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Functions of nearly maximal Gowers-Host-Kra norms on Euclidean spaces

by

Anh Martina Neuman

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Michael Christ, Chair

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Anh Martina Neuman

Abstract

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Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Michael Christ, Chair

Let $k \geq 2, n \geq 1$ be integers. Let $f : \mathbb{R}^n \rightarrow \mathbb{C}$. The k th Gowers-Host-Kra norm of f is defined recursively by

$$\|f\|_{U^k}^{2^k} = \int_{\mathbb{R}^n} \|T^h f \cdot \bar{f}\|_{U^{k-1}}^{2^{k-1}} dh$$

with $T^h f(x) = f(x+h)$ and $\|f\|_{U^1} = |\int_{\mathbb{R}^n} f(x) dx|$. These norms were introduced by Gowers [11] in his work on Szemerédi's theorem, and by Host-Kra [13] in ergodic setting. These norms are also discussed extensively in [17]. It's shown by Eisner and Tao in [10] that for every $k \geq 2$ there exist $A(k, n) < \infty$ and $p_k = 2^k/(k+1)$ such that $\|f\|_{U^k} \leq A(k, n)\|f\|_{p_k}$, with $p_k = 2^k/(k+1)$ for all $f \in L^{p_k}(\mathbb{R}^n)$. The optimal constant $A(k, n)$ and the extremizers for this inequality are known [10]. In this dissertation, it is shown that if the ratio $\|f\|_{U^k}/\|f\|_{p_k}$ is nearly maximal, then f is close in L^{p_k} norm to an extremizer.

I dedicate this dissertation to my mother, Nguyen thi Thu Trang.

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Chapter 1

Introduction

Let $k \geq 2, n \geq 1$ be integers and $f : \mathbb{R}^n \rightarrow \mathbb{C}$ be a measurable function. The k th Gowers-Host-Kra norm of f is defined recursively by:

$$\|f\|_{U^k}^{2^k} = \int_{\mathbb{R}^n} \|T^h f \cdot \bar{f}\|_{U^{k-1}}^{2^{k-1}} dh. \quad (1.1)$$

Here, $T^h f(x) = f(x + h)$, and $\|f\|_{U^1} = |\int_{\mathbb{R}^n} f(x) dx|$. These norms were introduced by Gowers [11] in his work on Szemerédi's theorem, and by Host-Kra [13] in ergodic setting.

They are also discussed in [17]. There is an alternative expression to (1.1) [17]:

$$\|f\|_{U^k}^{2^k} = \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} \mathcal{C}^{\omega_\alpha} f(x + \alpha \cdot \vec{h}) dx d\vec{h}. \quad (1.2)$$

Here $\alpha = (\alpha_1, \dots, \alpha_k) \in \{0,1\}^k$, $\vec{h} = (h_1, \dots, h_k) \in \mathbb{R}^{kn}$ and $\alpha \cdot \vec{h} = \sum_{i=1}^k \alpha_i h_i \in \mathbb{R}^n$.

$\mathcal{C}f := \bar{f}$ is the conjugation operator and ω_α is the parity of the number of ones in $\alpha \in \{0,1\}^k$.

For instance, if $k = 2$, then (1.2) becomes

$$\|f\|_{U^2}^4 = \int_{\mathbb{R}^{3n}} f(x) \overline{f(x+h_1)} \overline{f(x+h_2)} f(x+h_1+h_2) dx dh_1 dh_2.$$

It follows that, after some changes in variables,

$$\begin{aligned} \|f\|_{U^2}^4 &= \int_{\mathbb{R}^{3n}} f(x) \overline{f(y)} \overline{f(z)} f(x+y-z) dx dy dz = \int_{\mathbb{R}^{2n}} f(x) \overline{f(y)} (f * \bar{f})(x+y) dx dy \\ &= \int_{\mathbb{R}^{2n}} f(x) \overline{f(y-x)} (f * \bar{f})(y) dx dy = \int_{\mathbb{R}^n} (f * \bar{f})^2(y) dy = \int_{\mathbb{R}^n} |\hat{f}|^4(x) dx \end{aligned} \quad (1.3)$$

or $\|f\|_{U^2} = \|\hat{f}\|_4$. Hence the second Gowers-Host-Kra norm is an actual norm. The same has been shown for higher order Gowers-Host-Kra norms [17]. It's also shown in [10] that for every $k \geq 2, n \geq 1$ there exist $A(k, n) = A(k, 1)^n$ with $A(k, 1) = 2^{k/2^k} / (k+1)^{(k+1)/2^{k+1}}$ and $p_k = 2^k / (k+1)$ such that, for every $f \in L^{p_k}(\mathbb{R}^n)$,

$$\|f\|_{U^k} \leq A(k, n) \|f\|_{p_k}. \quad (1.4)$$

We call f an extremizer of (1.4) if equality is attained with this choice of f . See **Chapter 2** below for terminology used in this exposition. It's shown in [10] that this constant $A(k, n)$ is best possible and that equality is attained iff $f(x) = C \exp(-(x-c) \cdot M(x-c)) \exp(2\pi i P(x))$, with $C \in \mathbb{C}$, $c \in \mathbb{R}^n$, M is a positive-definite $n \times n$ matrix and $P : \mathbb{R}^n \rightarrow \mathbb{R}$ is a polynomial of degree at most $k-1$. In this exposition, we seek to answer the question, of what happens if the ratio $\|f\|_{U^k} / \|f\|_{p_k}$ is nearly maximized on $\mathbb{R}^n$. This question was not addressed in [10]. We set out to show that f must then be close in L^{p_k} norm to an extremizer. More precisely:

Theorem 1.1. Let $k \geq 2, n \geq 1$ be integers. For every $\epsilon > 0$ there exists $\delta > 0$ with the following property. Suppose $f \in L^{p_k}(\mathbb{R}^n)$ and

$$\|f\|_{U^k} \geq (1 - \delta)A(k, n)\|f\|_{p_k}. \quad (1.5)$$

Then there exists an extremizer $\mathcal{F}$ of (1.4) on $\mathbb{R}^n$ such that $\|\mathcal{F} - f\|_{p_k} \leq \epsilon\|f\|_{p_k}$.

Theorem 1.1 can also be equivalently stated in terms of sequences:

Theorem 1.2. Let $k \geq 2, n \geq 1$ be integers. Suppose $\{f_j\}_j$ is a sequence of functions on $\mathbb{R}^n$ such that $\|f_j\|_{p_k} = 1$ and

$$\|f_j\|_{U^k} \geq (1 - \delta_j)A(k, n)\|f_j\|_{p_k} = (1 - \delta_j)A(k, n)$$

with $\delta_j \rightarrow 0$ as $j \rightarrow \infty$. Then there exist an extremizer $\mathcal{F}$ of (1.4) on $\mathbb{R}^n$, $\lambda_j \in \mathbb{R}_{>0}, c_j \in \mathbb{R}^n$, real-valued polynomial of degree at most $k-1$, P_j , and positive definite $M_j \in M^{n \times n}(\mathbb{R})$ such that $\|\mathcal{F} - \lambda_j^{n/p_k} f_j e^{iP_j}(\lambda_j(M_j(\cdot - c_j)))\|_{p_k} \leq o_{\delta_j}(1)\|f_j\|_{p_k} = o_{\delta_j}(1)$.

Theorem 1.1 and **Theorem 1.2** are equivalent to each other, and hence we will not distinguish them in this discussion and simply refer to both as “**Theorem 1.1**”. Via the relation between the second Gowers-Host-Kra norm of f and the L^4 norm of $\hat{f}$ shown in (1.3), this near extremization question has been resolved in [7], as $p_2 = 4/3$, and hence $\|f\|_{U^2} \geq (1 - \delta)A(2, n)\|f\|_{4/3}$ becomes $\|\hat{f}\|_4 \geq (1 - \delta)A(2, n)\|f\|_{4/3}$.

The alternative expression (1.2) of the Gowers-Host-Kra norm leads to an inner product inequality. The Gowers inner product [17] of degree 2^k , $\mathcal{T}_k$, is defined as follows. For each $\alpha \in \{0, 1\}^k$, let $f_\alpha : \mathbb{R}^n \rightarrow \mathbb{C}$ be a measurable function. Then

$$\mathcal{T}_k(f_\alpha : \alpha \in \{0, 1\}^k) = \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0, 1\}^k} \mathcal{C}^{\omega_\alpha} f_\alpha(x + \alpha \cdot \vec{h}) dx d\vec{h}.$$

The mentioned inner product inequality is the following

$$|\mathcal{T}_k(f_\alpha : \alpha \in \{0, 1\}^k)| \leq A(k, n)^{2^k} \prod_{\alpha \in \{0, 1\}^k} \|f_\alpha\|_{p_k}. \quad (1.6)$$

This inequality can be easily deduced from the Gowers-Cauchy-Schwarz inequality [17]. See **Chapter 2** below. Observe that (1.4) is a consequence of (1.6) if $f_\alpha = f$ for all $\alpha \in \{0, 1\}^k$.

A more general version of (1.5) is then

$$|\mathcal{T}_k(f_\alpha : \alpha \in \{0, 1\}^k)| \geq (1 - \delta) A(k, n)^{2^k} \prod_{\alpha \in \{0, 1\}^k} \|f_\alpha\|_{p_k}. \quad (1.7)$$

We will not be investigating the near extremization question in this full generality; however, the arguments presented in the second part of this exposition, which spans from **Chapter 4** to **Chapter 10**, can be extended for the said case. The conclusion of **Theorem 1.1** is qualitative; the dependence of ϵ on δ is not made explicit in this exposition. This non-quantitative dependence $\epsilon = \epsilon(\delta)$ is a result of implicitness arisen in **Proposition 6.1** and **Section 8.4**; other steps could be made explicit without this implicitness. There will be two parts in this exposition. The first part, **Chapter 3**, is a very short proof of **Theorem 1.1** in the case of nonnegative functions, which utilizes a stability result of Young's convolution inequality in [8]. The second part, spanning from **Chapter 4** to **Chapter 10**, concludes

the theorem for general complex-valued functions and does not rely on the said stability result. The framework used in the second part parallels to that laid out in [8]. Another such inequality that is closely related to the Gowers product inequality is the following:

$$|\mathcal{T}_k(f_\alpha : \alpha \in \{0, 1\}^k)| \leq C(k, n, \vec{p}) \prod_{\alpha \in \{0, 1\}^k} \|f_\alpha\|_{p_\alpha}$$

with $\vec{p} = (p_\alpha : \alpha \in \{0, 1\}^k)$. The p_α satisfy the necessary scaling condition $\sum_{\alpha \in \{0, 1\}^k} p_\alpha^{-1} = k + 1$. In **Chapter 4** below, it's shown that the said inequality holds (in other words, $C(k, n, \vec{p}) < \infty$) if $\vec{p}$ lies in a small neighborhood of the point $\vec{P} = (2^k/(k+1), \dots, 2^k/(k+1))$ on the hyperplane $\sum_{\alpha \in \{0, 1\}^k} p_\alpha^{-1} = k + 1$. No general sufficient algebraic conditions on p_α are known as of now; however, see [1] for a geometric description. This discovery at least confirms that the characteristic polytope, as discussed in [1], of the Gowers inner product structure - which is a Brascamp-Lieb structure - is not a degenerate one-point set. The Gowers-Host-Kra norm inequality is an important step towards the study of the near extremization problems of these more general cases.

Chapter 2

Basics

The notation $\mathbb{N}$ denotes the set of nonnegative integers. The notation x, y or h indicates an element in a Euclidean space while the notation $\vec{x}, \vec{y}$ or $\vec{h}$ indicates a tuple of such elements. In particular, the notation $(x, \vec{h})$ always means $(x, \vec{h}) \in \mathbb{R}^{(k+1)n}$, with $x \in \mathbb{R}^n$, $\vec{h} = (h_1, \dots, h_k)$ and $h_i \in \mathbb{R}^n$, $i \in \{1, \dots, k\}$. The notation $crd(S)$ denotes the cardinality of a finite set S . The notation $\mathcal{L}(S)$ denotes the n -dimensional Lebesgue measure of a Euclidean set $S \in \mathbb{R}^n$. If S is a ball with center c and $r > 0$, then rS denotes the dilated set $\{r(x - c) \mid x \in S\}$. The notation $spt(f)$ denotes the essential support of a measurable function f on a Euclidean space. If f is continuous then this support is taken to be the closure of the set of points at which f does not vanish. The symbol $\cdot$ denotes several meanings of multiplication, which will be clear from context. The notation $\langle a, b \rangle_{\mathbb{R}^n}$ denotes the usual real inner product between two Euclidean elements a, b .

Let $a, b > 0$. The symbol $\asymp$ in $a \asymp b$ means the following: There exist $C > c > 0$ such that $ca \leq b \leq Ca$. In our context, c, C will be integral powers of 2, say $2^j a \leq b \leq 2^l a$; in some cases, j and l are chosen so that $2^j a$ and $2^l a$ are among the closest numbers of this type to b .

Let $S \subset \mathbb{R}^n$. Then

$$\tilde{S}^{k+1} = \{(x, \vec{h}) \in \mathbb{R}^{(k+1)n} \mid x, h_i, x + \alpha \cdot \vec{h} \in S, \forall i \in \{1, \dots, k\}, \forall \alpha \in \{0, 1\}^k\}.$$

By “ f is of unit norm” we mean $\|f\|_p = 1$, if the L^p norm is understood. By “a Gaussian on $\mathbb{R}^n$ ” we mean the function, $m \exp(-a|x - c|^2)$, with $m, a \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}^n$. By “centered Gaussian” we mean $m \exp(-a|x|^2)$. By “the standard centered Gaussian” we mean $m \exp(-|x|^2)$, with $m > 0$ chosen so that $\|\mathcal{G}\|_p = 1$, for some appropriate L^p norm. By “measurable” we mean “Lebesgue measurable.” By “a centered ball” we mean “a ball that is centered at the origin.” Finally, we employ some short-hand notations - for instance, if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ then $\{f > \lambda\}$ means the super-level set $\{x \in \mathbb{R}^n : f(x) > \lambda\}$ - and we make an effort to indicate parametric dependence of our constants whenever it’s important for our calculations. When the dimension $n = 1$, we simply write $C(\text{other parameters}, 1) = C(\text{other parameters})$.

We call (1.4) “the k th Gowers-Host-Kra norm inequality” and (1.6) “the k th Gowers inner product inequality.” Let $f_\alpha, f : \mathbb{R}^n \rightarrow \mathbb{C}$ be measurable functions, $\alpha \in \{0, 1\}^k$, and $\vec{f} = (f_\alpha :$

$\alpha \in \{0, 1\}^k$). We say f is an extremizer of the k th Gowers-Host-Kra norm inequality if:

$$\|f\|_{U^k} = A(k, n)\|f\|_{p_k}.$$

We say f is a $(1 - \delta)$ near extremizer of the k th Gowers-Host-Kra norm inequality if:

$$\|f\|_{U^k} \geq (1 - \delta)A(k, n)\|f\|_{p_k}$$

and we say $\{f_i\}_i$ is an extremizing sequence of the k th Gowers-Host-Kra norm inequality if

$$\|f_i\|_{p_k} = 1 \text{ and,}$$

$$\|f_i\|_{U^k} \geq (1 - \delta_i)A(k, n)\|f_i\|_{p_k} = (1 - \delta_i)A(k, n)$$

with $\delta_i \rightarrow 0$ as $i \rightarrow \infty$. We say $\vec{f}$ is an extremizing tuple of the k th Gowers inner product inequality if:

$$\mathcal{T}_k(\vec{f}) = \mathcal{T}_k(f_\alpha : \alpha \in \{0, 1\}^k) = A(k, n)^{2^k} \prod_{\alpha \in \{0, 1\}^k} \|f_\alpha\|_{p_k}.$$

We say $\vec{f}$ is a $(1 - \delta)$ near extremizing tuple of the k th Gowers inner product inequality if:

$$\mathcal{T}_k(\vec{f}) = \mathcal{T}_k(f_\alpha : \alpha \in \{0, 1\}^k) \geq (1 - \delta)A(k, n)^{2^k} \prod_{\alpha \in \{0, 1\}^k} \|f_\alpha\|_{p_k}.$$

Gowers-Cauchy-Schwarz inequality implies Gowers product inequality:

$$\mathcal{T}_k(f_\alpha : \alpha \in \{0, 1\}^k) \leq \prod_{\alpha \in \{0, 1\}^k} \|f_\alpha\|_{U^k} \leq A(k, n)^{2^k} \prod_{\alpha \in \{0, 1\}^k} \|f_\alpha\|_{p_k}.$$

The first inequality is the Gowers-Cauchy-Schwarz inequality; the second is the Gowers-Host-Kra norm inequality. We called the resulting inequality, the Gowers product inequality.

Relation of Gowers-Host-Kra norm inequality to sharp Young's inequality:

Specializing the version of the sharp Young's inequality given in [10] to the exponents $s = 2^{k-1}$, $r = p_{k-1}$ and $t = q = p_k$, we have:

$$\left(\int_{\mathbb{R}^n} \|(T^h f) \bar{g}\|_{L^{p_{k-1}}(\mathbb{R}^n)}^{2^{k-1}} dh \right)^{1/2^{k-1}} \leq \left(\frac{C_{2k/(k+1)}^2}{C_k} \right)^{nk/2^{k-1}} \|f\|_{L^{p_k}(\mathbb{R}^n)} \|g\|_{L^{p_k}(\mathbb{R}^n)}$$

with $C_t = \left(\frac{t^{1/t}}{t^{1/t'}} \right)^{1/2}$, $1/t + 1/t' = 1$. It's shown in [10] that $A(k) = \left(\frac{C_{2k/(k+1)}^2}{C_k} \right)^{k/2^k} C_{k-1}^{1/2}$ and that $A(k) \leq 1 = A(1)$ for all $k \geq 1$. Hence:

$$\left(\int_{\mathbb{R}^n} \|(T^h f) \bar{g}\|_{L^{p_{k-1}}(\mathbb{R}^n)}^{2^{k-1}} dh \right)^{1/2^{k-1}} \leq \left(\frac{A(k, n)}{A(k-1, n)^{1/2}} \right)^2 \|f\|_{L^{p_k}(\mathbb{R}^n)} \|g\|_{L^{p_k}(\mathbb{R}^n)}.$$

Now letting $f = g$, incorporating this with the $(k-1)$ th Gowers-Host-Kra norm inequality and using the definition of the Gowers-Host-Kra norms, we obtain:

$$\begin{aligned} \|f\|_{U^k(\mathbb{R}^n)}^2 &= \left(\int_{\mathbb{R}^n} \|(T^h f) \bar{f}\|_{U^{k-1}(\mathbb{R}^n)}^{2^{k-1}} dh \right)^{1/2^{k-1}} \\ &\leq A(k-1, n) \left(\frac{A(k, n)}{A(k-1, n)^{1/2}} \right)^2 \|f\|_{L^{p_k}(\mathbb{R}^n)}^2 = A(k, n)^2 \|f\|_{L^{p_k}(\mathbb{R}^n)}^2 \end{aligned}$$

which is the k th Gowers-Host-Kra norm inequality. Due to this relation between the two inequalities, the optimal constants $A(k, n)$ satisfy: $A(k, m \cdot n) = (A(k, m))^n$; in particular, $A(k, m) = (A(k))^m$.

Facts about Lorentz semi-norms: [16]

Let f be a nonnegative measurable function on $\mathbb{R}^n$ and let $f = \sum_{j \in \mathbb{Z}} 2^j F_j$ be the unique

decomposition discussed below in **Section 4.1**. We claim that:

$$\|f\|_{(q,\tilde{q})} = \left(\frac{\tilde{q}}{q} \int_0^\infty (t^{1/q} f^*(t))^{\tilde{q}} dt \right)^{1/\tilde{q}} \asymp \left(\sum_{j \in \mathbb{Z}} 2^{j\tilde{q}} (\mathcal{L}(\mathcal{F}_j))^{\tilde{q}/q} \right)^{1/\tilde{q}}$$

and,

$$\|f\|_{(q,\infty)} = \sup_{t>0} t^{1/q} f^*(t) \asymp \sup_{j: \mathcal{F}_j \neq \emptyset} 2^j (\mathcal{L}(\mathcal{F}_j))^{1/q}.$$

We prove this claim here. First note that if $f = 1_E$, E being a measurable set of $\mathbb{R}^n$, then

$\|f\|_{(q,\tilde{q})} = (\mathcal{L}(E))^{1/q}$, $1 \leq \tilde{q} \leq \infty$. Since $1_{\mathcal{F}_j} \leq F_j < 2 \cdot 1_{\mathcal{F}_j}$, we first consider the case

$f = \sum_{j=-l}^l 2^j 1_{E_j}$ with pairwise disjoint measurable sets E_j of $\mathbb{R}^n$ and $l \in \mathbb{Z}_{>0}$. Then:

$$\|f\|_{(q,\tilde{q})} = \left(\frac{q}{\tilde{q}} \right)^{1/\tilde{q}} \left(\sum_{j=-l}^l 2^{j\tilde{q}} (B_j^{\tilde{q}/q} - B_{j+1}^{\tilde{q}/q}) \right)^{1/\tilde{q}} \asymp \left(\sum_{j=-l}^l 2^{j\tilde{q}} (\mathcal{L}(E_j))^{\tilde{q}/q} \right)^{1/\tilde{q}}.$$

Here $B_j = \sum_{i=j}^l (\mathcal{L}(E_i))$ and $B_{l+1} := 0$. Note that the constants in the last approximation above doesn't depend on l . Then for the general case:

$$\|f\|_{(q,\tilde{q})} = \left\| \sum_{j \in \mathbb{Z}} 2^j F_j \right\|_{(q,\tilde{q})} = \lim_{l \rightarrow \infty} \left\| \sum_{j=-l}^l 2^j F_j \right\|_{(q,\tilde{q})} \asymp \lim_{l \rightarrow \infty} \left(\sum_{j=-l}^l 2^{j\tilde{q}} (\mathcal{L}(\mathcal{F}_j))^{\tilde{q}/q} \right)^{1/\tilde{q}}$$

for all sufficiently positively large l depending on f . Hence this ultimately allows us to write:

$$\|f\|_{(q,\tilde{q})} \asymp \lim_{l \rightarrow \infty} \left(\sum_{j=-l}^l 2^{j\tilde{q}} (\mathcal{L}(\mathcal{F}_j))^{\tilde{q}/q} \right)^{1/\tilde{q}} = \left(\sum_{j \in \mathbb{Z}} 2^{j\tilde{q}} (\mathcal{L}(\mathcal{F}_j))^{\tilde{q}/q} \right)^{1/\tilde{q}}.$$

Similarly, using the notations as above, if $\tilde{q} = \infty$ and if $f = \sum_{j=-l}^l 2^j 1_{E_j}$, then $\|f\|_{(q,\infty)} = \sup_{-l \leq j \leq l} 2^j B_j^{1/q} \asymp \sup_{-l \leq j \leq l} 2^j (\mathcal{L}(E_j))^{1/q}$. Then for the general case:

$$\begin{aligned} \|f\|_{(q,\infty)} &= \left\| \sum_{j \in \mathbb{Z}} 2^j F_j \right\|_{(q,\infty)} = \lim_{l \rightarrow \infty} \left\| \sum_{j=-l}^l 2^j F_j \right\|_{(q,\infty)} \\ &\asymp \lim_{l \rightarrow \infty} \left(\sup_{-l \leq j \leq l} 2^j (\mathcal{L}(\mathcal{F}_j))^{1/q} \right) \asymp \sup_{j: \mathcal{F}_j \neq \emptyset} 2^j (\mathcal{L}(\mathcal{F}_j))^{1/q}. \end{aligned}$$

The last approximation follows because the first approximation holds for all sufficiently positively large l .

Chapter 3

A short proof of Theorem 1.1 for nonnegative functions

[9] Let $\beta \in \{0, 1\}^{k+1}$, then $\beta = (\alpha, 0)$ or $\beta = (\alpha, 1)$ for some $\alpha \in \{0, 1\}^k$. Let $\vec{f} = (f_\beta : \beta \in \{0, 1\}^{k+1})$, and suppose $f_\beta \geq 0$ for all $\beta \in \{0, 1\}^{k+1}$. Then

$$\begin{aligned} \mathcal{T}_{k+1}(f_\beta : \beta \in \{0, 1\}^{k+1}) &= \int_{\mathbb{R}^n} \mathcal{T}_k(T^h f_{(\alpha, 0)} \cdot f_{(\alpha, 1)} : \alpha \in \{0, 1\}^k) dh \\ &\leq \int_{\mathbb{R}^n} \prod_{\alpha \in \{0, 1\}^k} \|T^h f_{(\alpha, 0)} \cdot f_{(\alpha, 1)}\|_{U^k} dh \leq A(k, n)^{2^k} \int_{\mathbb{R}^n} \prod_{\alpha \in \{0, 1\}^k} \|T^h f_{(\alpha, 0)} \cdot f_{(\alpha, 1)}\|_{p_k} dh \\ &\leq A(k, n)^{2^k} \prod_{\alpha \in \{0, 1\}^k} \left(\int_{\mathbb{R}^n} \|T^h f_{(\alpha, 0)} \cdot f_{(\alpha, 1)}\|_{p_k}^{2^k} dh \right)^{1/2^k}. \quad (3.1) \end{aligned}$$

The first inequality is due to the Gowers-Cauchy-Schwarz inequality, the second to the Gowers-Host-Kra norm inequality and the last to Hölder's inequality. Substituting $p_k =$

$2^k/(k+1)$ in (3.1):

$$\begin{aligned} \int_{\mathbb{R}^n} \|T^h f_{(\alpha,0)} \cdot f_{(\alpha,1)}\|_{p_k}^{2^k} dh &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f_{(\alpha,0)}^{p_k}(y+h) f_{(\alpha,1)}^{p_k}(y) dy \right)^{2^k/p_k} dh \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f_{(\alpha,0)}^{p_k}(y+h) f_{(\alpha,1)}^{p_k}(y) dy \right)^{k+1} dh = \|\tilde{f}_{(\alpha,0)}^{p_k} * f_{(\alpha,1)}^{p_k}\|_{k+1}^{k+1} \\ &\leq B(k+1, n)^{k+1} \|\tilde{f}_{(\alpha,0)}^{p_k}\|_q^{k+1} \|f_{(\alpha,1)}^{p_k}\|_q^{k+1} = B(k+1, n)^{k+1} \|f_{(\alpha,0)}\|_{p_k q}^{2^k} \|f_{(\alpha,1)}^{p_k}\|_{p_k q}^{2^k}. \end{aligned} \quad (3.2)$$

The sole inequality in the display (3.2) above follows from Young's convolution inequality.

Here $\tilde{f}(y) = f(-y)$, $2q^{-1} = (k+1)^{-1} + 1$, or equivalently, $q = p_{k+1}/p_k$ and $B(k+1, n)$ is the optimal constant in Young's convolution inequality for the involved exponents. Combining (3.1) and (3.2), we obtain the following majorization:

$$\begin{aligned} \mathcal{T}_{k+1}(f_\beta : \beta \in \{0,1\}^{k+1}) &\leq A(k, n)^{2^k} B(k+1, n)^{k+1} \prod_{\beta \in \{0,1\}^{k+1}} \|f_\beta\|_{p_{k+1}} \\ &= A(k+1, n)^{2^{k+1}} \prod_{\beta \in \{0,1\}^{k+1}} \|f_\beta\|_{p_{k+1}} \end{aligned}$$

as it happens that $A(k, n)^{2^k} B(k+1, n)^{k+1} = A(k+1, n)^{2^{k+1}}$. Now suppose that there exists

$\delta > 0$ such that

$$\mathcal{T}_{k+1}(f_\beta : \beta \in \{0,1\}^{k+1}) \geq (1 - \delta) A(k+1, n)^{2^{k+1}} \prod_{\beta \in \{0,1\}^{k+1}} \|f_\beta\|_{p_{k+1}}.$$

Then each inequality in (3.1) and (3.2) must hold in reverse, up to a factor of $1 - c(k)\delta$, for some small $c(k) > 0$. In particular, for each $\alpha \in \{0,1\}^k$,

$$\|\tilde{f}_{(\alpha,0)}^{p_k} * f_{(\alpha,1)}^{p_k}\|_{k+1}^{k+1} \geq (1 - c(k)\delta) B(k+1, n)^{k+1} \|f_{(\alpha,0)}^{p_k}\|_q^{k+1} \|f_{(\alpha,1)}^{p_k}\|_q^{k+1}. \quad (3.3)$$

Since $(\alpha, 0), (\alpha, 1) \in \{0,1\}^{k+1}$ if $\alpha \in \{0,1\}^k$, a stability result for Young's convolution inequality in [8] implies from (3.3) that, for each $\beta \in \{0,1\}^{k+1}$, there exists a Gaussian

function G_β such that:

$$\|G_\beta - f_\beta^{p_k}\|_q \leq o_\delta(1) \|f_\beta^{p_k}\|_q. \quad (3.4)$$

Since $\|G_\beta^{1/p_k} - f_\beta\|_{p_{k+1}} \leq C(k) \|G_\beta - f_\beta^{p_k}\|_q^{1/p_k}$, by a simple calculation, (3.4) leads to $\|G_\beta^{1/p_k} - f_\beta\|_{p_{k+1}} \leq o_\delta(1) \|f_\beta\|_{p_{k+1}}$.

