Title

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Publication Date

2024

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by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

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Spring 2024

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Abstract

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This thesis provides a detailed account of several novel methods and ideas developed by the author and collaborators to study the low regularity dynamics for a diverse selection of nonlinear PDE. In this manuscript, these techniques are applied to resolve several questions concerning the well-posedness of various families of quasilinear dispersive equations and free boundary problems arising in fluid mechanics.

After giving a brief overview of the main results in Chapter 1, we begin in Chapter 2 with a systematic analysis of the *incompressible free boundary Euler equations* on a time-dependent, compact fluid domain Ω_t,

$$\begin{cases} \partial_t v + v\cdot\nabla v = -\nabla p - ge_d \ \text{ on } \ \Omega_t, \\ \nabla\cdot v = 0 \ \text{ on } \ \Omega_t, \\ \partial_t + v\cdot\nabla \ \text{ is tangent to } \bigcup_t \{t\}\times\partial\Omega_t \subseteq \mathbb{R}^{d+1}, \\ p_{|\partial\Omega_t} = 0. \end{cases}$$

This system models, among other things, the dynamics of a fluid droplet under the influence of gravity. In this chapter, a complete local well-posedness theory in H^s-based Sobolev spaces is developed. Our well-posedness theory includes (i) Local well-posedness in the Hadamard sense, i.e., local existence, uniqueness, and the first proof of continuous dependence on the data, all in low regularity Sobolev spaces; (ii) Enhanced uniqueness: Our uniqueness

result holds at the level of the Lipschitz norm of the velocity and the $C^{1,\frac{1}{2}}$ regularity of the free surface; (iii) Stability bounds: We construct a nonlinear functional which measures, in a suitable sense, the distance between two solutions (even when defined on different domains) and we show that this distance is propagated by the flow; (iv) Energy estimates: We prove refined, essentially scale invariant energy estimates for solutions, relying on a newly constructed family of elliptic estimates; (v) Continuation criterion: We give the first proof of a sharp continuation criterion in the physically relevant pointwise norms, at the level of scaling. In essence, we show that solutions can be continued as long as the velocity is in $L^1_T W^{1,\infty}$ and the free surface is in $L^1_T C^{1,\frac{1}{2}}$, which is at the same level as the Beale-Kato-Majda criterion for the boundaryless case; (vi) A novel proof of the construction of regular solutions. Our entire approach is in the Eulerian framework and can be adapted to work in more general fluid domains.

In Chapter 3, we move to a systematic study of the so-called *general quasilinear ultrahyperbolic Schrödinger equation*,

$$\begin{cases} i\partial_t u + \partial_j g^{jk}(u,\overline{u})\partial_k u = F(u,\overline{u},\nabla u,\nabla\overline{u}), \qquad u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}^m, \\ u(0,x) = u_0(x). \end{cases}$$

Here, g is some real, symmetric, and uniformly non-degenerate metric and F is some smooth nonlinear function of its arguments. In this chapter, we develop novel techniques for establishing large data local well-posedness in low regularity Sobolev spaces for this equation. Our main result represents a definitive improvement over the landmark results of Kenig, Ponce, Rolvung, and Vega [84, 86, 87, 90], as it weakens the regularity and decay assumptions to the same scale of spaces considered by Marzuola, Metcalfe and Tataru in [106], but removes the uniform ellipticity assumption on the metric from their result. Our method has the additional benefit of being relatively simple and robust. In particular, it only relies on pseudodifferential calculus for classical symbols.

Finally, in Chapter 4, we turn our attention to a more specialized quasilinear dispersive model; namely, the *generalized derivative nonlinear Schrödinger equation* (GDNLS),

$$\begin{cases} i\partial_t u + \partial_x^2 u = i|u|^{2\sigma}\partial_x u, \\ u(0) = u_0. \end{cases}$$

We study this equation in the regime $\frac{1}{2} < \sigma < 1$ where the local theory is most difficult, and analyze this equation at both low and high regularity to establish the first global well-posedness result for this problem in H^s spaces. This appears to also be the first result of its kind for any quasilinear dispersive model where the nonlinearity is both rough and lacks the decay necessary for global smoothing-type estimates. These two features pose considerable difficulty when trying to apply standard tools for closing low-regularity estimates, such as Strichartz estimates, gauge transformations or maximal function estimates. To circumvent this issue, several new ideas are developed. In addition to establishing a suitable global theory, we also dramatically improve the local results in the high regularity regime compared to the previous literature.

To my parents.

Contents

Acknowledgments

I would first like to thank my advisor, Daniel Tataru. He was of tremendous help to me when I was first introduced to the subjects of dispersive PDE and free boundary problems. More importantly, through his mentorship, I developed standards for conducting research that only a small handful of other world-class mathematicians could have taught me. His unique and systematic approach to problem-solving is something I hope to emulate in the years to come.

Secondly, I would like to thank Mitchell Taylor, one of my good friends (and former roommate). Most of my recent works in PDE and also on the subject of phase retrieval (not discussed in this thesis) are joint with him. In our collaborations, he has brought forth a host of important, novel ideas that have set the foundations for many of our results. He is also very meticulous in removing the myriad of typos I seem to introduce to our draft papers, which I greatly appreciate.

I would also like to thank my friends and family for their love and support throughout this long and arduous process and for keeping me level-headed during the most stressful points of this degree. Additionally, I would like to thank my fellow graduate students as well as my office-mates Ovidiu Avadanei and Izak Oltman for making many of the past years very enjoyable.

All three of the works in this thesis are based on joint works and include the contents of the following preprints:

(i) [75] titled *Sharp Hadamard local well-posedness, enhanced uniqueness, and pointwise continuation criterion for the incompressible free boundary Euler equations.* This constitutes the majority of the material in Chapter 2 and is joint with Mihaela Ifrim, Daniel Tataru, and Mitchell Taylor.

(ii) [128] titled *Low regularity solutions for the general quasilinear ultrahyperbolic Schrödinger equation.* This constitutes the majority of the material in Chapter 3 and is joint with Mitchell Taylor.

(iii) [129] titled *Global well-posedness for the generalized derivative nonlinear Schrödinger equation.* This constitutes the majority of the material in Chapter 4 and is joint with Mitchell Taylor.

Finally, I am most fortunate to have many other fantastic collaborators and mathematical influences. In the realm of PDE, I thank Mihaela Ifrim and Sung-Jin Oh for their excellent mentorship over the years. In some sense, they both played the role of secondary advisor, which I am grateful for. I look forward to continuing my ongoing collaborations with both of them. I would also like to thank my other co-authors, Michael Christ, Daniel Freeman, Timur Oikhberg, and Xinwei Yu with whom I have produced many interesting results in the areas of stable phase retrieval and fluid mechanics. Not present in this thesis are my articles [27, 43, 74, 130, 131, 132, 133], which many of these co-authors either influenced or directly collaborated on.

During my time as a graduate student, I was supported by the Berkeley fellowship for the full duration of my degree. I was also supported partially by Prof. Daniel Tataru's NSF grant DMS-2054975 as well as by a Simons Investigator grant from the Simons Foundation. Some work in this thesis was carried out while in residence at the Simons Laufer Mathematical Sciences Institute (formerly MSRI) in Berkeley, California, during the summer of 2023, while participating in the program "Mathematical problems in fluid dynamics, Part II", which was supported by the National Science Foundation under Grant No. DMS-1928930.

Chapter 1

Introduction

The primary objective of this thesis is to present a collection of new techniques developed to understand the low regularity well-posedness of certain classes of quasilinear dispersive equations and free boundary problems arising in fluid mechanics. This text is divided into three main chapters, each focusing on a different model problem. In Chapter 2, we systematically study the Cauchy problem for the free boundary Euler equations. This is a system of nonlinear PDE that models the dynamics of an inviscid fluid in a time-dependent domain. In Chapters 3 and 4, we turn our attention to understanding two distinct classes of quasilinear dispersive equations. In such problems, one is generally tasked with understanding the dynamics of nonlinear wave interactions. Some fundamental models are the nonlinear Schrödinger equations, nonlinear wave equations, the Korteweg–De Vries equation, and so forth. In this thesis, we will be concerned primarily with dispersive equations of Schrödinger type. The models we consider here arise in many physical scenarios including water waves, integrable systems, quantum mechanics and magnetohydrodynamics.

In the literature, what one means by well-posedness tends to vary somewhat from one equation to another, but to heuristically describe the key elements that one would like in such a theory, let us begin by considering the following general model nonlinear evolution equation,

$$\begin{cases} \partial_t u + F(u, Du, \dots, D^k u) = 0, \text{ on } [0,T] \times \mathbb{R}^d, \\ u(0) := u_0, \text{ on } \mathbb{R}^d. \end{cases} \tag{1.0.1}$$

Here $[0,T]$ is some time interval, F is a smooth function of its arguments and u_0 is some suitable initial datum. Above, $D := (\partial_{x_1}, ..., \partial_{x_d})$.

Remark 1.0.1. Of course, the spatial domain $\mathbb{R}^d$ above can be replaced by more general domains (for instance, when we study the free boundary Euler equations later, it will be some time-dependent open subset of $\mathbb{R}^d$). Moreover, F need not be smooth (which we will have to contend with in Chapter 4). However, for the sake of our heuristic discussion, we specialize for now to the above situation.

At the bare minimum, well-posedness for (1.0.1) amounts to studying the following question: For some suitable function space X_0 (for instance, a Sobolev space) and $u_0 \in X_0$, can we construct a unique solution $u \in C([0,T];X_0)$ to (1.0.1) on some time interval whose size depends on u_0? Of course, whether this question is well-defined depends heavily on the nature of the PDE in question and on the function space X_0. It also turns out that in many situations, insisting that a solution u (with given initial data) be unique in the class $C([0,T];X_0)$ is too strong. It is often convenient to instead work with a stronger function space X_T which embeds continuously into $C([0,T];X_0)$ for which one can establish existence and uniqueness for (1.0.1) in X_T. In the context of nonlinear dispersive equations, this could take the form of a Strichartz space or local smoothing type space (which we will employ in Chapters 3 and 4).

Many equations of the form (1.0.1) come directly from real-world physical systems. Therefore, it is also of fundamental importance to understand how sensitive the dynamics of solutions are to small perturbations of the initial data. This is the question of continuous dependence. More precisely, if we have a sequence of initial data $u_0^n \in X_0$ with $u_0^n \to u_0 \in X_0$, does it follow that the corresponding solutions (after possibly restricting the time interval) $u^n \in X_T$ generated by u_0^n converge to the solution $u \in X_T$ generated by u_0? To build on this notion, it is also often of interest to quantify the strength of this dependence. For instance, one can ask whether we have a Lipschitz-type bound roughly of the form

$$\|v-u\|_{X_T} \leq C\|u_0-v_0\|_{X_0},$$

for solutions v and u generated by data v_0 and u_0, respectively. Here, $C>0$ is a constant which can in general depend on v and u. For many nonlinear equations, one can fruitfully treat the nonlinear part of the equation as a perturbative error term (at least on short time scales), to construct a solution by means of Picard iteration or the Contraction Mapping Theorem. This (by design) allows one to construct a data-to-solution map which is Lipschitz in the above sense. When it is possible to construct solutions in this way, it is common to call the corresponding problem semilinear. For instance, it is well known that the nonlinear

Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta)u = |u|^{p-1}u \ \text{ on } \ [0,T]\times\mathbb{R}^d, \\ u(0) := u_0 \ \text{ on } \ \mathbb{R}^d, \end{cases}$$

is semilinear in this sense when p is a positive odd integer, X_0 is a suitable Sobolev space and X_T is a suitable Strichartz space (see Chapter 4 for a discussion of these spaces). To show this, one can often exploit the various dispersive estimates that hold for the corresponding inhomogeneous linear flow

$$\begin{cases} (i\partial_t + \Delta)u = f \ \text{ on } \ [0,T]\times\mathbb{R}^d, \\ u(0) := u_0 \ \text{ on } \ \mathbb{R}^d. \end{cases}$$

When one does not have Lipschitz dependence on the initial data, but merely continuous dependence, it is common to refer to the Cauchy problem under consideration as quasilinear. These are problems in which one cannot (in contrast to semilinear problems) treat the nonlinear part of the equation perturbatively in the function spaces being considered. In most cases, this makes the question of local well-posedness considerably more challenging. For instance, in Chapter 4, we will consider a Schrödinger type equation in which (unlike the nonlinear Schrödinger equation above) one cannot (directly) apply standard Strichartz estimates or other dispersive tools in the analysis.

All three of the equations considered in this thesis are of quasilinear type. Despite the overarching theme of this thesis being one of well-posedness, the methods used to address each problem are rather diverse and vary considerably from equation to equation. The purpose of the remainder of this introduction is to provide an expository overview of each of the main results. We opt to postpone a more technical overview of each problem to their corresponding chapters. This manuscript is structured such that each chapter can be read independently of the other, in a relatively modular fashion.

1.1 Free boundary problems

Chapter 2 will be focused on the contents of the preprint [75], which is concerned with the well-posedness of a class of free boundary problems arising in fluid mechanics. Free boundary problems in this context are equations where the evolution of the fluid boundary is strongly coupled to that of the flow. Classical examples include the dynamics of water waves or a gaseous star. A large subset of these problems are modeled by the free boundary incompressible Euler equations

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p - g e_d, \\ \nabla \cdot v = 0, \end{cases} \tag{1.1.1}$$

which describes the motion of an inviscid fluid on some time-dependent domain $\Omega_t \subset \mathbb{R}^d$. Here, v is the fluid velocity, p is the pressure and $g \geq 0$ is the gravitational constant. The dynamics of v are coupled to the domain by the kinematic boundary condition, which asserts that the material derivative vector field

$$D_t := \partial_t + v \cdot \nabla \;\text{ is tangent to } \bigcup_t \{t\} \times \partial\Omega_t \subseteq \mathbb{R}^{d+1}. \tag{1.1.2}$$

Physically, this asserts that the free surface $\Gamma_t := \partial\Omega_t$ moves with the fluid velocity v. Additionally, we have the dynamic boundary condition

$$p|_{\Gamma_t} = \sigma\kappa, \tag{1.1.3}$$

where $\sigma \geq 0$ is the surface tension and κ is the mean curvature of the free surface. This represents a balance of forces at the fluid interface between the interior of the fluid and the atmosphere. In this thesis, we specialize in the case of zero surface tension (that is, $\sigma = 0$).

Of fundamental importance is understanding the Cauchy problem for the free boundary Euler equations, which roughly amounts to the question: Given an initial state $(v_0, \Gamma_0) \in H^s$ with v_0 divergence-free, can we find a unique solution $(v, \Gamma) \in C([0,T]; H^s)$ which persists on some non-trivial time interval $[0,T]$? For zero surface tension, this question turns out to be fundamentally tied to the sign of the *Taylor coefficient*, a, which is defined on the boundary Γ_t by

$$a := -\nabla p \cdot n_{\Gamma_t}. \tag{1.1.4}$$

It is a classical result of Ebin [40] that the free boundary Euler equations are ill-posed unless $a \geq 0$. Therefore, it is natural to assume a uniform lower bound $a > c_0 > 0$ on the Taylor term for the initial data (which one hopes to propagate for some time). Physically, this condition ensures that the pressure increases into the fluid. Geometrically, it asserts that p is a non-degenerate defining function for the free boundary hyper-surface Γ_t.

To understand the correct Sobolev regularities for studying this problem, our first clue comes from the natural scaling symmetry,

$$v_\lambda(t,x) = \lambda^{-\frac{1}{2}} v\left(\lambda^{\frac{1}{2}} t, \lambda x\right), \quad p_\lambda(t,x) = \lambda^{-1} p\left(\lambda^{\frac{1}{2}} t, \lambda x\right), \quad (\Gamma_\lambda)_t = \{\lambda^{-1}x : x \in \Gamma_{\lambda^{\frac{1}{2}} t}\},$$

which is exactly the scaling that leaves the Taylor term dimensionless. We remark that the scale-invariant Sobolev index is given by $s_c = \frac{d+1}{2}$, which naturally restricts our range of exponents to $s \geq s_c$. However, this does not tell the full story, as even in the boundaryless case a result of Bourgain-Li [17] shows that well-posedness holds only in the more restricted range

$$s > \frac{d}{2} + 1,$$

which is heuristically connected to another scaling law of the boundaryless problem; namely,

$$v(t,x) \mapsto \lambda^{-1} v(t, \lambda x).$$

This latter exponent range $s > \frac{d}{2} + 1$ is what is considered in this thesis. Now, we aim to present a schematic overview of the results obtained in this thesis and [75] related to this question. To avoid cumbersome topological issues and notation, we postpone providing completely precise statements until Chapter 2. In a nutshell, the results can be divided into three main theorems. The first is Hadamard well-posedness.

Theorem 1.1.1 (Ifrim, P., Tataru, Taylor, [75])**.** Fix $s > \frac{d}{2} + 1$. For any (v_0, Γ_0) in H^s with v_0 divergence-free, there exists a time $T > 0$, depending only on the data size and the lower bound in the Taylor term, for which there exists a unique solution $(v(t), \Gamma_t) \in C([0,T]; H^s)$ to the free boundary Euler equations. Moreover, the data-to-solution map is continuous with respect to the H^s topology.

This in particular establishes the first proof of local well-posedness for the free boundary Euler equations at the optimal regularity threshold on a compact fluid domain. It also provides the first proof of continuity of the data-to-solution map for this problem which was previously unknown at any regularity. This last issue is notoriously difficult to deal with due to the highly nonlinear character of the problem and because it requires one to compare solutions that are defined on different domains. One important ingredient to overcome this is the construction of a novel nonlinear distance functional, which is suitable for obtaining local L^2 stability type bounds for this problem. This construction as well as some consequences are very roughly summarized by the following theorem.

Theorem 1.1.2 (Ifrim, P., Tataru, Taylor, [75])**.** Suppose that (v, Γ_t) and $(v_h, \Gamma_{t,h})$ are sufficiently nearby solutions to the free boundary Euler equations in a time interval $[0, T]$ and satisfy the Taylor condition a,$a_h > c_0 > 0$. Then there exists a distance functional $(v, v_h) \mapsto D(v, v_h)$ which is propagated by the flow, with the bound

$$\frac{d}{dt} D(v, v_h) \lesssim_{A,A_h} (B + B_h) D(v, v_h)$$

where

$$B := \|v\|_{C^1(\Omega_t)} + \|\Gamma_t\|_{C^{1,\frac{1}{2}}} + \|D_t p\|_{C^1(\Omega_t)}, \qquad A := \|v\|_{C^{\frac{1}{2}+\varepsilon}(\Omega_t)} + \|\Gamma_t\|_{C^{1,\varepsilon}},$$

and B_h and A_h are the analogous quantities corresponding for (v_h, Γ_h).

Here, A is an implicit growth parameter used to control constants in fixed-time elliptic estimates and B is a dynamic control parameter that appears to linear order, which controls the growth of the distance functional in time. One can observe that this estimate is essentially scale-invariant. We postpone providing a precise description of $D(v, v_h)$ until Chapter 2, but it measures (in a suitably coercive manner) the L^2 distance between the two velocity functions v and v_h as well as the boundary hypersurfaces Γ and Γ_h. This bound plays a critical role in the proof of the continuous dependence result mentioned above. One other important consequence is the following new uniqueness result that holds at very limited regularity (i.e. at even lower regularity than our well-posedness result). It can roughly be stated as follows.

Theorem 1.1.3 (Ifrim, P., Tataru, Taylor, [75])**.** For every initial data $(v_0, \Gamma_0) \in C^1 \times C^{1,\frac{1}{2}}$, there is at most one solution (v, Γ) in the class $A \in L^\infty_T$ and $B \in L^1_T$.

Another fundamental question to consider are necessary conditions under which locally well-posed solutions can develop singularities. On this note, in Chapter 2 we also establish the first criterion in this direction which is on the same scale as the celebrated Beale-Kato-Majda criterion [11] for the Euler equations in the boundaryless case. Roughly speaking, our result is as follows:

Theorem 1.1.4 (Ifrim, P., Tataru, Taylor, [75])**.** Let $s > \frac{d}{2} + 1$. Then H^s solutions can be continued for as long as

$$\sup_{0 \le t < T} \|v(t)\|_{C^{\frac{1}{2}+\varepsilon}(\Omega_t)} + \|\Gamma(t)\|_{C^{1,\varepsilon}} < \infty, \qquad \int_0^T \|v(t)\|_{C^1(\Omega_t)} + \|\Gamma(t)\|_{C^{1,\frac{1}{2}}} dt < \infty.$$

This result gives a very definitive answer to a well-known question of Craig and Wayne [92]. It is also notable in that it is phrased only in terms of the natural dynamic variables (v, Γ) in point-wise norms, in contrast to almost all other works on this problem. This question has received quite a bit of attention recently. We note, for instance, a small sample of the much weaker results in this direction obtained in [35, 48, 157]. The proof of Theorem 1.1.4 relies on novel and very delicate scale-invariant energy estimates and a careful usage of the distance functional above. To establish the requisite energy estimates, we had to develop a new family of elliptic estimates that more precisely balance the contributions of the input function and the domain-dependent constants in the bounds for various elliptic operators (one example being the Dirichlet-Neumann operator), simultaneously, in both pointwise and L^2 based norms. The reader is referred to Section 2.5 in Chapter 2 for details. These estimates can be thought of as significant generalizations of the so-called tame estimates which have been fundamental in the analysis of many free boundary problems. See [5, 95], for instance.

1.2 Quasilinear Schrödinger equations

In both Chapters 3 and 4, we turn our attention to the realm of nonlinear dispersive equations. One of the most important classes of such equations are the so-called nonlinear Schrödinger equations. In one of the most general formulations, they take the form

$$\begin{cases} i\partial_t u + \partial_j g^{jk}(u, \overline{u})\partial_k u = F(u, \overline{u}, \nabla u, \nabla \overline{u}), \quad u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^m, \\ u(0, x) = u_0(x), \end{cases} \tag{1.2.1}$$

where g is some real, symmetric, and uniformly non-degenerate metric (which here, we allow to depend on u itself) and F is some nonlinear function of its arguments. Even at this level of generality, such an equation is ubiquitous in several physical systems. Some well-known examples come from the study of water waves [34] and the theory of completely integrable models [76, 137]. More recently, it was shown that the Hall magnetohydrodynamic equations without resistivity behave at leading order like a degenerate quasilinear Schrödinger system of the above type [80]. A very well-studied sub-class of the above is the nonlinear Schrödinger equation (NLS),

$$i\partial_t u + \Delta u = \pm |u|^{p-1} u, \quad p \geq 1, \tag{NLS}$$

which is a fundamental semilinear dispersive model. Another intensely studied example, which is more quasilinear, is given by the one dimensional generalized derivative nonlinear

Schrödinger equation (GDNLS)

$$i\partial_t u + \partial_x^2 u = i|u|^{2\sigma}\partial_x u, \quad \sigma \geq \frac{1}{2}. \tag{GDNLS}$$

When $\sigma = 1$, this equation is referred to as the derivative nonlinear Schrödinger equation (DNLS) and is physically motivated by the study of the one-dimensional compressible magneto-hydrodynamic equation in the presence of the Hall effect, and the propagation of circular polarized nonlinear Alfvén waves in magnetized plasmas.

In Chapter 3, we begin by studying the low regularity well-posedness problem for (1.2.1) in its most general form when F is a smooth function of its arguments. In particular, unlike with the models (NLS) and (GDNLS), we do not even assume that the principal operator $g^{jk}\partial_j\partial_k$ is elliptic. In this setting, (1.2.1) is often referred to as the *quasilinear ultrahyperbolic Schrödinger equation*. The contents of this chapter are based on the preprint [128].

Arguably, the first well-posedness results for this equation stem from the pioneering series [84, 86, 87, 90] of Kenig, Ponce, Rolvung and Vega (KPRV). This sequence ultimately culminates in a proof of large data well-posedness under a nontrapping assumption on the metric for systems of the form (1.2.1) in high regularity weighted Sobolev spaces of the form $H^s \cap L^2(\langle x\rangle^N dx)$. Here, s and N are suitably large, dimension-dependent parameters. In these fundamental works, [87] studies the well-posedness problem assuming ellipticity of the principal operator $\partial_i g^{ij}\partial_j$, while [84, 86, 90] consider more general symmetric, non-degenerate metrics, first in the constant coefficient case and then later for variable coefficients. The regularity and decay assumptions on the data in these results are rather strong. Given the physical motivation for this problem, it is therefore of considerable interest to weaken these assumptions as much as possible. One significant advance in this direction comes from the article [106] of Marzuola, Metcalfe and Tataru (MMT), which studies the problem in low regularity Sobolev spaces in the case that the principal operator $g^{jk}\partial_j\partial_k$ is elliptic. Instead of weighted Sobolev spaces, the data here comes from the much weaker space l^1H^s, $s > \frac{d}{2} + 2$. Here, l^1H^s is an appropriate translation invariant Sobolev-type space, imposing similar regularity requirements as H^s, but slightly stronger decay. See [106] or Section 3.2 for the precise definition. This additional decay is necessary in general. To understand why this is the case, it is instructive to inspect the leading part of the linearized flow which is given by

$$\begin{cases} i\partial_t v + \partial_j g^{jk}\partial_k v + b^j\partial_j v + \tilde{b}^j\partial_j\overline{v} = f, \\ v(0,x) = v_0. \end{cases} \tag{1.2.2}$$

A well-known necessary condition for L^2 well-posedness of the above linear system is that the first order coefficient $\mathrm{Re}(b^j)$ is integrable along the bicharacteristic (or Hamilton) flow of the principal differential operator $\partial_j g^{jk}\partial_k$. This is the well-known Mizohata condition, which is not guaranteed by the milder decay condition $u_0 \in H^s$. Therefore, the introduction of the space l^1H^s is natural. It is worth remarking that these spaces impose considerably weaker decay and regularity than the weighted Sobolev spaces considered by Kenig, Ponce, Rolvung, and Vega.

In the small data regime (i.e. $\|u_0\|_{l^1H^s} \ll 1$) where $g^{jk}(u_0,\overline{u_0})$ is close to a Euclidian metric and the first order coefficients b^j and $\tilde{b}^j$ are small in a suitable sense, this additional decay is essentially the only further requirement in establishing local well-posedness. In this case, the Hamilton trajectories are approximately straight lines and the first-order terms can be treated perturbatively through the use of local smoothing type estimates similar to those exhibited by the flat flow. However, in the large data regime, the Hamilton trajectories can a priori be confined to a compact set for an infinite length of time and moreover, the first-order coefficients can be large. In light of the Mizohata condition, to deal with the first issue, it is natural to impose a non-trapping condition on the (initial) metric, which ensures that all nontrivial bicharacteristics escape to spatial infinity at both ends. Dealing with the large first-order terms on the other hand requires considerable care and this issue is at the heart of the arguments in the works of KPRV and MMT and also our work discussed in Chapter 3.

The crucial role played by the Hamilton flow also suggests a natural regularity threshold to aim for in the study of the local well-posedness of the above system; namely $s > \frac{d}{2}+2$. From the perspective of Sobolev embeddings, this ensures that for (1.2.1) the metric g has at least C^2 regularity, which in particular ensures the Hamilton flow for the principal operator $\partial_j g^{jk}\partial_k$ is locally well-defined. This is important for making sense of the Mizohata condition mentioned above. The following theorem, which is the main result of our paper [128] shows that this regularity assumption is actually sufficient for constructing solutions to (1.2.1) in the large data regime,

Theorem 1.2.1 (P., Taylor, [128])**.** Let $s > \frac{d}{2}+2$ and suppose that the initial data $u_0 \in l^1H^s$ is such that $g(u_0)$ is a real, symmetric, uniformly non-degenerate, nontrapping metric.

Assume that F is smooth and vanishes at least quadratically at the origin. Then (1.2.1) is locally well-posed in l^1H^s.

Our result therefore represents a definitive improvement over the landmark results of Kenig, Ponce, Rolvung, and Vega [84, 86, 87, 90]. A detailed overview of the proof of this result can be found in Section 3.3. We remark that our method is very robust and also relatively simple, as unlike in some of the above-mentioned papers, it only relies on the use of pseudodifferential operators with classical symbols.

Finally, in Chapter 4, we turn our attention to a more specialized model quasilinear Schrödinger equation; namely, the generalized derivative nonlinear Schrödinger equation given by (GDNLS). The contents of this chapter are based on the preprint [129].

In stark contrast to (GDNLS), we begin by remarking that both the local and global well-posedness theory for the semilinear equation (NLS) is by now very well understood in many cases of interest. Indeed, if s_c is the critical index, i.e., the index for which the Sobolev norm $\|u\|_{\dot{H}^{s_c}}$ is invariant under the scaling symmetry of (NLS), one can often establish local well-posedness (and global well-posedness when the nonlinearity is defocusing) in H^s based spaces when $s \geq \max\{0, s_c\}$. See Tao's book [150] for an overview of some of the results in this direction. Like with (NLS), the (GDNLS) equations admit a one-parameter family of scaling symmetries,

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda^{\frac{1}{2\sigma}} u(\lambda^2 t, \lambda x), \quad \lambda > 0,$$

which makes the critical Sobolev index $s_c = \frac{1}{2} - \frac{1}{2\sigma}$. In particular, the problem is L^2_x critical when $\sigma = 1$ and subcritical when $\sigma < 1$. Moreover, (GDNLS) admits the following conserved quantities:

$$\begin{aligned}
M(u) &= \frac{1}{2}\int_{\mathbb{R}} |u|^2 dx,\\
P(u) &= \frac{1}{2} Re \int_{\mathbb{R}} i\overline{u} u_x dx,\\
E(u) &= \frac{1}{2}\int_{\mathbb{R}} |u_x|^2 dx + \frac{1}{2(\sigma+1)} Re \int_{\mathbb{R}} i|u|^{2\sigma}\overline{u} u_x dx,
\end{aligned}$$

which are the mass, momentum and energy, respectively. In terms of Sobolev regularity, these quantities correspond to L^2_x, $H^{\frac{1}{2}}_x$ and H^1_x, respectively. Motivated by the conserved energy above, a longstanding question until very recently was to understand in the case

$\sigma = 1$ whether the equation (GDNLS) is globally well-posed in H^1. This was resolved in [10]. Global well-posedness in L^2 was shown shortly after in [59]. Ultimately, the resolution of this problem rested heavily on the fact that when $\sigma = 1$ the equation is completely integrable, which allowed for the systematic use of methods from inverse scattering. Amusingly, until our preprint [129], the question of whether the equation (GDNLS) is globally well-posed in H^1 was unknown for any $\sigma \neq 1$. This is despite considerable efforts in the literature. In fact, the main motivation for introducing (GDNLS) was to better understand the global theory when $\sigma = 1$. It turns out that the obstructions in the case $\sigma > 1$ and $\sigma < 1$ are essentially dual. For the case $\sigma > 1$, the nonlinearity has enough decay (and regularity) to allow for one to obtain local H^1 solutions. This can be done with a contraction argument using a variety of now standard methods ranging from the global smoothing or maximal function type estimates of Kenig, Ponce, and Vega or even just Strichartz estimates (after performing a suitable gauge transformation to conjugate out the derivative nonlinearity). On the other hand, in this case, the problem is L^2 supercritical with respect to scaling. Therefore, like in the case $\sigma = 1$ (i.e. the L^2_x critical case), the energy and mass are in general not suitably coercive to control the H^1_x norm for long times. In fact, when $\sigma > 1$, finite-time blowup is expected, but has yet to be proved. On the other hand, in the case $\sigma < 1$, the problem is L^2_x subcritical. As a result, the conserved mass and energy above can be used to control the H^1_x norm of a solution globally in time. Therefore, the crux of the matter is in obtaining a suitable local well-posedness theory in the energy space H^1_x. However, the methods that work for this purpose in the case $\sigma \geq 1$ completely fail here. This is because the nonlinearity in (GDNLS) lacks the decay and regularity necessary for implementing either global smoothing type estimates or a gauge transformation to ameliorate the derivative nonlinearity. Nevertheless, we managed to prove the following result.

Theorem 1.2.2 (P., Taylor, [129])**.** Let $\sigma \in (\frac{\sqrt{3}}{2}, 1)$ and let $1 \leq s < 4\sigma$. Then (gDNLS) is globally well-posed in $H^1(\mathbb{R})$.

When $s = 1$, the key idea in this theorem is to introduce a family of partial gauge transformations adapted to each dyadic frequency scale for a suitable paralinearization of (GDNLS). Unlike in the case $\sigma \geq 1$, one cannot conjugate away the entire derivative nonlinearity due to the lack of decay of the coefficient $|u|^{2\sigma}$ (one would need it to be at least integrable). However, one useful strategy is to compromise and instead try to conjugate away only the portion of $|u|^{2\sigma}\partial_x u$ where the coefficient $|u|^{2\sigma}$ is "sufficiently large". What sufficiently large means in this context depends on the dyadic frequency localization scale of $\partial_x u$ and on the power σ. Because we cannot conjugate away the entire nonlinearity, we still face some derivative loss when trying to estimate the Strichartz norms of a solution u. However, this loss is weakened considerably because in the remaining nonlinearity, the coefficient $|u|^{2\sigma}$ is now small and can be used to compensate for some of the loss from the high-frequency factor $\partial_x u$. It turns out that this is weak enough to allow us to then make use of the maximal function estimates of Kenig, Ponce, and Vega to establish local well-posedness. The technical restriction $\sigma > \frac{\sqrt{3}}{2}$ comes from suitably balancing the losses from the partial gauge transformation (which get worse as σ gets smaller) with the gain from the maximal function estimates. For the sake of brevity, a detailed discussion will not be given here, but a more thorough overview can be found in Chapter 4. One nice consequence of this result is that it validates the soliton stability results in an article by Liu, Simpson, and Sulem [102] which were contingent on establishing $H^1(\mathbb{R})$ well-posedness for this problem for a small range of $\sigma < 1$.

When $\sigma < 1$, since the nonlinearity in (GDNLS) is quite rough, one also expects an upper bound on the scale of Sobolev spaces for which one can construct solutions. We managed to extend the range for which globally well-posed solutions exist to $s < 4\sigma$. This is significant, as the nonlinearity in (GDNLS) is only $C^{1,2\sigma-1}$-Hölder continuous. Put another way, this threshold is twice as large as the threshold one would get from a naive energy estimate. The proof of this rests on a modulation analysis, where near the characteristic hypersurface $\tau + \xi^2 = 0$ for the linear flow, one can use time derivatives (which in this region act like two spatial derivatives) to measure the regularity of a solution. This explains why the threshold of 4σ is twice the naïve threshold. Away from the characteristic set, the equation is essentially elliptic and it is relatively straightforward to deal with the nonlinearity in this region. Again, the reader is referred to Chapter 4 for a detailed overview.

Chapter 2

The free boundary Euler equations

2.1 Introduction

This chapter is concerned with the contents of the preprint [75]. Here, our goal is to study the dynamics of an inviscid fluid droplet in the absence of surface tension. At the time t, our fluid occupies a compact, connected, but not necessarily simply connected region $\overline{\Omega}_t \subseteq \mathbb{R}^d$, and its motion is governed by the incompressible Euler equations

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p - g e_d, \\ \nabla \cdot v = 0. \end{cases} \tag{2.1.1}$$

Here, v is the fluid velocity, p is the pressure, $g \geq 0$ is the gravitational constant, and e_d is the standard vertical basis vector. In the local theory of the droplet problem, the gravity can be freely neglected. However, it becomes important in the case of an unbounded fluid domain and in the case of a domain with a rigid bottom, so we retain it in (2.1.1) for completeness.

An essential role in the analysis of the droplet problem is played by the vector field

$$D_t := \partial_t + v \cdot \nabla,$$

which is called the *material derivative* and describes the particle trajectories. On the free boundary, we require the kinematic boundary condition

$$D_t \ \text{ is tangent to } \bigcup_t \{t\} \times \partial\Omega_t \subseteq \mathbb{R}^{d+1}, \tag{2.1.2}$$

which says that the domain Ω_t is transported along the material derivative (or equivalently, the particle trajectories), and that the normal velocity of $\Gamma_t := \partial\Omega_t$ is given by $v \cdot n_{\Gamma_t}$.

Additionally, we require the dynamic boundary condition

$$p|_{\Gamma_t} = 0, \tag{2.1.3}$$

which represents the balance of forces at the fluid interface in the absence of surface tension. Using the above boundary conditions, it is easy to see that the energy

$$E := \int_{\Omega_t} \left(\frac{|v|^2}{2} + gx \cdot e_d \right) dx$$

is formally conserved. Throughout the chapter, we will refer to the system (2.1.1)-(2.1.3) as the *free boundary (incompressible) Euler equations*.

