho}_s - \hat{\rho}_t) ds, \quad \text{where} \quad \Theta \equiv \frac{1 - \alpha}{1 - \alpha + \alpha \varepsilon}. \]

(B.3.54)

As we log-linearize around the steady state equilibrium with constant price, \( \hat{\rho}_t \) changes with a rate of \( \pi_t \),\(^3\) we have

\[ \pi_t = \frac{d\hat{\rho}_t}{dt} = \delta (\delta + \rho) \mathbb{E}_t \int_t^\infty e^{-(\delta + \rho)(s-t)} (\Theta \varphi_s + \hat{\rho}_s - \hat{\rho}_t) ds. \]

(B.3.55)

Since we now have equation (B.3.55) for instantaneous inflation \( \pi_t \), we manipulate this equation as:

\[
\begin{align*}
\pi_t + \delta \dot{\hat{\rho}}_t &= \delta (\delta + \rho) \mathbb{E}_t \int_t^\infty e^{-(\delta + \rho)(s-t)} (\Theta \varphi_s + \hat{\rho}_s) ds \\
&= \delta (\delta + \rho) e^{(\delta + \rho)t} \mathbb{E}_t \int_t^\infty e^{-(\delta + \rho)s} (\Theta \varphi_s + \hat{\rho}_s) ds \\
&= \delta (\delta + \rho) (\Theta \varphi_t + \hat{\rho}_t) dt + \delta (\delta + \rho) e^{(\delta + \rho)t} \mathbb{E}_t \int_{t+dt}^\infty e^{-(\delta + \rho)s} (\Theta \varphi_s + \hat{\rho}_s) ds,
\end{align*}
\]

(B.3.56)

where we can rewrite the first line of equation (B.3.56) at time \( t + dt \) instead of \( t \) as

\[
\begin{align*}
\pi_{t+dt} + \delta \dot{\hat{\rho}}_{t+dt} &= \delta (\delta + \rho) e^{(\delta + \rho)(t+dt)} \mathbb{E}_{t+dt} \int_{t+dt}^\infty e^{-(\delta + \rho)s} (\Theta \varphi_s + \hat{\rho}_s) ds \\
&= \delta (\delta + \rho) e^{(\delta + \rho)t} (1 + (\delta + \rho)dt) \mathbb{E}_{t+dt} \int_{t+dt}^\infty e^{-(\delta + \rho)s} (\Theta \varphi_s + \hat{\rho}_s) ds.
\end{align*}
\]

(B.3.57)

Due to the martingale representation theorem (see Oksendal (1995)), there exists a measurable process \( H_t \) such that following holds.

---

\(^3\)According to Woodford (2003) and Yun (2005), this assumption is reasonable as it becomes a part of optimal monetary policy in the presence of price dispersion \( \Delta_t \). In the case of positive inflation targets, see Coibion et al. (2012).
\[
\mathbb{E}_{t+dt} \int_{t+dt}^{\infty} e^{-(\delta+\rho)s}(\Theta \hat{\phi}_s + \hat{\rho}_s) ds = \mathbb{E}_t \int_{t+dt}^{\infty} e^{-(\delta+\rho)s}(\Theta \hat{\phi}_s + \hat{\rho}_s) ds + H_t dZ_t, \quad (B.3.58)
\]
which we plug into equation (B.3.57), to obtain
\[
\pi_{t+dt} + \delta \hat{\rho}_{t+dt} = \delta(\delta + \rho) \left( e^{(\delta+\rho)t} \mathbb{E}_t \int_{t+dt}^{\infty} e^{-(\delta+\rho)s}(\Theta \hat{\phi}_s + \hat{\rho}_s) ds + e^{(\delta+\rho)t} H_t dZ_t - (\Theta \hat{\phi}_{t+dt}) dt \right) \\
+ e^{(\delta+\rho)t}(\delta + \rho) dt \cdot \mathbb{E}_t \int_{t+dt}^{\infty} e^{-(\delta+\rho)s}(\Theta \hat{\phi}_s + \hat{\rho}_s) ds.
\]
We subtract equation (B.3.56) from equation (B.3.59) to get the following expression. We use \(dZ_t dt = 0\) to get the second equality. Also \(\sigma_{\pi,t}\) is defined as an instantaneous volatility of inflation fluctuation.

\[
\begin{align*}
\mathbf{d} \pi_t + \delta \mathbf{d} \pi_t dt &= \delta(\delta + \rho) \left( e^{(\delta+\rho)t} \mathbb{E}_t \int_{t}^{\infty} e^{-(\delta+\rho)s}(\Theta \hat{\phi}_s + \hat{\rho}_s) ds + e^{(\delta+\rho)t} H_t dZ_t - (\Theta \hat{\phi}_t + \hat{\rho}_t) dt \right) \\
&= \delta(\delta + \rho) e^{(\delta+\rho)t} H_t dZ_t - \delta(\delta + \rho) \Theta \hat{\phi}_t dt \\
&+ \delta(\delta + \rho) \left( (\delta + \rho) dt \cdot \mathbb{E}_t \int_{t}^{\infty} e^{-(\delta+\rho)s}(\Theta \hat{\phi}_s + \hat{\rho}_s - \hat{\rho}_t) ds \right). 
\end{align*}
\]
(\(B.3.60\))

Thus from equation (B.3.60) we get the following continuous time version of New Keynesian Phillips curve (NKPC).\(^4\)

\[
d \pi_t = \rho \pi_t dt - \delta(\delta + \rho) \Theta \hat{\phi}_t dt + \sigma_{\pi,t} dZ_t. \quad (B.3.61)
\]
We know in flexible price equilibrium, a real marginal cost is given as \(\overline{\phi} \), thus \(\hat{\phi}_t\) can be thought of log-deviation of the marginal cost from the flexible price case, which equals the log-deviation of real wage from the flexible price real wage. Therefore, we obtain:\(^5\)

\[
\hat{\phi}_t = \hat{w}_t \hat{p}_t = \frac{\hat{Q}_t}{(\epsilon - 1)(1 - \alpha)} \equiv \frac{\kappa}{\delta(\delta + \rho) \Theta} \hat{Q}_t. \quad (B.3.62)
\]

\(^4\)This form is exactly the same as the Phillips curve in Werning (2012) and Cochrane (2017) if we take expectation.

\(^5\)Here we use log-linearization result of Lemma 2 to represent the real marginal cost gap \(\hat{w}_t \hat{p}_t\) as a function of capitalists’ consumption gap \(\hat{C}_t = \hat{Q}_t\).
Finally plugging equation (B.3.62) into equation (B.3.61), we represent New-Keynesian Phillips curve in terms of asset price gap $\hat{Q}_t$. We know $\kappa > 0$ due to the Assumption 1.

\[ d\pi_t = (\rho\pi_t - \kappa \hat{Q}_t)dt + \sigma_{\pi,t}dZ_t, \quad \text{and} \quad \mathbb{E}_t d\pi_t = (\rho\pi_t - \kappa \hat{Q}_t)dt, \]  
\[ \text{(B.3.63)} \]

which proves the proposition 4.\(^6\)

\[ \square \]

### B.3.4 Section 2.3

#### Section 2.3.2

**Proof of Proposition 6.** This result is a direct consequence of Blanchard and Kahn (1980) and Buiter (1984).

#### Section 2.3.1

**Proof of Proposition 5.** From equation (2.3.9), \{\sigma_t^q\} process can be written in the following way.

\[ d\sigma_t^q = -\frac{\phi^2(\sigma_t^q)^2}{2(\sigma + \sigma_t^q)^3} dt - \phi \frac{\sigma_t^q}{\sigma + \sigma_t^q} dZ_t. \]  
\[ \text{(B.3.64)} \]

Using Ito’s lemma, we get the process for \((\sigma + \sigma_t^q)^2\) which is a martingale, as seen below.

\[ d(\sigma + \sigma_t^q)^2 = 2(\sigma + \sigma_t^q)d\sigma_t^q + (d\sigma_t^q)^2 \]
\[ = 2(\sigma + \sigma_t^q)\left(-\frac{\phi^2(\sigma_t^q)^2}{2(\sigma + \sigma_t^q)^3} dt - \phi \frac{\sigma_t^q - \sigma_{q,n}}{\sigma + \sigma_t^q} dZ_t\right) + \phi^2 \frac{(\sigma_t^q)^2}{(\sigma + \sigma_t^q)^2} dt \]  
\[ = -2\phi(\sigma_t^q) dZ_t. \]  
\[ \text{(B.3.65)} \]

Therefore, we would have \(\mathbb{E}_0((\sigma + \sigma_t^q)^2) = (\sigma + \sigma_0^q)^2\) where \(\mathbb{E}_0\) is an expectation operator with respect to the \(t = 0\) filtration. By the famous Doob’s martingale convergence theorem (as \((\sigma + \sigma_t^q)^2 \geq 0, \forall t\)), we know \(\sigma_t^q \xrightarrow{a.s.} \sigma^q = \sigma_{q,n} = 0\) since:

\[ d\sigma_t^q \xrightarrow{\mathbb{E}_0} -\frac{\phi^2(\sigma_t^q)^2}{2(\sigma + \sigma_t^q)^3} dt - \phi \frac{\sigma_t^q}{\sigma + \sigma_t^q} dZ_t. \]  
\[ \text{(B.3.66)} \]

Therefore, equation (B.3.66) proves \(\sigma_t^q \xrightarrow{a.s.} \sigma^q = \sigma_{q,n} = 0\). From equation (2.3.7) \(\sigma_t^q \xrightarrow{a.s.} \sigma^q = \sigma_{q,n} = 0\) leads to $\hat{Q}_t \xrightarrow{a.s.} 0$ and $\pi_t \xrightarrow{a.s.} 0$. Finally, we must have $\mathbb{E}(\max_t (\sigma_t^q)^2) = \infty$, otherwise the uniform integrability says $\mathbb{E}( (\sigma + \sigma_{t}^q)^2 ) = (\sigma + \sigma_{0}^q)^2$, which is a contradiction to our earlier result $\sigma_t^q \xrightarrow{a.s.} \sigma_{q,n}$ since $\sigma_{\infty}^q = \sigma_{q,n} = 0$ and $\sigma_{0}^q > \sigma_{q,n} = 0$ by assumption in Proposition 5.

\[ ^6\text{Since } \hat{y}_t = \xi \hat{Q}_t, \text{ Phillips curve can be represented in terms of output gap } \hat{y}_t \text{ as in Proposition 4.} \]
Section 2.4.4

Proof of Proposition 8. First we solve the capitalist’s problem in equation (2.4.15) with \( \tau_t \) subsidy rate on stock market investment.

\[
\max_{C_t, \theta_t} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \log C_t \, dts \cdot da_t = (a_t(i_t + \theta_t((1+\tau_t)i^m_t-i_t)) - p_tC_t - T_t) \, dt + \theta_t a_t(\sigma_t + \sigma^q_t) \, dZ_t.
\]

Putting all the relevant state variables \((i_t, \tau_t, T_t, p_t, i^m_t, \sigma_t, \sigma^q_t)\) into vector \(S_t\), then Hamilton-Jacobi-Bellman (HJB) equation can be written in the following way.

\[
\rho V(a_t, S_t, t) = \max_{C_t, \theta_t} \mathbb{E}_0 \log C_t + \frac{\partial V}{\partial a_t}(a_t(i_t + \theta_t((1+\tau_t)i^m_t-i_t)) - p_tC_t - T_t) \qquad \text{(B.3.67)}
\]

Following Merton (1971), we know the value function has the following form.

\[
V(a_t, S_t, t) = \frac{1}{\rho} \log a_t + f(S_t, t). \qquad \text{(B.3.69)}
\]

The first-order conditions for \(C_t\) and \(\theta_t\) are easy to compute as follows.

\[
\rho_t C_t = \rho a_t \quad \text{and} \quad \frac{(1 + \tau_t)i^m_t - i_t}{\sigma_t + \sigma^q_t} = \theta_t(\sigma_t + \sigma^q_t). \qquad \text{(B.3.70)}
\]

Thus compared to the case without \(\tau_t\) (equation (B.3.16)), each capitalist invests more in the stock market, as she gets a higher expected return per unit risk she bears. By plugging the guessed value function form (equation (B.3.69)) into equation (B.3.68), we get a partial differential equation (PDE) for the function \(f(S_t, t)\), verifying our form in equation (B.3.69) is a reasonable guess. Here we plug \(T_t = a_t\tau_t\theta_t i^m_t\) that holds in equilibrium into equation (B.3.68).

\[
\rho f(S_t, t) = \log \frac{\rho}{\rho_t} + \frac{1}{\rho}(i_t + \theta_t(i^m_t - i_t) - \rho) - \frac{1}{2} \theta_t^2(\sigma_t + \sigma^q_t)^2 + \frac{\partial f}{\partial t} \quad \text{with} \quad \theta_t = \frac{(1 + \tau_t)i^m_t - i_t}{(\sigma_t + \sigma^q_t)^2}. \qquad \text{(B.3.71)}
\]

Thus solving the partial differential equation in equation (B.3.71) restores the functional form \(f(S_t, t)\). In equilibrium, \(\theta_t = 1\) holds and it pins down the risk-premium level as follows.

\[
i^m_t = \frac{i_t + (\sigma_t + \sigma^q_t)^2}{1 + \tau_t}. \qquad \text{(B.3.72)}
\]
which is equation (2.4.16). A consumption for capitalists $C_t$ thus evolves with the following process.

$$\frac{dC_t}{C_t} = (i_t^m - \pi_t - \rho)dt + (\sigma_t + \sigma_t^q)dz_t$$

$$= \left(\frac{i_t + (\sigma_t + \sigma_t^q)^2}{1 + \tau_t} - \pi_t - \rho\right)dt + (\sigma_t + \sigma_t^q)dz_t,$$  \hspace{1cm} (B.3.73)

with which we obtain,

$$d\ln C_t = \left(\frac{i_t + (\sigma_t + \sigma_t^q)^2}{1 + \tau_t} - \pi_t - \frac{(\sigma_t + \sigma_t^q)^2}{2} - \rho\right)dt + (\sigma_t + \sigma_t^q)dz_t,$$  \hspace{1cm} (B.3.74)

from which we subtract the second equation (the process for $\ln C'_t$) in equation (B.3.41) and get the following $\hat{C}_t$ process.

$$d\hat{Q}_t = d\hat{C}_t = \left(\frac{i_t + (\sigma_t + \sigma_t^q)^2}{1 + \tau_t} - \pi_t - \frac{(\sigma_t + \sigma_t^q)^2}{2} - r_t^n - \frac{(\sigma_t)^2}{2}\right)dt + \sigma_t dz_t.$$  \hspace{1cm} (B.3.75)

In the ZLB situation described in Section 2.4.1, $i_t = 0$ for $t \leq T$ holds (ZLB) but since the economy gets after $T$, we have $\sigma_t = \bar{\sigma}$ and $r_t^n = r^n(\bar{\sigma}) = r$ for $t \leq T$ yields equation (2.4.17), thus proving Proposition 8.

**Proof of Proposition 9.** We start from equation (2.4.19), the condition that characterizes equilibrium stock market return $i_t^m$.

$$i_t^m = \frac{\gamma_t}{p_t A_t Q_t} = \frac{y_t - \frac{W_t}{p_t A_t Q_t}}{p_t A_t Q_t} + \frac{d(p_t A_t Q_t)}{dt} \frac{1}{p_t A_t Q_t} = \rho - \tau_t i_t^m + \frac{d(p_t A_t Q_t)}{dt} \frac{1}{p_t A_t Q_t}.$$  \hspace{1cm} (B.3.76)

Thus we get $(1 + \tau_t)i_t^m = \rho + \pi_t + g + \mu_t^q + \sigma_t^q$. Due to $(1 + \tau_t)i_t^m = i_t + (\sigma_t + \sigma_t^q)^2$ with the subsidy rate $\tau_t$, we can predict that $\mu_t^q$ remains unchanged compared with no subsidy case, given the levels of $i_t, \sigma_t^q$. Thus the policy does not change \{\hat{Q}_t\} process. To connect this intuition with the formulas, we start from the following process for $C_t$, which is different from equation (B.3.73) since here capitalists do not pay $T_t = a_t \theta_t \tau_t i_t^m$ amount of lump-sum taxes.

$$\frac{dC_t}{C_t} = ((1 + \tau_t)i_t^m - \pi_t - \rho)dt + (\sigma_t + \sigma_t^q)dz_t$$

$$= (i_t + (\sigma_t + \sigma_t^q)^2 - \pi_t - \rho)dt + (\sigma_t + \sigma_t^q)dz_t,$$  \hspace{1cm} (B.3.77)

where we use the equilibrium condition $(1 + \tau_t)i_t^m = i_t + (\sigma_t + \sigma_t^q)^2$ in equation (B.3.72). Since equation (B.3.77) is the same as equation (B.3.40), the dynamics of $C_t$ with $\tau_t = 0$, the policy that subsidizes stock market investment with the lump-sum tax imposed on workers.
does not have any effect on \( \{\hat{Q}_t, \pi_t\} \) and \( \{\hat{Q}_t, \pi_t\} \) process remains the same as the economy without \( \tau_t \).

\[ \square \]

**Section 2.4.4**

**Proof of Proposition 10.** A direct fiscal transfer \( T_t > 0 \) from capitalists to hand-to-mouth workers raises the total amount of dividends in the financial market, leading to a lower required capital gain and a higher \( \hat{Q}_t \) at the ZLB. Then stock market return \( i_t^m \) in this case can be written in the following way. Here we use the fact that \( T_t = \varphi_t p_t A_t Q_t \) holds in equilibrium.

\[
i_t^m = \frac{A_t N_{W,t} - \frac{W_t}{p_t} N_{W,t}}{A_t Q_t} + \frac{d(p_t A_t Q_t)}{p_t A_t Q_t} \frac{1}{dt} = \rho \frac{T_t}{p_t A_t Q_t} \frac{1}{>0} + \frac{d(p_t A_t Q_t)}{p_t A_t Q_t} \frac{1}{dt} \tag{B.3.78}
\]

Thus we see that a dividend yield rises by \( \varphi_t \), which leads to the case in which capital gain is reduced by \( \varphi_t \) given the level of \( i_t \) and \( \sigma_t^q \). Thus it can mitigate the recession during the ZLB as the asset price \( Q_t \) drops less. To derive equation (2.4.21), we start from the following capitalist’s optimization problem.

\[
\max \mathbb{E}_0 \int_0^\infty e^{\rho t} \log C_t \, dt \quad \text{s.t.} \quad da_t = (a_t(i_t + \theta_t(i_t^m - i_t)) - p_t C_t - T_t) dt + \theta_t a_t(\sigma_t + \sigma_t^q) dZ_t,
\]

where \( T_t = \varphi_t a_t \) holds in equilibrium. The equilibrium conditions for \( C_t \) and \( \theta_t \) are exactly the same as equation (B.3.16) with \( \sigma_t^p = 0 \), thus we would have \( C_t = \rho p_t A_t Q_t \) and \( i_t^m = i_t + (\sigma_t + \sigma_t^q)^2 \) in equilibrium. Thus in equilibrium, we get the following wealth process for capitalists.

\[
\frac{da_t}{a_t} = (i_t^m - \rho - \varphi_t) dt + (\sigma_t + \sigma_t^q) dZ_t, \tag{B.3.80}
\]

which leads to the following consumption process for \( C_t \).

\[
\frac{dC_t}{C_t} = (i_t^m - \pi_t - \varphi_t - \rho) dt + (\sigma_t + \sigma_t^q) dZ_t
\]

\[
= (i_t + (\sigma_t + \sigma_t^q)^2 - \pi_t - \varphi_t - \rho) dt + (\sigma_t + \sigma_t^q) dZ_t, \tag{B.3.81}
\]

with which we derive

\[
d\ln C_t = \left( i_t + \frac{(\sigma_t + \sigma_t^q)^2}{2} - \pi_t - \varphi_t - \rho \right) dt + (\sigma_t + \sigma_t^q) dZ_t. \tag{B.3.82}
\]
If we subtract the process for $C_t^n$ in equation (B.3.41), we get the following $\hat{C}_t$ process.

$$d\hat{Q}_t = d\hat{C}_t = \left( i_t - \pi_t - \varphi_t - (r_t^p - \frac{(\sigma_t + \sigma_t^n)^2}{2} + \frac{(\sigma_t)^2}{2}) \right) dt + \sigma_t^q dZ_t$$ (B.3.83)

$$= (i_t - \pi_t - \varphi_t - r_t^T) dt + \sigma_t^q dZ_t.$$

In the ZLB situation described in Section 2.4.1, $i_t = 0$ for $t \leq T$ holds (ZLB) but since the economy gets stabilized after $T$, we have $\sigma_t^q = \sigma_t^{q,n} = 0$ for $t \leq T$. Plugging these conditions into equation (B.3.75) with $\sigma_t = \bar{\sigma}$ and $r_t^n = r^n(\bar{\sigma}) = \bar{\bar{\rho}}$ for $t \leq T$ yields equation (2.4.21), thus proving Proposition 10.

\[\square\]

**B.3.5 Section 2.4.3**

**Proof of Proposition 7.** Central bank solves the following problem in the environment in Section 2.4.3.$^7$

\[
\min_{\sigma_1^{q,c}, \sigma_2^{q,L}, \hat{T}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \hat{Q}_t^2 dt, \text{ s.t.} \begin{cases} 
   d\hat{Q}_t = -\left( r_1^T (\sigma_1^{q,L}) \right) dt + (\sigma_1^{q,L}) dZ_t, & \text{for } t < \hat{T}, \\ 
   d\hat{Q}_t = -\left( r_2^T (\sigma_2^{q,L}) \right) dt + (\sigma_2^{q,L}) dZ_t, & \text{for } \hat{T} < t < \hat{T'}, \\ 
   d\hat{Q}_t = 0, & \text{for } t \geq \hat{T'}, \\ 
   r_1^T (\sigma_1^{q,L}) = \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{(\bar{\sigma} + \sigma_1^{q,L})^2}{2} < 0, \\ 
   r_2^T (\sigma_2^{q,L}) = \rho + g - \frac{\bar{\sigma}^2}{2} - \frac{(\sigma_2^{q,L})^2}{2} > 0, \end{cases}
\]

with $\hat{Q}_0 = r_1^T (\sigma_1^{q,L}) T + r_2^T (\sigma_2^{q,L}) (\hat{T'} - T)$. (B.3.84)

With $r_1^T (\sigma_1^{q,L}) < 0$, $r_2^T (\sigma_2^{q,L}) > 0$, a gap process is represented in the following ways (for $\hat{C}_t = \hat{Q}_t$).

$$d\hat{C}_t = \begin{cases} 
   -r_1^T (\sigma_1^{q,L}) dt + (\sigma_1^{q,L}) dZ_t, & \text{for } t \leq \hat{T}, \\ 
   -r_2^T (\sigma_2^{q,L}) dt + (\sigma_2^{q,L}) dZ_t, & \text{for } \hat{T} < t \leq \hat{T'}, \\ 
   0, & \text{for } t \geq \hat{T'}. \end{cases}$$ (B.3.85)

After $\hat{T}$, there is no movement of $\hat{C}_t$ at all. If we let $r_1^T$ be $r_1^T (\sigma_1^{q,L})$ for $s < T$ and $r_2^T (\sigma_2^{q,L})$ for $T \leq s \leq \hat{T}'$, then gap process can be written in the following integral form. Here $Z_t$,

\[\text{In the proof, we implicitly assume that } r_1^T (\sigma_1^{q,L}) < 0 \text{ and } r_2^T (\sigma_2^{q,L}) > 0 \text{ hold for optimal } \sigma_1^{q,L} \text{ and } \sigma_2^{q,L} \text{ so ZLB binds until } T.\]
\( W_{t-T} \) and \( U_{T-t} \) are independent brownian motion.

\[
\tilde{\mathbf{C}}_t = \begin{cases} 
\tilde{C}_{\text{det}}(t; \tilde{T'})^2 + (\sigma_{1q}^{qL})^2 t, & \text{for } t \leq T, \\
\tilde{C}_{\text{det}}(t; \tilde{T'})^2 + (\sigma_{1q}^{qL})^2 T + (\sigma_{2q}^{qL})^2 (t - T), & \text{for } T < t \leq \tilde{T'}, \\
(\sigma_{1q}^{qL})^2 T + (\sigma_{2q}^{qL})^2 (\tilde{T'} - T), & \text{for } \tilde{T'} < t.
\end{cases}
\]

We square each term and take the expectation operator with respect to the information at \( t = 0 \), when central bank solves its commitment problem. We get the following expressions.

\[
\mathbb{E}_0  \tilde{\mathbf{C}}_t^2 = \begin{cases} 
\mathbb{E}_0 \int_0^\infty e^{-\rho t} \tilde{C}_t^2 dt, & \text{for } t \leq T, \\
\int_0^{\tilde{T'}} e^{-\rho t} \tilde{C}_{\text{det}}(t; \tilde{T'})^2 dt + (\sigma_{1q}^{qL})^2 t + (\sigma_{1q}^{qL})^2 T + (\sigma_{2q}^{qL})^2 (t - T), & \text{for } T < t \leq \tilde{T'}, \\
(\sigma_{1q}^{qL})^2 T + (\sigma_{2q}^{qL})^2 (\tilde{T'} - T), & \text{for } \tilde{T'} < t.
\end{cases}
\]

If we plug these expressions into central bank’s loss function, then central bank’s commitment problem can be represented by the following optimization. Now central bank can control \( \sigma_{1q}^{qL}, \sigma_{2q}^{qL} \) in addition to its conventional monetary policy tool \( \{i_t\} \) (including \( \tilde{T'} \)).

\[
\min_{\mathbf{i} \geq 0, \sigma_{1q}^{qL}, \sigma_{2q}^{qL}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \tilde{\mathbf{C}}_t^2 dt = \min_{\tilde{T'}, \sigma_{1q}^{qL}, \sigma_{2q}^{qL}} \int_0^{\tilde{T'}} e^{-\rho t} \tilde{C}_{\text{det}}(t; \tilde{T'})^2 dt + (\sigma_{1q}^{qL})^2 \int_0^\infty e^{-\rho t} dt + (\sigma_{1q}^{qL})^2 T + (\sigma_{2q}^{qL})^2 (\tilde{T'} - T) \int_{\tilde{T'}}^\infty e^{-\rho t} dt
\]

\[
= \min_{\tilde{T'}, \sigma_{1q}^{qL}, \sigma_{2q}^{qL}} \int_0^{\tilde{T'}} e^{-\rho t} (t - T) dt + (\sigma_{2q}^{qL})^2 (\tilde{T'} - T) \int_{\tilde{T'}}^\infty e^{-\rho t} dt
\]

From deterministic fluctuation

From stochastic fluctuation

(B.3.88)

First we get the first-order condition for \( \tilde{T'} \).
\[ 2 r_2^T \left( \sigma_2^{q,L} \right) \int_0^{\hat{T}'} e^{-\rho t} \hat{C}_{\text{det}}(t; \hat{T}') \, dt + \left( \sigma_2^{q,L} \right)^2 \frac{1}{\rho} e^{-\rho \hat{T}'} = 0, \quad \text{(B.3.89)} \]

from which we have:

\[ \int_0^{\infty} e^{-\rho t} \hat{C}_{\text{det}}(t; \hat{T}') \, dt = \int_0^{\hat{T}'} e^{-\rho t} \hat{C}_{\text{det}}(t; \hat{T}' | \sigma_1^{q,L} < 0, \sigma_2^{q,L} < 0) \, dt < 0. \quad \text{(B.3.90)} \]

The above first-order condition for \( \hat{T}' \) shows that the central bank lowers the value of \( \hat{T}' \) in the optimum, compared to \( \hat{T} \), the duration for which it implements forward guidance only (with \( \sigma_1^{q,L} = \sigma_1^{q,n} \) and \( \sigma_2^{q,L} = \sigma_2^{q,n} \)), thus we would have \( \hat{T}' < \hat{T} \) at optimum. The reason is that in the case central bank only implements a forward guidance without financial market intervention, we have the following optimization condition, which is derived by plugging \( \sigma_1^{q,L} = 0 \) and \( \sigma_2^{q,L} = 0 \) into equation (B.3.89).

\[ \int_0^{\hat{T}'} e^{-\rho t} \hat{C}_{\text{det}}(t; \hat{T}' | \sigma_1^{q,L} = 0, \sigma_2^{q,L} = 0) \, dt = 0. \quad \text{(B.3.91)} \]

Since we know \( \hat{C}_{\text{det}}(t; \hat{T}' | \sigma_1^{q,L} = 0, \sigma_2^{q,L} = 0) < \hat{C}_{\text{det}}(t; \hat{T}' | \sigma_1^{q,L} < 0, \sigma_2^{q,L} < 0) \), from equation (B.3.84), we infer \( \hat{T}' < \hat{T} \) at optimum by comparing equation (B.3.91) with equation (B.3.90).

