

The Kolmogorov-Obukhov-She-Leveque Scaling in Turbulence

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In fond memory of Mark I. Vishik.

Abstract

We construct the 1962 Kolmogorov-Obukhov statistical theory of turbulence from the stochastic Navier-Stokes equations driven by generic noise. The intermittency corrections to the scaling exponents of the structure functions of turbulence are given by the She-Leveque intermittency corrections. We show how they are produced by She-Waymire log-Poisson processes, that are generated by the Feynmann-Kac formula from the stochastic Navier-Stokes equation. We find the Kolmogorov-Hopf equations and compute the invariant measures of turbulence for 1-point and 2-point statistics. Then projecting these measures we find the formulas for the probability distribution functions (PDFs) of the velocity differences in the structure functions. In the limit of zero intermittency, these PDFs reduce to the Generalized Hyperbolic Distributions of Barndorff-Nilsen.

1 Introduction

It has become clear, since Kolmogorov and Obukhov [16, 15, 21] proposed a statistical theory of turbulence based on dimensional arguments, that noise plays a big role in turbulent flow. The laminar flow is described by the deterministic Navier-Stokes equation

$$(1) \quad \begin{aligned} u_t + u \cdot \nabla u &= \nu \Delta u - \nabla p, \\ u(x, 0) &= u_0(x), \end{aligned}$$

with the incompressibility conditions

$$\nabla \cdot u = 0, \quad (2)$$

where $u(x, t)$, $x \in \mathbb{R}^3$, $t \in \mathbb{R}^+$ is the velocity of the fluid and ν is the kinematic viscosity. Eliminating the pressure p using (2) gives the equation

$$u_t + u \cdot \nabla u = \nu \Delta u + \nabla \{ \Delta^{-1} [\text{trace}(\nabla u)^2] \}. \quad (3)$$

The turbulence of the fluid is quantified by the dimensionless Reynolds number $R = \frac{UL}{\nu}$ where U is a typical velocity of the flow and L is a typical length scale associated with the flow. If R is small the flow is laminar and if R is sufficiently large the flow is turbulent. Many studies, theoretical [8], numerical and experimental, show that noise plays a big role in fully developed turbulence. For R large enough laminar flow, although it still exists, is unstable and the fluid instabilities magnify the small ambient noise that exists in any fluid flow. This noise is quelled if R is small enough and then laminar flow is stable. The upshot is that there is large noise driving fully developed turbulent flow and Kolmogorov's point of view is correct: turbulent velocity is a stochastic process described by the stochastic Navier-Stokes equation

$$(4) \quad \begin{aligned} du &= (\nu \Delta u - u \cdot \nabla u + \nabla \{ \Delta^{-1} [\text{trace}(\nabla u)^2] \}) dt + df_t(u, x, t), \\ u(x, 0) &= u_0(x). \end{aligned}$$

Here df_t denotes the stochastic forcing in fully developed turbulence.

Now the question becomes: what is the stochastic forcing (noise) in fully developed turbulence? Can its form be traced back to the fluid instabilities or is it of a general nature? It is fair to say that attempts made over the last two decades to trace the noise back to the fluid instabilities have failed and the experiments

indicate that the noise in fully developed turbulence is of a general nature. At the same time it is clear that the noise cannot be white both in space and time because it was shown by Walsh [29] that the solution of parabolic equation driven by such noise is only continuous in space in two or lower dimensions. In three dimensions and higher it is a distribution, discontinuous in space, contrary to the spatially continuous fluid velocity observed in fully developed turbulence.

It turns out that the noise in fully developed turbulence is generic but not white. It can be shown from general principle, see [10, 11], that with periodic boundary conditions the generic noise has the form,

$$\sum_{k \in \mathbb{Z}^3} c_k^{\frac{1}{2}} db_t^k e_k(x) + \sum_{k \neq 0} d_k |k|^{1/3} dt e_k(x) + u \sum_{k \neq 0} \int_{\mathbb{R}} h_k \bar{N}^k(dt, dz). \quad (5)$$

Here the noise is both additive and multiplicative. The first two terms form the additive noise and the third term consists of jumps multiplying the velocity u and forms the multiplicative part of the noise. In the additive noise, each Fourier component $e_k = e^{2\pi i k \cdot x}$ comes with its own independent Brownian motion b_t^k and a deterministic term $|k|^{1/3}$. The coefficients $c_k^{\frac{1}{2}}$ and d_k decay sufficiently fast so that the Fourier series converges. In the multiplicative (multiplied by u) noise, \bar{N} is a (compensated) jump measure, counting the number of jumps. The sizes of the jumps h_k (corresponding to jumps in the velocity gradient) do not decay, but for $t < \infty$, only finitely many h_k s, $|k| \leq m$, are nonzero.

The argument in [10] is the following. The additive noise describes the mean of the dissipation process in turbulent flow. It is a Fourier series with coefficients that are independent Brownian motions in the first term of the additive noise above. Thus it is white in time but it is not white in space. The coefficients $c_k^{\frac{1}{2}}$ make it converge in L^2 , the space where we make sense of the noise. None of the coefficients $c_k^{\frac{1}{2}}$ vanish. Thus the noise in fully developed turbulence is non-degenerate, and this is what experiments and simulations indicate. However, this is not a complete description of the mean, there frequently is a bias in the flow and associated are large deviation of the dissipation process. These large deviations are described by the second term in the additive noise that is derived from the large deviation principle or Cramer's theorem, see [11]. These two terms give a more complete description of the mean of the dissipation process and they must scale in the same way with the wave number k . This is the reason for the factor $|k|^{1/3}$ in the second term. It is clear from this argument that the first part of (5) is a completely generic additive dissipation noise in $L^2(\mathbb{T}^3)$.

The second, multiplicative part of the noise in (5) stems from the jumps (or near-jumps) in the gradient of the velocity ∇u . Large excursion of the velocity, or jumps in ∇u , are observed in turbulent flow and associated with these excursions are large dissipation events. These event are modeled by the last multiplicative term in (5). It consists of u multiplied by a sum of jumps associated with the k th wavenumber, where $h_k(\cdot)$ is the size of the jump and $\bar{N}(dt, \cdot)$ is the compensated number of jumps in an infinitesimal time interval dt . Compensation means that one has to subtract the associated Lévy (mean) measure, see [11].

The important feature to notice about the noise (5) is that it has no structure embedded in it. Let \mathcal{E} denote the energy of the Navier-Stokes equation

$$\mathcal{E} = \frac{1}{2|\Omega|} \int_{\Omega} |u(x,t)|^2 dx. \quad (6)$$

Here $|\Omega|$ denotes the volume of Ω and "mean" refers to the fact that we are dividing the energy by the volume. The mean energy dissipation is now defined to be

$$\varepsilon = -\frac{d}{dt} \mathcal{E}. \quad (7)$$

The noise is white in time and the coefficient $c_k^{1/2}$ and d_k barely make the Fourier series converge in L^2 so that the energy of the stochastic velocity is still defined. Thus the noise is not white in space but it is as close to being white as it can be and the energy still make sense. None of the coefficients $c_k^{1/2}$ and d_k vanish. Thus the noise is not degenerate and this seems to be a general feature of the noise in fully developed turbulence. The noise is active in all directions in infinite dimensional functions space and no Fourier component is missing in the noise. The multiplicative noise consists of the simplest jumps imaginable multiplying the velocity u . There is no structure in the jumps they depend on the wave number k , or we think about them as jumps in the Fourier coefficients, but then the k th jump imparts energy to every Fourier coefficient of u .