As is the case with all Euler flows, an important role in the above evolution is played by the *vorticity*, ω, defined by

$$\omega_{ij} = \partial_i v_j - \partial_j v_i.$$

By taking the curl of (2.1.1), the vorticity is easily seen to solve the following transport equation along the flow:

$$D_t\omega = -(\nabla v)^*\omega - \omega\nabla v. \tag{2.1.4}$$

If initially $\omega = 0$, then (2.1.4) guarantees that this condition is propagated dynamically. Such velocity fields are called *irrotational*, and the corresponding solutions to the free boundary incompressible Euler equations are called *water waves*.

By taking the divergence of (2.1.1), we obtain the following Laplace equation for the pressure:

$$\begin{cases} \Delta p = -\mathrm{tr}(\nabla v)^2 \ \text{ in } \Omega_t, \\ p = 0 \ \text{ on } \Gamma_t. \end{cases} \tag{2.1.5}$$

For regular enough v on sufficiently regular Ω_t, the equation (2.1.5) uniquely determines the pressure from the velocity and domain. A key role in the study of the free boundary Euler equations is played by the *Taylor coefficient*, a, which is defined on the boundary Γ_t by

$$a := -\nabla p \cdot n_{\Gamma_t}. \tag{2.1.6}$$

Indeed, a classical result of Ebin [40] asserts that the free boundary Euler equations are ill-posed unless $a \geq 0$. For this reason, we will always assume that the initial data for the free boundary Euler equations verifies the following:

Taylor sign condition. There is a $c_0 > 0$ such that $a_0 := -\nabla p_0 \cdot n_{\Gamma_0} > c_0$ on Γ_0.

For irrotational data on compact simply connected domains, the Taylor sign condition is automatic by the strong maximum principle [101]. See also [69, 159] for similar results on unbounded domains when $g > 0$. Geometrically, enforcing $a_0 > 0$ ensures that the initial pressure p_0 is a non-degenerate defining function for the initial boundary hypersurface Γ_0, and thus can be used to describe the regularity of the boundary. As part of our well-posedness theorem below, we prove that the Taylor sign condition is propagated by the flow on some non-trivial time interval.

Another important role in this chapter is played by the *material derivative of the Taylor coefficient*, $D_t a$, which turns out to be closely related to (a derivative of) the normal component of the velocity $v \cdot n_{\Gamma_t}$. We will elaborate further on this relation shortly when we discuss our choice of control parameters and good variables.

The Cauchy problem: scaling, Sobolev spaces and control parameters

A state for the free boundary Euler equations consists of a domain Ω and a velocity field v on Ω. A bounded connected domain Ω can be equally described by its boundary Γ. Hence, in the sequel, by a state we mean a pair (v, Γ).

Describing the time evolution of (v, Γ) along the free boundary incompressible Euler flow is most naturally done in a functional setting described via appropriate Sobolev norms. To understand the proper setting, it is very helpful to consider the scaling properties of our problem. The boundaryless incompressible Euler flow admits a two parameter scaling group. However, when considering the free boundary flow there is an additional constraint; namely, that the pointwise property $a \approx 1$ rests unchanged. At a technical level, this is reflected in the fact that the Taylor coefficient appears as a weight in the Sobolev norms which are used on Γ. Imposing this constraint leaves us with a one parameter family of scaling laws, which have the form

$$\begin{aligned} v_\lambda(t,x) &= \lambda^{-\frac{1}{2}} v\left(\lambda^{\frac{1}{2}} t, \lambda x\right), \\ p_\lambda(t,x) &= \lambda^{-1} p\left(\lambda^{\frac{1}{2}} t, \lambda x\right), \\ (\Gamma_\lambda)_t &= \{\lambda^{-1} x : x \in \Gamma_{\lambda^{\frac{1}{2}} t}\}. \end{aligned}$$

As noted earlier, the above transformations have the property that the Taylor coefficient has the dimensionless scaling,

$$a_\lambda(t,x) = a\left(\lambda^{\frac{1}{2}}t, \lambda x\right).$$

A first benefit we derive from the scaling law is to understand what are the matched Sobolev regularities for v and Γ. This leads us to the following definition.

Definition 2.1.1 (State space)**.** The *state space* $\mathbf{H}^s$ is the set of all pairs (v,Γ) such that Γ is the boundary of a bounded, connected domain Ω and such that the following properties are satisfied:

(i) (Regularity). $v \in H^s_{div}(\Omega)$ and $\Gamma \in H^s$, where $H^s_{div}(\Omega)$ denotes the space of divergence free vector fields in $H^s(\Omega)$.

(ii) (Taylor sign condition). $a := -\nabla p \cdot n_\Gamma > c_0 > 0$, where c_0 may depend on the choice of (v,Γ), and the pressure p is obtained from (v,Γ) by solving the elliptic equation (2.1.5) associated to (2.1.1) and (2.1.3).

For states (v,Γ) as above, we define their size by

$$\|(v,\Gamma)\|^2_{\mathbf{H}^s} := \|\Gamma\|^2_{H^s} + \|v\|^2_{H^s(\Omega)}.$$

Note, however, that $\mathbf{H}^s$ is not a linear space, so $\|\cdot\|_{\mathbf{H}^s}$ does not induce a norm topology in the usual sense. Heuristically, the state space $\mathbf{H}^s$ may be thought of as an infinite dimensional manifold, though a precise interpretation of this is beyond the scope of this thesis. For our purposes, it suffices to define a consistent notion of topology on $\mathbf{H}^s$. Although we will not describe the precise topology in the introduction, this topology will allow us to define the space $C([0,T];\mathbf{H}^s)$ of continuous functions with values in $\mathbf{H}^s$, as well as an appropriate notion of $\mathbf{H}^s$ continuity of the data-to-solution map $(v_0,\Gamma_0) \mapsto (v(t),\Gamma_t)$. Armed with these notions, it makes sense to talk about the Cauchy problem.

Problem 2.1.2 (Cauchy problem for the free boundary Euler equations)**.** Given an initial state $(v_0,\Gamma_0) \in \mathbf{H}^s$, find the unique solution $(v,\Gamma) \in C([0,T];\mathbf{H}^s)$ in some time interval $[0,T]$.

A natural question to ask is what are the exponents s for which the Cauchy problem is well-posed in $\mathbf{H}^s$. Our first clue in this direction comes from scaling, which leads us to the critical exponent

$$s_c = \frac{d+1}{2},$$

and implicitly the lower bound $s \geq s_c$. However, this does not tell the entire story, as even in the boundaryless case a result of Bourgain-Li [17] shows that well-posedness holds only in the more restricted range

$$s > \frac{d}{2} + 1,$$

which is heuristically connected to another scaling law of the boundaryless problem; namely,

$$v(t, x) \mapsto \lambda^{-1} v(t, \lambda x).$$

This latter exponent range $s > \frac{d}{2} + 1$ is exactly what we consider in our work. Specifically, in this chapter we solve the Cauchy problem for the free boundary incompressible Euler equations at the same regularity level as the incompressible Euler equations on a fixed domain.

The reader who is more familiar with the boundaryless case may ask at this point why we confine ourselves to L^2 based Sobolev spaces, instead of using the full range of indices L^p as in the boundaryless case. The reason for this is precisely the boundary, where a portion of the dynamics is concentrated. In particular, as a subset of our problem we have the irrotational case $\omega = 0$, when the flow may be fully interpreted as the flow of the free boundary. This case, commonly identified as water waves, yields a dispersive flow, where L^p based Sobolev spaces are disallowed if $p \neq 2$. This is not to say that exponents $p \neq 2$ do not play a central role in our analysis. Instead, we use them, particularly the case $p = \infty$, in the definition of our *control parameters*, which control the size and growth of our energy functionals. Precisely, our analysis involves two such control parameters, which ideally should be appropriately scale invariant, as follows:

(i) An "elliptic" control parameter $A^\sharp$, used to control implicit constants in fixed time elliptic estimates, given by

$$A^\sharp = \|v\|_{\dot{C}^{\frac{1}{2}}(\Omega)} + \|\Gamma\|_{Lip}, \tag{2.1.7}$$

which is exactly invariant under scaling.

(ii) A "dynamical" control parameter $B^\sharp$, used to control the growth of energy in time, given by

$$B^\sharp = \|v\|_{Lip(\Omega)} + \|\Gamma\|_{\dot{C}^{1,\frac{1}{2}}}. \tag{2.1.8}$$

This latter control parameter is 1/2 derivatives above scaling, and instead the scale invariant quantity is $\|B^\sharp\|_{L^1_t}$, which is what will actually appear in our continuation criterion later on.

With these control parameters in hand, we would like to have energy estimates in the scale invariant form

$$\frac{d}{dt}E^k(v,\Gamma) \lesssim_{A^\sharp} B^\sharp E^k(v,\Gamma), \tag{2.1.9}$$

where E^k denotes a suitable energy at the $\mathbf{H}^k$ regularity. As noted earlier, these are our ideal choices, but for our results we need to make some small adjustments and relax them a bit, as follows:

a) Working with $A^\sharp$ would require edge case elliptic estimates in Lipschitz domains, bringing forth a broad host of issues which are less central to our problem, if even possible to overcome. So, instead, we will simply add ε derivatives to the norms in $A^\sharp$.

b) In the case of $B^\sharp$, we do not want to lose the sharp scaling, which is exactly as in the Beale-Kato-Majda criteria in the boundaryless case. Therefore, we do not want to add extra derivatives as we did with $A^\sharp$. However, as we shall soon see, the quantity $\|D_t a\|_{L^\infty(\Gamma)}$ appears as a control parameter in the L^2 estimate for the linearized equation. As it turns out, in order to propagate our low regularity difference bounds, control of $\|D_t a\|_{L^\infty(\Gamma)}$ will be needed. However, for the energy estimates, a careful analysis will show that the control parameter $B^\sharp$ is sufficient, if we slightly modify the form of the estimate (2.1.9). In both cases, maintaining the sharp top order control parameter is non-trivial. In the difference estimates, it requires a careful analysis on intersections of domains (and hence, in particular, performing elliptic theory on Lipschitz domains) and in the energy estimates it requires (amongst several other things) finding a way to appropriately absorb the logarithmic divergences occurring in the endpoint elliptic estimates when attempting to control $\|D_t a\|_{L^\infty(\Gamma)}$ by $B^\sharp$. To deal with this latter issue, we will take some inspiration from the proof of Beale-Kato-Majda [11].

The issues mentioned above have well-known counterparts in the boundaryless Euler flow. In fact, *strong ill-posedness* of the boundaryless Euler equations has been recently proven in the "ideal" pointwise spaces C^1 and *Lip* [18, 41].

Historical comments

The local well-posedness problem for the free boundary Euler equations has a long history. For irrotational flows, the first rigorous local existence result in Sobolev spaces was obtained by Wu [159, 160], in the late 1990s. Since then, various methods have been introduced to shorten the proofs, lower the regularity threshold and allow for more complicated geometries. For a small sample of such results we cite Beyer and Günther in [12], Lannes in [95], Alazard, Burq and Zuily in [6, 5], Hunter, Ifrim and Tataru in [69], Ai in [2, 3] and Ai, Ifrim and Tataru in [4]. Although physically restrictive, the irrotationality assumption allows one to reduce the dynamics to a system of equations on the free boundary. Depending on the choices made, this typically culminates in either the Zakharov-Craig-Sulem formulation of the water waves problem used in [2, 3, 6, 5, 95], or the holomorphic coordinates formulation used in [4, 69]. In either case, the reduction to a system of equations on $\mathbb{R}^{d-1}$ greatly simplifies the analysis.

For the free boundary Euler equations with non-trivial vorticity, certain generalized systems based on the above irrotational reductions have been proposed [20, 163]. However, historically, the most successful approach has been to use Lagrangian coordinates to fix the domain. For an execution of this approach to proving local existence, the reader may consult the papers of Christodoulou and Lindblad [28], Coutand and Shkoller [32] and Lindblad [101]. One may also compare with the article [93] of Kukavica and Tuffaha, which uses the so-called *arbitrary Lagrangian-Eulerian change of variables*, as well as the more recent advances in the Lagrangian analysis presented in [9, 37].

In contrast to the above articles, we will utilize a *fully Eulerian* strategy to prove the local well-posedness of the free boundary Euler equations. In other words, we will work directly with the physical equations (2.1.1)-(2.1.3), and avoid the use of any non-trivial coordinates changes. On time-independent domains, both the Lagrangian and Eulerian approaches have been widely successful in analyzing fluid equations. However, for free boundary problems, the Eulerian approach has seen relatively little attention, due to the obvious difficulty in having the domain of the fluid itself serve as a time-dependent unknown. Our aim in this chapter is to directly confront this issue. Corollaries of our newly obtained insights include:

(i) The first proof of the continuity of the data-to-solution map for this problem.

(ii) An enhanced uniqueness result, requiring only pointwise norms of very limited regularity.

(iii) Refined low regularity energy estimates with geometrically natural pointwise control parameters.

(iv) A new, direct proof of existence for regular solutions.

(v) A method to obtain rough solutions as unique limits of regular solutions at a Sobolev regularity that matches the optimal result for the Euler equations on $\mathbb{R}^d$.

(vi) An essentially scale invariant continuation criterion akin to that of Beale-Kato-Majda for the incompressible Euler equations on the whole space.

We will elaborate further on the ideas for obtaining the above results in Section 2.1. For now, it is important to note that we are not the first to utilize an Eulerian approach to analyze the well-posedness of fluid equations in the free boundary setting. The pioneering work in this regard is the remarkable series of papers by Shatah and Zeng [140, 139, 141]. However, Shatah and Zeng primarily consider the free boundary Euler equations with surface tension. While they are able to produce a solution to the pure gravity problem in the zero surface tension limit, it seems that their construction at least requires bounded curvature, which corresponds to greater regularity assumptions on the data than we need here. For this reason, the overlap between their analysis and ours tends to be on a more philosophical level, which we will elaborate on further in Section 2.1. A more direct comparison is with the memoir [157] of Wang, Zhang, Zhao and Zheng. In [157], the authors construct solutions to the free boundary Euler equations in an unbounded graph domain at the same *Sobolev* regularity that we achieve here. That is, they prove existence and uniqueness of solutions in H^s for $s > \frac{d}{2}+1$. The approach in [157] is in the style of Alazard, Burq and Zuily [6, 5], though the addition of vorticity makes the execution much more technical. Our approach is completely different to the one that they follow and works well in more complicated fluid domains. Additionally, we prove properties (i)-(vi) above. We also remark that all other fully Eulerian approaches (see, e.g., [111, 110, 112]) follow Shatah and Zeng, and hence require the regularizing effect of surface tension and higher regularity. The one step towards a fully Eulerian proof without surface tension is the work [134] of de Poyferré, who proves energy estimates for the pure gravity shoreline problem. However, the energy estimates in [134] have H^s based control norms and no well-posedness proof is presented.

The goal of this thesis is twofold. First, we intend to present a comprehensive, Hadamard style well-posedness theory, with an aim towards proving sharp results. At the same time, we provide a novel, geometric analysis, which we argue is more direct and streamlined than previous works. For instance, our proofs do not require paralinearization or Chemin-Lerner spaces as in [157]. Moreover, our existence scheme is new and direct - it does not use Nash-Moser, the approach in [157], or go through the zero surface tension limit as in [140, 139, 141]. For this reason, we believe that the techniques introduced in this thesis will have a wide range of applicability.

Finally, we mention that the analysis we present here is for the case of a compact fluid domain. In the study of the free boundary Euler equations, it is also common to consider the case of an infinite ocean of either finite or infinite depth. The choice of compact fluid domain emphasizes the geometric nature of our problem, and removes the temptation to flatten the domain into a strip or a half-space. Although some changes need to be made, as with the analysis of the capillary problem [140, 139, 141] by Shatah and Zeng, the general strategy we use here can be adapted to all three geometries. That being said, to streamline the exposition, we do allow some of our estimates to depend on the domain volume, which is a conserved quantity for the droplet problem.

An overview of the main results

In a nutshell, our main result asserts that the free boundary incompressible Euler equations are well-posed in $\mathbf{H}^s$ for $s > \frac{d}{2} + 1$. However, simply stating this fails to convey the full strength of both the result and of its various aspects and consequences. Instead, it is more revealing to divide the result in a modular way into four independently interesting parts; namely, (a) uniqueness and stability, (b) well-posedness, (c) energy estimates and (d) the continuation criteria.

To set the stage for our results, let Ω_* be a bounded, connected domain with smooth boundary Γ_*. Given $\varepsilon, \delta > 0$, consider the collar neighborhood $\Lambda_* := \Lambda(\Gamma_*, \varepsilon, \delta)$ consisting of all hypersurfaces Γ which are δ-close to Γ_* in the $C^{1,\varepsilon}$ topology. As long as $\delta > 0$ is small enough, hypersurfaces in Λ_* can be written as graphs over Γ_*. This permits us to define Sobolev and Hölder norms on these hypersurfaces in a consistent fashion. To state our results, we will assume that a collar neighborhood Λ_* has been fixed, and consider solutions with initial data (v_0, Γ_0) having $\Gamma_0 \in \Lambda_*$. A more precise description of the functional setting will be given later, in Section 2.3. For now, we remark that, while the collar neighborhood is very useful in order to uniformly define the $\mathbf{H}^s$ norms, it is not needed at all for the definition of our control parameters.

Uniqueness and stability

We start by stating our uniqueness result, which requires the least in terms of notations and preliminaries. Here, of crucial importance are the control parameters

$$A := A_\varepsilon := \|v\|_{C_x^{\frac{1}{2}+\varepsilon}(\Omega_t)} + \|\Gamma_t\|_{C_x^{1,\varepsilon}},\ \varepsilon > 0, \tag{2.1.10}$$

and

$$B_{\mathrm{diff}} := \|v\|_{W_x^{1,\infty}(\Omega_t)} + \|D_t p\|_{W_x^{1,\infty}(\Omega_t)} + \|\Gamma_t\|_{C_x^{1,\frac{1}{2}}}, \tag{2.1.11}$$

which represent slight adjustments of the ideal control parameters $A^\sharp$ and $B^\sharp$, as discussed earlier. Using these control parameters, our main uniqueness result is as follows:

Theorem 2.1.3 (Uniqueness)**.** Let $\varepsilon, T > 0$ and let Ω_0 be a domain with boundary Γ_0 of $C^{1,\frac{1}{2}}$ regularity. Then for every divergence free initial data $v_0 \in W^{1,\infty}(\Omega_0)$, the free boundary Euler equations with the Taylor sign condition admit at most one solution (v, Γ_t) with $\Gamma_t \in \Lambda_*$ and

$$\sup_{0\le t\le T} A_\varepsilon(t) + \int_0^T B_{\mathit{diff}}(t)\, dt < \infty.$$

To the best of our knowledge, Theorem 2.1.3 is the first uniqueness result for the free boundary Euler equations which involves only low regularity pointwise norms. Indeed, as far as we are aware, all other papers on this subject are content to prove uniqueness in the same class of H^s spaces for which they prove existence.

While uniqueness is a fundamental property in its own right, in our work it can be seen as a corollary of a far more useful stability result, which we now explain. Let (v, Γ_t) and $(v_h, \Gamma_{t,h})$ be two solutions to the free boundary Euler equations with corresponding domains Ω_t and $\Omega_{t,h}$. An obvious objective is to show that if (v, Γ_t) and $(v_h, \Gamma_{t,h})$ are "close" at time zero, then they remain close on a suitable timescale. However, since the domains Ω_t and $\Omega_{t,h}$ are evolving in time, we cannot compare the solutions (v, Γ_t) and $(v_h, \Gamma_{t,h})$ in a linear way. To resolve this issue, we construct a nonlinear functional which quantifies the distance between solutions and is propagated by the flow.

To avoid comparing solutions whose corresponding domains are very different, we harmlessly restrict ourselves to solutions (v, Γ_t) and $(v_h, \Gamma_{t,h})$ evolving in the same collar neighborhood Λ_*. For such solutions we define the nonlinear distance functional

$$D((v,\Gamma),(v_h,\Gamma_h)) := \frac{1}{2}\int_{\tilde{\Omega}_t} |v - v_h|^2 dx + \frac{1}{2}\int_{\tilde{\Gamma}_t} b|p - p_h|^2\, dS. \tag{2.1.12}$$

Here, p and p_h are the pressures, $\tilde{\Gamma}_t$ is the boundary of $\tilde{\Omega}_t := \Omega_t \cap \Omega_{t,h}$ and b is a suitable weight function. Morally speaking, the first term on the right-hand side of (2.1.12) measures the L^2 distance between v and v_h. On the other hand, by the Taylor sign condition, p and p_h are non-degenerate defining functions for Γ_t and $\Gamma_{t,h}$, so the second term on the right-hand side of (2.1.12) gives a measure of the distance between Γ_t and $\Gamma_{t,h}$. In Section 2.4, we prove that (2.1.12) does indeed act as a proper measure of distance between solutions. More crucially, we prove that this distance is propagated by the flow, in the sense that

$$\frac{d}{dt} D((v,\Gamma),(v_h,\Gamma_h)) \lesssim_{A,A_h} (B_{\text{diff}} + B_{\text{diff},h}) D((v,\Gamma),(v_h,\Gamma_h)). \tag{2.1.13}$$

Here, A_h and $B_{\text{diff},h}$ are the control parameters (2.1.10) and (2.1.11) corresponding to the solution $(v_h, \Gamma_{t,h})$. An immediate corollary of the stability estimate (2.1.13) is the aforementioned Theorem 2.1.3. However, (2.1.13) will also prove to be useful in various other scenarios. For example, we will use it in our proof of the continuity of the data-to-solution map, as well as in the construction of rough solutions as unique limits of regular solutions.

Well-posedness

Our second main result is concerned with the well-posedness problem. To fix the notations, we start with a collar neighborhood Λ_* and $s > \frac{d}{2} + 1$. We then consider initial data $(v_0, \Gamma_0) \in \mathbf{H}^s$ with $\Gamma_0 \in \Lambda_*$. Viewing Γ_0 as a graph over Γ_*, we may unambiguously define its H^s norm. With this setup, we may state our well-posedness theorem as follows:

Theorem 2.1.4 (Hadamard local well-posedness)**.** Fix $s > \frac{d}{2} + 1$ and a collar Λ_*. For any (v_0, Γ_0) in $\mathbf{H}^s$ with $\Gamma_0 \in \Lambda_*$ there exists a time $T > 0$, depending only on $\|(v_0, \Gamma_0)\|_{\mathbf{H}^s}$ and the lower bound in the Taylor sign condition, for which there exists a unique solution $(v(t), \Gamma_t) \in C([0,T]; \mathbf{H}^s)$ to the free boundary Euler equations satisfying a proportional uniform lower bound in the Taylor sign condition. Moreover, the data-to-solution map is continuous with respect to the $\mathbf{H}^s$ topology.

The regularity of the velocity in Theorem 2.1.4 matches the optimal Sobolev regularity for the Euler equations on $\mathbb{R}^d$. Indeed, as shown by Bourgain and Li [17], the Euler equations are ill-posed in $H^s(\mathbb{R}^d)$ when $s = \frac{d}{2} + 1$.

We note crucially that our work is not the first to reach the $s > \frac{d}{2} + 1$ Sobolev threshold for the free boundary Euler equations. Indeed, this threshold was achieved for the first time in the recent memoir [157], in the case of an unbounded fluid domain with graph geometry. However, it is important to note that the approach in [157] is very different from ours, as it passes through a paralinearization and utilizes properties of strip-like domains and Chemin-Lerner spaces. In particular, the approach in [157] cannot be easily modified to the droplet problem, whereas our approach applies equally well in unbounded domains. Moreover, there is no mention of the continuity of the data-to-solution map in [157]. To the best of our knowledge, Theorem 2.1.4 gives the *first* proof of this important property for the free boundary Euler equations. In addition, our approach significantly refines the well-posedness theory by adding properties (ii)-(vi) above as well as introduces an entirely new set of techniques that we believe will have broad applications.

When it comes to free boundary problems, the continuity of the data-to-solution map – if justified – is usually proven by reformulating the problem on a fixed domain and then working with the standard notion of continuous dependence on fixed domains. As far as we are aware, the only exception to this appears in the work [140, 139, 141] of Shatah and Zeng, where continuous dependence is proven for the free boundary Euler equations with surface tension directly in the Eulerian setting. The drawback of Shatah and Zeng's proof, however, is that it relies crucially on the regularizing effect of surface tension, so is not applicable to the pure gravity problem. In particular, Shatah and Zeng do not construct a distance functional, as we do here. For this reason, our robust proof which simultaneously avoids domain flattenings and works on a quasilinear problem without regularizing effects can be seen as one of the main novelties of our result.

Energy estimates

Controlling the growth of solutions to our boundary value problem is essential for both local well-posedness and understanding potential blowup. This control is achieved via energy estimates. Due to the complex geometry of our problem, the first challenge is to construct good energy functionals.

Fix an integer $k \geq 0$. In light of Theorem 2.1.3 and the stability estimate (2.1.13), it is natural to try to construct an energy functional $E^k = E^k(v, \Gamma)$ satisfying $E^k(v, \Gamma) \approx_A \|(v, \Gamma)\|^2_{\mathbf{H}^k}$ and the estimate

$$\frac{d}{dt} E^k(v, \Gamma) \lesssim_A B_{\text{diff}} E^k(v, \Gamma).$$

Indeed, by Grönwall's inequality, this would yield the bound

$$\|(v,\Gamma)(t)\|_{\mathbf{H}^k}^2 \lesssim \exp\left(\int_0^t C_A B_{\text{diff}}(s)\, ds\right) \|(v,\Gamma)(0)\|_{\mathbf{H}^k}^2,$$

for some constant C_A depending only on A, the collar, and the verification of the Taylor sign condition. Morally speaking, such an estimate would then allow one to conclude that solutions to the free boundary Euler equations with the Taylor sign condition can be continued as long A remains bounded and $B_{\text{diff}} \in L_t^1$.

However, there is one issue with the above estimates. Note that the control parameter A in (2.1.10) depends only on the Hölder norms of our main variables (the surface and the velocity) at (nearly) the correct scale. However, the control parameter B_{diff} in (2.1.11) depends also on the auxiliary variable $D_t p$. From the point of view of the analysis of the free boundary Euler equations, this is completely natural. Indeed, even at the level of the linearized equation, one sees that the uniform norm of $\nabla D_t p$ (or more specifically the uniform norm of $D_t a$, but these are essentially equivalent) appears as a control parameter for the L^2 energy estimates in Proposition 2.2.2. On the other hand, for the purpose of providing a clear and physical description of how solutions to the free boundary Euler equations break down, we would ultimately like to use the control parameter $B := B^\sharp$ defined in (2.1.8), which depends only on the Hölder norms of Γ and v. To achieve this, our key observation is that, as long as $k > \frac{d}{2} + 1$, we can use a log of the energy to absorb endpoint losses, and hence prove an estimate of the form

$$\|D_t p\|_{W_x^{1,\infty}(\Omega_t)} \lesssim_A \log(1 + E^k) B. \tag{2.1.14}$$

An estimate akin to (2.1.14) is not to be expected in the difference estimates, as the distance functional is too low of regularity to absorb the logarithmic divergences inevitably arising from C^1 and $W^{1,\infty}$ elliptic estimates. With the above discussion in mind, the actual energy estimates we prove can be essentially stated as follows.

Theorem 2.1.5 (Energy estimates)**.** Fix a collar neighborhood Λ_*, let $s \in \mathbb{R}$ with $s > \frac{d}{2} + 1$ and let $k > \frac{d}{2} + 1$ be an integer. Then for Γ restricted to Λ_* there exists an energy functional $\mathbf{H}^k \ni (v,\Gamma) \mapsto E^k(v,\Gamma)$ such that

(i) (Energy coercivity).

$$E^k(v,\Gamma) \approx_A \|(v,\Gamma)\|_{\mathbf{H}^k}^2. \tag{2.1.15}$$

(ii) (Energy propagation). If, in addition to the above, $(v, \Gamma) = (v(t), \Gamma_t)$ is a solution to the free boundary incompressible Euler equations, then $E^k(t) := E^k(v(t), \Gamma_t)$ satisfies

$$\frac{d}{dt} E^k \lesssim_A B \log(1 + \|(v, \Gamma)\|_{\mathbf{H}^s}) E^k. \tag{2.1.16}$$

Here, A is as in (2.1.10) and $B = B^\sharp$.

By Grönwall's inequality, (2.1.15) and (2.1.16) yield the following single and double exponential bounds of the type

$$\begin{aligned} \|(v(t), \Gamma_t)\|_{\mathbf{H}^k}^2 &\lesssim_A \exp\left(\int_0^t C_A B \log(1 + \|(v, \Gamma)\|_{\mathbf{H}^s}) ds \right) \|(v_0, \Gamma_0)\|_{\mathbf{H}^k}^2, \\ \|(v(t), \Gamma_t)\|_{\mathbf{H}^k}^2 &\lesssim_A \exp\left(\log(1 + C_A \|(v_0, \Gamma_0)\|_{\mathbf{H}^k}^2) \exp \int_0^t C_A B \, ds \right), \end{aligned} \tag{2.1.17}$$

for all integers $k > \frac{d}{2} + 1$. We do not directly prove the analogue of Theorem 2.1.5 for noninteger exponents k. Nevertheless, as a consequence of our analysis in the last section of the chapter, we do obtain the bounds (2.1.17) also for noninteger k. This is achieved by using frequency envelopes in order to combine the distance functional and the energy estimates akin to a nonlinear Littlewood-Paley type theory. It is also worth noting that a similar double exponential growth rate for the $L^1_T L^\infty_x$ norm of the vorticity appears in the classical Beale-Kato-Majda [11] criteria as a consequence of trying to weaken the natural control parameters of the problem.

In order to understand the form of the energy functionals used in Theorem 2.1.5, a key step is to identify Alinhac style *good variables* for the problem, which are as follows:

(i) The vorticity ω, which is measured in $H^{k-1}(\Omega)$.

(ii) The Taylor coefficient a, which is measured in $H^{k-1}(\Gamma)$.

(iii) The material derivative $D_t a$ of the Taylor coefficient, which is measured in $H^{k-\frac{3}{2}}(\Gamma)$.

Our energy functionals are constructed as certain combinations of well-chosen norms of the above good variables. The general strategy for constructing these norms is to apply appropriate vector fields and elliptic operators to ω, a and $D_t a$ at the $\mathbf{H}^k$ regularity in such a way that the resulting variables solve the linearized equation to leading order. After this, the nonlinear energy E^k may be essentially defined as the linear energy evaluated at these good variables. As it turns out, after completing this process, we arrived at essentially the same energy as [134], which was derived by different means. However, as can be immediately inferred from our control norms, the way we treat the energy is very different from [134]. Indeed, without going into details, we mention that the proof of Theorem 2.1.5 requires not only a delicate analysis of the fine structure and cancellations present in the free boundary Euler equations, but also the use of a new family of refined elliptic estimates. Although we refrain from stating them here in the introduction, these elliptic estimates serve as an important part of the chapter. Moreover, since they are quite general, we believe that they will prove to be useful in other problems as well.

Low regularity continuation criterion

A very natural objective in the study of the Euler equations is to find a geometric characterization of how solutions break down. For the Euler equations without free boundary, this direction traces back to the famous paper of Beale, Kato and Majda [11]. In recent years, interest in sharp blow up criterion for the free boundary Euler equations has risen, and progress has been made by de Poyferré [35], Ginsberg [48], Wang and Zhang [156] and Wang, Zhang, Zhao and Zheng [157]. Here, we explain our rather definitive answer to this question, which is essentially a consequence of our local well-posedness result in Theorem 2.1.4 and the energy estimates in Theorem 2.1.5. However, to avoid topological issues, we must first introduce a notion of thickness for the fluid domain.

Definition 2.1.6. The fluid domain Ω has thickness at least $R > 0$ if for each $x \in \Gamma$, $B(x, R) \cap \Gamma$ is the graph of a $C^{1,\varepsilon}$ function which separates $B(x, R)$ into two connected components.

With this notion in hand, our continuation criterion reads as follows:

Theorem 2.1.7 (Continuation criterion)**.** A solution $(v, \Gamma) \in C(\mathbf{H}^s)$, $s > \frac{d}{2} + 1$, of the free boundary incompressible Euler equations with the Taylor sign condition can be continued for as long as the following properties hold:

a) (Uniform bound from below for the Taylor coefficient). There is a $c > 0$ such that

$$a \geq c > 0.$$

b) (Uniform thickness). There is an $R > 0$ such that Ω_t has thickness at least R.

c) (Control parameter bounds). The control parameters satisfy

$$A \in L_t^\infty, \qquad B \in L_t^1.$$

One may compare our continuation criteria for the free boundary problem with the classical Beale-Kato-Majda criteria for the boundaryless problem and note that they are essentially at the same level, with the natural addition of the $C^{1,\frac{1}{2}}$ boundary regularity bound. Another minor difference is that we use the Lipschitz bound on the velocity v rather than the uniform bound on the vorticity ω. One may ask whether it is possible to further relax our criterion in order to use only the vorticity bound. The major obstruction is that while in fixed domains the vorticity uniquely determines the velocity, in our case an appropriate boundary condition is also needed, which is best described via the $D_t a$ good variable. So, a potential conjecture might be that in order to use only the vorticity bound in the interior, one might have to compensate by adding a uniform bound on $D_t a$, as seen in the linear control parameter B_{lin} and in the difference estimates. That being said, in this thesis we have opted for a continuation criteria involving only the natural variables v and Γ and no auxiliary pressure related terms.

As mentioned above, several recent articles [35, 48, 156, 157] have focused on obtaining improved continuation criterion for the free boundary Euler equations. The most significant of these contributions is the memoir [157], which proves that H^k solutions to the free boundary Euler equations with the Taylor sign condition can be continued after $t = T$ as long as properties a) and b) in Theorem 2.1.7 hold and

$$\sup_{t\in[0,T]} \left(\|\kappa(t)\|_{(L^p\cap L^2)(\Gamma_t)} + \|v(t)\|_{W^{1,\infty}(\Omega_t)}\right) < \infty \text{ for some } p > 2d - 2. \tag{2.1.18}$$

Here, κ denotes the mean curvature of the surface. To motivate their result, [157] recalls a question of Craig and Wayne [92], which asks one to find (in the context of the irrotational water waves problem) the lowest Hölder regularity of the surface and velocity potential whose boundedness on $[0, T]$ implies that one can continue the solution past $t = T$. Although (2.1.18) makes significant progress on this question, it fails to achieve purely pointwise norms and is far from scale invariant. Moreover, the criterion (2.1.18) only applies to solutions which a priori live in integer based Sobolev spaces H^k. This limits the applicability of (2.1.18) to solutions with at least a half derivative of excess regularity. In contrast, Theorem 2.1.7 replaces the criterion $v \in L_T^\infty W_x^{1,\infty}$ by the sharp and scale invariant criterion $v \in L_T^1 W_x^{1,\infty}$, and only requires control of Hölder norms of the free surface at the correct scale. In particular, Theorem 2.1.7 gives a rather definitive answer to Craig and Wayne's question for the full free boundary Euler equations. For the state-of-the-art result for the two-dimensional irrotational water waves problem, see [4]. Also, note that Theorem 2.1.7 applies to solutions in all Sobolev spaces $\mathbf{H}^s$ with $s > \frac{d}{2} + 1$, not just to those in integer spaces. This improvement is by no means trivial; rather, it follows from a careful usage of our distance functional.

Outline of the chapter

This chapter has a modular structure, where, for the essential part, only the main results of each section are used later.

The linearized equations

The starting point for our analysis, in Section 2.2, is to derive the linearization of our problem in Eulerian coordinates. The linearized system will serve as a guide to several of the choices made in our nonlinear analysis. In particular, it will suggest the correct variables to use, as well as the form of our distance functional. Moreover, when proving energy estimates, the Alinhac style good variables we construct will be shown to solve the linearized equations to leading order. This is also where the control parameters A and B_{lin} (an enhanced version of B) make their first appearance.

Function spaces and the geometry of moving domains

Section 2.3 describes the appropriate functional setting for our analysis. We begin by setting up a basic framework for our problem, including introducing low regularity control neighborhoods which will allow us to establish uniform control over constants in Sobolev and elliptic estimates in certain topologies for an appropriate family of domains. After defining the function spaces and norms that we will be using, we define the state space $\mathbf{H}^s$ where we will seek solutions to the free boundary Euler equations. Unlike in problems on fixed domains, the state space $\mathbf{H}^s$ will not be linear. However, it will be equipped with an appropriate notion of convergence, allowing us to define continuity of functions with values in $\mathbf{H}^s$ as well continuity of the data-to-solution map.

Stability estimates and uniqueness

The aim of Section 2.4 is to construct a nonlinear distance functional which will allow us to track the distance between two solutions at very low regularity. The general scheme is akin to the difference bounds in a weaker topology which are common in the study of quasilinear problems on fixed domains. However, here there are fundamental difficulties to overcome, as we are seeking to not only compare functions on different domains, but also track the evolution in time of this distance. These difficulties are embedded into the nonlinear character of our distance functional; both careful choices and delicate estimates are required to propagate this distance forward in time. To the best of our knowledge, this is only the second time difference estimates have been successfully proven in the free boundary setting. The other successful execution, which conceptually inspired the present approach, was in the case of a compressible gas [36, 72], which is very different from the incompressible liquid we consider here. In particular, unlike in the gas case, the boundary of our fluid contains non-trivial energy, requiring interesting geometric insights to understand.