To characterize optimal \( \sigma_1^{q,L} \) and \( \sigma_2^{q,L} \), we need variational argument, as \( \sigma_1^{q,L} \) and \( \sigma_2^{q,L} \) affects the level of \( r_1^T(\sigma_1^{q,L}) \), \( r_2^T(\sigma_2^{q,L}) \), and \( \hat{C}_{\text{det}}(t; \hat{T}') \) eventually. In specific, we have the following conditions.

\[ \frac{\partial r_1^T(\sigma_1^{q,L})}{\partial \sigma_1^{q,L}} = -(\bar{\sigma} + \sigma_1^{q,L}) < 0, \quad \frac{\partial r_2^T(\sigma_2^{q,L})}{\partial \sigma_2^{q,L}} = -(\sigma + \sigma_2^{q,L}) < 0. \quad \text{(B.3.92)} \]

**Finding \( \sigma_1^{q,L} \):** If \( \sigma_1^{q,L} \) increases, then \( r_1^T(\sigma_1^{q,L}) \) falls, changing \( \hat{C}_{\text{det}}(t; \hat{T}') \)'s path as the following Figure B.6 indicates (from thick blue to dashed red in Figure B.6). When we differentiate

![Figure B.6: Variation along \( \sigma_1^{q,L} \) Increase to \( \sigma_1^{q,L,New} > \sigma_1^{q,L} \)](attachment)
$\hat{C}_{\text{det}}(t; \hat{T}')$ with respect to $\sigma_{1}^{q,L}$, we get the following conditions.

$$
\hat{C}_{\text{det}}(t; \hat{T}') = \int_{t}^{\hat{T}'} r_s^T ds, \quad \frac{\partial \hat{C}_{\text{det}}}{\partial \sigma_{1}^{q,L}} = \int_{t}^{T} - (\hat{\sigma} + \sigma_{1}^{q,L}) ds = -(\hat{\sigma} + \sigma_{1}^{q,L})(T - t), \forall t \leq T.
$$

(B.3.93)

To find optimal $\sigma_{1}^{q,L}$, we differentiate the objective function by $\sigma_{1}^{q,L}$ and get the following condition.

$$(\hat{\sigma} + \sigma_{1}^{q,L}) \int_{0}^{T} e^{-\rho t} \hat{C}_{\text{det}}(t; \hat{T}')(T - t) dt = (\sigma_{1}^{q,L}) \frac{1 - e^{-\rho T}}{\rho^2},$$

(B.3.94)

from which we can show $\sigma_{1}^{q,L} < 0$ must be satisfied at optimum, since:

$$
\int_{0}^{T} e^{-\rho t} \hat{C}_{\text{det}}(t; \hat{T}')(T - t) dt = \int_{0}^{T} \left( \int_{0}^{t} e^{-\rho s} \hat{C}_{\text{det}}(s; \hat{T}') ds \right) dt < 0.
$$

(B.3.95)

And equation (B.3.94) implies $\sigma_{1}^{q,L} < 0$ at optimum.

Finding $\sigma_{2}^{q,L}$: If $\sigma_{2}^{q,L}$ increases, then $r_{1}^{T}(\sigma_{2}^{q,L})$ falls, changing $\hat{C}_{\text{det}}(t; \hat{T}')$'s shape as the following Figure B.7 indicates (from thick blue to dashed red in Figure B.7). When we differentiate $\hat{C}_{\text{det}}(t; \hat{T}')$ with respect to $\sigma_{2}^{q,L}$, we get the following conditions.

$$
\frac{\partial \hat{C}_{\text{det}}}{\partial \sigma_{2}^{q,L}} = \begin{cases}
\int_{T}^{\hat{T}'} -(\hat{\sigma} + \sigma_{2}^{q,L}) ds = -(\hat{\sigma} + \sigma_{2}^{q,L})(\hat{T}' - T), \forall t < T, \\
\int_{t}^{\hat{T}'} -(\hat{\sigma} + \sigma_{2}^{q,L}) ds = -(\hat{\sigma} + \sigma_{2}^{q,L})(\hat{T}' - t), \quad T < \forall t < \hat{T}'.
\end{cases}
$$

(B.3.96)

To find optimal $\sigma_{2}^{q,L}$, we differentiate the objective function by $\sigma_{2}^{q,L}$ and get the following

Figure B.7: Variation along $\sigma_{2}^{q,L}$ Increase to $\sigma_{2}^{q,L,\text{New}} > \sigma_{2}^{q,L}$
conditions.

\[
\begin{align*}
(\sigma + \sigma_{2}^{q,L}) & \left( \int_{0}^{T} e^{-\rho t} \hat{C}_{det}(t; \hat{T}')(\hat{T}' - T)dt + \int_{T}^{\hat{T}'} e^{-\rho t} \hat{C}_{det}(t; \hat{T}')(\hat{T}' - t)dt \right) \\
& = \left( \sigma_{2}^{q,L} \right) \frac{e^{-\rho T} - e^{-\rho \hat{T}'} }{\rho^2},
\end{align*}
\]

(B.3.97)

with which the following equation (B.3.98) shows \(\sigma_{2}^{q,L} < 0\) holds at the optimum.

\[
\int_{0}^{T} e^{-\rho t} \hat{C}_{det}(t; \hat{T}')(\hat{T}' - T)dt + \int_{T}^{\hat{T}'} e^{-\rho t} \hat{C}_{det}(t; \hat{T}')(\hat{T}' - t)dt \\
< \int_{0}^{T} e^{-\rho t} \hat{C}_{det}(t; \hat{T}')(\hat{T}' - T)dt + \int_{T}^{\hat{T}'} e^{-\rho t} \hat{C}_{det}(t; \hat{T}')(\hat{T}' - t)dt \\
= (\hat{T}' - T) \int_{0}^{\hat{T}'} e^{-\rho t} \hat{C}_{det}(t; \hat{T}')dt < 0.
\]

(B.3.98)

Thus we proved that during high TFP volatility period \((t \leq T)\) and the low TFP volatility period with the forward guidance \((T \leq t \leq \hat{T}')\), a central bank wants to target financial volatility levels lower than their levels in flexible price economy \((\sigma_{1}^{q,L} < \sigma_{1}^{q,n} = 0 \text{ and } \sigma_{2}^{q,L} < \sigma_{2}^{q,n} = 0)\). This intervention lowers the required risk-premium and boost asset price level \(\hat{Q}_t\), thus raising the output.

\(\square\)

**First-order conditions for \(\hat{T}', \sigma_{1}^{q,L}, \sigma_{2}^{q,L} \):** A deterministic component of capitalists’ consumption gap \(\hat{C}_t\) process, \(\hat{C}_{det}(t; \hat{T}')\), is given as \((\text{with } r_{1}^{T}(\sigma_{1}^{q,L}) \text{ and } r_{2}^{T}(\sigma_{2}^{q,L})\) given in equation (2.4.9)):

\[
\hat{C}_{det}(t; \hat{T}') = \int_{t}^{\hat{T}'} r_{s} ds = \begin{cases} 
  r_{1}^{T}(\sigma_{1}^{q,L})(T - t) + r_{2}^{T}(\sigma_{2}^{q,L})(\hat{T}' - T), & \text{for } \forall t \leq T, \\
  r_{2}^{T}(\sigma_{2}^{q,L})(\hat{T}' - t), & \text{for } T \leq \forall t < \hat{T}',
\end{cases}
\]

(B.3.99)

based on which we obtain the following formula.

\[
\int_{0}^{\hat{T}'} e^{-\rho t} \hat{C}_{det}(t; \hat{T}')dt = \int_{0}^{T} e^{-\rho t} [r_{1}^{T}(\sigma_{1}^{q,L})(T - t) + r_{2}^{T}(\sigma_{2}^{q,L})(\hat{T}' - T)] dt \\
+ \int_{T}^{\hat{T}'} e^{-\rho t} r_{2}^{T}(\sigma_{2}^{q,L})(\hat{T}' - t) dt.
\]

(B.3.100)
And we will use the following integration results in this part.

\[
\begin{align*}
\int_0^T e^{-\rho t} (T - t) dt &= \frac{e^{-\rho T} T}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2}, \\
\int_T^{\hat{T}} e^{-\rho t} (\hat{T} - t) dt &= \frac{e^{-\rho \hat{T}}}{\rho^2} + \frac{\hat{T} - T}{\rho} e^{-\rho T} - \frac{e^{-\rho T}}{\rho^2}, \\
\int_0^T e^{-\rho t} (T - t)^2 dt &= -\frac{2}{\rho^3} e^{-\rho T} + \frac{T^2}{\rho} - \frac{2T}{\rho^2} + \frac{2}{\rho^3}, \\
\int_T^{\hat{T}} e^{-\rho t} (\hat{T} - t)^2 dt &= -\frac{2}{\rho^3} e^{-\rho \hat{T}} + \frac{(\hat{T} - T)^2}{\rho} e^{-\rho T} - \frac{2(\hat{T} - T)}{\rho^2} e^{-\rho T} + \frac{2}{\rho^3} e^{-\rho T}.
\end{align*}
\]

The first condition (first-order condition for \(\hat{\sigma}'\)) can be written as:

\[
2r_2^T (\sigma_2^{q,L}) \int_0^{\hat{T}} e^{-\rho t} \hat{C}_{det}(t; \hat{T}') dt + (\sigma_2^{q,L})^2 \frac{e^{-\rho \hat{T}'} \rho}{e^{\hat{T}'}} = 0, \tag{B.3.102}
\]

where

\[
\int_0^{\hat{T}} e^{-\rho t} \hat{C}_{det}(t; \hat{T}') dt = r_1^T (\sigma_1^{q,L}) \left[ \frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2} \right] + r_2^T (\sigma_2^{q,L})(\hat{T}' - T) \frac{1 - e^{-\rho T}}{\rho} + r_2^T (\sigma_2^{q,L}) \left[ \frac{e^{-\rho \hat{T}'} \rho}{e^{\hat{T}'}} - \frac{1}{\rho^2} e^{-\rho \hat{T}'} \right]. \tag{B.3.103}
\]

We plug all the integration and get the following expression for the first-order condition for \(\hat{T}'\).

\[
2r_2^T (\sigma_2^{q,L}) \left\{ r_1^T (\sigma_1^{q,L}) \left[ \frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2} \right] + r_2^T (\sigma_2^{q,L})(\hat{T}' - T) \frac{1 - e^{-\rho T}}{\rho} \right. \\
+ r_2^T (\sigma_2^{q,L}) \left[ \frac{e^{-\rho \hat{T}'} \rho}{e^{\hat{T}'}} - \frac{1}{\rho^2} e^{-\rho \hat{T}'} \right] \} + (\sigma_2^{q,L})^2 \frac{e^{-\rho \hat{T}'} \rho}{e^{\hat{T}'}} = 0. \tag{B.3.104}
\]

The above equation (B.3.104) has all of \(\{\sigma_1^{q,L}, \sigma_2^{q,L}, \hat{T}'\}\) as \(r_1^T (\sigma_1^{q,L})\) and \(r_2^T (\sigma_2^{q,L})\) are functions of \(\sigma_1^{q,L}, \sigma_2^{q,L}\), respectively.

The second condition (first-order condition for \(\sigma_1^{q,L}\)) can be written as:

\[
(\bar{\sigma} + \sigma_1^{q,L}) \int_0^T e^{-\rho t} \hat{C}_{det}(t; \hat{T}')(T - t) dt = (\sigma_1^{q,L}) \frac{1 - e^{-\rho T}}{\rho^2}, \tag{B.3.105}
\]

where

\[
\int_0^T e^{-\rho t} \hat{C}_{det}(t; \hat{T}')(T - t) dt = r_1^T (\sigma_1^{q,L}) \left[ -\frac{2}{\rho^3} e^{-\rho T} + \frac{T^2}{\rho^2} - \frac{2T}{\rho^3} + \frac{2}{\rho^3} \right] + r_2^T (\sigma_2^{q,L})(\hat{T}' - T) \left[ \frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2} \right]. \tag{B.3.106}
\]
Plugging equation (B.3.106) into equation (B.3.105), we get the following first-order condition for the $\sigma^{q,L}_1$.

$$
(\bar{\sigma} + \sigma^{q,L}_1) \left\{ r_1^T (\sigma^{q,L}_1) \left[ - \frac{2}{\rho^3} e^{-\rho T} + \frac{T^2}{\rho^2} - \frac{2T}{\rho^3} + \frac{2}{\rho^3} \right] + r_2^T (\sigma^{q,L}_2) (\hat{T}' - T) \left[ \frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2} \right] \right\} = (\sigma^{q,L}_1) \frac{1 - e^{-\rho T}}{\rho^2}.
$$

(B.3.107)

Finally, the first-order condition for the $\sigma^{q,L}_2$ becomes:

$$
(\bar{\sigma} + \sigma^{q,L}_2) \left\{ (\hat{T}' - T) \int_0^T e^{-\rho t} \hat{C}_{det}(t; \hat{T}') dt + \int_T^{\hat{T}'} e^{-\rho t} \hat{C}_{det}(t; \hat{T}') (\hat{T}' - t) dt \right\} = (\sigma^{q,L}_2) \frac{e^{-\rho T} - e^{-\rho \hat{T}'}}{\rho^2},
$$

(B.3.108)

where

$$
\int_0^T e^{-\rho t} \hat{C}_{det}(t; \hat{T}') dt = r_1^T (\sigma^{q,L}_1) \left[ \frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2} \right] + r_2^T (\sigma^{q,L}_2) (\hat{T}' - T) \frac{1 - e^{-\rho T}}{\rho},
$$

(B.3.109)

and

$$
\int_T^{\hat{T}'} e^{-\rho t} \hat{C}_{det}(t; \hat{T}') (\hat{T}' - t) dt = r_2^T (\sigma^{q,L}_2) \left[ - \frac{2e^{-\rho \hat{T}'}}{\rho^3} + \frac{(\hat{T}' - T)^2}{\rho} e^{-\rho T} - \frac{2(\hat{T}' - T)}{\rho^2} e^{-\rho T} + \frac{2e^{-\rho T}}{\rho^3} \right].
$$

(B.3.110)

Thus the first-order condition for the $\sigma^{q,L}_2$ can be written as:

$$
(\bar{\sigma} + \sigma^{q,L}_2) \left\{ \left[ r_1^T (\sigma^{q,L}_1) \left[ \frac{e^{-\rho T}}{\rho^2} + \frac{T}{\rho} - \frac{1}{\rho^2} \right] + r_2^T (\sigma^{q,L}_2) (\hat{T}' - T) \frac{1 - e^{-\rho T}}{\rho} \right] (\hat{T}' - T) \right. \\
\left. + r_2^T (\sigma^{q,L}_2) \left[ - \frac{2}{\rho^3} e^{-\rho \hat{T}'} + \frac{(\hat{T}' - T)^2}{\rho} e^{-\rho T} - \frac{2(\hat{T}' - T)}{\rho^2} e^{-\rho T} + \frac{2}{\rho^3} e^{-\rho T} \right] \right\} = (\sigma^{q,L}_2) \frac{e^{-\rho T} - e^{-\rho \hat{T}'}}{\rho^2}.
$$

(B.3.111)
B.4 Welfare Derivation

B.4.1 Efficient Steady State (Efficient Flexible Price Equilibrium) with a production subsidy

First-Best Allocation

A first-best allocation must be the solution of the following optimization problem.

$$\max_{c_t, n_{W,t}, c_{W,t}} \omega_1 \log \frac{c_t}{A_t} + \omega_2 \left( \frac{\left( \frac{c_{W,t}}{A_t} \right)^{1-\psi}}{1-\psi} - \frac{(n_{W,t})^{1+\chi_0}}{1+\chi_0} \right) \text{ s.t. } c_t + c_{W,t} = A_t n_{W,t}. \tag{B.4.1}$$

where $\omega_1 > 0$ and $\omega_2 > 0$ are two welfare weights attached to capitalists and workers, respectively, and we assume no price dispersion, thus $\Delta_t = 1$. For the expositional purposes, let us define $x_t \equiv n_{W,t}$ and $y_t \equiv \frac{c_{W,t}}{A_t}$, then the first-order condition for equation (B.4.1) can be written as

$$y_t^{-\psi} = x_t^{\chi_0}, \quad \frac{\omega_1}{\omega_2} = x_t^{\chi_0}(x_t - y_t). \tag{B.4.2}$$

Workers’ and Firms’ Problem

Now we introduce a production subsidy $\tau > 0$ given to the firms, assuming that it is financed through a lump-sum tax on workers. Our objective is to make sure our flexible price equilibrium (or steady-state) allocation $(n_{W,t}, c_{W,t}, c_{t})$ is efficient and satisfies equation (B.4.2). With the subsidy $\tau$, workers solve the following problem.

$$\max_{c_{W,t}, n_{W,t}} \left( \frac{c_{W,t}}{A_t} \right)^{1-\psi} - \frac{(n_{W,t})^{1+\chi_0}}{1+\chi_0} \text{ s.t. } p_t c_{W,t} = w_t n_{W,t} - p_t T_t, \tag{B.4.3}$$

where $T_t = \tau y_t$ is the (real) lump-sum tax amount imposed on workers. The equation (B.4.3)’s first order condition is written as:

$$(n_{W,t})^{\chi_0 + \psi} \left( \frac{w_t}{p_t A_t} - \tau \right)^{\psi} = \frac{w_t}{p_t A_t}. \tag{B.4.4}$$

where we express the $n_{W,t}$ that satisfies equation (B.4.4) as a function of a normalized real wage $\frac{w_t}{p_t A_t}$, assuming $n_{W,t} \equiv f_N(\frac{w_t}{p_t A_t})$. When $\tau = 0$, it returns to equation (B.3.12). Due to $\tau$, $n_{W,t}$ rises, compared to the amount implied by equation (B.3.12), since (i) workers feel poorer due to the lump-sum tax $T_t$, thus a higher marginal utility of consumption induces them to work more. (ii) eventually firms’ labor demand would rise, which raises the labor supply of workers.

Since we are dealing with the flexible price economy benchmark, each firm’s optimization is
changed with the introduction of $\tau$ in the following way, with $E_t = (N_{W,t})^{\alpha}$.

$$\max_{p_t(i)} (1 + \tau) p_t(i) \left( \frac{p_t(i)}{p_t} \right)^{-\epsilon} y_t - w_t \left( \frac{y_t}{A_t E_t} \right) \frac{1}{1 - \alpha} \left( \frac{p_t(i)}{p_t} \right)^{\frac{\epsilon}{1 - \alpha}},$$

(B.4.5)

where $p_t(i) = p_t$ for $\forall i$ at optimum and the solution of equation (B.4.5) features $w_{n,t} = (1 + \tau)(\epsilon - 1)(1 - \alpha)\frac{\epsilon}{\epsilon}$.

(B.4.6)

where we plug equation (B.4.6) into equation (B.4.4) and obtain

$$N_{W,t} = f_N \left( \frac{w_{n,t}^n}{p_t^n A_t} \right) = f_N \left( \frac{(1 + \tau)(\epsilon - 1)(1 - \alpha)}{\epsilon} \right),$$

(B.4.7)

Since our goal is to align the allocation implied by equation (B.4.7) with the first-best allocation implied by equation (B.4.2), $N_{W,t}$ and $C_{W,t}$ in equation (B.4.7) must satisfy equation (B.4.2) as follows.

$$\frac{(1 + \tau)(\epsilon - 1)(1 - \alpha)}{\epsilon} - \tau = f_N \left( \frac{(1 + \tau)(\epsilon - 1)(1 - \alpha)}{\epsilon} \right)^{\frac{x_0 + \varphi}{\varphi}}.$$  

(B.4.8)

Plugging equation (B.4.6) into equation (B.4.4), we get:

$$(N_{W,t}^n)^{x_0 + \varphi} \left( \frac{(1 + \tau)(\epsilon - 1)(1 - \alpha)}{\epsilon} - \tau \right) = \frac{(1 + \tau)(\epsilon - 1)(1 - \alpha)}{\epsilon}.$$  

(B.4.9)

Solving jointly equation (B.4.8) and equation (B.4.9), we conclude the optimal $\tau^*$ must satisfies the following familiar condition.

$$\frac{(1 + \tau^*)(\epsilon - 1)(1 - \alpha)}{\epsilon} = 1.$$  

(B.4.10)

Therefore, the optimal $\tau^* > 0$ eliminates mark-up of firms and restores efficiency. With $\tau = \tau^*$, normalized real wage becomes 1 and we get the following benchmark efficient allocation from equation (B.4.7).

$$N_{W,t}^n \equiv \bar{x} = (1 - \tau^*)^{-\frac{x_0}{x_0 + \varphi}}, \quad C_{W,t}^n \equiv \bar{y} = (1 - \tau^*)^{\frac{x_0}{x_0 + \varphi}}, \quad \frac{C_{t}^n}{A_t} = \bar{x} - \bar{y} = (1 - \tau^*)^{-\frac{x_0}{x_0 + \varphi}}\tau^*.$$  

(B.4.11)

The last step is to ensure the welfare weights $\omega_1 > 0$ and $\omega_2 > 0$ satisfy equation (B.4.2).

By plugging equation (B.4.11) into the second condition of equation (B.4.2), we obtain

$8$ Therefore, $\tau^*$ is a function of primitive parameters $\epsilon$ and $\alpha$.

$9$ Since $\omega_1$ and $\omega_2$ are chosen arbitrarily, we make sure that our allocation with a production subsidy can be on the efficient frontier, which is generated by a varying set of $\{\omega_1, \omega_2\}$. 
\[
\frac{\omega_1}{\omega_2} = (N_{W,t}^n x_0 (N_{W,t}^n - \frac{C_{W,t}^n}{A_t}) = (1 - \tau^*) \frac{(\chi_0 + 1)\psi}{\chi_0 + \psi} \cdot \tau^*.
\]

Thus, with \( \omega_1 > 0 \) and \( \omega_2 > 0 \) satisfying equation (B.4.12), our allocation with \( \tau = \tau^* \) is efficient. The next step is to approximate a joint welfare in equation (B.4.1) with those \( \omega_1, \omega_2 \) up to a second-order and express it in terms of gap variables and the price dispersion.