The structure of the turbulent velocity is created by the Navier-Stokes evolution acting on the generic noise (5). We consider the stochastic Navier-Stokes equation describing the turbulent velocity to understand this structure,

$$\begin{aligned} du = & (\nu \Delta u - u \cdot \nabla u + \nabla \Delta^{-1} \text{tr}(\nabla u)^2) dt + \sum_{k \in \mathbb{Z}^3} c_k^{\frac{1}{2}} db_t^k e_k(x) \\ (8) \quad & + \sum_{k \neq 0} d_k |k|^{1/3} dt e_k(x) + u \sum_{k \neq 0}^m \int_{\mathbb{R}} h_k \bar{N}^k(dt, dz), \end{aligned}$$

$$u(x, 0) = u_0(x),$$

$e_k = e^{2\pi i k \cdot x}$ s are Fourier component, b_t^k are independent Brownian motions and the coefficients $c_k^{\frac{1}{2}}$ and d_k decay sufficiently fast so that the Fourier series converges. The h_k s are sizes of the jumps in the multiplicative noise, see the discussion of the noise (5) above. The equation (8) defines the fluid velocity u as a stochastic process, taking its values in $L^2(\mathbb{T}^3)$ for each t .

2 The Kolmogorov-Obukhov Statistical Theory of Turbulence

It is clear, once we understand that the velocity in turbulent flow satisfies the stochastic Navier-Stokes equation (8), that any theory of turbulence that can be compared with simulations and experiments must be a statistical theory. In other words, the pointwise values of the fluid velocity are not deterministic and the only deterministic quantities that can be associated with the flow are statistical quantities. In practice this means that one must take averages of simulations and experiments and compare these with expectations of the corresponding stochastic process.

In 1941, Kolmogorov and Obukhov [16, 15, 21] proposed such a statistical theory. The main consequence and test of this theory was that the structure functions of the velocity differences of a turbulent fluid

$$E(|u(x, t) - u(x + l, t)|^p) = S_p = C_p l^{p/3}$$

should scale with the distance (lag variable) l between them, to the power $p/3$. E is the expectation computed by an ensemble average from simulations or experiments. This theory was immediately criticized by Landau for not taking into account the influence of the large flow structure on the constants C_p and later for not including the influence of the intermittency, in the velocity fluctuations, on the scaling exponents.

In 1962 Kolmogorov and Obukhov [17, 22] proposed a corrected theory where both of those issues were addressed. They also pointed out that the scaling exponents for the first two structure functions could be corrected by log-normal processes. For higher order structure functions the log-normal processes gave intermittency corrections inconsistent with contemporary simulations and experiments, see [1].

The correct intermittency corrections were found by She and Leveque [26] in 1994. She and Waymire [27] and Dubrulle [12] showed that these corrections are produced by log-Poisson processes. These log-Poissonian processes give the intermittency corrections that agree with modern direct Navier-Stokes simulations (DNS) and experiments.

The structure functions are obviously not the only statistical quantity that one would like to compute. The goal is to be able to compute the invariant measure for the one point, two point, etc., statistics. Then one can compute all the statistical quantities that can be simulated and measured. The mean, variance, skewness and flatness are quantities that one can compute from the one point invariant measure. The structure functions and the associated probability density functions (PDF) for the velocity differences can be computed from the two point invariant measure. This will be our goal and the surprising thing is that one can find a linear functional differential equation for these measures, see [10, 11], and solve them although the nonlinear stochastic Navier-Stokes equation (8) cannot be solved. We outline this below but more details can be found in [11]. Hopf [14] was the first to show that the characteristic function of the invariant measure satisfied a linear functional differential equation, the theory of such equations has recently become available, see [25].

3 The Stochastic Navier-Stokes Integral Equation

We write the stochastic Navier-Stokes equation (8) in integral form,

$$(9) \quad u = e^{K(t)} e^{\int_0^t dq} M_t u^0 + \sum_{k \neq 0} c_k^{1/2} \int_0^t e^{K(t-s)} e^{\int_s^t dq} M_{t-s} db_s^k e_k(x), \\ + \sum_{k \neq 0} d_k \int_0^t e^{K(t-s)} e^{\int_s^t dq} M_{t-s} |k|^{1/3} dt e_k(x),$$

where K is the linear (Navier-Stokes) operator

$$K = \nu \Delta + \nabla \Delta^{-1} tr(\nabla u \nabla) = \nu \Delta + D \quad (10)$$

and the multiplicative factor

$$M_t = \exp\left\{-\int u(B_s, s) dB_s - \frac{1}{2} \int_0^t |u(B_s, s)|^2 ds\right\},$$

is a Martingale, with $B_t \in \mathbb{R}^3$ an auxiliary Brownian motion. By Ito's formula and a computation similar to the one that produces the geometric Lévy process, see [23],

$$3 \int_s^t dq = \sum_{k \neq 0}^m \left\{ \int_0^t \int_{\mathbb{R}} \ln(1 + h_k) \bar{N}^k(ds, dz) + \int_0^t \int_{\mathbb{R}} (\ln(1 + h_k) - h_k) m_k(ds, dz) \right\},$$

where m_k is the Lévy measure of the jump process x_t^k . The operator K does not generate a semi-group because of its dependence on u but with some conditions on u , see below, it generates a flow. The notation $e^{K(t-s)}f(s)$ simply means that we solve the equation $f_t = Kf$, with initial data $f(s)$ for the time interval $[s, t]$.

The existence of unique turbulent solutions to the stochastic Navier-Stokes equations (8) can be proven in some cases. For example if the equation is driven by a strong swirling flow, see [9]. This result is not terribly surprising. If the initial data had the symmetry of the swirl then the deterministic problem would be two-dimensional and the global existence of the two-dimensional Navier-Stokes equation is well known. It is also well-known that if the initial data is close to such a two-dimensional flow then global existence can be extended to this case also, see [2, 3], for another such example.

In [9] the author obtained the global bound for the Sobolev space norm of u , based on $L^2(\mathbb{T}^3)$ with index $\frac{11}{6}^+ = \frac{11}{6} + \varepsilon$, ε small, for a swirling flow,

$$E(\|u\|_{\frac{11}{6}^+}^2(t)) \leq C, \quad (11)$$

where E denotes the expectation and the constant C is independent of t . The Sobolev space consists of Hölder continuous functions of Hölder index $1/3$, as pointed out by Onsager [24].

Suppose that

$$E(\|u\|_{\frac{3}{2}^+}^2) \leq C, \quad (12)$$

then the operator K generates a flow denoted by $e^{K(t)}$ and $\lim_{t \rightarrow \infty} e^{K(t)}f_0 = 0$, for $f_0 \in H^1(\mathbb{T}^3)$, see [11].

Then using the bound (11), we get an estimate on the spectrum of operator K . Recall from (10) that D is the pressure operator.

Lemma 3.1 *Suppose that (11) holds, then the pressure operator is bounded by the spectrum of the symmetric operator $D^T D$ with discrete spectrum λ_k^2 and satisfies the estimate*

$$-C|k|^{2/3} \leq -\lambda_k \leq |\nabla \Delta^{-1} \operatorname{tr} \nabla u \cdot \nabla P_k|_2 \leq \lambda_k \leq C|k|^{2/3}, \quad k \in \mathbb{Z}^3, \quad (13)$$

on the Hilbert space $H^{\frac{11}{6}+}(\mathbb{T}^3)$, in the inertial range, see below. P_k is the projection onto the k th eigenspace of the symmetric operator. Moreover, in the inertial range the operator K satisfies the bound

$$-C|k|^{2/3} - 4\nu\pi^2|k|^2 \leq |KP_k|_2 \leq C|k|^{2/3} + 4\nu\pi^2|k|^2, \quad k \in \mathbb{Z}^3. \quad (14)$$

We will use this estimate below in order to compute the structure functions of turbulence or the moments of the velocity difference at two points in the fluid, in the inertial range of turbulence, where $1/L \leq |k| \leq 1/\eta$, $k_o = 1/\eta = (\varepsilon/\nu^3)^{1/4}$, a constant. $\eta = 1/k_o$ is called the Kolmogorov length scale, ε is the energy dissipation rate (7) and L is a typical length scale associated with the large eddies in the flow. The above estimate implies that for a large Reynolds number where ν is small and $1/L \leq |k| \leq 1/\eta$, we can think of the spectrum of K growing as a constant times $|k|^{2/3}$, with the error $4\nu\pi^2|k|^2$, in the inertial range, see [11] for more details.