As a consequence of our stability estimates, we deduce uniqueness of solutions at very low regularity. Also, as we shall see in later sections, the low regularity distance bounds we prove will serve both as an essential building block in our construction of rough solutions as unique limits of regular solutions as well as in the proof of the continuity of the data-to-solution map.

Elliptic theory

The main goal of Section 2.5 is to introduce a new family of refined elliptic estimates which will be crucial for obtaining the sharp pointwise control norms in the higher energy bounds. The secondary objective of Section 2.5 is to define a relevant Littlewood-Paley theory, collect various "balanced" product, Moser and Sobolev type estimates, and note several identities for operators and functions defined on moving domains. For the most part, the material in Section 2.5 does not rely on any specific structure of the Euler equations, so should be applicable to other free boundary problems as well. In Section 2.6, we construct the regularization operators which we will need for our existence scheme and the frequency envelopes for states $(v, \Gamma) \in \mathbf{H}^s$ that we will use to establish the refined properties of the data-to-solution map.

Energy estimates

In Section 2.7 we establish energy estimates within the $\mathbf{H}^k$ scale of spaces. As a first step, we construct a coercive energy functional $(v, \Gamma) \mapsto E^k(v, \Gamma)$ associated to each integer $k > \frac{d}{2} + 1$. The scheme here is to identify Alinhac style "good variables" (w_k, s_k) which solve the linearized equation modulo perturbative source terms. We then define our energy as the sum of the rotational energy and the linearized energy evaluated at these good variables. To prove the energy estimates, we split the argument in a modular fashion into two parts. First, we prove the coercivity of our energy functional; that is, we show that $E^k(v, \Gamma) \approx \|(v, \Gamma)\|^2_{\mathbf{H}^k}$. After this, we track the time evolution of the energy, establishing control of $E^k(v, \Gamma)$ in terms of the initial data, with growth dictated by the pointwise control parameters A and B. Both steps of this argument are delicate. In particular, the former makes extensive use of the refined elliptic estimates from Section 2.5, and the latter requires us to identify and exploit various structural properties and fine cancellations present in the Euler equations.

Construction of regular solutions

Section 2.8 is devoted to the construction of regular solutions to the free boundary Euler equations. The overarching scheme we utilize is similar to [72], which analyzed the case of a compressible gas. However, we stress that the main difficulties in the incompressible liquid case are quite different than for the gas, especially near the free boundary, as the surface of a liquid carries a non-trivial energy. As a general overview, the scheme we utilize is constructive, employing a time discretization via an Euler type method together with a separate transport step to produce good approximate solutions. However, a naïve implementation of Euler's method loses derivatives. To overcome this, we ameliorate the derivative loss by an initial regularization of each iterate in our discretization. To ensure that the uniform energy bounds survive, such a regularization needs to be chosen carefully. For this, we employ a modular approach and try to decouple this process into two steps, where we regularize individually the domain and the velocity. We believe that this modular approach will serve as a recipe for a new and relatively simple method for constructing solutions to various free boundary problems. That being said, the execution of this scheme is still quite subtle, requiring several novel ideas in addition to those coming from [72].

Rough solutions and continuous dependence

The last section of the chapter aims to construct rough solutions as strong limits of smooth solutions. This is achieved by considering a family of dyadic regularizations of the initial data, which generate corresponding smooth solutions. For these smooth solutions we control on one hand higher Sobolev norms $\mathbf{H}^k$, using our energy estimates, and on the other hand the L^2 type distance between consecutive ones, from our difference estimates. Combining the high and the low regularity bounds directly yields rapid convergence in all $\mathbf{H}^l$ spaces for $l < k$. To gain strong convergence in $\mathbf{H}^k$, we use frequency envelopes to more accurately control both the low and the high Sobolev norms above. This allows us to bound differences in the strong $\mathbf{H}^k$ topology. Interpolation and a similar argument yields local existence in fractional Sobolev spaces as well as continuous dependence of the solutions in terms of the initial data in the strong topology. Finally, our main continuation result in Theorem 2.1.7 follows along similar lines, given the careful treatment of our control norms in the energy and difference estimates.

For problems on $\mathbb{R}^d$, the scheme outlined above for obtaining rough solutions from smooth solutions, good energy estimates and difference estimates is more classical; see the expository article [71]. However, as we shall see, the fact that solutions are all defined on different domains leads to some new subtleties in our free boundary setting.

2.2 The linearized equation

The first goal of this section is to formally derive the linearization of our problem, working entirely in Eulerian coordinates; this is the system of equations (2.2.6). Then, we prove Theorem 2.2.1, which asserts that the linearized system is well-posed in L^2, with energy bounds determined by our sharp control parameters. The key elements here are the linearized energy (2.2.9) and the basic energy estimate (2.2.10).

Conceptually, the linearized system is an essential piece of the puzzle. On a practical level, however, it is not immediately useful in proving well-posedness, as it is not clear that C^1 one parameter families of solutions exist in the first place. It is only a posteriori, after well-posedness is established, that the linearized energy estimates may be used to derive bounds for differences of solutions. Instead, we will use our understanding of the linearized system to guide us in our choice of distance functional in Section 2.4 and later in our choice of energy functionals in Section 2.7.

To derive the linearized system, we take a one parameter family of solutions (v_h, p_h) defined on domains $\Omega_{t,h}$, with $(v_0, p_0) := (v, p)$ and $\Omega_{t,0} := \Omega_t$. We define $w = \partial_h v_h|_{h=0}$ and $q = \partial_h p_h|_{h=0}$.

In Ω_t, the linearized equation is rather standard:

$$\begin{cases} \partial_t w + w \cdot \nabla v + v \cdot \nabla w = -\nabla q, \\ \nabla \cdot w = 0. \end{cases}$$

However, we also need to linearize the kinematic and dynamic boundary conditions on the surface Γ_t. For this, let us denote by $\Gamma_{t,h}$ the free surface at time t for the solution (v_h, p_h), so $\Gamma_{t,0} := \Gamma_t$. Fix a one parameter family of diffeomorphisms $\phi_h(t) : \Gamma_t \to \Gamma_{t,h}$, with $\phi_0(t) = Id_{\Gamma_t}$. The dynamic boundary condition (2.1.3) asserts that for every point $x \in \Gamma_t$,

$$p_h(t, \phi_h(t)(x)) = 0.$$

Differentiating in h and evaluating at $h = 0$ gives

$$q|_{\Gamma_t} = -\nabla p|_{\Gamma_t} \cdot \psi(t),$$

where $\psi(t) := \frac{\partial}{\partial h}\phi_h(t)|_{h=0}$. Using that $\nabla p|_{\Gamma_t}$ is normal to Γ_t we deduce that

$$q|_{\Gamma_t} = -\nabla p|_{\Gamma_t} \cdot n_{\Gamma_t} \psi(t) \cdot n_{\Gamma_t} =: as. \tag{2.2.1}$$

Here, we define $s := \psi(t) \cdot n_{\Gamma_t}$ which we loosely interpret as the normal velocity in the parameter h of the family $\Gamma_{t,h}$ at $h = 0$. We will use this as one of our linearized variables. Note that since $a > 0$, s does not depend on the choice of diffeomorphisms $\phi_h(t)$.

Next, we linearize the kinematic boundary condition. Analogously to $v \cdot n_{\Gamma_t}$ describing the normal velocity of the free surface, we expect $w \cdot n_{\Gamma_t}$ to describe the "normal velocity" of our linearized variable s. Therefore, up to a perturbative error, $D_t s$ should agree with $w \cdot n_{\Gamma_t}$. In fact, we obtain the relation

$$D_t s - w \cdot n_{\Gamma_t} = s(n_{\Gamma_t} \cdot \nabla v) \cdot n_{\Gamma_t}. \tag{2.2.2}$$

To derive (2.2.2), we note that (2.1.2) and (2.1.3) imply that

$$D_t p = 0 \quad \text{on } \Gamma_t. \tag{2.2.3}$$

This is the equation that we will linearize to obtain (2.2.2). As before, let $\phi_h(t) : \Gamma_t \to \Gamma_{t,h}$ be a diffeomorphism. We then have for $x \in \Gamma_t$,

$$[(\partial_t + v_h \cdot \nabla)p_h](t, \phi_h(t)(x)) = 0.$$

Taking h derivative and evaluating at $h = 0$ yields,

$$w \cdot \nabla p + D_t q + \nabla D_t p \cdot \psi = 0 \quad \text{on } \Gamma_t. \tag{2.2.4}$$

Using (2.2.1), and that $\nabla D_t p$ is normal to Γ_t by (2.2.3), we deduce (2.2.2) from (2.2.4) after some simple algebraic manipulation. Indeed, we have $\nabla p|_{\Gamma_t} = -an_{\Gamma_t}$. Then using the relation $q_{|\Gamma_t} = as$, we compute $D_t q = aD_t s + sD_t a$. This reduces (2.2.4) to

$$-aw \cdot n_{\Gamma_t} + aD_t s + sD_t a + s\nabla D_t p \cdot n_{\Gamma_t} = 0. \tag{2.2.5}$$

After division by a, the first two terms in (2.2.5) evidently align with the left-hand side of (2.2.2). The right-hand side of (2.2.2) appears by commuting the gradient with the material derivative in the last term of (2.2.5), and by using the fact that $\nabla p \cdot D_t n_{\Gamma_t} = 0$ to rewrite $sD_t a = -sD_t(\nabla p \cdot n_{\Gamma_t}) = -sD_t \nabla p \cdot n_{\Gamma_t}$.

Putting everything together, the linearized system takes the form:

$$\begin{cases} D_t w + \nabla q = -w \cdot \nabla v \quad \text{in } \Omega_t, \\ \nabla \cdot w = 0 \quad \text{in } \Omega_t, \\ D_t s - w \cdot n_{\Gamma_t} = s(n_{\Gamma_t} \cdot \nabla v) \cdot n_{\Gamma_t} \quad \text{on } \Gamma_t, \\ q = as \quad \text{on } \Gamma_t, \end{cases} \tag{2.2.6}$$

where the terms on the right-hand side can be viewed as perturbative source terms.

In order to study the well-posedness of the linearized system (2.2.6), we introduce an enhanced version B_{lin} of the control parameter $B^\sharp$:

$$B_{lin}(t) := \|a^{-1} D_t a\|_{L^\infty(\Gamma_t)} + \|\nabla v\|_{L^\infty(\Omega_t)}. \tag{2.2.7}$$

Using this, we may state our main linearized well-posedness result as follows.

Theorem 2.2.1. Let (v, Γ) be a solution to the free boundary incompressible Euler equations in a time interval $[0, T]$ so that $a > 0$, $A^\sharp$ stays uniformly bounded and $B_{lin} \in L^1_T$. Then the linearized system (2.2.6) for (w, s) is well-posed in $L^2(\Omega) \times L^2(\Gamma)$ in $[0, T]$.

Here we recall that Ω and Γ are time dependent. The rest of this section is devoted to the proof of this very simple theorem. The basic strategy is to construct a suitable energy functional and prove corresponding energy estimates. Once this is done, well-posedness follows via a standard duality argument, which is left for the reader. To execute this argument, one simply notes that the adjoint system is essentially identical to the direct system (2.2.6), modulo perturbative terms, and that the energy estimates are time reversible.

Below, we will work with a slightly more general system, since this is what will appear in the higher order energy bounds later on. We define the *generalized linearized system* as follows:

$$\begin{cases} D_t w + \nabla q = f \quad \text{in } \Omega_t, \\ \nabla \cdot w = 0 \quad \text{in } \Omega_t, \\ D_t s - w \cdot n_{\Gamma_t} = g \quad \text{on } \Gamma_t, \\ q = as \quad \text{on } \Gamma_t, \end{cases} \tag{2.2.8}$$

where we allow for arbitrary source terms f and g on the right-hand side of the first and third equation.

It remains to prove a suitable energy estimate for the system (2.2.8). The natural energy associated to this system is

$$E_{lin}(w,s)(t) = \frac{1}{2}\int_{\Omega_t} |w|^2\, dx + \frac{1}{2}\int_{\Gamma_t} as^2\, dS. \tag{2.2.9}$$

Using (2.2.9), the main energy estimate for the generalized linear system is as follows:

Proposition 2.2.2. Suppose $a > 0$. Then the system (2.2.8) satisfies the energy estimate

$$\frac{d}{dt}E_{lin}(w,s)(t) \leq B_{lin}E_{lin}(w,s)(t) + \langle as, g\rangle_{L^2(\Gamma_t)} + \langle w, f\rangle_{L^2(\Omega_t)}. \tag{2.2.10}$$

We note that the energy functional (2.2.9) is also the energy functional for the linearized system (2.2.6), and that this proposition yields energy estimates for (2.2.6), thereby concluding the proof of Theorem 2.2.1.

Proof. We will make use of the following standard Leibniz type formulas (see; for example, [39, Appendix A]).

Proposition 2.2.3. (i) Assume that the time-dependent domain Ω_t flows with Lipschitz velocity v. Then the time derivative of the time-dependent volume integral is given by

$$\frac{d}{dt}\int_{\Omega_t} f(t,x)\,dx = \int_{\Omega_t} D_t f + f\nabla\cdot v\,dx.$$

(ii) Assume that the time-dependent hypersurface Γ_t flows with divergence free velocity v. Then the time derivative of the time-dependent surface integral is given by

$$\frac{d}{dt}\int_{\Gamma_t} f(t,x)\,dS = \int_{\Gamma_t} D_t f - f(n_{\Gamma_t}\cdot\nabla v)\cdot n_{\Gamma_t}\,dS.$$

Now, to prove the energy estimate (2.2.10), we apply Proposition 2.2.3 to obtain

$$\begin{aligned}\frac{d}{dt}E_{lin}(w,s)(t) &= \int_{\Omega_t} D_t w\cdot w\,dx + \int_{\Gamma_t} asD_t s\,dS + \frac{1}{2}\int_{\Gamma_t} D_t a s^2\,dS \\ &\quad - \frac{1}{2}\int_{\Gamma_t}[n_{\Gamma_t}\cdot\nabla v\cdot n_{\Gamma_t}]as^2\,dS \\ &\leq \int_{\Omega_t} D_t w\cdot w\,dx + \int_{\Gamma_t} asD_t s\,dS + B_{lin}E_{lin}(w,s)(t).\end{aligned} \tag{2.2.11}$$

Integrating by parts, we obtain

$$\begin{aligned}\int_{\Omega_t} D_t w\cdot w\,dx + \int_{\Gamma_t} asD_t s\,dS &= \int_{\Omega_t} w\cdot f\,dx + \int_{\Gamma_t} asD_t s\,dS - \int_{\Gamma_t} qw\cdot n_{\Gamma_t}\,dS \\ &= \langle as, g\rangle_{L^2(\Gamma_t)} + \langle w, f\rangle_{L^2(\Omega_t)}.\end{aligned}$$

Combining this with (2.2.11) completes the proof. □

2.3 Analysis on moving domains

One difficulty when working directly on moving domains is that many of the standard Sobolev and elliptic estimates have domain dependent constants. It is therefore necessary to work in a framework which allows for uniform control of these constants in certain topologies. This section is devoted to dealing with this issue. Our approach in this regard is somewhat analogous to that of Shatah and Zeng [140, 139, 141] and de Poyferré [134, Section 3], but with the key difference being that our control neighborhoods will only be uniform in the pointwise C^1 or $C^{1,\varepsilon}$ topologies as opposed to the stronger L^2 based topologies considered in those papers. This will be essential for establishing the pointwise continuation criterion for solutions.

Function spaces

To begin, we precisely define the function spaces and norms that we will be using. Throughout, $\Omega \subseteq \mathbb{R}^d$ will denote a bounded, connected domain. We define $H^s(\Omega)$, $s \geq 0$, as the set of all $f \in L^2(\Omega)$ such that

$$\|f\|_{H^s(\Omega)} := \inf \left\{ \|F\|_{H^s(\mathbb{R}^d)} : F \in H^s(\mathbb{R}^d),\ F|_\Omega = f \right\} \tag{2.3.1}$$

is finite. Here, $\|\cdot\|_{H^s(\mathbb{R}^d)}$ is defined in the standard way, via the Fourier transform. We let $H_0^s(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ and identify $H^{-s}(\Omega)$ isometrically with the dual space $(H_0^s(\Omega))^*$. Importantly, with this definition of the H^s norm, the constants in Sobolev embedding theorems (either $H^s \to L^p$ or $H^s \to C^\alpha$) are independent of Ω. For regular enough domains and integer s, the norm defined in (2.3.1) is equivalent to the standard one. We will precisely quantify this equivalence later.

We next define the regularity of the boundary of a connected domain Ω, which is characterized in terms of the regularity of local coordinate parameterizations of $\partial\Omega$. Indeed, in general, an m-dimensional manifold $\mathcal{M} \subseteq \mathbb{R}^d$ is said to be of class $C^{k,\alpha}$ or H^s, $s > \frac{d}{2}$, if, locally in linear frames, $\mathcal{M}$ can be represented by graphs with the same regularity.

If $s > \frac{d+1}{2}$, then given Ω as above with boundary of class H^s, we can define what it means to be an H^r function on $\partial\Omega$ for $s \geq r \geq -s$. Indeed, these are simply the functions whose coordinate representatives are locally in $H^r(\mathbb{R}^{d-1})$. It is easy to see that the space of H^r functions on $\partial\Omega$, $s \geq r \geq -s$, can be made into a Banach space. Indeed, a norm can be chosen by taking a covering of $\partial\Omega$ by a finite number of coordinate patches and an adapted partition of unity. However, there is one problem with this approach. Although such a norm is well-defined up to equivalence, the precise value of the norm is dependent on the choice of local coordinates. Since we will be dealing with a family of domains, we need to make sure that we define norms on their boundaries in a consistent and uniform way.

Collar coordinates

As a first step towards resolving the above issue, we fix a bounded, connected reference domain Ω_* with smooth boundary $\Gamma_* := \partial\Omega_*$. We define H^s and $C^{k,\alpha}$ based norms on Γ_* by making an appropriate choice of local parameterizations of Γ_*. Letting $\delta > 0$ be a small positive constant, we define $N(\Gamma_*, \delta)$ to be the collection of all C^1 hypersurfaces Γ such that there exists a C^1 diffeomorphism $\Phi_\Gamma : \Gamma_* \to \Gamma$ with

$$\|\Phi_\Gamma - id_{\Gamma_*}\|_{C^1(\Gamma_*)} < \delta.$$

If $\delta > 0$ is small enough, we can represent hypersurfaces $\Gamma \in N(\Gamma_*, \delta)$ as graphs over Γ_*. Indeed, we denote the outward unit normal to Γ_* by n_{Γ_*}. Following [141, Section 2.1], if we have a smooth unit vector field $\nu : \Gamma_* \to \mathbb{S}^{d-1}$ which is suitably transversal to Γ_* (that is, $\nu \cdot n_{\Gamma_*} > 1 - c$ for some small $c > 0$), it follows from the implicit function theorem that there exists a $\delta > 0$, determined by Γ_* and ν, such that the map

$$\varphi : \Gamma_* \times [-\delta, \delta] \to \mathbb{R}^d, \quad \varphi(x, \mu) = x + \mu\nu(x)$$

is a C^1 diffeomorphism from its domain to a collar neighborhood of Γ_*. If $\delta > 0$ is small enough, the above coordinate system associates each hypersurface $\Gamma \in N(\Gamma_*, \delta)$ with a unique function $\eta_\Gamma : \Gamma_* \to \mathbb{R}$ such that

$$\Phi_\Gamma(x) := \varphi(x, \eta_\Gamma(x)) = x + \eta_\Gamma(x)\nu(x) \tag{2.3.2}$$

is a diffeomorphism in $C^1(\Gamma_*, \Gamma \subseteq \mathbb{R}^d)$. We can think of the map Φ_Γ as a way to represent Γ as a (global) graph over Γ_*. With this notation in hand, we can now define what it means to be a H^s hypersurface which is close to Γ_*.

Definition 2.3.1. For $\delta > 0$ small enough and $\alpha \in [0, 1)$, define the control neighborhood $\Lambda(\Gamma_*, \alpha, \delta)$ as the collection of all hypersurfaces $\Gamma \in N(\Gamma_*, \delta)$ such that the associated map $\eta_\Gamma : \Gamma_* \to \mathbb{R}$ satisfies

$$\|\eta_\Gamma\|_{C^{1,\alpha}(\Gamma_*)} < \delta.$$

Definition 2.3.2. Suppose $s \geq 0$, $\Gamma \in N(\Gamma_*, \delta)$ for $\delta > 0$ small enough, and the associated map $\eta_\Gamma : \Gamma_* \to \mathbb{R}$ satisfies $\eta_\Gamma \in H^s(\Gamma_*)$. We then define the H^s norm of Γ by

$$\|\Gamma\|_{H^s} := \|\eta_\Gamma\|_{H^s(\Gamma_*)}.$$

In the above definitions, $\|\eta_\Gamma\|_{C^{1,\alpha}(\Gamma_*)}$ and $\|\eta_\Gamma\|_{H^s(\Gamma_*)}$ are computed with respect to fixed, independent of Γ, local coordinates on Γ_*. In an analogous way, we define for $\gamma \in [0, 1)$ and integers $k \geq 0$, the $C^{k,\gamma}$ norm, $\|\Gamma\|_{C^{k,\gamma}}$. As was essentially noted in [141, Section 2.1], when $0 < \delta \ll 1$, each $\Gamma \in \Lambda(\Gamma_*, \alpha, \delta)$ is associated to a well-defined domain Ω.

Remark 2.3.3. One key point in Definition 2.3.1 is that we only require Γ be close to Γ_* in the $C^{1,\alpha}$ topology, as opposed to the stronger L^2 based topologies used in [134, 140, 139, 141]. In practice, we will want the control topology to be as weak as possible. For our purposes, we will typically take $\alpha = \varepsilon > 0$ for some arbitrarily small (but fixed) constant $\varepsilon > 0$.

Remark 2.3.4. A second key point in Definition 2.3.1 concerns the choice of the small parameter δ. This will not be arbitrarily small, but instead its size may also be chosen to depend on weaker topologies; namely, (i) the $C^{1,\varepsilon}$ norm of Γ_* and (ii) the thickness (see Definition 2.1.6) of the domain Ω. This will serve two purposes:

- To allow us to place any rough H^s boundary Γ within a suitable control neighborhood $\Lambda(\Gamma_*, \varepsilon, \delta)$.
- To allow us to obtain the robust continuation result in Theorem 2.1.7, which does not require any reference to control neighborhoods.

Following the discussion in the above two remarks, throughout the chapter we will often abbreviate $\Lambda(\Gamma_*, \varepsilon, \delta)$ by Λ_*, where the suppressed parameters $\varepsilon > 0$ and $\delta > 0$ are understood to be small but fixed universal parameters, which depend only on s and on the thickness of Ω.

State space

Fix a collar neighborhood Λ_* and $s > \frac{d}{2} + 1$. We define $\mathbf{H}^s$ as the set of all pairs (v, Γ) such that $\Gamma \in \Lambda_*$ is the boundary of a bounded, connected domain Ω and such that the following properties are satisfied:

(i) (Regularity). $v \in H^s_{div}(\Omega)$ and $\Gamma \in H^s$, where $H^s_{div}(\Omega)$ denotes the space of divergence free vector fields in $H^s(\Omega)$.

(ii) (Taylor sign condition). $a := -\nabla p \cdot n_\Gamma > c_0 > 0$, where c_0 may depend on the choice of (v, Γ), and the pressure p is obtained from (v, Γ) by solving the standard elliptic equation (2.1.5) associated to (2.1.1) and (2.1.3).

Given initial data (v_0, Γ_0) in the state space $\mathbf{H}^s$, our eventual goal will be to construct local solutions $(v(t), \Gamma_t)$ that evolve continuously in $\mathbf{H}^s$. To accomplish this, we must define a suitable notion of topology on our state space. This will enable us to establish two key properties of our flow; namely,

(i) Continuity of solutions with values in $\mathbf{H}^s$.

(ii) Continuous dependence of solutions $(v(t), \Gamma_t)$ as functions of the initial data (v_0, Γ_0).

Note that since $\mathbf{H}^s$ is not a linear space, the above two continuity properties require some explanation. To measure the size of individual states $(v, \Gamma) \in \mathbf{H}^s$, we define $\|(v, \Gamma)\|_{\mathbf{H}^s}^2 := \|\Gamma\|_{H^s}^2 + \|v\|_{H^s(\Omega)}^2$. However, since $\mathbf{H}^s$ is not a linear space, $\|\cdot\|_{\mathbf{H}^s}$ does not induce a norm topology in the usual sense. Hence, we still need an appropriate way of comparing different states. Motivated by [36, 72], we define convergence in $\mathbf{H}^s$ as follows.

Definition 2.3.5. We say that a sequence $(v_n, \Gamma_n) \in \mathbf{H}^s$ converges to $(v, \Gamma) \in \mathbf{H}^s$ if

(i) (Uniform Taylor sign condition). For some $c_0 > 0$ independent of n, we have

$$a_n, a > c_0 > 0.$$

(ii) (Domain convergence). $\Gamma_n \to \Gamma$ in H^s. That is, $\eta_{\Gamma_n} \to \eta_\Gamma$ in $H^s(\Gamma_*)$ where η_{Γ_n} and η_Γ correspond to the collar coordinate representations of Γ_n and Γ, respectively.

(iii) (Norm convergence). For every $\varepsilon > 0$ there exists a smooth divergence free function $\tilde{v}$ defined on a neighborhood $\tilde{\Omega}$ of $\overline{\Omega}$ with $\|\tilde{v}\|_{H^s(\tilde{\Omega})} < \infty$ and satisfying

$$\|v - \tilde{v}\|_{H^s(\Omega)} \leq \varepsilon$$

and

$$\limsup_{n \to \infty} \|v_n - \tilde{v}\|_{H^s(\Omega_n)} \leq \varepsilon.$$

With the above notion of convergence, it makes sense to define $C([0, T]; \mathbf{H}^s)$. We remark, however, that in [134, 140, 139, 141], $C([0, T]; \mathbf{H}^s)$ is defined in a slightly different way, via the existence of an extension to a continuous function with values in $H^s(\mathbb{R}^d)$. In Section 2.5, we construct a family of extension operators which depend continuously in a suitable sense on the domain, making the above two notions of continuity essentially interchangeable.

2.4 Difference estimates and uniqueness

Comparing different solutions is key to any well-posedness result. Since our problem is quasilinear, such a comparison cannot be achieved uniformly in the leading $\mathbf{H}^s$ topology, but instead only in weaker topologies. The main result of this section provides a Lipschitz bound for the distance between two solutions in the L^2 topology, akin to our bounds for the linearized equation. Notably, our distance bounds propagate at the level of our control parameters, which require for instance a Lipschitz bound on the velocity but no higher regularity. This is what will allow us to establish uniqueness of solutions under very weak regularity assumptions. Moreover, as we shall see shortly, these low regularity distance bounds also serve as an essential building block in our construction of rough solutions as unique limits of smooth solutions, as well as in our proof of the continuity of the data-to-solution map.

The fundamental difficulty in achieving our distance bounds is the need to compare states which live on different domains. To overcome this difficulty, we construct a "distance functional" which *simultaneously* captures the distance between (functions on) different domains and admits a time evolution that we are able to track. To the best of our knowledge, no such low regularity difference bounds or even uniqueness results were previously known for any incompressible free boundary Euler model. Instead, we take our cue from the work [72] of the first and the third authors, which considers a similar free boundary problem but for a compressible Euler model. We note, however, that the similarity between the uniqueness argument here and its counterpart in [72] is only at the conceptual level, as the two flows have very different behaviors both inside the domain and near the free boundary.

The distance functional

Our first objective is to use the linearized energy as a guide to construct a distance functional which will be suitable for comparing nearby solutions. We begin by fixing a collar neighborhood $\Lambda(\Gamma_*, \varepsilon, \delta)$, where $\varepsilon > 0$ and $\delta > 0$ are small. We then suppose that we have two states (v, Γ), (v_h, Γ_h) with respective domains Ω, Ω_h. We let η_Γ and η_{Γ_h} be the corresponding representations of Γ and Γ_h as graphs over Γ_*. Following the linearized energy estimate, we aim to define analogues of the linearized variables w and s, which heuristically should measure the L^2 distance between v and v_h and the distance between Γ and Γ_h, respectively. One technical caveat is that v and v_h are not defined on the same domain. For this reason, we define $\widetilde{\Omega} = \Omega \cap \Omega_h$. We can represent the free boundary $\widetilde{\Gamma}$ for $\widetilde{\Omega}$ as a graph over Γ_* via the function $\eta_{\widetilde{\Gamma}} = \eta_\Gamma \wedge \eta_{\Gamma_h}$. Note that although the graph representation $\eta_{\widetilde{\Gamma}}$ is well-defined, $\widetilde{\Gamma}$ is only Lipschitz in general, so will not be in $\Lambda(\Gamma_*, \varepsilon, \delta)$.

To measure the (signed) distance between Γ and Γ_h, we define $s_h^* : \Gamma_* \to \mathbb{R}$ by

$$s_h^*(x) = \eta_{\Gamma_h}(x) - \eta_\Gamma(x). \tag{2.4.1}$$

As will become evident below, although s_h^* correctly measures the distance between the free hypersurfaces, it has the "wrong" domain. To fix this, we define the variable $s_h : \widetilde{\Gamma} \to \mathbb{R}$ by pushing s_h^* forward to the hypersurface $\widetilde{\Gamma}$. In other words, for $x \in \widetilde{\Gamma}$, we define $s_h(x) = s_h^*(\pi(x))$, where π denotes the canonical projection, mapping the image of $\Gamma_* \times [-\delta, \delta]$ under φ back to Γ_*. For convenience, we also extend ν to a vector field X defined on the image of φ via $X(x) = \nu(\pi(x))$. We will not actually use the displacement function s_h directly in the difference estimates below. In particular, it will not act as our desired analogue of the linearized variable s. This is because its dynamics are somewhat awkward to work with. Instead of using s_h, it is far more convenient (and geometrically natural) to use the the pressure difference $p - p_h$ (along with a suitable weight to be defined below) to measure the distance between Γ and Γ_h. To motivate this, recall that for solutions to the free boundary Euler equations, the Taylor sign condition implies that p and p_h are non-degenerate defining functions for Γ_t and $\Gamma_{t,h}$ within a suitable collar neighborhood. Therefore, on the boundary of $\tilde{\Omega}_t = \Omega_t \cap \Omega_{t,h}$, $p - p_h$ should be proportional to the displacement function s_h. The dynamics of $p - p_h$ turn out to be much easier to work with than those of s_h, as terms involving $p - p_h$ will appear naturally when we use the free boundary Euler equations to compare solutions.

With the above motivation in mind and using the linearized equation as a guide, we define our distance functional as follows:

$$D((v, \Gamma), (v_h, \Gamma_h)) := D(v, v_h) := \frac{1}{2}\int_{\tilde{\Omega}} |v - v_h|^2 \, dx + \frac{1}{2}\int_{\tilde{\Gamma}} b|p - p_h|^2 \, dS, \tag{2.4.2}$$

where the weight function b is defined by

$$b := a^{-1} 1_{\tilde{\Gamma} \cap \Gamma} + a_h^{-1} 1_{\tilde{\Gamma} \cap \Gamma_h}.$$

As $p - p_h$ vanishes on $\Gamma \cap \Gamma_h$, we may rewrite the distance functional in the slightly more convenient form

$$D(v, v_h) = \frac{1}{2}\int_{\tilde{\Omega}} |v - v_h|^2 \, dx + \frac{1}{2}\int_{\mathcal{A}} a^{-1}|p - p_h|^2 \, dS + \frac{1}{2}\int_{\mathcal{A}_h} a_h^{-1}|p - p_h|^2 \, dS,$$

where $\mathcal{A} := \tilde{\Gamma} \cap \Gamma - \Gamma \cap \Gamma_h$ and $\mathcal{A}_h := \tilde{\Gamma} \cap \Gamma_h - \Gamma \cap \Gamma_h$.

Letting $\overline{F}$ denote the average of F along the flow φ between the free surfaces, the fundamental theorem of calculus implies that for $x \in \widetilde{\Gamma}$,

$$p_h(x) - p(x) = \begin{cases} -\overline{\nabla p_h \cdot X} s_h(x) & \text{if } x \in \mathcal{A}, \\ -\overline{\nabla p \cdot X} s_h(x) & \text{if } x \in \mathcal{A}_h. \end{cases} \tag{2.4.3}$$

Therefore, thanks to the Taylor sign condition and assuming the regularity $p, p_h \in C^{1,\varepsilon}$, we should have $|p - p_h| \approx |s_h|$ on $\widetilde{\Gamma}$ within a tight enough collar neighborhood. The precise manner in which we have this proportionality will be made clear shortly. Finally, note that, for solutions to the free boundary Euler equations, a simple computation yields the following equation for $v - v_h$ in $\tilde{\Omega}_t$:

$$\begin{cases} D_t(v - v_h) + \nabla(p - p_h) = (v_h - v) \cdot \nabla v_h, \\ \nabla \cdot (v - v_h) = 0. \end{cases} \tag{2.4.4}$$

Remark 2.4.1. Although it is not particularly important for the difference estimates, we note that the distance functional (2.4.2) makes sense for general (not necessarily dynamical) states (v, Γ) and (v_h, Γ_h). Indeed, given suitable states (v, Γ) and (v_h, Γ_h), we can always associate pressures p and p_h by solving the standard elliptic equation associated to (2.1.1) and (2.1.3). As we will see in Section 2.7, it is very important that our energy functional for the $\mathbf{H}^k$ energy bounds be defined for general states $(v, \Gamma) \in \mathbf{H}^k$.

Difference estimates

We are now ready to propagate difference bounds for two solutions to the free boundary Euler equations.

Theorem 2.4.2 (Difference Bounds)**.** Let $0 < \varepsilon, \delta \ll 1$ and let $\Lambda_* = \Lambda(\Gamma_*, \varepsilon, \delta)$ be a collar neighborhood. Suppose that (v, Γ_t) and $(v_h, \Gamma_{t,h})$ are solutions to the free boundary Euler equations that evolve in the collar in a time interval $[0, T]$ and satisfy a,$a_h > c_0 > 0$. Then we have the estimate

$$\frac{d}{dt} D(v, v_h) \lesssim_{A, A_h} (B + B_h) D(v, v_h)$$

where

$$B := \|v\|_{W^{1,\infty}(\Omega_t)} + \|\Gamma_t\|_{C^{1,\frac{1}{2}}} + \|D_t p\|_{W^{1,\infty}(\Omega_t)}, \qquad A := \|v\|_{C^{\frac{1}{2}+\varepsilon}(\Omega_t)} + \|\Gamma_t\|_{C^{1,\varepsilon}},$$

B_h and A_h are the analogous quantities corresponding to v_h, p_h, $D_t^h p_h$ and $\Gamma_{t,h}$ and we have implicitly assumed that our solutions have regularity $B, B_h \in L^1_T$ and $A, A_h \in L^\infty_T$.

Remark 2.4.3. It is worth remarking that all of the results in this section hold equally well if the control parameter B is replaced by

$$B_\varepsilon = \|v\|_{C^{1,\varepsilon}(\Omega_t)} + \|\Gamma_t\|_{C^{1,\frac{1}{2}}},$$

which depends solely on the regularity of v and Γ_t. This is because we will later prove an elliptic estimate of the form

$$\|D_t p\|_{W^{1,\infty}(\Omega_t)} \lesssim_A B_\varepsilon.$$

See Lemma 2.7.9 and Remark 2.7.10 for details. We prefer, however, to work with the control parameter B defined above as its L^1_T norm is scale invariant.

Proof. For simplicity of notation, we drop the t subscript for the domains below. We also use $\lesssim_A$ as a shorthand for $\lesssim_{A,A_h}$. To ensure that we can estimate expressions involving the pressure in terms of the control parameters A and B above, we need the bounds

$$\|p\|_{C^{1,\varepsilon}(\Omega)} \lesssim_A 1, \quad \|p\|_{C^{1,\frac{1}{2}}(\Omega)} \lesssim_A B, \tag{2.4.5}$$

as well as the analogous bounds for p_h. The proof that these bounds hold will be postponed until later when the requisite elliptic estimates are developed. See Lemma 2.7.5 and Lemma 2.7.9 for details. Now, to proceed with the difference estimate, we recall the identity

$$\frac{d}{dt}D(v,v_h) = \frac{1}{2}\frac{d}{dt}\int_{\tilde{\Omega}} |v-v_h|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_{\mathcal{A}} a^{-1}|p-p_h|^2\,dS + \frac{1}{2}\frac{d}{dt}\int_{\mathcal{A}_h} a_h^{-1}|p-p_h|^2\,dS. \tag{2.4.6}$$

To compute the first term, we would like to use Reynolds' transport theorem, as in Proposition 2.2.3. However, here we do not have a good velocity field $\tilde{v}$ so that $\tilde{\Omega}$ flows with velocity $\tilde{v}$. Constructing such a field seems to be at the very least impractical, so we will instead allow for a correction term which is a boundary integral. For this purpose, suppose that $D(t)$ is a time-dependent domain for which we may define at almost every point of the boundary a normal velocity v_b for the boundary. Note that if $D(t)$ were flowing with velocity v, then $v_b = v \cdot n_{\partial D(t)}$, where $n_{\partial D(t)}$ is the outward unit normal. For more general velocity fields v on $D(t)$, we have the following proposition.