This discussion so far applies to $k+1 \geq 3$, since $k \geq 2$. For the characterization of a near extremizing tuple $\vec{f} = (f_1, f_2, f_3, f_4)$ of the second Gowers inner product inequality, note that

$$\mathcal{T}_2(f_1, f_2, f_3, f_4) = \langle \tilde{f}_1 * f_2, \tilde{f}_3 * f_4 \rangle \leq \|\tilde{f}_1 * f_2\|_2 \|\tilde{f}_3 * f_4\|_2 \leq A(2, n)^4 \prod_{i=1}^4 \|f_i\|_{4/3}. \quad (3.5)$$

The first inequality is an application of the Cauchy-Schwarz inequality while the second follows from Young's convolution inequality. The appearance of the constant in (3.5) is due to the facts that $A(j, n)^{2^j} B(j+1, n)^{j+1} = A(j+1, n)^{2^{j+1}}$ actually holds for all $j \geq 1$ and that $A(1, n) = A(1)^n = 1$. If for some $\delta > 0$,

$$\mathcal{T}_2(f_1, f_2, f_3, f_4) \geq (1 - \delta) A(2, n)^4 \prod_{i=1}^4 \|f_i\|_{4/3}$$

then with a similar argument as above, we trade each inequality in (3.5) for a reverse inequality with a factor of $1 - c\delta$ and use the said stability result for Young's convolution inequality to conclude that, for each $i \in \{1, \dots, 4\}$, there exists a Gaussian G_i such that, $\|G_i - f_i\|_{4/3} \leq o_\delta(1) \|f_i\|_{4/3}$. As Gaussians are the only nonnegative maximizers of the Gowers-Host-Kra norm inequality; hence this concludes **Theorem 1.1** for the case of nonnegative near extremizers.

Chapter 4

Localization around a single scale

4.1 Normalization

Let $\Theta : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ be such that $\Theta(t) \rightarrow 0$ if $t \rightarrow \infty$. Let $f : \mathbb{R}^n \rightarrow \mathbb{C}$ and $f \in L^q$, $q \in [1, \infty)$.

We say that f is normalized (with respect to Θ) if the following two conditions occur:

$$\begin{aligned} \int_{\{|f|>\rho\}} |f|^q dx &\leq \Theta(\rho) \\ \int_{\{|f|<\rho\}} |f|^q dx &\leq \Theta(\rho^{-1}). \end{aligned}$$

We often say “ f is normalized” for short as the presence of such a function Θ will always be implied, and the selection of Θ will precede the selection of the involved near extremizers, so as to create no ambiguity. We will also allow the following relaxed version of normalization. Let $\nu > 0$ be a small number and Θ be as above. Then f is said to be ν -normalized, with respect to Θ , if $f = g + h$, g is normalized with respect to Θ and $\|h\|_q \leq \nu$. In our

application, $q = p_k = \frac{2^k}{k+1}$. From now on, whenever we say a “growth function” we mean a function $\Theta : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ satisfying $\Theta(\rho) \rightarrow 0$ as $\rho \rightarrow \infty$. Our claim is that, there exist a growth function Θ and a $\delta_0 > 0$, so that the following property occurs. If $0 < \delta \leq \delta_0$ and f is a $(1 - \delta)$ near extremizer of the Gowers-Host-Kra norm inequality, then there exists $\lambda > 0$, such that, $\tilde{f}(x) = \lambda^{n/p_k} f(\lambda x)$ is normalized, with respect to Θ . Towards this end, we will prove an equivalent result below.

Since the normalization condition concerns the absolute value of f and since $|f|$ is a $(1 - \delta)$ near extremizer if f is, as $\|f\|_{U^k} \leq \| |f| \|_{U^k}$, we consider only nonnegative functions. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be a measurable function. There exists a unique decomposition $f = \sum_{j \in \mathbb{Z}} 2^j F_j$ such that $1_{\mathcal{F}_j} \leq F_j < 2 \cdot 1_{\mathcal{F}_j}$, with the measurable sets $\mathcal{F}_j$ being pairwise disjoint up to null sets. This decomposition comes from the layer cake representation: $f(x) = \int_0^\infty 1_{\{f>t\}}(x) dt = \sum_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} 1_{\{f>t\}}(x) dt = \sum_{j \in \mathbb{Z}} 2^j F_j(x)$, with $F_j(x) = 2^{-j} \int_{2^j}^{2^{j+1}} 1_{\{f \geq t\}}(x) dt$. It's readily checked that $1_{\mathcal{F}_j} \leq F_j < 2 \cdot 1_{\mathcal{F}_j}$, with $\mathcal{F}_j = \{2^j \leq f < 2^{j+1}\}$. This decomposition will set up the stage for the use of Lorentz semi-norms which will be needed. In particular, we want to show:

Proposition 4.1. There exist positive functions ϕ, Φ satisfying $\lim_{t \rightarrow \infty} \phi(t) = 0$ and $\lim_{t \rightarrow 0} \Phi(t) = 0$, and positive constants δ_0, c_0 such that the following holds:

Let $0 \leq f \in L^{p_k}(\mathbb{R}^n)$. Let $f = \sum_{j \in \mathbb{Z}} 2^j F_j$ with the F_j related to f as indicated above.

Suppose that $\|f\|_{p_k} = 1$, and for some $0 < \delta \leq \delta_0$, we have

$$\|f\|_{U^k} \geq (1 - \delta)A(k, n)\|f\|_{p_k} = (1 - \delta)A(k, n).$$

Then there exist $c_0 \in \mathbb{R}_{>0}$ and $l \in \mathbb{Z}$ such that:

$$2^l(\mathcal{L}(\mathcal{F}_l))^{1/p_k} \geq c_0\|f\|_{p_k} = c_0 \quad (4.1)$$

$$\sum_{|j-l| \geq m} 2^{jp} \mathcal{L}(\mathcal{F}_j) \leq (\phi(m) + \Phi(\delta)) \cdot \|f\|_{p_k}^{p_k} = \phi(m) + \Phi(\delta). \quad (4.2)$$

Remark 4.1: Statement (4.1) ensures that there exists a scale 2^l so that the contribution of the quantity $2^l(\mathcal{L}(\mathcal{F}_l))^{1/p_k}$ to the total norm $\|f\|_{p_k}$ is non-negligible. Choose λ to be an integral power of 2 to dilate the set $\mathcal{F}_l$, so that its dilated version has measure as close to one as possible. Statement (4.2) is equivalent to saying, this dilated version of f is then ν -normalized with respect to an appropriate growth function Θ and a positive $\nu = \nu(\delta)$ that tends to zero as $\delta \rightarrow 0$.

Let $1 \leq q, \tilde{q} < \infty$ and $f = \sum_{j \in \mathbb{Z}} 2^j F_j$ be as above. Recall from **Chapter 2** that $\|f\|_{(q, \tilde{q})} \asymp (\sum_{j \in \mathbb{Z}} 2^{j\tilde{q}}(\mathcal{L}(\mathcal{F}_j))^{\tilde{q}/q})^{1/\tilde{q}}$ if $\tilde{q} = \infty$, and $\|f\|_{(q, \infty)} \asymp \sup_{j: \mathcal{F}_j \neq \emptyset} 2^j(\mathcal{L}(\mathcal{F}_j))^{1/q}$.

4.2 A digression

Let $\vec{f}$ denote $(f_\alpha : \alpha \in \{0, 1\}^k)$. We claim that, there exists $C(k, n) < \infty$ such that:

$$|\mathcal{T}_k(\vec{f})| = \left| \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0, 1\}^k} f_\alpha(x + \alpha \cdot \vec{h}) dx d\vec{h} \right| \leq C(k, n) \prod_{\alpha \in \{0, 1\}^k} \|f_\alpha\|_{(p_k, 2^k)}. \quad (4.3)$$

This is a consequence of Lorentz space interpolation. Indeed:

Let $F(x) = \int_{\mathbb{R}^{nk}} \prod_{\alpha \in \{0,1\}^k; \alpha \neq \vec{0}} f_\alpha(x + \alpha \cdot \vec{h}) d\vec{h}$ and $(q')^{-1} + q^{-1} = 1$. By Hölder's inequality for $1 \leq r, \tilde{r} \leq \infty$ for the Lorentz semi-norms [16],

$$|\mathcal{T}_k(\vec{f})| \leq C \|f_{\vec{0}}\|_{(r, \tilde{r})} \|F\|_{(r', \tilde{r}')} . \quad (4.4)$$

Let $Q_r(\vec{P})$ denote a closed cube centered at $\vec{P} = (p_k, \dots, p_k) \in \mathbb{R}^{2^k}$ with radius r and $\Omega = \{\vec{q} = (q_\alpha : \alpha \in \{0,1\}^k) \in Q_r(\vec{P}) : \sum_{\alpha \in \{0,1\}^k} (q_\alpha)^{-1} = k+1\}$. Then if $r = r(k)$ is sufficiently small, there exists a constant $C(k, n) < \infty$ such that for every $\vec{q} \in \Omega$,

$$|\mathcal{T}_k(\vec{f})| \leq C(k, n) \prod_{\alpha \in \{0,1\}^k} \|f_\alpha\|_{q_\alpha} . \quad (4.5)$$

A proof of (4.5) will be given at the end of this section. Assuming (4.5), then if $\vec{q} \in \Omega$,

$$\|F\|_{q'_{\vec{0}}} \leq C(k, n) \prod_{\alpha \in \{0,1\}^k; \alpha \neq \vec{0}} \|f_\alpha\|_{q_\alpha} . \quad (4.6)$$

If we vary one $q_\alpha, \alpha \neq \vec{0}$, while keeping other values $q_\beta, \beta \neq \vec{0}, \alpha$, fixed and apply interpolation for the Lorentz norms, we obtain from (4.6):

$$\|F\|_{(r'_\alpha, \infty)} \leq C(k, n) \|f_\alpha\|_{(p_k, \infty)} \prod_{\beta \in \{0,1\}^k; \beta \neq \vec{0}, \alpha} \|f_\beta\|_{q_\beta}$$

whenever $p_k^{-1} + r_\alpha^{-1} + \sum q_\beta^{-1} = k+1$ and $r_\alpha, q_\beta \in [p_k - 2^{-k}, p_k + 2^{-k}]$. We continue applying interpolation on other exponents q_β , one by one. Suppose that q_γ is the last exponent to be interpolated, then:

$$\|F\|_{(r'_\gamma, \infty)} \leq C(k, n) \|f_\gamma\|_{q_\gamma} \prod_{\beta \in \{0,1\}^k; \beta \neq \vec{0}, \gamma} \|f_\beta\|_{(p_k, \infty)}$$

whenever $r_\gamma^{-1} + q_\gamma^{-1} + (2^k - 2)p_k^{-1} = k + 1$ and $r_\gamma, q_\gamma \in [p_k - 2^{-k}, p_k + 2^{-k}]$. Applying interpolation one last time, we have:

$$\begin{aligned} \|F\|_{(p'_k, 1)} &\leq C(k, n) \|f_\gamma\|_{(p_k, 1)} \prod_{\alpha \in \{0, 1\}^k; \alpha \neq \vec{0}, \gamma} \|f_\alpha\|_{(p_k, \infty)} \\ \|F\|_{(p'_k, \infty)} &\leq C(k, n) \|f_\gamma\|_{(p_k, \infty)} \prod_{\alpha \in \{0, 1\}^k; \alpha \neq \vec{0}, \gamma} \|f_\alpha\|_{(p_k, \infty)}. \end{aligned}$$

Since γ can be any of the values $\alpha \in \{0, 1\}^k$, these calculations yield,

$$\|F\|_{(p'_k, 1)} \leq C(k, n) \prod_{\alpha \in \{0, 1\}^k; \alpha \neq \vec{0}} \|f_\alpha\|_{(p_k, r_\alpha)} \quad (4.7)$$

$$\|F\|_{(p'_k, \infty)} \leq C(k, n) \prod_{\alpha \in \{0, 1\}^k; \alpha \neq \vec{0}} \|f_\alpha\|_{(p_k, \infty)} \quad (4.8)$$

whenever $\sum r_\alpha^{-1} = 1$ and $r_\alpha \in \{1, \infty\}$. Then (4.4), (4.7) and (4.8) yield:

$$|\mathcal{T}_k(\vec{f})| \leq C \|f_{\vec{0}}\|_{(p_k, r_{\vec{0}})} \|F\|_{(p'_k, r'_{\vec{0}})} \leq C(k, n) \prod_{\alpha \in \{0, 1\}^k} \|f_\alpha\|_{(p_k, r_\alpha)} \quad (4.9)$$

whenever $\sum r_\alpha^{-1} = 1$ and $r_\alpha \in \{1, \infty\}$. To complete the proof of (4.3), we need the following short lemma:

Lemma 4.2. [9] Let $\mathcal{T}$ be a scalar-valued m -linear form on $\mathbb{R}^n$ and $q \in [1, \infty)$. Suppose that

$$|\mathcal{T}(\vec{f})| \leq C(m) \prod_{1 \leq i \leq m} \|f_i\|_{(q, r_i)} \quad (4.10)$$

whenever $r_i \in \{1, \infty\}$ and $\sum r_i^{-1} = 1$. Suppose furthermore that $\vec{q} = (q_i)_{i=1}^m$ with $\sum q_i^{-1} = 1$ and $q_i \in (1, \infty)$. Then

$$|\mathcal{T}(\vec{f})| \leq C(m, \vec{q}) \prod_{1 \leq i \leq m} \|f_i\|_{(q, q_i)}. \quad (4.11)$$

Proof: We use the described decomposition $f_i = \sum_{j \in \mathbb{Z}} 2^j F_{i,j}$, with $1_{\mathcal{F}_{i,j}} \leq \mathcal{L}(F_{i,j}) < 2 \cdot 1_{\mathcal{F}_{i,j}}$ and the measurable sets $\mathcal{F}_{i,j}$ pairwise disjoint. Let $S_{i,\lambda} = \{j : 2^\lambda \leq 2^j (\mathcal{L}(\mathcal{F}_{i,j}))^{1/p_k} < 2^{\lambda+1}\}$ for each $\lambda \in \mathbb{Z}$. Define $f_{i,\lambda} = \sum_{j \in S_{i,\lambda}} 2^j F_{i,j}$. Note that $f_i = \sum_{\lambda \in \mathbb{Z}} 2^\lambda f_{i,\lambda}$ and for $s \in [1, \infty)$:

$$\|f_i\|_{(q,s)} \asymp \left(\sum_{\lambda \in \mathbb{Z}} 2^{\lambda s} \text{crd}(S_{i,\lambda}) \right)^{1/s}. \quad (4.12)$$

If $s = \infty$:

$$\|f_i\|_{(q,\infty)} \asymp \lim_{t \rightarrow \infty} \left(\sum_{\lambda \in \mathbb{Z}} 2^{\lambda t} \text{crd}(S_{i,\lambda}) \right)^{1/t} = \sup_{\lambda: S_{i,\lambda} \neq \emptyset} 2^\lambda. \quad (4.13)$$

Let $\vec{\lambda} = (\lambda_i)_{i=1}^m \in \mathbb{Z}^m$. Then from (4.10), (4.12) and (4.13):

$$\begin{aligned} |\mathcal{T}(\vec{f})| &\leq C(m) \sum_{\vec{\lambda}} 2^{\sum_i \lambda_i} \min_{1 \leq i \leq m} (\text{crd}(S_{i,\lambda_i})) \\ &\leq C(m) \sum_{\vec{\lambda}} \theta((\text{crd}(S_{i,\lambda_i}))_{i=1}^m) \prod_{1 \leq i \leq m} 2^{\lambda_i} \text{crd}(S_{i,\lambda_i})^{1/q_i} \end{aligned} \quad (4.14)$$

with $\sum q_i^{-1} = 1$ and $\theta((t_i)_{i=1}^m) = (\min_i t_i) \cdot \prod_{1 \leq i \leq m} t_i^{-1/q_i}$. Consider all the vectors $\vec{\lambda}$ for which:

$$\text{crd}(S_{1,\lambda_1}) = \min_i \text{crd}(S_{i,\lambda_i}) \quad (4.15)$$

$$\text{crd}(S_{m,\lambda_m}) = \max_i \text{crd}(S_{i,\lambda_i}). \quad (4.16)$$

The following analysis applies the same way for other vectors $\vec{\lambda}$ with only minor changes in notation. For each $i \in \{1, \dots, m\}$, define a nonnegative integer n_i so that

$$2^{-n_i-1} \text{crd}(S_{1,\lambda_1}) \leq \text{crd}(S_{i,\lambda_i}) \leq 2^{-n_i} \text{crd}(S_{1,\lambda_1}). \quad (4.17)$$

Fix a tuple $\vec{n} = (n_i)_{i=1}^m \in \mathbb{N}^m$; we sum the right hand side of (4.14) over the $\vec{\lambda}$ for which all three (4.15), (4.16) and (4.17) are satisfied with this fixed $\vec{n}$. Define $I_1 = 1$ and I_i , $i \neq 0$,

is the largest index l such that $2^{-n_i-1} \text{crd}(S_{1,\lambda_1}) \leq \text{crd}(S_{i,l}) \leq 2^{-n_i} \text{crd}(S_{1,\lambda_1})$. With these choices of I_i , it's clear that $I_1 \mapsto I_i$ are bounded-to-one functions, uniformly in $\vec{n}$. Then for this fixed $\vec{n}$, the sum on the right hand side of (4.14) will be majorized by:

$$C \sum_{I_m} 2^{-n_m} \prod_{1 \leq i \leq m} 2^{n_i/q_i} \prod_{1 \leq t \leq m} 2^{I_t} \text{crd}(S_{t,I_t})^{1/q_t} \leq C 2^{-r \sum u_i} \sum_{I_m} \prod_{1 \leq t \leq m} 2^{I_t} \text{crd}(S_{t,I_t})^{1/q_t} \quad (4.18)$$

with $u_m = n_m$ and $u_i = n_m - n_i \geq 0$ and $0 < r = \min_i 1/q_i$. Once again, with these choices of I_i , the constant C in (4.18) is a universal constant that doesn't depend on $\vec{n}$. Applying Hölder's inequality to the right hand side sum in (4.18) to have it further majorized by:

$$C 2^{-r \sum u_i} \prod_{1 \leq i \leq m} \left(\sum_{I_m} 2^{I_i \cdot q_i} \text{crd}(S_{i,I_i}) \right)^{1/q_i} \leq C 2^{-r \sum u_i} \prod_{1 \leq i \leq m} \|f_i\|_{(q,q_i)}. \quad (4.19)$$

Sum the sum in the right hand side of (4.14) over $\vec{n} \in \mathbb{N}^m$ and utilize the bound in (4.19) and the convergence of geometric series, we conclude that (4.10) indeed implies (4.11).

Applying this **Lemma 4.2** to (4.9), with $q = p$ and $m = 2^k$ with indices $\alpha \in \{0,1\}^k$, we obtain (4.3) with the desired exponents $q_i = 2^k$.

Remark 4.2: The interpolation result of this discussion is a very restricted result that is suited to the task at hand. A more general interpolation result can be found in [12].

Proof of (4.5): [9] We prove the following by induction. For every $k \geq 2$, there exists r_k such that, if $\vec{p} = (p_\alpha : \alpha \in \{0, 1\}^k)$ is such that $\sum_\alpha p_\alpha^{-1} = k + 1$ and $|p_\alpha - 2^k/(k + 1)| \leq r_k$, for every $\alpha \in \{0, 1\}^k$, then

$$\mathcal{T}_k(\vec{f}) \leq \prod_{\alpha \in \{0, 1\}^k} \|f_\alpha\|_{p_\alpha}$$

for every $\vec{f} = (f_\alpha : \alpha \in \{0, 1\}^k)$ with each f_α nonnegative and measurable.

If $k = 2$ then the conclusion is a consequence of Young's and Hölder's inequalities:

$$\mathcal{T}_k(\vec{f}) = \langle f_1 * \tilde{f}_2, f_3 * \tilde{f}_4 \rangle_{\mathbb{R}} \leq \|f_1 * \tilde{f}_2\|_{p_{1,2}} \|f_3 * \tilde{f}_4\|_{p_{3,4}} \leq \prod_{i=1}^4 \|f_i\|_{p_i}.$$

Here, $\tilde{f}(x) = f(-x)$. The condition on $p_i, i \in \{1, \dots, 4\}$ is simply, $p_{1,2}^{-1} + p_{3,4}^{-1} = 1$, or equivalently, $\sum_{i=1}^4 p_i^{-1} = 2 + 1$.

Assume the conclusion holds for the case $k - 1$. Let $\vec{f} = (f_\alpha \geq 0 : \alpha \in \{0, 1\}^k)$. For each $\alpha \in \{0, 1\}^k$, there is a unique $\beta \in \{0, 1\}^{k-1}$ such that $\alpha = (\beta, 0)$ or $\alpha = (\beta, 1)$. Then for each $\beta \in \{0, 1\}^{k-1}$, let $g_\beta^t(x) = f_{(\beta, 0)}(x)f_{(\beta, 1)}(x + t)$. Write $\vec{g}^t = (g_\beta^t : \beta \in \{0, 1\}^{k-1})$, then

$$\begin{aligned} \mathcal{T}_k(\vec{f}) &= \int_{\mathbb{R}^n} \mathcal{T}_{k-1}(\vec{g}^t) dt \leq \int_{\mathbb{R}^n} \prod_{\beta \in \{0, 1\}^{k-1}} \|g_\beta^t\|_{p_\beta} dt \leq \prod_{\beta \in \{0, 1\}^{k-1}} \left(\int_{\mathbb{R}^n} \|g_\beta^t\|_{p_\beta}^{2^{k-1}} dt \right)^{1/2^{k-1}} \\ &\leq \prod_{\beta \in \{0, 1\}^{k-1}} \|f_{(\beta, 0)}\|_{p_{(\beta, 0)}} \|f_{(\beta, 1)}\|_{p_{(\beta, 1)}}. \end{aligned}$$

The first inequality follows from the induction hypothesis for some $r_{k-1} > 0$ such that, $\sum_{\beta \in \{0, 1\}^{k-1}} p_\beta^{-1} = k$ and $|p_\beta - 2^{k-1}/k| \leq r_{k-1}$, for every $\beta \in \{0, 1\}^{k-1}$. The second inequality follows from Minkowski's inequality, and the last from the sharp Young's inequality, with the following relations between the exponents: $1/2^{k-1} + 1/p_\beta = 1/p_{(\beta, 0)} + 1/p_{(\beta, 1)}$. Now it's easily observed that if $p_{(\beta, 0)} = 2^k/(k + 1) = p_{(\beta, 1)}$ then $p_\beta = 2^{k-1}/k$, if $\sum_{\beta \in \{0, 1\}^{k-1}} p_\beta^{-1} = k$

then $\sum_{\beta \in \{0,1\}^k} p_{(\beta,0)}^{-1} + p_{(\beta,1)}^{-1} = k + 1$, and for each $\beta \in \{0,1\}^{k-1}$, p_β is a continuous function of $p_{(\beta,0)}$ and $p_{(\beta,1)}$. Hence if for each $\beta \in \{0,1\}^{k-1}$, $p_{(\beta,0)}$ and $p_{(\beta,1)}$ are taken sufficiently close to $2^k/(k+1)$, so that p_β is within an r_{k-1} distance to $2^{k-1}/k$, then the conclusion for the case k follows from the induction hypothesis for the case $k-1$. The proof of (4.5) is now complete.

4.3 Proof of Proposition 4.1

Let $\eta = \eta(\delta) > 0$ be a small number to be chosen later. Define $S = S(\eta) = \{j \in \mathbb{Z} : 2^j(\mathcal{L}(\mathcal{F}_j))^{1/p_k} > \eta\}$. Statement (4.1) will follow if this set S is nonempty, which is indeed true if η is sufficiently small. To see this, consider $f_S = \sum_{j \in S} 2^j F_j$ and $f_{S^c} = \sum_{j \notin S} 2^j F_j$. Then:

$$\begin{aligned} \|f_{S^c}\|_{(p,2^k)}^{2^k} &= \left\| \sum_{j \notin S} 2^j F_j \right\|_{(p_k,2^k)}^{2^k} \asymp \sum_{j \notin S} 2^{j2^k} (\mathcal{L}(\mathcal{F}_j))^{2^k/p_k} \\ &\leq \max_{j \notin S} (2^j(\mathcal{L}(\mathcal{F}_j))^{1/p_k})^{2^k - p_k} \sum_{j \notin S} (2^j(\mathcal{L}(\mathcal{F}_j))^{1/p_k})^{p_k} \leq \eta^{2^k - p_k} \|f_{S^c}\|_{p_k}^{p_k} \end{aligned}$$

or

$$\|f_{S^c}\|_{(p_k,2^k)} \leq C \eta^{1-p_k/2^k} \|f_{S^c}\|_{p_k}^{p_k/2^k}. \quad (4.20)$$

As above, for a set $A \subset \mathbb{Z}$, we denote $f_A = \sum_{j \in A} 2^j F_j$ and define a 2^k tuple-valued function

on $\mathbb{R}^n$, $\vec{f}_A = (f_A, f, \dots, f)$. With this notation:

$$\begin{aligned} |\mathcal{T}_k(\vec{f}_{S^c})| &= \left| \int_{\mathbb{R}^{(k+1)n}} f_{S^c}(x) \prod_{\alpha \in \{0,1\}^k; \alpha \neq 0} f(x + \alpha \cdot \vec{h}) dx d\vec{h} \right| \leq C(k, n) \|f_{S^c}\|_{(p_k, 2^k)} \|f\|_{p_k}^{2^k-1} \\ &\leq C(k, n) \eta^{1-p_k/2^k} \|f_{S^c}\|_{p_k}^{p_k/2^k} \|f\|_{p_k}^{2^k-1}. \end{aligned} \quad (4.21)$$

The first inequality follows from (4.3) and the fact that $\|f\|_{(p_k, 2^k)} \leq \|f\|_{p_k}$ and the last inequality from (4.20). Recall that $\|f\|_{p_k} = 1$. Hence $S = \emptyset$ in the context of (4.21) would imply that:

$$\|f\|_{U^k}^{2^k} \leq C(k, n) \eta^{1-p_k/2^k} \|f\|_{p_k}^{2^k} = C(k, n) \eta^{1-p_k/2^k}. \quad (4.22)$$

Since $1 - p_k/2^k > 0$, the right hand side of (4.22) is small if η is small. From the near extremizing hypothesis, $\|f\|_{U^k} \geq A(k, n)(1-\delta)$. Hence, if η is sufficiently small then $S = S(\eta)$ must be nonempty; otherwise (4.22) implies $\|f\|_{U^k} = 0$, a contradiction. Note that the argument also shows, for given η , if δ is sufficiently small then $S = S(\eta) \neq \emptyset$. In addition, from the definition of S and Chebyshev's inequality,

$$crd(S) \leq C \eta^{-p_k} \|f\|_{p_k}^{p_k} = C \eta^{-p_k}. \quad (4.23)$$

We now prove statement (4.2). Let $\eta > 0$ be a small number, and consequently let $\delta > 0$ be small enough if needed, so that $S = S(\eta) \neq \emptyset$. Let $M = \max_{i, i' \in S} |i - i'|$. In order to prove (4.2), it suffices to show that M is bounded by a finite upper bound that does not depend on a particular $(1 - \delta)$ near extremizer f . Assume that S has more than one element. Note that since S is a set of integers, $M \geq crd(S)$. Let $N > 0$ be a large integer to be chosen

below, which will depend only on n, k and η - and consequently on δ . Suppose that with this choice of N we have $M \leq 10N\text{crd}(S)$, then we obtain the desired bound, since from (4.23):

$$M \leq CN\text{crd}(S) \leq C(k, n, \delta)\eta(\delta)^{-p}.$$

If $M > 10N\text{crd}(S)$, then there must exist integers $J > I$ such that: $S \cap (-\infty, I] \neq \emptyset$, $S \cap [J, \infty) \neq \emptyset$, $S \cap (I, J) = \emptyset$ and $J - I \geq M/(N\text{crd}(S))$. Indeed, choose an integer L such that $M/(2N\text{crd}(S)) \leq L \leq M/(N\text{crd}(S))$. Consider the intervals $I_j = [jL, (j+1)L) \subset (\inf(S), \sup(S))$ and let K be the set of all these indices j . Then $\text{crd}(K) \asymp M/L \asymp N\text{crd}(S)$. Hence $I_j \cap S = \emptyset$ for at least one of these intervals. Since

$$\|f_{S^c}\|_{(p_k, 2^k)}^{2^k} = \sum_{l \notin S} \|2^l F_l\|_{(p_k, 2^k)}^{2^k} \geq \sum_{j \in K: I_j \cap S = \emptyset} \left\| \sum_{l \in I_j} 2^l F_l \right\|_{(p_k, 2^k)}^{2^k}$$

and $\text{crd}(K) - \text{crd}(S) \geq c(N-1)\text{crd}(S) \geq cN-1$, there exists one such interval I_j such that,

$$\|f_{S^c}\|_{(p_k, 2^k)}^{2^k} \geq CN^{-1} \left\| \sum_{l \in I_j} 2^l F_l \right\|_{(p_k, 2^k)}^{2^k}.$$

Take such an interval, then $I = jL$ and $J = (j+1)L$ are our desired integers. Moreover, with this pair of (I, J) :

$$\left\| \sum_{I < l < J} 2^l F_l \right\|_{(p_k, 2^k)}^{2^k} \leq CN^{-1} \|f_{S^c}\|_{(p_k, 2^k)}^{2^k} \leq CN^{-1} \eta^{2^k - p_k}. \quad (4.24)$$

The second inequality follows from (4.20). Define $f_u = \sum_{i \geq J} 2^i F_i$, $f_d = \sum_{i \leq I} 2^i F_i$ and $f_b = f - f_u - f_d$. As (4.24) states, $\|f_b\|_{p_k} \leq CN^{-1} \eta^c$ for some $c > 0$. Now in terms of these f_u, f_d, f_b :

$$\|f\|_{U^k}^{2^k} = |\mathcal{T}_k(f_u + f_d + f_b, \dots, f_u + f_d + f_b)|.$$

Expanding the above sum, we will have $C(k)$ terms in which at least one of the components is f_b , $C(k)$ mixed terms in which the components consist of only f_u and f_d but no f_b , and two pure terms in which the components are either f_u or f_d , which we denote $\mathcal{T}_k(\vec{f}_u)$ and $\mathcal{T}_k(\vec{f}_d)$, respectively. We claim that the contribution of all the mixed terms is majorized by $C(k, n)2^{-\eta^{p_k} M/N}$. We will first need the following fact about the 2^k linear form $\mathcal{T}_k$ on indicators of sets. Recall that $\Omega = \{\vec{q} = (q_\alpha : \alpha \in \{0, 1\}^k) \in Q_r(\vec{P}) : \sum_{\alpha \in \{0, 1\}^k} (q_\alpha)^{-1} = k + 1\}$ and $Q_r(\vec{P})$ denotes a closed cube of sufficiently small size r centered at $\vec{P} = (p_k, \dots, p_k) \in \mathbb{R}^{2^k}$.