The proof of Lemma 3.1 and the bounds (13) and (14) is given in [10] and [11].

4 The Log-Poissonian Processes

The processes found by She and Leveque [26], and shown to be log-Poisson processes by She and Waymire [27] and by Dubrulle [12], are produced by applying the Feynman-Kac formula to the potential dq . Namely, $e^{\int_0^t dq} = e^{\sum_{k \neq 0}^m \int_0^t dq_k}$ and by setting $h_k = \beta - 1$ and computing the mean of N_t^k

$$E(N_t^k) = \int_{\mathbb{R}} m_k(t, dz) = -\frac{\gamma \ln |k|}{\beta - 1}, \quad (15)$$

we get that

$$\begin{aligned} 3 \int_0^t dq_k &= \int_0^t \int_{\mathbb{R}} \ln(1 + h_k) \bar{N}^k(ds, dz) + \int_0^t \int_{\mathbb{R}} (\ln(1 + h_k) - h_k) m_k(ds, dz) \\ &= N_k(t) \ln(\beta) + (\beta - 1) \left(\gamma \frac{\ln |k|}{\beta - 1} \right). \end{aligned}$$

This gives the term

$$e^{\int_0^t dq_k} = e^{(\gamma \ln |k| + N_k \ln \beta)/3} = (|k|^\gamma \beta^{N_k})^{1/3} = \left(|k|^\gamma \beta^{N_t^k} \right)^{1/3}, \quad (16)$$

in the (implicit) solution (9) of the stochastic Navier-Stokes equation. These are exactly the log-Poisson processes found by the above authors. Then we get

$$\ln E((e^{\gamma \ln |k| + N_k \ln \beta})^{\frac{p}{3}}) = \ln E((|k|^\gamma \beta^{N_k})^{\frac{p}{3}}) = \gamma \left(\frac{p}{3} - \frac{\beta^{p/3} - 1}{\beta - 1} \right) \ln |k| = -\tau_p \ln |k|,$$

for the logarithm of the p th moment, where τ_p are the intermittency corrections in (23). Now the expression

$$\tau_p = -\gamma \left(\frac{p}{3} - \frac{\beta^{p/3} - 1}{\beta - 1} \right)$$

implies that $\tau_0 = 0$ and $\tau_3 = 0$ independently of γ . The latter condition is required by the Kolmogorov 4/5th law, see [13]. However, to be consistent with the spectral theory of the operator D above, that moves energy around in quanta of $|k|^{2/3}$, we should set $\gamma = 2/3$. This means that the log-Poissonian processes also move energy in quanta of $|k|^{2/3}$ in Fourier space. However, $|k|^{2/3}$ is multiplied by β^{N_k} in (16) above, namely the number of jumps on the k th level contribute to the transfer of energy, and so far β is a free parameter. We follow [26] in making the assumption that determines β , see also [28]. The basic assumption is that the most singular structures in the turbulent fluid are one-dimensional vortex lines that the highest moments capture. Thus (assuming $0 < \beta < 1$) by the Lagrange transformation, see [26],

$$\tau_p = -\frac{2}{3} \left(\frac{p}{3} \right) + \frac{2}{3} \frac{1}{1-\beta} - \frac{2}{3} \frac{\beta^{p/3}}{1-\beta} \rightarrow -\frac{2}{3} \left(\frac{p}{3} \right) + C_o$$

as $p \rightarrow \infty$, where $C_o = 2$ is the codimension of the one-dimensional vortex lines and this implies that $\beta = 2/3$. We will make this choice of β .

Thus we see that the jumps multiplying u in the equation (8) produce the log-Poisson processes $(|k|^{2/3} (2/3)^{N_t^k})^{1/3}$ in the integral equation for u .

$$\begin{aligned} u &= e^{K(t)} \left(\prod_k^m |k|^{2/3} (2/3)^{N_t^k} \right)^{1/3} M_t u_0 \\ &+ \sum_{k \neq 0} c_k^{1/2} \int_0^t e^{K(t-s)} \left(\prod_j^m |j|^{2/3} (2/3)^{N_{t-s}^j} \right) M_{t-s} db_s^k e_k(x) \\ &+ \sum_{k \neq 0} d_k \int_0^t e^{K(t-s)} \left(\prod_j^m |j|^{2/3} (2/3)^{N_{t-s}^j} \right)^{1/3} M_{t-s} |k|^{1/3} dt e_k(x) \end{aligned}$$

since only the k th log-Poissonian processes are correlated with the k th Fourier component. This equation clearly shows how the intermittency in the velocity (in Equation (8)) causes intermittency in the dissipation through the Navier-Stokes evolution, if we recall how the discrete (Poisson) distribution picks the k th term (associated with e_k) out of the product.

5 The Kolmogorov-Obukhov-She-Leveque Scaling

In 1962 Kolmogorov and Obukhov [17, 22] proposed a corrected statistical theory of turbulence were both of the criticisms discussed in Section 2 were addressed. This was Landau's criticism for not taking into account the influence of the large flow structure on the constants C_p and the later criticism for not including the influence of the intermittency in the velocity fluctuations on the scaling exponents, see Section 4. They presented their refined similarity hypothesis

$$S_p = C'_p \langle \tilde{\epsilon}^{p/3} \rangle l^{p/3}, \quad (17)$$

where l is the lag variable and the averaged energy dissipation rate is

$$\tilde{\epsilon} = \frac{1}{\frac{4}{3}\pi l^3} \int_{|s| \leq l} \epsilon(x+s) ds, \quad (18)$$

ϵ being the mean energy dissipation rate (7). They also pointed out that the scaling exponents for the first two structure functions could be corrected by log-normal processes. However, for higher order structure functions the log-normal processes gave intermittency corrections inconsistent with contemporary simulations and experiments.

In the refined similarity hypothesis (17) the averaged dissipation rate $\tilde{\epsilon}$ will depend on the large flow structure, so its addition addresses Landau's objections at least partially. The assumption is that

$$\langle \tilde{\epsilon}^{p/3} \rangle \sim l^{\tau_p},$$

because of intermittency, where the τ_p are called the intermittency corrections (to the scaling). Consequently, intermittency corrections are produced,

$$S_p = C'_p \langle \tilde{\epsilon}^{p/3} \rangle l^{p/3} = C_p l^{p/3 + \tau_p} = C_p l^{\zeta_p},$$

where

$$\zeta_p = \frac{p}{3} + \tau_p \quad (19)$$

are the scaling exponents with intermittency corrections included, and the C_p s are not universal but depend on the large flow structure. We will see below that starting with the stochastic Navier-Stokes equation (8) this scaling hypothesis in fact holds.

The She-Leveque intermittency corrections are

$$\tau_p = -\frac{2p}{9} + 2(1 - (2/3)^{p/3}),$$

given by the log-Poissonian processes derived above. These intermittency corrections are consistent with contemporary simulations and experiments, see [1], [7], [26] and [28].

We will now show how the integral form (9) of the stochastic Navier-Stokes equation can be used to compute an estimate for the structure functions of turbulence.

In order to compute the structure functions of turbulence or the moments of the velocity difference at two points in the fluid, we need to estimate the operator K above, compare Equation (13). Recall the eigenvalues $\lambda_k > 0$ that are the square roots of the eigenvalues of the symmetric operator $D^T D$ above, with P_k the projector onto the corresponding eigenspace. Then the equation (14) can be reformulated as

$$\begin{aligned} -C|k|^{2/3} - 4\nu\pi^2|k|^2 \leq -\lambda_k - \nu 4\pi^2|k|^2 &\leq |KP_k|_2 \\ (20) \qquad \qquad \qquad &\leq \lambda_k + \nu 4\pi^2|k|^2 \leq C|k|^{2/3} + \nu 4\pi^2|k|^2, \end{aligned}$$

if u satisfies the bound

$$E(\|u\|_{11+})(t) \leq C. \tag{21}$$

For a large Reynolds number ν is small and since $|k|^2 \leq k_o^2$, $k_o = (\varepsilon/\nu^3)^{1/4}$, where k_o is the inverse of the Kolmogorov length, we can now think of the spectrum of K growing as a constant times $|k|^{2/3}$ in the inertial range. ε is the dissipation rate (7). The coefficient C is a constant times a Sobolov space norm of u , by the estimate (11), see [9]. The lower estimate in (20) is the relevant one for the forward cascade of energy.