Proposition 2.4.4. Given a velocity field v defined on a time-dependent domain $D(t)$ with Lipschitz boundary flowing with normal velocity v_b, we have

$$\frac{d}{dt}\int_{D(t)} f\,dx = \int_{D(t)} D_t f + \nabla\cdot v f\,dx + \int_{\partial D(t)} f(v_b - v\cdot n_{\partial D(t)})\,dS.$$

The proof is a straightforward application of the divergence theorem.

In our setting, we need to make a vector field choice on $\tilde{\Omega}_t$; this will simply be the velocity v, though we could have equally chosen v_h. We remark that in the corresponding argument in [72] the average of the two was used, in order to better symmetrize the problem. However, the argument here is slightly more robust, and such a choice is not needed.

For this choice of v, we examine the boundary weight $v \cdot n_{\partial D(t)} - v_b$ appearing in the above formula. For this we use the disjoint boundary decomposition

$$\tilde{\Gamma} = \mathcal{A} \cup \mathcal{A}_h \cup (\Gamma \cap \Gamma_h),$$

where the normal $n_{\tilde{\Gamma}}$ is given a.e. by

$$n_{\tilde{\Gamma}} = \begin{cases} n_\Gamma & \text{in } \mathcal{A} \cup (\Gamma \cap \Gamma_h), \\ n_{\Gamma_h} & \text{in } \mathcal{A}_h \cup (\Gamma \cap \Gamma_h), \end{cases}$$

with the two normals agreeing a.e. on $\Gamma \cap \Gamma_h$. Correspondingly, for almost every point on $\tilde{\Gamma}$ we have $|v_b - v \cdot n_{\tilde{\Gamma}}| \leq |v - v_h|$, as can be seen by working with the collar parameterization $\eta_\Gamma \wedge \eta_{\Gamma_h}$ for $\tilde{\Gamma}$ and the kinematic boundary conditions for Γ and Γ_h.

We now use Proposition 2.4.4 and the incompressibility of v for each of the three terms in (2.4.6). We begin by studying the first term, where we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\tilde{\Omega}} |v - v_h|^2 \, dx \leq \frac{1}{2}\int_{\tilde{\Omega}} D_t|v - v_h|^2 \, dx + \frac{1}{2}\int_{\tilde{\Gamma}} |v - v_h|^3 \, dS. \tag{2.4.7}$$

We note that, unlike in the case of the linearized equation, here we obtain a nonzero boundary term. However, this term has the redeeming feature that it is cubic in the difference $v - v_h$. To estimate it, we use a simple variant of the trace theorem. Indeed, as $\Gamma, \Gamma_h \in \Lambda_*$, we may find a smooth vector field X defined on $\mathbb{R}^d$ with C^k bounds uniform in Λ_* which is also uniformly transverse to $\tilde{\Gamma}$. By the divergence theorem, we then have

$$\begin{aligned} \frac{1}{2}\int_{\tilde{\Gamma}} |v - v_h|^3 \, dS \lesssim \int_{\tilde{\Gamma}} X \cdot n_{\tilde{\Gamma}} |v - v_h|^3 \, dS &\lesssim (B + B_h)\|v - v_h\|^2_{L^2(\tilde{\Omega})} \\ &\lesssim (B + B_h) D(v, v_h). \end{aligned} \tag{2.4.8}$$

Now, for the remaining term in (2.4.7), we use (2.4.4) and integrate by parts to obtain

$$\begin{aligned} \frac{1}{2}\int_{\tilde{\Omega}} D_t|v - v_h|^2 \, dx &= \int_{\tilde{\Omega}} (v - v_h) D_t(v - v_h) \, dx \\ &= -\int_{\tilde{\Gamma}} (p - p_h)(v - v_h) \cdot n_{\tilde{\Gamma}} \, dS + \int_{\tilde{\Omega}} (v - v_h) \cdot [(v_h - v) \cdot \nabla v_h] \, dx \\ &\leq -\int_{\tilde{\Gamma}} (p - p_h)(v - v_h) \cdot n_{\tilde{\Gamma}} \, dS + (B + B_h) D(v, v_h). \end{aligned} \tag{2.4.9}$$

Using the decomposition $\tilde{\Gamma} = \mathcal{A} \cup \mathcal{A}_h \cup (\Gamma \cap \Gamma_h)$ and using that $p - p_h = 0$ on $\Gamma \cap \Gamma_h$ by the dynamic boundary condition (2.1.3), we can write

$$
\begin{aligned}
-\int_{\tilde{\Gamma}} (p - p_h)(v - v_h) \cdot n_{\tilde{\Gamma}} \, dS &= -\int_{\mathcal{A}} (p - p_h)(v - v_h) \cdot n_\Gamma \, dS - \int_{\mathcal{A}_h} (p - p_h)(v - v_h) \cdot n_{\Gamma_h} \, dS \\
&= \int_{\mathcal{A}} a^{-1}(p - p_h)(v - v_h) \cdot \nabla p \, dS \\
&+ \int_{\mathcal{A}_h} a_h^{-1}(p - p_h)(v - v_h) \cdot \nabla p_h \, dS.
\end{aligned}
$$

Now, define

$$
J := \int_{\mathcal{A}} a^{-1}(p - p_h)(v - v_h) \cdot \nabla p \, dS + \frac{1}{2}\frac{d}{dt}\int_{\mathcal{A}} a^{-1}|p - p_h|^2 \, dS,
$$

and

$$
J_h := \int_{\mathcal{A}_h} a_h^{-1}(p - p_h)(v - v_h) \cdot \nabla p_h \, dS + \frac{1}{2}\frac{d}{dt}\int_{\mathcal{A}_h} a_h^{-1}|p - p_h|^2 \, dS.
$$

Combining (2.4.8) and (2.4.9), we obtain

$$
\frac{d}{dt} D(v, v_h) \lesssim (B + B_h) D(v, v_h) + J + J_h.
$$

It remains to show that

$$
J + J_h \lesssim_A (B + B_h) D(v, v_h).
$$

We show the details for J. The treatment of J_h will be virtually identical. We begin by using Proposition 2.2.3 to expand

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathcal{A}} a^{-1}|p - p_h|^2 \, dS = -\frac{1}{2}\int_{\mathcal{A}} a^{-2} D_t a |p - p_h|^2 \, dS - \frac{1}{2}\int_{\mathcal{A}} a^{-1}|p - p_h|^2 [n_\Gamma \cdot \nabla v \cdot n_\Gamma] \, dS \\
+ \int_{\mathcal{A}} a^{-1}(p - p_h) D_t (p - p_h) \, dS.
\end{aligned}
\tag{2.4.10}
$$

The validity of the identity (2.4.10) is justified by noting that $|p - p_h|^2$ vanishes to second order on $\Gamma \cap \Gamma_h$, so one can extend by zero to write the integral on the left-hand side as an integral over Γ, apply standard identities there, and then return to an integral over $\mathcal{A}$. From (2.4.10) and adding the first term in the definition of J, we obtain (noting that by the kinematic and dynamic boundary conditions, we have $D_t p = 0$ on $\mathcal{A}$),

$$
J \lesssim_A -\int_{\mathcal{A}} a^{-1}(p - p_h) D_t^h p_h dS + \int_{\mathcal{A}} a^{-1}(p - p_h)(v - v_h) \cdot \nabla (p - p_h) dS + BD(v, v_h).
$$

In the above, we used the standard identity (2.5.35) to control $D_t a$. For the first term on the right-hand side we use that $D_t^h p_h$ vanishes on Γ_h, (2.4.3), the fundamental theorem of calculus, the Taylor sign condition and (2.4.5), to estimate

$$|D_t^h p_h| \lesssim_A \|\nabla D_t^h p_h\|_{L^\infty}|s_h| \approx_A \|\nabla D_t^h p_h\|_{L^\infty}|p-p_h| \lesssim_A (B+B_h)|p-p_h|.$$

Hence,

$$\int_{\mathcal{A}} a^{-1}(p-p_h)D_t^h p_h dS \lesssim_A (B+B_h)D(v,v_h).$$

It remains to estimate the cubic term, and show that

$$\left|\int_{\mathcal{A}} a^{-1}(p-p_h)(v-v_h)\cdot\nabla(p-p_h)\, dS\right| \lesssim_A (B+B_h)D(v,v_h). \tag{2.4.11}$$

We will need to perform a more careful analysis here, so that only the pointwise control terms appear in the estimate. Note that if we had instead settled for L^2 based control parameters, this cubic term could be handled relatively easily.

We recall that $\mathcal{A} \subseteq \Gamma$. Given a point $x \in \mathcal{A}$, its distance to Γ_h is proportional to $|(p-p_h)(x)|$. We consider a locally finite Vitali type covering of the set $\mathcal{A}$ with countably many balls $B_j = B(x_j, r_j)$ of radius r_j proportional to $|(p-p_h)(x_j)|$, so that in particular we have $B_j \subseteq \Omega_h$. We denote by D_j the energy of the difference in the region B_j, i.e., the integral in (2.4.2) restricted to B_j. Then

$$\sum_j D_j \lesssim D((v,\Gamma),(v_h,\Gamma_h)).$$

Hence, by the uniform bound on a^{-1}, it would suffice to show that

$$\int_{\mathcal{A}\cap B_j} |(p-p_h)(v-v_h)\cdot\nabla(p-p_h)|\; dS \lesssim_A (B+B_h)D_j. \tag{2.4.12}$$

We will indeed show that this bound holds for the bulk of the expression on the left. However, for the remaining part we will return to a global argument. For $\mathcal{A}$ we just use the uniform Lipschitz bound in this analysis. We first note that in $\tilde{\Omega}\cap B_j$ we have

$$|p-p_h| \approx_A r_j,$$

which after integration yields a good bound for r_j within B_j:

$$\int_{\mathcal{A}\cap B_j} |p-p_h|^2\, dS \approx_A r_j^{d+1} \lesssim_A D_j. \tag{2.4.13}$$

Next we consider $v - v_h$, for which we use the $C^{\frac{1}{2}}$ norm, which is part of our control norm A, in order to estimate the surface integral by the ball integral. This yields

$$\int_{\mathcal{A}\cap B_j} |v - v_h|^2 \, dS \lesssim_A r_j^{-1} \int_{\tilde{\Omega}\cap B_j} |v - v_h|^2 \, dx + r_j^d A^2 \lesssim_A r_j^{-1} D_j + r_j^d A^2 \lesssim_A r_j^{-1} D_j. \quad (2.4.14)$$

It remains to consider $\nabla(p - p_h)$. Our starting point is the global bound

$$\|\nabla p\|_{C^{\frac{1}{2}}(\Omega)} + \|\nabla p_h\|_{C^{\frac{1}{2}}(\Omega_h)} \lesssim_A B + B_h, \quad (2.4.15)$$

which is noted in (2.4.5). This allows us to replace $\nabla(p - p_h)$ with its average $\overline{\nabla(p - p_h)}_j$ in any smaller ball $\tilde{B}_j \subseteq \tilde{\Omega} \cap B_j$ of comparable size, because

$$\|\nabla(p - p_h) - \overline{\nabla(p - p_h)}_j\|_{L^\infty(\tilde{\Omega}\cap B_j)} \lesssim_A r_j^{\frac{1}{2}}(B + B_h).$$

Putting everything together we arrive at

$$\int_{\mathcal{A}\cap B_j} |(p - p_h)(v - v_h) \cdot (\nabla(p - p_h) - \overline{\nabla(p - p_h)}_j)| \, dS \lesssim_A (B + B_h) D_j,$$

which represents the bulk of (2.4.12).

It remains to estimate the contribution of the local average of $\nabla(p - p_h)$. Here we view $p - p_h$ as a solution to the following Laplace equation in $\tilde{\Omega}$:

$$\begin{cases} \Delta(p - p_h) = -\mathrm{tr}(\nabla v)^2 + \mathrm{tr}(\nabla v_h)^2, \\ p - p_{h|\tilde{\Gamma}} = \tilde{g} := p 1_{\mathcal{A}_h} - p_h 1_{\mathcal{A}}. \end{cases}$$

We split the problem for $p - p_h$ into an inhomogeneous one with homogeneous boundary condition, and a homogeneous one with inhomogeneous boundary condition,

$$p - p_h = (p - p_h)_{inh} + (p - p_h)_{hom}.$$

For the inhomogeneous problem we can write the source term in divergence form to estimate

$$\|\mathrm{tr}(\nabla v)^2 - \mathrm{tr}(\nabla v_h)^2\|_{H^{-1}(\tilde{\Omega})} \lesssim (B + B_h) D^{\frac{1}{2}},$$

which by a simple energy estimate gives a global L^2 bound

$$\|\nabla(p - p_h)_{inh}\|_{L^2(\tilde{\Omega})} \lesssim_A (B + B_h) D^{\frac{1}{2}}.$$

This in turn yields a bound for the corresponding averages by Hölder's inequality,

$$\sum_j r_j^d |\overline{\nabla(p-p_h)_{inh,j}}|^2 \lesssim_A (B+B_h)^2 D.$$

The contribution of this into (2.4.11) is then estimated using (2.4.13) and (2.4.14) as follows:

$$\begin{aligned} J_{inh} := & \sum_j \int_{\mathcal{A}\cap B_j} |p-p_h||v-v_h||\overline{\nabla(p-p_h)_{inh,j}}|\, dS \\ \lesssim_A & \sum_j r_j^{\frac{d+1}{2}} \|v-v_h\|_{L^2(\mathcal{A}\cap B_j)} |\overline{\nabla(p-p_h)_{inh,j}}| \\ \lesssim_A & \sum_j D_j^{\frac{1}{2}} r_j^{\frac{d}{2}} |\overline{\nabla(p-p_h)_{inh,j}}| \\ \lesssim_A & (B+B_h)D, \end{aligned}$$

where in the last step we have used Cauchy-Schwarz with respect to j.

For the homogeneous term, on the other hand, we need to carefully examine the regularity of the Dirichlet data $\tilde{g}$. On one hand, by the definition of the distance D we have the L^2 bound

$$\|\tilde{g}\|^2_{L^2(\tilde{\Gamma})} \lesssim_A D. \tag{2.4.16}$$

On the other hand, by (2.4.15), on each of the two regions $\mathcal{A}_h$ respectively $\mathcal{A}$, we have formally

$$\|\tilde{g}\|_{C^{1,\frac{1}{2}}(\mathcal{A}_h)} + \|\tilde{g}\|_{C^{1,\frac{1}{2}}(\mathcal{A})} \lesssim_A B + B_h. \tag{2.4.17}$$

This bound has to be carefully interpreted, which we do within the proof of Lemma 2.4.5 below.

A formal interpolation between (2.4.16) and (2.4.17) would yield a $W^{1,6}(\tilde{\Gamma})$ bound for $\tilde{g}$. We make this bound rigorous in the following.

Lemma 2.4.5. The function $\tilde{g}$ above satisfies the bound

$$\|\tilde{g}\|_{W^{1,6}(\tilde{\Gamma})} \lesssim (B+B_h)^{\frac{2}{3}} D^{\frac{1}{6}}. \tag{2.4.18}$$

Proof. We begin by noting that the two components $g := p1_{\mathcal{A}_h}$ and $g_h := -p_h 1_{\mathcal{A}}$ of $\tilde{g}$ are nonzero on disjoint sets $\mathcal{A}_h$ respectively $\mathcal{A}$, and vanish on the corresponding boundaries $\partial\mathcal{A}_h$, respectively $\partial\mathcal{A}$. Hence, we can prove the bound (2.4.18) separately for the two components.

We consider g, which lives on $\mathcal{A}_h \subseteq \Gamma_h$. Here not only is Γ_h a Lipschitz surface, but it also has a $C^{1,\frac{1}{2}}$ bound of B_h (which is not the case for $\tilde{\Gamma}$).

Using a standard partition of unity we can reduce the problem to the case when Γ_h is a graph,

$$\Gamma_h = \{x_d = \phi(x')\},$$

where

$$\|\phi\|_{Lip} \lesssim_A 1, \qquad \|\phi\|_{C^{1,\frac{1}{2}}} \lesssim B_h. \tag{2.4.19}$$

We denote the Lipschitz projection of $\mathcal{A}_h$ by $\mathcal{P}\mathcal{A}_h \subseteq \mathbb{R}^{d-1}$. We can equivalently consider g as a function on $\mathcal{P}\mathcal{A}_h$, in which case the bound (2.4.18) becomes

$$\|\nabla g\|_{L^6(\mathcal{P}\mathcal{A}_h)} \lesssim_A (B + B_h)^{\frac{2}{3}} D^{\frac{1}{6}}. \tag{2.4.20}$$

We now summarize the information that we have on g as a function on $\mathcal{P}\mathcal{A}_h$:

(i) (L^2 control).

$$\|g\|^2_{L^2(\mathcal{P}\mathcal{A}_h)} \lesssim_A D,$$

which comes from (2.4.16).

(ii) (Hölder control).

$$\|\nabla g\|_{C^{\frac{1}{2}}(\mathcal{P}\mathcal{A}_h)} \lesssim_A B + B_h,$$

which is a consequence of (2.4.15), (2.4.19) and chain rule.

(iii) (Zero boundary data).

$$g = 0 \ \text{ on } \ \partial\mathcal{P}\mathcal{A}_h.$$

We will prove that these three properties imply the desired bound (2.4.20). The difficulty here is that we do not know that $\nabla g = 0$ on $\partial\mathcal{P}\mathcal{A}_h$; else we could simply extend g by 0 outside $\mathcal{P}\mathcal{A}_h$ and this becomes a standard interpolation bound. Further, we do not a priori control the regularity of the boundary $\partial\mathcal{P}\mathcal{A}_h$.

Without any loss of generality we assume that $g > 0$ on $\mathcal{P}\mathcal{A}_h$; else we split this set into connected components where g has constant sign, modulo a set where $\nabla g = 0$ a.e. To prove the desired bound we will use a well-chosen Vitali covering of the set $S = \mathcal{P}\mathcal{A}_h \setminus \{\nabla g = 0\}$ with balls. This choice is as follows: For each $x \in S$ we consider a ball $B_x = B(x, r_x)$ with radius $r_x = c^2(B+B_h)^{-2}|\nabla g(x)|^2$ where $c>0$ is a small universal constant, chosen so that $|\nabla g|$ is nearly constant on B_x, i.e.,

$$|\nabla g(y) - \nabla g(x)| \lesssim c|\nabla g(x)| \ll |\nabla g(x)|, \qquad y \in B_x.$$

The union of the balls B_x with $x \in S$ clearly covers S, so Vitali's lemma allows us to extract a countable disjoint subfamily of such balls $B_j = B_{x_j}$ so that

$$S \subseteq \bigcup 5B_j.$$

Since ∇g is almost constant on B_x and $g(x) > 0$, a key observation is that there must exist a nontrivial sector $C_x \subseteq B_x$ where

$$g > 0 \quad \text{in } C_x, \qquad |C_x| \approx |B_x|.$$

Since $g = 0$ on $\partial\mathcal{PA}_h$, it follows that we must have $C_x \subseteq S$; this is what allows us to bypass the lack of geometric information on the set $\mathcal{PA}_h$.

On C_x, the function g is almost linear with slope approximately $|\nabla g(x)|$. Therefore, we must have

$$\|g\|^2_{L^2(C_x)} \gtrsim r_x^{d+1}|\nabla g(x)|^2.$$

We will use this bound to estimate from above the L^6 norm of ∇g in each $5B_j$ as follows:

$$\begin{aligned}\|\nabla g\|^6_{L^6(5B_j)} &\lesssim r_{x_j}^{d-1}|\nabla g(x_j)|^6 \\ &\lesssim \|g\|^2_{L^2(C_j)} r_{x_j}^{-2}|\nabla g(x_j)|^4 \\ &\approx \|g\|^2_{L^2(C_j)}(B+B_h)^4.\end{aligned}$$

Now, we sum over j, using the disjointness of the balls B_j and thus of C_j. This gives

$$\sum_j \|\nabla g\|^6_{L^6(5B_j)} \lesssim \|g\|^2_{L^2(S)}(B+B_h)^4 \lesssim_A D(B+B_h)^4,$$

which concludes the proof of the lemma.

□

Now we use the bound in Lemma 2.4.5 to solve the homogeneous Dirichlet problem in $\tilde{\Omega}$ and to obtain the estimate

$$\|\nabla(p-p_h)^*_{hom}\|_{L^6(\tilde{\Gamma})} \lesssim (B+B_h)^{\frac{2}{3}} D^{\frac{1}{6}},$$

where $*$ stands for the nontangential maximal function. This bound is due to Verchota [155], but see also the further discussion by Jerison-Kenig [82, Theorem 5.6] as well as the case of C^1 boundaries considered earlier by Fabes-Jodeit-Rivière [42].

The exponent 6 is allowed above provided that the Lipschitz norm of the boundary is sufficiently small. Precisely, the upper limit of the allowed exponents goes to infinity as the corner size decreases to 0. The smallness of the intersection angle between Γ and Γ_h is a consequence of the $C^{1,\varepsilon}$ common regularity bound together with the use of a sufficiently refined collar region.

To use the nontangential maximal function bound, within the ball $B_j = B(x_j, r_j)$ we consider a smaller ball

$$\tilde{B}_j = B(x_j - \frac{1}{2} r_j n_j, \frac{1}{4} r_j).$$

For $y \in \tilde{B}_j$ we have

$$|\nabla(p - p_h)_{hom}(y)| \lesssim |\nabla(p - p_h)^*_{hom}(z)|, \qquad z \in \tilde{\Gamma} \cap \frac{1}{4} B_j.$$

Taking averages on the left and integrating on the right, we arrive at

$$r_j^{d-1} |\overline{\nabla(p - p_h)_{hom,j}}|^6 \lesssim_A \|\nabla(p - p_h)^*_{hom}\|^6_{L^6(\tilde{\Gamma} \cap \frac{1}{4} B_j)}.$$

Since the balls B_j are disjoint, summation in j yields

$$\sum_j r_j^{d-1} |\overline{\nabla(p - p_h)_{hom,j}}|^6 \lesssim (B + B_h)^4 D. \tag{2.4.21}$$

On the other hand, for $v - v_h$ we use the interpolation bound (2.4.8), which gives

$$\|v - v_h\|_{L^3(\tilde{\Gamma})} \lesssim (B + B_h)^{\frac{1}{3}} D^{\frac{1}{3}}. \tag{2.4.22}$$

We are now ready to estimate the corresponding contribution to (2.4.11) using also (2.4.13) and (2.4.14) as follows:

$$\begin{aligned}
J_{hom} &:= \sum_j \int_{\mathcal{A} \cap B_j} |p - p_h| |v - v_h| |\overline{\nabla(p - p_h)_{hom,j}}| \, dS \\
&\lesssim_A \sum_j r_j (r_j^{\frac{2(d-1)}{3}} \|v - v_h\|_{L^3(\mathcal{A} \cap B_j)}) |\overline{\nabla(p - p_h)_{hom,j}}| \\
&\lesssim_A \sum_j r_j^{\frac{d+1}{2}} \|v - v_h\|_{L^3(\mathcal{A} \cap B_j)} (r_j^{\frac{d-1}{6}} |\overline{\nabla(p - p_h)_{hom,j}}|) \\
&\lesssim_A (B + B_h) D.
\end{aligned}$$

At the last step we have applied Hölder's inequality in j with exponents 2, 3 and 6, using (2.4.13), (2.4.22) and (2.4.21). This completes the proof of (2.4.12) and therefore the proof of Theorem 2.4.2.

$\square$

One consequence of the difference bounds is the following uniqueness result.

Theorem 2.4.6 (Uniqueness)**.** Let $\varepsilon > 0$ and let Ω_0 be a bounded domain with boundary $\Gamma_0 \in \Lambda(\Gamma_*, \varepsilon, \delta)$. Then for $\Gamma_0 \in C^{1,\frac{1}{2}}$ and divergence free $v_0 \in W^{1,\infty}(\Omega_0)$ satisfying the Taylor sign condition, the free boundary Euler equations admit at most one solution (v, Γ_t) on a time interval $[0, T]$ with $\Gamma_t \in \Lambda(\Gamma_*, \varepsilon, \delta)$ and

$$\sup_{0 \le t \le T} \|v\|_{C_x^{\frac{1}{2}+\varepsilon}(\Omega_t)} + \int_0^T \|v\|_{W_x^{1,\infty}(\Omega_t)} + \|D_t p\|_{W_x^{1,\infty}(\Omega_t)} + \|\Gamma_t\|_{C_x^{1,\frac{1}{2}}} \, dt < \infty.$$

Proof. Suppose (v, Ω_t) and $(v_h, \Omega_{t,h})$ are a pair of solutions satisfying the conditions of the theorem with the same initial data. From the differences estimates, we immediately obtain $v = v_h$ on $\Omega_t \cap \Omega_{t,h}$. Next, we argue that the domain Ω_t coincides with $\Omega_{t,h}$. First, we note that the intersection is non-empty if $\delta > 0$ is small enough. We now show $\Omega_t \subseteq \Omega_{t,h}$. It suffices to show $\Omega_t \subseteq \overline{\Omega}_{t,h}$. If this is not true, then there is $x \in \Gamma_{t,h}$ such that $x \in \Omega_t$. Such a point must lie on $\partial(\Omega_t \cap \Omega_{t,h})$. Therefore, from the estimate for the distance functional, we have $p(x) = 0$. However, within a small enough collar neighborhood, the Taylor sign condition tells us that the level set $\{p = 0\}$ corresponds exactly to the free surface Γ_t. This is a contradiction to x being an interior point of Ω_t. Therefore $\Omega_t \subseteq \Omega_{t,h}$. The reverse inclusion follows by an identical argument. $\square$

2.5 Balanced elliptic estimates

In this section, we prove a collection of refined elliptic estimates which will be crucial for obtaining the sharp pointwise control norms in the higher energy bounds. These estimates will turn out to be quite general and should be applicable to other free boundary problems. In a sense, they can be seen as significant refinements of the so-called *tame estimates* which have been fundamental in the analysis of many water waves problems (see the discussion in [7, 95]), but are not nearly sufficient for our purposes. Indeed, as we will soon see, our proofs of the higher energy bounds require estimates for various elliptic operators which more precisely balance the contributions of the input function and the domain regularity, simultaneously, in both pointwise and L^2 based norms. This simultaneous balance cannot be achieved with the known tame estimates, which often only seem to balance the contributions in L^2 based norms or involve domain dependent constants in pointwise norms which are significantly off scale. The technical utility of our balanced estimates will become readily apparent in Section 2.7, where they will be used to efficiently dispatch with expressions involving relatively complicated iterated applications of the Dirichlet-to-Neumann operator and various other elliptic operators.

In the following, we will always assume that Ω is a bounded domain with boundary $\Gamma \in \Lambda_* := \Lambda(\Gamma_*, \varepsilon_0, \delta)$ for suitably small (but fixed) constants $\varepsilon_0, \delta > 0$. Most of the bounds in this section do not make reference to a particular velocity function, and so, the implicit constants in many of the estimates will only depend on the surface component of the control parameter A; namely, $A_\Gamma := \|\Gamma\|_{C^{1,\varepsilon_0}}$. Hence, for this section, by the relation $X \lesssim_A Y$, we mean $X \leq C(A_\Gamma)Y$ for some constant C depending exclusively on A_Γ. The only exception to this rule (which we will make note of explicitly) will be in Section 2.5, where we will use the full control parameter A to establish estimates for commutators of various elliptic operators with D_t. We will also harmlessly let A depend on the domain volume throughout, as the volume of the domain will be conserved in the dynamic problem.

Throughout the section, by a slight abuse of notation, we will follow the convention that a parameter ε may vary from line to line by a fixed scalar factor. Generally speaking, we will take $\varepsilon > 0$ to be any positive constant with $\varepsilon \ll \varepsilon_0$.

Extension operators in Λ_* and product type estimates on Ω

To establish the desired elliptic estimates, it will be convenient to have an extension operator which is bounded from $H^s(\Omega) \to H^s(\mathbb{R}^d)$ for $s \geq 0$, and $C^{k,\alpha}(\Omega) \to C^{k,\alpha}(\mathbb{R}^d)$ for a suitable range of k and α with bounds depending only on the implicit constant A. Among other things, this will enable us to recover many of the standard product type estimates which are well-known on $\mathbb{R}^d$. To this end, let $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant M. Let $\Omega = \{(x, y) \in \mathbb{R}^d : y > \varphi(x)\}$. Moreover, for $1 \leq p \leq \infty$ and an integer $k \geq 0$, let $W^{k,p}(\Omega)$ denote the usual Sobolev space consisting of distributions whose derivatives up to order k belong to $L^p(\Omega)$. It is a classical result of Stein [142, Theorem 5', p. 181] that there exists a linear operator $\mathcal{E}$ mapping functions on Ω to functions on $\mathbb{R}^d$ with the property that $\mathcal{E} : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ is well-defined and continuous for all $1 \leq p \leq \infty$ and integers k. Moreover, the norm of $\mathcal{E} : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ depends only on the dimension

d, the order of differentiability k and the Lipschitz constant M. The operator $\mathcal{E}$ is called *Stein's extension operator*. As one can see directly from its definition [142, Equation (24), p. 182], $\mathcal{E}$ also maps $C^1(\Omega) \to C^1(\mathbb{R}^d)$.

As explained in Section 3.3 of [142], a partition of unity argument allows one to construct an extension operator $\mathcal{E} = \mathcal{E}_\Omega$ on all Lipschitz domains Ω, with constant depending only on d, k, p, the number and size of the balls needed to cover the boundary, and the Lipschitz constant of the defining function on each ball. Since for a tight enough collar Λ_* one can use the same balls to cover all elements of Λ_* with control of the Lipschitz constant on each ball, this shows that Stein's extension operator has norm bounds that are uniform for domains with boundary in Λ_*.

In the above discussion, the definition of the $W^{k,p}$ norm was the usual one, defined by requiring the first k weak-derivatives to be in L^p. However, as noted earlier, we also define the H^s norm of a function f as the infimum of the H^s norms of all possible extensions of f to $\mathbb{R}^d$. Clearly, $\| \cdot \|_{W^{k,2}} \lesssim \| \cdot \|_{H^k}$ with constant independent of the domain. However, by the above, for domains with boundary in Λ_*, the reverse inequality also holds, with implicit constant depending on A_Γ.

From [108, Theorem B.8] we know that for any non-empty open subset Ω of $\mathbb{R}^d$ and any $s_0, s_1 \in \mathbb{R}$ we have the identification

$$(H^{s_0}(\Omega), H^{s_1}(\Omega))_{\theta,2} = H^s(\Omega), \text{ where } s = (1-\theta)s_0 + \theta s_1 \text{ and } 0 < \theta < 1,$$

with equivalent norms uniform in the collar. Thus, by interpolation, we have the following result.

Proposition 2.5.1. Let Ω be a bounded domain with boundary $\Gamma \in \Lambda_*$. Then for every $s \geq 0$ and $0 \leq \alpha \leq 1 + \varepsilon_0$, Stein's extension operator $\mathcal{E}$ satisfies

$$\|\mathcal{E}\|_{C^\alpha(\Omega) \to C^\alpha(\mathbb{R}^d)}, \qquad \|\mathcal{E}\|_{H^s(\Omega) \to H^s(\mathbb{R}^d)} \lesssim_A 1$$

uniformly in Λ_*.

Proof. The H^s case follows from interpolation between integer powers. For C^α, we first note from [108, Theorem A.1] (and higher order variants, c.f. [46, Lemma 6.37]) that there are extension operators with the above $C^\alpha \to C^\alpha$ bound. That Stein's operator has this property then follows by making use of such extensions and interpolating, similar to [103, p. 11-12]. □

Remark 2.5.2. As mentioned in [82, Proposition 2.17], by an interpolation argument, one can also prove that Stein's extension operator maps the Besov space $B^{p,q}_{\alpha}(\Omega)$ to $B^{p,q}_{\alpha}(\mathbb{R}^d)$ for all $\alpha > 0$, $1 \leq p, q \leq \infty$ and Lipschitz domains Ω. However, we will not require anything this precise.

Littlewood-Paley decomposition and paraproducts on Ω

Using the Stein extension operator, many of the standard paraproduct estimates on $\mathbb{R}^d$ pass over to Ω.

Littlewood-Paley decomposition

For a distribution u on $\mathbb{R}^d$, we will make use of the standard Littlewood-Paley decomposition

$$u = \sum_{k \geq 0} P_k u,$$

where for $k > 0$, P_k corresponds to a Fourier multiplier with smooth symbol supported in the dyadic frequency region $|\xi| \approx 2^k$ and P_0 corresponds to a multiplier localized to the unit ball. The notation $P_{<k}$, $P_{\leq k}$, $P_{\geq k}$ and $P_{>k}$ will have the usual meaning. Using the Stein extension operator, we may also consider Littlewood-Paley projections when u is defined only on Ω. In this case, we abuse notation, and write $P_k u$ instead of $P_k \mathcal{E} u$, with corresponding definitions for $P_{<k}$, $P_{\leq k}$, etc. We will also often write u_k, $u_{<k}$, etc. as shorthand for the above operators applied to u.

Paraproducts on Ω

The above decomposition allows us to make use of some of the standard tools of paradifferential calculus (see e.g. [14] and [109]) on $\mathbb{R}^d$ and apply them to functions defined on Ω. For bilinear expressions, we will make heavy use of the Littlewood-Paley trichotomy (now defined for functions on Ω with suitable regularity),

$$f \cdot g = T_f g + T_g f + \Pi(f, g),$$

where the above three terms correspond to the respective "low-high", "high-low" and "high-high" frequency interactions between f and g. More specifically, $T_f g$ is defined as

$$T_f g := \sum_k f_{<k-k_0} g_k,$$

where k_0 is some universal parameter independent of k. We will be able to take, e.g., $k_0 = 4$ for most purposes.

Bilinear estimates on Ω

One important consequence of the bounds for $\mathcal{E}$ and the corresponding inequality on $\mathbb{R}^d$ is the following algebra property for $H^s(\Omega)$, $s \geq 0$,

$$\|fg\|_{H^s(\Omega)} \lesssim_A \|f\|_{H^s(\Omega)}\|g\|_{L^\infty(\Omega)} + \|g\|_{H^s(\Omega)}\|f\|_{L^\infty(\Omega)}. \tag{2.5.1}$$

In our estimates for the elliptic problems below, the bilinear terms above will frequently appear in the form $\partial_i f \partial_j g$ where f is some function defined on $\mathbb{R}^d$ encoding the regularity of the domain and the desired uniform bound for g is below C^1. For this reason, in order to avoid negative Hölder norms inside a domain, we will need the following paraproduct type estimate, which we will use in the sequel.

Proposition 2.5.3 (Bilinear paraproduct type estimate on Ω)**.** Let either i) $s > 0$ and $\alpha_1, \alpha_2, \beta \in [0,1]$ or ii) $s = 0$, $\alpha_1 = \alpha_2 = 1$ and $\beta \in [0,1]$. Then we have for any $r \geq 0$,

$$\begin{aligned}\|\partial_i f \partial_j g\|_{H^s(\Omega)} \lesssim_A \|g\|_{H^{s+2-\alpha_1}(\Omega)}\|f\|_{C^{\alpha_1}(\Omega)} + \|f\|_{H^{s+r+1}(\Omega)} \sup_{k>0} 2^{-k(r+\alpha_2-1)}\|g_k^1\|_{C^{\alpha_2}(\Omega)} \\ + \|f\|_{C^{1,2\varepsilon}(\Omega)} \sup_{k>0} 2^{k(s+\beta-\varepsilon)}\|g_k^2\|_{H^{1-\beta}(\Omega)},\end{aligned}$$

where $g = g_k^1 + g_k^2$ is any sequence of partitions of g in $C^{\alpha_2}(\Omega) + H^{1-\beta}(\Omega)$.