Lemma 4.3. Suppose $\vec{p} = (p_\alpha : \alpha \in \{0, 1\}^k) \in \Omega$. Then there exist $\tau > 0$ and C such that, if $E_\alpha \subset \mathbb{R}^n$ are sets with finite measures, $\alpha \in \{0, 1\}^k$, then:

$$\mathcal{T}_k(1_{E_\alpha} : \alpha \in \{0, 1\}^k) \leq C(k, n) \rho(\mathcal{L}(E_\alpha) : \alpha \in \{0, 1\}^k)^\tau \prod_{\alpha \in \{0, 1\}^k} (\mathcal{L}(E_\alpha))^{1/p_\alpha} \quad (4.25)$$

with $\rho(c_\alpha : \alpha \in \{0, 1\}^k) = \min_{\beta \neq \gamma \in \{0, 1\}^k} \frac{c_\beta}{c_\gamma}$.

Proof. Select $\vec{q} = (q_\alpha : \alpha \in \{0, 1\}^k) \in \Omega$ that differs from $\vec{p} = (p_\alpha : \alpha \in \{0, 1\}^k)$ only in two components, for convenience, say $\alpha = \vec{0}$ and $\alpha = (1, \dots, 1) = \vec{1}$, so that $q_{\vec{0}}^{-1} = p_{\vec{0}}^{-1} + \tau$ and $q_{\vec{1}}^{-1} = p_{\vec{1}}^{-1} - \tau$ for some $\tau > 0$ small enough. Then from (4.5), for this pair $(\vec{0}, \vec{1})$ we obtain:

$$\mathcal{T}_k(1_{E_\alpha} : \alpha \in \{0, 1\}^k) \leq C(k, n) \prod_{\alpha \in \{0, 1\}^k} |E_\alpha|^{1/q_\alpha} \leq C(k, n) \left(\frac{\mathcal{L}(E_{\vec{0}})}{\mathcal{L}(E_{\vec{1}})} \right)^\tau \prod_{\alpha \in \{0, 1\}^k} (\mathcal{L}(E_\alpha))^{1/p_\alpha}.$$

Repeating this process to other pairs (β, γ) , $\beta, \gamma \in \{0, 1\}^k$, we obtain the conclusion (4.25).

Lemma 4.4. Using the terminology above, then the contribution of all the quantities $\mathcal{T}_k$ with mixed-term inputs is majorized by $C(k, n)2^{-\eta^{p_k} M/N}$ in absolute value.

Remark 4.3: This lemma confirms the intuition that if the components are not “compatible” then their multilinear product doesn’t contribute much.

Proof of Lemma 4.4: Take a typical mixed term $\mathcal{T}_k(f_u, \dots, f_d)$ in the expansion of $\mathcal{T}_k(f_u + f_d + f_b, \dots, f_u + f_d + f_b)$, whose at least two components are assumed to be different. The notation used here is only for convenience; the argument will be independent of which two components are different. Let $\mathbb{Z}^{2^k} \supset R = \{\vec{i} = (i_\alpha : \alpha \in \{0, 1\}^k) : i_{\vec{0}} \geq J, i_{\vec{1}} \leq I\}$. Let $\epsilon > 0$ be a small number and let $R_\epsilon = \{\vec{i} \in R : 2^{i_\alpha} \mathcal{L}(\mathcal{F}_{i_\alpha})^{1/p_k} \geq \epsilon, \forall \alpha \in \{0, 1\}^k\}$. We take ϵ sufficiently small so that $R_\epsilon \neq \emptyset$. By Chebyshev’s inequality,

$$\text{crd}(R_\epsilon) \leq C\epsilon^{-2^k p_k}. \quad (4.26)$$

By the definition of R_ϵ and the Gowers product inequality:

$$\sum_{\vec{i} \in R_\epsilon^c} 2^{\sum_{\alpha \in \{0, 1\}^k} i_\alpha} \mathcal{T}_k(1_{\mathcal{F}_{i_\alpha}} : \alpha \in \{0, 1\}^k) \leq C\epsilon. \quad (4.27)$$

Next we need to find an upper bound for the corresponding sum over $\vec{i} \in R_\epsilon$. Note that if $\vec{i} = (i_\alpha : \alpha \in \{0, 1\}^k) \in R_\epsilon$, then $2^{i_\alpha} (\mathcal{L}(\mathcal{F}_{i_\alpha}))^{1/p_k} \leq C$, since $\|f\|_{p_k} = 1$, and that $2^{i_{\vec{1}}} (\mathcal{L}(\mathcal{F}_{i_{\vec{1}}}))^{1/p_k} \geq \epsilon$, which then implies

$$\mathcal{L}(\mathcal{F}_{i_{\vec{1}}}) \geq \epsilon^{p_k} 2^{-i_{\vec{1}} p_k} \geq \epsilon^{p_k} 2^{-I p_k}. \quad (4.28)$$

Furthermore, since $i_{\vec{0}} \geq J \geq I + M/(N \text{crd}(S)) \geq I + c\eta^{p_k} M/N$,

$$\mathcal{L}(\mathcal{F}_{i_{\vec{0}}}) \leq C 2^{-i_{\vec{0}} p_k} \leq C 2^{-I p_k - c\eta^{p_k} M/N}. \quad (4.29)$$

(4.28) and (4.29) conclude:

$$\frac{\mathcal{L}(\mathcal{F}_{i_{\vec{0}}})}{\mathcal{L}(\mathcal{F}_{i_{\vec{1}}})} \leq C\epsilon^{-p_k} 2^{-c\eta^{p_k} M/N}. \quad (4.30)$$

Applying **Lemma 4.3** to the sets $\mathcal{L}(\mathcal{F}_{i_\alpha})$ and utilizing (4.26) and (4.30), we obtain a bound on the sum over $\vec{i} \in R_\epsilon$:

$$\begin{aligned} \sum_{\vec{i} \in R_\epsilon} 2^{\sum_{\alpha \in \{0,1\}^k} i_\alpha} \mathcal{T}_k((1_{\mathcal{F}_{i_\alpha}} : \alpha \in \{0,1\}^k)) &\leq C(k, n) \epsilon^{-p_k} 2^{-c\eta^{p_k} M/N} \text{crd}(R_\epsilon) \\ &\leq C(k, n) \epsilon^{-C(k)} 2^{-c\eta^{p_k} M/N}. \end{aligned} \quad (4.31)$$

(4.27) and (4.31) give:

$$|\mathcal{T}_k(f_u, \dots, f_d)| \leq C\epsilon + C(k, n) \epsilon^{-C(k)} 2^{-c\eta^{p_k} M/N}. \quad (4.32)$$

Choose $\epsilon = e^{-\tau\eta^{p_k} M/N}$ for a sufficiently small $\tau = \tau(k) > 0$; we have from (4.32):

$$|\mathcal{T}_k(f_u, \dots, f_d)| \leq C(k, n) 2^{-c\eta^{p_k} M/N}. \quad (4.33)$$

Recall that (4.24) implies $\|f_b\|_{p_k} \leq CN^{-1}\eta^c$. Then (4.32), the definition of the Gowers-Host-Kra norms and the Gowers-Host-Kra norm inequality give

$$\begin{aligned} \|f\|_{U^k}^{2^k} &\leq |\mathcal{T}_k(\vec{f}_u)| + |\mathcal{T}_k(\vec{f}_d)| + C(k, n) 2^{-\eta^{p_k} M/N} + CN^{-1}\eta^c \\ &\leq A(k, n) 2^k (\|f_u\|_{p_k}^{2^k} + \|f_d\|_{p_k}^{2^k}) + C(k, n) 2^{-\eta^{p_k} M/N} + CN^{-1}\eta^c. \end{aligned} \quad (4.34)$$

From the definition of f_u and f_d , we have $\min(\|f_u\|_{p_k}, \|f_d\|_{p_k}) \geq \eta$. Since $(\|f_u\|_{p_k}^{p_k} + \|f_d\|_{p_k}^{p_k})^{1/p_k} \leq \|f\|_{p_k} = 1$, $\max(\|f_u\|_{p_k}, \|f_d\|_{p_k}) \leq (1 - c\eta^{p_k})\|f\|_{p_k} = 1 - c\eta^{p_k}$. Hence:

$$\|f_u\|_{p_k}^{2^k} + \|f_d\|_{p_k}^{2^k} \leq \max(\|f_u\|_{p_k}, \|f_d\|_{p_k})^{2^k - p_k} (\|f_u\|_{p_k}^{p_k} + \|f_d\|_{p_k}^{p_k}) \leq 1 - c\eta^{p_k}. \quad (4.35)$$

Inserting the bound in (4.35) into (4.34) and utilizing the fact that f is a $(1 - \delta)$ near extremizer:

$$A(k, n)^{2^k} (1 - \delta) \leq A(k, n)^{2^k} (1 - c\eta^{p_k}) + C(k, n)2^{-\eta^{p_k} M/N} + CN^{-1}\eta^c$$

or,

$$2^{-c\eta^{p_k} M/N} \geq c(k, n)\eta^{p_k} - c(k, n)N^{-1} - C(k, n)\delta. \quad (4.36)$$

Now choose N large enough to be the nearest strictly positive integer $\asymp c\eta^{-p_k}$ for some small c , so that from (4.36):

$$2^{-c\eta^{2p_k} M} \geq c(k, n)\eta^{p_k} - C(k, n)\delta. \quad (4.37)$$

Choose δ_0 sufficiently small so that $c(k, n)\eta^{p_k} - C(k, n)\delta_0 > 0$, which yields, $\eta \geq C_0(k, n)\delta_0^{1/p_k}$ for some $C_0(k, n) > 0$. We replace the first term in the right hand side of (4.37) with this new lower bound of η ; the inequality then yields:

$$M \leq C\eta^{-2p_k}$$

with $C = C(k, n, \delta)$, which is a desired bound on M . This completes the proof of statement (4.2) and consequently, the proof of **Proposition 4.1**.

Chapter 5

Control of distribution functions

We wish to obtain two conclusions in this chapter; one is that, for a near extremizer, the contribution in L^{p_k} norm of the superposition of its super-level sets associated with values outside a large compact range is negligible, and the other is, the measure of a super-level set with value in the said compact range closely approximates the measure of a super-level set of an extremizer of the same value.

5.1 Precompactness of symmetric rearrangements

Note that if $f \geq 0$ is a $(1 - \delta)$ near extremizer then so is its symmetric rearrangement, f^* :

$$\begin{aligned} \|f^*\|_{U^k}^{2^k} &= \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} f^*(x + \alpha \cdot \vec{h}) \, dx d\vec{h} \geq \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} f(x + \alpha \cdot \vec{h}) \, dx d\vec{h} \\ &= \|f\|_{U^k}^{2^k} \geq (1 - \delta) A(k, n)^{n2^k} \|f\|_{p_k}^{2^k} = (1 - \delta) A(k, n)^{n2^k} \|f^*\|_{p_k}^{2^k}. \end{aligned}$$

The first inequality is due to the general rearrangement inequality and the second to the L^{p_k} -norm preservation property of symmetric rearrangements [14]. By definition, the normalization condition preserves measure. That means if Θ is a growth function and if $f \geq 0$ is normalized with respect to Θ then so is f^* :

$$\begin{aligned} \int_{\{f^* > \rho\}} f^*(x) dx &= \int_{\{|f| > \rho\}} |f|(x) dx \leq \Theta(\rho) \\ \int_{\{f^* < \rho\}} f^*(x) dx &= \int_{\{|f| < \rho\}} |f|(x) dx \leq \Theta(\rho^{-1}). \end{aligned}$$

Let $\{f_i\}_i$ be a sequence of measurable functions on $\mathbb{R}^n$. We say $\{f_i\}_i$ is a normalized extremizing sequence if $\|f_i\|_{p_k} = 1$, f_i is normalized with respect to Θ for all i and if there exists a sequence $\{\delta_i\}_i$ satisfying $\delta_i \rightarrow 0$ as $i \rightarrow \infty$, such that:

$$\|f_i\|_{U^k} \geq (1 - \delta_i)A(k, n)\|f_i\|_{p_k}$$

Suppose furthermore that $f_i \geq 0$ for all i and let $\{f_i^*\}_i$ be the corresponding symmetric rearrangement sequence of $\{f_i\}_i$. Then as noted, $\{f_i^*\}_i$ is also a normalized extremizing sequence.

Lemma 5.1. Let $\{f_i^*\}_i$ be as above. Let $\epsilon > 0$. Then there exist $r > 0, R > 0$ such that, for all sufficiently large j :

$$\int_{\{|x| \leq r\}} (f_i^*)^{p_k} dx \leq \epsilon \tag{5.1}$$

$$\int_{\{|x| \geq R\}} (f_i^*)^{p_k} dx \leq \epsilon. \tag{5.2}$$

Proof: Indeed, by normalization,

$$\begin{aligned} \int_{\{|x| \leq r\}} (f_i^*)^{p_k} dx &\leq \int_{\{|x| \leq r\} \cap \{f_i^* \leq \alpha\}} (f_i^*)^{p_k} dx + \int_{\{f_i^* \geq \alpha\}} (f_i^*)^{p_k} dx \\ &\leq c(n) \alpha^{p_k} r^n + \int_{\{f_i^* \geq \alpha\}} (f_i^*)^{p_k} dx \leq c(n) \alpha^{p_k} r^n + \Theta(\alpha). \end{aligned} \quad (5.3)$$

The third inequality follows from the fact that f_i^* is normalized. Choose α sufficiently large so that $\Theta(\alpha) \leq (1/2)\epsilon$ and with this α , r small enough so that $c(n) \alpha^{p_k} r^n \leq (1/2)\epsilon$. Putting all these upper bounds in (5.3), we obtain (5.1). The fact that f_i^* is radially symmetric and nonincreasing gives us,

$$c(n) |s|^n (f_i^*)^{p_k}(s) \leq \int_{\{|x| \leq s\}} (f_i^*)^{p_k} dx \leq 1.$$

Hence, if $R \leq |s| \leq R_0$:

$$(f_i^*)^{p_k}(s) \leq C(n) |s|^{-n} \leq C(n) R^{-n}. \quad (5.4)$$

In other words, f_i^* is bounded above by $C(n) R^{-n/p_k}$ in an annulus $\{R \leq |x| \leq R_0\}$, with $R_0 > R$, uniformly for all i , and hence, $\{|x| \geq R\} \subset \{(f_i^*)^{p_k} \leq C(n) R^{-n}\}$. Thus:

$$\int_{\{|x| \geq R\}} (f_i^*)^{p_k} dx \leq \int_{\{(f_i^*)^{p_k} \leq C(n) R^{-n}\}} (f_i^*)^{p_k} dx \leq \Theta((c(n) R^n)^{1/p_k}). \quad (5.5)$$

The last inequality in (5.5) follows from normalization. Now choose R sufficiently large so that $\Theta((c(n) R^n)^{1/p_k}) \leq \epsilon$. Then (5.5) implies (5.2). This completes the proof of the lemma.

Let $r > 0, R > 0$ and $\mathcal{C}_{r,R} = \{r \leq |x| \leq R\}$. The proof of **Lemma 5.1**, particularly (5.4), shows that the sequence $\{f_i^*\}_i$ restricted to a ray cutting the annulus $\mathcal{C}_{r,R}$, produces a

uniformly bounded, nonincreasing sequence of functions on an interval $[r, R]$. It follows from Helly's selection principle [15] that there exists a subsequence of $\{f_i^*\}_i$ that converges pointwise on every such ray. Since f_i^* is radial for all i , such a subsequence also converges pointwise on the annulus $\mathcal{C}_{r,R}$. We denote such a subsequence by $\{f_{i,r,R}^*\}_i$ and the corresponding pointwise limit by $\mathcal{F}_{r,R}$. It's clear that $\mathcal{F}_{r,R}$ is also radial. We select r to be an increasing function of ϵ and R a decreasing function of ϵ . Choose a sequence $\{\epsilon_i\}_i$ that decreases to zero and denote $r(\epsilon_i) = r_i$, $R(\epsilon_i) = R_i$. Then by a Cantor diagonal argument and passing to a subsequence of $\{f_{i,r_i,R_i}^*\}_i$, we obtain a sequence of nonincreasing sequence of radial functions $\{\mathcal{F}_{r_i,R_i}\}_i$ such that $\mathcal{F}_{r_i,R_i}$ is supported on the annulus $\mathcal{C}_{r_i,R_i}$, for all i , and $\mathcal{F}_{r_i,R_i} = \mathcal{F}_{r_n,R_n}$ on $\mathcal{C}_{r_n,R_n}$ if $i \geq n$. Let $\mathcal{F}$ be the pointwise limit of $\{\mathcal{F}_{r_i,R_i}\}_i$. Note that $\mathcal{F}_{r_n,R_n} \leq \mathcal{F}_{r_i,R_i} \leq \mathcal{F}$, if $i \geq n$. For convenience, we denote the subsequence of $\{f_{i,r_i,R_i}^*\}_i$ in the construction of $\mathcal{F}_{r_i,R_i}$ at every index i as simply $\{f_i^*\}_i$. We argue that $\mathcal{F}$ must be a centered Gaussian function on $\mathbb{R}^n$. Indeed, by **Lemma 5.1**, for all sufficiently large i ,

$$\int_{\{|x| \leq r_i\}} (f_i^*)^{p_k} dx + \int_{\{|x| \geq R_i\}} (f_i^*)^{p_k} dx \leq 2\epsilon_i. \quad (5.6)$$

Take $\eta > 0$. Then take I sufficiently large so that $2\epsilon_i < \eta$, if $i \geq I$. By letting $i \rightarrow \infty$, it follows from (5.6) and the fact that $\mathcal{F}_{r_i,R_i} = \mathcal{F}_{r_n,R_n}$ on $\mathcal{C}_{r_n,R_n}$ if $i \geq n \geq I$,

$$\begin{aligned}
 \int_{\mathbb{R}^n} |\mathcal{F}_{r_n,R_n} - \mathcal{F}_{r_i,R_i}|^{p_k} dx &\leq \int_{\{|x| \leq r_I\}} |\mathcal{F}_{r_n,R_n} - \mathcal{F}_{r_i,R_i}|^{p_k} + \int_{\{|x| \geq R_I\}} |\mathcal{F}_{r_n,R_n} - \mathcal{F}_{r_i,R_i}|^{p_k} \\
 &\quad + \int_{\{r_I \leq |x| \leq R_I\}} |\mathcal{F}_{r_n,R_n} - \mathcal{F}_{r_i,R_i}|^{p_k} dx \leq \eta.
 \end{aligned}$$

Hence $\{\mathcal{F}_{r_i,R_i}\}_i$ is a Cauchy sequence in L^{p_k} and thus $\mathcal{F}$ is also its L^{p_k} -limit. On the other

hand, by the Dominated Convergence Theorem, $\mathcal{F}_{r_i, R_i}$ is the pointwise limit and thus the L^{p_k} -limit of $\{f_i^*\}_i$ on $\mathcal{C}_{r_i, R_i}$. Hence $\mathcal{F}$ is the L^{p_k} -limit of $\{f_i^*\}_i$ on $\mathbb{R}^n$. By the Gowers-Host-Kra norm inequality, $\|\mathcal{F} - f_i^*\|_{U^k} \leq A(k, n)\|\mathcal{F} - f_i^*\|_{p_k}$, and hence, as $i \rightarrow \infty$, $\|f_i^*\|_{U^k} \rightarrow \|\mathcal{F}\|_{U^k}$. But since $\|f_i^*\|_{U^k} \geq (1 - \delta_i)A(k, n)\|f_i^*\|_{p_k}$, these imply that $\|\mathcal{F}\|_{U^k} = A(k, n)\|\mathcal{F}\|_{p_k}$. By the characterization of extremizers of the Gowers-Host-Kra norm inequality, $\mathcal{F}$ must be a Gaussian, which then is a centered Gaussian, as it is the pointwise limit of radial functions.

5.2 Control of distribution functions

Recall that if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a $(1 - \delta)$ near extremizer then so is $|f|$. Hence we only consider nonnegative near extremizers for now. Suppose $\|f\|_{U^k} \geq (1 - \delta)A(k, n)\|f\|_{p_k}$ and $\|\mathcal{F} - f^*\|_{p_k} \leq \delta$ with $\mathcal{F}$ being a centered Gaussian on $\mathbb{R}^n$. Denote, for $s > 0$, $F_s = \{f > s\}$, $F_s^* = \{f^* > s\}$ and $\mathcal{F}_s = \{\mathcal{F} > s\}$.

Lemma 5.2. Let $f, f^*, \mathcal{F}, F_s, F_s^*, \mathcal{F}_s$ be as above. Given $\delta > 0$. There exists $\eta = \eta(\delta) > 0$ satisfying $\eta(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, such that if $s \in [\eta, \|\mathcal{F}\|_\infty - \eta]$ then

$$\mathcal{L}(\mathcal{F}_s \Delta F_s^*) = o_\delta(1) \quad (5.7)$$

$$|\mathcal{L}(\mathcal{F}_s) - \mathcal{L}(F_s)| = o_\delta(1). \quad (5.8)$$

Proof: Let $\eta \in (0, 1)$ be a small number to be chosen below. Consider $F_s^* \setminus \mathcal{F}_{s-\eta}$. If $x \in F_s^* \setminus \mathcal{F}_{s-\eta}$ then $|(\mathcal{F} - f^*)(x)| \geq \eta$. Chebyshev's inequality gives, $\mathcal{L}(F_s^* \setminus \mathcal{F}_{s-\eta}) \leq$

$\eta^{-p_k} \|\mathcal{F} - f^*\|_{p_k}^{p_k} \leq \eta^{-p_k} \delta^{p_k}$, and, $\mathcal{L}(\mathcal{F}_{s+\eta} \setminus F_s^*) \leq \eta^{-p_k} \delta^{p_k}$. Hence:

$$\mathcal{L}(\mathcal{F}_{s+\eta} \Delta \mathcal{F}_{s-\eta}) \leq c \eta^{-p_k} \delta^{p_k}. \quad (5.9)$$

On the other hand, since $\mathcal{F}$ is a Gaussian and all the super-level sets of $\mathcal{F}$ are nested, centered ellipsoids, we also have

$$\mathcal{L}(\mathcal{F}_{s+\eta} \Delta \mathcal{F}_{s-\eta}) \leq C \eta^{1/2}. \quad (5.10)$$

Optimizing (5.9) and (5.10) to have:

$$\mathcal{L}(\mathcal{F}_{s+\eta} \Delta \mathcal{F}_{s-\eta}) \leq C \delta^{p_k/(2p_k+1)} \quad (5.11)$$

Since the super-level sets of f^* are also centered ellipsoids, if $\eta = \delta^{2p_k/(2p_k+1)}$ then

$$\mathcal{L}(\mathcal{F}_{s-\eta} \Delta F_s^*) \leq c \delta^{p_k/(2p_k+1)} \quad (5.12)$$

$$\mathcal{L}(\mathcal{F}_{s+\eta} \Delta F_s^*) \leq c \delta^{p_k/(2p_k+1)}. \quad (5.13)$$

(5.11), (5.12), (5.13) then imply $\mathcal{L}(\mathcal{F}_s \Delta F_s^*) \leq C \delta^{p_k/(2p_k+1)}$, and consequently, $|\mathcal{L}(\mathcal{F}_s) - \mathcal{L}(F_s)| \leq C \delta^{p_k/(2p_k+1)}$, which are the promises (5.9) and (5.10), respectively.

Remark 5.1: We can summarize the findings above in the following language:

Given a δ small, there exist $\eta(\delta)$ and $\epsilon(\eta, \delta)$ with the following properties. If $\delta \rightarrow 0$ then $\eta(\delta) \rightarrow 0$, and if η is fixed, $\delta \rightarrow 0$ then $\epsilon(\eta, \delta) \rightarrow 0$. If $\mathcal{F}_s, F_s$ are as above, then $\sup_{s \in [\eta(\delta), \|\mathcal{F}_s\|_\infty - \eta(\delta)]} |\mathcal{L}(\mathcal{F}_s) - \mathcal{L}(F_s)| \leq \epsilon(\eta, \delta)$. Moreover, by the description of $\epsilon(\eta, \delta)$, we have that if $\xi(\eta)$ is a positive continuous function satisfying $\xi(\eta) \rightarrow 0$ as $\eta \rightarrow 0$, then

there exists a function $\eta \mapsto \delta_0(\eta) > 0$ such that $\sup_{0 < \delta \leq \delta_0(\eta)} \epsilon(\eta, \delta) \leq \xi(\eta)$. In particular, $\lim_{\eta \rightarrow 0} \sup_{0 < \delta \leq \delta_0(\eta)} \epsilon(\eta, \delta) = 0$, for some positive function δ_0 . These properties of $\epsilon(\eta, \delta)$ will be needed below in **Chapter 5**.