**Derivation of a Quadratic Loss Function**

As we previously defined, let \( x_t \equiv N^W(t) \), \( y_t \equiv \frac{C_{W,t}}{A_t} \). Their steady-state values (flexible price equilibrium values) are the ones in equation (B.4.11). From the economy-wide resource constraint, we express

\[
\frac{C_t}{A_t} = \frac{N_{W,t}}{\Delta_t} - \frac{C_{W,t}}{A_t} = \frac{x_t}{\Delta_t} - y_t, \text{ where } \Delta = \left( \int_0^1 \left( \frac{\rho_t(i)}{\rho_t} \right) - \frac{t}{\Delta_t} d_i \right)^{1-\alpha}.
\]

With equation (B.4.13), we express our social welfare in equation (B.4.1) with \( \omega_1 \) and \( \omega_2 \) satisfying equation (B.4.12) as follows.

\[
U(x_t, y_t, \Delta_t) \equiv \omega_1 \log \left( \frac{x_t}{\Delta_t} - y_t \right) + \omega_2 \left( \frac{y_t^{1-\varphi}}{1-\varphi} - \frac{x_t^{1+\chi_0}}{1+\chi_0} \right), \tag{B.4.14}
\]

which achieves its maximum value \( \bar{U} \) when \( x_t = \bar{x}, y_t = \bar{y}, \Delta_t = 1 \). A second-order approximation of equation (B.4.14) around the efficient benchmark allocation \( (\bar{x}, \bar{y}, 1) \) in equation (B.4.11) results in the following expression.

\[
U_t - \bar{U} = U_\Delta \cdot \bar{\Delta} \cdot \bar{\Delta} + \frac{1}{2} U_{xx} \cdot \bar{x}^2 \cdot (\bar{x}_t)^2 + \frac{1}{2} U_{yy} \cdot \bar{y}^2 \cdot (\bar{y}_t)^2 + U_{xy} \cdot \bar{x} \cdot \bar{y} \cdot \bar{x}_t \cdot \bar{y}_t + h.o.t. \tag{B.4.15}
\]

where \( \bar{\Delta} = 1 \) since we do not have a trend inflation and partial derivatives \( (U_\Delta, U_{xx}, U_{yy}, U_{xy}) \) are evaluated at the benchmark point \( (\bar{x}, \bar{y}, 1) \) as follows.

\[
U_\Delta = -\omega_2 (1 - \tau^*) \frac{-(\chi_0 + 1)\psi}{\chi_0 + \psi},
\]

\[
U_{xx} = -\omega_2 (1 - \tau^*) \frac{-(\chi_0 + 1)\psi}{\chi_0 + \psi} \left( \frac{1}{\tau^* + \chi_0} \right),
\]

\[
U_{yy} = -\omega_2 (1 - \tau^*) \frac{-(\chi_0 + 1)\psi}{\chi_0 + \psi} \left( \frac{1}{\tau^* + \frac{\varphi}{1-\tau^*}} \right),
\]

\[
U_{xy} = \omega_2 (1 - \tau^*) \frac{-(\chi_0 + 1)\psi}{\chi_0 + \psi} \frac{1}{\tau^*},
\]

where we use the relation between \( \omega_1 \) and \( \omega_2 \) in equation (B.4.12) in the process of derivation. Since \( \omega_2 \) can be regarded a free parameter, we set \( \omega_2 \equiv 1 \).

\[\text{We have } U_x = U_y = 0 \text{ at } (\bar{x}, \bar{y}, 1) \text{, where } U_x \text{ and } U_y \text{ are the partial derivatives with respect to } x_t \text{ and } y_t, \text{ respectively.}\]
**Log-Linearization** With the price dispersion $\Delta_t$, the hand-to-mouth worker's problem with $\tau^*$ features the following solution.

\[
(N_{W,t})^{x_0+\varphi} \left( \frac{W_t}{p_t A_t} - \frac{\tau^*}{\Delta_t} \right)^\varphi = \frac{W_t}{p_t A_t}, \tag{B.4.17}
\]

\[
\frac{C_{W,t}}{A_t} = \left( \frac{W_t}{p_t A_t} - \frac{\tau^*}{\Delta_t} \right) N_{W,t}.
\]

Linearizing the first equation around the benchmark allocation yields

\[
\hat{N}_{W,t} = \frac{1 - \frac{\varphi}{1 - \tau^*} \left( \frac{W_t}{p_t} \right) \varphi}{x_0 + \varphi} - \frac{\varphi \tau^*}{1 - \tau^*} \frac{x_0}{x_0 + \varphi} \hat{\Delta}_t. \tag{B.4.18}
\]

Linearizing the second consumption equation yields

\[
\hat{C}_{W,t} = \frac{1 + \frac{x_0}{1 - \tau^*} \left( \frac{W_t}{p_t} \right) + \frac{\tau^* x_0}{1 - \tau^*} \frac{x_0}{x_0 + \varphi} \hat{\Delta}_t}{\tau^* - (x_0 + \frac{\varphi}{1 - \tau^*})} \hat{Q}_t + \frac{\tau^*}{1 - \tau^*} \frac{x_0}{x_0 + \varphi} \hat{\Delta}_t. \tag{B.4.19}
\]

Finally, by linearizing the economy-wide resource constraint (equation (B.4.13)) with $\hat{Q}_t = \hat{C}_t$ and solving jointly with equation (B.4.18) and equation (B.4.19), we can express gaps in real wage, labor supply, and workers' consumption as functions of gaps in asset price and price dispersion as follows.

\[
\hat{x}_t \equiv \hat{N}_{W,t} = \frac{\tau^* (1 - \frac{\varphi}{1 - \tau^*}) \hat{Q}_t + \tau^* \left( x_0 + \frac{\varphi}{1 - \tau^*} \right) \hat{\Delta}_t}{\tau^* - \left( x_0 + \frac{\varphi}{1 - \tau^*} \right)}\]

\[
\hat{y}_t \equiv \hat{C}_{W,t} = \frac{\tau^* \left( 1 + \frac{x_0}{1 - \tau^*} \right) \hat{Q}_t + \tau^* \left( \frac{x_0}{x_0 + \varphi} \hat{\Delta}_t \right)}{\tau^* - \left( x_0 + \frac{\varphi}{1 - \tau^*} \right)}.
\]

Plugging equation (B.4.16) into the second-order approximation equation (equation (B.4.15)), we get the following expression

\[
U_t - \tilde{U} = -(1 - \tau^*) \left( \frac{x_0 + \varphi}{x_0 + \varphi} \right) \hat{\Delta}_t - \frac{1}{2} (1 - \tau^*) \left( \frac{x_0 - \varphi}{x_0 + \varphi} \right) \left( \frac{1}{\tau^*} + x_0 \right) (1 - \tau^*) \left( \frac{x_0 - \varphi}{x_0 + \varphi} \right) \left( \frac{1}{\tau^*} \right) \hat{x}_t \hat{y}_t
\]

\[
= -(1 - \tau^*) \left( \frac{x_0 + \varphi}{x_0 + \varphi} \right) \hat{\Delta}_t - \frac{1}{2} (1 - \tau^*) \left( \frac{x_0 - \varphi}{x_0 + \varphi} \right) \left( \frac{1}{\tau^*} + x_0 \right) (1 - \tau^*) \left( \frac{x_0 - \varphi}{x_0 + \varphi} \right) \left( \frac{1}{\tau^*} \right) \hat{x}_t \hat{y}_t.
\]
Finally by plugging equation (B.4.20) into equation (B.4.21), we get the following expression for \( U_t - \bar{U} \):

\[
U_t - \bar{U} = \tilde{\eta}_\Delta \Delta t + \tilde{\eta}_q (\hat{Q}_t)^2 + h.o.t \text{ with } \tilde{\eta}_\Delta = -(1 - \tau^*) \frac{-\chi_0 (1 - \varphi)}{\chi_0 + \varphi} \text{ and, (B.4.22)}
\]

\[
\tilde{\eta}_q = -\frac{1}{2} (1 - \tau^*) \frac{-\chi_0 (1 - \varphi)}{\chi_0 + \varphi} \left( 1 + \chi_0 \right) \left( \frac{\tau^* - \left( \chi_0 + \varphi \right)}{\chi_0 + \varphi} \right)^2 - \frac{1}{2} (1 - \tau^*) \frac{-\chi_0 (1 - \varphi)}{\chi_0 + \varphi} \left( 1 + \chi_0 \right) \left( \frac{\tau^* - \left( \chi_0 + \varphi \right)}{\chi_0 + \varphi} \right)^2 \]

\[
+ (1 - \tau^*) \frac{-\chi_0 (1 - \varphi)}{\chi_0 + \varphi} \left( \frac{\tau^* - \left( \chi_0 + \varphi \right)}{\chi_0 + \varphi} \right) \left( \frac{\tau^* - \left( \chi_0 + \varphi \right)}{\chi_0 + \varphi} \right), \quad \text{ (B.4.23)}
\]

where \( \tilde{\eta}_\Delta < 0 \) and \( \tilde{\eta}_q < 0 \) with our calibrated parameters.

**Loss function** Finally, we express dynamic loss function as

\[
L_0(\{\hat{Q}_t, \Delta_t\}_{t \geq 0}) = \mathbb{E}_0 \int_0^\infty \left( \hat{Q}_t^2 + \tilde{\eta}_\Delta \frac{\Delta t}{\gamma_q} \right) dt, \quad \text{(B.4.24)}
\]

and furthermore, we know the following relation from Woodford (2003).

\[
\tilde{\Delta}_t = \frac{\epsilon}{2 \Theta} \text{Var}_t(\rho_t(i)), \quad \text{and} \quad \int_0^\infty e^{-\rho t} \text{Var}_t(\rho_t(i)) dt = \frac{1}{\delta(\delta + \rho)} \int_0^\infty e^{-\rho t} \pi_t^2 dt. \quad \text{(B.4.25)}
\]

By plugging equation (B.4.25) into equation (B.4.24) and expressing the loss function \( L \) as a function of \( \{\hat{Q}_t\} \) and \( \{\pi_t\} \), we finally obtain

\[
L(\{\hat{Q}_t, \pi_t\}_{t \geq 0}) = \mathbb{E}_0 \int_0^\infty \left( \hat{Q}_t^2 + \Gamma \pi_t^2 \right) dt, \quad \text{with } \Gamma \equiv \frac{1}{\delta(\delta + \rho)} \frac{\epsilon}{2 \Theta} \frac{\tilde{\eta}_\Delta}{\gamma_q} > 0, \quad \text{(B.4.26)}
\]

which is equation (2.4.1).
Appendix C

Appendices to Chapter 3

C.1 Derivation and Proofs for 3

C.1.1 Detailed Derivations in Section 3.2

Detailed Derivations in Section 3.2.4

An intermediate firm $\nu$ maximizes the discounted stream of profits

$$\max \sum_{j=0}^{\infty} E_t \left[ \theta^j Q_{t,t+j} \cdot \left[ (1 + \zeta^F) \cdot P_{t+j}(\nu) Y_{t+j}(\nu) - W_{t+j}(\nu) N_{t+j}(\nu) - R_{t+j}^K P_{t+j}^K K_{t+j-1}(\nu) \right] \right],$$

(C.1.1)

where $Q_{t,t+j}$ is the firm’s stochastic discount factor between periods $t$ and $t+j$ and $\zeta^F$ is a production subsidy. Solving for the optimal resetting price at period $t$, $P_t^*$, we obtain

$$P_t^* = \frac{E_t \left[ \sum_{j=0}^{\infty} \theta^j Q_{t,t+j} \left( \frac{P_{t+j}}{P_t} \right)^{\epsilon+1} Y_{t+j} \left( \frac{1 + \zeta^F}{\epsilon - 1} \right)} {\frac{MC_{t+j|t}(\nu)}{P_{t+j}}} \right]} {E_t \left[ \sum_{j=0}^{\infty} \theta^j Q_{t,t+j} \left( \frac{P_{t+j}}{P_t} \right)^{\epsilon} Y_{t+j} \right]},$$

(C.1.2)

where subindex $t+j|t$ represents the value of the variable conditional on the firm having reset its price last time at period $t$, and $MC_{t+j|t}(\nu)/P_t$ is the real marginal cost of production, defined as

$$MC_{t+j|t}(\nu) = \left( \frac{\tilde{R}_{t+j}^K}{P_{t+j}} \cdot \frac{P_{t+j}^K}{P_t} \right)^{\alpha} \left( \frac{W_{t+j|t}(\nu)}{P_{t+j} A_{t+j}} \right)^{1-\alpha}.$$

(C.1.3)

1It can be derived using the optimal demand formula for labor and capital (equation (3.2.14)).
C.1.2 Detailed Derivation for Aggregation

Using equations equation (3.2.9a), equation (3.2.10), equation (3.2.11) and equation (C.1.3) we can express firm-specific marginal costs as a function of the aggregate variables as in

\[
MC_{t+j}(\nu) = \left(1 - \alpha\right)^{\frac{1}{\eta} + \alpha} \left(\frac{C_{t+j}}{A_{t+j}N_{t+j}}\right)^{\frac{1}{\eta} + \alpha} \left(\frac{X_{t+j}}{A_{t+j}N_{t+j}}\right)^{\frac{1}{\eta} + \alpha} \left(\frac{Y_{t+j}}{A_{t+j}N_{t+j}}\right)^{\frac{1}{\eta} + \alpha} \left(\frac{P_{t+j}}{P_t}\right)^{\frac{1}{\eta} + \alpha} \left(\frac{P_t}{P_{t+j}}\right)^{\frac{1}{\eta} + \alpha}.
\]

(C.1.4)

Similarly, we integrate loan and labor demand across the continuum of firms and obtain the following expressions for the loan and labor aggregation conditions.

\[
\frac{K_t}{A_{t-1}N_{t-1}} = \alpha(1 - \alpha)^{\frac{1}{\eta} + \alpha} \cdot GA_t \cdot GN \cdot \left(\frac{C_t}{A_tN_t}\right)^{\frac{1}{\eta} + \alpha} \left(\frac{Y_t}{A_tN_t}\right)^{\frac{1}{\eta} + \alpha} \left(\frac{R^K_{t+j} P^K_{t+j}}{P_t}\right)^{\frac{1}{\eta} + \alpha} \Delta_t,
\]

(C.1.5)

\[
\frac{N_t}{N_t} = (1 - \alpha)^{\frac{1}{\eta} + \alpha} \left(\frac{C_t}{A_tN_t}\right)^{-\alpha} \left(\frac{Y_t}{A_tN_t}\right)^{-\alpha} \left(\frac{R^K_{t+j} P^K_{t+j}}{P_t}\right)^{\alpha} \Delta_t^{-1},
\]

(C.1.6)

where \(\Delta_t\) is a measure of price-dispersion that can be recursively defined as

\[
\Delta_t = (1 - \theta) \left(\frac{P^*_t}{P_t}\right)^{\epsilon} + \theta \Pi_t^{\frac{1}{\eta} + \alpha} \Delta_{t-1}.
\]

(C.1.7)

Plugging the equation (C.1.4) and the expressions for \(Q_{t+j}\) into the optimal resetting price equation (equation (C.1.2)), we obtain

\[
\left(\frac{P^*_t}{P_t}\right)^{1+\epsilon} \left(\frac{1}{\eta} + \alpha\right)
\]

\[
= \mathbb{E}_t \left[ \sum_{j=0}^{\infty} (\theta \beta)^j \left(\frac{1}{\eta} + \alpha\right) \left(\frac{C_{t+j}}{A_{t+j}N_{t+j}}\right)^{-\alpha} \left(\frac{Y_{t+j}}{A_{t+j}N_{t+j}}\right)^{-\alpha} \left(\frac{P_{t+j}}{P_t}\right)^{\epsilon} \left(\frac{R^K_{t+j} P^K_{t+j}}{P_t}\right)^{\alpha} \right].
\]

(C.1.8)

We can simplify this expression as

\[
\left(\frac{P^*_t}{P_t}\right)^{1+\epsilon} = \left(\frac{F_t}{H_t}\right)^{1+\epsilon}.
\]

(C.1.9)

\(^2\)We assume \(\frac{(1+\epsilon)}{\epsilon-1} = 1\) at the efficient steady state
where $F_t$ and $H_t$ are recursively written as

\[
F_t = (1 - \alpha)^{\frac{1}{1+\alpha}} \left( \frac{C_t}{A_t \bar{N}_t} \right)^{\alpha} \left( \frac{Y_t}{A_t \bar{N}_t} \right)^{\alpha} \left( \frac{\bar{R}_t^K}{P_t} \right)^{\alpha} + \theta \beta \mathbb{E}_t \left[ \epsilon \left( \frac{n+1}{n+1} \right) F_{t+1} \right],
\]

\[
H_t = \left( \frac{C_t}{A_t \bar{N}_t} \right)^{-1} \left( \frac{Y_t}{A_t \bar{N}_t} \right) + \theta \beta \mathbb{E}_t \left[ \pi_{t+1}^{-1} H_{t+1} \right].
\]

Using equation (3.2.13) and equation (C.1.10), we obtain the following equilibrium condition for price-resetting in our framework.

\[
\frac{F_t}{H_t} = \left( \frac{1 - \theta}{1 - \theta \pi_t^{-1}} \right)^{\left( \frac{1}{\alpha} \right) \left[ 1 + \epsilon \left( \frac{\alpha}{1+\alpha} \right) \right]}.
\]

We now rewrite equation (3.2.9b) as

\[
1 = \beta \cdot \mathbb{E}_t \left[ \frac{R^S_{t+1}}{\Pi_{t+1} GA_{t+1} GN} \cdot \left( \frac{C_t}{A_t \bar{N}_t} \right) \cdot \left( \frac{C_{t+1}}{A_{t+1} \bar{N}_{t+1}} \right) \right].
\]

Since $R^S_{t+1}$ depends on bonds return $R^{HB}_{t+1}$ and loans return $R^K_{t+1}$ while shares of savings that flow into bonds $(1 - \lambda^K_t)$ and loans $(\lambda^K_t)$ are endogenous, we start from analyzing $R^{HB}_{t+1}$.

We can rewrite the aggregate return indices as functions of the bond yields \{YD^f_t\} as

\[
R^f_t = \sum_{f=0}^{F-1} \lambda_{t-1}^{f+1} \left( \frac{YD^f_t}{YD^{f+1}_{t-1}} \right)^{-(f+1)}, \quad j \in \{H, G, CB\},
\]

and also the household’s bond portfolio share as

\[
\lambda^{HB,f}_t = \left( \mathbb{E}_t \left[ \beta \cdot \frac{\bar{Z} f}{\Pi_{t+1} \cdot GA_{t+1} \cdot GN}, \left( \frac{C_t}{A_t \bar{N}_t} \right) \cdot \left( \frac{YD^f_{t+1}}{YD^f_t} \right)^{-(f+1)} \right] \right)^{\kappa_B}, \quad \forall f,
\]

\[
\Phi^B_t = \left( \sum_{j=1}^{F} \mathbb{E}_t \left[ \beta \cdot \frac{\bar{Z}^j}{\Pi_{t+1} \cdot GA_{t+1} \cdot GN}, \left( \frac{C_t}{A_t \bar{N}_t} \right) \cdot \left( \frac{YD^{j-1}_{t+1}}{YD^j_t} \right)^{-(j+1)} \right] \right)^{\frac{1}{\kappa_B}}.
\]

Now we find the equilibrium condition for the bond shares of the agents. Using bond market
equilibrium condition (equation (3.2.16)), we obtain
\[ \lambda_{HB,f}^t = \frac{B_t^G + B_t^{CB,f}}{B_t^n + B_t^{CB}} = \frac{\lambda_{t}^G B_t^G + \lambda_{CB,f} B_t^{CB}}{B_t^n + B_t^{CB}}. \]  
(C.1.12)

We can rearrange the previous expression as
\[ \lambda_{CB,f}^t = \lambda_{HB,f}^t + \left( \lambda_{HB,f}^t - \lambda_t^G \right) \cdot \frac{B_t^G}{B_t^{CB}}. \]  
(C.1.13)

Summing across maturities from \( f = 2 \) to \( F \), and using \( \sum_{f=2}^{F} \lambda_t^j = 1 - \lambda_t^1 \), \( j \in \{ H, G, CB \} \) we obtain
\[ \sum_{f=2}^{F} \lambda_t^{CB,f} = 1 - \lambda_{HB,1}^t + \left( \lambda_{G,1}^t - \lambda_{HB,1}^t \right) \cdot \frac{B_t^G}{B_t^{CB}}. \]  
(C.1.14)

Plugging equation (C.1.14) into equation (C.1.13) and after some rearrangements, we obtain
\[ \lambda_t^{CB,f} = \frac{\lambda_{t}^{HB,1} \left( \lambda_t^{CB,1} - \lambda_t^G \right) - \lambda_t^G \left( \lambda_t^{CB,1} - \lambda_{t}^{HB,1} \right)}{\lambda_{t}^{HB,1} - \lambda_t^G}, \]  
(C.1.15) \( f > 1 \).

Now, we can obtain an expression for central bank’s bond holdings using equation (C.1.14) as
\[ B_t^{CB} = \left( \frac{\lambda_{t}^{HB,1} - \lambda_t^G}{\lambda_t^{CB,1} - \lambda_{t}^{HB,1}} \right) \cdot B_t^G. \]  
(C.1.16)

Combining equation (3.2.16) and equation (C.1.16) we obtain
\[ \frac{B_t^H}{A_t N_t} = - \left( \frac{\lambda_t^{CB,1} - \lambda_t^G}{\lambda_t^{CB,1} - \lambda_{t}^{HB,1}} \right) \cdot \frac{B_t^G}{A_t N_t}. \]  
(C.1.17)

Combining \( L_t = \lambda_t^K S_t \) and \( B_t^H = (1 - \lambda_t^K) S_t \) with \( L_t = P_t^K K_t \), we obtain
\[ \frac{B_t^H}{A_t N_t P_t} = \frac{1}{GA_t \cdot GN} \left( 1 - \lambda_t^K \right) \left( \frac{P_t^K}{P_t} \right) \left( \frac{K_t}{A_{t-1} N_{t-1}} \right). \]  
(C.1.18)

Using \( B_t^H = - \left( B_t^G + B_t^{CB} \right) \), equation (C.1.16), and bond-market equilibrium condition (equation (3.2.16)), we get the following equation, which is equation (2.3.18).
\[ - \left( \frac{\lambda_t^{CB,1} - \lambda_t^G}{\lambda_t^{CB,1} - \lambda_{t}^{HB,1}} \right) \cdot \frac{B_t^G}{A_t N_t P_t} = \frac{1}{GA_t \cdot GN} \left( 1 - \lambda_t^K \right) \left( \frac{P_t^K}{P_t} \right) \left( \frac{K_t}{A_{t-1} N_{t-1}} \right). \]  
(C.1.19)
**Conventional Policy in Section 3.2.7**

Using bond market equilibrium (equation (3.2.16)) with \( \sum_{f=2}^{F} \lambda_t^{HB,f} = 1 - \lambda_t^{HB,1} \), we get

\[
B_t^H = -\frac{\sum_{i=2}^{F} (B_t^{G,i} + B_t^{CB,i})}{1 - \lambda_t^{HB,1}}. \tag{C.1.20}
\]

With equation (C.1.20), we obtain the equilibrium set of equations:

\[
\frac{\lambda_t^{HB,f}}{1 - \lambda_t^{HB,1}} = \frac{B_t^{G,f}}{A_tN_tP_t} + \frac{B_t^{CB,f}}{ANP}, \quad \forall f > 1. \tag{C.1.21}
\]

Combining equation (C.1.18) and equation (C.1.20) yields the following equilibrium equation:

\[
-\sum_{i=2}^{F} \left( \frac{B_t^{G,i}}{A_tN_tP_t} + \frac{B_t^{CB,i}}{ANP} \right) = \frac{1}{GA_t \cdot GN} \left( \frac{1 - \lambda_t^K}{\lambda_t^K} \right) \left( \frac{P_t^K}{P_t} \right) \left( \frac{K_t}{A_tN_t} \right), \tag{C.1.22}
\]

where normalized bond positions of the central bank are exogenously given.

Finally, combining equation (C.1.21) and equation (C.1.22) we finally obtain

\[
- \left( \frac{B_t^{G,f}}{A_tN_tP_t} + \frac{B_t^{CB,f}}{ANP} \right) \cdot \left( \lambda_t^{HB,f} \right)^{-1} = \frac{1}{GA_t \cdot GN} \left( \frac{1 - \lambda_t^K}{\lambda_t^K} \right) \left( \frac{P_t^K}{P_t} \right) \left( \frac{K_t}{A_tN_t} \right), \quad \forall f > 1. \tag{C.1.23}
\]

**Steady-State Derivations in Section 3.3.1**

In the steady state, the central bank decides the level of bond holdings of each maturity \( B_t^{CB,f} \) that it wants to hold. It can be calibrated to match the data of central bank’s balance sheet. Given \( \{ \lambda_t^{CB,f} \} \) and the size of its portfolio \( B_t^{CB} \), which is \( \zeta^B \) fraction of total government bond issuance satisfying \( B_t^{CB} = \zeta^B \cdot B_t^{G} \), we can obtain an steady state expression for the household bond shares as

\[
\lambda_t^{HB,f} = \frac{\lambda_t^{G,f} + \lambda_t^{CB,f} \cdot \zeta^B}{1 + \zeta^B}. \tag{C.1.24}
\]

From the definition of \( R_t^{HB} \) we have

\[
\sum_{f=1}^{F} \lambda_t^{HB,f} \cdot \left( \frac{R_t^f}{R_t^{HB}} \right) = 1.
\]
together with equation (C.2.7) and equation (C.2.8) rearranged as:

$$\lambda_{HB,f} = \left( \frac{z^f \cdot R_f}{\Phi^B} \right)_{\tilde{\Phi}^B}^\kappa, \ \forall f, \ \text{with} \ \tilde{\Phi}^B = \left[ \sum_{j=1}^F \left( \frac{z^j \cdot R_j}{R_{HB}} \right)^\kappa \right]^{1/\kappa}.$$  \hspace{1cm} (C.1.25)

The above equation (C.1.24) and equation (C.1.25) jointly determine the steady state yields and household shares. Unfortunately, there is no analytical expression for them and we have to solve for the steady state values numerically. How we proceed, relying on simple iterations:

1. Assume some initial guess for \( \left\{ \frac{R_{f,guess}}{R_{HB}} \right\}_{f=1}^F \)

2. Construct \( \tilde{\Phi}^{B,old} \) using previous guess with \( \tilde{\Phi}^B \) in equation (C.1.25)

3. Update estimates on \( \left\{ \frac{R_f}{R_{BH}} \right\}_{f=1}^F \) with the following rules

\[
\frac{R_{1,new}}{R_{HB}} = 1 - \sum_{f=2}^F \lambda_{HB,f} \left( \frac{R_f}{R_{BH}} \right)_{\lambda_{HB,1}}, \quad \frac{R_{f,new}}{R_{BH}} = \left( \lambda_{HB,f} \right)^{\frac{1}{k}} \left( \frac{z^f}{\tilde{\Phi}^{B,old}} \right), \ f > 1
\]

4. Construct new household shares \( \lambda_{HB,f,new} \) by plugging \( \left\{ \frac{R_{f,new}}{R_{HB}} \right\}_{f=1}^F \) into equation (C.1.25). Compute the discrepancy between these shares and the true ones found in equation (C.1.24). If the error is big, set \( \frac{R_{f,guess}}{R_{HB}} = \frac{R_{f,new}}{R_{HB}} \) and repeat from step 2 until convergence.