Proof. By Proposition 2.5.1, it suffices to prove these estimates for f, g defined on $\mathbb{R}^d$. We prove the estimate for $0 < \alpha_1, \alpha_2 < 1$ and $s > 0$ as the other cases are more easily dealt with. We recall that for $0 < \alpha < 1$, the C^α norm on $\mathbb{R}^d$ can be characterized by the equivalent Besov norm,

$$\|u\|_{C^\alpha(\mathbb{R}^d)} \approx \|P_{\leq 0} u\|_{L^\infty(\mathbb{R}^d)} + \sup_{j>0} 2^{\alpha j}\|P_j u\|_{L^\infty(\mathbb{R}^d)}. \tag{2.5.2}$$

We now decompose $\partial_i f \partial_j g$ into paraproducts,

$$\partial_i f \partial_j g = T_{\partial_i f}\partial_j g + T_{\partial_j g}\partial_i f + \Pi(\partial_i f, \partial_j g). \tag{2.5.3}$$

We then have the standard estimate

$$\|T_{\partial_i f}\partial_j g\|_{H^s(\mathbb{R}^d)} \lesssim \|f\|_{C^{\alpha_1}(\mathbb{R}^d)}\|\partial_j g\|_{H^{s+1-\alpha_1}(\mathbb{R}^d)},$$

which follows by shifting $1-\alpha_1$ derivatives off of the low frequency factor and onto the high frequency factor in each term. Using the hypothesis $s>0$, the high-high paraproduct may be estimated by the same term. For the remaining low-high interaction, we write

$$T_{\partial_j g}\partial_i f = \sum_k P_{<k-4}\partial_j g P_k \partial_i f = \sum_k P_{<k-4}\partial_j(g_k^1)P_k\partial_i f + \sum_k P_{<k-4}\partial_j(g_k^2)P_k\partial_i f.$$

Using standard Bernstein type inequalities and square summing, the first term on the right can be easily controlled by

$$\|\partial_i f\|_{H^{s+r}(\mathbb{R}^d)} \sup_{k>0} 2^{-k(r+\alpha_2-1)}\|g_k^1\|_{C^{\alpha_2}(\mathbb{R}^d)},$$

while the latter can be controlled by

$$\|f\|_{C^{1,2\varepsilon}(\mathbb{R}^d)} \sup_{k>0} 2^{k(s+\beta-\varepsilon)}\|g_k^2\|_{H^{1-\beta}(\mathbb{R}^d)}.$$

□

The following corollary of the above proposition will be used heavily in the higher energy bounds to control product terms on Ω with suitable pointwise control norms.

Corollary 2.5.4. Let s and α_1, α_2 be as in Proposition 2.5.3. Assume that $f \in H^{s+2-\alpha_2}(\Omega)\cap C^{\alpha_1}(\Omega)$ and $g \in H^{s+2-\alpha_1}(\Omega)\cap C^{\alpha_2}(\Omega)$. Then we have

$$\|\partial_i f \partial_j g\|_{H^s(\Omega)} \lesssim_A \|g\|_{H^{s+2-\alpha_1}(\Omega)}\|f\|_{C^{\alpha_1}(\Omega)} + \|f\|_{H^{s+2-\alpha_2}(\Omega)}\|g\|_{C^{\alpha_2}(\Omega)}.$$

Proof. This follows immediately from Proposition 2.5.3 by taking $g_j^2 = 0$ and $r = 1-\alpha_2$. □

Generalized Moser type estimate

Next, we prove a Moser type estimate with the same flavor as the above bilinear estimate. The main purpose of this estimate will be to suitably control (extensions of) compositions of functions on Ω with diffeomorphisms of $\mathbb{R}^d$. This will be important for obtaining more refined elliptic estimates where we need to use such diffeomorphisms to flatten the boundary.

Proposition 2.5.5 (Balanced Moser estimate)**.** Let $d \geq 1$ be an integer and let $G : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism with $\|DG\|_{C^\varepsilon}, \|DG^{-1}\|_{C^\varepsilon} \lesssim_A 1$. Let $s \geq 0, r \geq 0$ and $\alpha, \beta \in [0,1]$. Then for every $F \in H^s(\mathbb{R}^d)$ and partition $F = F_j^1 + F_j^2 \in C^\alpha(\mathbb{R}^d) + H^{1-\beta}(\mathbb{R}^d)$, we have

$$\begin{aligned}\|F(G)\|_{H^s(\mathbb{R}^d)} \lesssim_A \|F\|_{H^s(\mathbb{R}^d)} + \|G - Id\|_{H^{s+r}} \sup_{j>0} 2^{-j(\alpha+r-1)} \|F_j^1\|_{C^\alpha(\mathbb{R}^d)} \\ + \sup_{j>0} 2^{j(s+\beta-1-\varepsilon)} \|F_j^2\|_{H^{1-\beta}(\mathbb{R}^d)}.\end{aligned}$$

Remark 2.5.6. The same estimate holds for $F \in H^s(\Omega)$ by replacing F with its Stein extension.

Proof. The case $0 \leq s \leq 1$ is a consequence of the following standard fact.

Proposition 2.5.7 (Theorem 3.23 of [108])**.** Let $0 \leq s \leq 1$ and let $G : \mathbb{R}^d \to \mathbb{R}^d$ be a diffeomorphism with $\|DG\|_{L^\infty} \lesssim_A 1$ and $\|DG^{-1}\|_{L^\infty} \lesssim_A 1$. Then for every $F \in H^s(\mathbb{R}^d)$, we have

$$\|F(G)\|_{H^s(\mathbb{R}^d)} \approx_A \|F\|_{H^s(\mathbb{R}^d)}.$$

Now, assume $s > 1$. We begin by performing a Littlewood-Paley decomposition,

$$\|F(G)\|^2_{H^s(\mathbb{R}^d)} \lesssim_{j_0} \|F(G)\|^2_{L^2(\mathbb{R}^d)} + \sum_{j>j_0} 2^{2js} \|P_j(F(G))\|^2_{L^2(\mathbb{R}^d)},$$

where $j_0 > 0$ is some fixed constant depending only on A, to be chosen later. We have

$$2^{js}\|P_j(F(G))\|_{L^2(\mathbb{R}^d)} \lesssim 2^{js}\|P_j(F_{<j'}(G))\|_{L^2(\mathbb{R}^d)} + 2^{js}\|P_j(F_{\geq j'}(G))\|_{L^2(\mathbb{R}^d)},$$

where $F_{<j'} := P_{<j'}F$, $F_{\geq j'} := F - F_{<j'}$ and $j' := j - j_1$ with j_1 being some parameter depending only on s which will also be chosen later. For the latter term, by a change of variables and since $s > 0$, we have

$$\sum_{j>j_0} 2^{2js}\|P_j(F_{\geq j'}(G))\|^2_{L^2(\mathbb{R}^d)} \lesssim_A \sum_{j>j_0} \sum_{k\geq j'} 2^{2(j-k)s} 2^{2ks} \|P_k F\|^2_{L^2(\mathbb{R}^d)} \lesssim_A \|F\|^2_{H^s(\mathbb{R}^d)}.$$

On the other hand, using the fundamental theorem of calculus, we obtain

$$\begin{aligned}2^{js}\|P_j(F_{<j'}(G))\|_{L^2(\mathbb{R}^d)} \lesssim 2^{js} \sup_{\tau\in[0,1]} \|P_j\left(DF_{<j'}(G_\tau)P_{\geq j'}G\right)\|_{L^2(\mathbb{R}^d)} \\ + 2^{js}\|P_j\left(F_{<j'}(P_{<j'}G)\right)\|_{L^2(\mathbb{R}^d)},\end{aligned} \tag{2.5.4}$$

where

$$G_\tau = \tau P_{<j'}G + (1-\tau)G.$$

Now, as $\|DG\|_{\dot{C}^\varepsilon}, \|DG^{-1}\|_{\dot{C}^\varepsilon} \lesssim_A 1$, it follows that $P_{<j'}G$ and G_τ (for $\tau \in [0,1]$) are invertible with $\|P_{<j'}DG\|_{L^\infty}, \|DG_\tau\|_{L^\infty} \lesssim_A 1$ as long as j_0 is large enough (depending only on A and the collar). Now, to control the first term on the right-hand side of (2.5.4), we split $F_{<j'} = (F_j^1)_{<j'} + (F_j^2)_{<j'}$ and estimate (using the estimate for G_τ^{-1}),

$$\begin{aligned} 2^{js} \sup_{\tau\in[0,1]} \|P_j\left(DF_{<j'}(G_\tau)P_{\geq j'}G\right)\|_{L^2(\mathbb{R}^d)} &\lesssim_A 2^{-j(r+\alpha-1)}\|F_j^1\|_{C^\alpha(\mathbb{R}^d)} 2^{j(s+r)}\|P_{\geq j'}G\|_{L^2(\mathbb{R}^d)} \\ &\quad + 2^{j(s-1+\beta-\varepsilon)}\|F_j^2\|_{H^{1-\beta}(\mathbb{R}^d)}. \end{aligned} \tag{2.5.5}$$

Square summing (and possibly relabelling ε) gives

$$\begin{aligned} \left(\sum_{j>j_0} 2^{2js} \sup_{\tau\in[0,1]} \|P_j\left(DF_{<j'}(G_\tau)P_{\geq j'}G\right)\|^2_{L^2(\mathbb{R}^d)}\right)^{\frac{1}{2}} &\lesssim_A \sup_{j>0} 2^{-j(r+\alpha-1)}\|F_j^1\|_{C^\alpha(\mathbb{R}^d)}\|G - Id\|_{H^{s+r}} \\ &\quad + \sup_{j>0} 2^{j(s-1+\beta-\varepsilon)}\|F_j^2\|_{H^{1-\beta}(\mathbb{R}^d)}. \end{aligned}$$

Next, we control the second term on the right-hand side of (2.5.4), which is a bit easier. Let k be the largest integer strictly less than s so that $0 < s - k \leq 1$. If $j_1 := j - j'$ is large enough (depending only on k), we have by the chain rule and straightforward paraproduct analysis,

$$2^{js}\|P_j F_{<j'}(P_{<j'}G)\|_{L^2(\mathbb{R}^d)} \lesssim_A 2^{j(s-k)}\|\tilde{P}_j(D^k F_{<j'}(P_{<j'}G))\|_{L^2(\mathbb{R}^d)},$$

where $\tilde{P}_j$ is a slightly fattened Littlewood-Paley projection. We then use the fundamental theorem of calculus to obtain

$$\begin{aligned} 2^{j(s-k)}\|\tilde{P}_j(D^k F_{<j'}(P_{<j'}G))\|_{L^2(\mathbb{R}^d)} &\lesssim_A 2^{j(s-k)} \sup_{\tau\in[0,1]} \|\tilde{P}_j(D^{k+1}F_{<j'}(G_\tau)P_{\geq j'}G)\|_{L^2(\mathbb{R}^d)} \\ &\quad + 2^{j(s-k)}\|\tilde{P}_j(D^k F_{<j'}(G))\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

For the first term, we have simply

$$\begin{aligned} 2^{j(s-k)} \sup_{\tau\in[0,1]} \|\tilde{P}_j(D^{k+1}F_{<j'}(G_\tau)P_{\geq j'}G)\|_{L^2(\mathbb{R}^d)} &\lesssim_A 2^{j(s-k-1-\varepsilon)}\|D^{k+1}F_{<j'}\|_{L^2(\mathbb{R}^d)} \\ &\lesssim 2^{-j\varepsilon}\|F\|_{H^s(\mathbb{R}^d)}. \end{aligned}$$

For the second term, we have

$$2^{j(s-k)}\|\tilde{P}_j(D^k F_{<j'}(G))\|_{L^2(\mathbb{R}^d)} \lesssim_A 2^{j(s-k)}\|D^k F_{\geq j'}\|_{L^2(\mathbb{R}^d)} + \|\tilde{P}_j((D^k F)(G))\|_{H^{s-k}(\mathbb{R}^d)}.$$

Since $0 < s - k \leq 1$, we obtain from Proposition 2.5.7,

$$\left(\sum_{j>j_0} 2^{2j(s-k)}\|\tilde{P}_j(D^k F_{<j'}(P_{<j'}G))\|^2_{L^2(\mathbb{R}^d)}\right)^{\frac{1}{2}} \lesssim_A \|F\|_{H^s(\mathbb{R}^d)},$$

where we used that $s-k \leq 1$ to control $\|(D^k F)(G)\|_{H^{s-k}(\mathbb{R}^d)}$ and that $s-k>0$ to control the l^2 sum of $2^{j(s-k)}\|D^k F_{\geq j'}\|_{L^2(\mathbb{R}^d)}$. Combining everything together completes the proof. □

We also note a much cruder variant of the above proposition where we measure G only in pointwise norms and F in Sobolev based norms. This will only be needed in our construction of regularization operators later on.

Proposition 2.5.8 (Crude Moser estimate)**.** Under the assumptions of Proposition 2.5.5, the following bound holds for every $F \in H^s(\mathbb{R}^d)$,

$$\|F(G)\|_{H^s(\mathbb{R}^d)} \lesssim_A \|F\|_{H^s(\mathbb{R}^d)} + \|G - Id\|_{C^{s+r+\varepsilon}(\mathbb{R}^d)}\|F\|_{H^{1-r}(\mathbb{R}^d)}.$$

Proof. The proof follows almost identical reasoning to Proposition 2.5.5. The only difference is that we do not partition F in (2.5.5) and instead estimate

$$\|(DF_{<j'})(G_\tau)\|_{L^2(\mathbb{R}^d)} \lesssim_A \|DF_{<j'}\|_{L^2(\mathbb{R}^d)} \lesssim 2^{jr}\|F\|_{H^{1-r}(\mathbb{R}^d)}.$$

We then invoke Bernstein's inequality to obtain

$$2^{j(r+s)}\|P_{\geq j}G\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{-j\varepsilon}\|G - Id\|_{C^{s+r+\varepsilon}},$$

and conclude by summing in j. □

Local coordinate parameterizations and Sobolev norms in Λ_*

With the above estimates in hand, we can begin the process of proving refined versions of the various elliptic, trace and product type estimates on Γ that will be important for establishing our higher energy estimates. Our goal in this subsection is to construct a family of coordinate neighborhoods for Γ_* which will act as a "universal" set of coordinate neighborhoods which we can use to flatten the boundary of nearby hypersurfaces $\Gamma \in \Lambda_*$. We will also use these local coordinates to define Sobolev type norms on Γ which are suitable for proving uniform estimates later in this section. To achieve this, we slightly modify the construction from [140, Appendix A] (but note the difference in our definitions of Λ_*).

Local coordinates and partition of unity

As in [140, Appendix A], since Γ_* is compact, for any $\sigma > 0$ we can choose $x_i \in \mathbb{R}^d$ and $r, r_i \in (0, \frac{1}{2}]$, $i = 1, \ldots, m$, such that we have the following two properties:

(i) $B(\Gamma_*, r) \subseteq \cup_{i=1}^m R_i(r_i)$, where $B(S, \varepsilon)$ denotes the ε neighborhood of S and $R_i(\cdot) := \tilde{R}_i(\cdot) \times I_i(\cdot) \subseteq \mathbb{R}^d$ is a rotated cylinder with perpendicular vertical segment centered at x_i with the given equal radius and length.

(ii) For each i, $z = (\tilde{z}, z_d)$ being the natural Euclidean coordinates on R_i, there exists a function $f_{*i} : \tilde{R}_i(2r_i) \to I_i$ such that

$$\|f_{*i}\|_{C^0} < \sigma r_i, \quad \|Df_{*i}\|_{C^0} < \sigma \text{ and } \Omega_* \cap R_i(2r_i) = \{z_d > f_{*i}(\tilde{z})\}. \tag{2.5.6}$$

When $\delta > 0$ is small enough, for every $\Gamma \in \Lambda_*$ with corresponding bounded domain Ω, (i) holds with Γ_* replaced by Γ. Moreover, there exist functions $f_i : \tilde{R}_i(2r_i) \to I_i$ satisfying (ii) with Ω_* replaced by Ω such that we can control the Sobolev and Hölder type norms of f_i by the corresponding norms of Γ. Specifically, we have

$$\|f_i\|_{H^s} \lesssim_A 1 + \|\Gamma\|_{H^s}, \qquad \|f_i\|_{C^{k,\alpha}} \lesssim_A 1 + \|\Gamma\|_{C^{k,\alpha}}$$

for $s \geq 0$, integer $k \geq 0$ and $\alpha \in [0, 1)$. Indeed, by performing a computation in local coordinates, the above Sobolev bound follows from the Moser estimate in Proposition 2.5.5 and the pointwise bound can be verified directly from the chain rule and interpolation. Using these coordinate representations, we intend to construct local coordinate maps on each $\tilde{R}_i(2r_i)$ for Ω which flatten Γ and have uniform estimates in Λ_*. In some of the estimates in this section, by a slight abuse of notation, we write $\|\Gamma\|$ when we really mean $1 + \|\Gamma\|$ in order to declutter the notation. This will not affect any of the analysis for the dynamic problem.

On each $\tilde{R}_i(2r_i)$, let $\phi_i = \gamma_i f_i$, where $\gamma_i(\tilde{z}) = \overline{\gamma}\left(\frac{|\tilde{z}|}{r_i}\right)$ and $\overline{\gamma} : [0, \infty) \to [0, 1]$ is a smooth cutoff supported on $[0, \frac{3}{2}]$ and equal to 1 on $[0, \frac{5}{4}]$. We can extend ϕ_i to a function on $\mathbb{R}^d$ which gains half a degree of regularity in H^s norms and is bounded in suitable pointwise norms. Indeed, let $\tilde{z} \in \mathbb{R}^{d-1}$ and $s \geq \frac{1}{2}$. We define an extension Φ_i of ϕ_i by

$$\Phi_i(z) = \int_{\mathbb{R}^{d-1}} \widehat{\phi}_i(\xi') e^{-(1+|\xi'|^2)z_d^2} e^{2\pi i \xi' \cdot \tilde{z}} d\xi' \text{ for } z = (\tilde{z}, z_d) \in \mathbb{R}^d.$$

We first observe that for each integer $k \geq 0$ and $\alpha \in [0, 1)$, $\|\Phi_i\|_{C^{k,\alpha}(\mathbb{R}^d)} \lesssim_{k,\alpha} \|\phi_i\|_{C^{k,\alpha}(\mathbb{R}^{d-1})}$. One also has the same bounds for $W^{k,\infty}$ for each $k \geq 0$. To see this, we observe that Φ_i can be rewritten as the convolution

$$\Phi_i(z) = c_d e^{-z_d^2} \int_{\mathbb{R}^{d-1}} \phi_i(\tilde{z} + z_d y) e^{-|y|^2} dy,$$

where c_d is a dimensional constant. In this form, the above bounds are easily checked. We also have $\|\Phi_i\|_{H^{s+\frac{1}{2}}(\mathbb{R}^d)} \approx_s \|\phi_i\|_{H^s(\mathbb{R}^{d-1})}$ for every $s \geq 0$, which follows from inspecting the Fourier transform of Φ_i, in a similar fashion as [108, Lemma 3.36].

From the above, we see that if $\sigma > 0$ from (2.5.6) is small enough, then the map

$$H_i(\tilde{z}, z_d) := (\tilde{z}, z_d + \Phi_i(\tilde{z}, z_d))$$

is a diffeomorphism from $\mathbb{R}^d \to \mathbb{R}^d$ with $\|H_i - Id\|_{C^{k,\alpha}} \lesssim_A \|\Gamma\|_{C^{k,\alpha}}$ and $\|H_i - Id\|_{H^{s+\frac{1}{2}}} \lesssim_A \|\Gamma\|_{H^s}$ for $s \geq 0$, integer $k \geq 0$ and $\alpha \in [0, 1)$. Moreover, for the inverse function $G_i := H_i^{-1}$, the same bounds hold for $G_i - Id$ and its d'th component g_i satisfies the bounds $|\partial_{z_d} g_i| + |(\partial_{z_d} g_i)^{-1}| \lesssim_A 1$. Finally, if $\sigma > 0$ is small enough and Λ_* is a tight enough collar neighborhood we have, in the C^1 topology,

$$\|H_i - Id\|_{C^1} + \|G_i - Id\|_{C^1} \lesssim_A \rho,$$

where $\rho > 0$ is some positive constant which can be made as small as we like (depending on σ and Λ_*). We then have for some uniform $\delta_* > 0$,

$$\left(\tilde{R}_i\left(\frac{5}{4}r_i\right) \times I_i\left(\frac{5}{4}\delta_* r_i\right)\right) \cap \Omega = \left(\tilde{R}_i\left(\frac{5}{4}r_i\right) \times I_i\left(\frac{5}{4}\delta_* r_i\right)\right) \cap \{g_i > 0\}.$$

Partition of unity. Here, we construct a partition of unity for Ω with bounds uniform in Λ_*. We follow essentially the procedure from [140, Appendix A]. Let γ be a smooth cutoff defined on $[0, \infty)$ satisfying $0 \leq \gamma \leq 1$ with γ supported in $[0, \frac{5}{4})$ and equal to 1 on $[0, \frac{9}{8}]$. Moreover, let ζ be a smooth function defined on $[0, \infty)$ taking values in $[\frac{1}{3}, \infty)$ with $\zeta = \frac{1}{3}$ on $[0, \frac{1}{3}]$ and $\zeta(x) = x$ for $x \geq \frac{2}{3}$. Define

$$\tilde{\gamma}_{*i}(z) := \gamma(\frac{|\tilde{z}|}{r_i})\gamma(\frac{|z_d|}{\delta_* r_i}), \qquad \eta = \zeta \circ \sum_i (\tilde{\gamma}_{*i} \circ G_i).$$

We then define a partition of unity via

$$\gamma_{*i} := \frac{\tilde{\gamma}_{*i}(G_i)}{\eta}, \qquad \gamma_{*0} := (1 - \sum_i \gamma_{*i})\mathbb{1}_\Omega. \tag{2.5.7}$$

We see that $\sum_{i \geq 0} \gamma_{*i} = 1$ on Ω and $0 \leq \gamma_{*i} \leq 1$ for each $i \geq 0$. Moreover, by the Moser and Sobolev product estimates, we have

$$\|\gamma_{*i}\|_{H^{s+\frac{1}{2}}} \lesssim_A \|\Gamma\|_{H^s}$$

for $s \geq 0$.

Sobolev spaces on hypersurfaces in Λ_*

We can use the above partition of unity to define $C^{k,\alpha}$ and H^s spaces on hypersurfaces $\Gamma \in \Lambda_*$. Indeed, if Γ is C^1 and in H^s, we may define what it means to be in $H^r(\Gamma)$ for $0 \leq r \leq s$ through the inner product,

$$\langle f, g\rangle_{H^r(\Gamma)} := \sum_{i\geq 1} \langle \phi_i f_i, \phi_i g_i\rangle_{H^r(\mathbb{R}^{d-1})},$$

where $\phi_i := \gamma_{*i} \circ H_i(\tilde{z}, 0)$ (note that this is not the same ϕ_i as in the previous subsection), $f_i := f \circ H_i(\tilde{z}, 0)$ and $g_i := g \circ H_i(\tilde{z}, 0)$. If Γ is $C^{k,\alpha}$ we may also define

$$\|f\|_{C^{k,\alpha}(\Gamma)} := \sup_{i\geq 1} \|\phi_i f_i\|_{C^{k,\alpha}(\mathbb{R}^{d-1})}.$$

Finally, for a function v defined on Ω, we write $v_i = \gamma_{*i} v$ and $u_i = v_i(H_i)$.

Using the above and the full generality afforded by Proposition 2.5.5, we prove a refined product type estimate on the boundary Γ. Precisely, we have the following.

Proposition 2.5.9 (Product estimates on the boundary)**.** Let Ω be a bounded domain with boundary $\Gamma \in \Lambda_*$. If f, g are functions on Γ and $g = g_j^1 + g_j^2$ is any sequence of partitions, then for $s \geq 0$ and $r \geq 1$ we have

$$\|fg\|_{H^s(\Gamma)} \lesssim_A \|f\|_{L^\infty(\Gamma)}\|g\|_{H^s(\Gamma)} + (\|f\|_{H^{s+r-1}(\Gamma)} + \|f\|_{L^\infty(\Gamma)}\|\Gamma\|_{H^{s+r}}) \sup_{j>0} 2^{-j(r-1)}\|g_j^1\|_{L^\infty(\Gamma)}$$
$$+ (1 + \|f\|_{C^{2\varepsilon}(\Gamma)}) \sup_{j>0} 2^{j(s-\varepsilon)}\|g_j^2\|_{L^2(\Gamma)}.$$

Remark 2.5.10. If we take $r = 1$ and $g_j^1 = g$, we recover something resembling the standard algebra property,

$$\|fg\|_{H^s(\Gamma)} \lesssim_A \|f\|_{L^\infty(\Gamma)}\|g\|_{L^\infty(\Gamma)}\|\Gamma\|_{H^{s+1}} + \|f\|_{H^s(\Gamma)}\|g\|_{L^\infty(\Gamma)} + \|g\|_{H^s(\Gamma)}\|f\|_{L^\infty(\Gamma)}, \quad (2.5.8)$$

but with the twist being the additional explicit presence of the H^{s+1} norm of the surface on the right-hand side. We also remark that the proof below will allow for the first term on the right of (2.5.8) to be replaced by $(\|f\|_{W^{1,\infty}(\Gamma)}\|g\|_{L^\infty(\Gamma)} + \|f\|_{L^\infty(\Gamma)}\|g\|_{W^{1,\infty}(\Gamma)})\|\Gamma\|_{H^s}$, which is perhaps more natural, but we will never actually need this.

Proof. Let $(\gamma_{*i})_i$ be the partition of unity for Ω defined in (2.5.7). As before, we write $\phi_i(\tilde{z}) := \gamma_{*i}(H_i(\tilde{z}, 0))$, which is smooth with domain independent bounds since G_i and H_i

are inverse. Similarly, we write $f_i = f(H_i(\tilde{z},0))$ and $g_i = g(H_i(\tilde{z},0))$, which are functions defined on the support of ϕ_i. By definition, it suffices to control $\|\phi_i f_i g_i\|_{H^s(\mathbb{R}^{d-1})}$ for each $i \geq 1$. To begin with, let $j' = j-4$ and let P_j and $P_{<j'}$ denote Littlewood-Paley projections on $\mathbb{R}^{d-1}$. Moreover, define $\tilde{\phi}_i$ to be a smooth compactly supported function equal to 1 on the support of γ_{*i} with support properties chosen so that $\tilde{\phi}_i$ is supported in the region where f_i is well-defined. Then a simple paraproduct estimate using the Littlewood-Paley trichotomy gives

$$\|\phi_i f_i g_i\|_{H^s(\mathbb{R}^{d-1})} \lesssim_A \|f\|_{L^\infty(\Gamma)}\|\phi_i g_i\|_{H^s(\mathbb{R}^{d-1})} + \left(\sum_{j>0} 2^{2js}\|P_{<j'}(\phi_i g_i)P_j(f_i\tilde{\phi}_i)\|^2_{L^2(\mathbb{R}^{d-1})}\right)^{\frac{1}{2}}.$$

For the latter term in the above, we estimate

$$\begin{aligned}\left(\sum_{j>0} 2^{2js}\|P_{<j'}(\phi_i g_i)P_j(f_i\tilde{\phi}_i)\|^2_{L^2(\mathbb{R}^{d-1})}\right)^{\frac{1}{2}} &\lesssim_A \|f_i\tilde{\phi}_i\|_{H^{s+r-1}(\mathbb{R}^{d-1})}\sup_{j>0} 2^{-j(r-1)}\|g_j^1\|_{L^\infty(\Gamma)} \\ &\quad + (1+\|f\|_{C^{2\varepsilon}(\Gamma)})\sup_{j>0} 2^{j(s-\varepsilon)}\|g_j^2\|_{L^2(\Gamma)}.\end{aligned}$$

We are then reduced to showing

$$\|f_i\tilde{\phi}_i\|_{H^{s+r-1}(\mathbb{R}^{d-1})} \lesssim_A \|f\|_{H^{s+r-1}(\Gamma)} + \|f\|_{L^\infty(\Gamma)}\|\Gamma\|_{H^{s+r}}.$$

For this, we note that

$$\|f_i\tilde{\phi}_i\|_{H^{s+r-1}(\mathbb{R}^{d-1})} \leq \sum_{j\geq 1}\|\tilde{\phi}_i\gamma_{*j}(H_i(\tilde{z},0))f_i\|_{H^{s+r-1}(\mathbb{R}^{d-1})}.$$

Let us write $\varphi_{ij} := G_j \circ H_i$. Then we have

$$\|\tilde{\phi}_i\gamma_{*j}(H_i(\tilde{z},0))f_i\|_{H^{s+r-1}(\mathbb{R}^{d-1})} = \|(\phi_j f_j)(\varphi_{ij}(\tilde{z},0))\tilde{\phi}_i\|_{H^{s+r-1}(\mathbb{R}^{d-1})}.$$

We note that φ_{ij} is a diffeomorphism having the same bounds as G_j and H_i. By using the extension Φ from earlier, we may assume that $\phi_j f_j$ is defined on $\mathbb{R}^d$ with $\|\phi_j f_j\|_{H^{s+r-\frac{1}{2}}(\mathbb{R}^d)} \lesssim \|\phi_j f_j\|_{H^{s+r-1}(\mathbb{R}^{d-1})}$ and $\|\phi_j f_j\|_{L^\infty(\mathbb{R}^d)} \lesssim \|\phi_j f_j\|_{L^\infty(\mathbb{R}^{d-1})}$. Therefore, by the trace estimate on $\mathbb{R}^{d-1}$, the fact that φ_{ij} is a diffeomorphism and the balanced Moser estimate, we have

$$\begin{aligned}\|\tilde{\phi}_i(\phi_j f_j)(\varphi_{ij}(\tilde{z},0))\|_{H^{s+r-1}(\mathbb{R}^{d-1})} &\lesssim_A \|(\phi_j f_j)\circ\varphi_{ij}\|_{H^{s+r-\frac{1}{2}}(\mathbb{R}^d)} \\ &\lesssim_A \|\phi_j f_j\|_{H^{s+r-1}(\mathbb{R}^{d-1})} + \|\Gamma\|_{H^{s+r}}\|f\|_{L^\infty(\Gamma)}.\end{aligned}$$

Since, by definition, we have

$$\|\phi_j f_j\|_{H^{s+r-1}(\mathbb{R}^{d-1})} \leq \|f\|_{H^{s+r-1}(\Gamma)},$$

the proof is complete. □

Trace estimates

Now, we prove a refined version of the trace theorem for Γ.

Proposition 2.5.11 (Balanced trace estimate)**.** Let Ω be a bounded domain with boundary $\Gamma \in \Lambda_*$. For every $s > \frac{1}{2}$, $r \geq 0$, $\alpha, \beta \in [0,1]$ and every sequence of partitions $v = v_j^1 + v_j^2$, we have

$$\|v_{|\Gamma}\|_{H^{s-\frac{1}{2}}(\Gamma)} \lesssim_A \|v\|_{H^s(\Omega)} + \|\Gamma\|_{H^{s+r-\frac{1}{2}}} \sup_{j>0} 2^{-j(r+\alpha-1)} \|v_j^1\|_{C^\alpha(\Omega)} + \sup_{j>0} 2^{j(s-1+\beta-\varepsilon)} \|v_j^2\|_{H^{1-\beta}(\Omega)}.$$

Proof. For $i \geq 1$, define $\tilde{v}_i = \gamma_{*i} \mathcal{E} v$ where $\mathcal{E}$ is the Stein extension operator for Ω. It suffices to prove the estimate with the left-hand side replaced by $\|\tilde{v}_i(H_i(\tilde{z},0))\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})}$. Using the trace theorem on $\mathbb{R}^{d-1}$, we have

$$\|\tilde{v}_i(H_i(\tilde{z},0))\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})} \lesssim \|\tilde{u}_i\|_{H^s(\mathbb{R}^d)},$$

where $\tilde{u}_i := \tilde{v}_i \circ H_i$. We then use Proposition 2.5.5 and the operator bounds for $\mathcal{E}$ in Proposition 2.5.1 to conclude. □

An extension operator depending continuously on the domain

Another use of the above local coordinates is to construct a family of extension operators which depend continuously in a suitable sense on the domain. This will be important for establishing our continuous dependence result later on. Potentially, something akin to the Stein extension operator could work here, but we opt for the following simpler construction where the dependence on the domain is more transparent.

Proposition 2.5.12. Fix a collar neighborhood Λ_* and let $s > \frac{d}{2} + 1$. For each bounded domain Ω with H^s boundary $\Gamma \in \Lambda_*$ there exists an extension operator $E_\Omega : H^s(\Omega) \to H^s(\mathbb{R}^d)$ such that for all $v \in H^s(\Omega)$,

$$\|E_\Omega v\|_{H^s(\mathbb{R}^d)} + \|\Gamma\|_{H^s} \approx_{A, \|v\|_{C^{\frac{1}{2}}(\Omega)}} \|(v,\Gamma)\|_{\mathbf{H}^s}, \qquad \|E_\Omega v\|_{H^s(\mathbb{R}^d)} \lesssim_A \|\Gamma\|_{H^{s-\frac{1}{2}}} \|v\|_{H^s(\Omega)}, \quad (2.5.9)$$

where the dependence on $\|v\|_{C^{\frac{1}{2}}(\Omega)}$ is polynomial. Moreover, if Ω_n is a sequence of domains with $\Gamma_n \to \Gamma$ in H^s, then for every $v \in H^s(\mathbb{R}^d)$, there holds

$$\|E_{\Omega_n} v_{|\Omega_n} - E_\Omega v_{|\Omega}\|_{H^s(\mathbb{R}^d)} \to 0. \qquad (2.5.10)$$

Remark 2.5.13. One can loosely think of (2.5.10) as a strong operator topology convergence for this family of extensions.

Proof. Given a family of domains Ω_n and Ω with boundaries $\Gamma_n, \Gamma \in \Lambda_*$, denote by γ_{*i}^n and γ_{*i} the corresponding partitions of unity, so that

$$v = \sum_i \gamma_{*i}^n v \quad \text{on } \Omega_n \text{ and } v = \sum_i \gamma_{*i} v \quad \text{on } \Omega.$$

Define $u_i^n = (\gamma_{*i}^n v) \circ H_i^n$ on $\mathbb{R}_+^d$. Let k be the largest integer less than or equal to s, and define the half-space extension

$$\begin{cases} \tilde{u}_i^n(\tilde{z}, z_d) = \sum_{j=1}^{k+1} c_j u_i^n(\tilde{z}, -\frac{z_d}{j}) & \text{if } z_d < 0, \\ \tilde{u}_i^n(\tilde{z}, z_d) = u_i^n(\tilde{z}, z_d) & \text{if } z_d \geq 0, \end{cases}$$

where $c_1, \dots, c_{k+1}$ are gotten as in [46, Lemma 6.37] by solving an appropriate Vandermonde system. It is standard to verify that we have $\tilde{u}_i^n \in H^s(\mathbb{R}^d)$.

We define the Ω_n extension of v by

$$\tilde{v}_n = \sum_i \tilde{u}_i^n \circ G_i^n,$$

and similarly let $\tilde{v}$ by the Ω extension of v. To verify the continuous dependence property, we want to verify that if $\Gamma_n \to \Gamma$ in H^s, then $\tilde{v}_n \to \tilde{v}$ in $H^s(\mathbb{R}^d)$. For this, it suffices to prove that $\tilde{u}_i^n \circ G_i^n \to \tilde{u}_i \circ G_i$ in $H^s(\mathbb{R}^d)$ for each i. We note that

$$\|\tilde{u}_i^n \circ G_i^n - \tilde{u}_i \circ G_i\|_{H^s(\mathbb{R}^d)} \leq \|(\tilde{u}_i^n - \tilde{u}_i) \circ G_i^n\|_{H^s(\mathbb{R}^d)} + \|\tilde{u}_i \circ G_i^n - \tilde{u}_i \circ G_i\|_{H^s(\mathbb{R}^d)}. \tag{2.5.11}$$

The first term on the right-hand side of (2.5.11) can be shown to go to zero by using standard Moser estimates. The latter term goes to zero by arguing similarly to the proof that translation is continuous in L^p spaces (using a simple density argument to replace $\tilde{u}_i$ by a smooth function).

Finally, the bounds (2.5.9) follow from the definition of the extension and Proposition 2.5.5.

□

Pointwise elliptic estimates

Here we establish variants of the $C^{2,\alpha}$ and $C^{1,\alpha}$ estimates for the Dirichlet problem which adequately track the dependence on the domain regularity. In our analysis later, we will mostly use the $C^{1,\alpha}$ estimates with $\alpha = \frac{1}{2}$ or $\alpha = \varepsilon$. However, the $C^{2,\alpha}$ estimates will be relevant for proving bounds for our regularization operators, which are defined in Section 2.6.