Lemma 5.3. Let $f^*, \mathcal{F}, F_s$ and η be as above. There exists $C > 0$ so that,

$$\left\| \int_0^\eta 1_{F_s} ds \right\|_{p_k} \leq C \|\mathcal{F} - f^*\|_{p_k} + C\eta(\log(1/\eta))^C$$

and,

$$\left\| \int_{\|\mathcal{F}\|_\infty - \eta}^\infty 1_{F_s} ds \right\|_{p_k} \leq C \|\mathcal{F} - f^*\|_{p_k} + O(\eta).$$

Proof: Indeed,

$$\begin{aligned} \left\| \int_0^\eta 1_{F_s} ds \right\|_{p_k} &= \left\| \int_0^\eta 1_{F_s^*} ds \right\|_{p_k} = \|\min(f, \eta)\|_{p_k} \leq \|\mathcal{F} - f^*\|_{p_k} + \|\min(\mathcal{F}, \eta)\|_{p_k} \\ &\leq \|\mathcal{F} - f^*\|_{p_k} + \eta(\log(1/\eta))^C. \end{aligned}$$

Likewise:

$$\begin{aligned} \left\| \int_{\|\mathcal{F}\|_\infty - \eta}^\infty 1_{F_s} ds \right\|_{p_k} &= \left\| \int_{\|\mathcal{F}\|_\infty - \eta}^\infty 1_{F_s^*} ds \right\|_{p_k} = \|\max(0, f^* - (\|\mathcal{F}\|_\infty - \eta))\|_{p_k} \\ &\leq \|\mathcal{F} - f^*\|_{p_k} + \|\max(0, \mathcal{F} - (\|\mathcal{F}\|_\infty - \eta))\|_{p_k} \leq \|\mathcal{F} - f^*\|_{p_k} + O(\eta). \end{aligned}$$

Remark 5.2: One can appreciate the usefulness of the normalization condition in obtaining the decay properties (5.1) and (5.2) and consequently the precompactness of $\{f_i^*\}_i$ in L^{p_k} , which then allows us to extract an extremizer $\mathcal{F}$. Moreover, one observes that the only needed hypotheses for **Lemma 5.2** and **Lemma 5.3** are that f^* is radially symmetric and

decreasing, $\mathcal{F}$ is a centered Gaussian, and $\|\mathcal{F} - f\|_{p_k} \leq \delta, \|\mathcal{F} - f^*\|_{p_k} \leq \delta$; the specific fact that f is a near extremizer for the Gowers-Host-Kra norm inequality did not enter. This observation was first used by Christ to obtain similar conclusions about near extremizing triple (f, g, h) of Young's inequality on Euclidean spaces [8].

Chapter 6

Control of super-level sets

Let $n = 1$. Let f be a $(1 - \delta)$ near extremizer of the Gowers-Host-Kra norm inequality and $\mathcal{F}$ a centered Gaussian such that $\|f\|_{p_k} = 1$ and $\|\mathcal{F} - f^*\|_{p_k} \leq \delta$. In this chapter, we only consider $f \geq 0$. Since f is of unit norm, so are f^* and $\mathcal{F}$. For $t > 0$, let $\Omega(t) = [t, \|\mathcal{F}\|_\infty - t]$. Recall that for $s > 0$, $F_s = \{f > s\}$, $F_s^* = \{f^* > s\}$, $\mathcal{F}_s = \{\mathcal{F} > s\}$.

Proposition 6.1. For every $\epsilon > 0$ there exists $\delta > 0$ such that for every $(1 - \delta)$ near extremizer f and every $\nu_\alpha \in \Omega(\epsilon)$, $\alpha \in \{0, 1\}^k$,

$$\mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \geq (1 - \epsilon) \mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k). \quad (6.1)$$

Remark 6.1: We will prove the statement in the following form:

For every $\epsilon > 0$ there exists $\zeta > 0$ and $\delta > 0$ such that if f is a $(1 - \delta)$ near extremizer of

the Gowers-Host-Kra norm inequality and if $\nu_\alpha \in \Omega(\zeta)$, $\alpha \in \{0, 1\}^k$ then,

$$\mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \geq (1 - \epsilon) \mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k). \quad (6.2)$$

(6.2) are equivalent to (6.1) if $\zeta = o_\epsilon(1)$ and $\epsilon = o_\zeta(1)$, which will be the case.

Assume **Proposition 6.1** for a moment. Assume $\nu_\alpha = \nu$ for all $\alpha \in \{0, 1\}^k$. Then by definition, $\mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) = \mathcal{T}_k(1_{F_\nu} : \alpha \in \{0, 1\}^k) = \|1_{F_\nu}\|_{U^k}^{2^k}$ and $\mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k) = \|1_{F_\nu^*}\|_{U^k}^{2^k}$. Then (6.2) becomes $\|1_{F_\nu}\|_{U^k} \geq (1 - \epsilon) \|1_{F_\nu^*}\|_{U^k}$, which entails that the sets F_{ν_α} will then be nearly intervals, by the following result in [6]:

Proposition 6.2. [6] Let $n \geq 1$. For every $\epsilon > 0$ there exists $\delta > 0$ such that if $E \subset \mathbb{R}^n$ is a measurable set with finite measure and $\|1_E\|_{U^k} \geq (1 - \delta) \|1_{E^*}\|_{U^k}$ then there exists an ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ such that $\mathcal{L}(E \Delta \mathcal{E}) < \epsilon \mathcal{L}(E)$.

In what follows, $\vec{\nu} = (\nu_\alpha : \alpha \in \{0, 1\}^k)$.

6.1 Proof of Proposition 6.1

Set-up:

Let $t \in (0, 1)$. By the Gowers product inequality

$$\begin{aligned} \int_{\mathbb{R}^{\{0,1\}^k} \setminus \Omega(t)^{\{0,1\}^k}} \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) d\vec{\nu} \\ \leq A(k)^{2^k} \int_{\mathbb{R}^{\{0,1\}^k} \setminus \Omega(t)^{\{0,1\}^k}} \prod_{\alpha \in \{0,1\}^k} \|1_{F_{\nu_\alpha}}\|_{p_k} d\vec{\nu}. \end{aligned} \quad (6.3)$$

Moreover $\mathbb{R}^{\{0,1\}^k} \setminus \Omega(t)^{\{0,1\}^k} \subset \cup_{\alpha \in \{0,1\}^k} (\mathbb{R} \setminus \Omega(t)) \times \mathbb{R}^{\{0,1\}^k - 1}$. Hence an immediate consequence of **Lemma 5.3**, Fubini's theorem and (6.3) is that there exists a universal $C(k) > 0$ such that for all small $t > 0$:

$$\begin{aligned} \int_{\mathbb{R}^{\{0,1\}^k} \setminus \Omega(t)^{\{0,1\}^k}} \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) d\vec{\nu} &\leq A(k)^{2^k} \prod_{\alpha \in \{0,1\}^k} \left\| \int_{\mathbb{R} \setminus \Omega(t)} 1_{F_{\nu_\alpha}} d\nu_\alpha \right\|_{p_k} \\ &\leq C(k)\delta + C(k)t(\log(1/t))^C. \end{aligned} \quad (6.4)$$

Define:

$$H(\nu_{\vec{0}}) = \mathcal{T}_k((1_{F_{\nu_{\vec{0}}}}, 1_{F_{\nu_\alpha}}) : \alpha \in \{0, 1\}^k; \alpha \neq \vec{0})$$

$$\mathcal{H}(\nu_{\vec{0}}) = \mathcal{T}_k((1_{\mathcal{F}_{\nu_{\vec{0}}}}, 1_{\mathcal{F}_{\nu_\alpha}}) : \alpha \in \{0, 1\}^k; \alpha \neq \vec{0}).$$

In other words, we fix all the values $\nu_\alpha, \alpha \neq \vec{0}$ and consider $H, \mathcal{H}$ as nonnegative functions of only $\nu_{\vec{0}}$. Since level sets are nested, both H and $\mathcal{H}$ are non-increasing. Moreover, since $\mathcal{F}$ is a centered Gaussian of unit norm, if $\nu_\alpha \in \Omega(t)$, for all $\alpha \in \{0, 1\}^k$, then $\mathcal{H}$ is bounded below by a strictly positive function of t as long as $0 < t < \|\mathcal{F}\|_\infty$. Similarly, $\mathcal{T}_k(1_{\mathcal{F}_{\nu_\alpha}} :$

$\alpha \in \{0, 1\}^k \geq \phi(t)$, with ϕ being a strictly positive continuous function satisfying $\phi \rightarrow 0$ as $t \rightarrow 0$; this function ϕ will be needed below. Moreover, $\mathcal{H}$ is also Lipschitz continuous with a Lipschitz constant $L(t)$ that is independent of a specific $\vec{\nu}$, as long as $\nu_\alpha \in \Omega(t)$, for all $\alpha \in \{0, 1\}^k$. Now:

$$\begin{aligned}
 \int_{\Omega(t)^{\{0,1\}^k}} \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) d\vec{\nu} &= \|f\|_{U^k}^2 - \int_{\mathbb{R}^{\{0,1\}^k} \setminus \Omega(t)^{\{0,1\}^k}} \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) d\vec{\nu} \\
 &\geq (1 - \delta) \|\mathcal{F}\|_{U^k} - C(k)\delta - C(k)t(\log(1/t))^C \\
 &\geq (1 - \delta) \int_{\Omega(t)^{\{0,1\}^k}} \mathcal{T}_k(1_{\mathcal{F}_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) d\vec{\nu} \\
 &\quad - C(k)\delta - C(k)t(\log(1/t))^C. \quad (6.5)
 \end{aligned}$$

The first inequality follows from (6.4) and the near extremizing hypothesis. The second inequality follows from the definition of the Gowers-Host-Kra norms. Recall from *Remark 5.1*, that if $s \in \Omega(t)$, then:

$$\sup_{s \in \Omega(t)} |\mathcal{L}(\mathcal{F}_s) - \mathcal{L}(F_s)| \leq \epsilon(t, \delta) \quad (6.6)$$

and $\epsilon(t, \delta)$ has the following properties:

1. $\epsilon(t, \delta) \rightarrow 0$ if $\delta \rightarrow 0$ and t is fixed.
2. For every positive continuous function $\xi(t)$ satisfying $\xi(t) \rightarrow 0$ as $t \rightarrow 0$, there exists $\delta_0(t) > 0$ such that $\epsilon(t, \delta) \leq \xi(t)$ for all $0 < \delta \leq \delta_0(t)$; hence $\lim_{t \rightarrow 0} \sup_{0 < \delta \leq \delta_0(t)} \epsilon(t, \delta) = 0$.

It follows from (6.6) and the Gowers product inequality, that if $\vec{\nu} \in \Omega(t)^{\{0,1\}^k}$ then:

$$\begin{aligned} & |\mathcal{T}_k(1_{\mathcal{F}_{\nu_\alpha}} : \alpha \in \{0,1\}^k) - \mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0,1\}^k)| \\ & \leq C(k) \sup_{s \in \Omega(t)} |(\mathcal{L}(\mathcal{F}_s))^{1/p_k} - (\mathcal{L}(F_s))^{1/p_k}| \cdot ((\mathcal{L}(\mathcal{F}_t))^{1/p_k} + \epsilon(t, \delta))^{2^k-1} \leq C(k, t) \epsilon(t, \delta). \end{aligned} \quad (6.7)$$

The constant $C(k, t)$ depends on $\mathcal{L}(\mathcal{F}_t)$ and satisfies $C(k, t) \rightarrow \infty$ if $t \rightarrow 0$. Combining (6.7) and the general rearrangement inequality, we obtain:

$$\begin{aligned} \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) & \leq \mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0,1\}^k) \\ & \leq \mathcal{T}_k(1_{\mathcal{F}_{\nu_\alpha}} : \alpha \in \{0,1\}^k) + C(k, t) \epsilon(t, \delta). \end{aligned} \quad (6.8)$$

Take two numbers $\rho, \eta \in (0, 1)$ satisfying $\rho \leq \eta$, then $\Omega(\eta) \subset \Omega(\rho)$. Integrating $\vec{\nu}$ in (6.8) over $\Omega(\rho)^{\{0,1\}^k} \setminus \Omega(\eta)^{\{0,1\}^k}$ gives:

$$\begin{aligned} \int_{\Omega(\rho)^{\{0,1\}^k} \setminus \Omega(\eta)^{\{0,1\}^k}} \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) d\vec{\nu} & \leq \int_{\Omega(\rho)^{\{0,1\}^k} \setminus \Omega(\eta)^{\{0,1\}^k}} \mathcal{T}_k(1_{\mathcal{F}_{\nu_\alpha}} : \alpha \in \{0,1\}^k) d\vec{\nu} \\ & \quad + C(k, \rho) \epsilon(\rho, \delta). \end{aligned} \quad (6.9)$$

Here again $C(k, \rho) \rightarrow \infty$ if $\rho \rightarrow 0$. Substituting $t = \rho$ in (6.5) to have:

$$\begin{aligned} \int_{\Omega(\rho)^{\{0,1\}^k}} \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) d\vec{\nu} & \geq (1 - \delta) \int_{\Omega(\rho)^{\{0,1\}^k}} \mathcal{T}_k(1_{\mathcal{F}_{\nu_\alpha}} : \alpha \in \{0,1\}^k) d\vec{\nu} \\ & \quad - C(k) \delta - C(k) \rho (\log(1/\rho))^C. \end{aligned} \quad (6.10)$$

Subtracting (6.9) from (6.10):

$$\begin{aligned} \int_{\Omega(\eta)^{\{0,1\}^k}} \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) d\vec{\nu} & \geq \int_{\Omega(\eta)^{\{0,1\}^k}} \mathcal{T}_k(1_{\mathcal{F}_{\nu_\alpha}} : \alpha \in \{0,1\}^k) d\vec{\nu} \\ & \quad - C(k) \delta \phi(\eta) - C(k) \delta - C(k) \rho (\log(1/\rho))^C - C(k, \rho) \epsilon(\rho, \delta). \end{aligned} \quad (6.11)$$

Let $\xi(t)$ be a positive continuous function satisfying $\xi(t) \rightarrow 0$ as $t \rightarrow 0$. We now require in our selection of ρ, η , that $C(k)\rho(\log(1/\rho))^C + C(k)\phi(\eta) \leq \xi(\eta)$. By the second property of $\epsilon(t, \delta)$, there exists a positive function $\delta_0(t)$ such that for all $0 < \delta \leq \delta_0(t)$, $t \in (0, 1)$, the two following conditions are satisfied:

$$\epsilon(t, \delta) \leq \xi(t)$$

$$\epsilon(t, \delta) \leq C(k, t)^{-1}\xi(t)$$

with $C(k, \cdot)$ having the same meaning as in (6.11). Since $\xi(t) \rightarrow 0$ as $t \rightarrow 0$, $c_1\xi(\rho) \leq \xi(\eta) \leq c_2\xi(\rho)$, if $\rho \leq \eta$ are both sufficiently small. These observations together with the conditions allow us to rewrite (6.11) as:

$$\int_{\Omega(\eta)^{\{0,1\}^k}} (\mathcal{T}_k(1_{\mathcal{F}_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) - \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k)) d\vec{\nu} \leq C(k)\delta + C(k)\xi(\eta). \quad (6.12)$$

Note that if we re-define $\epsilon(s, \delta) = C(k)\delta + C(k)\xi(s)$, then this new $\epsilon(s, \delta)$ still satisfies the two properties of the error function ϵ mentioned above. We rewrite (6.8) and (6.12) respectively in this new language, with $\nu_\alpha \in \Omega(t)$, $\alpha \in \{0, 1\}^k$:

$$\mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \leq \mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k) \leq \mathcal{T}_k(1_{\mathcal{F}_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) + C(k)\epsilon(t, \delta) \quad (6.13)$$

$$\int_{\Omega(t)^{\{0,1\}^k}} (\mathcal{T}_k(1_{\mathcal{F}_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) - \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k)) d\vec{\nu} \leq \epsilon(t, \delta). \quad (6.14)$$

By Fubini's theorem, (6.14) is simply,

$$\begin{aligned} \int_{\Omega(t)^{\{0,1\}^k}} (\mathcal{T}_k(1_{\mathcal{F}_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) - \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k)) d\vec{\nu} \\ = \int_{\Omega(t)^{\{0,1\}^k}} (\mathcal{H}(\nu_{\vec{0}}) - H(\nu_{\vec{0}})) d\vec{\nu} \leq \epsilon(t, \delta). \end{aligned} \quad (6.15)$$

(6.13) implies, if $\nu_\alpha \in \Omega(t)$, for all $\alpha \in \{0, 1\}^k$:

$$\mathcal{H}(\nu_{\vec{0}}) - H(\nu_{\vec{0}}) \geq -C(k)\epsilon(t, \delta). \quad (6.16)$$

Integrating (6.16) over $\Omega(t)$, we have uniformly for $(\nu_\alpha : \alpha \in \{0, 1\}^k; \alpha \neq \vec{0}) \in \Omega(t)^{\{0,1\}^k \setminus \{\vec{0}\}}$:

$$\int_t^{\|\mathcal{F}\|_\infty - t} (\mathcal{H}(\nu_{\vec{0}}) - H(\nu_{\vec{0}})) d\nu_{\vec{0}} \geq -C(k)\epsilon(t, \delta). \quad (6.17)$$

A process:

Recall that $\mathcal{H}$ is Lipschitz continuous with a Lipschitz constant majorized by a quantity $L(t)$ that is independent of $\vec{\nu}$, as long as $\nu_\alpha \in \Omega(t)$, for all $\alpha \in \{0, 1\}^k$. Let $K = K(t) = \max(1, L(t))$. Note that $K(t) \rightarrow \infty$ as $t \rightarrow 0$, since $\mathcal{L}(\mathcal{F}_{\|\mathcal{F}\|_\infty - t}) \rightarrow 0$ and $\mathcal{L}(\mathcal{F}_t) \rightarrow \infty$ as $t \rightarrow 0$. So far we have two parameters $\rho \leq \eta$. The parameter ρ is an auxiliary one whose ultimate use was to define the new error function ϵ ; our main parameter is the parameter η . We now will define new parameters in terms of η . Let $r = r(\eta)$ be a small quantity to be chosen below, and suppose that there exists $\nu'_{\vec{0}} \in [\eta, \|\mathcal{F}\|_\infty - \eta - r]$ such that:

$$H(\nu'_{\vec{0}}) \leq \mathcal{H}(\nu'_{\vec{0}}) - r. \quad (6.18)$$

Then, we claim, a similar property will hold for a sub-range of $\nu_{\vec{0}} \in [\nu'_{\vec{0}}, \nu'_{\vec{0}} + cK^{-1}r]$:

$$H(\nu_{\vec{0}}) \leq H(\nu'_{\vec{0}}) \leq \mathcal{H}(\nu'_{\vec{0}}) - r \leq \mathcal{H}(\nu_{\vec{0}}) - cr. \quad (6.19)$$

Indeed, the non-increasing property of H gives the first inequality in (6.19); the second is just (6.18), and the Lipschitz continuity of $\mathcal{H}$ over the selected interval gives the last.

Furthermore, if (6.18) happens, we can increase the lower bound in (6.17):

$$\begin{aligned}
 & \int_{\eta}^{\|\mathcal{F}\|_{\infty}-\eta} (\mathcal{H}(\nu_{\vec{0}}) - H(\nu_{\vec{0}})) d\nu_{\vec{0}} \\
 &= \int_{[\nu'_{\vec{0}}, \nu'_{\vec{0}} + cK^{-1}r]} + \int_{[\eta, \|\mathcal{F}\|_{\infty}-\eta] \setminus [\nu'_{\vec{0}}, \nu'_{\vec{0}} + cK^{-1}r]} (\mathcal{H}(\nu_{\vec{0}}) - H(\nu_{\vec{0}})) d\nu_{\vec{0}} \\
 &\geq cr^2 K^{-1} - C(k)\epsilon(\eta, \delta). \quad (6.20)
 \end{aligned}$$

The inequality in (6.20) follows from applying the respective lower bounds as in (6.17), with $t = \eta$, and (6.19). It will be in our desire that $r(\eta) \rightarrow 0$ as $\eta \rightarrow 0$ but at a rate $o(\eta)$. To this end, take $q \in (0, 1)$. We consider only sufficiently small values of δ so that $qcK^{-1}r^2 \geq C\epsilon(\eta, \delta)$. This is possible by the first property of ϵ described above. This allows us to rewrite (6.20) as:

$$\int_{\eta}^{\|\mathcal{F}\|_{\infty}-\eta} (\mathcal{H}(\nu_{\vec{0}}) - H(\nu_{\vec{0}})) d\nu_{\vec{0}} \geq c(k)K^{-1}r^2. \quad (6.21)$$

Let $\mathcal{S}$ be the set of $(\nu_{\alpha} : \alpha \in \{0, 1\}^k; \alpha \neq \vec{0}) \in \Omega(\eta)^{\{0, 1\}^k \setminus \{\vec{0}\}}$ for which there exists at least one $\nu'_{\vec{0}}$ such that (6.18) is satisfied. By (6.15), with $t = \eta$, (6.21) and Markov's inequality:

$$\mathcal{L}(\mathcal{S}) \leq C(k)Kr^{-2}\epsilon(\eta, \delta). \quad (6.22)$$

Select the parameter $\delta = \delta(\eta)$ so that $\delta \rightarrow 0$ as $\eta \rightarrow 0$, with a rate sufficiently rapid so that $K(\eta)\epsilon(\eta, \delta(\eta)) \rightarrow 0$ as $\eta \rightarrow 0$; this is possible due to the first property of $\epsilon(\eta, \delta(\eta))$ mentioned above. Then choose $r = r(\eta) \rightarrow 0$ satisfying $r \rightarrow 0$ as $\eta \rightarrow 0$, with a rate sufficiently slow so that $K^2(\eta)(r^{-2}(\eta)\epsilon(\eta, \delta(\eta)))^{1/2^k} \rightarrow 0$ as $\eta \rightarrow 0$. Since $K(\eta) \rightarrow \infty$ as $\eta \rightarrow 0$ and $K^2(\eta)(r^{-2}(\eta)\epsilon(\eta, \delta(\eta)))^{1/2^k} \rightarrow 0$, (6.22) implies $\mathcal{L}(\mathcal{S}) \rightarrow 0$ as $\eta \rightarrow 0$. Introduce

another parameter $\zeta = \zeta(\eta)$ satisfying $\zeta \rightarrow 0$ as $\eta \rightarrow 0$ and $\zeta \geq \eta$, so that $\Omega(\zeta) \subset \Omega(\eta)$.

Let $S = S(\eta)$ to be the set of all $\vec{\nu} = (\nu_{\vec{0}}, (\nu_\alpha : \alpha \in \{0, 1\}^k; \alpha \neq \vec{0})) \in \Omega(\zeta)^{\{0, 1\}^k}$ so that $(\nu_\alpha : \alpha \in \{0, 1\}^k; \alpha \neq \vec{0}) \in \mathcal{S}$. It's clear from definition and (6.22) that $\mathcal{L}(S) \rightarrow 0$ as $\eta \rightarrow 0$.

By the definition of S , if $\vec{\nu} \in \Omega(\zeta)^{\{0, 1\}^k} \setminus S$ then,

$$\mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \geq \mathcal{T}_k(1_{\mathcal{F}_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) - r. \quad (6.23)$$

Recall that $\mathcal{T}_k(1_{\mathcal{F}_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \geq \phi(\zeta)$ if $\nu_\alpha \in \Omega(\zeta)$ and ϕ is a positive continuous function.

Then (6.23) and the general rearrangement inequality imply:

$$\mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k) \geq \phi(\zeta) - r. \quad (6.24)$$

(6.13), with $t = \zeta$, and (6.23) furthermore imply:

$$\begin{aligned} \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) &\geq \mathcal{T}_k(1_{\mathcal{F}_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) - r \\ &\geq \mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k) - r - C(k)\epsilon(\zeta, \delta). \end{aligned} \quad (6.25)$$

We now further require $\zeta = \zeta(\eta)$ tending to zero as $\eta \rightarrow 0$ with a rate sufficiently slow so that $\phi(\zeta)$ also tends to zero slowly and,

$$\frac{r + C(k)\epsilon(\eta, \delta)}{\phi(\zeta) - r} \rightarrow 0 \quad (6.26)$$

$$\frac{K(\eta)^2 r^{-2} \epsilon(\eta, \delta)}{\phi(\zeta) - r} \rightarrow 0. \quad (6.27)$$

(6.24), (6.25) and (6.26) imply:

$$\begin{aligned} \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) &\geq \mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k) - \frac{r + C(k)\epsilon(\eta, \delta)}{\phi(\zeta) - r} \cdot (\phi(\zeta) - r) \\ &\geq (1 - o_\eta(1)) \mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k). \end{aligned} \quad (6.28)$$

Note that (6.28) is precisely (6.2) for the case $\vec{\nu} \in \Omega(\zeta)^{\{0,1\}^k} \setminus S$. We now investigate the exceptional set S . Hence, in addition to the above requirements on ζ , we also enforce $\zeta(\eta) \rightarrow 0$ as $\eta \rightarrow 0$ with a rate sufficiently slow so that $(\mathcal{L}(S))^{1/2^k} / \zeta(\eta) \rightarrow 0$ as $\eta \rightarrow 0$. This allows us to find, if η is sufficiently small, for every $\vec{\nu} \in S \cap \Omega(2\zeta)^{\{0,1\}^k}$ two vectors $\vec{\nu}' \neq \vec{\nu}'' \in \Omega(\zeta)^{\{0,1\}^k} \setminus S$ such that $\nu_\alpha - 2(\mathcal{L}(S))^{1/2^k} \leq \nu''_\alpha \leq \nu_\alpha \leq \nu'_\alpha \leq \nu_\alpha + 2(\mathcal{L}(S))^{1/2^k}$, for all $\alpha \in \{0,1\}^k$. Then for these vectors $\vec{\nu} \in \Omega(2\zeta)^{\{0,1\}^k} \subset \Omega(\eta)^{\{0,1\}^k}$:

$$\begin{aligned} \mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k) &\geq \mathcal{T}_k(1_{F_{\nu'_\alpha}} : \alpha \in \{0,1\}^k) \\ &\geq \mathcal{T}_k(1_{\mathcal{F}_{\nu'_\alpha}} : \alpha \in \{0,1\}^k) - r \geq \mathcal{T}_k(1_{\mathcal{F}_{\nu''_\alpha}} : \alpha \in \{0,1\}^k) - C(\zeta)(\mathcal{L}(S))^{1/2^k} - r \\ &\geq \mathcal{T}_k(1_{F_{\nu''_\alpha}^*} : \alpha \in \{0,1\}^k) - C(\zeta)(\mathcal{L}(S))^{1/2^k} - r - C(k)\epsilon(\eta, \delta). \end{aligned} \quad (6.29)$$

Since level sets are nested, $\mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0,1\}^k)$ is non-increasing in terms of each variable ν_α ; that explains the first inequality in (6.29). The second inequality comes from the definition of the set S . Since $\Omega(2\zeta)$ is a compact set and $\mathcal{F}$ is a centered Gaussian, $\mathcal{T}_k(1_{\mathcal{F}_{\nu_\alpha}} : \alpha \in \{0,1\}^k)$ is Lipschitz continuous with a Lipschitz constant $C(\zeta)$ in each variable $\nu_\alpha \in \Omega(2\zeta)$; hence the third inequality follows. Finally the last comes from (6.13). It remains to show:

$$C(\zeta)(\mathcal{L}(S))^{1/2^k} + r + C(k)\epsilon(\eta, \delta) \leq o_\eta(1)\mathcal{T}_k(1_{F_{\nu''_\alpha}^*} : \alpha \in \{0,1\}^k). \quad (6.30)$$

Indeed, from the definition of $C(\zeta)$ we have $C(\zeta) \leq CK(\zeta) \leq CK(\eta)$, and from the definition of S , we have,

$$C(\zeta)(\mathcal{L}(S))^{1/2^k} \leq C(k)K(\eta)(K(\eta)r^{-2}\epsilon(\eta, \delta))^{1/2^k} \leq C(k)K^2(\eta)(r^{-2}\epsilon(\eta, \delta))^{1/2^k}.$$

Hence (6.24), (6.26), (6.27) and (6.29) then imply (6.30):

$$\begin{aligned} C(\zeta)(\mathcal{L}(S))^{1/2^k} + r + C(k)\epsilon(\eta, \delta) &\leq \frac{C(k)K^2(\eta)(r^{-2}\epsilon(\eta, \delta))^{1/2^k}}{\phi(\zeta) - r} \cdot (\phi(\zeta) - r) \\ &+ \frac{r + C(k)\epsilon(\eta, \delta)}{\phi(\zeta) - r} \cdot (\phi(\zeta) - r) \leq o_\eta(1)\mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k). \end{aligned}$$

Note that $\mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k)$ has a non-increasing property in terms of each variable ν_α , similarly to that of $\mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k)$. This fact, (6.29) and (6.30) together give us the desired conclusion for $\vec{\nu} \in S$:

$$\mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \geq (1 - o_\eta(1))\mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k). \quad (6.31)$$

Finally (6.28) and (6.31) together give the desired conclusion of (6.2).