Using equation (C.2.32) and equation (C.2.43) we obtain

$$R_{HB} = \frac{\beta^{-1} \Pi \cdot GA \cdot GN}{1 - \lambda^K} - \frac{\lambda^K}{1 - \lambda^K} R^K. \hspace{1cm} (C.1.26)$$

We can rewrite \( R^G \) as

$$R^G = \Xi \cdot R_{HB}, \ \Xi = \sum_{f=1}^F \lambda^{G,f} \cdot \left( \frac{R_f}{R_{HB}} \right),$$

and using equation (C.1.26) it becomes

$$R^G = \Xi \cdot \left[ \frac{\beta^{-1} \Pi \cdot GA \cdot GN}{1 - \lambda^K} - \frac{\lambda^K}{1 - \lambda^K} R^K \right]. \hspace{1cm} (C.1.27)$$

We obtain an expression for price dispersion as

$$\Delta = \left[ \frac{1 - \theta}{1 - \theta \Pi^{\xi_{-1}}(z^f_{\xi})} \left( \frac{1 - \theta \Pi^{\xi_{-1}}(\eta)}{1 - \theta} \right) \right]^\xi_{-1}(\eta_{\xi}) \hspace{1cm} (C.1.28)$$
From the capital producer’s optimization (equation (C.2.44)), we obtain an expression for $P^K$

$$\frac{P^K}{P} = \beta^{-1} \cdot GA \cdot GN - (1 - \delta). \quad (C.1.29)$$

The equilibrium government bonds are obtained from its budget constraint (equation (C.2.20)) and written as

$$\frac{B^G}{P \bar{NA}} = \left(1 - \frac{R^G}{\Pi \cdot GA \cdot GN}\right)^{-1} \left[\zeta^G + \zeta^F - \zeta^T\right] \left(\frac{Y}{A\bar{N}}\right). \quad (C.1.30)$$

The model needs government to be a borrower, so $B^G < 0$ at steady-state. Also, we would like to match the data in which the government runs primary deficit $\zeta^G + \zeta^F - \zeta^T > 0$. The only way to achieve that is by having $R^G < \Pi \cdot GA \cdot GN$. Plugging $B^{CB} = \zeta^{CB} \cdot B^G$ and equation (C.1.29) into equation (C.1.19) yields

$$\frac{K}{A\bar{N}} = - (1 + \zeta^{CB}) \cdot GA \cdot GN \cdot \left(\frac{1}{\beta^{-1} \cdot GA \cdot GN - (1 - \delta)}\right) \left(\frac{\chi^K}{1 - \chi^K}\right) \cdot \left(\frac{B^G}{A\bar{NP}}\right). \quad (C.1.31)$$

By plugging the previous equation (C.1.30) into the market clearing condition (equation (C.2.31)), we obtain

$$\frac{K}{A\bar{N}} = \xi^K \left(1 - \frac{R^G}{\Pi \cdot GA \cdot GN}\right)^{-1} \left(\frac{\chi^K}{1 - \chi^K}\right) \left(\frac{Y}{A\bar{N}}\right), \quad (C.1.32)$$

with $\xi^K = (1 + \zeta^{CB}) \left[\zeta^G + \zeta^F - \zeta^T\right] \left(\frac{\beta \cdot GA \cdot GN}{GA \cdot GN - \beta (1 - \delta)}\right). \quad (C.1.33)$

By plugging the previous equation (C.1.32) into the market clearing condition (equation (C.2.31)), we obtain the following relation between consumption and output.

$$\frac{C}{A\bar{N}} = \left[1 - \xi^C \left(1 - \frac{R^G}{\Pi \cdot GA \cdot GN}\right)^{-1} \left(\frac{\chi^K}{1 - \chi^K}\right)\right] \left(\frac{Y}{A\bar{N}}\right), \quad (C.1.34)$$

with $\xi^C = \left[1 - \frac{1 - \delta}{GA \cdot GN}\right] \xi^K$. 

The steady state representation of firms’ pricing (equation (C.2.14) and equation (C.2.15))
can be written as

\[ F = \xi^F \left[ (1 - \zeta^G) - \xi^C \left( 1 - \frac{R^G}{\Pi G A N} \right)^{-1} \left( \frac{\lambda^K}{1 - \lambda^K} \right) \right]^{-}(\frac{\eta_1}{\eta + 1})^\alpha \left( \frac{Y}{AN} \right)^{(1 - \alpha)(\frac{\eta_1}{\eta + 1})} (R^K)^{\alpha \frac{\eta_1}{\eta + 1}}. \]

(C.1.35)

with \( \xi^F = (1 - \alpha)^{\frac{\eta_1}{\eta + 1}} \left[ 1 - \theta \beta \Pi^{\frac{\eta_1}{\eta + 1}} \right]^{-1} \left( \frac{1 + \xi^F}{\epsilon - 1} \right) \left[ \frac{G A \cdot G N - (1 - \delta) \beta}{\Pi \cdot G A \cdot G N} \right] \alpha \left( \frac{\eta_1}{\eta + 1} \right)^{\alpha \frac{\eta_1}{\eta + 1}} \).

\[ H = [1 - \theta \beta \Pi^{-1}]^{-1} \left[ (1 - \zeta^G) - \xi^C \left( 1 - \frac{R^G}{\Pi G A \cdot G N} \right)^{-1} \left( \frac{\lambda^K}{1 - \lambda^K} \right) \right]^{-1}. \]

(C.1.36)

Plugging equation (C.1.35) and equation (C.1.36) into firms’ optimal price-resetting equation (equation (C.1.11)) and rearranging the resulting equation, we obtain

\[ \frac{Y}{AN} = \xi^Y \left[ (1 - \zeta^G) - \xi^C \left( 1 - \frac{R^G}{\Pi G A \cdot G N} \right)^{-1} \left( \frac{\lambda^K}{1 - \lambda^K} \right) \right]^{-}(\frac{\eta_1}{\eta + 1}) \left( R^K \right)^{(\frac{\eta_1}{\eta + 1})}, \]

(C.1.37)

with \( \xi^Y = (\xi^F)^{-}(\frac{\eta_1}{\eta + 1})(1 - \theta \beta \Pi^{-1}) \left[ (1 - \zeta^G) - \xi^C \left( 1 - \frac{R^G}{\Pi G A \cdot G N} \right)^{-1} \left( \frac{\lambda^K}{1 - \lambda^K} \right) \right]^{-}(\frac{\eta_1}{\eta + 1}). \)

(C.1.38)

Finally, plugging equation (C.1.32), equation (C.1.34) and equation (C.1.37) into loan aggregation equation and rearranging properly, we obtain the following relation.

\[ \xi^{R^K} = R^K \left( 1 - \frac{R^G}{\Pi \cdot G A \cdot G N} \right)^{-1} \left( \frac{\lambda^K}{1 - \lambda^K} \right) \]

(C.1.39)

\[ = \alpha(1 - \alpha)^{\frac{1}{\eta + 1}} \cdot G A \cdot G N \cdot \frac{\eta + a}{(1 - \alpha)(\eta + 1)} \left( \frac{G A \cdot G N - (1 - \delta) \beta}{\Pi \cdot G A \cdot G N} \right)^{\alpha \frac{\eta_1}{\eta + 1}} \left( \xi^Y \right)^{(1 - \alpha)(\frac{\eta_1}{\eta + 1})}. \]

As \( \xi^{R^K} \) is constant, after plugging equation equation (C.1.27) into equation (C.1.39) and rearranging the equation, we obtain

\[ R^K = \left[ 1 - \frac{\Xi \xi^{R^K}}{\Pi \cdot G A \cdot G N} \right]^{-1} \left[ \frac{1 - \lambda^K}{\lambda^K} - \beta^{-1} \Xi \right] \xi^{R^K}. \]

(C.1.40)

Finally, by plugging equation (C.1.40) into equation (C.1.26), we get

\[ R^{HB} = \frac{\beta^{-1} \Pi \cdot G A \cdot G N}{1 - \lambda^K} \left[ 1 - \frac{\Xi \cdot \xi^{R^K}}{\Pi \cdot G A \cdot G N} \right]^{-1} \left[ 1 - \frac{\beta^{-1} \Xi}{1 - \lambda^K} \right] \xi^{R^K}. \]

(C.1.41)

Equations equation (C.1.40) and equation (C.1.41) plugged into equation (C.2.40) form a non-linear equation of the unknown \( \lambda^K \). We obtain its value by relying on computational methods, and then we can back out the rest of the steady state variables. Once we back out \( R^K \) and \( \lambda^K \), we can back out \( R^{HB} \) using equation (C.1.26). After that, we can then simply
back out bond returns as

\[ R' = R^{HB} \cdot \left( \frac{R'_f}{R^{HB}} \right). \]

Now that we have found the bond returns, we can recursively obtain the bond yields using

\[ YD' = \left[ R' \cdot (YD'^{-f-1})^{-1} \right]^{1/f}, \]

where \( YD^0 = 1 \), which we use to get started with the recursion from \( f = 1 \) to \( F \).

**Log-linearization**

We start by log-linearizing the equations that are common to the conventional policy model and the QE one, then derive the ones that are different.

Log-linearize equations equation (C.2.26), equation (C.2.27) and equation (C.2.28) to obtain

\[ \hat{g}_t = \hat{\varepsilon}_t^A, \quad \hat{\xi}_t^G = \frac{\hat{a}_t^G}{1 + \hat{a}_t^G} \cdot \hat{u}_t^G, \quad \hat{\xi}_t^T = \frac{\hat{a}_t^T}{1 + \hat{a}_t^T} \cdot \hat{u}_t^T. \tag{C.1.42} \]

Equations (equation (C.2.1) and equation (C.2.2)) with the help of equation (C.1.42) can be linearized as

\[ \hat{y}_t = \mathbb{E}_t \left[ \hat{y}_{t+1} - \left( 1 - \xi_t^G \right) \cdot \frac{Y}{C} \left( \hat{y}_t - \frac{1}{1 + \hat{a}_t^G} \cdot \hat{u}_t^G \right) + \left( 1 - \delta \right) \frac{K}{GA \cdot GN} \left( \hat{k}_t - \hat{\varepsilon}_t^A \right) - \frac{K}{C} \hat{k}_{t+1} \right], \tag{C.1.43} \]

\[ \hat{c}_t = \mathbb{E}_t \left[ \hat{c}_{t+1} - \left( \hat{r}^S_{t+1} - \hat{\pi}_{t+1} \right) \right], \tag{C.1.44} \]

where I used equation (C.1.42) to solve for \( \hat{\xi}_t^G \) and \( \hat{g}_t \).

Plugging equation (C.1.43) into equation (C.1.44) and using equation (C.2.29), we obtain the following dynamic IS equation for output \( \hat{y}_t \).

\[ \hat{y}_t = \mathbb{E}_t \left[ \hat{y}_{t+1} - \left( \hat{1} - \xi_t^G \right) \cdot \frac{Y}{C} \left( \hat{y}_t - \frac{1}{1 + \hat{a}_t^G} \cdot \hat{u}_t^G \right) + \left( 1 - \delta \right) \frac{K}{GA \cdot GN} \left( \hat{k}_t - \hat{\varepsilon}_t^A \right) - \frac{K}{C} \hat{k}_{t+1} \right. \]

\[ \left. - \left( \hat{1} - \xi_t^G \right) \cdot \frac{K}{C} \hat{k}_{t+2} - \left( \hat{1} - \xi_t^G \right) \cdot \frac{C}{Y} \left( \hat{r}^S_{t+1} - \hat{\pi}_{t+1} \right) + \frac{1 - \rho_G}{1 + \hat{a}_t^G} \cdot \hat{u}_t^G \right]. \tag{C.1.45} \]

Linearizing the household’s bond portfolio conditions (equation (C.2.7) and equation (C.2.8)) yields

\[ \hat{\lambda}^{HB,f}_t = \kappa^B \mathbb{E}_t \left[ \hat{z}^f_{t+1} - \hat{g}_t \hat{a}_{t+1} + \hat{c}_t - \hat{c}_{t+1} - (f - 1)\hat{y}d_{t+1}^f + f\hat{y}d_t^f - \hat{\phi}_t^B \right]. \tag{C.1.46} \]
\[ \hat{\phi}_t^B = \mathbb{E}_t \left( -\hat{\pi}_{t+1} - \hat{g}_a_{t+1} + \hat{c}_t - \hat{c}_{t+1} \right) + \sum_{j=1}^{F} \left[ \frac{\beta z^j (YD^{j-1})^{-j}}{\prod \cdot GA \cdot GN \cdot \Phi^B (YD^j)^{-j}} \right] \kappa^B \hat{z}^j_t \]

\[ + \sum_{j=1}^{F} \left[ \frac{\beta z^j (YD^{j-1})^{-j}}{\prod \cdot GA \cdot GN \cdot \Phi^B (YD^j)^{-j}} \right] \kappa^B \hat{y}^j_t, \quad (C.1.47) \]

\[ - \sum_{j=0}^{F-1} \left[ \frac{\beta z^{j+1} (YD^j)^{-j}}{\prod \cdot GA \cdot GN \cdot \Phi^B (YD^j)^{-j}} \right] \kappa^B \mathbb{E}_t (\hat{y}^j_{t+1}). \quad (C.1.48) \]

Combining equation (C.1.46) and equation (C.1.48), we obtain the following expression for \( \hat{\lambda}_{t}^{HB,f} \):

\[ \hat{\lambda}_{t}^{HB,f} = \sum_{j=1}^{F} \psi_{1}^{fj} \hat{z}_t + \sum_{j=1}^{F} \psi_{2}^{fj} \hat{y}^j_t + \sum_{j=1}^{F} \psi_{3}^{fj} \mathbb{E}_t \left[ \hat{y}^j_{t+1} \right], \quad (C.1.49) \]

where

\[ \psi_{1}^{fj} = \begin{cases} 
1 - \left[ \frac{\beta \cdot z^j (YD^{j-1})^{-j}}{\prod \cdot GA \cdot GN \cdot \Phi^B (YD^j)^{-j}} \right] \kappa^B , & \text{if } f = j, \\
- \left[ \frac{\beta \cdot z^j (YD^{j-1})^{-j}}{\prod \cdot GA \cdot GN \cdot \Phi^B (YD^j)^{-j}} \right] \kappa^B , & \text{if } f \neq j, 
\end{cases} \]

\[ \psi_{2}^{fj} = j \cdot \psi_{1}^{fj}, \]

\[ \psi_{3}^{fj} = \begin{cases} 
-j \cdot \left[ 1 - \left[ \frac{\beta \cdot z^{j+1} (YD^j)^{-j}}{\prod \cdot GA \cdot GN \cdot \Phi^B (YD^j)^{-j}} \right] \kappa^B \right] \kappa^B , & \text{if } j = f - 1, \\
j \cdot \left[ \frac{\beta \cdot z^{j+1} (YD^j)^{-j}}{\prod \cdot GA \cdot GN \cdot \Phi^B (YD^j)^{-j}} \right] \kappa^B , & \text{if } j \neq f - 1, \\
0 , & \text{if } j = F. \]

We can put the system of \( F \) equation in matrix format as

\[ \overrightarrow{\lambda}_{t}^{HB} = \psi_{1} \overrightarrow{z}_t + \psi_{2} \overrightarrow{y}_t + \psi_{3} \mathbb{E}_t \left[ \overrightarrow{y}^j_{t+1} \right], \quad (C.1.50) \]

where \( \{\psi_{1}, \psi_{2}, \psi_{3}\} \) are matrices containing elements of \( \{\psi_{1}^{fj}, \psi_{2}^{fj}, \psi_{3}^{fj}\} \), with \( f \) representing rows and \( j \) columns.
Linearizing equations equation (C.2.21) and equation (C.2.29) yields the following expression:

\[ \hat{\lambda}_t^G = \Xi \cdot \hat{u}_t^B, \quad \hat{\mu}_t^B = \Upsilon^B \cdot \hat{u}_t^B. \]

where \( \Xi \) is a matrix whose elements \( \Xi_{fj} \) (f-rows, j-columns) are

\[
\Xi_{fj} = \begin{cases} 
0, & \text{if } f = 1 \& j = f, \\
1 - \lambda^G f, & \text{if } f \geq 2 \& j = f, \\
-\lambda^G j, & \text{if } j \neq f,
\end{cases}
\]

and similarly \( \Upsilon^B \) is a matrix containing elements \( \tau_f^B \) from equation (C.2.23). By defining \( \Xi = \Xi \cdot \Upsilon^B \), we can combine the previous two equations to obtain

\[ \hat{\lambda}_t^G = \Xi \cdot \hat{u}_t^B. \] (C.1.51)

Linearizing equation (C.2.25) with the help of equation (C.1.51) yields

\[ \hat{b}_t^G = \Xi \cdot \hat{u}_t^B + \hat{F}_{x1} \cdot \hat{b}_t^G, \] (C.1.52)

where \( \hat{F}_{x1} \) is a unit vector of size \( F \).

Log-linearizing the household’s stochastic discount factor yields the following formula.

\[ \hat{q}_{t,t+1} = \hat{c}_t - \hat{r}_{t+1} - \hat{g}_{t+1} - \hat{d}^a_{t+1}. \] (C.1.53)

Log-linearizing \( \Phi^S_T \) in the household’s portfolio between loans and bonds (equation (C.2.41)), we obtain

\[
\hat{\lambda}_t^K = \kappa^S \left( \frac{(z^B R^H B)^{\kappa^S}}{(z^B R^H B)^{\kappa^S} + (z^K R^K)^{\kappa^S}} \right) (\hat{z}_t^K + E_t [\hat{r}^K_{t+1}]) + \frac{(z^K R^K)^{\kappa^S}}{(z^B R^H B)^{\kappa^S} + (z^K R^K)^{\kappa^S}} (\hat{z}_t^K + E_t [\hat{r}^K_{t+1}] - \hat{r}^H B_{t+1}).
\] (C.1.54)

Log-linearizing the household’s portfolio decision between loans and bonds (equation (C.2.40)) and making use of the previous expression (equation (C.1.54)), we obtain

\[
\hat{\lambda}_t^K = \kappa^S (1 - \lambda^K) (\hat{z}_t^K + E_t [\hat{r}^K_{t+1}]) - \lambda^K R^K (\hat{r}^H B_{t+1} - \hat{r}^K_{t+1}).
\] (C.1.55)

By linearizing the formula for the effective savings rate of the household (equation (C.2.12)), we obtain

\[
\hat{r}_t^S = \frac{\lambda^K (R^K - R^H B)}{R^S} \hat{\lambda}_{t-1}^K + \frac{(1 - \lambda^K) R^H B}{R^S} \hat{r}_{t+1}^H B + \frac{\lambda^K R^K}{R^S} \hat{r}_t^K.
\] (C.1.56)
Log-linearizing the effective bond rates (equation (C.2.11)) of household, government, and the central bank yields

$$\hat{r}_t^j = \sum_{f=1}^{F} \frac{\lambda_{j,f}^f (YD^f - 1)^{(f-1)}}{R_j^f (YD^f)^{-f}} \cdot \left[ \hat{\lambda}_{t-1}^f - (f-1) \hat{y}_d^f + f \hat{y}_{d_{t-1}}^f \right], \; j \in \{HB, G, CB\},$$

(C.1.57)

with which we can express these equations on matrix format as

$$\hat{r}_t^j = \Psi_{j,4} \cdot \hat{\lambda}_{t-1}^j - \Psi_{j,5} \cdot \hat{y}_d^j + \Psi_{j,6} \cdot \hat{y}_{d_{t-1}}, \; j \in \{HB, G, CB\},$$

(C.1.58)

where \{\Psi_{j,4}, \Psi_{j,5}, \Psi_{j,6}\} are 1xF-sized matrices whose elements are defined as follows.

$$\Psi_{j,4}^1 = \frac{\lambda_{j,f}^f (YD^f - 1)^{(f-1)}}{R_j^f (YD^f)^{-f}},$$

$$\Psi_{j,5}^1 = \left\{ \begin{array}{ll} \frac{\lambda_{j,f+1}^f (YD^f)^{-f}}{R_j^f (YD^f)^{-(f+1)}}, & \text{if } f < F, \; j \in \{HB, G, CB\} \\ 0, & \text{if } f = F \end{array} \right\},$$

$$\Psi_{j,6}^1 = \frac{\lambda_{j,f}^f (YD^f - 1)^{(f-1)}}{R_j^f (YD^f)^{-f}} f.$$

By plugging equation (C.1.51) into \(\hat{r}^G\) in equation (C.1.58), we obtain

$$\hat{r}_t^G = \Psi_{G,4} \cdot \hat{\lambda}_{t-1}^G - \Psi_{G,5} \cdot \hat{y}_d^G + \Psi_{G,6} \cdot \hat{y}_{d_{t-1}}^G.$$

(C.1.59)

By plugging equation (C.1.50) into \(\hat{r}^{HB}\) in equation (C.1.58), we obtain

$$\hat{r}_t^{HB} = \Psi_{HB,4} \cdot \hat{\lambda}_{t-1}^{HB} + \left[ \Psi_{HB,4} \hat{\psi}^2 + \Psi_{HB,6} \right] \cdot \hat{y}_d^{t-1} + \Psi_{HB,4} \cdot \hat{\psi}^3 \cdot \hat{z}_{t-1} - \Psi_{HB,5} \cdot \hat{y}_d^t.$$

(C.1.60)

Taking the expectation operator \(E_t\) on the previous equation (C.1.60), we obtain

$$E_t \left[ \hat{r}_{t+1}^{HB} \right] = \Psi_{HB,4} \cdot \hat{\lambda}_{t}^{HB} + \left[ \Psi_{HB,4} \hat{\psi}^2 + \Psi_{HB,6} \right] \cdot \hat{y}_d^{t} + \left[ \Psi_{HB,4} \hat{\psi}^3 - \Psi_{HB,5} \right] E_t \left[ \hat{y}_d^{t+1} \right].$$

(C.1.61)

By plugging equation (C.1.55) and equation (C.1.61) into equation (C.1.56), we now obtain the expected effective savings rate as follows.\(^3\)

$$E_t \left[ \hat{r}_{t+1}^{S} \right] = \Psi_7 \cdot \hat{\lambda}_{t}^{S} + \Psi_8 \cdot \hat{y}_d^{t} + \Psi_9 E_t \left[ \hat{y}_d^{t+1} \right] + \Psi_10 \cdot \hat{r}_{t+1}^{K} + \Psi_11 \cdot \hat{z}_{t}^{K}.$$

(C.1.62)

\(^3\)Since \(\hat{r}_{t+1}^{K}\) is determined at quarter \(t\), thus \(E_t (\hat{r}_{t+1}^{K}) = \hat{r}_{t+1}^{K}\) holds.
where
\[
\psi^7 = \psi^{HB,4} \psi^1 \left[ \frac{(1 + \kappa^S \lambda^K)(1 - \lambda^K) R^{HB} - \kappa^S (1 - \lambda^K) \lambda^K R^K}{R^S} \right],
\]
\[
\psi^8 = \left[ \psi^{HB,4} \psi^2 + \psi^{HB,6} \right] \left[ \frac{(1 + \kappa^S \lambda^K)(1 - \lambda^K) R^{HB} - \kappa^S (1 - \lambda^K) \lambda^K R^K}{R^S} \right],
\]
\[
\psi^9 = \left[ \psi^{HB,4} \psi^3 - \psi^{HB,5} \right] \left[ \frac{(1 + \kappa^S \lambda^K)(1 - \lambda^K) R^{HB} - \kappa^S (1 - \lambda^K) \lambda^K R^K}{R^S} \right],
\]
\[
\psi^{10} = \frac{1 + \kappa^S (1 - \lambda^K)}{R^S} \lambda^K R^K - \frac{\kappa^S (1 - \lambda^K) \lambda^K R^{HB}}{R^S},
\]
\[
\psi^{11} = \frac{\kappa^S \lambda^K (1 - \lambda^K)}{R^S} (R^K - R^{HB}).
\]

Plugging back the expression of the household’s expected bonds rate (equation (C.1.61)) into her portfolio decision between loans and bonds (equation (C.1.55)), we obtain
\[
\hat{\lambda}_t^K = \kappa^S (1 - \lambda^K) (\hat{z}_t^K + \hat{r}_{t+1}^K) - \psi^{12} \cdot \hat{z}_t - \psi^{13} \cdot \hat{d}_t - \psi^{14} \cdot \mathbb{E}_t \left[ \hat{y}_d_{t+1} \right], \quad (C.1.63)
\]
where
\[
\psi^{12} = \kappa^S (1 - \lambda^K) \psi^{HB,4} \psi^1,
\]
\[
\psi^{13} = \kappa^S (1 - \lambda^K) \left[ \psi^{HB,4} \psi^2 + \psi^{HB,6} \right],
\]
\[
\psi^{14} = \kappa^S (1 - \lambda^K) \left[ \psi^{HB,4} \psi^3 - \psi^{HB,5} \right].
\]

If we linearize loan aggregation equation (equation (C.2.19)), we obtain
\[
\hat{k}_t = \hat{g}_a_t + \left( \frac{\eta + 1}{\eta + \alpha} \right) \hat{y}_t - \left( \frac{\eta (1 - \alpha)}{\eta + \alpha} \right) \mathbb{E}_t \left[ \hat{q}_{t+1} + \hat{r}_{t+1}^K + \hat{\beta}_t^K - \hat{\varepsilon}_t \right]. \quad (C.1.64)
\]

By plugging equation (C.1.53) into equation (C.1.64) and using equation (C.1.42) with rearranging, we obtain
\[
p^K_t = \left( \frac{\eta + 1}{\eta (1 - \alpha)} \right) \hat{y}_t - \left( \frac{\eta + \alpha}{\eta (1 - \alpha)} \right) \left[ \hat{k}_t - \hat{\varepsilon}_t^A \right] + \mathbb{E}_t \left[ \hat{c}_{t+1} + \hat{\pi}_{t+1} - \hat{r}_{t+1}^K \right]. \quad (C.1.65)
\]

---

4We have the first-order price dispersion $\hat{\Delta}_t$, generated from the positive trend inflation. We ignore its roles in most cases other than the welfare derivation. For this issue, see Woodford (2003), Coibion et al. (2012), and Carreras et al. (2016).
Combining equation (C.2.29), equation (C.1.43), and equation (C.1.65) we obtain

\[ p^K_t = \left( \frac{\eta + 1}{\eta(1 - \alpha)} \right) \hat{y}_t - \left( \frac{\eta + \alpha}{\eta(1 - \alpha)} \right) \left[ \hat{k}_t - \hat{\epsilon}_t^A \right] + \left[ \frac{a^G}{1 + a^G} \frac{\gamma}{C} \right] \left[ \hat{\pi}_t [\hat{y}_t + 1] - \frac{\rho_G}{1 + a^G} \hat{\pi}_t^G \right] \\
+ \left[ \frac{1 - \delta}{GA \cdot GN} \right] \hat{k}_{t+1} - \frac{K}{C} \hat{\pi}_t \left[ k_{t+2} \right] + \hat{\pi}_t [\hat{\pi}_{t+1}] - \hat{p}_t^K. \] (C.1.66)