As will become apparent later, to obtain the desired pointwise elliptic estimates, it is crucial to use a domain flattening map whose Jacobian has determinant 1. This will be especially necessary for the $C^{1,\alpha}$ estimate, as we must preserve the divergence form of the equation. For this reason, instead of the map H_i, we will use the more familiar domain flattening map

$$F_i(z) = (\tilde{z}, z_d + \phi_i(\tilde{z})), \tag{2.5.12}$$

whose Jacobian has determinant 1. The tradeoff when using the flattening F_i is that it does not exhibit a $\frac{1}{2}$ gain in regularity for the H^s norm on the interior compared to the boundary, but this will not matter for this section because all domain dependent coefficients will be placed in L^∞ based norms. We let $\Psi_i := F_i^{-1}$, and begin with the $C^{2,\alpha}$ estimates.

Proposition 2.5.14 ($C^{2,\alpha}$ estimates for the inhomogeneous Dirichlet problem)**.** Let $0 < \alpha < 1$ and let Ω be a bounded domain with boundary $\Gamma \in \Lambda_*$ having $C^{2,\alpha}$ regularity. Consider the boundary value problem

$$\begin{cases} \Delta v = g & \text{in } \Omega, \\ v = \psi & \text{on } \Gamma. \end{cases}$$

Then v satisfies the estimate

$$\|v\|_{C^{2,\alpha}(\Omega)} \lesssim_A \|\Gamma\|_{C^{2,\alpha}} \|v\|_{W^{1,\infty}(\Omega)} + \|g\|_{C^\alpha(\Omega)} + \|\psi\|_{C^{2,\alpha}(\Gamma)}.$$

Proof. We write $v_i = \gamma_{*i} v$, $h_i = \Delta v_i$, $f_i = h_i \circ F_i$ and $v_i = u_i \circ \Psi_i$. Omitting some of the subscripts for notational convenience, we see that $u := u_i$ satisfies the equation

$$\begin{cases} \Delta u = \partial_k((\delta^{jk} - a^{jk})\partial_j u) + f, \\ u_{|z_d=0} = (\gamma_{*i}\psi)(H_i(\tilde{z}, 0)), \end{cases} \tag{2.5.13}$$

where $a^{jk} = (\Psi^j_{x_l} \Psi^k_{x_l})(F_i)$ with repeated indices summed over. Note that to compute the boundary term in (2.5.13) we used that $F_i(\tilde{z}, 0) = H_i(\tilde{z}, 0)$. By the well-known Schauder estimates for the half-space, we obtain

$$\|u\|_{C^{2,\alpha}} \lesssim_A \|(\delta^{jk} - a^{jk})\partial_j u\|_{C^{1,\alpha}} + \|f\|_{C^\alpha} + \|(\gamma_{*i}\psi)(H_i(\tilde{z}, 0))\|_{C^{2,\alpha}}. \tag{2.5.14}$$

Using the Besov characterization (2.5.2) and the paradifferential expansion (2.5.3), it is straightforward to estimate

$$\|(\delta^{jk} - a^{jk})\partial_j u\|_{C^{1,\alpha}} \lesssim \|\delta^{jk} - a^{jk}\|_{C^\varepsilon} \|u\|_{C^{2,\alpha}} + \|\Gamma\|_{C^{2,\alpha}} \|v\|_{W^{1,\infty}(\Omega)}. \tag{2.5.15}$$

As a^{ij} is close to the identity in C^ε, this simplifies the estimate (2.5.14) to

$$\|u\|_{C^{2,\alpha}} \lesssim_A \|\Gamma\|_{C^{2,\alpha}}\|v\|_{W^{1,\infty}(\Omega)} + \|f\|_{C^\alpha} + \|(\gamma_{*i}\psi)(H_i(\tilde{z},0))\|_{C^{2,\alpha}}. \tag{2.5.16}$$

Clearly, we have $\|f\|_{C^\alpha} \lesssim_A \|h\|_{C^\alpha(\Omega)}$. On the other hand, we have

$$\|u(\Psi_i)\|_{\dot{C}^{2,\alpha}} \lesssim_A \|(D\Psi_i)^*(D^2u)(\Psi_i)D\Psi_i\|_{\dot{C}^\alpha} + \|(Du)(\Psi_i)D^2\Psi_i\|_{\dot{C}^\alpha}. \tag{2.5.17}$$

We can estimate both terms above by the right-hand side of (2.5.16). We show how to do this for the first term, as the second term is similar. For this, we may assume that u is defined on all of $\mathbb{R}^d$ by using a suitable extension operator from the half-space to $\mathbb{R}^d$. Then we write as usual $u_{<j}$ to mean $P_{<j}u$ and $u_{\geq j} := u - u_{<j}$. By the Besov characterization of C^α, we need to estimate

$$\sup_{j>0} 2^{j\alpha}\|P_j((D\Psi_i)^*(D^2u)(\Psi_i)D\Psi_i)\|_{L^\infty}.$$

By the standard Littlewood-Paley trichotomy, we first obtain,

$$2^{j\alpha}\|P_j((D\Psi_i)^*(D^2u)(\Psi_i)D\Psi_i)\|_{L^\infty} \lesssim_A \|u\|_{C^{2,\alpha}} + 2^{j\alpha}\|D^2u\|_{L^\infty}\|\tilde{P}_jB(D\Psi_i, D\Psi_i)\|_{L^\infty},$$

where B is a suitable bilinear form. For the latter term, we split $u = u_{<j} + u_{\geq j}$ and estimate using Bernstein's inequality,

$$\begin{aligned} 2^{j\alpha}\|D^2u\|_{L^\infty}\|\tilde{P}_jB(D\Psi_i, D\Psi_i)\|_{L^\infty} &\lesssim_A \|v\|_{W^{1,\infty}(\Omega)}2^{j(1+\alpha)}\|\tilde{P}_jB(D\Psi_i, D\Psi_i)\|_{L^\infty} + \|u\|_{C^{2,\alpha}} \\ &\lesssim_A \|\Gamma\|_{C^{2,\alpha}}\|v\|_{W^{1,\infty}(\Omega)} + \|u\|_{C^{2,\alpha}}. \end{aligned}$$

The other term in (2.5.17) is similarly handled. Combining the above, we obtain

$$\|v_i\|_{C^{2,\alpha}(\Omega)} \lesssim_A \|\Gamma\|_{C^{2,\alpha}}\|v\|_{W^{1,\infty}(\Omega)} + \|h\|_{C^\alpha(\Omega)} + \|(\gamma_{*i}\psi)(H_i(\tilde{z},0))\|_{C^{2,\alpha}}.$$

Expanding

$$h = \Delta(\gamma_{*i}v) = \Delta\gamma_{*i}v + 2\nabla\gamma_{*i}\cdot\nabla v + \gamma_{*i}\Delta v$$

we obtain

$$\|h\|_{C^\alpha(\Omega)} \lesssim_A \|\Gamma\|_{C^{2,\alpha}}\|v\|_{W^{1,\infty}(\Omega)} + \|\nabla\gamma_{*i}\cdot\nabla v\|_{C^\alpha(\Omega)} + \|g\|_{C^\alpha(\Omega)}.$$

The second term on the right-hand side can be estimated crudely by

$$\|\nabla\gamma_{*i}\cdot\nabla v\|_{C^\alpha(\Omega)} \lesssim_A \|v\|_{C^{1,\alpha}(\Omega)} + \|\Gamma\|_{C^{2,\alpha}}\|v\|_{W^{1,\infty}(\Omega)}.$$

Finally, by estimating the term $\|v\|_{C^{1,\alpha}(\Omega)} \lesssim \delta_0\|v\|_{C^{2,\alpha}(\Omega)} + C(\delta_0)\|v\|_{C^0(\Omega)}$ for some δ_0 sufficiently small and absorbing the first term into the left-hand side of the estimate, we conclude the proof. □

By very similar reasoning and the corresponding estimate in the half-space (see Theorem 8.33 in [46]) we also have a $C^{1,\alpha}$ variant if the source term g is replaced by $\nabla \cdot g$. More precisely, we have the following.

Proposition 2.5.15 ($C^{1,\alpha}$ estimates for the Dirichlet problem)**.** Let Ω be a bounded $C^{1,\alpha}$ domain with $0 < \alpha < 1$ and with boundary $\Gamma \in \Lambda_*$. Consider the boundary value problem

$$\begin{cases} \Delta v = \nabla \cdot g_1 + g_2 & \text{in } \Omega, \\ v = \psi & \text{on } \partial\Omega. \end{cases}$$

Then v satisfies the estimate

$$\|v\|_{C^{1,\alpha}(\Omega)} \lesssim_A \|\Gamma\|_{C^{1,\alpha}}(\|v\|_{W^{1,\infty}(\Omega)} + \|g_1\|_{L^\infty(\Omega)}) + \|g_1\|_{C^\alpha(\Omega)} + \|g_2\|_{L^\infty(\Omega)} + \|\psi\|_{C^{1,\alpha}(\Gamma)}.$$

Interpolating and using the straightforward estimate

$$\|v\|_{L^\infty(\Omega)} \lesssim_A \|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)} + \|\psi\|_{L^\infty(\Gamma)},$$

we deduce also

$$\|v\|_{C^{1,\varepsilon}(\Omega)} \lesssim_A \|g_1\|_{C^\varepsilon(\Omega)} + \|g_2\|_{L^\infty(\Omega)} + \|\psi\|_{C^{1,\varepsilon}(\Gamma)} \tag{2.5.18}$$

and

$$\begin{aligned} \|v\|_{C^{1,\alpha}(\Omega)} \lesssim_A \|\Gamma\|_{C^{1,\alpha}}(\|g_1\|_{C^\varepsilon(\Omega)} + \|g_2\|_{L^\infty(\Omega)} + \|\psi\|_{C^{1,\varepsilon}(\Gamma)}) + \|g_1\|_{C^\alpha(\Omega)} + \|g_2\|_{L^\infty(\Omega)} \\ + \|\psi\|_{C^{1,\alpha}(\Gamma)}. \end{aligned}$$

Proof. Much of the proof is similar to the $C^{2,\alpha}$ estimate. We only outline the slight changes. First, we note that

$$\begin{aligned} \Delta v_i &= \partial_j(\partial_j\gamma_{*i}v) + \partial_j\gamma_{*i}\partial_j v + \gamma_{*i}\nabla \cdot g_1 + \gamma_{*i}g_2 \\ &= \partial_j(\partial_j\gamma_{*i}v) + \nabla \cdot (\gamma_{*i}g_1) + \partial_j\gamma_{*i}\partial_j v - \nabla\gamma_{*i} \cdot g_1 + \gamma_{*i}g_2 =: \nabla \cdot h_1 + h_2. \end{aligned}$$

Hence, localizing with γ_{*i} preserves the divergence source term to leading order. More precisely, h_2 will be suitable for estimating in L^∞ in the sense that $\|h_2\|_{L^\infty} \lesssim_A \|v\|_{W^{1,\infty}(\Omega)} + \|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)}$. The next step is to perform the domain flattening procedure. The most important point here is that since the Jacobian determinant of F_i is 1, the corresponding equation for u (using the notation from the proof of Proposition 2.5.14) becomes

$$\begin{cases} \partial_k(a^{jk}\partial_j u) = \nabla \cdot \tilde{h}_1 + \tilde{h}_2 & \text{in } \Omega, \\ u_{|z_d=0} = (\gamma_{*i}\psi)(H_i(\tilde{z}, 0)) & \text{on } \partial\Omega, \end{cases}$$

where

$$\tilde{h}_1 := (h_1 \cdot D\Psi_i)(F_i), \qquad \tilde{h}_2 := h_2(F_i).$$

In other words, the divergence structure of the equation is preserved. From this point, the proof follows the same line of reasoning as the $C^{2,\alpha}$ estimates by writing an equation for Δu. The difference is that we use the $C^{1,\alpha}$ norm and the corresponding estimate for the Laplace equation in the half-space when the equation has the above divergence form. □

When g_1 and g_2 are zero in the above proposition, we can interpolate using the maximum principle for $\mathcal{H}$ and the $C^{1,\varepsilon}$ bound above to obtain C^α bounds for the harmonic extension with constant depending only on A_Γ.

Corollary 2.5.16. Let $0 \leq \alpha < 1$. The following low regularity bound for $\mathcal{H}$ holds uniformly for domains Ω with boundary $\Gamma \in \Lambda_*$,

$$\|\mathcal{H}g\|_{C^\alpha(\Omega)} \lesssim_A \|g\|_{C^\alpha(\Gamma)}.$$

Proof. By the above and the maximum principle, we have $C^{1,\varepsilon}(\Gamma) \to C^{1,\varepsilon}(\Omega)$ and $C^0(\Gamma) \to C^0(\Omega)$ bounds for $\mathcal{H}$ that are uniform in Λ_*. By [103, Example 5.15] we also know that $(C^0(\mathbb{R}^n), C^{1,\varepsilon}(\mathbb{R}^n))_{\theta,\infty} = C^\alpha(\mathbb{R}^n)$ for an appropriate choice of θ. Therefore, we just have to transfer the interpolation properties on $\mathbb{R}^n$ for $n = d$ and $n = d-1$ to Ω and Γ, respectively, with constants uniform in the collar. For Ω, we argue as in Proposition 2.5.1, and on Γ we simply unravel the definition of our function spaces via the partition of unity. □

Remark 2.5.17. Of course, we note that Corollary 2.5.16 avoids C^1 and Lipschitz regularity, as these do not fall into the interpolation scale.

L^2 based balanced elliptic estimates

In this subsection, we will prove H^s type estimates for various elliptic problems. In the following analysis, we will always be using the coordinate maps H_i and G_i (as opposed to F_i and Ψ_i from the pointwise estimates) to flatten the boundary since we will now need the $\frac{1}{2}$ gain of regularity on Ω in H^s based norms given by this flattening.

The Dirichlet problem

We begin our analysis by proving estimates for the inhomogeneous Dirichlet problem

$$\begin{cases} \Delta v = g & \text{in } \Omega, \\ v = \psi & \text{on } \Gamma. \end{cases}$$

We first recall two baseline estimates which will be used heavily in the derivation of the higher regularity bounds below. The first is when $\psi = 0$, in which case v satisfies the H^1 estimate

$$\|v\|_{H^1(\Omega)} \lesssim_A \|g\|_{H^{-1}(\Omega)}. \tag{2.5.19}$$

On the other hand, for $\frac{1}{2} < s \leq 1$ and $g = 0$, we have

$$\|v\|_{H^s(\Omega)} \lesssim_A \|\psi\|_{H^{s-\frac{1}{2}}(\Gamma)}. \tag{2.5.20}$$

The bound (2.5.19) is completely standard. The bound (2.5.20) was established by Jerison and Kenig in [82], and even holds, in an appropriate sense, at the endpoint $s = \frac{1}{2}$. For our purposes, we will only need the range $\frac{1}{2} < s \leq 1$, but we do need to quantify the dependence of the implicit constant in [82] on the domain. As noted in [158], the implicit domain dependent constant is, as expected, solely dependent on the Lipschitz character of Ω, so is controlled uniformly in the collar. Formally, [158] only quantifies the domain dependence for the inhomogeneous problem $g \neq 0$, $\psi = 0$, but the analogous homogeneous estimate follows immediately from this and the existence of an extension operator $E : H^{s-\frac{1}{2}}(\Gamma) \to H^s(\Omega)$ for $\frac{1}{2} < s \leq 1$ with norm uniform in Λ_*. In this low regularity range of s, such an operator can be constructed by using the partition of unity for Ω and the construction in [140]. We omit the details.

In a small number of places in the higher energy bounds, the following elliptic estimates which hold on C^{1,ε_0} (but not quite Lipschitz) domains will be convenient for simplifying the analysis.

Proposition 2.5.18. For every $0 < s < \frac{1}{2} + \varepsilon_0$, there holds

$$\|\Delta^{-1} g\|_{H^{s+1}(\Omega)} \lesssim_A \|g\|_{H^{s-1}(\Omega)}, \qquad \|\mathcal{H}\psi\|_{H^{s+1}(\Omega)} \lesssim_A \|\psi\|_{H^{s+\frac{1}{2}}(\Gamma)}.$$

Proposition 2.5.18 is well-known to specialists; see, e.g., [114]. We remark that bounds of this type hold in the range $s < \frac{1}{2}$ when the domain is Lipschitz; the excess regularity given by a C^{1,ε_0} domain is required to extend the range to $s < \frac{1}{2} + \varepsilon_0$.

Next, we move to the higher regularity estimates for the Dirichlet problem.

Proposition 2.5.19 (Higher regularity bounds for the inhomogeneous Dirichlet problem)**.** Let Ω be a bounded domain with boundary $\Gamma \in \Lambda_*$. Suppose that v solves the Dirichlet problem

$$\begin{cases} \Delta v = g & \text{in } \Omega, \\ v = \psi & \text{on } \partial\Omega, \end{cases}$$

and let $s \geq 2$. Then for $r \geq 0$, $\alpha \in [0,1], \beta \in [0,1]$ and any sequence of partitions $v := v_j^1 + v_j^2$, we have

$$\begin{aligned} \|v\|_{H^s(\Omega)} \lesssim_A \|g\|_{H^{s-2}(\Omega)} + \|\psi\|_{H^{s-\frac{1}{2}}(\Gamma)} + \|\Gamma\|_{H^{s+r-\frac{1}{2}}} \sup_{j>0} 2^{-j(\alpha-1+r)} \|v_j^1\|_{C^\alpha(\Omega)} \\ + \sup_{j>0} 2^{j(s-1+\beta-\varepsilon)} \|v_j^2\|_{H^{1-\beta}(\Omega)}. \end{aligned}$$

Proof. Using the partition of unity, it suffices to estimate $v_i := \gamma_{*i} v$ for each $i \geq 0$. Since the case $i = 0$ is essentially an interior regularity estimate, we focus on the case $i \geq 1$. We define

$$h := \Delta v_i = g\gamma_{*i} + v\Delta\gamma_{*i} + 2\nabla v \cdot \nabla\gamma_{*i}.$$

Using the map $H_i = G_i^{-1}$, we can write a variable coefficient equation for $u := v_i \circ H_i$,

$$\begin{cases} -\Delta u = (a^{ij} - \delta^{ij})\partial_i\partial_j u + b_j \partial_j u - f, \\ u|_{\{z_d=0\}} = (\gamma_{*i}\psi)(H_i(\tilde{z}, 0)). \end{cases}$$

Here (dropping the i index from the partition and now using it as a dummy index), we wrote $a^{lm} := (G^l_{x_k} G^m_{x_k}) \circ H$ (where k is summed over), $b_j := (\Delta G^j) \circ H$ and $f = h \circ H$. As a first step, we prove the following estimate for u:

$$\begin{aligned} \|u\|_{H^s} \lesssim_A \|f\|_{H^{s-2}} + \|\psi\|_{H^{s-\frac{1}{2}}(\Gamma)} + \|\Gamma\|_{H^{s+r-\frac{1}{2}}} \sup_{j>0} 2^{-j(\alpha-1+r)} \|v_j^1\|_{C^\alpha(\Omega)} \\ + \sup_{j>0} 2^{j(s-1+\beta-\varepsilon)} \|v_j^2\|_{H^{1-\beta}(\Omega)}. \end{aligned} \tag{2.5.21}$$

For this, we use the standard elliptic regularity for the half-space to obtain

$$\begin{aligned} \|u\|_{H^s} \lesssim_A \|u\|_{L^2} + \|f\|_{H^{s-2}} + \|b_i\partial_i u\|_{H^{s-2}} + \|(a^{ij} - \delta^{ij})\partial_i\partial_j u\|_{H^{s-2}} \\ + \|(\gamma_{*i}\psi)(H_i(\tilde{z}, 0))\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})}. \end{aligned} \tag{2.5.22}$$

By definition, the last term on the right-hand side is controlled by $\|\psi\|_{H^{s-\frac{1}{2}}(\Gamma)}$. Moreover, by a change of variables and the baseline estimates (2.5.19) and (2.5.20), we can control, crudely,

$$\begin{aligned} \|u\|_{L^2} \lesssim_A \|v_i\|_{L^2(\Omega)} \lesssim_A \|h\|_{L^2(\Omega)} + \|\psi\|_{H^{\frac{1}{2}}(\Gamma)} \lesssim_A \|f\|_{L^2} + \|\psi\|_{H^{\frac{1}{2}}(\Gamma)} \\ \lesssim_A \|f\|_{H^{s-2}} + \|\psi\|_{H^{s-\frac{1}{2}}(\Gamma)}. \end{aligned} \tag{2.5.23}$$

For the purpose of estimating the third and fourth terms on the right-hand side, we may assume that $u \in H^s(\mathbb{R}^d)$ with compact support instead of just $u \in H^s(\mathbb{R}^d_+)$ by using any suitable extension for the half-space. We then recall that in a suitably refined collar, we have

$$\|a^{ij} - \delta^{ij}\|_{L^\infty} + \|DG - I\|_{L^\infty} \ll_A 1.$$

Next, we define a partition of u as follows: First write $v_i = \gamma_{*i} v_j^1 + \gamma_{*i} v_j^2$ and then $u = v_i \circ H_i = (\gamma_{*i} v_j^1) \circ H_i + (\gamma_{*i} v_j^2) \circ H_i =: u_j^1 + u_j^2$. To prove (2.5.21), it suffices now by interpolation and the above estimates to prove the estimate

$$\|b_i \partial_i u\|_{H^{s-2}} + \|(a^{ij} - \delta^{ij})\partial_i \partial_j u\|_{H^{s-2}} \lesssim_A \|u\|_{H^{s-\varepsilon}} + \|DG - I\|_{L^\infty} \|u\|_{H^s} + \text{RHS}(2.5.21). \quad (2.5.24)$$

We show the details for $b_i \partial_i u$ since it is the more difficult of the two terms to deal with (as it involves two derivatives applied to the domain flattening map) and because the estimate for $(a^{ij} - \delta^{ij})\partial_j \partial_i u$ follows from a similar analysis. Our first aim is to establish the bound

$$\|b_i \partial_i u\|_{H^{s-2}} \lesssim_A \|(\nabla u)(G) \cdot \Delta G\|_{H^{s-2}} + \text{RHS}(2.5.24), \quad (2.5.25)$$

which, to leading order, is essentially like doing an H^{s-2} "change of variables". This bound follows immediately from Proposition 2.5.7 for $2 \leq s \leq 3$, so we restrict to $s \geq 3$. To simplify notation a bit, we write $w := b_i \partial_i u$. We begin by applying Proposition 2.5.5 to obtain

$$\|w\|_{H^{s-2}} \lesssim_A \|(\nabla u)(G) \cdot \Delta G\|_{H^{s-2}} + \|\Gamma\|_{H^{s+r-\frac{1}{2}}} \sup_{j>0} 2^{-j(1+r)} \|w_j^1\|_{L^\infty} + \sup_{j>0} 2^{j(s-2-\varepsilon)} \|w_j^2\|_{L^2}, \quad (2.5.26)$$

where $w = w_j^1 + w_j^2$ is a well-chosen partition which needs to be picked so that we can estimate the latter two terms above by RHS(2.5.24). We take

$$w_j^1 := (\Delta P_{<j} G \cdot (\nabla P_{<j} u_j^1)(G))(H),$$
$$w_j^2 := (\Delta P_{<j} G \cdot (\nabla P_{<j} u_j^2)(G) + \Delta P_{<j} G \cdot (\nabla P_{\geq j} u)(G) + \Delta P_{\geq j} G \cdot (\nabla u)(G))(H).$$

It is then easily verified using the above and (2.5.26) that we have

$$\|w\|_{H^{s-2}} \lesssim_A \|(\nabla u)(G) \cdot \Delta G\|_{H^{s-2}} + \sup_{j>0} 2^{j(s-2-\varepsilon)} \|\Delta P_{\geq j} G \cdot (\nabla u)(G)\|_{L^2} + \text{RHS}(2.5.24).$$

To estimate the latter term on the right, we use that $s - 2 - \varepsilon > 0$ to estimate

$$2^{j(s-2-\varepsilon)} \|\Delta P_{\geq j} G \cdot (\nabla u)(G)\|_{L^2} \leq \sup_{l \geq 0} 2^{l(s-2-\varepsilon)} \|\Delta P_l G \cdot (\nabla u)(G)\|_{L^2}.$$

Then splitting $u = P_{<l} u_l^1 + (P_{<l} u_l^2 + P_{\geq l} u)$, a change of variables and a simple application of the Bernstein inequalities allows us to control the above term by the right-hand side of

(2.5.24). This establishes (2.5.25) for $s \geq 3$. Finally, for each $s \geq 2$, it remains to estimate $\|(\nabla u)(G)\Delta G\|_{H^{s-2}}$ by the right-hand side of (2.5.24). From a simple paradifferential analysis as in Proposition 2.5.3, we have

$$\begin{aligned}\|(\nabla u)(G)\cdot\Delta G\|_{H^{s-2}} &\lesssim_A \|(\nabla u)(G)\|_{H^{s-1-\varepsilon}} + \|T_{(\nabla u)(G)}\Delta G\|_{H^{s-2}}\\ &\lesssim_A \|(\nabla u)(G)\|_{H^{s-1-\varepsilon}} + \mathrm{RHS}(2.5.24),\end{aligned}$$

where, above, to estimate the latter term in the first line, we estimated each summand $P_{<j-4}(\nabla u(G))P_j\Delta G$ in the paradifferential expansion of $T_{(\nabla u)(G)}\Delta G$ using the partition $u = P_{<j}u_j^1 + \left(P_{<j}u_j^2 + P_{\geq j}u\right)$ and Bernstein's inequality. Then, using Proposition 2.5.5 and this same partition, we have easily

$$\|(\nabla u)(G)\|_{H^{s-1-\varepsilon}} \lesssim_A \mathrm{RHS}(2.5.24).$$

This establishes the bound (2.5.24) for $b_i\partial_i u$. The bound for $(a^{ij}-\delta^{ij})\partial_i\partial_j u$ follows similar reasoning, but is easier because it involves only one derivative applied to the domain flattening map, and therefore the initial change of variables performed above is not needed. This concludes the estimate (2.5.21). Our next step is replace u on the left-hand side of (2.5.21) with v_i and replace f on the right-hand side with g. Recall first that $v_i = u\circ G_i$ and $f = h\circ H_i$. We may assume that v_i and u are defined on $\mathbb{R}^d$ using Stein's extension or a suitable half-space extension in the case of u. Therefore, using the partition $u = u_j^1 + u_j^2$ as defined earlier and Proposition 2.5.5 we obtain

$$\|v_i\|_{H^s(\Omega)} \lesssim_A \|u\|_{H^s} + \|\Gamma\|_{H^{s+r-\frac{1}{2}}} \sup_{j>0} 2^{-j(\alpha-1+r)}\|v_j^1\|_{C^\alpha(\Omega)} + \sup_{j>0} 2^{j(s+\beta-1-\varepsilon)}\|v_j^2\|_{H^{1-\beta}(\Omega)},$$

where we used that $\|G-Id\|_{H^{s+r}} \lesssim_A \|\Gamma\|_{H^{s+r-\frac{1}{2}}}$.

To conclude we now need only show that

$$\begin{aligned}\|f\|_{H^{s-2}} \lesssim_A \|g\|_{H^{s-2}(\Omega)} &+ \|\Gamma\|_{H^{s+r-\frac{1}{2}}} \sup_{j>0} 2^{-j(\alpha-1+r)}\|v_j^1\|_{C^\alpha(\Omega)} + \sup_{j>0} 2^{j(s+\beta-1-\varepsilon)}\|v_j^2\|_{H^{1-\beta}(\Omega)}\\ &+ \sup_i \|v_i\|_{H^{s-\varepsilon}(\Omega)}.\end{aligned} \tag{2.5.27}$$

Expanding out $h = \Delta(v\gamma_{*i})$ and using again a paradifferential expansion similar to Proposition 2.5.3, the identity $g := \Delta v$ and the splitting $v = v_j^1 + v_j^2$ we observe first that

$$\begin{aligned}\|h\|_{H^{s-2}(\Omega)} \lesssim_A \|g\|_{H^{s-2}(\Omega)} &+ \|\Gamma\|_{H^{s+r-\frac{1}{2}}} \sup_{j>0} 2^{-j(\alpha-1+r)}\|v_j^1\|_{C^\alpha(\Omega)}\\ &+ \sup_{j>0} 2^{j(s+\beta-1-\varepsilon)}\|v_j^2\|_{H^{1-\beta}(\Omega)} + \sup_i \|v_i\|_{H^{s-\varepsilon}(\Omega)}.\end{aligned}$$

Therefore, we need to only show (2.5.27) with g replaced by h. For this, we first extend h to a function $\tilde{h} := \mathcal{E}\Delta(\gamma_{*i}v)$ on $\mathbb{R}^d$ using Stein's extension. Then, using the partition $\tilde{h} = h_j^1 + h_j^2$ with $h_j^1 = \mathcal{E}\Delta P_{<j}(v_j^1\gamma_{*i})$ and $h_j^2 = \mathcal{E}\Delta P_{<j}(v_j^2\gamma_{*i}) + \mathcal{E}\Delta P_{\geq j}(v\gamma_{*i})$ together with Proposition 2.5.5, we obtain (2.5.27) and conclude the proof. □

We also note a much cruder variant of the above estimate which will be useful for constructing regularization operators later on. As with the corresponding Moser bound in Proposition 2.5.8, the proposition below could be optimized considerably, but such optimizations will not be needed in this chapter.

Proposition 2.5.20 (Cruder variant of the Dirichlet estimates)**.** Let Γ, v, ψ, g and $s \geq 2$ be as in Proposition 2.5.19, and assume that $\psi = 0$. Then for every $\delta > 0$, we have the estimate

$$\|v\|_{H^s(\Omega)} \lesssim_{A,\delta} \|g\|_{H^{s-2}(\Omega)} + \|\Gamma\|_{C^{s+\delta}}\|v\|_{H^1(\Omega)}.$$

Proof. We only give a sketch of the proof since it is essentially a much simpler version of Proposition 2.5.19. One starts by using the cruder flattening (2.5.12) as in the pointwise elliptic estimates and writing the corresponding equation for u (using the notation in (2.5.13)). This flattening is a bit more convenient for this estimate because the source terms in (2.5.13) are simpler. Moreover, we will only need to measure Γ in pointwise norms, and therefore will not need the $\frac{1}{2}$ gain of regularity from the flattening in Proposition 2.5.19. As in the proof of Proposition 2.5.19, we then obtain the preliminary bound

$$\|u\|_{H^s} \lesssim_A \|f\|_{H^{s-2}} + \|(\delta^{jk} - a^{jk})\partial_j u\|_{H^{s-1}}.$$

Using simple paraproduct type estimates and a change of variables, it is straightforward to then estimate

$$\|u\|_{H^s} \lesssim_{A,\delta} \|f\|_{H^{s-2}} + \|\Gamma\|_{C^{s+\delta}}\|v\|_{H^1(\Omega)}. \tag{2.5.28}$$

Then, to conclude, one estimates using Proposition 2.5.8 with $r = 0$ and $r = 2$,

$$\|v_i\|_{H^s} \lesssim_{A,\delta} \|u\|_{H^s} + \|\Gamma\|_{C^{s+\delta}}\|v\|_{H^1(\Omega)}, \qquad \|f\|_{H^{s-2}} \lesssim_A \|h\|_{H^{s-2}(\Omega)} + \|\Gamma\|_{C^{s+\delta}}\|v\|_{H^1(\Omega)},$$

and then performs a simple paraproduct analysis to finally estimate

$$\|h\|_{H^{s-2}(\Omega)} \lesssim_A \|g\|_{H^{s-2}(\Omega)} + \|\Gamma\|_{C^{s+\delta}}\|v\|_{H^1(\Omega)} + \|v\|_{H^{s-\varepsilon}(\Omega)}.$$

Combining the above and interpolating finishes the proof. □

Harmonic extension bounds

By taking $g = 0$ in Proposition 2.5.19, we obtain the following corollary for the harmonic extension operator $\mathcal{H}$.

Proposition 2.5.21 (Harmonic extension bounds)**.** Let Ω be a bounded domain with boundary $\Gamma \in \Lambda_*$. Then the following bound holds for the harmonic extension operator $\mathcal{H}$ when $s \geq 2$, $r \geq 0$, $\beta \in [0, \frac{1}{2})$ and $\alpha \in [0, 1)$,

$$\begin{aligned}\|\mathcal{H}\psi\|_{H^s(\Omega)} \lesssim_A \|\psi\|_{H^{s-\frac{1}{2}}(\Gamma)} + \|\Gamma\|_{H^{s+r-\frac{1}{2}}} \sup_{j>0} 2^{-j(\alpha-1+r)} \|\psi_j^1\|_{C^\alpha(\Gamma)} \\ + \sup_{j>0} 2^{j(s-1+\beta-\varepsilon)} \|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)}.\end{aligned}$$

Here, $\psi = \psi_j^1 + \psi_j^2$ is any sequence of partitions.

Proof. First, Proposition 2.5.19 yields the estimate

$$\begin{aligned}\|\mathcal{H}\psi\|_{H^s(\Omega)} \lesssim_A \|\psi\|_{H^{s-\frac{1}{2}}(\Gamma)} + \|\Gamma\|_{H^{s+r-\frac{1}{2}}} \sup_{j>0} 2^{-j(\alpha-1+r)} \|\phi_j^1\|_{C^\alpha(\Omega)} \\ + \sup_{j>0} 2^{j(s-1-\varepsilon)} \|\phi_j^2\|_{H^1(\Omega)},\end{aligned}$$

where $\phi_j^1 = P_{<j}\mathcal{H}\psi_j^1$ and $\phi_j^2 = P_{<j}\mathcal{H}\psi_j^2 + P_{\geq j}\mathcal{H}\psi$. From the C^α bounds for $\mathcal{H}$ in Corollary 2.5.16 (which hold only for $\alpha \in [0, 1)$), we have $\|\phi_j^1\|_{C^\alpha(\Omega)} \lesssim \|\psi_j^1\|_{C^\alpha(\Gamma)}$. On the other hand, from (2.5.20), we obtain

$$\sup_{j>0} 2^{j(s-1-\varepsilon)} \|\phi_j^2\|_{H^1(\Omega)} \lesssim_A \|\mathcal{H}\psi\|_{H^{s-\varepsilon}(\Omega)} + \sup_{j>0} 2^{j(s-1+\beta-\varepsilon)} \|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)}.$$

The proof then concludes by interpolation and again (2.5.20). □

Curvature estimate

With the above local coordinates, we can control the surface regularity in terms of the mean curvature. The following estimate is a slight refinement of Lemma 4.7 as well as Propositions A.2 and A.3 in [140].

Proposition 2.5.22 (Curvature estimate)**.** Let $s \geq 2$. The following estimates for $\|\Gamma\|_{H^s}$ and the normal n_Γ hold:

$$\|\Gamma\|_{H^s} + \|n_\Gamma\|_{H^{s-1}(\Gamma)} \lesssim_A 1 + \|\kappa\|_{H^{s-2}(\Gamma)}.$$

Proof. We only sketch the details as the proof is similar to [140]. As in their proof, let $\{f_i \in H^s(\tilde{R}_i(2r_i))\}$ be the local coordinate functions associated to Γ defined earlier. Let $\gamma : [0,\infty) \to [0,1]$ be a smooth cutoff function supported on $[0, \frac{3}{2}]$ with $\gamma = 1$ on $[0, \frac{5}{4}]$. On each $\tilde{R}_i(2r_i)$, we let

$$\gamma_i(\tilde{z}) = \gamma(\frac{|\tilde{z}|}{r_i}), \qquad \kappa_i(\tilde{z}) = \gamma_i(\tilde{z})\kappa(\tilde{z}, f_i(\tilde{z})), \qquad g_i = \gamma_i f_i.$$

Using the mean curvature formula

$$\kappa(\tilde{z}, f(\tilde{z})) = -\partial_j(\frac{\partial_j f}{\sqrt{1+|\nabla f|^2}}) = -\frac{\Delta f}{(1+|\nabla f|^2)^{\frac{1}{2}}} + \frac{\partial_j f \partial_k f \partial_{jk} f}{(1+|\nabla f|^2)^{\frac{3}{2}}},$$

we obtain the following elliptic equation for g_i:

$$\begin{aligned} -\Delta g_i = &-\frac{\partial_{j_1} f_i \partial_{j_2} f_i}{(1+|\nabla f_i|^2)}\partial_{j_1 j_2} g_i + (1+|\nabla f_i|^2)^{\frac{1}{2}}\kappa_i - \Delta\gamma_i f_i - 2D\gamma_i \cdot Df_i \\ &+ \frac{\partial_{j_1} f_i \partial_{j_2} f_i}{1+|\nabla f_i|^2}(\partial_{j_1 j_2}\gamma_i f_i + \partial_{j_1}\gamma_i \partial_{j_2} f_i + \partial_{j_2}\gamma_i \partial_{j_1} f_i). \end{aligned}$$

As $\|Df_i\|_{L^\infty} \ll 1$ the first term on the right-hand side can be viewed perturbatively. A paradifferential type analysis similar to the estimate for u in Proposition 2.5.19 together with standard Moser and product type estimates then gives

$$\|g_i\|_{H^s} \lesssim_A \delta\|g_i\|_{H^s} + \|f_i\|_{H^{s-\varepsilon}} + \|\kappa\|_{H^{s-2}(\Gamma)}$$

for some $\delta > 0$ small enough (depending on Λ_*). We then obtain

$$\|g_i\|_{H^s} \lesssim_A \|f_i\|_{H^{s-\varepsilon}} + \|\kappa\|_{H^{s-2}(\Gamma)},$$

and so, we obtain,

$$\sup_i \|f_i\|_{H^s} \lesssim_A 1 + \|\kappa\|_{H^{s-2}(\Gamma)},$$

which completes the proof. □

Estimates for the Dirichlet-to-Neumann operator

Here, we use the above estimates to prove refined bounds for the Dirichlet-to-Neumann operator which is defined by $\mathcal{N} := n_\Gamma \cdot (\nabla\mathcal{H})_{|\Gamma}$. We begin with the following baseline ellipticity estimate.