Remark 6.2: It's important for us to analyze the subset S , even when its measure is at most $o_\eta(1)$, in order to prepare our next discussions. As noted above, the conclusion of **Proposition 6.2** is only applicable when we have the conclusion of **Proposition 6.1** for the diagonal case $\nu_\alpha = \nu$, for all $\alpha \in \{0, 1\}^k$. Hence we can't afford to bypass even a subset of measure zero.

Chapter 7

Conclusion for one dimension

7.1 Preparation

Let $n = 1$. Recall that in the proof of **Proposition 6.1**, the parameter $\delta = \delta(\eta)$ satisfies $\delta \rightarrow 0$ as $\eta \rightarrow 0$. We can choose $\delta(\eta)$ to be a one-to-one function, in which case it allows us to rephrase the findings of the previous two chapters as follows:

Let $\mathcal{F}$ be a centered Gaussian. For every δ there exists $\eta = \eta(\delta)$ satisfying $\eta \rightarrow 0$ as $\delta \rightarrow 0$ such that the following occurs. Suppose $f \in L^{p_k}(\mathbb{R})$ is a nonnegative $(1 - \delta)$ near extremizer with $\|\mathcal{F} - f^*\|_{p_k} \leq \delta \|f\|_{p_k}$. Then for every $s \in \Omega(\eta) = [\eta, \|\mathcal{F}\|_\infty - \eta]$, there exists an interval I_s such that $\mathcal{L}(I_s \Delta F_s) = o_\delta(1) \mathcal{L}(F_s)$, $\mathcal{L}(\mathcal{F}_s \Delta F_s^*) = o_\delta(1)$ and consequently $|\mathcal{L}(\mathcal{F}_s) - \mathcal{L}(F_s)| = o_\delta(1)$.

As hinted, to further analyze the distribution of f , we replace the measurable set F_s with one such corresponding interval I_s . One conclusion of this chapter is that, if I_s and $I_{s'}$ are two intervals such that $s, s' \in \Omega(\eta)$ and $c_s, c_{s'}$ are centers of $I_s, I_{s'}$ respectively, then c_s must be close to $c_{s'}$ in an appropriate sense that will be made clear. To this end, we first argue that a selected tuple $(1_{I_{\nu_\alpha}} : \alpha \in \{0, 1\}^k)$ nearly achieves equality in the rearrangement inequality:

Lemma 7.1. Let δ, η be as above. Let $\vec{\nu} = (\nu_\alpha : \alpha \in \{0, 1\}^k)$ with $\nu_\alpha \in \Omega(\eta)$. There exists $\delta_0 > 0$ such that if $\delta \leq \delta_0$ then:

$$\mathcal{T}_k(1_{I_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \geq (1 - o_\delta(1)) \mathcal{T}_k(1_{I_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k).$$

Proof: Since $\mathcal{L}(I_s^* \Delta F_s^*) \leq \mathcal{L}(I_s \Delta F_s)$, $\mathcal{L}(I_s^* \Delta F_s^*) = o_\delta(1) \mathcal{L}(F_s)$ if $s \in \Omega(\eta)$. **Proposition 6.1** and the Gowers product inequality then imply the following three inequalities:

$$\begin{aligned} |\mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) - \mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k)| &= o_\delta(1) \prod_{\alpha \in \{0, 1\}^k} (\mathcal{L}(F_{\nu_\alpha}))^{1/p_k} \\ |\mathcal{T}_k(1_{F_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) - \mathcal{T}_k(1_{I_{\nu_\alpha}} : \alpha \in \{0, 1\}^k)| &= o_\delta(1) \prod_{\alpha \in \{0, 1\}^k} (\mathcal{L}(F_{\nu_\alpha}))^{1/p_k} \\ |\mathcal{T}_k(1_{F_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k) - \mathcal{T}_k(1_{I_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k)| &= o_\delta(1) \prod_{\alpha \in \{0, 1\}^k} (\mathcal{L}(F_{\nu_\alpha}))^{1/p_k} \end{aligned}$$

which then give us:

$$\mathcal{T}_k(1_{I_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \geq \mathcal{T}_k(1_{I_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k) - o_\delta(1) \prod_{\alpha \in \{0, 1\}^k} (\mathcal{L}(F_{\nu_\alpha}))^{1/p_k}. \quad (7.1)$$

Since $\mathcal{L}(I_{\nu_\alpha}) \geq (1 - o_\delta(1)) \mathcal{L}(\mathcal{F}_{\nu_\alpha}) \geq (1 - o_\delta(1)) \mathcal{L}(\mathcal{F}_{\|\mathcal{F}\|_\infty - \eta}) \geq \mathcal{L}(I_{\|\mathcal{F}\|_\infty - \eta})$, this means

$I_{\|\mathcal{F}\|_\infty - \eta}^* \subset I_{\nu_\alpha}^*$, for $\nu_\alpha \in \Omega(\eta)$ and $\alpha \in \{0, 1\}^k$. Define $\psi(\eta) = \mathcal{T}_k(1_{I_{\|\mathcal{F}\|_\infty - \eta}^*}, \dots, 1_{I_{\|\mathcal{F}\|_\infty - \eta}^*})$.

Then the said set inclusion implies:

$$\mathcal{T}_k(1_{I_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k) \geq \mathcal{T}_k(1_{I_{\|\mathcal{F}\|_\infty - \eta}^*}, \dots, 1_{I_{\|\mathcal{F}\|_\infty - \eta}^*}) = \psi(\eta). \quad (7.2)$$

Note that $\mathcal{L}(\mathcal{F}_\eta) = C(\log(1/\eta))^C$ and that $\psi(\eta)$ stays strictly bounded below as long as $\eta > 0$. We select $\eta = \eta(\delta) \rightarrow 0$ sufficiently slowly as $\delta \rightarrow 0$ and $\delta_0 > 0$ so that if $\delta \leq \delta_0$ then $o_\delta(1)(\log(1/\eta(\delta)))^C = o_\delta(1)$. Then (7.2) gives us:

$$\begin{aligned} -o_\delta(1) \prod_{\alpha \in \{0, 1\}^k} (\mathcal{L}(F_{\nu_\alpha}))^{1/p_k} &\geq -o_\delta(1)(\mathcal{L}(\mathcal{F}_\eta))^{k+1} \geq -o_\delta(1)(\log(1/\eta))^C \geq -o_\delta(1)\psi(\eta) \\ &= -o_\delta(1)\mathcal{T}_k(1_{I_{\|\mathcal{F}\|_\infty - \eta}^*}, \dots, 1_{I_{\|\mathcal{F}\|_\infty - \eta}^*}) \geq -o_\delta(1)\mathcal{T}_k(1_{I_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k). \end{aligned} \quad (7.3)$$

A combination of (7.1) and (7.3) then give the desired conclusion.

We will show in **Section 7.4** below that **Lemma 7.1** implies the centers $c_{\nu_\alpha}, c_{\nu_\beta}$ are close to each other, for $\alpha, \beta \in \{0, 1\}^k$ and $\nu_\alpha, \nu_\beta \in \Omega(\eta)$. We now replace a near extremizer with a superposition of interval approximations of its super-level sets. Suppose there exist a sequence $\delta_i \rightarrow 0$ as $i \rightarrow \infty$, a nonnegative sequence of functions $\{f_i\}_i$ and a centered Gaussian extremizer $\mathcal{F}$, such that $\|f_i\|_{U^k} \geq (1 - \delta_i)A(k)\|f_i\|_{p_k}$ and $\|\mathcal{F} - f_i^*\|_{p_k} \leq \delta_i\|f_i\|_{p_k}$. Assume $\|f_i\|_{p_k} = 1$ for all i . Let $s \in \mathbb{R}_{>0}$. Denote $F_{i,s} = \{f_i > s\}$ and $F_{i,s}^* = \{f_i^* > s\}$. Consider another sequence $\eta_i \rightarrow 0$ as $i \rightarrow \infty$ and suppose further that, if $s \in [\eta_i, \|\mathcal{F}\|_\infty - \eta_i]$, then there exists an interval $I_{i,s}$ such that $\mathcal{L}(I_{i,s} \Delta F_{i,s}) \leq \delta_i \mathcal{L}(F_{i,s})$. Let $h_i(x) = \int_{\eta_i}^{\|\mathcal{F}\|_\infty - \eta_i} 1_{I_{i,s}}(x) ds$.

Lemma 7.2. Let f_i and h_i be as above. Then $\|f_i - h_i\|_{p_k} \rightarrow 0$ as $i \rightarrow \infty$.

Proof: By **Lemma 5.3** and Minkowski's integral inequality,

$$\begin{aligned}
 \|f_i - h_i\|_{p_k} &\leq \left\| \int_{\eta_i}^{\|\mathcal{F}\|_{\infty} - \eta_i} 1_{F_{i,s}} ds - \int_{\eta_i}^{\|\mathcal{F}\|_{\infty} - \eta_i} 1_{I_{i,s}} ds \right\|_{p_k} + C\eta_i(\log(1/\eta_i))^C \\
 &\leq \int_{\eta_i}^{\|\mathcal{F}\|_{\infty} - \eta_i} \|1_{F_{i,s}} - 1_{I_{i,s}}\|_{p_k} ds + C\eta_i(\log(1/\eta_i))^C \\
 &= \int_{\eta_i}^{\|\mathcal{F}\|_{\infty} - \eta_i} (\mathcal{L}(I_{i,s} \Delta F_{i,s}))^{1/p_k} ds + C\eta_i(\log(1/\eta_i))^C \\
 &\leq \delta_i \int_{\eta_i}^{\|\mathcal{F}\|_{\infty} - \eta_i} (\mathcal{L}(F_{i,s}))^{1/p_k} ds + C\eta_i(\log(1/\eta_i))^C \leq \delta_i + C\eta_i(\log(1/\eta_i))^C.
 \end{aligned}$$

The last inequality is due to $\|f_i\|_{p_k} = 1$. Let $i \rightarrow \infty$, we get the conclusion.

By **Lemma 7.2**, if we can establish that $\{h_i\}_i$ is precompact in L^{p_k} , then we have the same result for $\{f_i\}_i$. We show here a related compactness result which will be needed.

Lemma 7.3. Let $a < b \in \mathbb{R}$ and $1 \leq q < \infty$. Let $[-B, B]$ be a closed interval. For each $i \in \mathbb{Z}_{>0}$ and $s \in \mathbb{R}$ let $I_{i,s} \subset [-B, B]$ be an interval. Suppose further that the function $(x, s) \mapsto 1_{I_{i,s}}(x)$ is measurable. Then $\{\int_a^b 1_{I_{i,s}}(x) ds\}_i$ is precompact in L^q .

Proof: Let $g_i(x) = \int_a^b 1_{I_{i,s}}(x) ds$. It's clear that, there exists $C > 0$ such that $\|g_i\|_q \leq C$.

Moreover, $\int_{\{|x|>r\}} |g_i(x)|^q dx = 0$, for every $r > B$, and $\lim_{h \rightarrow 0} \|T^h g_i - g_i\|_q = 0$ uniformly in i . Hence by the Fréchet-Kolmogorov theorem [3], $\{g_i\}_i$ is precompact in L^q .

7.2 A monotonicity result

Let $I = [-1, 1]$ and $J = [-\eta - 1, 1 + \eta]$ for some $\eta \in [0, \frac{2}{k-1}]$. Define:

$$\phi(t) = \int_{\mathbb{R}^{k+1}} 1_{J+t}(x) \prod_{\alpha \in \{0,1\}^k; \alpha \neq \vec{0}} 1_I(x + \alpha \cdot \vec{h}) d\vec{h} dx.$$

$\phi(t)$ is a continuous, nonnegative even function of t and has a compact support. Furthermore, let:

$$H(x) = \int_{\mathbb{R}^k} \prod_{\alpha \in \{0,1\}^k; \alpha \neq \vec{0}} 1_I(x + \alpha \cdot \vec{h}) d\vec{h}.$$

Then $\phi(t) = \int_{\mathbb{R}} 1_{J+t} \cdot H(x) dx$. H is also a continuous, nonnegative even function whose support is the interval $[-\frac{k+1}{k-1}, \frac{k+1}{k-1}]$. Indeed, the interval $[-1, 1]$ is clearly contained in the support of H . Suppose $x > 1$ and $x \in \text{spt}(H)$. Then there exists $\vec{h} = (h_i)_i \in \mathbb{R}^k$ such that $|x + \alpha \cdot \vec{h}| \leq 1$ for all $\alpha \in \{0, 1\}^k$ (there exists, in fact, a set of such $\vec{h}$ of positive measure). Let $j \in \{1, \dots, k\}$ and define $\beta \in \{0, 1\}^k$ by $\beta_j = 1$ and $\beta_i = 0$ if $i \neq j$. Then $|x + \beta \cdot \vec{h}| = |x + h_j| \leq 1$ implies $h_j \leq 1 - x$. Since this holds for every $j \in \{1, \dots, k\}$, $k^{-1} \sum_{i=1}^k h_i \leq 1 - x$. Now consider $\alpha = \vec{1} = (1, \dots, 1)$. Then $|x + \vec{1} \cdot \vec{h}| = |x + \sum_{i=1}^k h_i| \leq 1$ implies $k^{-1} \sum_{i=1}^k h_i \geq -k^{-1}(1 + x)$. By transitivity, $-k^{-1}(1 + x) \leq 1 - x$, or equivalently $x \leq \frac{k+1}{k-1}$. Similarly, if $x < -1$ and $x \in \text{spt}(H)$ then $x \geq -\frac{k+1}{k-1}$.

Now suppose $x \in [-\frac{k+1}{k-1}, \frac{k+1}{k-1}]$. We define $\vec{h} = (h_i)_i \in \mathbb{R}^k$ such that $h_i \leq \frac{1-x}{k}$ if $1 < x \leq \frac{k+1}{k-1}$ and $h_i \leq \frac{x-1}{k}$ if $-\frac{k+1}{k-1} \leq x < -1$. For the former case, $-1 \leq x(1-k) + k \leq x + \alpha \cdot \vec{h} \leq x + 1 - x = 1$, for all $\alpha \in \{0, 1\}^k$; hence $x \in \text{spt}(H)$. We also obtain the same conclusion for the latter case.

We claim that $\phi(t)$ is a strictly decreasing function of $t \geq 0$ in its support. We make a quick remark that this claim and the fact that $\text{spt}(H) = [-\frac{k+1}{k-1}, \frac{k+1}{k-1}]$ are why we take $\eta \in [0, \frac{2}{k-1}]$. Since if $\eta > \frac{2}{k-1}$ then there exist $t_1 > t_0$ sufficiently small, so that they are both in the support of ϕ , say, $t_1 = (\eta - \frac{2}{k-1})/10 > t_0 = (\eta - \frac{2}{k-1})/20$, such that,

$$\begin{aligned} \phi(t_1) &= \int_{\mathbb{R}^{k+1}} 1_{J+t_1}(x) \prod_{\alpha \in \{0,1\}^k; \alpha \neq 0} 1_I(x + \alpha \cdot \vec{h}) d\vec{h} dx \\ &= \int_{\mathbb{R}^{k+1}} 1_{J+t_0}(x) \prod_{\alpha \in \{0,1\}^k; \alpha \neq 0} 1_I(x + \alpha \cdot \vec{h}) d\vec{h} dx = \phi(t_0). \end{aligned}$$

With our choice of η , $\text{spt}(\phi) \subset [-\frac{2(k+1)}{k-1}, \frac{2(k+1)}{k-1}]$. Note that:

$$(d/dt)\phi(t) = \int_{\mathbb{R}} (\delta_{t+1+\eta} - \delta_{t-1-\eta}) \cdot H(x) dx = H(t+1+\eta) - H(t-1-\eta).$$

Furthermore, with our choice of η , $t-1-\eta$ always lies in the support of H . Hence our claim on the monotonicity of ϕ will follow if we can show $H(x)$ is strictly decreasing for $x \geq 0$ in its support. Let $\mathcal{C} = \{(x, \vec{h}) \in \mathbb{R}^{k+1} : x + \alpha \cdot \vec{h} \in I, \forall \alpha \in \{0,1\}^k\}$. Then $\mathcal{C}$ is a compact, convex, balanced subset of $\mathbb{R}^{k+1}$. Let $\mathcal{C}_x = \{\vec{h} \in \mathbb{R}^k : (x, \vec{h}) \in \mathcal{C}\}$. We observe that $H(x) = \mathcal{L}(\mathcal{C}_x)$.

Lemma 7.4. Let H be as above. Then $H(0) > H(x)$, for all $x > 0$.

Lemma 7.4, if assumed true, will imply that $H(x) > H(y)$ if $0 \leq x < y$ and x, y both lie in the support of K . Indeed, let $0 < x < y$ be in the interior of the support of H , then $H(x) = \mathcal{L}(\mathcal{C}_x) > 0$ and $H(y) = \mathcal{L}(\mathcal{C}_y) > 0$. Let $t \in (0, 1)$ be such that $x = (1-t) \cdot 0 + ty$. By the convexity of $\mathcal{C}$, $\mathcal{C}_x \supset (1-t)\mathcal{C}_0 + t\mathcal{C}_y$. Then by the Brunn-Minkowski inequality,

$\mathcal{L}(C_x) \geq (\mathcal{L}(C_0))^{1-t} (\mathcal{L}(C_y))^t$, and by **Lemma 7.4**:

$$H(x) \geq H(0)^{1-t} H(y)^t > H(y)^{1-t} H(y)^t = H(y).$$

7.3 Proof of Lemma 7.4

We will deduce **Lemma 7.4** from the following more general result. The setup is as follows:

Let $M, N \in \mathbb{Z}_{>0}$. For $i \in \{1, \dots, M\}$, let J_i be a closed interval centered at 0, $\mathcal{L}(J_i) = l_i \in \mathbb{R}_{>0}$, and $L_i : \mathbb{R}^N \rightarrow \mathbb{R}$ be surjective linear mappings. For $\vec{t} = (t_i)_i \in \mathbb{R}^M$, we let:

$$\Psi(\vec{t}) = \int_{\mathbb{R}^N} \prod_{i=1}^M 1_{J_i+t_i}(L_i(\vec{y})) d\vec{y}.$$

We make two observations. First, if $\vec{v} \in \mathbb{R}^N$:

$$\int_{\mathbb{R}^N} \prod_{i=1}^M 1_{J_i}(L_i(\vec{y} + \vec{v})) d\vec{y} = \int_{\mathbb{R}^N} \prod_{i=1}^M 1_{J_i}(L_i(\vec{y})) d\vec{y}.$$

Secondly, by the general rearrangement inequality, for $\vec{t} = (t_i)_i \in \mathbb{R}^M$:

$$\int_{\mathbb{R}^N} \prod_{i=1}^M 1_{J_i+t_i}(L_i(\vec{y})) d\vec{y} \leq \int_{\mathbb{R}^N} \prod_{i=1}^M 1_{J_i}(L_i(\vec{y})) d\vec{y}$$

or equivalently, $\Psi(\vec{t}) \leq \Psi(\vec{0})$. It then follows that, for $\vec{t} \in \mathbb{R}^M$, $\Psi(\vec{t}) = \Psi(\vec{0})$ if there exists $\vec{v} \in \mathbb{R}^N$ such that $L_i(\vec{v}) = t_i$, $i \in \{1, \dots, M\}$. It will be shown that this is also a necessary condition, provided that $(L_i, l_i)_{i=1}^M$ is **an admissible tuple**:

Admissibility: For $i \in \{1, \dots, M\}$, let J_i be a closed interval centered at 0, $\mathcal{L}(J_i) = l_i \in \mathbb{R}_{>0}$ and $L_i : \mathbb{R}^N \rightarrow \mathbb{R}$ be surjective linear mappings. Let $\vec{l} = (l_i)_i \in (\mathbb{R}_{>0})^M$. Define $\mathcal{K}_{\vec{l}} = \{\vec{x} \in \mathbb{R}^N : |L_i(\vec{x})| \leq l_i, \forall i \in \{1, \dots, M\}\}$. We said that $(L_i, l_i)_i$ is **an admissible tuple** if for every $m \in \{1, \dots, M\}$ there exists $\vec{x}_m \in \mathcal{K}_{\vec{l}}$ such that $|L_m(\vec{x}_m)| = l_m$.

Lemma 7.5. [9] Let L_i, J_i, l_i and Ψ be as above. Suppose $(L_i, l_i)_{i=1}^M$ is an admissible tuple. Let $\vec{t} \in \mathbb{R}^M$. Then $\Psi(\vec{0}) \geq \Psi(\vec{t})$ and equality holds iff there exists $\vec{v} \in \mathbb{R}^N$ such that $L_i(\vec{v}) = t_i$ for all $i \in \{1, \dots, M\}$.

Proof: Let $K = \{(\vec{x}, \vec{t}) \in \mathbb{R}^N \times \mathbb{R}^M : L_i(\vec{x}) \in J_i + t_i, \forall i \in \{1, \dots, M\}\}$. For each $\vec{t} \in \mathbb{R}^M$, let $K(\vec{t}) = \{\vec{x} \in \mathbb{R}^N : (\vec{x}, \vec{t}) \in K\}$. It's clear that K is convex, and if $(\vec{x}, \vec{t}) \in K$ then $|L_i(\vec{x}) - t_i| \leq l_i$. Since $\Psi(\vec{t})$ represents the N -dimensional volume of $K(\vec{t})$ and J_i are centered at 0, $\Psi(\vec{t}) = \Psi(-\vec{t})$ or $\mathcal{L}(K(\vec{t})) = \mathcal{L}(K(-\vec{t}))$. Moreover,

$$K(\vec{0}) \supset (1/2)K(\vec{t}) + (1/2)K(-\vec{t}). \quad (7.4)$$

Suppose that $\Psi(\vec{t}) = \Psi(\vec{0})$ for some $\vec{t} \in \mathbb{R}^M$, which implies $\Psi(\vec{0}) = \mathcal{L}(K(\vec{0})) = \Psi(\vec{t}) = (\mathcal{L}(K(\vec{t})))^{1/2}(\mathcal{L}(K(-\vec{t})))^{1/2}$. Then (7.4) and the Brunn-Minkowski inequality imply:

$$\mathcal{L}(K(\vec{0})) \geq \mathcal{L}((1/2)K(\vec{t}) + (1/2)K(-\vec{t})) \geq (\mathcal{L}(K(\vec{t})))^{1/2}(\mathcal{L}(K(-\vec{t})))^{1/2} = \mathcal{L}(K(\vec{0}))$$

which yields $\mathcal{L}((1/2)K(\vec{t}) + (1/2)K(-\vec{t})) = (\mathcal{L}(K(\vec{t})))^{1/2}(\mathcal{L}(K(-\vec{t})))^{1/2}$. Hence by the characterization of equality case of the Brunn-Minkowski inequality, there exists $\vec{v} \in \mathbb{R}^N$ such that $K(\vec{t}) + \vec{v} = K(\vec{0})$. By definition, this means, if $\vec{x} \in \mathbb{R}^N$ and $i \in \{1, \dots, M\}$:

$$L_i(\vec{x}) \in J_i + t_i \iff L_i(\vec{x} + \vec{v}) \in J_i. \quad (7.5)$$

Let $\vec{z} = \vec{x} + \vec{v}$, then by (7.5), $|L_i(\vec{z})| \leq l_i$ implies,

$$|L_i(\vec{z}) - t_i - L_i(\vec{v})| \leq l_i. \quad (7.6)$$

Fix $m \in \{1, \dots, M\}$. Admissibility assumption implies that there exist $\vec{x}_{m,\pm} \in \mathbb{R}^N$ such that $|L_i(\vec{x}_{m,\pm})| \leq l_i$ and $L_m(\vec{x}_{m,\pm}) = \pm l_m$. In particular, (7.6) implies

$$|L_m(\vec{x}_{m,\pm}) - t_m - L_m(\vec{v})| \leq l_m. \quad (7.7)$$

Suppose $t_m + L_m(\vec{v}) < 0$. Then $L_m(\vec{x}_{m,+}) - t_m - L_m(\vec{v}) = l_m - (t_m + L_m(\vec{v})) > l_m$, which poses a contradiction to (7.7). Similarly, $t_m + L_m(\vec{v}) > 0$ will also imply the contradiction to (7.7) since $L_m(\vec{x}_{m,-}) - (t_m + L_m(\vec{v})) = -l_m - (t_m + L_m(\vec{v})) < -l_m$. Hence $t_m = L_m(-\vec{v})$ for every $m \in \{1, \dots, M\}$. We've completed the proof of the lemma as the “if” direction is apparent by the discussion in the beginning of this section.

To finish the proof of **Lemma 7.4**, let $M = 2^k - 1$, $N = k$. We note that for each $\vec{0} \neq \alpha \in \{0, 1\}^k$, $L_\alpha : \mathbb{R}^k \rightarrow \mathbb{R}$ defined by $L_\alpha(\vec{h}) = \alpha \cdot \vec{h}$ is a surjective linear mapping. Moreover, for each $\vec{0} \neq \alpha \in \{0, 1\}^k$, let $|\alpha| = \sum_{i=1}^k \alpha_i$ and $\vec{h}_\alpha \in \mathbb{R}^k$ be such that $(\vec{h}_\alpha)_i = 1/|\alpha|$ if $\alpha_i = 1$ and $(\vec{h}_\alpha)_i = 0$ otherwise. Then it's easily checked that $L_\alpha(\pm \vec{h}_\alpha) = \pm 1$ and $|L_\beta(\vec{h}_\alpha)| \leq 1$ if $\alpha \neq \beta \in \{0, 1\}^k \setminus \{\vec{0}\}$. Hence the tuple $((L_\alpha, 1)_\alpha : \alpha \in \{0, 1\}^k \setminus \{\vec{0}\})$ is admissible. Note that with $\vec{x} = (x)_i \in \mathbb{R}^{\{0,1\}^k \setminus \{\vec{0}\}}$:

$$H(x) = \int_{\mathbb{R}^k} \prod_{\alpha \in \{0,1\}^k; \alpha \neq \vec{0}} 1_I(x + \alpha \cdot \vec{h}) d\vec{h} = \int_{\mathbb{R}^k} \prod_{\alpha \in \{0,1\}^k; \alpha \neq \vec{0}} 1_{I-x}(\alpha \cdot \vec{h}) d\vec{h} = \Psi(-\vec{x}).$$

Then from **Lemma 7.6**, $H(0) = \Psi(\vec{0}) \geq \Psi(-\vec{x}) = H(x)$. If $H(0) = H(x)$ for some $x > 0$ then there must exist $\vec{v} \in \mathbb{R}^k$ such that $L_\alpha(\vec{v}) = \alpha \cdot \vec{v} = -x$, for all $\vec{0} \neq \alpha \in \{0, 1\}^k$, which is clearly impossible. Hence $H(0) > H(x)$ for all $x > 0$, as we wish to conclude.

7.4 Alignment of super-level sets

Section 7.2 concludes that, if I and J are two intervals “compatible” in size, and if $\mathcal{T}_k(J, I, \dots, I) = \int_{\mathbb{R}^{k+1}} 1_I(x) \prod_{\alpha \in \{0, 1\}^k; \alpha \neq \vec{0}} 1_I(x + \alpha \cdot \vec{h}) d\vec{h} dx$ is nearly maximized over all tuples of intervals of the same sizes, then the centers of I and J must be “close” to each other. The compatibility condition is given by allowing $\mathcal{L}(J) = \mathcal{L}(I) + \eta$ with $\eta \in [0, \frac{2}{k-1} \mathcal{L}(I)]$.

Let $0 < \epsilon \ll \frac{2}{k-1}$. Then the discussion in the beginning of **Section 7.1** and **Lemma 7.1** conclude that there exist $\delta > 0$ and $\eta = \eta(\delta) > 0$ satisfying the following properties:

- 1) If f is a nonnegative $(1 - \delta)$ near extremizer and $\|\mathcal{F} - f\|_{p_k} \leq \delta$ for a centered Gaussian $\mathcal{F}$, then for every $s \in \Omega(\eta) = [\eta, \|\mathcal{F}\|_\infty - \eta]$ there exists an interval I_s such that $\mathcal{L}(I_s \Delta F_s) < c\epsilon \mathcal{L}(F_s)$ and $|\mathcal{L}(\mathcal{F}_s) - \mathcal{L}(F_s)| < c\epsilon$.
- 2) Furthermore, $\mathcal{T}_k(1_{I_{\nu_\alpha}} : \alpha \in \{0, 1\}^k) \geq (1 - \epsilon) \mathcal{T}_k(1_{I_{\nu_\alpha}^*} : \alpha \in \{0, 1\}^k)$, if $\nu_\alpha \in \Omega(\eta)$, $\alpha \in \{0, 1\}^k$.