Now estimates of the structure functions are possible and we get the following result. Suppose that the coefficients c_k and d_k in equation (4) satisfy the conditions $\sum_{k \in \mathbb{Z}^3 \setminus \{0\}} c_k < \infty$ and $\sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |k|^{1/3} |d_k| < \infty$. Then the scaling of the structure functions of (8) is

$$S_p \sim C_p |x - y|^{\zeta_p}, \tag{22}$$

where

$$\zeta_p = \frac{p}{3} + \tau_p = \frac{p}{9} + 2(1 - (2/3)^{p/3}), \quad (23)$$

$\frac{p}{3}$ being the Kolmogorov-Obukhov '41 scaling and τ_p the She-Leveque intermittency corrections, when the lag variable $|x - y|$ is small.

The values in equation (23) agree with experimental values in [7], they are in agreement with Kolmogorov-Obukhov scaling hypothesis with intermittency corrections, computed by She and Leveque, but disagree with the log-normal distribution [17, 22], for the intermittency corrections.

The estimate of the first structure function is straight-forward,

$$\begin{aligned} S_1(x, y, t) &= E(|u(x, t) - u(y, t)|) = \\ &= 2 \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} d_k \int_0^t e^{-\lambda_k(t-s)} |k|^{1/3} ds E([e^{\gamma \ln |k| + N_k \ln(\beta)}]^{1/3}) \sin(\pi k \cdot (x - y)) \\ (24) \quad &\leq \frac{2}{C} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |d_k| \frac{(1 - e^{-\lambda_k t})}{|k|^{\zeta_1}} |\sin(\pi k \cdot (x - y))|. \end{aligned}$$

We have estimated the spectrum of $K(t)$ by $-\lambda_k = -C|k|^{2/3}$ in the second line (we use this approximation, $\mathbf{v} = 0$, throughout the computations) and also used the expectation of the Poisson jump process

$$E([e^{\gamma \ln |k| + N_k \ln(\beta)}]^{1/3}) = \frac{1}{|k|^{\tau_1}},$$

from Section 4. We used the lower estimate in (20) and this makes the estimate in (24) be an overestimate of the efficiency of the cascade. The measure of the discrete process must be written as

$$\sum_{l=-\infty}^{\infty} \delta_{l,k} \prod_{j \neq l}^m \delta_{N_t^j} \sum_{j=0}^{\infty} \binom{m_l}{j} \frac{m_l^j}{j!} e^{(-m_l)}, \quad (25)$$

where $\delta_{l,k} = 0, l \neq k, 1, l = k$ is the Kronecker delta function, because N_t^k depends on the k th Fourier component e_k (or db_t^k and $|k|^{1/3} dt$) but is independent of the components with different wavenumbers. The δ functions in the product imply that the probabilities of all the N_t^j 's, $j \neq k$ concentrate at 0.

Now, if $\sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |d_k| < \infty$, then we get a stationary state as $t \rightarrow \infty$

$$S_1(x, y, \infty) \leq \frac{2}{C} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{|d_k|}{|k|^{\zeta_1}} |\sin(\pi k \cdot (x - y))|,$$

and for $|x - y|$ small,

$$S_1(x, y, \infty) \sim \frac{2\pi^{\zeta_1}}{C} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |d_k| |x - y|^{\zeta_1},$$

where $\zeta_1 = 1/3 + \tau_1 \approx 0.37$.

A similar computation gives the second structure function,

$$\begin{aligned} S_2 &= E(|u(x, t) - u(y, t)|^2) \\ &\leq \frac{2}{C} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} c_k \frac{1 - e^{-2\lambda_k t}}{|k|^{\zeta_2}} \sin^2(\pi k \cdot (x - y)) \\ &\quad + \frac{4}{C^2} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} d_k^2 \frac{(1 - e^{-\lambda_k t})^2}{|k|^{\zeta_2}} \sin^2(\pi k \cdot (x - y)), \end{aligned}$$

again by using the lower estimate in (20). As $t \rightarrow \infty$, we get

$$S_2(x, y, \infty) \sim \frac{4\pi^{\zeta_2}}{C^2} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} [d_k^2 + \left(\frac{C}{2}\right)c_k] |x - y|^{\zeta_2},$$

when $|x - y|$ is small, where $\zeta_2 = 2/3 + \tau_2 \approx 0.696$.

Similarly

$$\begin{aligned} S_3 &= E(|u(x, t) - u(y, t)|^3) \\ &\leq \frac{2^3}{C^3} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{[|d_k|^3 (1 - e^{-\lambda_k t})^3 + 3(C/2)c_k |d_k| (1 - e^{-2\lambda_k t})(1 - e^{-\lambda_k t})]}{|k|} \\ &\quad \times |\sin^3(\pi k \cdot (x - y))|, \end{aligned}$$

and

$$S_3(x, y, \infty) \sim \frac{2^3 \pi}{C^3} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} [|d_k|^3 + 3(C/2)c_k |d_k|] |x - y|,$$

where $\zeta_3 = 1$.

All the structure functions are computed in a similar manner, for the p th struc-

Table 1: Moments of a Gaussian

Order	Raw moment	Central moment	Cumulant
1	M	0	M
2	$M^2 + \sigma^2$	σ^2	σ^2
3	$M^3 + 3M\sigma^2$	0	0
4	$M^4 + 6M^2\sigma^2 + 3\sigma^4$	$3\sigma^4$	0
5	$M^5 + 10M^3\sigma^2 + 15M\sigma^4$	0	0
6	$M^6 + 15M^4\sigma^2 + 45M^2\sigma^4 + 15\sigma^6$	$15\sigma^6$	0
7	$M^7 + 21M^5\sigma^2 + 105M^3\sigma^4 + 105M\sigma^6$	0	0
8	$M^8 + 28M^6\sigma^2 + 210M^4\sigma^4 + 420M^2\sigma^6 + 105\sigma^8$	$105\sigma^8$	0

ture functions, we get that S_p is estimated by

$$\begin{aligned}
S_p &\leq \frac{2^p}{C^p} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{\sigma^p \cdot (-i\sqrt{2})^p U\left(-\frac{1}{2}p, \frac{1}{2}, -\frac{1}{2}(M/\sigma)^2\right)}{|k|^{\zeta_p}} |\sin^p(\pi k \cdot (x-y))| \\
&= \frac{2^p}{C^p} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{1}{|k|^{\zeta_p}} \frac{2^{(p+1)/2} \mu \sigma^{p-1}}{\sqrt{\pi}} \Gamma\left(1 + \frac{p}{2}\right) {}_1F_1\left(\frac{1-p}{2}, \frac{3}{2}, -\frac{\mu^2}{2\sigma^2}\right) |\sin^p(\pi k \cdot l)|, \quad p \text{ odd}, \\
&= \frac{2^p}{C^p} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{1}{|k|^{\zeta_p}} \frac{2^{p/2} \sigma^p}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right) {}_1F_1\left(-\frac{p}{2}, \frac{1}{2}, -\frac{\mu^2}{2\sigma^2}\right) |\sin^p(\pi k \cdot l)|, \quad p \text{ even},
\end{aligned}$$

where U is the confluent hypergeometric function, $l = (x-y)$ is the lag variable, $M = |d_k|(1 - e^{-\lambda_k t})$ and $\sigma = \sqrt{(C/2)c_k(1 - e^{-2\lambda_k t})}$ and ${}_1F_1(a, b, z) = \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{(n)} n!}$, $a^{(n)} = a(a+1)(a+2)\cdots(a+n-1)$, is the generalized hypergeometric series. Thus the coefficients of S_p are given by the raw moments of a Gaussian, the first few of which are listed in Table 1. Now $S_p(x, y, \infty)$ is

$$S_p \sim \frac{2^p \pi^{\zeta_p}}{C^p} \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} ((C/2)c_k)^{p/2} \cdot (-i\sqrt{2})^p U\left(-\frac{1}{2}p, \frac{1}{2}, -\frac{d_k^2}{Cc_k}\right) |x-y|^{\zeta_p},$$

to leading order for $|x-y|$ small. We also obtain Kolmogorov's 4/5 law, see [13],

$$S_3 = -\frac{4}{5} \varepsilon(0) |x-y|$$

to leading order, where ε is the mean energy dissipation rate (7).