If we linearize the supply block (equation (C.2.14), equation (C.2.15), and equation (C.2.16)), we obtain

\[ \hat{f}_t = 1 - \theta \beta \hat{\pi}^e \left( \frac{\eta + 1}{\eta(1 - \alpha)} \right) \left[ \hat{y}_t + \alpha \hat{\pi}_t \left[ \hat{y}_{t,t+1} + \hat{p}^K_{t,t+1} + \hat{\pi}^K_t - \hat{c}_t \right] \right] \\
+ \theta \beta \hat{\pi}^e \left( \frac{\eta + 1}{\eta(1 - \alpha)} \right) \left[ \frac{\eta + 1}{\eta(1 - \alpha)} \right] \left[ \hat{\pi}_{t+1} + \hat{f}_{t+1} \right]. \] (C.1.67)

\[ \hat{h}_t = 1 - \theta \beta \hat{\pi}^e - \left[ \hat{y}_t - \hat{c}_t \right] + \theta \beta \hat{\pi}^e - \left[ \hat{y}_t + \hat{\pi}_{t+1} + \hat{h}_{t+1} \right]. \] (C.1.68)

\[ \hat{f}_t - \hat{h}_t = \left[ 1 + \epsilon \left( \frac{1 - \alpha}{\eta(1 - \alpha)} \right) \right] \left( \frac{\theta \hat{\pi}^e}{1 - \theta \hat{\pi}^e} \right) \hat{\pi}_t. \] (C.1.69)

Plugging equation (C.1.64) into equation (C.1.70) and using equation (C.1.42), we obtain

\[ \hat{f}_t = 1 - \theta \beta \hat{\pi}^e \left( \frac{\eta + 1}{\eta(1 - \alpha)} \right) \left[ \hat{y}_t - \alpha \left( \hat{k}_t - \hat{\epsilon}_t^A \right) \right] + \theta \beta \hat{\pi}^e \left( \frac{\eta + 1}{\eta(1 - \alpha)} \right) \left[ \frac{\eta + 1}{\eta(1 - \alpha)} \right] \left[ \hat{\pi}_{t+1} + \hat{f}_{t+1} \right]. \] (C.1.70)

Plugging equation (C.1.43) into equation (C.1.68)

\[ \hat{h}_t = 1 - \theta \beta \hat{\pi}^e - \left[ \hat{y}_t - \hat{c}_t \right] + \theta \beta \hat{\pi}^e - \left[ \hat{y}_t + \hat{\pi}_{t+1} + \hat{h}_{t+1} \right]. \] (C.1.71)

Linearizing the government’s budget constraint (equation (C.2.20)) yields

\[ \hat{b}^G_t = \frac{R^G}{A \cdot GN} \left[ \hat{r}^G_t - \hat{\pi}_t - \hat{g}_t = \hat{b}_{t-1} \right] \]
\[ - \left[ \xi^G + \xi^F - \xi^T \right] \left( \frac{Y}{B/P} \right) \left[ \hat{y}_t + \left( \frac{\zeta^G}{\xi^G + \xi^F - \xi^T} \right) \right] \left( \frac{a^G}{1 + a^G} \right) \hat{u}^G_t \]
\[ - \left( \frac{\zeta^T}{\xi^G + \xi^F - \xi^T} \right) \right] \left( \frac{a^T}{1 + a^T} \right) \hat{u}^T_t. \] (C.1.72)

Using the steady state equilibrium condition (equation (C.1.30)) with equation (C.1.42)
and equation (C.1.59), we can express the previous equation (C.1.73) as

\[
\hat{b}_t^G = \frac{R^G_{\Psi}}{\prod GAGN} \left[ \psi^{G,4} \cdot \hat{\mu}_t^{\Psi,4} + \psi^{G,5} \cdot \hat{\nu}_t^{\Psi,5} - \psi^{G,6} \cdot \hat{\nu}_t^{\Psi,6} - \hat{\pi}_t - \hat{\xi}_t^A + \hat{b}_{t-1}^G \right] + \left(1 - \frac{R^G_{\Psi}}{\prod GAGN} \right) \left[ \hat{\mu}_t + \left( \frac{\zeta^G}{\zeta^G + \zeta^F - \zeta^T} \right) \left( \frac{\hat{a}^G}{1 + \hat{a}^G} \right) \hat{b}_t^G - \left( \frac{\zeta^T}{\zeta^G + \zeta^F - \zeta^T} \right) \left( \frac{\hat{a}^T}{1 + \hat{a}^T} \right) \hat{b}_t^G \right].
\] (C.1.74)

Linearizing the labor aggregation condition (equation (C.2.18)) yields

\[
\hat{h}_t = -\alpha \left( \frac{\eta}{\eta + \alpha} \right) \hat{c}_t + \left( \frac{\eta}{\eta + \alpha} \right) \hat{y}_t + \alpha \left( \frac{\eta}{\eta + \alpha} \right) \left[ \mathbb{E}_t (\hat{q}_{t,t+1}) + \hat{r}_{t+1}^K + \hat{\beta}_{t+1} \right].
\] (C.1.75)

Linearizing the capital producer's optimization condition (equation (C.2.13)) yields

\[
0 = \mathbb{E}_t \left[ \hat{q}_{t,t+1} + \hat{\pi}_{t+1} + \left( \frac{\hat{p}_t^K / \hat{P}}{1 - \hat{\delta} + \hat{p}_t^K / \hat{P}} \right) \hat{\beta}_{t+1} \right].
\] (C.1.76)

By plugging equation (C.1.44) and equation (C.1.53) into the previous equation (C.1.76) and rearranging, we get

\[
\mathbb{E}_t \left[ \hat{r}_{t+1}^K - \hat{\pi}_{t+1} \right] = \left( \frac{\hat{p}_t^K / \hat{P}}{1 - \hat{\delta} + \hat{p}_t^K / \hat{P}} \right) \mathbb{E}_t \left[ \hat{\beta}_{t+1} \right].
\] (C.1.77)

Plugging expressions on the effective savings rate (equation (C.1.62)) and the rental price of capital (equation (C.1.66)) into equation (C.1.77) we obtain

\[
\hat{r}_{t+1}^K = -\psi^{15} \cdot \hat{z}_t - \psi^{16} \hat{z}_{t-K} - \psi^{17} \hat{y}_t \hat{d}_t - \psi^{18} \mathbb{E}_t \left[ \hat{y}_t \hat{d}_{t+1} \right] + \psi^{19} \mathbb{E}_t \left[ \hat{\pi}_{t+1} \right] + \psi^{20} \mathbb{E}_t \left[ \hat{\pi}_{t+2} \right]
+ \psi^{21} \mathbb{E}_t \left[ \hat{y}_{t+1} \right] + \psi^{22} \mathbb{E}_t \left[ \hat{y}_{t+2} \right] - \psi^{23} \hat{\kappa}_{t+1}
+ \psi^{24} \mathbb{E}_t \left[ \hat{\kappa}_{t+2} \right] - \psi^{25} \mathbb{E}_t \left[ \hat{\kappa}_{t+3} \right] - \psi^{26} \hat{b}_t^G - \psi^{27} \mathbb{E}_t \left[ \hat{r}_{t+2}^K \right],
\] (C.1.78)

where we defined

\[
\psi^{15} = \left( \psi^{10} \right)^{-1} \psi^7, \psi^{16} = \left( \psi^{10} \right)^{-1} \psi^{11}, \psi^{17} = \left( \psi^{10} \right)^{-1} \psi^8, \psi^{18} = \left( \psi^{10} \right)^{-1} \psi^6, \psi^{19} = \left( \psi^{10} \right)^{-1},
\psi^{20} = \left( \psi^{10} \right)^{-1} \left( \frac{\hat{p}_t^K / \hat{P}}{1 - \hat{\delta} + \hat{p}_t^K / \hat{P}} \right), \psi^{21} = \psi^{20} \left( \frac{\eta + 1}{\eta (1 - \alpha)} \right), \psi^{22} = \psi^{20} \left[ \frac{\hat{a}^G}{1 + \hat{a}^G} \cdot \frac{\hat{r}_{t+1}^K}{\hat{C}} \right],
\psi^{23} = \psi^{20} \left( \frac{\eta + \alpha}{\eta (1 - \alpha)} \right), \psi^{24} = \psi^{20} \left[ \frac{1 - \hat{\delta}}{GAGN} \cdot \frac{K}{\hat{C}} \right], \psi^{25} = \psi^{20} \frac{K}{\hat{C}}, \psi^{26} = \psi^{22} \left( \hat{p}_G \right)^2.
\]

Finally, plugging the effective savings rate (equation (C.1.62)) into the Euler equation (equa-
\[
\hat{y}_t = \mathbb{E}_t \left[ \hat{y}_{t+1} + \psi^{27} \pi_{t+1} - \psi^{28} \hat{z}_t - \psi^{29} \hat{z}_t^K - \psi^{30} \hat{y}_t + \psi^{31} \mathbb{E}_t \left[ \hat{y}_{t+1} \right] 
- \psi^{32} \hat{r}_{t+1} - \psi^{33} (\hat{k}_t - \hat{\epsilon}_t^A) + \psi^{34} \hat{k}_{t+1} - \psi^{35} \hat{k}_{t+2} + \psi^{36} \hat{\nu}_t^G \right],
\]

(C.1.79)

where we defined

\[
\begin{align*}
\psi^{27} &= (1 - \zeta^G)^{-1} \frac{C}{Y}, \\
\psi^{28} &= \psi^{27} \psi^7, \\
\psi^{29} &= \psi^{27} \psi^{11}, \\
\psi^{30} &= \psi^{27} \psi^8, \\
\psi^{31} &= \psi^{27} \psi^9, \\
\psi^{32} &= \psi^{27} \psi^{10}, \\
\psi^{33} &= \frac{(1 - \zeta^G)^{-1} (1 - \delta) K}{GA \cdot GN} \frac{1}{Y}, \\
\psi^{34} &= (1 - \zeta^G)^{-1} \left[ 1 + \frac{1 - \delta}{GA \cdot GN} \right] K \frac{1}{Y}, \\
\psi^{35} &= (1 - \zeta^G)^{-1} \frac{K}{Y}, \\
\psi^{36} &= \frac{1 - \rho_G}{1 + a^G}.
\end{align*}
\]

### Log-linearization: Conventional Policy Specific Derivations

Linearizing bond market equilibrium condition (equation (C.1.23)) using equation (C.1.42), we obtain

\[
\lambda_{HB,f}^{HB,f} = \left( \frac{B_{G,f}^G}{B_{G,f}^G + B_{CB,f}^G} \right) \hat{b}_{t,f}^{g,f} + \epsilon_t^A + \frac{1}{1 - \lambda^K} \hat{\lambda}_t^K - \hat{\beta}_t^K - \hat{k}_t, \quad f \geq 2.
\]

(C.1.80)

From \( \lambda_{HB,1}^{HB,1} = 1 - \sum_{f=2}^{F} \lambda_t^{HB,f} \) we obtain

\[
\hat{\lambda}_t^{HB,1} = - \sum_{f=2}^{F} \lambda_t^{HB,f} \hat{\lambda}_t^{HB,f}.
\]

(C.1.81)

We can rearrange the previous expressions (equation (C.1.80) and equation (C.1.81)) in the matrix form as

\[
\Theta^1 \cdot \hat{\lambda}_t^{HB} = \Theta^2 \cdot \hat{b}_t^g + \Theta^3 \cdot \epsilon_t^A - \Theta^3 \cdot \hat{\beta}_t^K - \Theta^3 \cdot \hat{k}_t + \Theta^4 \cdot \hat{\lambda}_t^K.
\]

(C.1.82)
where \( \{\Theta^1, \Theta^2\} \) are \( FxF \)-sized matrices with elements \( \Theta^1_{jf} \) (row \( j \), column \( f \)) and \( \{\Theta^3, \Theta^4\} \) are \( Fx1 \) vectors with \( j \)-element \( \Theta^3_{j1} \). We define their elements as

\[
\Theta^1_{jf} = \begin{cases} 1, & \text{if } j = f, \\ \frac{\lambda^H_{B,f}}{\lambda^H_{B,1}}, & \text{if } j = 1 & \& f > 1, \end{cases}
\]

\[
\Theta^2_{jf} = \begin{cases} B^G_{f,f} + B^C_{B,f}, & \text{if } j > 1 & \& j = f, \\ 0, & \text{otherwise}, \end{cases}
\]

\[
\Theta^3_{j1} = \begin{cases} 0, & \text{if } j = 1, \\ 1, & \text{otherwise}, \end{cases}
\]

\[
\Theta^4 = \frac{1}{1 - \lambda^K} \cdot \Theta^3.
\]

By inverting \( \Theta^1 \) in equation (C.1.82), we can rewrite equation (C.1.82) as

\[
\hat{\lambda}^H_t = \Theta^5 \cdot b^g_t + \Theta^6 \cdot \hat{\varepsilon}_t^A - \Theta^6 \cdot \hat{\rho}_t^K - \Theta^6 \cdot \hat{k}_t + \Theta^7 \cdot \hat{\lambda}_t^K,
\]

where

\[
\Theta^5 = (\Theta^1)^{-1} \cdot \Theta^2, \quad \Theta^6 = (\Theta^1)^{-1} \cdot \Theta^3, \quad \Theta^7 = (\Theta^1)^{-1} \cdot \Theta^4.
\]

Plugging the government’s bond portfolio (equation (C.1.52)), the household’s loan share (equation (C.1.63)), and the rental price of capital (equation (C.1.66)) into equation (C.1.83), we obtain

\[
\hat{\lambda}^H_t = \Theta^8 b^g_t - \Theta^9 \hat{y}_t - \Theta^{10} \hat{\pi}_t + \Theta^{11} \hat{k}_t - \Theta^{12} \hat{\pi}_{t+1} + \Theta^{13} \hat{k}_{t+1} + \Theta^{14} \hat{\pi}_{t+1} - \Theta^{15} \hat{y}_t d_t - \Theta^{16} \hat{\pi}_t \left[ \hat{y}_t d_{t+1} \right] - \Theta^{17} \hat{z}_t^A + \Theta^{18} \hat{z}_t^K - \Theta^{19} \hat{u}_t^G + \Theta^{20} \hat{u}_t^B.
\]

where we defined

\[
\Theta^8 = \Theta^5 \cdot 1_{Fx1}, \quad \Theta^9 = \Theta^6 \left( \frac{\eta + 1}{\eta(1 - \alpha)} \right), \quad \Theta^{10} = \Theta^6 \left( \frac{\rho^G G \bar{C}}{1 + \rho^G G} \right), \quad \Theta^{11} = \alpha \Theta^9,
\]

\[
\Theta^{12} = \Theta^6 \left( \frac{1 - \delta}{G \bar{A} \cdot G \bar{N}} \cdot \frac{K}{\bar{C}} \right), \quad \Theta^{13} = \Theta^6 \frac{K}{\bar{C}}, \quad \Theta^{14} = \Theta^6 + \kappa^S \cdot (1 - \lambda^K) \cdot \Theta^7, \quad \Theta^{15} = \Theta^7 \psi^4, \quad \Theta^{16} = \Theta^7 \psi^4, \quad \Theta^{17} = \Theta^7 \psi^4, \quad \Theta^{18} = \kappa^S \cdot (1 - \lambda^K) \cdot \Theta^7, \quad \Theta^{19} = \Theta^6 \frac{\rho^G}{1 + \rho^G G}, \quad \Theta^{20} = \Theta^5 \Xi.
\]

By plugging the household’s optimal portfolio across maturities (equation (C.1.50)) into equation (C.1.84), we obtain

\[
\hat{y}^t d_t = \Theta^{21} b^g_t - \Theta^{22} \hat{y}_t - \Theta^{23} \hat{\pi}_t \left[ \hat{y}_{t+1} \right] - \Theta^{24} \hat{\pi}_t \left[ \hat{\pi}_{t+1} \right] + \Theta^{25} \hat{k}_t - \Theta^{26} \hat{k}_{t+1} + \Theta^{27} \hat{\pi}_t \left[ \hat{k}_{t+2} \right]
\]

\[
+ \Theta^{28} \hat{\rho}_t^K - \Theta^{29} \hat{\pi}_t \left[ \hat{y}_t d_{t+1} \right] - \Theta^{30} \hat{z}_t^A + \Theta^{31} \hat{z}_t^K - \Theta^{32} \hat{\varepsilon}_t^A + \Theta^{33} \hat{u}_t^G + \Theta^{34} \hat{u}_t^B.
\]
where we defined
\[
\Theta^{21} = [\Theta^{15} + \Psi^2]^{-1} \Theta^8,
\]
\[
\Theta^{22} = [\Theta^{15} + \Psi^2]^{-1} \Theta^9,
\]
\[
\Theta^{23} = [\Theta^{15} + \Psi^2]^{-1} \Theta^{10},
\]
\[
\Theta^{24} = [\Theta^{15} + \Psi^2]^{-1} \Theta^6,
\]
\[
\Theta^{25} = [\Theta^{15} + \Psi^2]^{-1} \Theta^{11},
\]
\[
\Theta^{26} = [\Theta^{15} + \Psi^2]^{-1} \Theta^{12},
\]
\[
\Theta^{27} = [\Theta^{15} + \Psi^2]^{-1} \Theta^{13},
\]
\[
\Theta^{28} = [\Theta^{15} + \Psi^2]^{-1} \Theta^{14},
\]
\[
\Theta^{29} = [\Theta^{15} + \Psi^2]^{-1} (\Theta^{16} + \Psi^3),
\]
\[
\Theta^{30} = [\Theta^{15} + \Psi^2]^{-1} (\Theta^{17} + \Psi^1),
\]
\[
\Theta^{31} = [\Theta^{15} + \Psi^2]^{-1} \Theta^{18},
\]
\[
\Theta^{32} = [\Theta^{15} + \Psi^2]^{-1} \Theta^{19},
\]
\[
\Theta^{33} = [\Theta^{15} + \Psi^2]^{-1} \Theta^{20}.
\]

**Log-linearization: YCC Policy Specific Derivations**

Linearizing the Taylor rule for \(f\)-maturity bond (equation (3.2.19c)) yields
\[
\hat{y}^{\text{GP},f}_t = \gamma^f_{SP} \hat{y}^{\text{SP},f}_t + (1 - \gamma^f_{SP}) \left[ \gamma^f_{\pi} \hat{\pi}_t + \varepsilon_{YD,t}^{f} \right], \quad f \geq 2. \tag{C.1.86}
\]

We define a \((F - 1) \times (F - 1)\) matrix \(\Gamma^{SP}\) with \(\Gamma^{SP}_{ff} = \gamma^{f+1}_{SP}\) for \(f = 1 \sim F - 1\) and \(\Gamma^{SP}_{ij} = 0\) for \(i \neq j\). Also we define \(\mathcal{T}^{YD}_{(f \geq 2)}\), a \((F - 1) \times L\) matrix with \(\mathcal{T}^{YD}_{(f \geq 2), f,l} = \tau^{YD}_{f+1,l}\) (row \(f\), column \(l\))^5 and a vector of Taylor coefficients \(\overrightarrow{\gamma}_{\pi(f \geq 2)} = [\gamma^2_{\pi}, \ldots, \gamma^F_{\pi}]^{T}\). If we construct such vectors as
\[
\overrightarrow{y^{\text{GP},f}_t}_{(f \geq 2)} = \left[ \hat{y}^{\text{GP},2}_t, \ldots, \hat{y}^{\text{GP},F}_t \right]^{T}, \quad \overrightarrow{y^{\text{SP},f}_t}_{(f \geq 2)} = \left[ \hat{y}^{\text{SP},2}_t, \ldots, \hat{y}^{\text{SP},F}_t \right]^{T}, \tag{C.1.87}
\]
then above equation (C.1.86) can be written in vector form as
\[
\overrightarrow{y^{\text{GP},f}_t}_{(f \geq 2)} = \Gamma^{SP} \overrightarrow{y^{\text{SP},f}_t}_{(f \geq 2)} + (I - \Gamma^{SP}) \cdot \left[ \overrightarrow{\gamma}_{\pi(f \geq 2)} \cdot \hat{\pi}_t + \mathcal{T}^{YD}_{(f \geq 2)} \cdot \varepsilon_{YD,t}^{f} \right], \tag{C.1.88}
\]
where \(l\) is the identity matrix of size \(F - 1\). Since \(\overrightarrow{y^{\text{SP},f}_t}_{(f \geq 2)}\) is the yield vector that prevails in the counterfactual scenario where the current yield is determined by the conventional

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^5For large \(F\), we reduce the state-space using a fewer number of monetary policy shocks than \(F\).
monetary policy, it follows equation (C.1.85), expecting that the economy is driven by the YCC monetary policy. Thus, we can represent its dynamics as

\[
\hat{y}_{d_t}^{SP} = \Theta^{21} \hat{b}_t^G - \Theta^{22} \hat{y}_t - \Theta^{23} \hat{E}_t [\hat{y}_{t+1}] + \Theta^{24} \hat{E}_t [\hat{\pi}_{t+1}] + \Theta^{25} \hat{k}_t - \Theta^{26} \hat{k}_{t+1} + \Theta^{27} \hat{E}_t [\hat{k}_{t+2}]
\]

\[+ \Theta^{28} \hat{r}_t^K - \Theta^{29} \hat{E}_t [\hat{y}_{d_t}^{SP}] - \Theta^{30} \hat{z}_t - \Theta^{31} \hat{z}_t^K - \Theta^{32} \hat{\epsilon}_t^A + \Theta^{33} \hat{u}_t^G + \Theta^{34} \hat{u}_t^B,
\]

(C.1.89)

where coefficients \( \Theta^i \) for \( i = 21 \sim 34 \) are the same as the conventional policy case, and \( \hat{y}_{d_t}^{SP} \) and \( \hat{y}_{d_t}^{GP} \) are defined as \(^6\)

\[
\hat{y}_{d_t}^{SP} = [\hat{y}_{d_t}^{GP}, \hat{y}_{d_t}^{SP} (f \geq 2)]', \quad \hat{y}_{d_t}^{SP} = [\hat{y}_{d_t}^{GP}, \hat{y}_{d_t}^{SP} (f \geq 2)]'.
\]

(C.1.90)

where \( \hat{y}_{d_t}^{GP} \) follows the Taylor rule rule in equation (3.2.19a) and equation (3.2.19b). Now that \( \hat{y}_{d_t}^{GP} \), not \( \hat{y}_{d_t}^{SP} \) governs agents’ intertemporal decisions, equation (C.1.79) becomes

\[
\hat{y}_t = \hat{E}_t [\hat{y}_{t+1} + \Psi^{27} \hat{\pi}_{t+1} - \Psi^{28} \hat{z}_t - \Psi^{29} \hat{z}_t^K - \Psi^{30} \hat{y}_{d_t} - \Psi^{31} \hat{E}_t [\hat{y}_{d_t}^{GP}]
\]

\[+ \Psi^{32} \hat{k}_{t+1} - \Psi^{33} (\hat{k}_t - \hat{\epsilon}_t^A) + \Psi^{34} \hat{k}_{t+1} - \Psi^{35} \hat{k}_{t+2} + \Psi^{36} \hat{u}_t^G].
\]

(C.1.91)

---

\(^6\)There is no \( \hat{y}_{d_t}^{SP,1} \) in our formulation in equation (3.2.19c). Therefore, we use \( \hat{y}_{d_t}^{GP,1} \) instead of \( \hat{y}_{d_t}^{GP,1} \) in constructing \( \hat{y}_{d_t}^{SP} \).
C.2 Summary of Equilibrium Equations

Equilibrium Equations: Conventional Policy

(i). \( \frac{C_t}{A_t N_t} = (1 - \zeta^G_t) \left( \frac{Y_t}{A_t N_t} \right) + \left( 1 - \frac{\delta}{G A_t \cdot G N} \right) \left( \frac{K_t}{A_{t-1} N_{t-1}} \right) - \left( \frac{K_{t+1}}{A_t N_t} \right) \) (C.2.1)

(ii). \( 1 = \beta \cdot E_t \left[ \frac{R_{t+1}^S}{\Pi_{t+1} \cdot G A_{t+1} \cdot G N} \left( \frac{C_t}{A_t N_t} \right) \right] \) (C.2.2)

(iii). \( \lambda^H_{t,1} = 1 - \sum_{f=2}^{F} \lambda^H_{t,f} \) (C.2.3)

(iv). \(- \left( \frac{B^G,f_t}{A_t N_t P_t} + \frac{B^{CB,f_t}}{ANP} \right) (\lambda^H_{t,f})^{-1} = \frac{1}{G A_t \cdot G N} \left( 1 - \frac{\lambda^K_t}{\lambda^K_t} \right) \left( \frac{P^K_t}{P_t} \right) \left( \frac{K_t}{A_{t-1} N_{t-1}} \right), \forall f > 1 \) (C.2.4)

(v). \( YD^1_t = \max \{ YD^1_{ts}, 1 \} \) (C.2.5)

(vi). \( YD^1_{ts} = \mathbb{Y}D^1 \cdot \left( \frac{\Pi_t}{\Pi} \right)^{\gamma_x} \left( \frac{Y_t}{Y} \right)^{\gamma_y} \cdot \exp \left( \tilde{\xi}_t^{YD^1} \right) \) (C.2.6)

(vii). \( \lambda^H_{t,f} = \left( \frac{E_t \left[ \frac{\beta z_f^t}{\Pi_{t+1} \cdot G A_{t+1} \cdot G N} \left( \frac{C_t}{A_t N_t} \right) \right]}{\Phi^B_t} \right)^{\kappa_B} \left( \frac{YD^1_{t+1}^{f-1}}{YD^1_t^f} \right)^{-f}, \forall f \) (C.2.7)