Lemma 2.5.23. The Dirichlet-to-Neumann map on Γ satisfies

$$\|\psi\|_{H^1(\Gamma)} \lesssim_A \|\mathcal{N}\psi\|_{L^2(\Gamma)} + \|\psi\|_{L^2(\Gamma)}.$$

Proof. Let $v = \mathcal{H}\psi$. We begin by proving the standard estimate

$$\int_\Gamma |\nabla v|^2 dS \lesssim_A \|\mathcal{N}\psi\|_{L^2(\Gamma)}^2 + \|\psi\|_{L^2(\Gamma)}\|\psi\|_{H^1(\Gamma)}. \tag{2.5.29}$$

Let X be a smooth vector field on $\mathbb{R}^d$ which is uniformly transversal to all hypersurfaces in Λ_*. That is, $X \cdot n_\Gamma \gtrsim_A 1$ and $|DX| \lesssim_A 1$. Integration by parts then gives

$$\begin{aligned}\int_\Gamma |\nabla v|^2\, dS &\lesssim_A \int_\Gamma n_\Gamma \cdot X|\nabla v|^2\, dS \\ &\lesssim_A \|\nabla v\|_{L^2(\Omega)}^2 + 2\int_\Omega X_j \partial_j \nabla v \cdot \nabla v\, dx \\ &\lesssim_A \|\nabla v\|_{L^2(\Omega)}^2 + 2\int_\Gamma (X \cdot \nabla v)\mathcal{N}\psi\, dS.\end{aligned}$$

For the first term, we have from the $H^{\frac{1}{2}} \to H^1$ harmonic extension bound and straightforward interpolation,

$$\|v\|_{H^1(\Omega)}^2 \lesssim_A \|\psi\|_{H^{\frac{1}{2}}(\Gamma)}^2 \lesssim_A \|\psi\|_{L^2(\Gamma)}\|\psi\|_{H^1(\Gamma)}.$$

Combining this with the Cauchy Schwarz inequality for the second term, we obtain (2.5.29). Using the partition of unity $(\gamma_{*i})_i$, it straightforward to then estimate

$$\|\psi\|_{H^1(\Gamma)} \lesssim_A \|\psi\|_{L^2(\Gamma)} + \|\nabla^\top v\|_{L^2(\Gamma)} \lesssim_A \|\psi\|_{L^2(\Gamma)} + \|\nabla v\|_{L^2(\Gamma)},$$

where $\nabla^\top$ denotes the projection of ∇ onto the tangent space of Γ. Combining this with (2.5.29) and Cauchy Schwarz concludes the proof. □

We will also need the reverse inequality.

Lemma 2.5.24. The Dirichlet-to-Neumann map on Γ satisfies

$$\|\mathcal{N}\psi\|_{L^2(\Gamma)} \lesssim_A \|\psi\|_{H^1(\Gamma)}.$$

Proof. Using the same notation as in the above lemma and essentially the same argument, we have the estimate

$$\begin{aligned}&\int_\Gamma (X \cdot n_\Gamma)|\nabla^\top \psi|^2\, dS + \int_\Gamma (X \cdot n_\Gamma)|\mathcal{N}\psi|^2\, dS = \int_\Gamma (X \cdot n_\Gamma)|\nabla v|^2\, dS \\ &\geq -C\|\psi\|_{H^1(\Gamma)}^2 + 2\int_\Gamma (X \cdot \nabla v)\mathcal{N}\psi\, dS\end{aligned}$$

for some constant C depending only on A. Writing $X^\top := X - (X \cdot n_\Gamma)n_\Gamma$, we obtain

$$\int_\Gamma (X \cdot n_\Gamma)|\mathcal{N}\psi|^2\, dS \leq C\|\psi\|_{H^1(\Gamma)}^2 + \int_\Gamma (X \cdot n_\Gamma)|\nabla^\top \psi|^2\, dS - 2\int_\Gamma X^\top \cdot \nabla v \mathcal{N}\psi\, dS,$$

which by Cauchy Schwarz completes the proof. □

Next, we prove higher regularity versions of these bounds. The first bound below amounts essentially to elliptic regularity estimates for the Neumann boundary value problem.

Proposition 2.5.25 (Ellipticity for the Dirichlet-to-Neumann operator I)**.** Let $s \geq \frac{3}{2}$, $\alpha \in [0,1)$ and $\beta \in [0, \frac{1}{2})$. Then we have

$$\begin{aligned}\|\psi\|_{H^s(\Gamma)} \lesssim_A \|\psi\|_{L^2(\Gamma)} + \|\mathcal{N}\psi\|_{H^{s-1}(\Gamma)} + \|\Gamma\|_{H^{s+r}} \sup_{j>0} 2^{-j(r+\alpha-1)}\|\psi_j^1\|_{C^\alpha(\Gamma)} \\ + \sup_{j>0} 2^{j(s+\beta-\frac{1}{2}-\varepsilon)}\|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)}.\end{aligned} \tag{2.5.30}$$

Proof. The proof of this is very similar to the Dirichlet problem, so we only sketch the details. Indeed, write $v := \mathcal{H}\psi$. By Proposition 2.5.11, (2.5.20) and the $C^\alpha \to C^\alpha$ bound for $\mathcal{H}$, it suffices to control v in $H^{s+\frac{1}{2}}(\Omega)$ by the right-hand side of (2.5.30). As with the Dirichlet problem, the procedure is to write the Laplace equation for $u = v_i \circ H_i$ and to reduce matters to the standard estimate for the Neumann problem on the half-space (which is available since $s > 1$). The only added technicality is that there are extra source terms coming from the Neumann data (in contrast to the source terms which do not appear for the Dirichlet problem with zero boundary data). By using Proposition 2.5.11 and an analysis similar to Proposition 2.5.19, it is straightforward to obtain the preliminary estimate

$$\begin{aligned}\|\psi\|_{H^s(\Gamma)} \lesssim_A \|v\|_{H^1(\Omega)} + \|\mathcal{N}\psi\|_{H^{s-1}(\Gamma)} + \|\Gamma\|_{H^{s+r}} \sup_{j>0} 2^{-j(r+\alpha-1)}\|v_j^1\|_{C^\alpha(\Omega)} \\ + \sup_{j>0} 2^{j(s+\beta-\frac{1}{2}-\varepsilon)}\|v_j^2\|_{H^{1-\beta}(\Omega)},\end{aligned}$$

where $v := v_j^1 + v_j^2$ is any partition of v. The first term $\|v\|_{H^1(\Omega)}$ is harmless and can be controlled by $\|\psi\|_{L^2(\Gamma)} + \|\mathcal{N}\psi\|_{L^2(\Gamma)}$ using the $H^{\frac{1}{2}} \to H^1$ bound for $\mathcal{H}$ and Lemma 2.5.23. We then take $v_j^1 = \mathcal{H}\psi_j^1$ and $v_j^2 = \mathcal{H}\psi_j^2$ and use again the $C^\alpha \to C^\alpha$ bounds for $\mathcal{H}$ and (2.5.20) to conclude. □

We will also need the following iterated version of the ellipticity bound above.

Proposition 2.5.26 (Ellipticity for the Dirichlet-to-Neumann operator II)**.** Let $s \geq \frac{1}{2}$ and let $k \geq 1$ be an integer. Then using the same notation as the previous proposition, we have the bound

$$\begin{aligned}\|\psi\|_{H^{s+k}(\Gamma)} \lesssim_A \|\psi\|_{L^2(\Gamma)} + \|\mathcal{N}^k\psi\|_{H^s(\Gamma)} + \|\Gamma\|_{H^{s+k+r}} \sup_{j>0} 2^{-j(\alpha-1+r)}\|\psi_j^1\|_{C^\alpha(\Gamma)} \\ + \sup_{j>0} 2^{j(s+k-\frac{1}{2}+\beta-\varepsilon)}\|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)}.\end{aligned}$$

Proof. Lemma 2.5.23 and Proposition 2.5.25 give us this bound for $k = 1$. For $k \geq 2$, we may assume inductively that the corresponding estimate holds for all $1 \leq m \leq k-1$. We begin by applying Proposition 2.5.25 to obtain

$$\begin{aligned}\|\psi\|_{H^{s+k}(\Gamma)} \lesssim_A \|\psi\|_{L^2(\Gamma)} + \|\mathcal{N}\psi\|_{H^{s+k-1}(\Gamma)} + \|\Gamma\|_{H^{s+k+r}} \sup_{j>0} 2^{-j(\alpha-1+r)} \|\psi_j^1\|_{C^\alpha(\Gamma)} \\ + \sup_{j>0} 2^{j(s+k-\frac{1}{2}+\beta-\varepsilon)} \|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)}.\end{aligned} \tag{2.5.31}$$

Using the inductive hypothesis, we have

$$\begin{aligned}\|\mathcal{N}\psi\|_{H^{s+k-1}(\Gamma)} \lesssim_A \|\mathcal{N}\psi\|_{L^2(\Gamma)} + \|\mathcal{N}^k\psi\|_{H^s(\Gamma)} + \|\Gamma\|_{H^{s+k+r}} \sup_{j>0} 2^{-jr} \|\phi_j^1\|_{L^\infty(\Gamma)} \\ + \sup_{j>0} 2^{j(s+k-1-2\varepsilon)} \|\phi_j^2\|_{H^\varepsilon(\Gamma)},\end{aligned}$$

where $\mathcal{N}\psi := \phi_j^1 + \phi_j^2$ is any partition of $\mathcal{N}\psi$. By Lemma 2.5.24, the first term on the right can be controlled by $\|\psi\|_{H^1(\Gamma)}$ which can be dispensed with by interpolation (between L^2 and $H^{1+\varepsilon}$ to ensure the domain dependent contributions in the estimate are harmless). Therefore, to conclude, we need to choose ϕ_j^1 and ϕ_j^2 so that the latter two terms on the right-hand side of the above are controlled by the right-hand side of (2.5.31). Using v, v_j^1 and v_j^2 from the previous proposition, we can take $\phi_j^1 = \nabla_n P_{<j} v_j^1$ and $\phi_j^2 = \nabla_n P_{<j} v_j^2 + \nabla_n P_{\geq j} v$. The proof then concludes in a similar way to Proposition 2.5.25. We omit the details. □

For our energy estimates, we will also need good bounds for the following div-curl system.

Proposition 2.5.27 (div-curl estimate with Neumann type data)**.** Let $v \in H^s(\Omega)$ be a vector field defined on Ω and let $s > \frac{3}{2}$, $\alpha, \beta \in [0,1]$. Let $v := v_j^1 + v_j^2$ be any partition of v. Moreover, let $\mathcal{B}v$ denote either the Neumann trace of v, $n_\Gamma \cdot \nabla v$ or the boundary value $\nabla^\top v \cdot n_\Gamma$. Then if v solves the div-curl system,

$$\begin{cases} \nabla \cdot v = f, \\ \nabla \times v = \omega, \\ \mathcal{B}v = g, \end{cases}$$

then v satisfies the estimate,

$$\begin{aligned}\|v\|_{H^s(\Omega)} \lesssim_A \|f\|_{H^{s-1}(\Omega)} + \|\omega\|_{H^{s-1}(\Omega)} + \|g\|_{H^{s-\frac{3}{2}}(\Gamma)} + \|v\|_{L^2(\Omega)} \\ + \|\Gamma\|_{H^{s+r-\frac{1}{2}}} \sup_{j>0} 2^{-j(r+\alpha-1)} \|v_j^1\|_{C^\alpha(\Omega)} + \sup_{j>0} 2^{j(s-1+\beta-\varepsilon)} \|v_j^2\|_{H^{1-\beta}(\Omega)}.\end{aligned}$$

Proof. The proof is very similar to the Dirichlet and Neumann problems in that one flattens the boundary and reduces to the corresponding estimate on the half-space with source terms depending on essentially f, ω, g and the domain regularity. We omit the details of the domain flattening as it is similar to Proposition 2.5.19. However, for the sake of clarity, it is instructive to explain the div-curl estimate in the case when Ω is the half-space $\{z_d < 0\}$ (particularly in the case of the latter boundary condition involving $\nabla^\top v \cdot n_\Gamma$). We show that it is in essence a statement about elliptic regularity for the Neumann problem. In such a setting, n_Γ takes the form e_d. We compute for each (Euclidean) component v_j of a vector field v on Ω,

$$\Delta v_j = \partial_i \omega_{ij} + \partial_j f.$$

Therefore, in the case of boundary data given by $\mathcal{B}v = n_\Gamma \cdot \nabla v$, the div-curl estimate is simply given by elliptic regularity for the Neumann problem. To understand the case of the other boundary value $\nabla^\top v \cdot n_\Gamma$, we note that the full Neumann data for v is determined by this boundary value and the curl and divergence of v. If $j \neq d$, this is seen from the identity

$$\partial_d v_j = \partial_j v_d + \omega_{dj}.$$

So, by the trace theorem and elliptic regularity for the Neumann problem, we have the desired control of v_j for $j \neq d$. If $j = d$, we have

$$\partial_d v_d = f - \sum_{i=1}^{d-1} \partial_i v_i,$$

which by the trace theorem and the estimate for v_i with $i \neq d$ gives us the estimate for v_d. □

We importantly do not claim that the above div-curl system is well-posed. In fact, the problem is generally over-determined (as, for instance, the curl and divergence fix Δv, which forbids certain choices of Neumann data). Fortunately, we will only need the above estimate in our analysis later when we prove energy estimates and to a lesser extent in our construction of regular solutions. We will not need any existence type statement for the above system, however.

Next, to complement the ellipticity estimates for $\mathcal{N}$, we will also need the reverse estimates which control powers of $\mathcal{N}$ applied to a function in terms of the corresponding Sobolev norms of that function. As a preliminary step, we state the following proposition.

Proposition 2.5.28 (Normal derivative trace bound)**.** Let $s > 0$, $r \geq 0$ and $\alpha, \beta \in [0,1]$. The normal trace operator $\nabla_n := n_\Gamma \cdot (\nabla)_{|\Gamma}$ satisfies the bound

$$\begin{aligned}\|\nabla_n v\|_{H^s(\Gamma)} \lesssim_A \|v\|_{H^{s+\frac{3}{2}}(\Omega)} &+ \|\Gamma\|_{H^{s+r+1}} \sup_{j>0} 2^{-j(r-1+\alpha)} \|v_j^1\|_{C^\alpha(\Omega)} \\ &+ \sup_{j>0} 2^{j(s+\beta+\frac{1}{2}-\varepsilon)} \|v_j^2\|_{H^{1-\beta}(\Omega)}.\end{aligned}$$

Proof. Using the partition $\nabla v = w_j^1 + w_j^2$ where $w_j^1 := \nabla P_{<j} v_j^1$ and $w_j^2 = \nabla P_{<j} v_j^2 + \nabla P_{\geq j} v$ together with the inequalities $\|n_\Gamma\|_{H^{s+r}(\Gamma)} \lesssim_A \|\Gamma\|_{H^{s+r+1}}$ and $\|n_\Gamma\|_{C^\varepsilon(\Gamma)} \lesssim_A 1$, we obtain from Proposition 2.5.9 and Proposition 2.5.11 (after possibly relabelling ε),

$$\begin{aligned}\|\nabla_n v\|_{H^s(\Gamma)} &\lesssim_A \|(\nabla v)_{|\Gamma}\|_{H^s(\Gamma)} + \|\Gamma\|_{H^{s+r+1}} \sup_{j>0} 2^{-jr} \|w_j^1\|_{L^\infty(\Omega)} + \sup_{j>0} 2^{j(s-2\varepsilon)} \|w^2_{j|\Gamma}\|_{L^2(\Gamma)} \\ &\lesssim_A \|v\|_{H^{s+\frac{3}{2}}(\Omega)} + \|\Gamma\|_{H^{s+r+1}} \sup_{j>0} 2^{-jr} \|w_j^1\|_{L^\infty(\Omega)} + \sup_{j>0} 2^{j(s-2\varepsilon)} \|w_j^2\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)}.\end{aligned}$$

By estimating

$$\|w_j^1\|_{L^\infty(\Omega)} \lesssim_A 2^{j(1-\alpha)} \|v_j^1\|_{C^\alpha(\Omega)}$$

and

$$2^{j(s-2\varepsilon)} \|w_j^2\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)} \lesssim_A \|v\|_{H^{s+\frac{3}{2}}(\Omega)} + 2^{j(s+\frac{1}{2}+\beta-\varepsilon)} \|v_j^2\|_{H^{1-\beta}(\Omega)},$$

we complete the proof. □

We can use Proposition 2.5.28 and the balanced bounds for $\mathcal{H}$ to prove a refined version of the $H^{s+1}(\Gamma) \to H^s(\Gamma)$ bound for $\mathcal{N}$.

Proposition 2.5.29 (Dirichlet-to-Neumann operator bound I)**.** Let $s \geq \frac{1}{2}$, $r \geq 0$, $\alpha \in [0,1)$ and $\beta \in [0, \frac{1}{2})$. Then

$$\begin{aligned}\|\mathcal{N}\psi\|_{H^s(\Gamma)} \lesssim_A \|\psi\|_{H^{s+1}(\Gamma)} &+ \|\Gamma\|_{H^{s+1+r}} \sup_{j>0} 2^{-j(r-1+\alpha)} \|\psi_j^1\|_{C^\alpha(\Gamma)} \\ &+ \sup_{j>0} 2^{j(s+\frac{1}{2}+\beta-\varepsilon)} \|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)}\end{aligned}$$

for any sequence of partitions $\psi = \psi_j^1 + \psi_j^2$.

Proof. The proof begins by writing $\mathcal{N} = \nabla_n \mathcal{H}$ and applying Proposition 2.5.28 to obtain

$$\begin{aligned}\|\mathcal{N}\psi\|_{H^s(\Gamma)} \lesssim_A \|\mathcal{H}\psi\|_{H^{s+\frac{3}{2}}(\Omega)} &+ \|\Gamma\|_{H^{s+1+r}} \sup_{j>0} 2^{-j(r-1+\alpha)} \|\mathcal{H}\psi_j^1\|_{C^\alpha(\Omega)} \\ &+ \sup_{j>0} 2^{j(s+\frac{1}{2}+\beta-\varepsilon)} \|\mathcal{H}\psi_j^2\|_{H^{1-\beta}(\Omega)}.\end{aligned}$$

Using the $C^\alpha \to C^\alpha$ bounds for $\mathcal{H}$, (2.5.20) and Proposition 2.5.21, we conclude the proof. □

Similarly to the ellipticity estimate for $\mathcal{N}$, we will need a higher order version of the above estimate as well.

Proposition 2.5.30 (Dirichlet-to-Neumann operator bound II)**.** Let $m \geq 1$ be an integer, let $s \geq \frac{1}{2}$ and let $r \geq 0$, $\alpha \in [0,1)$ and $\beta \in [0, \frac{1}{2})$. Then we have the bound

$$\|\mathcal{N}^m \psi\|_{H^s(\Gamma)} \lesssim_A \|\psi\|_{H^{s+m}(\Gamma)} + \|\Gamma\|_{H^{s+r+m}} \sup_{j>0} 2^{-j(r+\alpha-1)} \|\psi_j^1\|_{C^\alpha(\Gamma)} + \sup_{j>0} 2^{j(s-\frac{1}{2}+m+\beta-\varepsilon)} \|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)}$$

and the closely related bound when $s \geq \frac{3}{2}$,

$$\|\mathcal{H}\mathcal{N}^m \psi\|_{H^{s+\frac{1}{2}}(\Omega)} \lesssim_A \|\psi\|_{H^{s+m}(\Gamma)} + \|\Gamma\|_{H^{s+r+m}} \sup_{j>0} 2^{-j(r+\alpha-1)} \|\psi_j^1\|_{C^\alpha(\Gamma)} + \sup_{j>0} 2^{j(s-\frac{1}{2}+m+\beta-\varepsilon)} \|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)} \tag{2.5.32}$$

for any partition $\psi = \psi_j^1 + \psi_j^2$.

Proof. We begin with the first bound. The previous proposition handles the case $m = 1$. Suppose $m > 1$ and let us suppose inductively that the bound holds for all integers greater than or equal to 1 and strictly less than m. Then we have from the inductive hypothesis,

$$\|\mathcal{N}^m \psi\|_{H^s(\Gamma)} \lesssim_A \|\mathcal{N}\psi\|_{H^{s+m-1}(\Gamma)} + \|\Gamma\|_{H^{s+m+r}} \sup_{j>0} 2^{-jr} \|\phi_j^1\|_{L^\infty(\Gamma)} + \sup_{j>0} 2^{j(s-1+m-\varepsilon)} \|\phi_j^2\|_{H^\varepsilon(\Gamma)}, \tag{2.5.33}$$

where $\mathcal{N}\psi := \phi_j^1 + \phi_j^2$ is the same partition of $\mathcal{N}\psi$ as in the proof of Proposition 2.5.26. Applying the inductive hypothesis again to the first term on the right and arguing the same way as in Proposition 2.5.26 to control the latter two terms in favour of ψ, ψ_j^1 and ψ_j^2 concludes the proof of the first estimate. To obtain the latter estimate, we proceed in a similar way as above. For the case $m = 1$, we can use Proposition 2.5.21 to control $\|\mathcal{H}\mathcal{N}\psi\|_{H^{s+\frac{1}{2}}(\Omega)}$ by the right-hand side of (2.5.33). Then one concludes the bound for all $m \geq 1$ by induction as above. □

Next, we note a bound for the operator $\nabla^\top$ which follows from similar reasoning to the above.

Proposition 2.5.31. Let $s \geq \frac{1}{2}$, $r \geq 0$, $\alpha \in [0,1)$ and $\beta \in [0, \frac{1}{2})$. Then

$$\begin{aligned}\|\nabla^\top \psi\|_{H^s(\Gamma)} \lesssim_A \|\psi\|_{H^{s+1}(\Gamma)} + \|\Gamma\|_{H^{s+1+r}} \sup_{j>0} 2^{-j(r-1+\alpha)} \|\psi_j^1\|_{C^\alpha(\Gamma)} \\ + \sup_{j>0} 2^{j(s+\frac{1}{2}+\beta-\varepsilon)} \|\psi_j^2\|_{H^{\frac{1}{2}-\beta}(\Gamma)}\end{aligned} \tag{2.5.34}$$

for any sequence of partitions $\psi = \psi_j^1 + \psi_j^2$.

Proof. By writing
$$\nabla^\top \psi = \nabla \mathcal{H}\psi - n_\Gamma \mathcal{N}\psi,$$
the proof follows essentially the same line of reasoning as the proofs of Proposition 2.5.28 and Proposition 2.5.29. We omit the details. □

Finally, we note a bound for $\mathcal{N}^m \nabla_n$ which will be needed frequently in the higher energy bounds.

Corollary 2.5.32. Let $\alpha, \beta \in [0,1]$, $s \geq \frac{1}{2}$ and $r \geq 0$. We have

$$\begin{aligned}\|\mathcal{N}^m \nabla_n v\|_{H^s(\Gamma)} \lesssim_A \|v\|_{H^{s+m+\frac{3}{2}}(\Omega)} + \|\Gamma\|_{H^{s+1+m+r}} \sup_{j>0} 2^{-j(r+\alpha-1)} \|v_j^1\|_{C^\alpha(\Omega)} \\ + \sup_{j>0} 2^{j(s+\beta+\frac{1}{2}+m-\varepsilon)} \|v_j^2\|_{H^{1-\beta}(\Omega)}\end{aligned}$$

where $v = v_j^1 + v_j^2$ is any sequence of partitions of v.

Proof. We omit most of the details. The proof proceeds by first using Proposition 2.5.30 with the partition $\nabla_n v = n_\Gamma \cdot w^1_{j|\Gamma} + n_\Gamma \cdot w^2_{j|\Gamma}$ in $L^\infty(\Gamma) + H^\varepsilon(\Gamma)$ where w_j^1 and w_j^2 are as in the proof of Proposition 2.5.28 and then using Proposition 2.5.28 to estimate $\nabla_n v$ in H^{s+m}. □

Moving surface identities

In this section, we suppose that Ω_t is a one parameter family of domains with boundaries $\Gamma_t \in \Lambda_*$ which flow with a velocity vector field v that is not necessarily divergence free. Our purpose is to collect various identities and commutator estimates involving the material derivative $D_t := \partial_t + v \cdot \nabla$ and functions on Γ_t. We begin by recalling several algebraic identities, many of which were proven in [140].

(i) (Material derivative of the normal).

$$D_t n_{\Gamma_t} = -\left((\nabla v)^*(n_{\Gamma_t})\right)^\top . \tag{2.5.35}$$

(ii) (Leibniz rule for $\mathcal{N}$).

$$\mathcal{N}(fg) = f\mathcal{N}g + g\mathcal{N}f - 2\nabla_n\Delta^{-1}(\nabla\mathcal{H}f \cdot \nabla\mathcal{H}g). \tag{2.5.36}$$

(iii) (Commutator with ∇).

$$[D_t, \nabla]g = -(\nabla v)^*(\nabla g). \tag{2.5.37}$$

(iv) (Commutator with Δ^{-1}).

$$[D_t, \Delta^{-1}]g = \Delta^{-1}\left(2\nabla v \cdot \nabla^2\Delta^{-1}g + \Delta v \cdot \nabla\Delta^{-1}g\right). \tag{2.5.38}$$

(v) (Commutator with $\mathcal{H}$).

$$S_0 f := [D_t, \mathcal{H}]f = \Delta^{-1}(2\nabla v \cdot \nabla^2\mathcal{H}f + \nabla\mathcal{H}f \cdot \Delta v). \tag{2.5.39}$$

(vi) (Commutator with $\mathcal{N}$).

$$S_1 f := [D_t, \mathcal{N}]f = D_t n_{\Gamma_t} \cdot \nabla\mathcal{H}f - n_{\Gamma_t} \cdot ((\nabla v)^*(\nabla\mathcal{H}f)) + n_{\Gamma_t} \cdot \nabla([D_t, \mathcal{H}]f). \tag{2.5.40}$$

We also have the general Leibniz type formula,

$$\frac{d}{dt}\int_{\Gamma_t} f dS = \int_{\Gamma_t} D_t f + f(\mathcal{D} \cdot v^\top - \kappa v^\perp)\, dS, \tag{2.5.41}$$

where $\mathcal{D}$ is the covariant derivative.

Balanced commutator estimates

Using the above identities, we now establish refined estimates for commutators involving D_t and the Dirichlet-to-Neumann operator. If we assume that v is divergence free, it is a straightforward calculation to verify that $S_0\psi$ can be rewritten in the form

$$S_0\psi = \Delta^{-1}\nabla \cdot \mathcal{B}(\nabla v, \nabla\mathcal{H}\psi), \tag{2.5.42}$$

where $\mathcal{B}$ is an $\mathbb{R}^d$-valued bilinear form. Using (2.5.40), we can write the commutator $[D_t, \mathcal{N}]$ as follows:

$$S_1\psi := [D_t, \mathcal{N}]\psi = \nabla_n S_0\psi - \nabla\mathcal{H}\psi \cdot (\nabla_n v) - \nabla^\top\psi \cdot \nabla v \cdot n_{\Gamma_t}.$$

In the higher energy bounds, we will need an estimate for higher order commutators S_k, given by

$$S_k\psi := [D_t, \mathcal{N}^k]\psi = \sum_{l+m=k-1} \mathcal{N}^l[D_t, \mathcal{N}]\mathcal{N}^m\psi, \tag{2.5.43}$$

where l, m are non-negative integers and $k \in \mathbb{N}$. From now on, let us write $A = \|v\|_{C^{\frac{1}{2}+\varepsilon}(\Omega)} + \|\Gamma\|_{C^{1,\varepsilon}}$. For $s \geq \frac{1}{2}$, we have the following refined estimates for S_k when v is divergence free, which will be useful for estimating $S_k D_t a$ and $S_k a$, respectively, in the higher energy bounds.

Proposition 2.5.33. Suppose that the flow velocity v is divergence free and let $s \geq \frac{1}{2}$, $k \geq 1$. Then we have the following bounds for S_k.

(i) (Variant 1). For any sequence of partitions $\psi = \psi_j^1 + \psi_j^2$, there holds

$$\begin{aligned}\|S_k\psi\|_{H^s(\Gamma)} \lesssim_A &\|v\|_{W^{1,\infty}(\Omega)}\|\psi\|_{H^{s+k}(\Gamma)} + \|v\|_{H^{s+\frac{3}{2}+k}(\Omega)}\|\psi\|_{L^\infty(\Gamma)} + \|\Gamma\|_{H^{s+\frac{3}{2}+k}}\|\psi\|_{L^\infty(\Gamma)} \\ &+\|v\|_{W^{1,\infty}(\Omega)}\|\Gamma\|_{H^{s+k+\frac{3}{2}}} \sup_{j>0} 2^{-\frac{j}{2}}\|\psi_j^1\|_{L^\infty(\Gamma)} \\ &+ \|v\|_{W^{1,\infty}(\Omega)} \sup_{j>0} 2^{j(s+k-\varepsilon)}\|\psi_j^2\|_{H^\varepsilon(\Gamma)}.\end{aligned}$$

(ii) (Variant 2).

$$\begin{aligned}\|S_k\psi\|_{H^s(\Gamma)} \lesssim_A &\|v\|_{W^{1,\infty}(\Omega)}\|\psi\|_{H^{s+k}(\Gamma)} + \|\Gamma\|_{H^{s+k+1}}(\|\psi\|_{C^{\frac{1}{2}}(\Gamma)} + \|v\|_{W^{1,\infty}(\Omega)}\|\psi\|_{L^\infty(\Gamma)}) \\ &+\|v\|_{H^{s+k+1}(\Omega)}\|\psi\|_{C^{\frac{1}{2}}(\Gamma)}.\end{aligned}$$

Proof. We will focus on the first estimate as the second one is similar. From (2.5.43), we need to prove the estimate in (i) with the left-hand side replaced with $\mathcal{N}^l[D_t,\mathcal{N}]\mathcal{N}^m\psi$ where $l+m = k-1$. We will focus first on the term $\mathcal{N}^l(\nabla_n S_0\mathcal{N}^m\psi)$ which is the most difficult to deal with. Let us write $G := \mathcal{B}(\nabla v, \nabla\mathcal{H}\mathcal{N}^m\psi)$ for notational convenience. We begin by applying Corollary 2.5.32 and then Proposition 2.5.19 to obtain (using the identity (2.5.42)),

$$\begin{aligned}\|\mathcal{N}^l(\nabla_n S_0\mathcal{N}^m\psi)\|_{H^s(\Gamma)} \lesssim_A \|G\|_{H^{s+l+\frac{1}{2}}(\Omega)} + \|\Gamma\|_{H^{s+\frac{3}{2}+k}} \sup_{j>0} 2^{-j(m+\frac{3}{2})}\|\Delta^{-1}\nabla\cdot G_j^1\|_{W^{1,\infty}(\Omega)} \\ + \sup_{j>0} 2^{j(s+l+\frac{1}{2}-\varepsilon)}\|\Delta^{-1}\nabla\cdot G_j^2\|_{H^1(\Omega)},\end{aligned}$$

where $G = G_j^1 + G_j^2$ is a partition of G defined by taking $G_j^1 = \mathcal{B}(\nabla P_{<j}v, \nabla P_{<j}\mathcal{H}\mathcal{N}_{<j}^m\psi)$, where $\mathcal{N}_{<j} := \nabla_n P_{<j}\mathcal{H}$. Using the $C^{1,\varepsilon}$ estimate for Δ^{-1} and the maximum principle for $\mathcal{H}$, it is straightforward to control

$$2^{-j(m+\frac{3}{2})}\|\Delta^{-1}\nabla\cdot G_j^1\|_{W^{1,\infty}(\Omega)} \lesssim_A \|v\|_{C^{\frac{1}{2}+\varepsilon}(\Omega)}\|\psi\|_{L^\infty(\Gamma)} \lesssim_A \|\psi\|_{L^\infty(\Gamma)}.$$

Moreover, using the $H^{-1}\to H_0^1$ estimate for Δ^{-1}, we can control the other term by

$$\begin{aligned}2^{j(s+l+\frac{1}{2}-\varepsilon)}\|\Delta^{-1}\nabla\cdot G_j^2\|_{H^1(\Omega)} \lesssim_A 2^{j(s+l+\frac{1}{2}-\varepsilon)}\|v\|_{W^{1,\infty}(\Omega)}\|\nabla P_{<j}\mathcal{H}\mathcal{N}_{<j}^m\psi - \nabla\mathcal{H}\mathcal{N}^m\psi\|_{L^2(\Omega)} \\ +\|v\|_{H^{s+\frac{3}{2}+k}(\Omega)}\|\psi\|_{L^\infty(\Gamma)}.\end{aligned}$$

Finally, it is straightforward (albeit somewhat technical) to verify that the terms on the right-hand side above can be controlled by the right-hand side of (i) using the $H^{\varepsilon} \to H^{\frac{1}{2}+\varepsilon}$ bound (2.5.20), Proposition 2.5.30, Proposition 2.5.9 with $g_j^2 = g$ (and the fact that $\|n_\Gamma\|_{C^{\varepsilon}(\Gamma)} \lesssim_A 1$) as well as the $H^{\frac{1}{2}+\varepsilon} \to H^{\varepsilon}$ trace estimates. Now, we turn to estimating $\|G\|_{H^{s+l+\frac{1}{2}}(\Omega)}$. By performing a paradifferential expansion as in Proposition 2.5.3, it is easy to see that

$$\|G\|_{H^{s+1+\frac{1}{2}}(\Omega)} \lesssim_A \|v\|_{W^{1,\infty}(\Omega)} \|\mathcal{H}\mathcal{N}^m\psi\|_{H^{s+l+\frac{3}{2}}(\Omega)} + \|T_{\nabla\mathcal{H}\mathcal{N}^m\psi}\nabla v\|_{H^{s+l+\frac{1}{2}}(\Omega)}.$$

Using Proposition 2.5.21 and Proposition 2.5.30, the first term on the right can be controlled by the right-hand side of (i). For the latter term, we need to control the l^2 sum of

$$2^{j(s+l+\frac{1}{2})}\|P_j\nabla v P_{<j-4}\nabla\mathcal{H}\mathcal{N}^m\psi\|_{L^2(\Omega)}.$$

For this, we estimate

$$\begin{aligned} 2^{j(s+l+\frac{1}{2})}\|P_j\nabla v P_{<j-4}\nabla\mathcal{H}\mathcal{N}^m\psi\|_{L^2(\Omega)} \lesssim_A\ & 2^{j(s+k+\frac{1}{2})}\|P_j\nabla v\|_{L^2(\Omega)}\|\psi\|_{L^\infty(\Gamma)} \\ & + 2^{j(s+l+\frac{1}{2})}\|v\|_{W^{1,\infty}(\Omega)}\|P_{<j-4}\nabla\mathcal{H}(\mathcal{N}^m - \mathcal{N}^m_{<j})\psi\|_{L^2(\Omega)}. \end{aligned}$$

The first term on the right when summed in l^2 is controlled by the right-hand side of (i). The same is true for the latter term after making use of (2.5.20) and Proposition 2.5.30. This concludes the full estimate for $\mathcal{N}^l(\nabla_n S_0 \mathcal{N}^m\psi)$. The other terms in $\mathcal{N}^l[D_t, \mathcal{N}]\mathcal{N}^m\psi$ are dealt with similarly. □

2.6 Regularization operators

Let Ω_* be a smooth, bounded domain with boundary Γ_*. In the following, we let Ω be a bounded domain with boundary $\Gamma \in \Lambda(\Gamma_*, \varepsilon, \delta)$ where $\varepsilon > 0$ and $\delta > 0$ are small positive constants. As usual, we will abbreviate the above set of hypersurfaces by Λ_* and consider the volume of the associated domains as part of our implicit constants. We recall from (2.3.2) that we have the diffeomorphism from Γ_* to Γ given by

$$\Phi_\Gamma(x) = x + \eta_\Gamma(x)\nu(x)$$

which parameterizes Γ as a graph over Γ_*. When constructing solutions to the free boundary Euler equations (and also when proving refined energy estimates), it will be important to have a good regularization operator at each dyadic scale which preserves divergence free functions. More precisely, beyond the obvious regularization properties (to be outlined below in more detail), our operators will need to have the following properties.