6 An Infinite-dimensional Ito Process and the Invariant Measure of Turbulence

The stochastic Navier-Stokes equation (9) was written in integral form above as

$$(26) \quad \begin{aligned} u = e^{Kt} P_t M_t u^0 + \sum_{k \neq 0} c_k^{1/2} \int_0^t e^{K(t-s)} P_{t-s} M_{t-s} db_s^k e_k(x) \\ + \sum_{k \neq 0} d_k \int_0^t e^{K(t-s)} P_{t-s} M_{t-s} |k|^{1/3} ds e_k(x) \end{aligned}$$

by the Feynman-Kac formula, where is the operator K generates the flow e^{Kt} , where $K = \nu \Delta + \nabla \Delta^{-1} \text{trace} \nabla u \nabla$ and

$$M_t = \exp\left\{-\int u(B_s, s) dB_s - \frac{1}{2} \int_0^t |u(B_s, s)|^2 ds\right\}$$

is a Martingale with $B_t \in \mathbb{R}^3$ an auxiliary Brownian motion, see Section 3 and [11].

$$P_t = \prod_k^m (|k|^{2/3} (2/3)^{N_t^k})^{1/3},$$

by the computation of how the log-Poisson processes are produced, from the stochastic Navier-Stokes equation, by the Feynman-Kac formula (16) above. Now let $C^{1/2}$, $D \in L(H)$ be linear operators on $H = L^2(\mathbb{T}^3)$, defined by

$$C^{1/2} u = \sum_{k \neq 0} C_k^{1/2} \hat{u}_k e_k, \quad Du = \sum_{k \neq 0} D_k \hat{u}_k e_k$$

for $u = \sum_{k \neq 0} \hat{u}_k e_k \in L^2(\mathbb{T}^3)$, $C_k^{1/2}$ and D_k are 3 by 3 diagonal matrices with entries $c_{k,j}^{1/2}$ and $d_{k,j}$, $j = 1, 2, 3$ on the diagonal, and $\bar{D}_k = |k|^{1/3} D_k$.

Next we define the variance

$$Q_t = \int_0^t e^{K(s)} P_s M_s C M_s P_s^* e^{K^*(s)} ds \quad (27)$$

and drift

$$E_t = \int_0^t e^{K(s)} P_s M_s \bar{D} ds \quad (28)$$

operators. The Kolmogorov-Hopf equation for the invariant measure can now be written as

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \text{tr}[Q_t \Delta \phi] + \text{tr}[E_t \nabla \phi], \quad (29)$$

where ϕ is a bounded function of u . This equation is simply the Kolmogorov backward equation for the infinite-dimensional Ito process (26).

Then the solution of the Kolmogorov-Hopf equation (29) can be written in the form

$$\begin{aligned} R_t \phi(z) &= \int_H \phi(y) \mathcal{N}_{(e^{Kt} P_t M_t z + E_t I, Q_t)} * \mathbb{P}_{P_t}(dy) \\ &= \int_H \phi(e^{Kt} P_t M_t z + E_t I + y) \mathcal{N}_{(0, Q_t)} * \mathbb{P}_{P_t}(dy) \end{aligned}$$

where $z = u_0$ and \mathbb{P}_{P_t} is the discrete Poisson law (25) of the log-Poisson process P_t . \mathcal{N}_{m, Q_t} denotes the infinite-dimensional normal distribution on H with mean m and variance Q_t , see [25], $I = \sum e_k$, and $E_t I \in H$.

6.1 The Invariant Measure of Turbulence

We can now write a formula for the invariant measure of turbulence.

Theorem 6.1 *The invariant measure of the stochastic Navier-Stokes equation on $H_c = H^{3/2+}(\mathbb{T}^3)$ has the form*

$$\mu(dx) = e^{\langle Q^{-1/2} E I, Q^{-1/2} x \rangle - \frac{1}{2} |Q^{-1/2} E I|^2} \mathcal{N}_{(0, Q)}(dx) \sum_k \delta_{k,l} \prod_{j \neq l}^m \delta_{N_t^j} \sum_{j=0}^{\infty} p_{m_l}^j \delta_{(N_t^l - j)}. \quad (30)$$

Here $|x| = \langle x, x \rangle^{1/2}$ where $\langle \cdot, \cdot \rangle$ is the inner product on H , $Q = Q_\infty$, $E = E_\infty$, $m_k = \ln |k|^{2/3}$ is the mean of the log-Poisson processes (15) and $p_{m_k}^j = \frac{(m_k)^j e^{-m_k}}{j!}$ is the probability of $N_\infty^k = N_k$ having exactly j jumps, $\delta_{k,l}$ is the Kronecker delta function.

Suppose that the operator Q is trace-class, $E(Q^{1/2} H) \subset Q^{1/2}(H)$ and that $e^{Kt} P_t M_t(H) \subset Q_t^{1/2}(H)$, $t > 0$, where $H = H_c$, then, with u given, the invariant measure μ is unique, ergodic and strongly mixing. We know that the above invariant measure is unique for the strong swirl [9] and strong rotation [2, 3] but it depends on u , and its uniqueness for general turbulent flows depends on the uniqueness of u .

The proof of Theorem 6.1 uses the above machinery and is analogous to the proof of Theorem 8.20 in [25], see [11] for details.

7 The Invariant Measure for the Velocity Differences

We will now find the Kolmogorov-Hopf functional differential equation for the invariant measure of the Navier-Stokes equation for the velocity differences

$$z = u - w = u(x, t) - u(y, t).$$

The previous measure was the measure determining the 1-point statistics but the measure for the velocity difference will determine the 2-point statistics. We are simplifying this a little using isotropy; namely, in general the velocity difference is a tensor. The linearized Navier-Stokes operator is now

$$\bar{K} = \nu \Delta - u \cdot \nabla + \nabla \Delta^{-1} \text{tr}((\nabla u + \nabla w) \nabla),$$

but otherwise the derivation is similar to the derivation of the 1-point measure above. The formula for the 2-point measure is the same (30), but now the operator K depends on the two points x and y and therefore the variance (27) and the drift (28), will also depend on these two points. In fact the measure depends on the lag variable $x - y$. A better way of capturing the dependence on the lag variable is to write the difference of the inertial terms as

$$-u \cdot \nabla w + w \cdot \nabla u = -u \cdot \nabla(u - w) - (u - w) \cdot \nabla u + (u - w) \cdot \nabla(u - w)$$

This produces the new operator

$$\tilde{K} = \nu \Delta - u \cdot \nabla + z \cdot \nabla - \nabla u + \nabla \Delta^{-1} \text{tr}((\nabla u + \nabla w) \nabla) = K - u \cdot \nabla + z \cdot \nabla - \nabla u$$

with the understanding that now K is a function of $(\frac{u+w}{2})$ through the pressure term. The last three terms are removed by a combination of Feynman-Kac and the Cameron-Martin formula (Girsanov's theorem) and we get the martingale

$$M_t = \exp\left\{ \int_0^t u(x - B_{-s} + y, s) \cdot dB_{-s} + \int_0^t z(B_s) \cdot dB_s - \frac{1}{2} \int_0^t |u(x - B_{-s} + y, s) + z(B_s, s)|^2 ds \right\}$$

after a time reversal of the auxiliary Brownian motion B_t see [20]. The computation of the measure follows the procedure for the computation of the measure for the 1-point statistics. The difference of the two equations (for u and w) is written

as an integral equation

$$(31) \quad z = e^{K(t)} e^{-\int_0^t \nabla u \, ds} e^{\int_0^t dq} M_t z^0 + \sum_{k \neq 0} c_k^{1/2} \int_0^t e^{K(t-s)} e^{-\int_s^t \nabla u \, dr} e^{\int_s^t dq} M_{t-s} db_s^k e_k(x) + \sum_{k \neq 0} d_k \int_0^t e^{K(t-s)} e^{-\int_s^t \nabla u \, dr} e^{\int_s^t dq} M_{t-s} |k|^{1/3} ds e_k(x)$$

by the Feynman-Kac formula and Girsanov's theorem where K is the operator

$$K = \nu \Delta + \nabla \Delta^{-1} \text{tr}((\nabla u + \nabla w) \nabla), \quad (32)$$

and

$$P_t = e^{\int_0^t \nabla u \, ds} e^{\int_0^t dq} M_t = e^{\int_0^t \nabla u \, ds} \prod_k |k|^{2/3} (2/3)^{N_t^k} M_t.$$