(viii). \( \Phi^B_t = \left[ \sum_{j=1}^{F} E_t \left[ \frac{\beta z^j_t}{\Pi_{t+1} \cdot G A_{t+1} \cdot G N} \left( \frac{C_t}{A_t N_t} \right) \left( \frac{YD^1_{t+1}}{YD^1_t} \right)^{(j-1)} \right] \right]^{\kappa_B} \) (C.2.8)

(ix). \( \lambda^K_t = \left( \frac{z^K_t E_t \left[ Q_{t,t+1} R^K_{t+1} \right]}{\Phi^K_t} \right)^{\kappa_S} \) (C.2.9)

(x). \( \Phi^K_t = \left[ \left( E_t \left[ Q_{t,t+1} R^K_{t+1} \right] \right)^{\kappa_S} + \left( z^K_t E_t \left[ Q_{t,t+1} R^K_{t+1} \right] \right)^{\kappa_S} \right]^{\frac{1}{\kappa_S}} \) (C.2.10)

(xi). \( R^f_t = \sum_{j=0}^{F-1} \lambda^K_{t,f+1} \left( \frac{YD^1_t^f}{YD^1_t^{(f+1)}} \right)^{-(f+1)} \cdot j \in \{ HB, G, CB \} \) (C.2.11)

(xii). \( R^S_t = (1 - \lambda^K_{t-1}) R^K_{t+1} + \lambda^K_{t-1} R^K_{t} \) (C.2.12)

(xiii). \( 1 = \mathbb{E}_t \left[ Q_{t,t+1} \Pi_{t+1} \left[ (1 - \delta) + \frac{P^K_{t+1}}{P_{t+1}} \right] \right] \) (C.2.13)
(xiv). \(F_t = (1 - \alpha) \frac{1}{1 - \alpha} \left( 1 + \zeta^G_{t-1} \varepsilon \right) \left( \frac{C_t}{A_t N_t} \right)^{-\alpha} \left( \frac{Y_t}{A_t N_t} \right)^{\frac{n+1}{\eta + 1}} \left( R_t^K \frac{P_t^K}{P_t} \right)^{\alpha} \left( \frac{n+1}{\eta + 1} \right) + \theta \beta \sum \{ \Pi_t \} F_{t+1} \]  
\quad \text{(C.2.14)}

(xv). \(H_t = \left( \frac{C_t}{A_t N_t} \right)^{-1} \frac{Y_t}{A_t N_t} + \theta \beta \sum \{ \Pi_t \} \left[ \Pi_{t+1}^{-1} H_{t+1} \right] \)  
\quad \text{(C.2.15)}

(xvi). \( \frac{F_t}{H_t} = \left( \frac{1 - \theta}{1 - \theta \sum \{ \Pi_t^x \}} \right)^{\frac{1}{x}} \left( \left[ 1 + \varepsilon \left( \frac{1 - \alpha}{\eta + 1} \right) \right] \right) \)  
\quad \text{(C.2.16)}

(xvii). \( \Delta_t = (1 - \theta) \left( \frac{1 - \theta \sum \{ \Pi_t^x \}}{1 - \delta} \right)^{\frac{1}{x}} + \theta \sum \{ \Pi_t \} \Delta_{t-1} \)  
\quad \text{(C.2.17)}

(xviii). \( \frac{N_t}{N_t} = (1 - \alpha) \left( \frac{n}{n + 1} \right) \left( \frac{C_t}{A_t N_t} \right)^{-\alpha} \left( \frac{Y_t}{A_t N_t} \right)^{\frac{n+1}{\eta + 1}} \left( R_t^K \frac{P_t^K}{P_t} \right)^{\alpha} \left( \frac{n+1}{\eta + 1} \right) \Delta_t \)  
\quad \text{(C.2.18)}

(xix). \( \frac{K_t}{A_t - 1 \bar{N}_t - 1} = \alpha(1 - \alpha)^{\frac{1}{1 - \alpha}} \cdot G A_t \cdot GN \cdot \left( \frac{C_t}{A_t N_t} \right)^{-\alpha} \left( \frac{Y_t}{A_t N_t} \right)^{\frac{n+1}{\eta + 1}} \left( R_t^K \frac{P_t^K}{P_t} \right)^{\alpha} \left( \frac{n+1}{\eta + 1} \right) \Delta_t \)  
\quad \text{(C.2.19)}

(xx). \( \frac{B_t^G}{P_t A_t + N_t} = \frac{R_t^G}{P_t A_t + N_t} \cdot \frac{B_t^G}{P_t - 1 A_t - 1 \bar{N}_t - 1} + \left[ \zeta^G_{t} + \zeta^F_{t} - \zeta^T_{t} \right] \left( \frac{Y_t}{A_t N_t} \right) \)  
\quad \text{(C.2.20)}

(xxii). \( \frac{\lambda^G_{t+1}}{1 + \sum_{l=2}^F a^B,l \exp \left( \tilde{u}^B,l t \right)} \)  
\quad \text{(C.2.21)}

(ii). \( \frac{\lambda^{G,f}_{t+1}}{1 + \sum_{l=2}^F a^B,l \exp \left( \tilde{u}^B,f t \right)} \)  
\quad \text{(C.2.22)}

(xiiii). \( \tilde{u}^{B,f}_{t} = \sum_{j=1}^J (C_{t+j}^B u^B_{t+j}) \)  
\quad \text{(C.2.23)}

(xxiv). \( u^B_{t+j} = \rho u^B_{t+j} + \varepsilon^B_{t+j} \)  
\quad \text{(C.2.24)}

(xxv). \( B_t^{G,f} = \lambda^G_{t} B_t^G, \quad \forall f = 1, \ldots, F \)  
\quad \text{(C.2.25)}

(xxvi). \( G A_t = \exp(\mu + \varepsilon^A) \)  
\quad \text{(C.2.26)}

(xxvii). \( \zeta^G_t = \frac{1}{1 + \alpha \exp \left( -u_t^G \right)} \)  
\quad \text{(C.2.27)}

(xxviii). \( \zeta^T_t = \frac{1}{1 + \alpha \exp \left( -u_t^T \right)} \)  
\quad \text{(C.2.28)}

(xxix). \( u_t^G = \rho G \cdot u_t^G + \varepsilon^G \)  
\quad \text{(C.2.29)}

(xxx). \( u_t^T = \rho T \cdot u_t^T + \varepsilon_t \)  
\quad \text{(C.2.30)}
Equilibrium Equations: YCC Policy

\[(i). \quad \frac{C_t}{A_t \bar{N}_t} = (1 - \zeta^G_t) \left[ \frac{Y_t}{A_t \bar{N}_t} + \left( 1 - \frac{\delta}{\bar{G}A_t \cdot GN} \right) \left( \frac{K_t}{A_{t-1} \bar{N}_{t-1}} \right) - \left( \frac{K_{t+1}}{A_t \bar{N}_t} \right) \right] \quad (C.2.31)\]

\[(ii). \quad 1 = \beta E_t \left[ \frac{R^S_{t+1}}{\Pi_{t+1} \cdot \bar{G}A_{t+1} \cdot GN} \left( \frac{C_{t+1}}{A_{t+1} \bar{N}_{t+1}} \right) \right] \quad (C.2.32)\]

\[(iii). \quad \frac{\lambda^{CB,1}_t - \lambda^G_t}{\lambda^{CB,1}_t - \lambda^{HB,1}_t} \frac{B_t^G}{A_t \bar{N}_t P_t} = \frac{1}{\bar{G}A_t \cdot GN} \left( 1 - \frac{\lambda^K_t}{\lambda^K_t} \right) \left( \frac{P^K_t}{P_t} \right) \left( \frac{K_t}{A_{t-1} \bar{N}_{t-1}} \right) \quad (C.2.33)\]

\[(iv). \quad \lambda^{CB,f}_t = \frac{\lambda^{HB,f}_t \cdot \left( 1 - \sum_{i \neq f,1} \lambda^{CB,i}_t - \lambda^{G,1}_t \right)}{\lambda^{HB,1}_t + \lambda^{HB,f}_t} \left( \frac{\lambda^{G,1}_t - \lambda^{G,f}_t}{\lambda^{HB,1}_t + \lambda^{HB,f}_t} \right), \; f \geq 2 \quad (C.2.34)\]

\[(v). \quad YD^1_t = \max \{ YD^1_{t^*} \} \quad (C.2.35)\]

\[(vi). \quad YD^1_{t^*} = \bar{Y}D^1 \cdot \left[ \frac{\Pi_t}{\Pi} \right] \left( \frac{Y_t}{Y} \right) \left( \frac{\gamma^1_t}{\gamma^1_{t^*}} \right) \cdot \exp \left( \tilde{\varepsilon}_{YD^1} \right) \quad (C.2.36)\]

\[(vii). \quad YD^{SP,f}_t = \bar{Y}D^{SP,f} \cdot \left[ \frac{YD^{SP,f}_t - \gamma^f_{SP}}{\bar{Y}D^{SP,f}} \right] \left[ \left( \frac{\Pi_t}{\Pi} \right) \left( \frac{Y_t}{Y} \right) \left( \frac{\gamma^f_t}{\gamma^f_{t^*}} \right) \cdot \exp \left( \tilde{\varepsilon}_{YD^f} \right) \right]^{-1} \quad , \; f \geq 2 \quad (C.2.37)\]

\[(viii). \quad \lambda^{HB,f}_t = \left( \frac{\beta z^i_t}{\Pi_{t+1} \cdot \bar{G}A_{t+1} \cdot GN} \left( \frac{C_{t+1}}{A_{t+1} \bar{N}_{t+1}} \right) \left( \frac{YD_{t+1}^{i-1} \gamma^i_{t^*}}{YD_{t}^{i-1} \gamma^i_t} \right) \right) \quad , \; \forall f \quad (C.2.38)\]

\[(ix). \quad \Phi^S_t = \sum_{j=1}^{F} \left[ \bar{E}_t \left( \frac{C_t}{A_t \bar{N}_t} \right) \left( \frac{YD_{t+1}^{j-1} \gamma^j_{t^*}}{YD_{t}^{j-1} \gamma^j_t} \right) \right]^{1/\kappa_S} \quad (C.2.39)\]

\[(x). \quad \lambda^K_t = \left( \frac{z^K_t \cdot \bar{E}_t \left[ Q_{t+1} R^K_{t+1} \right]}{\Phi^S_t} \right)^{1/\kappa_S} \quad (C.2.40)\]

\[(xi). \quad \Phi^S_t = \left[ \left( \frac{\bar{E}_t \left[ Q_{t+1} R^K_{t+1} \right]}{\kappa_S} + \left( z^K_t \cdot \bar{E}_t \left[ Q_{t+1} R^K_{t+1} \right] \right)^{1/\kappa_S} \right) \right]^{1/\kappa_S} \quad (C.2.41)\]

\[(xii). \quad R^i_t = \sum_{f=0}^{F-1} \lambda^{i,f+1}_{t-1} \frac{(YD^i_t)^{-f}}{(YD^i_{t+1})^{-(f+1)}} \quad j \in \{ HB, G, CB \} \quad (C.2.42)\]

\[(xiii). \quad R^S_t = (1 - \lambda^K_{t-1}) R^{HB}_t + \lambda^K_{t-1} R^K_t \quad (C.2.43)\]

\[(xiv). \quad 1 = \bar{E}_t \left[ Q_{t+1} \Pi_{t+1} \left( 1 - \delta + \frac{P^K_{t+1}}{P_{t+1}} \right) \right] \quad (C.2.44)\]
(xv). \[ F_t = (1 - \alpha)^{\frac{1-\alpha}{\eta+\alpha}} \left( \frac{1 + \zeta}{\epsilon - 1} \right) \left( \frac{C_t}{A_t \bar{N}_t} \right)^{-\alpha \left( \frac{n+1}{\eta+\alpha} \right)} \left( \frac{Y_t}{A_t \bar{N}_t} \right)^{n+1} \left( \eta + \alpha \right) \left( \bar{E}_t \left[ Q_{t,t+1} R_{t+1}^K \right] \frac{P_t^K}{P_t} \right)^\alpha \left( \frac{n+1}{\eta+\alpha} \right) \]

\[ + \theta \beta \bar{E}_t \left[ \Pi_{t+1} F_{t+1} \right] \] (C.2.45)

(xvi). \[ H_t = \left( \frac{C_t}{A_t \bar{N}_t} \right)^{-1} \frac{Y_t}{A_t \bar{N}_t} + \theta \beta \bar{E}_t \left[ \Pi_{t+1} H_{t+1} \right] \] (C.2.46)

(xvii). \[ \frac{F_t}{H_t} = \left( \frac{1 - \theta}{1 - \theta \Pi_t} \right) \left( \frac{1}{1 + \epsilon} \right) \left[ 1 + \epsilon \left( \frac{1-\alpha}{\eta+\alpha} \right) \right] \] (C.2.47)

(xviii). \[ \Delta_t = (1 - \theta) \left( \frac{1 - \theta \Pi_t^{-1}}{1 - \theta} \right) \left( \frac{1}{1 + \epsilon} \right) \left( \frac{1}{\eta + \alpha} \right) \Delta_{t-1} \] (C.2.48)

(ix). \[ \frac{N_t}{\bar{N}_t} = (1 - \alpha)^{\frac{n}{\eta+\alpha}} \left( \frac{C_t}{A_t \bar{N}_t} \right)^{-\alpha \left( \frac{n}{\eta+\alpha} \right)} \left( \frac{Y_t}{A_t \bar{N}_t} \right)^{\frac{n}{\eta+\alpha}} \left( \bar{E}_t \left[ Q_{t,t+1} R_{t+1}^K \right] \frac{P_t^K}{P_t} \right)^\alpha \left( \frac{n}{\eta+\alpha} \right) \Delta_t \] (C.2.49)

(xx). \[ \frac{K_t}{A_{t-1} \bar{N}_{t-1}} \]

\[ = \alpha (1 - \alpha)^{\frac{1-\alpha}{\eta+\alpha}} \cdot GA_t \cdot GN \left( \frac{C_t}{A_t \bar{N}_t} \right)^{\frac{n(1-\alpha)}{\eta+\alpha}} \left( \frac{Y_t}{A_t \bar{N}_t} \right)^{\frac{n+1}{\eta+\alpha}} \left( \bar{E}_t \left[ Q_{t,t+1} R_{t+1}^K \right] \frac{P_t^K}{P_t} \right)^\alpha \left( \frac{n(1-\alpha)}{\eta+\alpha} \right) \Delta_t \] (C.2.50)

(xi). \[ \frac{B_t^G}{P_t A_t \bar{N}_t} = \frac{R_t^G}{\Pi_t \cdot GA_t \cdot GN \frac{P_t}{A_{t-1} \bar{N}_{t-1}}} - \left[ \zeta_t^G + \zeta_t^F - \zeta_t^T \right] \left( \frac{Y_t}{A_t \bar{N}_t} \right) \] (C.2.51)

(xxii). \[ \tilde{\epsilon}_t^{YD,f} = \sum_{l=1}^{L} \tau_{t,l}^{YD} \tilde{\epsilon}_t^{YD,l} \] (C.2.52)
C.2.1 Calibrating \( \{z^f\} \) and \( z^K \) in the Steady State

**Calibration of \( \{z^f\} \)** Here we explain how to calibrate \( \{z^f\} \) to match the yield curve. Based on data on yields of bonds with different maturities, we calculate each \( f \)-maturity bond’s average holding returns \( \{R^f\} \), which we would use as our calibration target.

1. Compute the return ratio \( \{R^f/R^1\} \)
2. Back out steady state bond shares \( \{\lambda^{HB,f}\} \) using equation (C.1.24)
3. Normalize \( z^1 = 1 \) and obtain initial guess for \( \{z^{i,guess}\} \). Set \( z^{i,old} = z^{i,guess} \) in the iteration below
4. Construct \( \tilde{\Phi}^{old} \) using the following formula, where the return ratios \( \{R^f/R^1\} \) across maturities are obtained from the data

\[
\tilde{\Phi}^{old} = \left[ 1 + \sum_{f=2}^{F} \left[ z^f \left( \frac{R^f}{R^1} \right)^{\kappa_B} \right] \right]^{\frac{1}{\kappa_B}}
\]

5. Back out new \( z^{f,new}, f = 2, \ldots, F \) estimates using:

\[
z^{f,new} = (\lambda^{HB,f})^{\frac{1}{\kappa_B}} \left( \frac{R^f}{R^1} \right)^{-1} \tilde{\Phi}^{old}
\]

6. If difference with \( \tilde{\Phi}^{old} \) is large, set \( z^{f,old} = z^{f,new} \) and start again from the step 4

**Calibration of \( z^K \)** For the calibration of \( z^K \), first, we need to obtain data on \( \{R^{HB}, R^K\} \).

1. Guess \( z^{K,guess} \) and set \( z^{K,old} = z^{K,guess} \)
2. Solve for the steady state values of the model using \( z^{K,old} \). The reason is that we do not get the data on \( \lambda^K \), thus we use the model-dependent value of it
3. Construct \( \tilde{\Phi}^{old} \) using the following formula, where the ratio is the one obtained from the data

\[
\tilde{\Phi}^{old} = \left[ 1 + \left[ z^K \left( \frac{R^K}{R^{HB}} \right)^\kappa_S \right]^{\frac{1}{\kappa_S}} \right]
\]

4. Back out new \( z^{K,new} \) estimates as

\[
z^{K,new} = (\lambda^K)^{\frac{1}{\kappa_S}} \left( \frac{R^K}{R^{HB}} \right)^{-1} \tilde{\Phi}^{old}
\]

where \( \lambda^K \) comes from the model solution of the step 2
5. If difference with \( z^{K,old} \) is large, set \( z^{K,old} = z^{K,new} \) and start again from step 4.
Summary of Conventional Policy Linearized Equations  Those are the essential equations to solve the model, other variables can be found on equations above.

(i). \( \hat{y}_t = \mathbb{E}_t \left[ \hat{y}_{t+1} + \psi^{27} \hat{\pi}_{t+1} - \psi^{28} \hat{Z}_t - \psi^{29} \hat{Z}^K_t - \psi^{30} \hat{y}_t - \psi^{31} \mathbb{E}_t \left[ \hat{y}_{t+1} \right] \right] \)

\( \left. - \psi^{32} \hat{r}_{t+1}^K - \psi^{33} \left( \hat{k}_t - \hat{\epsilon}_A^t \right) + \psi^{34} \hat{r}_{t+1} - \psi^{35} \hat{r}_{t+2} + \psi^{36} \hat{u}_t^G \right] \)

(ii). \( \hat{y}^d_t = \Theta^{21} \hat{b}_t^G - \Theta^{22} \hat{y}_t - \Theta^{23} \mathbb{E}_t \left[ \hat{\pi}_{t+1} \right] - \Theta^{24} \mathbb{E}_t \left[ \hat{\pi}_{t+1} \right] + \Theta^{25} \hat{k}_t - \Theta^{26} \hat{k}_{t+1} + \Theta^{27} \mathbb{E}_t \left[ \hat{k}_{t+2} \right] \)

\( \left. + \Theta^{28} \hat{r}_{t+1}^K - \Theta^{29} \mathbb{E}_t \left[ \hat{y}_{t+1} \right] - \Theta^{30} \hat{Z}_t^+ + \Theta^{31} \hat{Z}^K_t - \Theta^{32} \hat{\epsilon}_A^t + \Theta^{33} \hat{\epsilon}_G^t + \Theta^{34} \hat{u}_t^B \right] \)

(iii). \( \hat{y}_t^{1*} = \max \left\{ \hat{y}_t^{1*}, 0 \right\} \)

(iv). \( \hat{y}_t^{1*} = \gamma_{\pi} \hat{\pi}_t + \gamma_{\eta} \hat{\eta}_t + \hat{\epsilon}_t^{YD} + \hat{\epsilon}_t^{YD} = \sum_{i=1}^{L} T_{YD}^{i} \epsilon_t^{YD} \)

(v). \( \hat{r}_{t+1}^K = -\psi^{15} \hat{Z}_t - \psi^{16} \hat{Z}^K_t - \psi^{17} \hat{y}_t - \psi^{18} \mathbb{E}_t \left[ \hat{y}_{t+1} \right] + \psi^{19} \mathbb{E}_t \left[ \hat{\pi}_{t+1} \right] + \psi^{20} \mathbb{E}_t \left[ \hat{\pi}_{t+2} \right] \)

\( \left. + \psi^{21} \mathbb{E}_t \left[ \hat{y}_{t+1} \right] + \psi^{22} \mathbb{E}_t \left[ \hat{y}_{t+2} \right] - \psi^{23} \hat{k}_{t+1} \right] \)

\( \left. + \psi^{24} \mathbb{E}_t \left[ \hat{k}_{t+2} \right] - \psi^{25} \mathbb{E}_t \left[ \hat{k}_{t+3} \right] - \psi^{26} \hat{u}_t^G - \psi^{27} \mathbb{E}_t \left[ \hat{r}_{t+1}^K \right] \right] \)

(vi). \( \hat{b}_t^G = \frac{R^G}{N \cdot GA \cdot GN} \cdot \left[ \psi^{G,4} \hat{B}_{t-1}^B - \psi^{G,5} \hat{y}_t - \psi^{G,6} \hat{y}_t - \psi^{G,7} \hat{y}_t - \hat{\pi}_t - \hat{\epsilon}_A^t + \hat{b}_t^{B-1} \right] \)

\( \left. + \left( 1 - \frac{R^G}{N \cdot GA \cdot GN} \right) \right] \left[ \hat{y}_t + \left( \frac{\zeta^G}{\zeta^G + \zeta^F - \zeta^T} \right) \frac{a^G}{1 + a^G} \hat{\theta}_t^G - \left( \frac{\zeta^T}{\zeta^G + \zeta^F - \zeta^T} \right) \frac{a^T}{1 + a^T} \hat{\theta}_t^T \right] \)

(vii). \( \hat{k}_t = \left[ 1 - \theta \beta \Pi^{\left( \frac{n+1}{n+\alpha} \right)} \right] \left( -\frac{n+1}{n\left( 1 - \alpha \right)} \right) \left[ \hat{y}_t - \alpha \left( \hat{k}_t - \hat{\epsilon}_A^t \right) \right] \)

\( \left. + \theta \beta \Pi^{\left( \frac{n+1}{n+\alpha} \right)} \mathbb{E}_t \left[ \left( \frac{n+1}{n+\alpha} \right) \hat{\pi}_{t+1} + \hat{\pi}_{t+1} \right] \right] \)

(viii). \( \hat{h}_t = \left[ 1 - \theta \beta \Pi^{\left( \frac{n+1}{n+\alpha} \right)} \right] \left[ 1 - \left( -1 - \zeta^G \right) \frac{Y}{C} \right] \hat{y}_t + \left[ \left( 1 - \zeta^G \right) \frac{Y}{C} \right] \frac{1}{1 + \hat{u}_t^G} \)

\( \left. \left. - \left[ \frac{1 - \delta \Pi^{\left( \frac{n+1}{n+\alpha} \right)} \left( \hat{k}_t - \hat{\epsilon}_A^t \right) + \frac{K}{C} \hat{k}_{t+1} \right] + \theta \beta \Pi^{\left( \frac{n+1}{n+\alpha} \right)} \mathbb{E}_t \left[ \left( \epsilon - 1 \right) \hat{\pi}_{t+1} + \hat{h}_{t+1} \right] \right] \right] \)

(ix). \( \hat{r}_t - \hat{h}_t = \left[ 1 + \epsilon \left( \frac{1 - \alpha}{n + \alpha} \right) \right] \left( \frac{\theta \Pi^{\left( \frac{n+1}{n+\alpha} \right)} \left( \frac{n+1}{n+\alpha} \right) \hat{\pi}_{t+1} \right) \)

\( \left( \frac{n+1}{n+\alpha} \right) \)

(x). \( u_t^{B,j} = \rho_B \cdot u_{t-1}^{B,j} + \epsilon_t^{B,j}, \quad \forall j = 1, \ldots, J \)

(xi). \( u_t^G = \rho_G \cdot u_{t-1}^G + \epsilon_t^G \)

(xii). \( u_t^T = \rho_T \cdot u_{t-1}^G + \epsilon_t^T \)
Summary of YCC Policy Linearized Equations  Those are the essential equation to solve the model, other variables can be found on equations above.