Let $\mathcal{F}_s$ denotes the super-level set of the Gaussian $\mathcal{F}$ associated with the value s . There exist $s_1 = \eta < s_2 < \dots < s_N = \|\mathcal{F}\|_\infty - \eta \in [\eta, \|\mathcal{F}\|_\infty - \eta]$ with $N = N(\eta)$ such that $\mathcal{L}(\mathcal{F}_{s_i} \Delta \mathcal{F}_{s_{i+1}}) \asymp \frac{1}{k-1} \mathcal{L}(\mathcal{F}_{s_i})$ for all $i \in \{1, \dots, N-1\}$. Since super-level sets are nested,

$\mathcal{L}(\mathcal{F}_s \Delta \mathcal{F}_{s_i}) \leq \frac{c}{k-1} \mathcal{L}(\mathcal{F}_{s_i})$, c being some sufficiently small constant, if $s_i \leq s \leq s_{i+1}$, and consequently, $|\mathcal{L}(I_s) - \mathcal{L}(I_{s_i})| \leq \frac{c}{k-1} \mathcal{L}(I_{s_i})$, $i \in \{1, \dots, N-1\}$. That is to say, the size compatibility condition is also satisfied by the sub-intervals with $s \in [s_i, s_{i+1}] \subset \Omega(\eta)$. Recall that c_s denotes the center of an interval I_s . If ϵ is sufficiently small, then by the second property, the previous paragraph and the satisfaction of the compatibility condition, $|c_s - c_{s_i}| = o_\epsilon(1) \mathcal{L}(I_{s_i})$ if $s_i \leq s \leq s_{i+1}$, $i \in \{1, \dots, N-1\}$. If $s \in \Omega(\eta)$, then s must lie in one such interval $[s_i, s_{i+1}]$; hence $|c_s - c_\eta| = o_\epsilon(1) \mathcal{L}(I_\eta) = o_\epsilon(1)(\log(1/\eta))^C$ for $s \in \Omega(\eta)$ since $\mathcal{L}(\mathcal{F}_\eta) = C(\log(1/\eta))^C$. Note that if $\epsilon \rightarrow 0$ then $\delta \rightarrow 0$, and if $\eta' > \eta$ then $\Omega(\eta) \subset \Omega(\eta')$. We now require $\eta = \eta(\delta) \rightarrow 0$ sufficiently slow so that $|c_s - c_\eta| = o_\epsilon(1) \mathcal{L}(I_\eta) = o_\epsilon(1)(\log(1/\eta))^C = o_\delta(1)$ as $\delta \rightarrow 0$. The intervals I_s might change as the parameters change, but the size estimates still hold. We now obtain the following lemma:

Lemma 7.7. Let $\delta, \eta = \eta(\delta)$ and c_s be as above. Then $|c_s - c_\eta| = o_\delta(1)$ if $s \in \Omega(\eta)$.

7.5 A compactness result

Suppose we have a nonnegative extremizing sequence $\{f_i\}_i$ such that $\|f_i\|_{p_k} = 1$ and

$$\|f_i\|_{U^k} \geq (1 - \delta_i) A(k) \|f_i\|_{p_k} = (1 - \delta_i) A(k)$$

and that $\|\mathcal{F} - f_i^*\|_{p_k} \leq \delta_i$ for some centered Gaussian $\mathcal{F}$ with $\delta_i \rightarrow 0$ as $i \rightarrow \infty$. Recall that in **Section 7.1** we defined for such a sequence $\{f_i\}_i$ an associate superposition sequence $h_i(x) = \int_{\eta_i}^{\|\mathcal{F}\|_{\infty} - \eta_i} 1_{I_{i,s}}(x) ds$. We select $\eta_i = \eta(\delta_i)$ satisfying $\eta_i \rightarrow 0$ as $i \rightarrow \infty$ such that

$|c_s - c_{\eta_i}| = o_{\delta_i}(1)$ if $s \in \Omega(\eta_i)$, as in **Section 7.4**. The conclusions of **Lemma 7.2** and **Lemma 7.3** stay unchanged with this selection of $\{\eta_i\}_i$.

Remark 7.1: It's possible to select I_{i,s_i} so that $(x, s_i) \mapsto 1_{I_{i,s_i}}(x)$ is a measurable function. Fix i and denote η_i as η , f_i as f , F_{i,s_i} as F_s and I_{i,s_i} as I_s . The set $E = \cup_{s \in \Omega(\eta)} \{s\} \times F_s$ is a measurable subset of $\Omega(\eta) \times \mathbb{R}$. Let $a_s < b_s$ denote the endpoints of I_s , $s \in \Omega(\eta)$, then it comes down to the ability to select the endpoints a_s, b_s of the intervals I_s in a measurable manner. We suppose for a moment that f is a continuous function. Decompose the range of values $\Omega(\eta)$ into a finite number $C_f(\eta)$ of smaller sub-ranges of values $\Omega'(\eta)$ such that $\mathcal{L}(F_s \Delta F_{s'}) = o_\eta(1)$ for each pair $s, s' \in \Omega'(\eta)$. Fix one such sub-range $\Omega'(\eta)$ and let F_{s_t}, F_{s_b} denote the super-level sets of f associated with the largest and smallest values of the range, respectively. By replacing F_{s_t} with I_{s_t} and F_{s_b} with I_{s_b} , we assume F_{s_t} and F_{s_b} are both intervals. Let a_{s_t} denote the left endpoint of F_{s_t} and a_{s_b} the left endpoint of F_{s_b} ; note that $|a_{s_t} - a_{s_b}| = C_f o_\eta(1)$. Consider the part of the graph of f inside the rectangle box $\Omega'(\eta) \times [a_{s_t}, a_{s_b}]$; call this set R_η . By the Measurable Choice Theorem [2], there exists a measurable function $s \mapsto a(s)$, so that $(s, a(s)) \in R_\eta$, for every $s \in \Omega'(\eta)$. Take $a(s)$ to be the left endpoint of our interval I_s . Proceed similarly to obtain a measurable function $s \mapsto b(s)$ for the right endpoint of I_s . Note that by construction $\mathcal{L}(I_s \Delta F_s) = o_\eta(1)\mathcal{L}(F_s)$, if $s \in \Omega'(\eta)$. Continue this procedure for each of these sub-ranges $\Omega'(\eta)$ and concatenate the obtained left endpoint and right endpoint functions to obtain a measurable function $(x, s) \mapsto 1_{I_s}(x)$, $s \in \Omega(\eta)$. For the general case we approximate f with a positive continuous function g

so that $\|f - g\|_{p_k} = \rho(\eta)$ with $\rho \ll \eta$ in order for us to have $\mathcal{L}(G_s \Delta F_s) = o_\eta(1)$, for a.e $s \in \Omega(\eta)$. Then apply the described procedure with G_s in place of F_s .

Another and easier way is to construct a piecewise constant function $(x, s) \mapsto I_s(x)$; ie, $I_s(x) = I_{s'}(x)$ if $s, s' \in \Omega'(\eta)$. This ensures measurability but might come at the expense of increasing the (still finite) number of sub-ranges so that $\mathcal{L}(I_s \Delta F_s) = o_\eta(1)\mathcal{L}(F_s)$ is still guaranteed.

Proposition 7.8. Let $\{h_i\}_i$ be as above. There exists $\{a_i\}_i$ such that $\{h_i(\cdot - a_i)\}_i$ is precompact in L^{p_k} .

Proof: Let $a_i = c_{\eta_i}$ with c_{η_i} being the center of the interval I_{η_i} . Define:

$$g_i(x) = h_i(x - a_i) = \int_{\eta_i}^{\|\mathcal{F}\|_\infty - \eta_i} 1_{I_{i,s}}(x - a_i) ds.$$

Let $\epsilon > 0$ be a small number and select $\eta > 0$ such that:

$$\left\| \int_0^\eta 1_{\mathcal{F}_s} ds \right\|_{p_k} + \left\| \int_{\|\mathcal{F}\|_\infty - \eta}^{\|\mathcal{F}\|_\infty} 1_{\mathcal{F}_s} ds \right\|_{p_k} \leq \epsilon. \quad (7.8)$$

For simplicity, we assume that $\eta_i \leq \eta$ for all i . Recall that $|\mathcal{L}(\mathcal{F}_s) - \mathcal{L}(I_{i,s})| = o_{\delta_i}(1)\mathcal{L}(\mathcal{F}_s)$ if $s \in \Omega(\eta_i)$. Then (7.8) entails:

$$\left\| \int_{\eta_i}^\eta 1_{I_{i,s}}(\cdot - a_i) ds \right\|_{p_k} + \left\| \int_{\|\mathcal{F}\|_\infty - \eta_i}^{\|\mathcal{F}\|_\infty - \eta} 1_{I_{i,s}}(\cdot - a_i) ds \right\|_{p_k} \leq C\epsilon. \quad (7.9)$$

The fact that $|\mathcal{L}(\mathcal{F}_s) - \mathcal{L}(I_{i,s})| = o_{\delta_i}(1)\mathcal{L}(\mathcal{F}_s)$ also implies that the intervals $I_{i,s}(x - a_i)$ are contained within some compact interval $[-B, B]$ if $s \in \Omega(\eta) = [\eta, \|\mathcal{F}\|_\infty - \eta] \subset \Omega(\eta_i) =$

$[\eta_i, \|\mathcal{F}\|_\infty - \eta_i]$. By **Lemma 7.3**, there exist a subsequence of $\{\int_\eta^{\|\mathcal{F}\|_\infty - \eta} 1_{I_{i,s}}(\cdot - a_i) ds\}_i$, for which we still use the same subscript, and $\mathcal{G} \in L^{p_k}(\mathbb{R})$, such that,

$$\lim_{i \rightarrow \infty} \left\| \int_\eta^{\|\mathcal{F}\|_\infty - \eta} 1_{I_{i,s}}(\cdot - a_i) ds - \mathcal{G} \right\|_{p_k} = 0. \quad (7.10)$$

Combining (7.9) with (7.10), we have:

$$\limsup_{i \rightarrow \infty} \|h_i(\cdot - a_i) - \mathcal{G}\|_{p_k} \leq C\epsilon.$$

Since this holds for every ϵ , we have the conclusion.

From the conclusions of **Lemma 7.2** and **Proposition 7.8**:

$$\|f_i(\cdot - a_i) - \mathcal{G}\|_{p_k} \leq \|(f_i - h_i)(\cdot - a_i)\|_{p_k} + \|h_i(\cdot - a_i) - \mathcal{G}\|_{p_k} \rightarrow 0 \quad (7.11)$$

as $i \rightarrow \infty$, for some $\mathcal{G} \in L^{p_k}(\mathbb{R})$. Since $\{f_i\}_i$ is an extremizing sequence:

$$\|f_i(\cdot - a_i)\|_{U^k} \geq (1 - \delta_i)A(k)\|f_i(\cdot - a_i)\|_{p_k} \quad (7.12)$$

(7.11), (7.12) and Gowers-Host-Kra norm inequality give:

$$\|\mathcal{G}\|_{U^k} = A(k)\|\mathcal{G}\|_{p_k}$$

which means $\mathcal{G}$ must be a Gaussian, by the characterization of nonnegative extremizers of the Gowers-Host-Kra norm inequality. We've now finished the proof of **Theorem 1.1** for nonnegative near extremizers in one dimension.

7.6 A remark on admissibility

Admissibility plays a central role, as demonstrated, in obtaining our result in **Section 7.3**. Admissibility condition highlights the interrelations between the intervals involved, in terms of their centers and lengths. For instance, it's equivalent to the condition that the functional Ψ strictly decreases once a subset of M centered intervals J_i is translated. Admissibility can be considered a boundary case of strict admissibility, whose definition speaks of the following: The lengths l_i must be selected so as to the condition $|L_m(\vec{x})| \leq l_m$ is not redundant for any particular m . In the case $N = 2$, $M = 3$ and $L_1((x_1, x_2)) = x_1, L_2((x_1, x_2)) = x_2, L_3((x_1, x_2)) = x_1 + x_2$, we recover the classic Riesz-Sobolev inequality [14]. In this case, strict admissibility is equivalent to the strict admissibility polygon condition given by Burchard [4], which is: $l_i < l_j + l_k$ for any permutation (i, j, k) of $(1, 2, 3)$. Indeed, if the set $\{(x_1, x_2) : |x_1| \leq l_1, |x_2| \leq l_2, |x_1 + x_2| \leq l_3\}$ is a nonempty proper subset of $\{(x_1, x_2) : |x_1| \leq l_1, |x_2| \leq l_2\}$, i.e. the condition $|L_3((x_1, x_2))| \leq l_3$ is not redundant, then $l_3 > l_2 + l_1$, and conversely.

Chapter 8

Nonnegative extremizers of Gowers product inequality

Our induction step to higher dimensions will need a complete characterization of an arbitrary nonnegative extremizing tuple $\vec{f} = (f_\alpha : \alpha \in \{0, 1\}^k)$ of the Gowers product inequality in one dimension. In fact, we obtain a characterization for all dimensions.

Let $m, n \geq 1$ be integers. For $1 \leq i \leq m$, let H, H_i be vector spaces, $B_i : H \rightarrow H_i$ be a surjective linear mapping, with $\ker(B_i) \cap \ker(B_j) = \{0\}$, $1 \leq i \neq j \leq m$, and $1 \leq p_i \leq \infty$. Then $(\vec{B}, \vec{p}) = ((B_1, \dots, B_m), (p_1, \dots, p_m))$ is called a Brascamp-Lieb datum [1]. A Brascamp-Lieb inequality is an inequality of the form:

$$\int_H \prod_{1 \leq i \leq m} f_i \circ B_i(x) dx \leq BL(\vec{B}, \vec{p}) \|f_i\|_{L^{p_i}(H_i)}.$$

$BL(\vec{B}, \vec{p})$ is the smallest constant such that the above inequality is satisfied for all input tuples $\vec{f} = (f_1, \dots, f_m)$ with measurable $f_i : H_i \rightarrow \mathbb{R}_{\geq 0}$. An extremizing tuple $\vec{f}$ is an input tuple with which the equal sign happens. We quote the following result:

Theorem 8.1. [1] Let $(\vec{B}, \vec{p})$ be an extremizable Brascamp-Lieb datum with $1 \leq p_i < \infty$ for all i . Suppose also that $B_i^* H_i \cap B_j^* H_j = \{0\}$ whenever $1 \leq i < j \leq m$. Then if $\vec{f} = (f_i)$ is an extremizing input, then all the f_i are Gaussians, thus there exist real numbers $C, c_i > 0$, positive definite transformations $M_i : H_i \rightarrow H_i$, and points $x_i \in H_i$ such that $f_i(x) = c_i \exp(-C \langle A_i(x - x_i), (x - x_i) \rangle_{H_i})$. Moreover, $x_i = B_i w$, for some $w \in H$.

Our example of $(\vec{B}, \vec{p}) = ((B_\alpha : \alpha \in \{0, 1\}^k), (p_k, \dots, p_k))$ with $B_\alpha : \mathbb{R}^{(k+1)n} \rightarrow \mathbb{R}^n$ defined by $B_\alpha(x, \vec{h}) = x + \alpha \cdot \vec{h}$ is a Brascamp-Lieb datum. Hence the quoted result allows us to say, if $\vec{f} = (f_\alpha : \alpha \in \{0, 1\}^k)$, $f_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, is a nonnegative extremizing tuple for the inequality,

$$|\mathcal{T}_k(\vec{f})| = |\mathcal{T}_k(f_\alpha : \alpha \in \{0, 1\}^k)| \leq A(k, n)^{2^k} \prod_{\alpha \in \{0, 1\}^k} \|f_\alpha\|_{p_k} = A(k, n)^{2^k} \quad (8.1)$$

then each f_α is a Gaussian of the described form, provided that the hypotheses are met.

Indeed, for each $\alpha \in \{0, 1\}^k$, define $B_\alpha^* : \mathbb{R}^n \rightarrow \mathbb{R}^{(k+1)n}$ by $B_\alpha^*(x) = (x, \vec{X}^\alpha)$ with $\vec{X}^\alpha = (X_1^\alpha, \dots, X_k^\alpha)$ and $\mathbb{R}^n \ni X_i^\alpha = x$ if $\alpha_i = 1$ and $X_i^\alpha = 0$ otherwise. It's easy to check that B_α^* is indeed the adjoint of B_α and that $B_\alpha^* \mathbb{R}^n \cap B_\beta^* \mathbb{R}^n = \{0\}$, $\alpha \neq \beta \in \{0, 1\}^k$. As indicated by the Gowers-Host-Kra norm inequality, the datum $((B_\alpha : \alpha \in \{0, 1\}^k), (p_k, \dots, p_k))$ is extrem-

izable. Hence $f_\alpha(x) = m_\alpha \exp(-C\langle M_\alpha(x - c_\alpha), (x - c_\alpha) \rangle_{\mathbb{R}^n})$, with $C, m_\alpha > 0$, $M_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$ a positive definite transformation, and $c_\alpha = c_0 + \alpha \cdot \vec{c}$, for some $(c_0, \vec{c}) \in \mathbb{R}^{(k+1)n}$.

Furthermore, in our case, we claim that there exists a positive definite transformation M such that $M_\alpha = M$ for all $\alpha \in \{0, 1\}^k$. To see this, note that if $\vec{f} = (f_\alpha : \alpha \in \{0, 1\}^k)$ is an extremizing tuple then so is its symmetric rearrangement tuple, $\vec{f}^* = (f_\alpha^* : \alpha \in \{0, 1\}^k)$; this is simply a consequence of the general rearrangement inequality, $\mathcal{T}_k(\vec{f}) = \mathcal{T}_k(f_\alpha : \alpha \in \{0, 1\}^k) \leq \mathcal{T}_k(f_\alpha^* : \alpha \in \{0, 1\}^k) = \mathcal{T}_k(\vec{f}^*)$ and the fact that $\|f_\alpha\|_{p_k} = \|f_\alpha^*\|_{p_k}$. It then suffices to assume $f_\alpha(x) = m_\alpha \exp(-C\langle M_\alpha x, x \rangle_{\mathbb{R}^n})$. For $k = 2$, the Cauchy-Schwarz inequality, Young's convolution inequality and the fact that $A(2, n) = ((C_{4/3}^2/C_2)^{1/2})^n A(1, n)^{1/2} = ((C_{4/3}^2/C_2)^{1/2})^n$ give us:

$$\mathcal{T}_2(f_1, f_2, f_3, f_4) = \int_{\mathbb{R}^n} (f_1 * f_2) \cdot (f_3 * f_4)(x) dx \leq \|f_1 * f_2\|_2 \|f_3 * f_4\|_2 \leq A(2)^4 \prod_{i=1}^4 \|f_i\|_{4/3}.$$

The equal sign is a simple consequence of change of variables. The conclusion for the case $k = 2$ then follows from the characterization of extremizers of the Cauchy-Schwarz inequality and of Young's convolution inequality. Assume the claim is true for index k or lower. Then:

$$\begin{aligned} \mathcal{T}_{k+1}(f_\gamma : \gamma \in \{0, 1\}^{k+1}) &= \int_{\mathbb{R}^n} \mathcal{T}_k(T^h f_{(\alpha,1)} \cdot f_{(\alpha,0)} : \alpha \in \{0, 1\}^k) dh \\ &\leq \int_{\mathbb{R}^n} \prod_{\alpha \in \{0,1\}^k} \|T^h f_{(\alpha,1)} \cdot f_{(\alpha,0)}\|_{U^k} dh = A(k, n)^{2^k} \int_{\mathbb{R}^n} \prod_{\alpha \in \{0,1\}^k} \|T^h f_{(\alpha,1)} \cdot f_{(\alpha,0)}\|_{p_k} dh \\ &\leq A(k, n)^{2^k} \prod_{\alpha \in \{0,1\}^k} \left(\int_{\mathbb{R}^n} \|T^h f_{(\alpha,1)} \cdot f_{(\alpha,0)}\|_{p_k}^{2^k} dh \right)^{1/2^k} \end{aligned}$$

$$\begin{aligned} &\leq A(k+1, n)^{2^{k+1}} \prod_{\alpha \in \{0,1\}^k} \|f_{(\alpha,0)}\|_{p_{k+1}} \|f_{(\alpha,1)}\|_{p_{k+1}} \\ &= A(k+1, n)^{2^{k+1}} \prod_{\gamma \in \{0,1\}^{k+1}} \|f_\gamma\|_{p_{k+1}}. \end{aligned}$$

The first inequality is the Gowers product inequality. The second equality is due to the fact that Gaussians are extremizers of the Gowers-Host-Kra norm inequality. The third inequality follows from Hölder's inequality and the fourth from the sharp Young's inequality, as discussed in **Chapter 2**. If $(f_\gamma : \gamma \in \{0,1\}^{k+1})$ is an extremizing tuple with $f_\gamma(x) = m_\gamma \exp(-C\langle M_\gamma x, x \rangle_{\mathbb{R}^n})$, this forces the first and third inequalities in the display to become equalities. Due to the Gowers-Host-Kra norm inequality and the fact that all the integrands involved are continuous, equal sign in the first inequality happens only when for all $h \in \mathbb{R}^n$ and $\alpha \in \{0,1\}^k$,

$$\mathcal{T}_k(T^h f_{(\alpha,1)} \cdot f_{(\alpha,0)}) = A(k, n)^{2^k} \prod_{\alpha \in \{0,1\}^k} \|T^h f_{(\alpha,1)} \cdot f_{(\alpha,0)}\|_{p_k}. \quad (8.2)$$

$T^h f_{(\alpha,1)} \cdot f_{(\alpha,0)}$ is still a Gaussian; hence by the induction hypothesis, (8.2) implies in particular for $h = 0$:

$$M_{(\alpha,0)} + M_{(\alpha,1)} = M_{(\beta,0)} + M_{(\beta,1)} \quad (8.3)$$

for all $\alpha, \beta \in \{0,1\}^k$. On the other hand, equal sign in the third inequality gives, for each $\alpha \in \{0,1\}^k$:

$$\left(\int_{\mathbb{R}^n} \|T^h f_{(\alpha,1)} \cdot f_{(\alpha,0)}\|_{p_k}^{2^k} dh \right)^{1/2^k} = A(k, n)^{2^k} \|f_{(\alpha,0)}\|_{p_k} \|f_{(\alpha,1)}\|_{p_k}. \quad (8.4)$$

The characterization of extremizers of the sharp Young's inequality [10] and (8.4) implies there exist $m_0, m_1 \in \mathbb{R}_{>0}$, a positive definite transformation $M : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $c \in \mathbb{R}^n$ such

that $\exp(-p_k \langle M(x - c), (x - c) \rangle_{\mathbb{R}^n}) = m_0 f_{(\alpha,0)}(x) = m_1 f_{(\alpha,1)}(x)$, which in turns implies

$$M_{(\alpha,0)} = M_{(\alpha,1)} = M \quad (8.5)$$

for all $\alpha \in \{0, 1\}^k$. Then (8.3) and (8.5) conclude $M_\gamma = M_{\gamma'}$ for all $\gamma, \gamma' \in \{0, 1\}^{k+1}$ and hence the induction step. In particular, we obtain:

Corollary 8.2. Let $k \geq 2, n \geq 1$ be integers and $\vec{f} = (f_\alpha : \alpha \in \{0, 1\}^k)$ with measurable $f_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, $\|f_\alpha\|_{p_k} = 1$. If

$$\mathcal{T}_k(\vec{f}) = \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} f_\alpha(x + \alpha \cdot \vec{h}) dx d\vec{h} = A(k, n)^{2^k} \prod_{\alpha \in \{0,1\}^k} \|f_\alpha\|_{p_k}$$

then $f_\alpha = m \exp(-\langle M(x - c_\alpha), (x - c_\alpha) \rangle_{\mathbb{R}^n})$, with $m > 0$, $M : \mathbb{R}^n \rightarrow \mathbb{R}^n$ a positive definite transformation, and $c_\alpha = c_0 + \alpha \cdot \vec{c}$, for some $(c_0, \vec{c}) \in \mathbb{R}^{(k+1)n}$.

Remark 8.1: Another proof for the fact $c_\alpha = c_0 + \alpha \cdot \vec{c}$, $\alpha \in \{0, 1\}^k$, for some $(c_0, \vec{c}) \in \mathbb{R}^{k+1}$ (hence only applicable for $n = 1$) is as follows. If $f_\alpha(x) = m \exp(-a(x - c_\alpha)^2)$ and

$$\mathcal{T}_k(\vec{f}) = \int \mathcal{T}_k(1_{F_{\alpha,v_\alpha}} : \alpha \in \{0, 1\}^k) d\vec{v} = A(k)^{2^k} = \mathcal{T}_k(\vec{f}^*) = \int \mathcal{T}_k(1_{F_{\alpha,v_\alpha}^*} : \alpha \in \{0, 1\}^k) d\vec{v}$$

then this entails $\mathcal{T}_k(1_{F_{\alpha,v}} : \alpha \in \{0, 1\}^k) = \mathcal{T}_k(1_{F_{\alpha,v}^*} : \alpha \in \{0, 1\}^k)$ for all $v \in (0, m)$. Note that $(F_{\alpha,v})^* = F_{\alpha,v}^*$. Fix v and let $\mathcal{L}(F_{\alpha,v}) = \mathcal{L}(F_{\alpha,v}^*) = l_\alpha$. Then apply **Lemma 7.5** to the admissible tuple $(B_\alpha, l_\alpha)_{\alpha \in \{0,1\}^k}$ to obtain the desired conclusion.

8.1 Gaussian near extremizers in one dimension

We now characterize Gaussian near extremizers of the Gowers product inequality in one dimension, which will be needed in **Chapter 9** below. Given a Gaussian tuple $\vec{f} = (f_\alpha = ca_\alpha^{1/2p_k} \exp(-a_\alpha(x - c_\alpha)^2) : \alpha \in \{0, 1\}^k)$, so that $\|f_\alpha\|_{p_k} = 1$. Suppose that for some $\delta > 0$ small,

$$\mathcal{T}_k(\vec{f}) = \mathcal{T}_k(f_\alpha : \alpha \in \{0, 1\}^k) \geq (1 - \delta)A(k)^{2^k} \prod_{\alpha \in \{0, 1\}^k} \|f_\alpha\|_{p_k} = (1 - \delta)A(k)^{2^k}.$$

We claim that there exist $a, \Gamma = \Gamma(k) > 0$ and a nonnegative function $\eta = \eta_k(\delta)$ that is increasing for small values of δ such that $|a_\alpha/a - \Gamma| \leq \eta$, for all $\alpha \in \{0, 1\}^k$. As before, we can first assume that $c_\alpha = 0$ for all $\alpha \in \{0, 1\}^k$. We start with the induction step for the case $k + 1$ and assume the claim is true for index k or lower. If:

$$\mathcal{T}_{k+1}(f_\beta : \beta \in \{0, 1\}^{k+1}) \geq (1 - \delta)A(k + 1)^{2^{k+1}}.$$

It then follows from the calculations in **Chapter 3** that there exists $c(k) > 0$ such that for each $\alpha \in \{0, 1\}^k$,

$$\|f_{(\alpha, 0)}^{p_k} * f_{(\alpha, 1)}^{p_k}\|_{k+1}^{k+1} \geq (1 - c(k)\delta)B(k + 1)^{k+1} \|f_{(\alpha, 0)}^{p_k}\|_q^{k+1} \|f_{(\alpha, 1)}^{p_k}\|_q^{k+1}.$$

Here, $q = p_{k+1}/p_k$ and $B(k + 1)$ is the optimal constant of Young's convolution inequality for the involved exponents. Since all the functions involved are centered Gaussians in one dimension and $p_k = 2^k/(k + 1)$, it's a simple calculation to show that there exist $\delta_0 > 0$ sufficiently small, a function $\eta = \eta_k(\delta)$ that is increasing for $0 < \delta \leq \delta_0$ and $\Gamma = \Gamma(k)$, such that

$|a_{(\alpha,0)}/a_{(\alpha,1)} - \Gamma| \leq \eta$, for all $\alpha \in \{0,1\}^k$. By symmetry, this means that if $\beta, \beta' \in \{0,1\}^{k+1}$ are such that $\beta_i = \beta'_i$ for all but one single index $i \in \{1, \dots, k+1\}$, then $|a_\beta/a_{\beta'} - \Gamma| \leq \eta$; in other words, there exists $a > 0$ such that $|a_\beta/a - \Gamma| \leq \eta$ for all $\beta \in \{0,1\}^{k+1}$. The case $k = 2$ is proved using similar arguments.