The variance is

$$Q_t = \int_0^t e^{K(s)} P_s C P_s^* e^{K^*(s)} ds \quad (33)$$

and the drift is

$$E_t = \int_0^t e^{K(s)} P_s \bar{D} ds, \quad (34)$$

where $\bar{D} = (|k|^{1/3} D_k)$. The Kolmogorov-Hopf equation for the Ito processes (31) now becomes

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \text{tr}[Q_t \Delta \phi] + \text{tr}[E_t \nabla \phi], \quad (35)$$

with $\phi(z)$ is a bounded function of z . It is also the Kolmogorov backward equation of the Ito process (31). Then the solution of the Kolmogorov-Hopf equation (35) can be written in the form

$$(36) \quad R_t \phi(z) = \int_H \phi(y) \mathcal{N}_{(e^{K(t)} P_t z + E_t I, Q_t)} * \mathcal{N}_{(0, 2\nu)} * \mathbb{P}_{P_t}(dy) = \int_H \phi(e^{K(t)} P_t z + E_t I + y) \mathcal{N}_{(0, Q_t)} * \mathcal{N}_{(0, 2\nu)} * \mathbb{P}_{P_t}(dy)$$

where \mathbb{P}_{P_t} is the discrete Poisson law (25) of the log-Poisson process P_t . Here $z = z_0$, \mathcal{N}_{m, Q_t} denotes the infinite-dimensional normal distribution on H with mean m and variance Q_t , $I = \sum e_k, E_t I \in H$ and $\mathcal{N}_{(0, 2\nu)}$ the law of the three-dimensional Brownian motion in the Martingale M_t . If Q_t is of trace-class $Q_t \in L^+(H)$, then R_t is Markovian.

Theorem 7.1 *The invariant measure for the velocity differences (two-point statistics) of the Navier-Stokes equation on $H_c = H^{3/2^+}(\mathbb{T}^3)$ has the form*

$$(37) \quad \begin{aligned} \mu(dx, dy) &= e^{\langle Q^{-1/2}EI, Q^{-1/2}x \rangle - \frac{1}{2}|Q^{-1/2}EI|^2} \mathcal{N}_{(0, Q)}(dx) * \mathcal{N}_{(0, 2\nu)}(dy) \\ &\times \sum_k \delta_{k,l} \prod_{j \neq l}^m \delta_{N_l^j} \sum_{j=0}^{\infty} p_{m_l}^j \delta_{(N_l-j)} \end{aligned}$$

where $Q = Q_\infty$, $E = E_\infty$. Here $|x| = \langle x, x \rangle^{1/2}$ where $\langle \cdot, \cdot \rangle$ is the inner product on H , $m_k = \ln |k|^{2/3}$ is the mean of the log-Poisson processes (15) and $p_{m_k}^j = \frac{(m_k)^j e^{-m_k}}{j!}$ is the probability of $N_\infty^k = N_k$ having exactly j jumps, $\delta_{k,l}$ is the Kronecker delta function.

Suppose that the operator Q is trace-class, $E(Q^{1/2}H) \subset Q^{1/2}(H)$ and that

$$e^{K(t)} P_t(H) \subset Q_t^{1/2}(H), \quad t > 0,$$

where $H = L^2(\mathbb{T}^3)$, then, given u , the invariant measure μ is unique, ergodic and strongly mixing. The proof of Theorem 7.1 is similar to the proof of Theorem 6.1, see [11] for details.

It is easy to check that the moments of the invariant measure for the two-point statistics give the estimates for the structure functions above. The variable in the latter three-dimensional Gaussian $\mathcal{N}_{(0, 2\nu)}(dy)$ in the invariant measure is the lag variable.

The same comments as above apply to the measure (37) as the invariant measure for the one-point statistics (30). It is unique for the strong swirl [9] and strong rotation [2, 3] but its uniqueness for general turbulent flows depends on the uniqueness of u .

8 The Differential Equation for the PDF

We must compute the probability distribution function (PDF) for the velocity differences, from the invariant measure (30), in order to compare with PDFs constructed from simulations and experiments. The simplest way of doing this is to derive the differential equation for the density function from the Kolmogorov-Hopf equation (35). We start with the Kolmogorov-Hopf equation

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \text{tr}[Q_t \Delta \phi] + \text{tr}[E_t \nabla \phi] \quad (38)$$

where Q_t and E_t are respectively the variance (27) and drift (28), computed with the operator K in (32). This can be done by redefining the underlying infinite-dimensional Ito process appropriately, see [11]. We have to take the trace of the functional variables to get the equation for the PDF. The resulting equation is

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \Delta \phi + \frac{1}{\sqrt{2t}} c \cdot \nabla \phi \quad (39)$$

where $\hat{c}(|k|) = (Q_t^{-1/2} E_t)_k$ are the Fourier coefficients of c , after we scale by the variance Q_t . Now scaling the equation by $-2t$ and sending $t \rightarrow \infty$ gives the equation

$$\frac{1}{2} \Delta \phi + c \cdot \nabla \phi = \phi, \quad (40)$$

with a trivial rescaling of time. This is the (stationary) equation for the distribution function. Now the PDF is for the absolute value of the velocity differences $w = |u(x,t) - u(y,t)|$, by the Equation (45) below, so the angle derivatives of w do not appear, and $\hat{c} = (Q^{-1/2} E)_k \sim \bar{c} |k|^{1/3} / |k|^{1/3} = \bar{c}$ for k large, if we ignore the intermittency corrections τ_p in (19). We will discuss below what this assumption means and how the intermittency corrections are restored. Thus, taking the trace of the spatial (lag) variables also, we get that $c = \frac{\bar{c}}{w}$. In polar coordinates $\Delta \phi = \phi_{ww} + \frac{2}{w} \phi_w$, in three dimensions. Thus (40) becomes

$$\frac{1}{2} \phi_{ww} + \frac{1 + \bar{c}}{w} \phi_w = \phi. \quad (41)$$

This is the stationary equation satisfied by the PDF.

The above computation is clarified by the following example. Consider the equation

$$\phi_t = \phi_{xx} + \frac{c}{\sqrt{2t}} \phi_x$$

where $\phi = \frac{e^{-(x-a)^2/b}}{\sqrt{\pi b}}$ is a Gaussian. It is easy to check that this equation holds if $a_t = -\frac{c}{\sqrt{2t}}$ and $b_t = 4$, so $a = -c\sqrt{2t}$ and $b = 4t$. Thus invariant measure is produced by scaling out t ,

$$\phi(y) dy = \frac{e^{-\frac{(y+c)^2}{2}}}{\sqrt{2\pi}} dy = \frac{e^{-\frac{(y-\frac{a}{\sqrt{b/2}})^2}{2}}}{\sqrt{2\pi}} dy = \phi(x,t) dx.$$

where $y = x/\sqrt{2t}$. This invariant measure satisfies the stationary equation (40).