(i). \[ \hat{y}_t = \mathbb{E}_t \left[ \hat{y}_{t+1} + \psi^{27} \hat{\pi}_{t+1} - \psi^{28} \hat{z}_t - \psi^{29} \hat{2}_t^K - \psi^{30} \hat{y}_d^G - \psi^{31} \mathbb{E}_t \left[ \hat{y}_d^{G,t+1} \right] \right. \]

\[ \left. - \psi^{32} \hat{r}^K_{t+1} - \psi^{33} (\hat{k}_t - \hat{\epsilon}_t^A) + \psi^{34} \hat{k}_{t+1} - \psi^{35} \hat{k}_{t+2} + \psi^{36} \hat{u}_t^G \right] \]

(ii). \[ \hat{y}_d^G_t = \Theta \beta^{21} \hat{b}_t^G - \Theta \beta^{22} \hat{y}_t - \Theta \beta^{23} \mathbb{E}_t \left[ \hat{y}_{t+1} \right] - \Theta \beta^{24} \mathbb{E}_t \left[ \hat{\pi}_{t+1} \right] - \Theta \beta^{25} \hat{k}_t - \Theta \beta^{26} \hat{k}_{t+1} + \Theta \beta^{27} \mathbb{E}_t \left[ \hat{k}_{t+2} \right] \]

\[ + \Theta \beta^{28} \hat{r}^K_{t+1} - \Theta \beta^{29} \mathbb{E}_t \left[ \hat{y}_d^{G,t+1} \right] - \Theta \beta^{30} \hat{z}_t + \Theta \beta^{31} \hat{2}_t^K - \Theta \beta^{32} \hat{\epsilon}_t^A + \Theta \beta^{33} \hat{u}_t^G + \Theta \beta^{34} \hat{u}_t^G \]

(iii). \[ \hat{y}_d^{G,1} = \max \left\{ \hat{y}_d^{1*}, 0 \right\} = \hat{y}_d^{SP,1} \]

(iv). \[ \hat{y}_d^{1*} = \gamma^G \hat{\pi}_t + \gamma^G \hat{y}_t + \hat{\epsilon}_t^{G,YD}, \quad \hat{\epsilon}_t^{G,YD} = \sum_{l=1}^L \kappa_{YD}^l \hat{\epsilon}_t^{YD} \]

(v). \[ \hat{y}_d^{G,f} = \gamma^G_{SP} \hat{y}_d^{SP,f} + \left( 1 - \gamma^G_{SP} \right) \left[ \gamma^G \hat{\pi}_t + \gamma^G \hat{y}_t + \hat{\epsilon}_t^{G,YD} \right], \quad \hat{\epsilon}_t^{G,YD} = \sum_{l=1}^L \kappa_{YD}^l \hat{\epsilon}_t^{YD}, \quad f \geq 2 \]

(vi). \[ \hat{r}_t^{K} = -\psi^{15} \hat{z}_t - \psi^{16} \hat{2}_t^K - \psi^{17} \hat{y}_d^{G,t} - \psi^{18} \mathbb{E}_t \left[ \hat{y}_d^{G,t+1} \right] + \psi^{19} \mathbb{E}_t \left[ \hat{\pi}_{t+1} \right] + \psi^{20} \mathbb{E}_t \left[ \hat{r}_{t+2} \right] \]

\[ + \psi^{21} \mathbb{E}_t \left[ \hat{y}_{t+1} \right] + \psi^{22} \mathbb{E}_t \left[ \hat{y}_{t+2} \right] - \psi^{23} \hat{k}_{t+1} + \psi^{24} \mathbb{E}_t \left[ \hat{k}_{t+2} \right] - \psi^{25} \mathbb{E}_t \left[ \hat{k}_{t+3} \right] - \psi^{26} \hat{u}_t^G - \psi^{27} \hat{u}_t^G \]

(vii). \[ \hat{b}_t^G = \frac{R_t^G}{\prod \cdot G \cdot A \cdot G} \cdot \left[ \psi^{G,4} \hat{u}_t^{B,t-1} - \psi^{G,5} \hat{y}_d^{G,t} - \psi^{G,6} \hat{y}_d^{G,t-1} - \hat{\pi}_t + \hat{\epsilon}_t^A + \hat{b}_t^G \right] \]

\[ + \left( 1 - \frac{R_t^G}{\prod \cdot G \cdot A \cdot G} \right) \left[ \hat{y}_t - \frac{\zeta^G}{\zeta^G + \zeta^T} \left( \frac{\hat{a}_G}{\hat{a}_G + \hat{a}_T} \right) \hat{a}_G^G \right. \]

\[ \left. - \left( \frac{\zeta^T}{\zeta^G + \zeta^T} \right) \left( \frac{\hat{a}_T}{\hat{a}_G + \hat{a}_T} \right) \hat{a}_T^G \right] \]

(viii). \[ \hat{\pi}_t = \left[ 1 - \theta \beta \prod \left( \frac{\hat{\pi}_{t+1}^{\delta}}{\hat{\pi}_{t+1}^{\alpha}} \right) \right] \left( \frac{\eta + 1}{\eta (1 - \alpha)} \right) \left[ \hat{y}_t - \alpha (\hat{k}_t - \hat{\epsilon}_t^A) \right] \]

\[ + \theta \beta \prod \left( \frac{\hat{\pi}_{t+1}^{\delta}}{\hat{\pi}_{t+1}^{\alpha}} \right) \mathbb{E}_t \left[ \epsilon \left( \frac{\eta + 1}{\eta + \alpha} \right) \hat{\pi}_{t+1} + \hat{\pi}_{t+1} \right] \]

(ix). \[ \hat{h}_t = \left[ 1 - \theta \beta \prod^{\delta-1} \right] \left[ \left( 1 - \frac{1 - \zeta^G}{\zeta^C} \right) \hat{y}_t + \left( \frac{1 - \zeta^G}{\zeta^C} \right) \frac{1}{1 + \hat{a}_G^G} \hat{a}_G^G \right. \]

\[ \left. - \frac{1 - \delta}{\prod \cdot G \cdot A \cdot G} \mathbb{E}_t \left[ \hat{k}_t - \hat{\epsilon}_t^A + \frac{K}{C} \hat{k}_{t+1} \right] + \theta \beta \prod^{\delta-1} \mathbb{E}_t \left[ \left( \epsilon - 1 \right) \hat{\pi}_{t+1} + \hat{h}_{t+1} \right] \right] \]

(x). \[ \hat{t}_t - \hat{h}_t = \left[ 1 + \epsilon \left( \frac{1 - \alpha}{\eta + \alpha} \right) \right] \left( \frac{\theta \prod^{\delta-1}}{1 - \theta \prod^{\delta-1}} \right) \hat{\pi}_t \]

(xi). \[ u_t^{B,j} = \rho_B \cdot u_{t-1}^{B,j} + \hat{\epsilon}_t^{B,j} \quad \forall j = 1, \ldots, J \]

(xii). \[ u_t^G = \rho_G \cdot u_{t-1}^G + \hat{\epsilon}_t^G, \quad u_t^T = \rho_T \cdot u_{t-1}^T + \hat{\epsilon}_t^T \]
C.3 Welfare

C.3.1 Deriving a second-order welfare

In order to approximate welfare up to a second-order, we cannot discard $\hat{\Delta}_t$, which is price dispersion’s log-deviation from its steady-state value. Since we have a positive steady-state inflation, price dispersion becomes of the first-order importance, in contrast to the standard New-Keynesian models in which zero trend inflation is assumed usually and price dispersion is the second-order term. In handling this issue when calculating the welfare cost of inflation, we closely follow the pioneering work of Woodford (2003) and Coibion et al. (2012). First, we express the necessary building blocks with the price dispersion gap term, $\hat{\Delta}_t$.

**Step 1:** For any variable $X$, we define $\bar{X}$ as its steady-state value (with the positive trend inflation $\Pi > 1$) and $\bar{X}^F$ as its flexible price steady-state value. Also define (small) letter $\tilde{x}$ as log-deviation of $X$ around $\bar{X}^F$, and $\hat{x}$ as log-deviation of $X$ around $\bar{X}$.

**Efficient (flexible-price) steady state** With optimal production subsidy $\zeta^F = (\epsilon - 1)^{-1}$ that eliminates the monopolistic competition distortion, there is no distortion in the flexible-price steady state economy anymore.\(^7\) In particular, individual firm’s optimal price resetting condition (equation (C.1.2)) becomes

\[
1 = \frac{P_t^*}{P_t} = \left(\frac{1 + \zeta^F}{\epsilon - 1}\right)^{-1} \frac{MC_t}{P_t} = MC_t/P_t,
\]

(C.3.1)

where we use the fact that all firms become identical, and thus $MC_t(\nu) = MC_t$ for all $\nu \in [0, 1]$. Therefore, the real marginal cost becomes 1 for all firms. Plugging the unit real marginal cost (equation (C.1.3)) into the individual firm’s labor demand (equation (3.2.14)) with $W_t(\nu) = W_t$ for $\forall \nu$, we obtain

\[
n_t = (1 - \alpha)y_t \left(\frac{\tilde{R}_t^K P_t^K}{P_t} / P_t\right)^{\alpha} \left(\frac{W_t}{P_t A_t}\right)^{-\alpha} = (1 - \alpha)y_t \left(\frac{W_t}{P_t A_t}\right)^{-1},
\]

(C.3.2)

which, with the household’s intra-temporal consumption-labor decision (equation (3.2.9a)), becomes:

\[
\frac{n_t^{p-1}}{c_t^{1/\gamma}} = (1 - \alpha)y_t, \quad (C.3.3)
\]

which is exactly the social efficiency condition that ensures the household’s marginal rate of substitution matches with the marginal rate of technical substitution. Therefore, at the flexible-price steady state, the new constant $\Phi$, which will turn out to enter in the per-period

\(^7\)A capital producing firm is perfectly competitive and therefore, our economy features no friction if it were not nominal rigidity nor trend inflation, and satisfies the first welfare theorem in the flexible price steady state.
welfare later, can be calculated as

$$\Phi \equiv (\bar{n}^F)^{1+\frac{1}{\alpha}} = (1 - \alpha)\bar{y}^F = (1 - \alpha)\bar{c}^F,$$

where $\bar{n}^F$, $\bar{y}^F$, and $\bar{c}^F$ are values of normalized labor, output, and consumption, respectively.

**Step 2:** With equation (C.2.49) and equation (C.2.50), we obtain

$$\left(\frac{N_t}{\bar{N}_t}\right)^{1-\alpha} \left(\frac{K_t}{A_{t-1}N_{t-1}}\right)^{\alpha} = \alpha^\alpha(1 - \alpha)^{1-\alpha}(GA_t \cdot GN)^{\alpha} \left(\frac{Y_t}{A_tN_t}\right)\Delta_t^{(1-\alpha)\frac{n+\alpha}{\eta+\alpha}},$$

which is the aggregate production function with price dispersion $\Delta_t$. Plugging steady-state (with trend-inflation) capital (equation (C.1.32)) and output (equation (C.1.37)) equations into equation (C.3.5) yields the formula for the steady-state labor, which is given as

$$\frac{N}{N} = \xi^N\xi^Y \left[\left(1 - \frac{R^G}{\Pi GAGN}\right)^{-1} \frac{\lambda^K}{1 - \lambda^K}\right]^{\frac{n}{\eta+\alpha}} \times \left[(1 - \xi^G) - \xi^C \left(1 - \frac{R^G}{\Pi GAGN}\right)^{-1} \frac{\lambda^K}{1 - \lambda^K}\right]^{\frac{n}{\eta+\alpha}} (R^K)^{\frac{-\alpha}{\eta+\alpha}},$$

with $\xi^N = (1 - \alpha)\left[\frac{1 - \theta}{1 - \theta \Pi^{\frac{n+\alpha}{\eta+\alpha}}}\right]^{\frac{n+\alpha}{\eta+\alpha}} \left(1 - \theta\Pi^{\frac{n+\alpha}{\eta+\alpha}}\right)\frac{1 - \theta\Pi^{\frac{n+\alpha}{\eta+\alpha}}}{1 - \theta}\left(\frac{\xi^K}{\alpha \cdot GA \cdot GN}\right)^{\frac{-\alpha}{\eta+\alpha}}.$

From equation (C.1.39), equation (C.1.40), and equation (C.1.41), we observe that equilibrium steady state values of $R^K$, $\lambda^K$, and $R^G$ do not depend on $\theta$, a degree of price-stickiness. However,

$$\frac{\xi^N\xi^Y}{(1 - \alpha)^{\frac{n}{\eta+\alpha}}} = \left[\frac{GAGN - \beta (1 - \delta)}{\Pi \cdot GA \cdot GN}\right]^{\frac{-\alpha}{\eta+\alpha}} \left(\frac{\xi^K}{\alpha GAGN}\right)^{\frac{-\alpha}{\eta+\alpha}} \left(\frac{1 - \theta\Pi^{\frac{n+\alpha}{\eta+\alpha}}}{1 - \theta\Pi^{\frac{n+\alpha}{\eta+\alpha}}\Pi^{\frac{n+\alpha}{\eta+\alpha}}(1 - \alpha)^{\frac{n+\alpha}{\eta+\alpha}}} \right)$$

is dependent on $\theta$, we see that $\bar{n} \neq \bar{n}^F$ and define $\log X_n \equiv \log \bar{n} - \log \bar{n}^F$, which will turn out to be useful later when we calculate the household’s first-order labor cost.

**Step 3: Price dispersion with a positive trend inflation**

**Delta method** Before we start, we would use this approximation throughout this section. For a random variable $X$ with $E(X) = \mu_X$, we have

$$\text{Var}(f(X)) = f'(\mu_X)^2 \cdot \text{Var}(X) + \text{h.o.t.}$$

**Price dispersion** We use lower-case $p_t$ and $p_t(\nu)$ as logarithms of $P_t$ and $P_t(\nu)$. By applying

\footnote{Due to the positive trend inflation $\Pi > 1$, we have non-zero price dispersion at the steady state.}
delta method to $P_{t}^{1-\epsilon} = \mathbb{E}_{\nu}(P_{t}(\nu)^{1-\epsilon})$, we obtain

$$p_{t} = \int_{0}^{1} p_{t}(\nu) d\nu + \frac{1}{2} \left( 1 - \epsilon \right) \frac{\text{Var}_{\nu}(P_{t}(\nu)^{1-\epsilon})}{\mathbb{E}_{\nu}(P_{t}(\nu)^{1-\epsilon})^{2}} + \text{h.o.t.}$$  \hspace{1cm} (C.3.9)

where we define $\bar{p}_{t} \equiv \mathbb{E}_{\nu}(p_{t}(\nu))$. Applying delta method to $\text{Var}_{\nu}(P_{t}(\nu)^{1-\epsilon})$ term, we have

$$\text{Var}_{\nu}(P_{t}(\nu)^{1-\epsilon}) = (1 - \epsilon)^{2} \cdot \left[ \exp((1 - \epsilon)\bar{p}_{t}) \right]^{2} \cdot \text{Var}_{\nu}(p_{t}(\nu)),$$  \hspace{1cm} (C.3.10)

where we define $D_{t} \equiv \text{Var}_{\nu}(p_{t}(\nu))$. Applying delta method to $\mathbb{E}_{\nu}(P_{t}(\nu)^{1-\epsilon})$, we obtain

$$\mathbb{E}_{\nu}(P_{t}(\nu)^{1-\epsilon}) = \exp((1 - \epsilon)\bar{p}_{t}) \left[ 1 + \frac{(1 - \epsilon)^{2}}{2} D_{t} \right].$$  \hspace{1cm} (C.3.11)

Plugging equation (C.3.10) and equation (C.3.11) into equation (C.3.9), we obtain

$$p_{t} = \bar{p}_{t} + \frac{1 - \epsilon}{2} \cdot \frac{D_{t}}{\left[ 1 + \frac{(1 - \epsilon)^{2}}{2} D_{t} \right]^{2}},$$  \hspace{1cm} (C.3.12)

which we linear-approximate around $D_{t} = \bar{D}$ and get\(^{9,10}\)

$$p_{t} - \bar{p}_{t} = \frac{1 - \epsilon}{2} \cdot \frac{\bar{D}}{\left[ 1 + \frac{(1 - \epsilon)^{2}}{2} \bar{D} \right]^{2}} + \frac{1 - \epsilon}{2} \cdot \frac{1 - \frac{(1 - \epsilon)^{2}}{2} \bar{D}}{\left[ 1 + \frac{(1 - \epsilon)^{2}}{2} \bar{D} \right]^{3}} \cdot (D_{t} - \bar{D})$$

$$= \Theta_{1}^{\delta} + \Theta_{2}^{\delta}(D_{t} - \bar{D}).$$  \hspace{1cm} (C.3.13)

Now from our original definition of price dispersion $\Delta_{t}$ (equation (??)), we take logarithm on both sides, linear-approximate around $\bar{D}$, and plug equation (C.3.13) into it to attain

$$\ln \Delta_{t} = \ln \int_{0}^{1} \left( \frac{P_{t}(\nu)}{\bar{P}_{t}} \right)^{-\frac{\epsilon(\eta + 1)}{\eta + \alpha}} d\nu$$

$$= \frac{\epsilon(\eta + 1)}{\eta + \alpha} (p_{t} - \bar{p}_{t}) + \ln \left( 1 + \frac{1}{2} \left( \frac{\epsilon(\eta + 1)}{\eta + \alpha} \right)^{2} \bar{D} \right) + \frac{1}{2} \left( \frac{\epsilon(\eta + 1)}{\eta + \alpha} \right)^{2} \left( D_{t} - \bar{D} \right)$$

$$= \Theta_{1}^{\Delta} + \Theta_{2}^{\Delta} \cdot (D_{t} - \bar{D}) + \text{h.o.t},$$  \hspace{1cm} (C.3.14)

\(^{9}\)In textbook New-Keynesian models, we assume zero inflation at the steady state, which yields $\bar{D} = 0$.

\(^{10}\)\(\Theta_{1}^{\delta}\) and \(\Theta_{2}^{\delta}\) are defined as in equation (C.3.13).
where

\[ \Theta_1^\Delta \equiv \frac{\epsilon(\eta + 1)}{\eta + \alpha} \cdot \frac{1 - \epsilon}{2} \cdot \frac{\bar{D}}{1 + (\frac{1 - \epsilon}{2})^2 \bar{D}} + \ln \left( 1 + \frac{1}{2} \frac{\epsilon(\eta + 1)}{\eta + \alpha} \right)^2 \bar{D}, \quad (C.3.15) \]

\[ \Theta_2^\Delta \equiv \frac{\epsilon(\eta + 1)}{\eta + \alpha} \cdot \frac{1 - \epsilon}{2} \cdot \frac{1 - (\frac{1 - \epsilon}{2})^2 \bar{D}}{1 + (\frac{1 - \epsilon}{2})^2 \bar{D}} + \frac{1}{2} \left( \frac{\epsilon(\eta + 1)}{\eta + \alpha} \right)^2 \bar{D}. \quad (C.3.16) \]

If we define \( b_t \) as the logarithm of the newly price-resetting firm’s relative price \( P_t^*/P_t \) and \( \bar{b} \) as its steady state value, we might have \( \bar{b} \neq 0 \) due to the trend inflation. Combining equation (C.1.9) and equation (C.1.11) and linearizing, we obtain

\[ b_t \equiv p_t^* - p_t = \bar{b} + \frac{\theta\pi^{-1}}{1 - \theta\pi^{-1}} \hat{\pi}_t = \bar{b} + M \cdot \hat{\pi}_t, \quad \text{with} \quad \bar{b} = \frac{1}{\epsilon - 1} \ln \left( \frac{1 - \theta}{1 - \theta\pi^{-1}} \right). \quad (C.3.17) \]

With \( D_t = \text{Var}_\nu(p_t(\nu)) = \mathbb{E}_\nu((p_t(\nu) - p_t + p_t - \bar{p}_t)^2) \), we can write it as

\[
D_t = \int_0^{1 - \theta} (p_t^* - p_t)^2 d\nu + 2 \left( \int_0^{1 - \theta} (p_t^* - p_t) d\nu \right) (p_t - \bar{p}_t) + (1 - \theta)(p_t - \bar{p}_t)^2 \\
+ \int_{1 - \theta}^1 (p_{t-1}(\nu) - \bar{p}_t)^2 d\nu \\
= (1 - \theta)(p_t^* - p_t)^2 + 2(1 - \theta)(p_t^* - p_t)(p_t - \bar{p}_t) + (1 - \theta)(p_t - \bar{p}_t)^2 \\
+ \theta D_{t-1} + \theta(\bar{p}_t - \bar{p}_{t-1})^2, \quad (C.3.18)
\]

where we use

\[
\int_{1 - \theta}^1 (p_{t-1}(\nu) - \bar{p}_t)^2 d\nu = \theta D_{t-1} + \theta(\bar{p}_{t-1} - \bar{p}_t)^2. \quad (C.3.19)
\]

**Conjecture** Following Coibion et al. (2012), we conjecture the dynamics of \( D_t \) up to a second-order as

\[ D_t - \bar{D} = \kappa_D \hat{\pi}_t + Z_D(\hat{\pi}_t)^2 + F_D(D_{t-1} - \bar{D}) + G_D(D_{t-1} - \bar{D})\hat{\pi}_t + H_D(D_{t-1} - \bar{D})^2. \quad (C.3.20) \]

With no trend inflation, we would have \( \pi = 0 \) and \( \bar{D} = 0 \), thus \( D_t \) becomes the second-order variable around 0 and we would have \( \kappa_D = 0 \). However with steady-state inflation \( \pi > 0 \) and the price dispersion measure \( \bar{D} > 0 \), as we see in equation (C.3.20), \( D_t \) includes \( \hat{\pi}_t \) term as one of its components, even though \( \kappa_D \) is of the first-order. Our objective here is to derive equation (C.3.20) from firms’ optimal pricing behaviors and the price dispersion’s effects on the aggregate price itself. Plugging equation (C.3.13) and equation (C.3.17) into equation (C.3.18) and replace \((D_t - \bar{D})\) with the conjectured form in equation (C.3.20) up

---

\footnote{Following Coibion et al. (2012), we assume \( \kappa_D \) is of the same order as the shock processes, so that the first term becomes a second-order. Then our log-linearized model derivation without price dispersion term is valid.}
to a second-order,\textsuperscript{12} and comparing coefficients, we obtain the following set of coefficients:

\[
\begin{align*}
\bar{D} &= (\bar{b} + \Theta_0^0)^2 + \frac{\theta}{1-\theta}(\bar{\pi})^2 \quad \text{(Steady-state value of } D_t), \\
\kappa_D &= \left[1 - 2(1-\theta)\Theta_2^0(\bar{b} + \Theta_0^0) + 2\theta \Theta_2^0 \bar{\pi}\right]^{-1} \left[2(1-\theta)M(\bar{b} + \Theta_0^0) + 2\theta \bar{\pi}\right], \\
Z_D &= \left[1 - 2(1-\theta)\Theta_2^0(\bar{b} + \Theta_0^0) + 2\theta \Theta_2^0 \bar{\pi}\right]^{-1} \\
&\quad \times \left[(1-\theta)M^2 + 2(1-\theta)M \Theta_2^0 \kappa_D + (\Theta_2^0)^2 (\kappa_D)^2 + \theta - 2\theta \Theta_2^0 \kappa_D\right], \\
F_D &= \left[1 - 2(1-\theta)\Theta_2^0(\bar{b} + \Theta_0^0) + 2\theta \Theta_2^0 \bar{\pi}\right]^{-1} \left[\theta + 2\theta \Theta_2^0 \bar{\pi}\right], \\
G_D &= \left[1 - 2(1-\theta)\Theta_2^0(\bar{b} + \Theta_0^0) + 2\theta \Theta_2^0 \bar{\pi}\right]^{-1} \left[2(1-\theta)M \Theta_2^0 F_D + 2(\Theta_2^0)^2 \kappa_D F_D - 2\Theta_2^0 F_D \\
&\quad + 2\theta \Theta_2^0 - 2\theta (\Theta_2^0)^2 F_D\right], \\
H_D &= \left[1 - 2(1-\theta)\Theta_2^0(\bar{b} + \Theta_0^0) + 2\theta \Theta_2^0 \bar{\pi}\right]^{-1} \left[(\Theta_2^0)^2 (F_D)^2 + \theta (\Theta_2^0)^2 - 2\theta (\Theta_2^0)^2 F_D\right].
\end{align*}
\]

**Consumption utility** We can second-order approximate the utility of consumption as

\[
u(c_t) = \log c_t = u(\bar{c}^F) + u'_{\bar{c}} \cdot \bar{c}^F \cdot \frac{c_t - \bar{c}^F}{\bar{c}^F} + \frac{1}{2} u''_{\bar{c}} \cdot (\bar{c}^F)^2 \cdot \left(\frac{c_t - \bar{c}^F}{\bar{c}^F}\right)^2 + \text{h.o.t}
\]

\[
= u(\bar{c}^F) + \bar{c}_t + \text{h.o.t.}
\] \hspace{1cm} \text{(C.3.22)}

**Step 4: Labor aggregation and cost**

By applying Delta method (equation (C.3.8)) to the labor aggregator, which is

\[
\left(\frac{N_t}{N_t}\right)^{\eta+1} = \int_0^1 \left(\frac{N_t(\nu)}{N_t}\right)^{\eta+1} \nu, \quad \text{(C.3.23)}
\]

we can obtain\textsuperscript{13}

\[
\bar{n}_t - \mathbb{E}_\nu(\bar{n}_t(\nu)) = \frac{\frac{1}{2} \left(\frac{\eta+1}{\eta}\right) \nabla}{1 + \frac{1}{2} \left(\frac{\eta+1}{\eta}\right)^2 \nabla} + \frac{1}{2} \left(\frac{\eta+1}{\eta}\right) \frac{1 - \frac{1}{2} \left(\frac{\eta+1}{\eta}\right)^2 \nabla}{1 + \frac{1}{2} \left(\frac{\eta+1}{\eta}\right)^2 \nabla} \cdot (\nabla_t - \nabla) \quad \text{(C.3.24)}
\]

where $\nabla_t \equiv \text{Var}_\nu(\log n_t(\nu))$. A second-order approximation to the firm $\nu$-specific labor cost

\textsuperscript{12}In the right-hand side of the expression, $(p_t - \bar{p}_t)^2$ appears and has a second-order term $(D_t - \bar{D})^2$ from equation (C.3.13), and we use equation (C.3.20) to replace this term with terms related to $(\bar{\pi}_t)^2$, $(D_t - 1 - \bar{D})^2$, and $\bar{\pi}_t(D_t - 1 - \bar{D})$.