We now claim that if δ is small enough then there exist $c_0 \in \mathbb{R}$ and $\vec{c} \in \mathbb{R}^k$ such that $|c_\alpha - (c_0 + \alpha \cdot \vec{c})| = o_\delta(1)$. To see this, we now assume $a_\alpha = a > 0$ for all $\alpha \in \{0,1\}^k$ as permitted by the above reasoning. For each $\alpha \in \{0,1\}^k$, let $\vec{\xi}^\alpha \in \mathbb{R}^{k+1}$ be such that $B_\alpha(\vec{\xi}^\alpha) = c_\alpha$. Define $\mathbb{R}^{k+1} \ni \vec{T}$ so that $\sum_{\alpha \in \{0,1\}^k} B_\alpha^* B_\alpha \vec{T} = \sum_{\alpha \in \{0,1\}^k} B_\alpha^* B_\alpha \vec{\xi}^\alpha$ (it's an easy calculation that $|\det(\sum_{\alpha \in \{0,1\}^k} B_\alpha^* B_\alpha)| > 0$). Here, B_α is defined as above with $n = 1$. Then by a change of variables:

$$\begin{aligned} \int_{\mathbb{R}^{k+1}} \prod_{\alpha \in \{0,1\}^k} f_\alpha \circ B_\alpha(\vec{x}) d\vec{x} &= C \int_{\mathbb{R}^{k+1}} \exp \left\{ -a \sum_{\alpha \in \{0,1\}^k} (B_\alpha(\vec{x} - \vec{\xi}^\alpha))^2 \right\} d\vec{x} \\ &= C \int_{\mathbb{R}^{k+1}} \exp \left\{ -a \sum_{\alpha \in \{0,1\}^k} (B_\alpha(\vec{x} - (\vec{\xi}^\alpha - \vec{T})))^2 \right\} d\vec{x} \\ &= C \exp \left\{ -a \sum_{\alpha \in \{0,1\}^k} (B_\alpha(\vec{\xi}^\alpha - \vec{T}))^2 \right\} \int_{\mathbb{R}^{k+1}} \exp \left\{ -a \sum_{\alpha \in \{0,1\}^k} (B_\alpha(\vec{x}))^2 \right\} d\vec{x} \\ &\leq C \int_{\mathbb{R}^{k+1}} \exp \left\{ -a \sum_{\alpha \in \{0,1\}^k} (B_\alpha(\vec{x}))^2 \right\} d\vec{x}. \end{aligned} \quad (8.6)$$

Note that by definition of $\vec{T}$, $\int_{\mathbb{R}^{k+1}} \exp \left\{ 2a \sum_{\alpha \in \{0,1\}^k} B_\alpha(\vec{x}) \cdot B_\alpha(\vec{\xi}^\alpha - \vec{T}) \right\} d\vec{x} = 1$, hence the last equality in (8.6) follows. It's also now clear from (8.6) and the fact that its last expression is $\|c \exp(-ax^2)\|_{U^k} = A(k)^{2k} \|c \exp(-ax^2)\|_{p_k}^{2k} = A(k)^{2k}$, that if $\mathcal{T}_k(\vec{f}) = \int_{\mathbb{R}^{k+1}} \prod_{\alpha \in \{0,1\}^k} f_\alpha \circ B_\alpha(\vec{x}) d\vec{x}$ is nearing its optimal value then $|c_\alpha - B_\alpha(\vec{T})| = o_\delta(1)$ for all

$\alpha \in \{0, 1\}^k$. Hence the claim follows.

Remark 8.2: If $\vec{f} = (f_\alpha : \alpha \in \{0, 1\}^k)$ is such that $\mathcal{T}_k(\vec{f}) \geq (1 - \delta)A(k)^{2^k} \prod_{\alpha \in \{0, 1\}^k} \|f_\alpha\|_{p_k}$, then it follows from the Gowers-Cauchy-Schwarz inequality that for each $\alpha \in \{0, 1\}^k$, $\|f_\alpha\|_{U^k} \geq (1 - o_\delta(1))A(k)\|f_\alpha\|_{p_k}$. If furthermore, $f_\alpha \geq 0$ then by the result for dimension one, there exists a Gaussian g_α such that $\|g_\alpha - f_\alpha\|_{p_k} = o_\delta(1)\|f_\alpha\|_{p_k}$. Then from the Gowers product inequality, $\vec{g} = (g_\alpha : \alpha \in \{0, 1\}^k)$ is also a near extremizing tuple: $\mathcal{T}_k(\vec{g}) \geq (1 - o_\delta(1))A(k)^{2^k} \prod_{\alpha \in \{0, 1\}^k} \|g_\alpha\|_{p_k}$, and the analytic descriptions of the g_α are given above. We note that all of these arguments can be generalized to higher dimensions. For now, a characterization of nonnegative near extremizing tuples for the Gowers product inequality is sufficient for an induction process in **Chapter 9** below. We also note that the arguments given in this section are stronger in the sense that they establish analytic properties of near extremizers, not just extremizers.

Chapter 9

Extension to higher dimensions

9.1 An additive relation

We first quote a result in [8]:

Proposition 9.1. [8] Let $n \geq 1$. There exists a positive constant $K = K(n) > 0$ with the following property. Let B be a ball of positive, finite radius. Let $\alpha, \beta, \gamma : \mathbb{R}^n \rightarrow \mathbb{C}$ be measurable functions. Let $\tau \in (0, \infty)$ and $\delta \in (0, 1]$. Suppose that,

$$\mathcal{L}(\{(x, y) \in B^2 : |\alpha(x) + \beta(y) - \gamma(x + y)| > \tau\}) < \delta(\mathcal{L}(B))^2.$$

Then there exists an affine function $L : \mathbb{R}^n \rightarrow \mathbb{C}$ such that

$$\mathcal{L}(\{x \in B : |\alpha(x) - L(x)| > K\tau\}) < K\delta\mathcal{L}(B).$$

Using this proposition, we prove the following:

Proposition 9.2. Let $n \geq 1$ and $l \geq 2$ be an integer. Let $C_i > 0$ be such that $C_i \asymp C_j$ for $i, j \in \{1, \dots, l\}$. There exists $K > 0$ with the following property. Let B_i be a ball in $\mathbb{R}^n$ such that $\mathcal{L}(B_i) = C_i$ and let $f_i : \mathbb{R}^n \rightarrow \mathbb{C}$ be a measurable function, $i \in \{1, \dots, l\}$. Let $\tau \in (0, \infty)$ and $\delta \in (0, 1]$. Suppose that,

$$|f_{l+1}(x_1 + \dots + x_l) - \sum_{i=1}^l f_i(x_i)| \leq \tau \quad (9.1)$$

for all $(x_1, \dots, x_l) \in B_1 \times \dots \times B_l$ outside a subset of measure $\delta \prod_{i=1}^l \mathcal{L}(B_i)$. Then there exist an affine function $a : \mathbb{R}^n \rightarrow \mathbb{C}$ and a positive function η satisfying $\lim_{\delta \rightarrow 0} \eta(\delta) = 0$, such that

$$|f_1(x) - a(x)| \leq K\tau$$

for all $x \in B_1$ outside a subset of measure $K\eta(\delta)\mathcal{L}(B_1)$.

Proof: Since the conclusion doesn't change after applying a finite number of translations and dilations, we assume that $B_1 = \dots = B_l = B$; here B is a ball of positive radius. Then (9.1) gives, for $j \geq 1$ and for all $(x_1, \dots, x_l) \in \times_{i=1}^l B$ outside a subset of measure $\delta(\mathcal{L}(B))^j$, the following holds:

$$f_{l+1}(x_1 + x_2 + \sum_{i=3}^l x_i) - \sum_{i=3}^l f_i(x_i) = f_1(x_1) + f_2(x_2) + O(\tau). \quad (9.2)$$

That means there exists a positive function η satisfying $\lim_{\delta \rightarrow 0} \eta(\delta) = 0$ such that for all $(x_3, \dots, x_l) \in \times_{i=3}^l B$ outside a subset of measure $\eta(\delta)(\mathcal{L}(B))^{l-2}$ such that for all $(x_1, x_2) \in$

$B \times B$ except for a subset of measure $\eta(\delta)(\mathcal{L}(B))^2$, (9.2) holds. Take such a point $(x_3, \dots, x_l) \in \times_{i=3}^l B$. For this point, define $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{C}$ by $\tilde{f}_{l+1}(u) = f_{l+1}(u + \sum_{i=3}^l x_i) - \sum_{i=3}^l f_i(x_i)$. Then (9.2) becomes:

$$\tilde{f}_{l+1}(x_1 + x_2) = f_1(x_1) + f_2(x_2) + O(\tau)$$

for all $(x_1, x_2) \in B \times B$ outside a subset of measure at least $(1 - \eta(\delta))(\mathcal{L}(B))^2$. Now apply **Proposition 9.1** to $\tilde{f}_{l+1}, f_1, f_2$ - and an appropriate translation and dilation if necessary - to obtain an affine function $a : \mathbb{R}^n \rightarrow \mathbb{C}$ such that $f_1(x) = a(x) + O(\tau)$ for all $x \in B_1$ outside a subset of measure $K\eta(\delta)\mathcal{L}(B_1)$.

9.2 Extension to higher dimensions

Let $n \geq 1$. Assume **Theorem 1.1** is true for nonnegative functions and for dimensions n and lower. Let $f(x, s) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ be a $(1 - \delta)$ near extremizer of the Gowers-Host-Kra norm inequality. Assume $\|f\|_{p_k} = 1$. Define $F : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ by $F(x) = \|f(x, \cdot)\|_{p_k}$. Define $f_x : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ by $f_x(s) = \frac{f(x, s)}{F(x)}$ if $F(x) \notin \{0, \infty\}$, and $f_x(s) \equiv 0$ if $F(x) \in \{0, \infty\}$ - which happens only for a null subset of $\text{spt}(F) \subset \mathbb{R}^n$, outside of which, $\|f_x\|_{p_k} = \|f\|_{p_k} = 1$. Thus, $\|F\|_{p_k} = \|f\|_{p_k} = 1$. From the definition of the Gowers-Host-Kra norms and Fubini's theorem,

$$\|f\|_{U^k}^{2^k} = \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) dx d\vec{h}.$$

The Gowers-Cauchy-Schwarz inequality then gives,

$$\begin{aligned}
 (1 - \delta)A(k, n + 1)^{2^k} &\leq \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) dx d\vec{h} \\
 &\leq A(k)^{2^k} \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) dx d\vec{h} = A(k)^{2^k} \|F\|_{U^k}^{2^k}. \quad (9.3)
 \end{aligned}$$

Since $A(k, m) = A(k)^m$, (9.3) implies $\|F\|_{U^k} \geq (1 - \delta)A(k, n)^{2^k}$. Then by the inductive assumption, there exists a Gaussian $\mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$ such that $\|\mathcal{F}\|_{p_k} = 1$ and $\|\mathcal{F} - F\|_{p_k} = o_\delta(1)$.

Let $B_R \subset \mathbb{R}^n$ denote a centered ball of radius R . Take $\eta > 0$ small. There exist $\delta = \delta(\eta) > 0$ small and B_R with $R = R(\eta) > 0$ large such that $\delta \rightarrow 0$ and $R \rightarrow \infty$ as $\eta \rightarrow 0$ with the following extra property. Let $\mathcal{F}^\dagger$ denote the standard centered Gaussian on $\mathbb{R}^n$. Suppose $F^\dagger$ is such that $\|F^\dagger\|_{p_k} = 1$ and $\|\mathcal{F}^\dagger - F^\dagger\|_{p_k} \leq \delta$. Then:

$$\begin{aligned}
 \int_{\mathbb{R}^{(k+1)n} \setminus \tilde{B}_R^{k+1}} \prod_{\alpha \in \{0,1\}^k} F^\dagger(x + \alpha \cdot \vec{h}) dx d\vec{h} \\
 = \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} F^\dagger \chi_{\mathbb{R}^n \setminus B_R}(x + \alpha \cdot \vec{h}) dx d\vec{h} < \eta. \quad (9.4)
 \end{aligned}$$

Indeed, since $\mathcal{F}^\dagger$ is the standard centered Gaussian, for every $\eta > 0$, there exists B_R with R sufficiently large so that:

$$\begin{aligned}
 \int_{\mathbb{R}^{(k+1)n} \setminus \tilde{B}_R^{k+1}} \prod_{\alpha \in \{0,1\}^k} \mathcal{F}^\dagger(x + \alpha \cdot \vec{h}) dx d\vec{h} \\
 = \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} \mathcal{F}^\dagger \chi_{\mathbb{R}^n \setminus B_R}(x + \alpha \cdot \vec{h}) dx d\vec{h} < \eta/2. \quad (9.5)
 \end{aligned}$$

Applying the Gowers product inequality to have:

$$\begin{aligned} \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} |\mathcal{F}^\dagger - F^\dagger| \chi_{\mathbb{R}^n \setminus B_R}(x + \alpha \cdot \vec{h}) dx d\vec{h} \\ \leq A(k, n)^{2^k} \prod_{\alpha \in \{0,1\}^k} \|\mathcal{F}^\dagger - F^\dagger\|_{p_k} < A(k, n)^{2^k} \delta. \end{aligned} \quad (9.6)$$

Simply take δ small enough so that $A(k, n)^{2^k} \delta < \eta/2$. Then (9.4) follows from (9.5) and (9.6).

Suppose now that our Gaussian $\mathcal{F}$ is indeed the standard centered one; this can be done by applying an affine transformation and scaling by a suitable constant. Then there exists a sufficiently large R so that

$$\int_{\mathbb{R}^{(k+1)n} \setminus \tilde{B}_R^{k+1}} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) dx d\vec{h} < \eta. \quad (9.7)$$

Now by the Gowers product inequality:

$$\mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) \leq A(k)^{2^k} \prod_{\alpha \in \{0,1\}^k} \|f_{x+\alpha \cdot \vec{h}}\|_{p_k} = A(k)^{2^k}$$

which then with (9.7), implies

$$\int_{\mathbb{R}^{(k+1)n} \setminus \tilde{B}_R^{k+1}} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) dx d\vec{h} < \eta A(k)^{2^k}. \quad (9.8)$$

Note that if $R \rightarrow \infty$ and $\delta \rightarrow 0$ then $\eta = \eta(R, \delta) \rightarrow 0$ in (9.7) and (9.8). We've set up the case for restricting our analysis in $\tilde{B}_R^{k+1}$ with R big enough.

Lemma 9.3. Let f, F, f_x, R be as above. Then there exist $\Omega \subset \tilde{B}_R^{k+1}$ and $\omega \subset B_R$ such that

$\mathcal{L}(\Omega) + \mathcal{L}(\omega) = o_\delta(1)$ satisfying:

For $(x, \vec{h}) \in \tilde{B}_R^{k+1} \setminus \Omega$ and $\alpha \in \{0, 1\}^k$,

$$\|\phi_{x+\alpha \cdot \vec{h}} - f_{x+\alpha \cdot \vec{h}}\|_{p_k} = o_\delta(1) \quad (9.9)$$

with $\phi_{x+\alpha \cdot \vec{h}}(s) = ca(x + \alpha \cdot \vec{h})^{1/2p} \exp(-a(x + \alpha \cdot \vec{h})(s - d(x + \alpha \cdot \vec{h}))^2)$. The functions $a : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$ and $d : \mathbb{R}^n \rightarrow \mathbb{R}$ are measurable and satisfy the following properties:

There exists a scalar $a > 0$ such that for $x \in B_R \setminus \omega$,

$$|a(x) - a| = o_\delta(1). \quad (9.10)$$

For $(x, \vec{h}) \in \tilde{B}_R^{k+1} \setminus \Omega$,

$$|d(x + \alpha \cdot \vec{h}) - (1, \alpha) \cdot (d(x), d(h_1), \dots, d(h_k))| = o_\delta(1). \quad (9.11)$$

Lastly,

$$\int_{\Omega} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \, dx d\vec{h} = o_\delta(1). \quad (9.12)$$

Proof: From the near extremizing hypothesis and the Gowers-Host-Kra norm inequality,

$$\begin{aligned} \|f\|_{U^k}^{2^k} &= \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) \, dx d\vec{h} \geq (1 - \delta) A(k, n+1)^{2^k} \\ &= (1 - \delta) A(k)^{2^k} A(k, n)^{2^k} \|F\|_{p_k}^{2^k} \geq (1 - \delta) A(k)^{2^k} \|F\|_{U^k}^{2^k} \\ &= (1 - \delta) A(k)^{2^k} \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \, dx d\vec{h}. \end{aligned} \quad (9.13)$$

Take R appropriately so that, as in (9.7) and (9.8),

$$\int_{\mathbb{R}^{(k+1)n} \setminus \tilde{B}_R^{k+1}} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \, dx d\vec{h} = o_\delta(1) \quad (9.14)$$

$$\int_{\mathbb{R}^{(k+1)n} \setminus \tilde{B}_R^{k+1}} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) dx d\vec{h} = o_\delta(1). \quad (9.15)$$

There might still exist $\Omega_0 \subset \tilde{B}_R^{k+1}$ such that,

$$\int_{\Omega_0} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) dx d\vec{h} = o_\delta(1) \quad (9.16)$$

and hence, due to the Gowers product inequality,

$$\int_{\Omega_0} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) dx d\vec{h} = o_\delta(1) A(k)^{2^k}. \quad (9.17)$$

Since F is $o_\delta(1)$ close in L^{p_k} norm to the standard centered Gaussian on $\mathbb{R}^n$, $\mathcal{F}$, (9.16) implies a similar inequality for $\mathcal{F}$:

$$\int_{\Omega_0} \prod_{\alpha \in \{0,1\}^k} \mathcal{F}(x + \alpha \cdot \vec{h}) dx d\vec{h} = o_\delta(1)$$

which then implies,

$$\mathcal{L}(\Omega_0) = o_\delta(1). \quad (9.18)$$

Now (9.13), (9.14), (9.15) and (9.17) imply that Ω_0 also has the following properties:

$$\begin{aligned} & \int_{\tilde{B}_R^{k+1} \setminus \Omega_0} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) \mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) dx d\vec{h} \\ & \geq (1 - o_\delta(1)) A(k)^{2^k} \int_{\tilde{B}_R^{k+1} \setminus \Omega_0} \prod_{\alpha \in \{0,1\}^k} F(x + \alpha \cdot \vec{h}) dx d\vec{h}. \end{aligned} \quad (9.19)$$

Since $\mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) \leq A(k)^{2^k}$ for all $(x, \vec{h}) \in \mathbb{R}^{(k+1)n}$, by the Gowers inner product inequality, (9.17) and (9.19) imply that for a.e $(x, \vec{h}) \in \tilde{B}_R^{k+1} \setminus \Omega_0$,

$$\mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) \geq (1 - o_\delta(1)) A(k)^{2^k}$$

which, by the Gowers-Cauchy-Schwarz inequality, entails

$$\prod_{\alpha \in \{0,1\}^k} \|f_{x+\alpha \cdot \vec{h}}\|_{U^k} \geq \mathcal{T}_k(f_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) \geq (1 - o_\delta(1))A(k)^{2^k} \prod_{\alpha \in \{0,1\}^k} \|f_{x+\alpha \cdot \vec{h}}\|_{p_k}$$

which then gives that, for each $\alpha \in \{0,1\}^k$ and a.e $(x, \vec{h}) \in \tilde{B}_R^{k+1} \setminus \Omega_0$:

$$\|f_{x+\alpha \cdot \vec{h}}\|_{U^k} \geq (1 - o_\delta(1))A(k)\|f_{x+\alpha \cdot \vec{h}}\|_{p_k}. \quad (9.20)$$

Excluding a null subset if necessary, then the inductive hypothesis for dimension $n = 1$ and

(9.20) imply that, if $(x, \vec{h}) \in \tilde{B}_R^{k+1} \setminus \Omega_0$ then $f_{x+\alpha \cdot \vec{h}}$ is $o_\delta(1)$ close in L^{p_k} norm to a Gaussian

$$\phi_{x+\alpha \cdot \vec{h}}(s) = ca(x + \alpha \cdot \vec{h})^{1/2p} \exp(-a(x + \alpha \cdot \vec{h})(s - d(x + \alpha \cdot \vec{h}))^2).$$

For $x \in B_R$, let $E^x = \{\vec{h} \in B_R^k : (x, \vec{h}) \in \tilde{B}_R^{k+1} \setminus \Omega_0\}$ and $\omega_0 = \{x \in B_R : \mathcal{L}(E^x) \leq (1 - \delta')(\mathcal{L}(B_R))^k\}$, with $\delta' = \delta'(\delta)$ satisfying $\delta' \rightarrow 0$ sufficiently slow compared to $\delta \rightarrow 0$ so that $\mathcal{L}(\omega_0) = o_\delta(1)\mathcal{L}(B_R)$ and for every $x \in B_R \setminus \omega_0$, $\mathcal{L}(E^x) \geq (1 - o_\delta(1))(\mathcal{L}(B_R))^k$. From the definitions of ω_0 and Ω_0 , if $x \in B_R \setminus \omega_0$ and $\vec{h} \in E^x$, we have a decomposition,

$$f_{x+\alpha \cdot \vec{h}} = \phi_{x+\alpha \cdot \vec{h}} + \rho_{x+\alpha \cdot \vec{h}} \quad (9.21)$$

for some Gaussian $\phi_{x+\alpha \cdot \vec{h}}(s) = ca(x + \alpha \cdot \vec{h})^{1/2p} \exp(-a(x + \alpha \cdot \vec{h})(s - d(x + \alpha \cdot \vec{h}))^2)$, and

$\|\rho_{x+\alpha \cdot \vec{h}}\|_{p_k} = o_\delta(1)$. This decomposition can be done so that $(x, \vec{h}) \mapsto a(x + \alpha \cdot \vec{h})$, $(x, \vec{h}) \mapsto d(x + \alpha \cdot \vec{h})$, $((x, \vec{h}), s) \mapsto \rho_{x+\alpha \cdot \vec{h}}(s)$ are all measurable. Take $x \in B_R \setminus \omega_0$ and $\vec{h} \in E^x$. By (9.20), (9.21) and the Gowers product inequality:

$$\mathcal{T}_k(\phi_{x+\alpha \cdot \vec{h}} : \alpha \in \{0,1\}^k) \geq (1 - o_\delta(1))A(k)^{2^k}. \quad (9.22)$$

By the result in **Section 8.1**, it follows from (9.22) that for every $\alpha \in \{0, 1\}^k$, $x \in B_R \setminus \omega_0$ and $\vec{h} \in E^x$,

$$a(x) = a(x + \alpha \cdot \vec{h}) + o_\delta(1) \quad (9.23)$$

$$|d(x + \alpha \cdot \vec{h}) - (1, \alpha) \cdot (d(x), d(h_1), \dots, d(h_k))| = o_\delta(1). \quad (9.24)$$

Since $\mathcal{L}(B_R \setminus \omega_0) = (1 - o_\delta(1))\mathcal{L}(B_R)$, there exists $x_0 \in B_R \setminus \omega_0$ such that $|x_0| = o_\delta(1)$. Moreover, $\mathcal{L}(E^{x_0}) \geq (1 - o_\delta(1))(\mathcal{L}(B_R))^{k+1}$ implies that the set V of values $x_0 + \alpha \cdot \vec{h}$ for some $\alpha \in \{0, 1\}^k$ and $\vec{h} \in E^{x_0}$ must take up a measure of $(1 - o_\delta(1))\mathcal{L}(B_R)$ in B_R . We define ω to be the complement of the intersection of $B_R \setminus \omega_0$ and V . Then by (9.23), for every $x \in B_R \setminus \omega$,

$$a(x) = a(x_0) + o_\delta(1) = a + o_\delta(1) \quad (9.25)$$

for some $a > 0$. Define Ω by letting $\tilde{B}_R^{k+1} \setminus \Omega$ to be the set $(x, \vec{h}) \in \tilde{B}_R^{k+1}$ with $x \in \omega$ and $\vec{h} \in E^x$, then $\mathcal{L}(\Omega) + \mathcal{L}(\omega) = o_\delta(1)$. Hence (9.16) is still retained with Ω in place of Ω_0 :

$$\int_{\Omega} \prod_{\alpha \in \{0, 1\}^k} F(x + \alpha \cdot \vec{h}) dx d\vec{h} = o_\delta(1)$$

which is (9.12). Moreover, if $\alpha \in \{0, 1\}^k$ and $(x, \vec{h}) \in \tilde{B}_R^{k+1} \setminus \Omega$,

$$\|\rho_{x+\alpha \cdot \vec{h}}\|_{p_k} = o_\delta(1)$$

which is (9.9). Finally, (9.24) and (9.25) are (9.11) and (9.10), respectively. Hence this completes the proof of **Lemma 9.3**.

Remark 9.1: It's possible to select a decomposition in (9.21) so that $(x, \vec{h}) \mapsto a(x + \alpha \cdot \vec{h})$, $(x, \vec{h}) \mapsto d(x + \alpha \cdot \vec{h})$, $((x, \vec{h}), s) \mapsto \rho_{x+\alpha \cdot \vec{h}}(s)$ are all measurable. Let $\delta > 0$ be as above, then since $\mathcal{L}(\omega) = o_\delta(1)$, at each $x \in B_R \setminus \omega$, we can define $\phi_x = ca(x)^{1/2p} \exp(-a(x)(s - d(x))^2)$ in a way that both $a(x)$ and $d(x)$ are locally piecewise constant in a sufficiently small neighborhood N_x of x , so that $\|\phi_y - f_y\|_{p_k} = o_\delta(1)$ for all $y \in N_x$. Then it's clear that the decomposition $f_x = \phi_x + \rho_x$ satisfies the conditions $\|\rho_x\|_{p_k} = o_\delta(1)$ and $x \mapsto \rho$ is measurable, if $x \in B_R \setminus \omega$.

Theorem 9.4. Let $f(x, s) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ be a $(1 - \delta)$ near extremizer of the Gowers-Host-Kra norm inequality. Then there exists a Gaussian $\mathcal{G} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}_{> 0}$ such that $\|\mathcal{G}\|_{p_k} = 1$ and $\|\mathcal{G} - f\|_{p_k} = o_\delta(1)$.