9 The PDF for the Turbulent Velocity Differences

It is now possible to compute the probability density function (PDF) for the velocity differences in turbulence. The form of the equation (41) suggests that we should look for a solution of the form $f = x^a K_\lambda$ where K_λ is a modified Bessel's function of the second kind, satisfying the equation,

$$K_{xx} + \frac{1}{x}K_x - \left(1 + \frac{\lambda^2}{x^2}\right)K = 0.$$

A substitution of this ansatz into the equation (41) gives $a = -\bar{c}$ and $\lambda = \bar{c}$. The solution is the generalized hyperbolic distribution, see Barndorff-Nielsen [4]. It has an algebraic cusp at the origin and exponential tails and is constructed by multiplying the modified Bessel's function of the second kind K_λ , by $x^{-\lambda}$. For the zeroth moment we get a distinguished solution for $\lambda = \bar{c} = 1 + \frac{1}{2}$, namely the Normal Inverse Gaussian (NIG) distribution that was also investigated by Barndorff-Nielsen [5] and used by Barndorff-Nielsen, Blæsild and Schmiegel to model PDF of velocity increments for several data sets in [6].

The PDF of the NIG is

$$\frac{\alpha \delta K_1 \left(\alpha \sqrt{\delta^2 + (x - \mu)^2} \right)}{\pi \sqrt{\delta^2 + (x - \mu)^2}} e^{\delta \gamma + \beta(x - \mu)}. \quad (42)$$

The parameters are:

$$\begin{aligned} \alpha & \text{ heavyness of the tail, } \beta & \text{ asymmetry, } \delta & \text{ scaling,} \\ \mu & \text{ centering, } \gamma = \sqrt{\alpha^2 - \beta^2}. \end{aligned}$$

The NIG distribution has very nice properties that are summarized in [6]. In particular its characteristic function and all of its moments are easily computed. The cumulant generating function $\mu z + \delta(\gamma - \sqrt{\alpha^2 - (\beta + z)^2})$ is particularly simple for the NIG and this make the moments easy to compute, see [6]. The first few moments and the characteristic function of the NIG distribution are:

	Mean	$\mu + \delta\beta/\gamma$
	Variance	$\delta\alpha^2/\gamma^3$
(43)	Skewness	$3\beta/(\alpha\sqrt{\delta\gamma})$
	Excess kurtosis or flatness	$3(1 + 4\beta^2/\alpha^2)/(\delta\gamma)$
	Characteristic Function	$e^{i\mu z + \delta(\gamma - \sqrt{\alpha^2 - (\beta + iz)^2})}$.

Thus we see that the probability density function of the velocity increment is a normalized inverse Gaussian (NIG) distribution that is a generalized hyperbolic distributions with index 1. Using the invariances of the NIG it is given by the four-parameter formula

$$f_j(x, \alpha, \beta, \delta, \mu) = \frac{\alpha\delta e^{\delta\gamma} K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\pi\sqrt{\delta^2 + (x-\mu)^2}} e^{\beta(x-\mu)}, \quad j = 1, 2, \quad (44)$$

where, α measures how heavy the exponential part of the tail of the distribution is, β measures how skew the distribution is, δ is a scaling parameter and μ determines the location (center) of the distribution, $\gamma = \sqrt{\alpha^2 - \beta^2}$. K_1 is the modified Bessel's function of the second kind with index 1. Now the 1st moment of the velocity differences is

$$(45) \quad \begin{aligned} E(\delta_j u) &= E([u(x+s, \cdot) - u(x, \cdot)] \cdot r) = E(|u(x+s, \cdot) - u(x, \cdot)| |r| \cos(\theta)) \\ &= \int_{-\infty}^{\infty} (x f_j)(x, \alpha, \beta, \delta, \mu) dx, \end{aligned}$$

where $j = 1$, if $r = \hat{s}$ is the longitudinal direction (that is the direction along the lag vector s), and $j = 2$, if $r = \hat{t}$ where $t \perp s$ is a transversal direction, \hat{r} and \hat{t} being unit vectors. θ is the angle between the vectors $[u(x+s, \cdot) - u(x, \cdot)]$ and r , and the absolute value of the former is the reason why the angle derivatives wash out in (41). The PDF is symmetric in the transversal direction, then $\beta = \mu = 0$. In that case there are only two independent adjustable parameters, α is the exponential decay at $x = \pm\infty$ and δ is the "peakedness" at the origin. In the nonsymmetric case, there are two more independent adjustable parameters, the skewness parameter β and the centering parameter μ .

The PDF for the velocity increments has the asymptotics,

$$f_j \sim \frac{\delta e^{\delta\gamma}}{\pi} \frac{e^{\beta(x-\mu)}}{(\delta^2 + (x-\mu)^2)}$$

for $(x-\mu)$ small. This is the algebraic (rational) cusp at the origin. The exponential tails are,

$$f_j \sim \frac{\sqrt{2}\delta\alpha e^{\delta\gamma - \beta\mu}}{\pi^{3/2}} \frac{e^{-\alpha|x| + \beta x}}{|x|^{3/2}}$$

for $|x|$ large.

The exponential tails of the PDF are caused by occasional sharp velocity gradients (rounded-off shocks), whereas the cusp at the origin is caused by the random

and gentle fluid motion in the center of the ramps leading up to the sharp velocity gradients, see Kraichnan [18].

For large values of the lag variable, the NIG distribution must also approximate a Gaussian. It turns out to do just that. Letting $\alpha, \delta \rightarrow \infty$, in the formulas for $f_j(x)$ above, in such a way that $\delta/\alpha \rightarrow \sigma$, we get that

$$f_j \rightarrow \frac{e^{-\frac{(x-\mu)^2}{2\sigma}}}{\sqrt{2\pi\sigma}} e^{\beta(x-\mu)}.$$

The parameter α, β, δ and μ depend on the lag variable in the two-point PDF as discussed above, but they also depend on the intermittency correction τ_p in (19). It turns out that the distribution functions for all of the moments can be expressed by the NIG distribution function, see [10]. However, since the intermittency corrections are different for the different moments, the NIG distributions for the different moments have different parameters. Thus the moments of the velocity differences are not the moments of the same NIG distributions, because of the intermittency correction. In fact, the invariant measure (37) has both a continuous and a discrete part and because of this each moment comes with its own PDF. All of these PDF are solutions of the stationary equation (41) and they can be expressed in terms of NIG distributions. However, their parameters α, β, δ and μ all depend on the particular moment for which one is computing the PDF. Thus these parameters are different for the different moments. This is the point of view taken in [10] and [11] and it gives a good fit with the PDFs of all the moments computed from simulations and experiments.

10 Intermittency

Making the parameter α, β, δ and μ depend on the intermittency corrections τ_p for the different moments gives a satisfactory fit with experimental data and simulations, but it is not fully satisfying theoretically. In particular it does not explain how these parameters depend on τ_p . We now present a derivation to show how the PDFs for the different moments depend on the intermittency corrections τ_p for the p th moment. The starting point is to write the Kolmogorov-Hopf equation

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \text{tr}[Q_t^p \Delta \phi] + \text{tr}[E_t^p \nabla \phi] \quad (46)$$

where Q_t^p and E_t^p are respectively the variance (27) and drift (28), computed with the operator K in (32), but for the p th moment $\phi = z^p$, where $z = \delta u$ is the velocity

difference. We have to take the trace of the functional variables to get the equation for the PDF. The resulting equation is

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \Delta \phi + \frac{1}{\sqrt{2t}} c_p \cdot \nabla \phi \quad (47)$$

where $\hat{c}_p(|k|) = ((Q_t^p)^{-1/2} E_t^p)_k$ are the Fourier coefficients of c_p , after we scale by the variance Q_t^p . Now scaling the equation by $-2t$ and sending $t \rightarrow \infty$ gives the equation

$$\frac{1}{2} \Delta \phi + c_p \cdot \nabla \phi = \phi, \quad (48)$$

with a trivial rescaling of time. This is the (stationary) equation for the distribution function. Now the PDF is for the absolute value of the velocity differences $w = |u(x, t) - u(y, t)|$, by the Equation (45) above, so the angle derivatives of w do not appear, and $\hat{c}_p = (Q_p^{-1/2} E_p)_k \sim \bar{c}_p |k|^{p/3+\tau_p} / |k|^{p/3+\tau_{2p}/2} = \bar{c}_p |k|^{\tau_p - \tau_{2p}/2}$ for k large, where the τ_p and τ_{2p} are intermittency corrections for z^p and z^{2p} respectively. Thus, taking the trace of the spatial (lag) variables also, we get that $c_p = \frac{\bar{c}_p}{w^{1-d}}$, where $d = \tau_p - \tau_{2p}/2$. In polar coordinates $\Delta \phi = \phi_{ww} + \frac{2}{w} \phi_w$, in three dimensions. Thus (48) becomes

$$\frac{1}{2} \phi_{ww} + \frac{1 + \bar{c}_p w^d}{w} \phi_w = \phi. \quad (49)$$

This is the stationary equation satisfied by the PDF, corresponding to the p th (and $2p$ th) moment.