\textsuperscript{13}In the flexible-price steady-state, there is no heterogeneity among firms and thus, $\bar{c}^F(\nu) = \bar{c}^F$ for $\forall \nu \in [0, 1]$. 

around the flexible-price steady state yields

\[
\frac{\eta}{\eta + 1} \left( \frac{N_t}{\bar{N}_t} \right)^{\frac{n+1}{n}} = \frac{\eta}{\eta + 1} (\bar{F})^{\frac{n+1}{n}} + \Phi \left[ \bar{n}_t(\nu) + \frac{1}{2} \left( \frac{\eta + 1}{n} \right) \bar{n}_t(\nu)^2 \right] + h.o.t \tag{C.3.25}
\]

where a constant \( \Phi \) is from equation (C.3.4). Aggregating equation (C.3.25) over firms \( \nu \in [0, 1] \) and plugging equation (C.3.19) results in

\[
\frac{\eta}{\eta + 1} \int_0^1 \left( \frac{N_t}{\bar{N}_t} \right)^{\frac{n+1}{n}} d\nu - \frac{\eta}{\eta + 1} (\bar{F})^{\frac{n+1}{n}} = \Phi \left[ E_\nu(\bar{n}_t(\nu)) + \frac{1}{2} \left( \frac{\eta + 1}{n} \right) \int_0^1 \bar{n}_t(\nu)^2 d\nu \right]
\]

\[
= -\Phi \left( \Theta_1^n - \frac{1}{2} \left( \frac{\eta + 1}{n} \right) (\Theta_0^n)^2 \right) + \Phi \left[ (1 - \left( \frac{\eta + 1}{n} \right) \Theta_1^n) \bar{n}_t + \frac{1}{2} \left( \frac{\eta + 1}{n} \right) \bar{n}_t^2 \right.
\]

\[
+ \frac{1}{2} \left( \frac{\eta + 1}{n} \right) (\Theta_0^n)^2 \left( \text{Var}_\nu(\bar{n}_t(\nu)) - \nabla \right)^2 - \frac{\eta + 1}{n} \Theta_2^n \bar{n}_t \left( \text{Var}_\nu(\bar{n}_t(\nu)) - \nabla \right)
\]

\[
+ \left( \frac{1}{2} \left( \frac{\eta + 1}{n} \right) (1 + 2 \Theta_1^n \Theta_2^n - \Theta_0^n) \right) \left( \text{Var}_\nu(\bar{n}_t(\nu)) - \nabla \right) + \frac{1}{2} \left( \frac{\eta + 1}{n} \right) \nabla \right]. \tag{C.3.26}
\]

**Labor dispersion** From individual firm’s labor and capital demand (equation (3.2.14)) and the household’s intra-marginal condition (equation (3.2.9a)), we obtain

\[
\tilde{k}_t(\nu) = \left( 1 + \frac{1}{n} \right) \bar{n}_t(\nu) + \text{aggregate}, \tag{C.3.27}
\]

where ‘aggregate’ stands for aggregate variables. Therefore, we obtain

\[
\tilde{y}_t(\nu) = \left( 1 + \frac{\alpha}{n} \right) \bar{n}_t(\nu) + \text{aggregate}, \tag{C.3.28}
\]

by plugging equation (C.3.27) into an individual firm’s production function \( \tilde{y}_t(\nu) = \alpha \tilde{k}_t(\nu) + (1 - \alpha) \bar{n}_t(\nu) \). From the Dixit-Stiglitz good demand (equation (3.2.11)) and with equation (C.3.28), we can get

\[
\text{Var}_\nu(\bar{n}_t(\nu)) = \left( \frac{\epsilon}{1 + \frac{\alpha}{n}} \right)^2 \text{Var}_\nu(p_t(\nu)), \text{ with } \nabla = \left( \frac{\epsilon}{1 + \frac{\alpha}{n}} \right)^2 \tilde{D}. \tag{C.3.29}
\]

**Step 5: Constructing a welfare function:** Combining the consumption utility (equa-
tion (C.3.22) and the labor disutility (equation (C.3.26)), we can construct welfare as

\[ E(U_t - \bar{U}^F) = E \left[ \hat{c}_t + \Phi \left( \Theta_1^n - \frac{1}{2} \left( \frac{\eta + 1}{\eta} \right) (\Theta_1^n)^2 \right) - \Phi \left\{ \left( 1 - \left( \frac{\eta + 1}{\eta} \right) \Theta_1^n \right) \tilde{n}_t + \frac{1}{2} \left( \frac{\eta + 1}{\eta} \right) \tilde{n}_t^2 \right. \right. \\
\left. \left. + \frac{1}{2} \left( \frac{\eta + 1}{\eta} \right) (\Theta_2^n)^2 (\text{Var}_\nu(\tilde{n}_t(\nu)) - \nabla)^2 - \frac{\eta + 1}{\eta} \Theta_2^n \tilde{n}_t (\text{Var}_\nu(\tilde{n}_t(\nu)) - \nabla) \right) \\
+ \left( \frac{1}{2} \left( \frac{\eta + 1}{\eta} \right) (1 + 2\Theta_1^n\Theta_2^n - \Theta_2^n) (\text{Var}_\nu(\tilde{n}_t(\nu)) - \nabla) + \frac{1}{2} \left( \frac{\eta + 1}{\eta} \right) \nabla \right\} \right] \].

(C.3.30)

with the flexible-price steady state utility given as

\[ \bar{U}^F = \log \bar{c}^F - \frac{\eta}{\eta + 1} (\bar{n}^F)^{\frac{\eta + 1}{\eta}} \]

\[ = \frac{1}{\eta + 1} \log \left[ (1 - \zeta^G) - \xi^C \left( 1 - \frac{R^G}{\Pi \cdot GA \cdot GN} \right)^{-1} \frac{\lambda^K}{1 - \lambda^K} \right] - \frac{\alpha}{1 - \alpha} \log(R^K) + \log(\xi^{Y,F}) \]

(C.3.31)

\[ - \frac{\eta}{\eta + 1} (\xi^{Y,F}, \xi^{N,F})^{\frac{\eta + 1}{\eta}} \left[ \left( 1 - \frac{R^G}{\Pi GAGN} \right)^{-1} \frac{\lambda^K R^K}{1 - \lambda^K} \right]^{-\frac{\alpha(n + 1)}{(1 - \alpha)n}} \]

\[ \times \left[ (1 - \zeta^G) - \xi^C \left( 1 - \frac{R^G}{\Pi GAGN} \right)^{-1} \frac{\lambda^K}{1 - \lambda^K} \right]^{-1} \]

where \( \xi^{Y,F} \) and \( \xi^{N,F} \) are values of \( \xi^Y \) (equation (C.1.38)) and \( \xi^N \) (equation (C.3.6)), when \( \theta = 0 \), satisfying

\[ (\xi^{Y,F}, \xi^{N,F})^{\frac{\eta + 1}{\eta}} = (1 - \alpha) \left( \frac{\xi^K}{\alpha \cdot GA \cdot GN} \cdot \frac{GA \cdot GN - \beta (1 - \delta)}{\Pi \cdot GA \cdot GN} \right)^{-\frac{\alpha(n + 1)}{(1 - \alpha)n}} \]  

(C.3.32)

\[ \xi^{Y,F} = (1 - \alpha) \frac{\alpha}{n + 1} \left( \frac{GA \cdot GN - \beta (1 - \delta)}{\Pi \cdot GA \cdot GN} \right)^{-\frac{\alpha}{n + 1}} \]  

(C.3.33)

If we define \( \log X_c^{14} \) as the log-difference in consumption between our steady state (with trend-inflation) and flexible-price steady state, we first check if \( \log X_c^{14} \) is determined by exogenous parameters and trend-inflation \( \Pi \). From equation (C.1.34), we see that \( \log X_c = \log X_y \), where \( \log X_y \) is the log-difference in output between our steady state (with trend-inflation) and the flexible-price steady state.

From equation (C.1.37), with the fact that \( R^G, R^K, \) and \( \lambda^K \) are all independent of price stickiness \( \theta \), \( \log X_y \) can be expressed as (by comparing our steady state value of \( Y \) and the

\[ ^{14} \text{Then}, \hat{c}_t = \bar{c} + \log X_c \text{ holds.} \]
corresponding value when there is no price stickiness \((\theta = 0)\)

\[
\log X_y = -\frac{(\eta + \alpha)}{(1 - \alpha)(\eta + 1)} \log \left( \frac{1 - \theta \beta \Pi^{\epsilon - 1}}{1 - \theta \beta \Pi^{\frac{1 + \alpha}{\eta + 1}}} \right) + \frac{\eta + \alpha + \epsilon(1 - \alpha)}{(\epsilon - 1)(1 - \alpha)(\eta + 1)} \log \left( \frac{1 - \theta}{1 - \theta \Pi^{\epsilon - 1}} \right)
\]

(C.3.34)

From equation (C.3.6) and equation (C.3.7), we also can calculate \(\log X_n = \log \bar{n} - \log \bar{n}^F\) as

\[
\log X_n = \frac{\eta + \alpha}{(\eta + 1)(1 - \alpha)} \left[ \log \left( \frac{1 - \theta \Pi^{\epsilon - 1}}{1 - \theta \beta \Pi^{\frac{1 + \alpha}{\eta + 1}}} \right) + \log \left( \frac{1 - \theta \beta \Pi^{\frac{1 + \alpha}{\eta + 1}}}{1 - \theta \beta \Pi^{\frac{1 + \alpha}{\eta + 1}}} \right) \right]
\]

(C.3.35)

With \(\tilde{c}_t = \hat{c}_t + \log X_y\), \(\tilde{n}_t = \hat{n}_t + \log X_n\), and the stationarity assumption (following Coibion et al. (2012)), we can get

\[
\mathbb{E} \left[ \tilde{c}_t - \Phi \left( 1 - \left( \frac{\eta + 1}{\eta} \right) \Theta_1^\eta \right) \tilde{n}_t \right] = \log X_y - \Phi \left( 1 - \left( \frac{\eta + 1}{\eta} \right) \Theta_1^\eta \right) \log X_n.
\]

(C.3.36)

Second order terms: With \(\tilde{n}_t = \hat{n}_t + \log X_n\), second-order terms can be collected as

\[
- \Phi \left[ \frac{1}{2} \left( \frac{\eta + 1}{\eta} \right) \mathbb{E} \left( \hat{n}_t^2 \right) + \frac{1}{2} \left( \frac{\eta + 1}{\eta} \right) \left( \Theta_2^\eta \right)^2 \mathbb{E} \left( \left( \text{Var}_\nu(\hat{n}_t(\nu)) - \bar{\nu} \right)^2 \right) \right.
\]

\[
- \frac{\eta + 1}{\eta} \Theta_2^\eta \mathbb{E} \left( \hat{n}_t \left( \text{Var}_\nu(\hat{n}_t(\nu)) - \bar{\nu} \right) \right)
\]

\[
+ \left( \frac{1}{2} \left( \frac{\eta + 1}{\eta} \right) \left( 1 + 2 \Theta_1^\eta \Theta_2^\eta \right) - \Theta_2^\eta - \frac{\eta + 1}{\eta} \Theta_2^\eta \log X_n \right) \mathbb{E} \left( \text{Var}_\nu(\hat{n}_t(\nu)) - \bar{\nu} \right) \mathbb{E} \left( \text{Var}_\nu(\hat{n}_t(\nu)) - \bar{\nu} \right)
\]

which, after we can plug equation (C.3.29) into, becomes

\[
- \Phi \left[ \frac{1}{2} \left( \frac{\eta + 1}{\eta} \right) \text{Var}(\hat{n}_t) + \frac{1}{2} \left( \frac{\eta + 1}{\eta} \right) \left( \Theta_2^\eta \right)^2 \left( \frac{\epsilon}{1 + \frac{1 - \frac{1}{\eta}}{\epsilon}} \right)^4 \mathbb{E}(D_t - \bar{D})^2 \right.
\]

\[
- \frac{\eta + 1}{\eta} \Theta_2^\eta \left( \frac{\epsilon}{1 + \frac{1 - \frac{1}{\eta}}{\epsilon}} \right)^2 \text{Cov}(\hat{n}_t, D_t)
\]

\[
+ \left( \frac{1}{2} \left( \frac{\eta + 1}{\eta} \right) \left( 1 + 2 \Theta_1^\eta \Theta_2^\eta \right) - \Theta_2^\eta \left( 1 + \frac{\eta + 1}{\eta} \log X_n \right) \right) \left( \frac{\epsilon}{1 + \frac{1 - \frac{1}{\eta}}{\epsilon}} \right)^2 \mathbb{E}(D_t - \bar{D}) \right].
\]

(C.3.37)

Finally, by plugging equation (C.3.20) into equation (C.3.38), we get the following proposition. Sine \(\kappa_D\) is of the same order as shock processes, up to a second-order, we can ignore covariance terms and the square term of \(D_t\). Therefore, a 2\(^{nd}\)-order approximation to the
expected per-period welfare would be given as

\[ \mathbb{E}U_t - \bar{U}^F = \Omega_0 + \Omega_n \text{Var}(\hat{n}_t) + \Omega_\pi \text{Var}(\hat{\pi}_t), \]  

(C.3.39)

with

\[ \Omega_0 = \log X_y - \Phi \left( 1 - \left( \frac{\eta + 1}{\eta} \right) \Theta_1^n \right) \log X_n + \Phi \left( \Theta_1^n - \frac{1}{2} \left( \frac{\eta + 1}{\eta} \right) \left( \Theta_1^n \right)^2 \right) \]

\[ - \Phi \frac{1}{2} \frac{\eta + 1}{\eta} \left( \log X_n \right)^2 - \Phi \frac{1}{2} \left( \frac{\eta + 1}{\eta} \right) \left( \frac{\epsilon}{1 + \frac{\theta}{\eta}} \right)^2 \bar{D}, \]

\[ \Omega_n = -\Phi \frac{1}{2} \left( \frac{\eta + 1}{\eta} \right), \]  

(C.3.40)

\[ \Omega_\pi = -\Phi \left[ \left( \frac{1}{2} \left( \frac{\eta + 1}{\eta} \right) \left( 1 + 2 \Theta_1^n \Theta_2^n \right) - \Theta_2^n \left( 1 + \frac{\eta + 1}{\eta} \log X_n \right) \right) \left( \frac{\epsilon}{1 + \frac{\theta}{\eta}} \right)^2 \frac{Z_D}{1 - F_D} \right], \]

where

\[ \log X_y = -\frac{(\eta + \alpha)}{(1 - \alpha)(\eta + 1)} \log \left( \frac{1 - \theta \beta \Pi^{\epsilon - 1}}{1 - \theta \beta \Pi^{\epsilon - \eta + \alpha}} \right) + \frac{\eta + \alpha + \epsilon(1 - \alpha)}{(\epsilon - 1)(1 - \alpha)(\eta + 1)} \log \left( \frac{1 - \theta}{1 - \theta \beta \Pi^{\epsilon - 1}} \right), \]  

(C.3.41)

\[ \log X_n = \frac{\eta + \alpha}{(\eta + 1)(1 - \alpha)} \left[ \log \left( \frac{1 - \theta \Pi^{\epsilon - 1}}{1 - \theta \beta \Pi^{\epsilon - 1}} \right) + \log \left( \frac{1 - \theta \beta \Pi^{1 + \eta - \alpha}}{1 - \theta \Pi^{1 + \eta - \alpha}} \right) \right]. \]  

(C.3.42)

Coefficients \( \Theta_1^n, \Theta_2^n \) are given in equation (C.3.24) and \( \bar{D} \) is given by jointly solving equation (C.3.13) and equation (C.3.21). \( \kappa_D, Z_D, F_D, G_D, H_D \) are given in equation (C.3.21).
C.4 Additional Figures and Tables

Figure C.1: Markets, Agents, and Mechanisms: Household invests her wealth in the bond market or issues loans to intermediate good producers, which in turn rely on loans issued by the household to rent capital from the capital producer. There are bonds of $f = 1 \sim F$ number of maturities issued by the government. With the conventional monetary policy, central bank controls the shortest maturity yield while not adjusting a purchase amount for longer-term ones, whereas with the YCC monetary policy, central bank controls all the yields to target business-cycle variables (in our model, inflation targeting).
<table>
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<tr>
<th>Parameters</th>
<th>Values</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Households</strong></td>
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</tr>
<tr>
<td>$\beta$</td>
<td>0.998</td>
<td>Discount factor</td>
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<tr>
<td>$\eta$</td>
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<td>Frisch labor elasticity</td>
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<tr>
<td>$GN$</td>
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<td>Population growth rate</td>
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<tr>
<td><strong>Intermediate good firms</strong></td>
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<tr>
<td>$\mu$</td>
<td>0.00375</td>
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<tr>
<td>$GA$</td>
<td>1.003757</td>
<td>Gross technology growth rate</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.25</td>
<td>Capital income share</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>7</td>
<td>Elasticity of substitution between differentiated goods</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.55</td>
<td>Calvo price stickiness parameter</td>
</tr>
<tr>
<td>$\sigma_A$</td>
<td>0.0090</td>
<td>Standard deviation of technology shock</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.025</td>
<td>Capital depreciation rate</td>
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<tr>
<td><strong>Term structure</strong></td>
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<tr>
<td>$\kappa_B$</td>
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<td>Bond maturity shape (volatility) parameter (Fréchet)</td>
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<tr>
<td>$\kappa_S$</td>
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<td>Capital shape (volatility) parameter (Fréchet)</td>
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<td>$\rho_f$</td>
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<td>Autoregressive coefficient: maturity scale (mean) ($z_f^t$)</td>
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<tr>
<td>$\rho_z^K$</td>
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<td>Autoregressive coefficient: capital scale (mean) ($z^K_t$)</td>
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<td>Standard deviation: maturity scale (mean) ($z_f^t$)</td>
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<td>$\sigma_z^K$</td>
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<td>Standard deviation: capital scale (mean) ($z^K_t$)</td>
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<tr>
<td>$\zeta^F$</td>
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<td>Government subsidy to firms (optimal)</td>
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<td>$\zeta^G$</td>
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<td>Government expenditure per GDP</td>
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<td>$\sigma^G$</td>
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<td>Government expenditure coefficient</td>
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<td>$\zeta^T$</td>
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<td>Government tax revenue per GDP</td>
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<tr>
<td>$\rho_G$</td>
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<td>Autoregressive coefficient: government expenditure shock</td>
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<tr>
<td>$\rho_T$</td>
<td>0.97</td>
<td>Autoregressive coefficient: government tax revenue shock</td>
</tr>
<tr>
<td>$\sigma_G$</td>
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<td>Standard deviation: government expenditure shock</td>
</tr>
<tr>
<td>$\sigma_T$</td>
<td>0.0037</td>
<td>Standard deviation: government tax revenue shock</td>
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<td><strong>Central bank</strong></td>
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<td>$\zeta^{CB}$</td>
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<td>Central bank's balance sheet per issued bond values</td>
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<td>$\bar{\pi}$</td>
<td>$0.02 = 0.005$</td>
<td>Trend inflation (steady-state inflation)</td>
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<td>$\gamma_{1}^{f}$</td>
<td>1.5</td>
<td>Taylor coefficient of $YD_{t}^{f}$: to inflation</td>
</tr>
<tr>
<td>$\gamma_{2}^{f}$</td>
<td>1.5</td>
<td>Taylor coefficient of $YD_{t}^{f,\geq 2}$: to inflation</td>
</tr>
<tr>
<td>$\gamma_{1}^{y}$</td>
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<td>Taylor coefficient: to output</td>
</tr>
<tr>
<td>$\gamma_{2}^{y}$</td>
<td>1.5</td>
<td>Taylor coefficient of $YD_{t}^{f,\geq 2}$: to output</td>
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<td>$\sigma^{YD_{t}}$</td>
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<td>Standard deviation: monetary shock (for $YD_{t}^{f}$)</td>
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<td>$\sigma^{YD_{t}^{f,\geq 2}}$</td>
<td>$10^{-8}$</td>
<td>Standard deviation: monetary shock (for $YD_{t}^{f,\geq 2}$)</td>
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<tr>
<td>$\tau^{YD}_{f}$</td>
<td>$I_{f \times f}$</td>
<td>State reduction matrix (for $YD_{t}^{f,\geq 2}$)</td>
</tr>
</tbody>
</table>

Table C.1: Parameter values
Figure C.2: Calibrated scale parameters of the Fréchet distribution: $z^f$
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z^f$</td>
<td></td>
<td>See Figure C.2</td>
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<tr>
<td>$z^K$</td>
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<td>Capital scale (mean) parameter</td>
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<td>$C_N$</td>
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<td>Normalized consumption</td>
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<tr>
<td>$K_N$</td>
<td>4.0118</td>
<td>Normalized output</td>
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<tr>
<td>$C_N$</td>
<td>21.0701</td>
<td>Normalized capital</td>
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<tr>
<td>$Y_N$</td>
<td>0.6208</td>
<td>Consumption per GDP</td>
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<tr>
<td>$K_N$</td>
<td>5.2521</td>
<td>Capital per GDP</td>
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<td>$\hat{p}_K$</td>
<td>0.0459</td>
<td>Normalized rental price of capital</td>
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<tr>
<td>$\lambda^{HB,f}$</td>
<td>See Figure 3.1</td>
<td>Household’s bond portfolio</td>
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<tr>
<td>$\lambda^K$</td>
<td>0.1720</td>
<td>Household’s loan share out of total savings</td>
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<tr>
<td>$R^K$</td>
<td>1.0852</td>
<td>Household’s loan rates</td>
</tr>
<tr>
<td>$\gamma_D^f$</td>
<td>See Figure 3.1</td>
<td>Equilibrium yield curve</td>
</tr>
</tbody>
</table>

Table C.2: Steady-state values with parameters in Table C.1
Figure C.3: Variations in $\kappa_B$ (scale parameter): when $\kappa_B \to \infty$, we return to the expectation hypothesis case, where all the discounted expected returns are equalized, and thus obtain a flat yield curve in the steady state.
Figure C.4: Variations in central bank’s bond portfolio across maturities: central bank’s relative purchase of bonds with different maturities is negatively related with yields, in line with literatures documenting that the central bank’s bond purchase (such as QEs and LSAPs in general) reduces an yield for the bond of targeted maturity in segregated markets.
Figure C.5: Variations in deficit ratio $\zeta^F + \zeta^G - \zeta^T$: a higher deficit ratio ends up hurting the economy: given that it is sustained only when the government issues more treasury bonds\footnote{If government issues more treasury debts to finance a higher deficit given output, it will raise the government’s effective bond return $R^G$, which forces government to issue more bonds and then pushes up $R^G$, ad infinitum, which is not sustained in the long run.} or its effective bond rate $R^G$ falls, a higher deficit ratio reduces output, consumption, and capital, which leads to drops in the deficit size (nominal) and the government’s bond issuance, pushing down its bond return $R^G$. A credit spread rises in response.
Figure C.6: Variations in deficit ratio $\zeta^F + \zeta^G - \zeta^T$: a higher deficit ratio ends up hurting the economy: given that it is sustained only when the government issues more treasury bonds\textsuperscript{16} or its effective bond rate $R^G$ falls, a higher deficit ratio reduces output, consumption, and capital, which leads to drops in the deficit size (nominal) and the government’s bond issuance, pushing down its bond return $R^G$. A credit spread rises in response.

\textsuperscript{16}If government issues more treasury debts to finance a higher deficit given output, it will raise the government’s effective bond return $R^G$, which forces government to issue more bonds and then pushes up $R^G$, ad infinitum, which is not sustained in the long run.
Figure C.7: Variations in scale parameter $z^K$: given calibrated $\{z^f\}$ and for $z^K \in [0.2, 2]$, a higher $z^K$ tends to push up $\lambda^K$, the household’s capital loan share out of her total savings, thus bringing up capital, output, consumption in the steady-state. It reduces an average marginal propensity to consume (MPC). Interestingly, a positive $z^K$ shock shifts down the entire yield curve, as well as the capital return (the loan rate $R^K$), from the household’s endogenous fund reallocation, resulting in a higher credit spread.\(^{17}\) As $R^G$ falls, government bond share with respect to GDP also falls.

\(^{17}\)Therefore, the bond market experiences larger drops in yields than the loan market experiences a falling loan rate.
Figure C.8: Variations in scale parameter $z^K$: given calibrated $\{z^f\}$ and for $z^K \in [0.2, 2]$, a higher $z^K$ tends to push up $\lambda^K$, the household's capital loan share out of her total savings, thus bringing up capital, output, consumption in the steady-state. It reduces an average marginal propensity to consume (MPC). Interestingly, a positive $z^K$ shock shifts down the entire yield curve, as well as the capital return (the loan rate $R^K$), from the household’s endogenous fund reallocation, resulting in a higher credit spread.\(^{18}\) As $R^G$ falls, government bond share with respect to GDP also falls.

\(^{18}\)Therefore, the bond market experiences larger drops in yields than the loan market experiences a falling loan rate.
Figure C.9: Variations in shape parameter $\kappa_S$: given calibrated $\{z_f\}$ and $z^K$ values and for $\kappa_S \in [0.5, 3]$, a higher $\kappa_S$ tends to reduce $\lambda^K$, the household’s capital loan share out of total savings. It pushes down capital (as we have a higher $R^K$, rental rate of capital for firms), output, and consumption while raising an average marginal propensity to consume (MPC). Credit spreads increase while a higher $R^K$ dragging government’s bond return $R^G$ and the entire yield curve up. Government ends up issuing more bonds per output.
Figure C.10: Variations in shape parameter $\kappa_S$: given calibrated $\{z^f\}$ and $z^K$ values and for $\kappa_S \in [0.5, 3]$, a higher $\kappa_S$ tends to reduce $\lambda^K$, the household’s capital loan share out of total savings. It pushes down capital (as we have a higher $R^K$, rental rate of capital for firms), output, and consumption while raising an average marginal propensity to consume (MPC). Credit spreads increase while a higher $R^K$ dragging government’s bond return $R^G$ and the entire yield curve up. Government ends up issuing more bonds per output.
A.3. Section 3.4.3

Figure C.11: Impulse response to $z^K_t$ shock: a positive $z^K$ shock incentivizes the household to issue more loans, raising aggregate capital and pushing down the capital return. It raises output and inflation, thus monetary policy rate rises in response. YCC policy turns out to be better-stabilizing.

\(^{19}\)Note that an inflection point arises in the inflation path with the conventional policy, as rising output and aggregate demand push up inflation, while a lower capital return (and wage) tends to bring it down (two countervailing forces).
Figure C.12: Impulse response to $\varepsilon_t^A$ shock: a positive technology growth shock generates similar effects to the prior literature,\textsuperscript{20} where output rises and inflation falls down. A rising output raises firms’ capital demand and brings up the capital return, while the capital level actually drops with a better technology. With YCC policy, (normalized) output ironically falls: as inflation falls, all yields shift down, bringing down both capital return and wage, compared to the conventional case. Then, the household reduces her labor supply, and normalized output falls. However, actual output (which is not normalized) increases in response to the technology shock even in the YCC policy case.

\textsuperscript{20}For example, see Ireland (2004).
Figure C.13: Impulse response to $\varepsilon_{t}^{YD}$ shock: a usual contractionary monetary policy shock pushes down output, inflation, and capital. As firms reduce their inputs demand, capital return and wage fall, which brings inflation down. On the other hand, YCC policy almost perfectly insulates the economy from the shock. As the policy shock hits the economy, central bank shifts up the entire yield curve, which prevents input prices (capital return and wage) from falling, and inflation slightly increases. Even though a higher real effective savings rate reduces aggregate demand, a higher wage raises the aggregate labor supply, thus output remains almost unchanged.
Figure C.14: Impulse response to $z^K_t$ shock with ZLB: a big negative shock to $z^K$ induces the household to issue less loans to intermediate firms and invest more in bond markets. Bond rates fall and the policy rate gets constrained by ZLB. Output, capital, inflation, and capital return all jump down in response. YCC policy is effective in stabilizing the economy, while generating a longer ZLB episode as in Figure 3.5.