Proof: Let $\psi_x(s) = ca^{1/2p} \exp(-a(s - d(x))^2)$, with $a > 0$ and the function d be as above; $x \in B_R$. By (9.22), (9.25) and the continuity of exponential functions, for $x \in B_R \setminus \omega$,

$$\|\psi_x - f_x\|_{p_k} = o_\delta(1). \quad (9.26)$$

An application of **Proposition 9.2** to (9.11) gives for every $x \in B_R$ except for a subset of measure $o_\delta(1)\mathcal{L}(B_R)$:

$$d(x) = L(x) + o_\delta(1). \quad (9.27)$$

Here L is an affine function on $\mathbb{R}^n$, which we can take to be real-valued, as is the center function d . Let ω' be the union of ω and this new subset of measure $o_\delta(1)\mathcal{L}(B_R)$; it's clear

$\mathcal{L}(\omega') = o_\delta(1)$. Let $\varsigma_x(s) = ca^{1/2p} \exp(-a(s - L(x))^2)$. Then (9.27) implies for $x \in B_R \setminus \omega'$:

$$\|\psi_x - \varsigma_x\|_{p_k} = o_\delta(1). \quad (9.28)$$

(9.26) and (9.28) then give $\|f_x - \varsigma_x\|_{p_k} = o_\delta(1)$, if $x \in B_R \setminus \omega'$. Let $\mathcal{G}(x, s) = \mathcal{F}(x)\varsigma_x(s)$. It's clear that $\|\mathcal{G}\|_{p_k} = 1$. Recall that $f(x, s) = F(x)f_x(s)$. Then:

$$\|\mathcal{G} - f\|_{p_k}^{p_k} \leq C \int_{\mathbb{R}^n} F^{p_k}(x) \|f_x - \psi_x\|_{p_k}^{p_k} dx + C \int_{\mathbb{R}^n} F^{p_k}(x) \|\psi_x - \varsigma_x\|_{p_k}^{p_k} dx + C \|\mathcal{F} - F\|_{p_k}^{p_k}. \quad (9.29)$$

The presence of the last term in (9.29) is due to the fact that $\varsigma_x(s)$ is a Schwartz function in terms of s and of L^∞ in terms of x . Moreover, since $\|\mathcal{F} - F\|_{p_k} = o_\delta(1)$, the contribution of this last term is of size $o_\delta(1)$ in absolute value. For the first term and the second term, we split them as follows, respectively:

$$\begin{aligned} \int_{\mathbb{R}^n} F^{p_k}(x) \|f_x - \psi_x\|_{p_k}^{p_k} dx &\leq \int_{B_R \setminus \omega'} |\mathcal{F}^{p_k}(x) - F^{p_k}(x)| \|f_x - \psi_x\|_{p_k}^{p_k} dx \\ &\quad + \int_{B_R \setminus \omega'} \mathcal{F}^{p_k}(x) \|f_x - \psi_x\|_{p_k}^{p_k} dx + C \int_{(\mathbb{R}^n \setminus B_R) \cup \omega'} F^{p_k}(x) dx \end{aligned} \quad (9.30)$$

and,

$$\begin{aligned} \int_{\mathbb{R}^n} F^{p_k}(x) \|\psi_x - \varsigma_x\|_{p_k}^{p_k} dx &\leq \int_{B_R \setminus \omega'} |\mathcal{F}^{p_k}(x) - F^{p_k}(x)| \|\psi_x - \varsigma_x\|_{p_k}^{p_k} dx + \int_{B_R \setminus \omega'} \mathcal{F}^{p_k}(x) \|\psi_x - \varsigma_x\|_{p_k}^{p_k} dx \\ &\quad + C \int_{(\mathbb{R}^n \setminus B_R) \cup \omega'} F^{p_k}(x) dx. \end{aligned} \quad (9.31)$$

The first and second terms in (9.30) and (9.31) has the size of $o_\delta(1)$ in absolute value due to (9.26), (9.28) and the fact that $\|\mathcal{F} - F\|_{p_k} = o_\delta(1)$. The third terms can be further dominated by the sum $C \int_{(\mathbb{R}^n \setminus B_R) \cup \omega'} \mathcal{F}^{p_k}(x) dx + \|\mathcal{F} - F\|_{p_k}$, the first term of which is of size

$o_\delta(1)$ in absolute value by the choice of R and the fact that $\mathcal{L}(\omega') = o_\delta(1)$. All of these yield

$$\|\mathcal{G} - f\|_{p_k} = o_\delta(1).$$

With this theorem, the extension step to higher dimensions for nonnegative near extremizers is now complete.

Chapter 10

Complex-valued Case

10.1 Preparation

Let $f : \mathbb{R}^n \rightarrow \mathbb{C}$ be a $(1 - \delta)$ near extremizer of Gowers-Host-Kra norm inequality. We write $f(x) = |f|(x)a(x)$, with $a(x) = e^{i2\pi q(x)}$ and $q : \mathbb{R}^n \rightarrow \mathbb{R}/\mathbb{Z}$. Assume $\|f\|_{p_k} = 1$. We first make a few observations. Recall that if f is a $(1 - \delta)$ near extremizer then so is $|f|$:

$$\||f|\|_{U^k} \geq \|f\|_{U^k} \geq (1 - \delta)A(k, n)\|f\|_{p_k} = (1 - \delta)A(k, n)\||f|\|_{p_k} = (1 - \delta)A(k, n). \quad (10.1)$$

Then by the previous chapters, $|f|$ is $o_\delta(1)$ close in L^{p_k} to a Gaussian extremizer, which by an affine change of variables, we can assume to be the standard centered Gaussian $\mathcal{F}^\dagger$ on $\mathbb{R}^n$: $\|\mathcal{F}^\dagger - |f|\|_{p_k} = o_\delta(1)$ and $\|\mathcal{F}^\dagger\|_{p_k} = 1$. Now (10.1) and Gowers-Host-Kra norm inequality imply,

$$A(k, n)\||f|\|_{p_k} = A(k, n) \geq \||f|\|_{U^k} \geq \|f\|_{U^k} \geq (1 - \delta)A(k, n)$$

which entails $\|f\|_{U^k} \geq (1 - o_\delta(1))\|f\|_{U^k}$. Then since $\|f\|_{U^k} > 0$:

$$\begin{aligned} \|f\|_{U^k}^{2^k} &= \operatorname{Re}(\|f\|_{U^k}^{2^k}) = \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} |f|(x + \alpha \cdot \vec{h}) \operatorname{Re} \left(\prod_{\alpha \in \{0,1\}^k} \mathcal{C}^{\omega_\alpha} a(x + \alpha \cdot \vec{h}) \right) dx d\vec{h} \\ &\geq (1 - o_\delta(1))\|f\|_{U^k}^{2^k} = (1 - o_\delta(1)) \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} |f|(x + \alpha \cdot \vec{h}) dx d\vec{h}. \end{aligned} \quad (10.2)$$

On the other hand, from the Gowers product inequality,

$$\begin{aligned} \left| \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} \mathcal{F}^\dagger(x + \alpha \cdot \vec{h}) dx d\vec{h} - \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} |f|(x + \alpha \cdot \vec{h}) dx d\vec{h} \right| \\ \leq A(k, n)^{2^k} \|\mathcal{F}^\dagger - |f|\|_{p_k}^{2^k} = o_\delta(1) \end{aligned} \quad (10.3)$$

and similarly, since $|a| = 1$ on $\mathbb{R}^n$,

$$\begin{aligned} \left| \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} \mathcal{F}^\dagger(x + \alpha \cdot \vec{h}) \operatorname{Re} \left(\prod_{\alpha \in \{0,1\}^k} \mathcal{C}^{\omega_\alpha} a(x + \alpha \cdot \vec{h}) \right) dx d\vec{h} \right. \\ \left. - \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} |f|(x + \alpha \cdot \vec{h}) \operatorname{Re} \left(\prod_{\alpha \in \{0,1\}^k} \mathcal{C}^{\omega_\alpha} a(x + \alpha \cdot \vec{h}) \right) dx d\vec{h} \right| = o_\delta(1). \end{aligned} \quad (10.4)$$

Then (10.3) and (10.4) allow us to replace $|f|$ with $\mathcal{F}^\dagger$ in (10.2):

$$\begin{aligned} \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} \mathcal{F}^\dagger(x + \alpha \cdot \vec{h}) \operatorname{Re} \left(\prod_{\alpha \in \{0,1\}^k} \mathcal{C}^{\omega_\alpha} a(x + \alpha \cdot \vec{h}) \right) dx d\vec{h} \\ \geq (1 - o_\delta(1)) \int_{\mathbb{R}^{(k+1)n}} \prod_{\alpha \in \{0,1\}^k} \mathcal{F}^\dagger(x + \alpha \cdot \vec{h}) dx d\vec{h}. \end{aligned} \quad (10.5)$$

Let $B_R \subset \mathbb{R}^n$ denote a centered ball of radius R . We can find a sufficiently large positive R , so that $\int_{\mathbb{R}^n \setminus B_R} (\mathcal{F}^\dagger)^{p_k} = o_\delta(1)$, and $\int_{\mathbb{R}^n \setminus B_R} |f|^{p_k} = o_\delta(1)$, since $\|\mathcal{F}^\dagger - |f|\|_{p_k} = o_\delta(1)$. Moreover, we can select this R so that the following properties are also satisfied:

$$\int_{\mathbb{R}^{(k+1)n} \setminus \tilde{B}_R^{k+1}} \prod_{\alpha \in \{0,1\}^k} \mathcal{F}^\dagger(x + \alpha \cdot \vec{h}) dx d\vec{h} = o_\delta(1) \quad (10.6)$$

$$\int_{\mathbb{R}^{(k+1)n} \setminus \tilde{B}_R^{k+1}} \prod_{\alpha \in \{0,1\}^k} \mathcal{F}^\dagger(x + \alpha \cdot \vec{h}) \operatorname{Re} \left(\prod_{\alpha \in \{0,1\}^k} \mathcal{C}^{\omega_\alpha} a(x + \alpha \cdot \vec{h}) \right) dx d\vec{h} = o_\delta(1). \quad (10.7)$$

Then (10.5), (10.6) and (10.7) allow us to reduce our analysis within a bounded region:

$$\begin{aligned} \int_{\tilde{B}_R^{k+1}} \prod_{\alpha \in \{0,1\}^k} \mathcal{F}^\dagger(x + \alpha \cdot \vec{h}) \operatorname{Re} \left(\prod_{\alpha \in \{0,1\}^k} \mathcal{C}^{\omega_\alpha} a(x + \alpha \cdot \vec{h}) \right) dx d\vec{h} \\ \geq (1 - o_\delta(1)) \int_{\tilde{B}_R^{k+1}} \prod_{\alpha \in \{0,1\}^k} \mathcal{F}^\dagger(x + \alpha \cdot \vec{h}) dx d\vec{h}. \end{aligned} \quad (10.8)$$

Since $|a| = 1$ on $\mathbb{R}^n$, it follows from (10.8) that:

$$\left| \operatorname{Re} \left(\prod_{\alpha \in \{0,1\}^k} \mathcal{C}^{\omega_\alpha} a(x + \alpha \cdot \vec{h}) \right) - 1 \right| = o_\delta(1)$$

or,

$$\prod_{\alpha \in \{0,1\}^k} \mathcal{C}^{\omega_\alpha} a(x + \alpha \cdot \vec{h}) = 1 + o_\delta(1)$$

for $(x, \vec{h})$ in a subset of $\tilde{B}_R^{k+1}$ of measure $(1 - o_\delta(1))\mathcal{L}(\tilde{B}_R^{k+1})$. Since $a(x) = e^{i2\pi q(x)}$, this last display is simply

$$e^{i2\pi \Delta_{h_k} \dots \Delta_{h_1} q(x)} = 1 + o_\delta(1). \quad (10.9)$$

Here $\Delta_h f(x) = f(x + h) - f(x)$.

Theorem 10.1. Let $n, k \geq 1$. There exists $K > 0$ with the following property.

Let $B \subset \mathbb{R}^n$ be a centered ball of positive radius and $\psi : B \rightarrow \mathbb{R}/\mathbb{Z}$ be a measurable function.

Let $\eta, \tau > 0$ be small numbers. Suppose

$$\mathcal{L}(\{(x, \vec{h}) \in \tilde{B}^{k+1} : |e^{i2\pi \Delta_{h_1} \dots \Delta_{h_k} \psi(x)} - 1| > \tau\}) < \eta \mathcal{L}(\tilde{B}^{k+1}). \quad (10.10)$$

Then there exist a polynomial $P : \mathbb{R}^n \rightarrow \mathbb{R}$ of degree at most $k - 1$ and a positive function ρ satisfying $\lim_{\tau \rightarrow 0} \rho(\tau) = 0$ such that

$$\mathcal{L}(\{x \in B : |e^{i2\pi\psi(x)}e^{-i2\pi P(x)} - 1| > \rho(\tau)\}) < K\eta\mathcal{L}(B). \quad (10.11)$$

10.2 Proof of Theorem 10.1

Let $\tilde{B}^2 = \{(x, h_1) \in \mathbb{R}^{2n} : x \in B, h_1 \in B \text{ and } x + h_1 \in B\}$. When $k = 1$, (10.10) becomes

$$e^{i2\pi(\psi(x+h_1)-\psi(x))} = 1 + O(\tau) \quad (10.12)$$

which happens for a subset of $\tilde{B}^2$ of measure $(1 - \eta)\mathcal{L}(\tilde{B}^2)$. Then there exists $c \in B$ such that $|c| = o_\eta(1)$ and that c satisfies (10.12) for $h_1 \in B$ outside a subset of measure at most $O(\eta)\mathcal{L}(B)$. That means

$$e^{i2\pi\psi(x)} = (1 + O(\tau))e^{i2\pi\psi(c)}$$

for $x \in B$ outside a subset of measure at most $O(\eta)\mathcal{L}(B)$. Hence we can take the constant polynomial $P \equiv \psi(c)$. Assuming the conclusion is true for the case $k - 1$, we now prove it for the case of index k .

Rewrite (10.10) as

$$1 + O(\tau) = e^{i2\pi\Delta_{h_1} \dots \Delta_{h_k} \psi(x)} = e^{i2\pi\Delta_{h_1} \dots \Delta_{h_{k-1}}(\Delta_{h_k} \psi(x))}. \quad (10.13)$$

This holds for $(x, \vec{h}) \in \tilde{B}^{k+1}$ outside a subset of measure at most $\eta\mathcal{L}(B)$. Applying the induction hypothesis to $\Delta_{h_k} \psi(x)$ in (10.13), we conclude that there exists a polynomial in

x , $P_{h_k}(x) = \sum_{j=0}^{k-2} \sum_{|\gamma|=j} a_\gamma(h_k) x^\gamma$ satisfying, for $(x, h_k) \in \tilde{B}^2$ outside a subset of measure at most $O(\eta)\mathcal{L}(\tilde{B}^2)$,

$$e^{i2\pi\Delta_{h_k}\psi(x)} e^{-i2\pi P_{h_k}(x)} = 1 + o_\tau(1). \quad (10.14)$$

The coefficient functions a_γ can be selected to be measurable functions in terms of h_k (in fact, these coefficient functions can be selected to be locally piecewise constant, in a manner that is described in *Remark 9.1*). To resolve (10.14), we prove a sub-claim:

Claim 10.1: Let ψ, B, η, τ be as above. Let $P_t(x) = \sum_{j=0}^{k-1} \sum_{|\gamma|=j} a_\gamma(t) x^\gamma$, with $a_\gamma : \mathbb{R}^n \rightarrow \mathbb{R}$ being measurable functions. Suppose for $(x, t) \in \tilde{B}^2$ outside a set of measure at most $O(\eta)\mathcal{L}(\tilde{B}^2)$ the following occurs:

$$e^{i2\pi\Delta_t\psi(x)} e^{-i2\pi P_t(x)} = 1 + O(\tau). \quad (10.15)$$

Then there exists a polynomial Q of degree at most k such that, for $x \in B$ outside a subset of measure at most $O(\eta)\mathcal{L}(B)$,

$$e^{i2\pi\psi(x)} e^{-i2\pi Q(x)} = 1 + o_\tau(1). \quad (10.16)$$

Proof of Claim 10.1: We again use induction. If $k = 1$ then (10.15) becomes

$$e^{i2\pi\psi(x+t)} e^{-i2\pi\psi(x)} e^{-i2\pi a_0(t)} = 1 + O(\tau). \quad (10.17)$$

Suppose (10.17) holds for $(x, t) \in \tilde{B}^2$ outside a subset of measure at most $\eta\mathcal{L}(\tilde{B}^2)$, we borrow the following result in [8]:

Proposition 10.2. [8] Let $n \geq 1$. There exists a positive constant $K = K(n) > 0$ with the following property. Let B be a ball of positive, finite radius. Let $\delta > 0$ and $\eta \in (0, 1/2]$. Let $f_1, f_2, f_3 : 2B \rightarrow \mathbb{C}$ be measurable functions that vanish only on sets of Lebesgue measure zero. Suppose that

$$\mathcal{L}(\{(x, y) \in B^2 : |f_1(x)f_2(y)f_3^{-1}(x+y) - 1| > \eta\}) < \delta(\mathcal{L}(B))^2.$$

Then for each index j there exists a real-linear function $L_j : \mathbb{R}^n \rightarrow \mathbb{C}$ such that

$$\mathcal{L}(\{x \in B : |f_j(x)e^{-L_j(x)}| > K\eta^{1/K}\}) < K\delta\mathcal{L}(B).$$

Applying **Proposition 10.2** verbatim, there exist a constant $K = K(n)$, an affine function $L : \mathbb{R}^n \rightarrow \mathbb{R}$ and a subset of B of measure at most $K\eta\mathcal{L}(B)$ outside of which, $|e^{i2\pi\psi(x)}e^{-i2\pi L(x)}| < K\tau^{1/K}$. Hence the claim holds for the base case $k = 1$. Assume the claim is true for the case $k - 1$, that is, if (10.15) holds for a polynomial P_t of degree at most $k - 1$ in x , for $(x, t) \in \tilde{B}^2$ outside a subset of measure at most $O(\eta)\mathcal{L}(\tilde{B}^2)$, then (10.16) follows, with a polynomial Q of degree at most k , in terms of x .

[5] Suppose now that P_t in (10.15) is a polynomial of degree at most k in x . The leading terms in x in P_t are $\sum_{|\gamma|=k} a_\gamma(t)x^\gamma$. Suppose also for the moment that the leading coefficient functions, for $|\gamma| = k$, is, $a_\gamma(t) = \langle A_\gamma, t \rangle_{\mathbb{R}^n}$, for some elements $A_\gamma \in \mathbb{R}^n$:

$$\sum_{|\gamma|=k} a_\gamma(t)x^\gamma = \sum_{|\gamma|=k} \langle A_\gamma, t \rangle_{\mathbb{R}^n} x^\gamma. \quad (10.18)$$

Define:

$$q(x) = \sum_{|\beta|=k+1} \left(\sum_{i=1}^n \beta_i^{-1} (A_\beta)_i \right) x^\beta. \quad (10.19)$$

The relations between indices γ in (10.18) and β in (10.19), and correspondingly, A_γ and A_β , are as follows. Each β has the form $\beta = (\beta_1, \dots, \beta_i, \dots, \beta_n) = (\gamma_1, \dots, \gamma_i + 1, \dots, \gamma_n)$, for some $i \in \{1, \dots, n\}$, and for each β that is arised from a γ in this way, $A_\beta = A_\gamma$. Note that q is a polynomial of degree at most $k + 1$, and the leading x -terms of $\Delta_t q$ are,

$$\sum_{|\beta|=k+1} \left(\sum_{i=1}^n (A_\beta)_i t_i x_i^{\beta_i-1} \prod_{j \neq i} x_j^{\beta_j} \right) = \sum_{|\gamma|=k} \langle A_\gamma, t \rangle_{\mathbb{R}^n} x^\gamma \quad (10.20)$$

which is to say, $\Delta_t q$ is a polynomial of degree at most k in terms of x . Consider $\Psi(x) = \psi(x) - q(x)$. Then (10.15) yields, for $(x, t) \in \tilde{B}^2$ outside a subset of measure at most $O(\eta)\mathcal{L}(\tilde{B}^2)$,

$$e^{i2\pi\Delta_t\Psi(x)} = e^{i2\pi\Delta_t(\psi-q)(x)} = (1 + O(\tau))e^{i2\pi P_t(x)}e^{-i2\pi\Delta_t q(x)} = (1 + O(\tau))e^{i2\pi S_t(x)}. \quad (10.21)$$

Note that (10.18) and (10.20) imply that in S_t in (10.21) takes the following form:

$$S_t(x) = \sum_{j=0}^{k-1} \sum_{|\mu|=j} c_\mu(t) x^\mu.$$

Hence by the induction hypothesis, we conclude from (10.21) that there exists a polynomial Q of degree at most k such that $e^{i2\pi\Psi(x)} = (1 + o_\tau(1))e^{i2\pi Q(x)}$, and in turn, $e^{i2\pi\psi(x)} = (1 + o_\tau(1))e^{i2\pi(Q(x)+q(x))}$, for $x \in B$ outside a subset of measure at most $O(\eta)\mathcal{L}(B)$. It's obvious that $Q + q$ is a polynomial of degree at most $k + 1$; we now close the induction loop.

In the above proof of *Claim 10.1*, we need to make an assumption that if $|\gamma| = k$, $a_\gamma(t) = \langle A_\gamma, t \rangle_{\mathbb{R}^n}$ for some $A_\gamma \in \mathbb{R}^n$, for $t \in B$ outside a subset of measure at most $O(\eta)\mathcal{L}(B)$. Now we show that such condition must indeed occur:

Claim 10.2: Let ψ, η, τ be as above. Given $P_t(x) = \sum_{j=0}^k \sum_{|\gamma|=j} a_\gamma(t) x^\gamma$, with $a_\gamma : \mathbb{R}^n \rightarrow \mathbb{R}$ being measurable functions. Suppose for $(x, t) \in \tilde{B}^2$ outside a subset of measure at most $O(\eta)\mathcal{L}(\tilde{B}^2)$, the following happens:

$$e^{i2\pi\Delta_t\psi(x)} = (1 + O(\tau))e^{i2\pi P_t(x)}. \quad (10.22)$$

Then, for $|\gamma| = k$, there exist $A_\gamma \in \mathbb{R}^n$ such that $a_\gamma(t) = \langle A_\gamma, t \rangle_{\mathbb{R}^n}$, for $t \in B$ outside of a subset of measure at most $O(\eta)\mathcal{L}(B)$.

Proof of Claim 10.2: (10.22) gives us,

$$\psi(x+t) - \psi(x) \equiv \sum_{j=0}^k \sum_{|\gamma|=j} a_\gamma(t) x^\gamma + o_\tau(1). \quad (10.23)$$

By “ $u \equiv v$ ”, we mean $u - v \in \mathbb{Z}$. Let $\mathcal{A}^x = \{t \in B : (x, t) \in \tilde{B}^2 \text{ and (10.23) applies.}\}$, for $x \in B$. Let $\mathcal{A}$ denote the set of all $x \in (1/2)B$ such that $\mathcal{L}(\mathcal{A}^x) > (1 - O(\eta))\mathcal{L}(B)$. Then there exists $c > 0$ such that, for all sufficiently small η , $\mathcal{L}(\mathcal{A}) > c$. Then uniformly for $x \in \mathcal{A}$, as t varies in $\mathcal{A}^x$, these values $x+t \in B$ occupy a subset of B of measure at least $(1 - O(\eta))\mathcal{L}(B)$. Fix $x \in \mathcal{A}$, this set of values $x+t$ has a nonempty intersection with $\mathcal{A}$. Fix one such value $x+t \in \mathcal{A}$. Then $\mathcal{L}(\mathcal{A}^{x+t}) > (1 - O(\eta))\mathcal{L}(B)$. That means, the set of values $t+s$, as s varies in $\mathcal{A}^{x+t}$, occupy a subset of B of measure at least $(1 - O(\eta))\mathcal{L}(B)$. This set of values $t+s$ will then have a nonempty intersection with $\mathcal{A}^x$. In other words, we can

select x, t, s so that the following argument is applicable:

Consider $\psi(x+t+s) - \psi(x+t)$. From (10.23) we can write the said difference in two ways.

Firstly:

$$\begin{aligned} \psi((x+t)+s) - \psi(x+t) &\equiv \sum_{j=0}^k \sum_{|\gamma|=j} a_\gamma(s)(x+t)^\gamma + o_\tau(1) = \sum_{|\gamma|=k} a_\gamma(s)(x+t)^\gamma + O(x^{k-1}) + o_\tau(1) \\ &= \sum_{|\gamma|=k} a_\gamma(s)x^\gamma + O(x^{k-1}) + o_\tau(1). \end{aligned} \quad (10.24)$$

By “ $O(x^{k-1})$ ” we mean a linear combination of monomials in terms of x of degree at most $k-1$. Secondly:

$$\begin{aligned} \psi(x+t+s) - \psi(x+t) &= [\psi(x+(t+s)) - \psi(x)] - [\psi(x+t) - \psi(x)] \\ &\equiv \sum_{|\gamma|=k} [a_\gamma(t+s) - a_\gamma(t)]x^\gamma + O(x^{k-1}) + o_\tau(1). \end{aligned} \quad (10.25)$$

Since $\mathcal{L}(\mathcal{A}) > c$, a comparison between (10.24) and (10.25) shows that, for $|\gamma| = k$:

$$a_\gamma(t+s) = a_\gamma(t) + a_\gamma(s). \quad (10.26)$$

By the argument presented above, (10.26) is satisfied for $t, s \in B$ except for a subset of measure at most $O(\eta)\mathcal{L}(B)$. Let $\{\rho_i\}_i$ be a sequence tending to zero. Then from (10.26), for a.e. $(t, s) \in B \times B$ outside a subset of measure at most $O(\eta)\mathcal{L}(B \times B)$ and for every ρ_i , we have, $|a_\gamma(t+s) - a_\gamma(t) - a_\gamma(s)| < \rho_i$. By **Proposition 9.1** and its proof given in [8], there exist affine functions $L_{\rho_i}(t) = \langle A_{\rho_i}, t \rangle_{\mathbb{R}^n} + b_{\rho_i}$, $A_{\rho_i} \in \mathbb{R}^n$, $b_{\rho_i} \in \mathbb{R}$, and a subset $U \subset B$, such that $a_\gamma(t) - L_{\rho_i}(t) = O(\rho_i)$ for a.e. $t \in U$. The relative complement of U has measure at

most $O(\eta)\mathcal{L}(B)$. The implicit constants in the notations $O(\rho_i), O(\eta)\mathcal{L}(B)$ are independent of i . Hence, $L_{\rho_i}(t)$ converges for every $t \in U$, and since L_{ρ_i} are affine functions on $\mathbb{R}^n$, this in turn implies $A_{\rho_i} \rightarrow A \in \mathbb{R}^n$ and $b_{\rho_i} \rightarrow b \in \mathbb{R}$ as $i \rightarrow \infty$. Hence $a_\gamma(t) = \langle A, t \rangle_{\mathbb{R}^n} + b$ for $t \in B$ outside of a subset of measure at most $O(\eta)\mathcal{L}(B)$. But then (10.26) gives $b = 0$, hence the proof of *Claim 10.2* is now complete.

We have also finished the proof of **Theorem 10.1**.

10.3 Production of an extremizer

We combine the results the previous chapters, (10.9) and **Theorem 10.1** to conclude that there exist a centered ball B_R , with R sufficiently large, so that

$$\int_{\mathbb{R}^n \setminus B_R} |f|^{p_k} = o_\delta(1) \quad (10.27)$$

and a polynomial P of degree at most $k - 1$ so that,

$$\|(e^{i2\pi P} - a) \cdot \chi_{B_R}\|_{p_k} = o_\delta(1). \quad (10.28)$$

Let $\mathcal{G} = \mathcal{F}^\dagger e^{i2\pi P}$, with $\mathcal{F}^\dagger$ is as in **Section 10.1**. Then $\mathcal{G}$ is an extremizer of (1.4). Observe:

$$\|\mathcal{G} - f\|_{p_k}^{p_k} = \int_{\mathbb{R}^n \setminus B_R} |\mathcal{G} - f|^{p_k} + \int_{B_R} |\mathcal{G} - f|^{p_k}. \quad (10.29)$$

By (10.27), the facts that $f(x) = |f|(x)a(x)$ and $|e^{i2\pi P} - a| \leq C$ uniformly on $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n \setminus B_R} ||f|e^{i2\pi P} - |f|a|^{p_k} = o_\delta(1).$$

which together with the fact that $\|\mathcal{F}^\dagger - |f|\|_{p_k}^{p_k} = o_\delta(1)$ allows us to conclude the following for the first term in (10.29):

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_R} |\mathcal{F}^\dagger e^{i2\pi P} - |f||a|^{p_k}| &\leq C(k) \int_{\mathbb{R}^n \setminus B_R} |\mathcal{F}^\dagger e^{i2\pi P} - |f|e^{i2\pi P}|^{p_k} + C(k) \int_{\mathbb{R}^n \setminus B_R} ||f|e^{i2\pi P} - |f||a|^{p_k}| \\ &\leq C(k) \|\mathcal{F}^\dagger - |f|\|_{p_k}^{p_k} + o_\delta(1) = o_\delta(1). \end{aligned}$$

In the same spirit, we split the second term in (10.29):

$$\int_{B_R} |\mathcal{G} - f|^{p_k} \leq C(k) \int_{B_R} |\mathcal{F}^\dagger e^{i2\pi P} - \mathcal{F}^\dagger a|^{p_k} + C(k) \int_{B_R} |\mathcal{F}^\dagger a - |f||a|^{p_k}|. \quad (10.30)$$

The second term in (10.30) is at most $o_\delta(1)$ in absolute value, again by $\|\mathcal{F}^\dagger - |f|\|_{p_k}^{p_k} = o_\delta(1)$ and $|a| = 1$ on $\mathbb{R}^n$. It follows from (10.28) that the contribution of first term in (10.30) is also at most $o_\delta(1)$ in absolute value:

$$\int_{B_R} |\mathcal{F}^\dagger e^{i2\pi P} - \mathcal{F}^\dagger a|^{p_k} \leq C(k) \int_{B_R} |e^{i2\pi P} - a|^{p_k} = o_\delta(1).$$

Hence we conclude that $\|\mathcal{G} - f\|_{p_k} = o_\delta(1)$.

We have now obtained the conclusion for **Theorem 1.1** for a general measurable near extremizer $f : \mathbb{R}^n \rightarrow \mathbb{C}$, $n \geq 1$, of the k th Gowers-Host-Kra norm inequality for $k \geq 2$.

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