Now the PDF is computed using the same method as in Section 9. We scale time by 2 to get the equivalent equation

$$\phi_{ww} + 2 \frac{1 + \bar{c}_p w^d}{w} \phi_w = \phi. \quad (50)$$

Then we look for a solution of (49) of the form $g(x) = x^a f(x)$, where $x = |w|$, and get the equation

$$f'' + \frac{1 + 2\bar{c}_p(x^d - 1)}{x} f' - \left(1 + \frac{(2\bar{c}_p^2 + \bar{c}_p)x^d + \frac{1}{4} - \bar{c}_p^2}{x^2}\right) f = 0, \quad (51)$$

by choosing $a = -\bar{c}_p - \frac{1}{2}$. When $d = 0$, (51) is a modified Bessel's equation of the second kind and we get the Generalized Hyperbolic Distribution, see [11], with index $\lambda = \bar{c} + \frac{1}{2}$ and a PDF $g = \frac{K_\lambda}{x^\lambda}$. Thus without intermittency the PDF of the

velocity differences is a Generalized Hyperbolic distribution with index $\bar{c} + \frac{1}{2}$ and f satisfies the equation,

$$f'' + \frac{1}{x}f' - \left(1 + \frac{(\bar{c} + \frac{1}{2})^2}{x^2}\right)f = 0. \quad (52)$$

In particular, when $\bar{c} = \frac{1}{2}$, we get the NIG distribution in Section 9.

In general the intermittency index d is

$$d_p = \tau_p - \frac{1}{2}\tau_{2p} = 1 - 2\left(\frac{2}{3}\right)^{p/3} + \left(\frac{2}{3}\right)^{2p/3}$$

Thus $d_0 = 0$ and $d_\infty = 1$. In the latter case (51) becomes

$$f'' + \frac{1 + 2\bar{c}_p(x-1)}{x}f' - \left(1 + \frac{(2\bar{c}_p^2 + \bar{c}_p)x + \frac{1}{4} - \bar{c}_p^2}{x^2}\right)f = 0. \quad (53)$$

The equation (51) can be solved numerically or by a Laurent series. They do not converge to a modified Bessel's function of the second kind K_λ . The NIG was an approximation, without intermittency, to the true PDFs given by $f_p(x)/x$. We get a sequence of PDF one for the p th and the $2p$ th moment, for each p , and not even $p = 1$ (and $2p = 2$) give a NIG or a Generalized Hypberbolic Distribution. When $\bar{c} = \frac{1}{2}$, the equation (53) becomes

$$f'' + f' - \left(1 + \frac{1}{x}\right)f = 0. \quad (54)$$

The solution of (54) has the form

$$f(x) = c_1 x e^{\left\{-(1+\sqrt{5})\frac{x}{2}\right\}} {}_1F_1[1 + 1/\sqrt{5}, 2, \sqrt{5}x] + c_2 x e^{\left\{-(1+\sqrt{5})\frac{x}{2}\right\}} U[1 + 1/\sqrt{5}, 2, \sqrt{5}x],$$

where ${}_1F_1$ is the hypergeometric function and U the confluent hypergeometric function. Decay at $\pm\infty$ requires $c_1 = 0$, so the resulting (unnormalized) PDF is

$$g(x) = \frac{f(x)}{x} = e^{\left\{-(1+\sqrt{5})\frac{x}{2}\right\}} U[1 + 1/\sqrt{5}, 2, \sqrt{5}x],$$

the PDF of the velocity differences with infinite $p = \infty$ intermittency.

The PDFs computed by use of (51) are probably not the easiest way to fit to simulations and experimental data. The NIG with varying parameters, in Section 9, is easier. The best way, theoretically and practically, may be to take a convolution of the Generalized Hyperbolic Distributions with index $\bar{c} + \frac{1}{2}$ and a discrete distribution as was done in the invariant measures (30) and (37). This works but the computations are involved and will be published elsewhere.

11 Conclusion

We have seen that the Navier-Stokes equation, for all but the largest scales in turbulent flow, can be expressed as a stochastic Navier-Stokes equation (8). The stochastic forcing results from instabilities of the flow, that magnify small ambient noise and saturate its growth into large stochastic forcing. This has been modeled before by a Reynolds decomposition and by a coarse graining of the flow. The stochastic force is generic and is determined by the general principles of probability with a minimum of physical inputs. It consists of two components additive noise and multiplicative noise and the additive component is determined by the Central Limit Theorem and the Large Deviation Principle. The physical input is that these two term must produce similar scalings because they are caused by the same dissipative processes. This determines the rate in the large deviation principle. The multiplicative noise multiplies the fluid velocity and models jumps (vorticity concentrations) in the velocity gradient. It is expressed by a generic Poisson process where only the rate needs to be given. This rate is determined by the spectral analysis of the (linearized) Navier-Stokes operator and the requirement, following [26], that the dimension of the most singular vorticity structure (filaments) is one. Thus the stochastic forcing is generic and determined with two mild physical inputs.

The stochastic Navier-Stokes equation can be expressed as an integral equation (9) and the log-Poissonian processes found by She and Leveque and explored by She and Waymire and Dubrulle are produced from the multiplicative noise by the Feynman-Kac formula. This give a satisfying mathematical derivation of the intermittency phenomena that had earlier been derived from empirical considerations. Moreover, the integral equations show how the Navier-Stokes evolution and the log-Poissonian intermittency processes act on the dissipation processes, to product the intermittency in the dissipation. This is a mathematical derivation of the experimental observation that intermittent dissipation processes accompany intermittent velocity variations. Using the integral equation we get an upper estimate on all the structure functions of the velocity differences in turbulence. The evidence from simulations and experiments is that this upper bound is reached in turbulent flow. Why the inertial cascade achieves this maximal efficiency in the energy transfer remains to be explained.

We then built on Hopf's [14] ideas to compute the invariant measure of turbulent flow. This measure can be computed because it solves a linear functional differential equation, see [25]. It turns out to be an infinite-dimensional Gaussian multiplied by a (discrete) Poisson distributions. This Poisson distribution corre-

sponds to the intermittency and the log-Poisson processes. Then by taking the trace of the invariant measure we get the PDF of the velocity differences. We derive the functional differential equation (PDE) for the PDF. This PDE (40) can also be solved and the solutions turns out to be the Generalized Hyperbolic Distributions, see [11], a special case of which is the normalized inverse Gaussian (NIG) distributions of Barndorff-Nilsen [5]. Their parameters are easily computed and in [10] and [11] this is done for both simulations and experiments. However, in the NIG distribution the parameters vary with the intermittency corrections. We also compute PDFs with the intermittency corrections for the p th and $2p$ th moments included. These are theoretically more satisfying but harder to use to fit simulations and experimental data. In the limit of $p \rightarrow \infty$ we get a PDF with intermittency given by the confluent hypergeometric function multiplied by an exponential.

It is interesting to notice that although the solution of the Navier-Stokes equation may not be unique or smooth the invariant measure of the velocity differences (37) may still be well defined by Leray's [19] existence theory. Moreover, different velocities produce equivalent measures so the statistical observables of turbulence can be unique although the turbulent velocity may not be. Indeed although the measure depends on the velocity u , because of ergodicity of the measure, the measures for different velocities are all the same.

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