(i) (Extension property). There is a $\delta_0 > 0$ such that the following holds: If Ω_j is a domain containing Ω with boundary $\Gamma_j \in \Lambda_*$ such that $\|\mathrm{dist}(x, \Omega)\|_{L^\infty(\Omega_j)} < \delta_0 2^{-j}$ then there is an associated regularization $\Psi_{\leq j} v$ at the dyadic scale 2^j, defined on Ω_j.

(ii) (Regularization is divergence free). Given Ω_j as above, the regularization $\Psi_{\leq j} v$ satisfies $\nabla \cdot \Psi_{\leq j} v = 0$ on Ω_j. Here, v is a divergence free function on Ω.

Remark 2.6.1. The first point will be convenient later for comparing velocities defined on different domains, which are sufficiently close. The second point is important as our regularization operators will not necessarily commute with derivatives (but will commute with derivatives up to lower order terms).

A more precise description of the above regularization operators is given by the following proposition.

Proposition 2.6.2. Fix α_0, let v, Ω and Ω_j be as above and let $A = \|\Gamma\|_{C^{1,\varepsilon}}$. Then there exists a regularization operator $\Psi_{\leq j}$ which is bounded from $H^s_{div}(\Omega) \to H^s_{div}(\Omega_j)$ for every $s \geq 0$ with the following properties.

(i) (Regularization bounds).

$$\|\Psi_{\leq j} v\|_{H^{s+\alpha}(\Omega_j)} \lesssim_A 2^{j\alpha} \|v\|_{H^s(\Omega)}, \qquad 0 \leq \alpha.$$

(ii) (Difference bounds).

$$\|(\Psi_{\leq j+1} - \Psi_{\leq j}) v\|_{H^{s-\alpha}(\Omega_{j+1})} \lesssim_A 2^{-j\alpha} \|v\|_{H^s(\Omega)}, \qquad 0 \leq \alpha \leq \min\{s, \alpha_0\}.$$

(iii) (Error bounds).

$$\|(I - \Psi_{\leq j}) v\|_{H^{s-\alpha}(\Omega)} \lesssim_A 2^{-j\alpha} \|v\|_{H^s(\Omega)}, \qquad 0 \leq \alpha \leq \min\{s, \alpha_0\}.$$

Proof. We begin with a preliminary step of constructing a regularization operator $\Phi_{\leq j}$ with the above three properties which maps $H^s(\Omega)$ to $H^s(\tilde{\Omega}_j)$ where $\tilde{\Omega}_j$ is a neighborhood of Ω_j, but does not necessarily preserve divergence free functions. To do this, we aim to construct a suitable kernel K^j such that

$$\Phi_{\leq j} v(x) = \int_\Omega K^j(x, y) v(y)\, dy.$$

Here, the kernel $K^j(x,y)$ is of the form

$$K^j(x,y) = \sum_{k=0}^{n} K_k^j(x,y)\chi_k(x),$$

where $(\chi_k)_{k=0}^n$ is a partition of unity of a neighborhood of Ω, obtained by selecting an open cover $\{U_k\}_{k=0}^n$ so that there are vectors $(e_k)_{k=1}^n$ all of the same length with e_k outward oriented and uniformly transversal to $\Gamma \cap U_k$. The remaining set U_0 is then chosen to cover the portion of Ω away from the boundary. Let $e_0 = 0$ and take e_k with $k \in \{1, \dots, n\}$ as above. Such a smooth partition of unity can be constructed with bounds depending only on the properties of Λ_*. To construct K^j we consider a smooth bump function ϕ_k with the following properties:

(i) The support of ϕ_k satisfies $\text{supp}\phi_k \subseteq B(e_k, \delta_1)$, $\delta_1 \ll 1$.

(ii) The average of ϕ_k is 1, i.e., $\int_{\mathbb{R}^d} \phi_k(z)\, dz = 1$.

(iii) ϕ_k has zero moments up to some sufficiently large order N, i.e., $\int_{\mathbb{R}^d} z^\alpha \phi_k(z) dz = 0$, $1 \leq |\alpha| \leq N$.

Then, for each $j > 0$, we consider a regularizing kernel

$$K_{0,k}^j(z) := 2^{jd}\phi_k(2^j z).$$

We then define $K_k^j(x,y) := K_{0,k}^j(x-y)$ for $y \in \Omega$. Note that for fixed $x \in U_k$, $K_k^j(x,y)$ is non-zero only if $2^j(x-y) \in B(e_k, \delta_1)$, i.e., y is within distance $2^{-j}\delta_1$ of $x - 2^{-j}e_k$. This is what will allow us to view our kernel K^j not only for $x \in \Omega$ but also for x in a $\mathcal{O}(2^{-j})$ enlargement of Ω. With this in mind, one can check that the family of kernels K^j satisfy the following:

(i) $K^j : \tilde{\Omega}_j \times \Omega \to \mathbb{R}$, where $\tilde{\Omega}_j := \{x \in \mathbb{R}^d : d(x,\Omega) \leq c2^{-j}\}$ with a small universal constant c.

(ii) $|\partial_x^\alpha \partial_y^\beta K^j(x,y)| \lesssim 2^{j(d+|\alpha|+|\beta|)}$, for multi-indices α, β.

(iii) $\int_\Omega K^j(x,y)\, dy = 1$.

(iv) $\int_\Omega K^j(x,y)(x-y)^\alpha\, dy = 0$, $1 \leq |\alpha| \leq N$.

From the definition of K^j, we see that $\Phi_{\leq j}v$ is defined on a neighborhood of Ω_j if δ_0 from property (i) above is small enough. It is then a straightforward matter to verify that $\Phi_{\leq j}$ satisfies the regularization, difference and error bounds in Proposition 2.6.2 when s and α are integers (the latter two bounds requiring the moment conditions, with $N = N(\alpha_0)$). The general bound follows by interpolation.

It remains to construct the regularization operator $\Psi_{\leq j}$ which preserves divergence free functions. We first note that without loss of generality we may assume that $\Gamma_j \in \Lambda_*$ with the regularization bound

$$\|\Gamma_j\|_{C^{k,\beta}} \lesssim_{A,k,\beta} 2^{j(\beta+k-1-\varepsilon)} \tag{2.6.1}$$

for each integer $k \geq 1$ and real number $0 \leq \beta < 1$. Indeed, for large enough j, by working in local coordinates and using standard mollification techniques we can use the uniform $C^{1,\varepsilon}$ regularity of η_Γ to construct a surface $\tilde{\Gamma}_j \in \Lambda_*$ with the bounds (2.6.1) such that $\tilde{\Gamma}_j$ is within distance $\lesssim_A 2^{-j(1+\varepsilon)}$ of Γ. For some small $c > 0$, we can then define a surface Γ_j via the parameterization $\eta_{\Gamma_j} := \eta_{\tilde{\Gamma}_j} + c2^{-j}$. This defines a domain whose boundary has the required regularization bound and which, if δ_0 is small enough, contains all domains within a $\delta_0 2^{-j}$ neighborhood of Ω. Therefore, it suffices to construct $\Psi_{\leq j}$ in the case when Γ_j satisfies (2.6.1). We make this assumption for the remainder of the construction.

Next, we correct $\Phi_{\leq j}v$ by a gradient potential. We define for $v \in H^s_{div}(\Omega)$,

$$\Psi_{\leq j}v := \Phi_{\leq j}v - \nabla\Delta^{-1}_{\Omega_j}(\nabla \cdot \Phi_{\leq j}v),$$

where $\Delta^{-1}_{\Omega_j}$ is the solution operator for the Dirichlet problem with zero boundary data associated to the domain Ω_j.

To prove the regularization bounds for $\Psi_{\leq j}$, we note that because v is divergence free, we have

$$\nabla \cdot \Phi_{\leq j}v(x) = \sum_{k=0}^{n} \int \phi_k(y)\nabla\chi_k(x) \cdot (v(x - 2^{-j}y) - v(x))\, dy. \tag{2.6.2}$$

In other words, no derivatives fall on v or the kernel when taking the divergence. From the above formula, one can easily verify the following bounds for $\nabla \cdot \Phi_{\leq j}v$ for every $s_1, s_2 \geq 0$:

$$\|\nabla \cdot \Phi_{\leq j}v\|_{H^{s_1}(\Omega_j)} \lesssim_A 2^{-js_2}\|v\|_{H^{s_1+s_2}(\Omega)}.$$

To establish the regularization property of $\Psi_{\leq j}$, we use this and (2.6.1) together with the balanced Dirichlet estimate Proposition 2.5.20 to obtain

$$\|\nabla\Delta_{\Omega_j}^{-1}(\nabla\cdot\Phi_{\leq j}v)\|_{H^{s+\alpha}(\Omega_j)}\lesssim_A 2^{j\alpha}\|v\|_{H^s(\Omega)}.$$

Therefore, the regularization bound $\|\Psi_{\leq j}v\|_{H^{s+\alpha}(\Omega_j)}\lesssim_A 2^{j\alpha}\|v\|_{H^s(\Omega)}$ follows immediately. The bounds for $\Psi_{\leq j+1}v-\Psi_{\leq j}v$ and $I-\Psi_{\leq j}v$ are analogous. □

Finally, we note the pointwise analogues of the above estimates.

Proposition 2.6.3. Given the assumptions of Proposition 2.6.2, the regularization operator $\Psi_{\leq j}$ satisfies the following pointwise bounds for $0\leq\alpha<2$:

$$\|\Psi_{\leq j}v\|_{C^\alpha(\Omega_j)}\lesssim_A 2^{j\beta}\|v\|_{C^{\alpha-\beta}(\Omega)},$$

for $0\leq\beta\leq\alpha$, and

$$\|(I-\Psi_{\leq j})v\|_{C^\alpha(\Omega)}+\|(\Psi_{\leq j+1}-\Psi_{\leq j})v\|_{C^\alpha(\Omega_{j+1})}\lesssim_A 2^{-j\beta}\|v\|_{C^{\alpha+\beta}(\Omega)},$$

for $\beta\geq 0$.

Proof. The corresponding bounds for $\Phi_{\leq j}$ are straightforward to directly verify. To estimate the gradient correction, we again may assume without loss of generality the bound (2.6.1) and then use the pointwise estimates from Proposition 2.5.14 and Proposition 2.5.15. □

Frequency envelopes

Let $\Gamma\in\Lambda_*$ and let $s>\frac{d}{2}+1$. Suppose that $v\in H^s(\Omega)$ and suppose that $\Gamma\in H^s$ is parameterized in collar coordinates by $x\mapsto x+\eta_\Gamma(x)\nu(x)$. At this point, we define $A:=\|\Gamma\|_{C^{1,\varepsilon}}+\|v\|_{C^{\frac{1}{2}}(\Omega)}$. Using the extension operator from Proposition 2.5.12, we have the following Littlewood-Paley decomposition for a function v defined on Ω:

$$v=\sum_{j\geq 0}P_jv,$$

where by abuse of notation P_jv is interpreted to mean $P_jE_\Omega v$ where E_Ω is as in Proposition 2.5.12 and P_0 is to be interpreted as $P_{\leq 0}$. We also have a corresponding Littlewood-Paley type decomposition for functions on Γ_*. Indeed, denote by $\langle D\rangle_*:=(I-\Delta_{\Gamma_*})^{\frac{1}{2}}$. For functions on Γ_*, we then write for $j>0$, $P_j:=\varphi(2^{-j}\langle D\rangle_*)-\varphi(2^{-j+1}\langle D\rangle_*)$ and $P_0:=\varphi(\langle D\rangle_*)$ where

$\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi = 1$ on the unit ball and with support in $B_2(0)$. We then have from Proposition 2.5.12 the almost orthogonality

$$\|(v, \Gamma)\|_{\mathbf{H}^s}^2 \approx_A \sum_{j \geq 0} 2^{2js} \left(\|P_j v\|_{L^2(\mathbb{R}^d)}^2 + \|P_j \eta_\Gamma\|_{L^2(\Gamma_*)}^2 \right).$$

The above equivalence will allow us to define $\mathbf{H}^s$ frequency envelopes for states $(v, \Gamma) \in \mathbf{H}^s$ with the l^2 decay required to establish our continuous dependence result as well as the continuity of solutions with values in $\mathbf{H}^s$ later on.

Remark 2.6.4. To define the Littlewood-Paley decomposition above, we use the extension E_Ω from Proposition 2.5.12 (as opposed to, e.g., the Stein extension) because of its transparent continuous dependence on the domain. This will be important for establishing continuous dependence of solutions to the free boundary Euler equations with respect to the data when we have to compare frequency envelopes for different initial data.

Definition 2.6.5 (Frequency envelopes)**.** Let $s > \frac{d}{2} + 1$, $\Gamma \in \Lambda_*$ and $(v, \Gamma) \in \mathbf{H}^s$. An $\mathbf{H}^s$ *frequency envelope* for the pair (v, Γ) is a positive sequence c_j such that for each $j \geq 0$,

$$\|P_j v\|_{H^s(\mathbb{R}^d)} + \|P_j \eta_\Gamma\|_{H^s(\Gamma_*)} \lesssim_A c_j \|(v, \Gamma)\|_{\mathbf{H}^s}, \qquad \|c_j\|_{l^2} \lesssim_A 1.$$

We say that the sequence $(c_j)_j$ is admissible if $c_0 \approx_A 1$ and it is slowly varying,

$$c_j \leq 2^{\delta|j-k|} c_k, \qquad j, k \geq 0, \qquad 0 < \delta \ll 1.$$

We can always define an admissible frequency envelope by the formula

$$c_j = 2^{-\delta j} + (1 + \|(v, \Gamma)\|_{\mathbf{H}^s})^{-1} \max_k 2^{-\delta|j-k|} \left(\|P_k v\|_{H^s(\mathbb{R}^d)} + \|P_k \eta_\Gamma\|_{H^s(\Gamma_*)} \right). \tag{2.6.3}$$

Unless otherwise stated, we will take this as our formula for c_j. The following proposition will be useful in our construction of rough solutions later on as well as for proving continuity of the data-to-solution map.

Proposition 2.6.6. Let $\Gamma \in \Lambda_*$ and let $s > \frac{d}{2} + 1$. Suppose that $(v, \Gamma) \in \mathbf{H}^s$ and let $(c_j)_j$ be its associated admissible frequency envelope. Then there exists a family of regularized domains Ω_j with boundaries $\Gamma_j \in \Lambda_*$ and $\Gamma_j \in H^s$ along with associated divergence free regularizations $v_j := \Psi_{\leq j} v$ defined on a 2^{-j} enlargement of $\Omega_j \cup \Omega$ such that the following holds.

(i) (Good pointwise approximation).

$$(v_j, \Gamma_j) \to (v, \Gamma) \quad in\ C^1 \times C^{1,\frac{1}{2}} \quad as\ j \to \infty.$$

(ii) (Uniform bound).

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^s} \lesssim_A \|(v, \Gamma)\|_{\mathbf{H}^s}.$$

(iii) (Higher regularity).

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^{s+\alpha}} \lesssim_A 2^{j\alpha} c_j \|(v, \Gamma)\|_{\mathbf{H}^s}, \quad \alpha > 1.$$

(iv) (Low frequency difference bounds). On a 2^{-j} enlarged neighborhood of $\Omega_j \cup \Omega_{j+1}$, there holds

$$\|(v_j, \eta_{\Gamma_j}) - (v_{j+1}, \eta_{\Gamma_{j+1}})\|_{L^2 \times L^2} \lesssim_A 2^{-js} c_j \|(v, \Gamma)\|_{\mathbf{H}^s}.$$

Proof. We define Γ_j by the graph parameterization $\eta_{\Gamma_j} = P_{\leq j} \eta_\Gamma$ (using the projections defined above). By Sobolev embedding, we have $|\eta_{\Gamma_j} - \eta_\Gamma| \lesssim 2^{-\frac{3}{2}j}$, and so the existence of the required divergence free regularization $v_j := \Psi_{\leq j} v$ comes from Proposition 2.6.2.

Next, we turn to verifying the above four properties. We focus on the bounds for v_j as the bounds for Γ_j are similar (and simpler). Properties (i) and (ii) are clear from Sobolev embedding and Proposition 2.6.2. Next, we turn to property (iii). We begin by establishing this property for $\Phi_{\leq j} v$ and then we will upgrade to the full divergence free regularization $v_j = \Psi_{\leq j} v$. We write w^l as shorthand for $P_l w$ and begin by splitting

$$\|\Phi_{\leq j} v\|_{H^{s+\alpha}} \leq \sum_{l \leq j} \|\Phi_{\leq j} v^l\|_{H^{s+\alpha}} + \sum_{l > j} \|\Phi_{\leq j} v^l\|_{H^{s+\alpha}}.$$

For $l \leq j$, we estimate

$$\|\Phi_{\leq j} v^l\|_{H^{s+\alpha}} \lesssim_A \|v^l\|_{H^{s+\alpha}} \lesssim_A 2^{l\alpha} c_l \|(v, \Gamma)\|_{\mathbf{H}^s} \lesssim_A 2^{j\alpha} c_j 2^{(\alpha-\delta)(l-j)} \|(v, \Gamma)\|_{\mathbf{H}^s}.$$

For $l > j$, we estimate

$$\|\Phi_{\leq j} v^l\|_{H^{s+\alpha}} \lesssim_A 2^{j(\alpha+s)} \|v^l\|_{L^2} \lesssim_A 2^{j\alpha} c_j 2^{(j-l)(s-\delta)} \|(v, \Gamma)\|_{\mathbf{H}^s}.$$

Summing up each contribution gives

$$\|\Phi_{\leq j} v\|_{H^{s+\alpha}} \lesssim_A 2^{j\alpha} c_j \|(v, \Gamma)\|_{\mathbf{H}^s}.$$

To obtain the corresponding bound for $\Psi_{\leq j}$, we simply note that by Proposition 2.5.20,

$$\|\nabla \Delta^{-1} \nabla \cdot \Phi_{\leq j} v\|_{H^{s+\alpha}} \lesssim_A \|\Phi_{\leq j} v\|_{H^{s+\alpha}} + 2^{j(s+\alpha-\varepsilon)} \|\nabla \cdot \Phi_{\leq j} v\|_{L^2}.$$

By (2.6.2), we have $2^{j(s+\alpha)}\|\nabla\cdot\Phi_{\leq j}v\|_{L^2}\lesssim_A 2^{j\alpha}\|v\|_{H^s}$. Therefore, if we choose δ in the definition of c_j so that $2^{-j\varepsilon}\leq c_j$, we have

$$\|\Psi_{\leq j}v\|_{H^{s+\alpha}}\lesssim_A 2^{j\alpha}c_j\|(v,\Gamma)\|_{\mathbf{H}^s}.$$

This establishes property (iii) for $\Psi_{\leq j}v$. The proof of property (iv) is similar except now one can use the difference and error bounds in Proposition 2.6.2. We omit the details. □

2.7 Higher energy bounds

Let $k>\frac{d}{2}+1$ be an integer. Our aim in this section is to establish control of the $\mathbf{H}^k$ norm of (v,Γ) in terms of the initial data where the growth of these norms is dictated by the pointwise control parameters A and B below. To accomplish this, we will first construct a coercive energy functional $(v,\Gamma)\mapsto E^k(v,\Gamma)$ associated to each integer $k>\frac{d}{2}+1$ and then we will prove energy estimates for $E^k(v,\Gamma)$ to obtain estimates for $\|(v,\Gamma)\|_{\mathbf{H}^k}$ when (v,Γ) is a solution to the free boundary Euler equations. More precisely, we prove the following theorem.

Theorem 2.7.1. Let $s\in\mathbb{R}$ with $s>\frac{d}{2}+1$ and let $k>\frac{d}{2}+1$ be an integer. Fix a collar neighborhood $\Lambda(\Gamma_*,\varepsilon,\delta)$ with $\delta>0$ sufficiently small. Then for Γ restricted to Λ_* there exists an energy functional $(v,\Gamma)\mapsto E^k(v,\Gamma)$ such that

(i) (Energy coercivity).
$$E^k(v,\Gamma)\approx_A 1+\|(v,\Gamma)\|^2_{\mathbf{H}^k}. \tag{2.7.1}$$

(ii) (Energy propagation). If, in addition to the above, $(v,\Gamma)=(v(t),\Gamma_t)$ is a solution to the free boundary Euler equations, then $E^k(t):=E^k(v(t),\Gamma_t)$ satisfies
$$\frac{d}{dt}E^k\lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})E^k.$$

Here, $A:=1+|\Omega|+\|v\|_{C^{\frac{1}{2}+\varepsilon}(\Omega)}+\|\Gamma\|_{C^{1,\varepsilon}}$ and $B:=1+\|v\|_{W^{1,\infty}(\Omega)}+\|\Gamma\|_{C^{1,\frac{1}{2}}}$.

By Grönwall's inequality, this gives the single and double exponential bounds

$$\|(v(t),\Gamma_t)\|^2_{\mathbf{H}^k}\lesssim_A \exp\left(\int_0^t C_AB(s)\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})ds\right)(1+\|(v_0,\Gamma_0)\|^2_{\mathbf{H}^k}).$$
$$\|(v(t),\Gamma_t)\|^2_{\mathbf{H}^k}\lesssim_A \exp\left(\log(C_A(1+\|(v_0,\Gamma_0)\|^2_{\mathbf{H}^k}))\exp\int_0^t C_AB(s)\,ds\right)$$

for all integers $k>\frac{d}{2}+1$.

Remark 2.7.2. It is important to note that the first part of Theorem 2.7.1 does not make any reference to the dynamical problem.

Constructing the energy functional

Before establishing the above theorem, we motivate our choice of energy. At this point, the discussion will be heuristic only. There are two quantities to control; namely, the H^k norms of v and Γ. However, these are coupled via the nonlinear evolution, so they must be measured in tandem. We achieve this by working instead with well-chosen *good variables*, which are selected as follows:

i) The vorticity ω. If v is a divergence free vector field on Ω, then in Euclidean coordinates, we have the following relation for Δv_i:

$$\Delta v_i = -\partial_j \omega_{ij},$$

where ω denotes the curl of v. Therefore, v is controlled by ω and a suitable boundary value. However, it turns out to be simpler to view v as the solution to a div-curl system, again with a boundary condition whose choice will be addressed shortly.

ii) The Taylor coefficient a. This variable is used to describe the regularity of the boundary. Precisely, as we will see later, we have the approximate relation

$$\mathcal{N} a \approx a\kappa$$

where κ represents the mean curvature of Γ. Thus, as long as the Taylor sign condition remains satisfied, the H^k norm of Γ should be comparable at leading order to the H^{k-1} norm of a.

iii) The material derivative of the Taylor coefficient, $D_t a$. At leading order this provides information about v via the approximate paradifferential relation

$$D_t a \approx \mathcal{N} T_n v,$$

for a suitable representation of the paraproduct above. This will provide the needed boundary condition for the div-curl system for v.

Thus, at the principal level we have the correspondence

$$v \leftrightarrow (\omega, D_t a), \qquad \Gamma \leftrightarrow a,$$

which will be the basis for our coercivity property. For the first part, it is better to think of v as solving a div-curl system. One might try to think of a rotational/irrotational decomposition $v = v_{rot} + v_{ir}$, where the two components solve div-curl systems as follows:

$$\begin{cases} \operatorname{curl} v_{rot} = \omega, \\ \nabla \cdot v_{rot} = 0, \\ v_{rot} \cdot n_\Gamma = 0 \quad \text{on } \Gamma, \end{cases} \qquad \begin{cases} \operatorname{curl} v_{ir} = 0, \\ \nabla \cdot v_{ir} = 0, \\ v_{ir} \cdot n_\Gamma = v \cdot n_\Gamma \quad \text{on } \Gamma. \end{cases}$$

Unfortunately, such a decomposition is not well-suited for our present problem, essentially due to the fact that in our setting n_Γ has less regularity than v on the free boundary; namely, H^{k-1} versus $H^{k-\frac{1}{2}}$. Hence, we cannot use such a decomposition directly, though a paradifferential form of it will appear later in our existence proof. Instead, we will bypass this difficulty by associating the $D_t a$ variable with $\nabla^\top v \cdot n_\Gamma$, the normal component of the tangential derivatives on the boundary, which will then play the role of the boundary condition in the div-curl system for v. This, in turn, yields the v part of the coercivity bound.

Now we turn our attention to the dynamical side, which ultimately determines the choice of the good variables. There we separate the good variables differently, into the vorticity $\omega \in H^{k-1}(\Omega)$ on one hand, which will provide the interior component of the energy, and the pair $(a, D_t a)$ in $H^{k-1}(\Gamma) \times H^{k-\frac{3}{2}}(\Gamma)$, which carries the boundary component of the energy. For the vorticity, this is immediately clear from the equation

$$D_t \omega_{ij} = -\omega_{ik} \partial_j v_k + \omega_{jk} \partial_i v_k, \tag{2.7.2}$$

which results from taking curl of (2.1.1). Based on the transport structure of the vorticity, it is natural to include the quantity $\|\omega\|^2_{H^{k-1}(\Omega)}$ as a component of the energy. On the other hand, it turns out that $\|(a, D_t a)\|^2_{H^{k-1}(\Gamma) \times H^{k-\frac{3}{2}}(\Gamma)}$ can be controlled by the linearized energy $E_{lin}(w_k, s_k)$, where s_k and w_k solve the linearized equation to leading order with

$$\begin{cases} w_k = \nabla \mathcal{H} \mathcal{N}^{k-2} D_t a, \\ s_k = \mathcal{N}^{k-1} a. \end{cases}$$

The derivation for this is a bit more involved than for the vorticity and will be handled later.

With the above discussion in mind, we define our energy as follows:

$$E^k(v, \Gamma) := 1 + \|v\|^2_{L^2(\Omega)} + \|\omega\|^2_{H^{k-1}(\Omega)} + E_{lin}(w_k, s_k). \tag{2.7.3}$$

In the sequel, we will sometimes refer to $\|\omega\|^2_{H^{k-1}(\Omega)}$ as the rotational part of the energy, denoted by $E^k_r(v,\Gamma)$, and $E_{lin}(w_k,s_k)$ as the irrotational part of the energy, denoted by $E^k_i(v,\Gamma)$.

Remark 2.7.3. This definition of the energy has to be interpreted in a suitable way when v and Γ do not solve the free boundary Euler equations. Indeed, it is important that, a priori, the definition of the energy functional does not depend on the dynamics of the problem. Therefore, for a bounded connected domain Ω with $(v,\Gamma)\in\mathbf{H}^k$, we define p through the boundary condition $p_{|\Gamma}=0$ and the Laplace equation

$$\Delta p = -\mathrm{tr}(\nabla v)^2.$$

The Taylor sign term is then defined via

$$a := -n_\Gamma\cdot\nabla p_{|\Gamma}.$$

Moreover, we define D_tp through the Dirichlet boundary condition $D_tp_{|\Gamma}=0$ and ΔD_tp given by

$$\Delta D_tp = 4\mathrm{tr}(\nabla^2 p\cdot\nabla v)+2\mathrm{tr}((\nabla v)^3)+\Delta v\cdot\nabla p =: F. \tag{2.7.4}$$

In other words, $D_tp=\Delta^{-1}F$. This is the definition of D_tp which is compatible with the dynamical problem. We then define $D_t\nabla p$ by

$$D_t\nabla p := -\nabla v\cdot\nabla p+\nabla D_tp$$

and then D_ta by

$$D_ta := -n_\Gamma\cdot D_t\nabla p_{|\Gamma}.$$

With these definitions, the energy functional (2.7.3) is well-defined, irrespective of whether the state (v,Γ) evolves dynamically.

Remark 2.7.4. We note that the energy functional (2.7.3) is essentially the same as that from [134]. The main difference, so far, is in the derivation of this energy. Indeed, our approach was to identify Alinhac style good unknowns, whereas [134] first derives a wave-type equation for a and then applies powers of the Dirichlet-to-Neumann operator to this equation, as if it were a vector field. However, as can be immediately inferred from the low regularity of our control norms, the way we treat the energy is very different from [134].

Coercivity of the energy functional

We begin by establishing the coercivity part of Theorem 2.7.1. That is, we want to show that

$$E^k(v,\Gamma) \approx_A 1 + \|(v,\Gamma)\|^2_{\mathbf{H}^k}.$$

We begin by collecting some preliminary estimates for the various quantities that will appear in our analysis.

L^∞ estimates for coercivity

Here we will establish some L^∞ based estimates for p and D_tp in terms of the control parameter A. The A control parameter involves only the physical variables v and Γ. The variables p and D_tp are related to these variables through solving a suitable Laplace equation. We will therefore need to make use of the Schauder type estimates in Proposition 2.5.15 to control these terms (in suitable pointwise norms) by A. For this, we have the following lemma.

Lemma 2.7.5. Given the assumptions of Theorem 2.7.1, the following pointwise estimates for p and D_tp hold.

(i) ($C^{1,\varepsilon}$ estimate for p).

$$\|p\|_{C^{1,\varepsilon}(\Omega)} \lesssim_A 1.$$

(ii) (Partition bound for D_tp). There exists a sequence of partitions $D_tp =: F_j^1 + F_j^2$ such that

$$\|F_j^1\|_{W^{1,\infty}(\Omega)} \lesssim_A 2^{j(\frac{1}{2}-\varepsilon)}, \quad \|F_j^2\|_{H^1(\Omega)} \lesssim_A 2^{-j(k-1-\varepsilon)}(\|v\|_{H^{k-\varepsilon}(\Omega)} + \|p\|_{H^{k+\frac{1}{2}-\varepsilon}(\Omega)}).$$

One can loosely think of the partition of D_tp in the second part of Lemma 2.7.5 as a splitting of D_tp into low and high frequency parts at a dyadic scale 2^j. The high frequency part will typically be best estimated in L^2 based norms, and the low frequency part in L^∞ based norms. In particular, one can think of the estimate for F_j^1 as an estimate for the "low frequency part" of D_tp in $C^{\frac{1}{2}+\varepsilon}$. This will serve as a substitute for what would be a $C^{\frac{1}{2}+\varepsilon}$ estimate for the inhomogenenous Dirichlet problem, which is not available to us (except for harmonic functions). The usefulness of this will be made more transparent later.

Proof. We begin with some notation. For any integer $l > 0$, we write $\Phi_l := \Phi_{\leq l+1} - \Phi_{\leq l}$ and $\Psi_l := \Psi_{\leq l+1} - \Psi_{\leq l}$. We also write Φ_0 and Ψ_0 to mean $\Phi_{\leq 0}$ and $\Psi_{\leq 0}$, respectively. For a vector or scalar valued function f defined on Ω, we write f^l and $f^{\le l}$ as shorthand for $\Phi_l f$ and $\Phi_{\le l} f$, respectively. If in addition, f is a divergence free vector field, we instead use f^l and $f^{\le l}$ to mean $\Psi_l f$ and $\Psi_{\le l} f$, respectively. This will ensure that the divergence free structure of f is preserved. We abuse notation and write $f^l g^{\le l}$ to mean

$$f^l g^{\le l} := \sum_{l\ge 0} \sum_{0\le m\le l} f^l g^m - \frac{1}{2}\sum_{l\ge 0} f^l g^l.$$

This definition ensures (with the convention that $\Phi_0 = \Phi_{\le 0}$ and $\Psi_0 = \Psi_{\le 0}$) that we have the decomposition

$$fg = f^l g^{\le l} + f^{\le l} g^l, \tag{2.7.5}$$

which can be thought of as a kind of crude bilinear paraproduct decomposition where $f^l g^{\le l}$ selects the portion of fg where f is at higher or comparable frequency compared to g. Likewise, we can define trilinear expressions of the form $f^l g^{\le l} h^{\le l}$ in such a way that we have $fgh = f^l g^{\le l} h^{\le l} + f^{\le l} g^l h^{\le l} + f^{\le l} g^{\le l} h^l$, and similarly for quadrilinear expressions. Now, we begin with the first part of the lemma. Expanding using (2.7.5) we see that

$$p = -\Delta^{-1}\mathrm{tr}(\nabla v)^2 = -2\Delta^{-1}\partial_j(v_i^l \partial_i v_j^{\le l}). \tag{2.7.6}$$

Importantly, because v^l is divergence free, we were able to write $\mathrm{tr}(\nabla v)^2$ as the divergence of a bilinear expression in v and ∇v, where the high frequency factor is undifferentiated. This will allow us to make use of the lower regularity $C^{1,\alpha}$ estimates in Proposition 2.5.15 and simultaneously allow us to rebalance derivatives in the bilinear expression for v. This theme of writing multilinear expressions in divergence form with the highest frequency factor undifferentiated will appear several times in the sequel in more complicated forms. In this case, we have from Proposition 2.5.15,

$$\|p\|_{C^{1,\varepsilon}(\Omega)} \lesssim_A \|v_i^l \partial_i v_j^{\le l}\|_{C^{\varepsilon}(\Omega)} \lesssim_A \|v\|^2_{C^{\frac{1}{2}+\varepsilon}(\Omega)} \lesssim_A 1.$$

Next, we turn to the estimate for $D_t p$, which is the more difficult part. From (2.7.4), we can write in Euclidean coordinates,

$$D_t p = 4\Delta^{-1}(\partial_i\partial_j p \partial_i v_j) + 2\Delta^{-1}(\partial_j v_k \partial_k v_i \partial_i v_j) + \Delta^{-1}(\partial_i \partial_i v_j \partial_j p). \tag{2.7.7}$$

In order to make full use of Proposition 2.5.15, we will again need to write $D_t p$ in the form $\Delta^{-1}\nabla\cdot f$ for some vector field f in a way which allows us to also rebalance derivatives, as we did in the estimate for p. We start by estimating the first term in (2.7.7). We first write $\partial_i\partial_j p\partial_i v_j = \nabla\cdot(\partial_i p\partial_i v)$ and use the partition

$$\Delta^{-1}\nabla\cdot(\partial_i p\partial_i v) = T_j^1 + T_j^2,$$

where $T_j^1 = \Delta^{-1}\nabla\cdot(\partial_i p\partial_i\Phi_{<j}v)$. From Proposition 2.5.15 and the $C^{1,\varepsilon}$ estimate for p above, we have

$$\|T_j^1\|_{W^{1,\infty}(\Omega)} \lesssim_A \|\nabla p\|_{C^\varepsilon(\Omega)}\|\nabla\Phi_{<j}v\|_{L^\infty(\Omega)} + \|\nabla p\|_{L^\infty(\Omega)}\|\nabla\Phi_{<j}v\|_{C^\varepsilon(\Omega)} \lesssim_A 2^{j(\frac{1}{2}-\varepsilon)}.$$

We also see from (2.5.19),

$$\|T_j^2\|_{H^1(\Omega)} \lesssim_A 2^{-j(k-1-\varepsilon)}\|\nabla p\|_{L^\infty(\Omega)}\|v\|_{H^{k-\varepsilon}(\Omega)} \lesssim_A 2^{-j(k-1-\varepsilon)}\|v\|_{H^{k-\varepsilon}(\Omega)}.$$

Next, we turn to the second term in (2.7.7). We start by performing a trilinear frequency decomposition. Using the symmetry of the indices, we have

$$\partial_j v_k\partial_k v_i\partial_i v_j = 3\partial_j v_k^l\partial_k v_i^{\leq l}\partial_i v_j^{\leq l}. \tag{2.7.8}$$

To best balance derivatives, we would like to write this in the form $\nabla\cdot\mathcal{T}(v^l,\nabla v^{\leq l},\nabla v^{\leq l})$ where $\mathcal{T}$ is an appropriate trilinear expression. To do this, we can use the symmetry of the expression and the fact that v is divergence free to write

$$\begin{aligned}\partial_j v_k^l\partial_k v_i^{\leq l}\partial_i v_j^{\leq l} &= \partial_j(v_k^l\partial_k v_i^{\leq l}\partial_i v_j^{\leq l}) - v_k^l\partial_k\partial_j v_i^{\leq l}\partial_i v_j^{\leq l}\\ &= \partial_j(v_k^l\partial_k v_i^{\leq l}\partial_i v_j^{\leq l}) - \frac{1}{2}v_k^l\partial_k(\partial_j v_i^{\leq l}\partial_i v_j^{\leq l})\\ &= \partial_j(v_k^l\partial_k v_i^{\leq l}\partial_i v_j^{\leq l}) - \frac{1}{2}\partial_k(v_k^l\partial_j v_i^{\leq l}\partial_i v_j^{\leq l}).\end{aligned} \tag{2.7.9}$$

We partition the last line above into $Q_j^1 + Q_j^2$ where

$$Q_j^1 := \partial_m(v_k^l\partial_k\Phi_{<j}v_i^{\leq l}\partial_i v_m^{\leq l}) - \frac{1}{2}\partial_k(v_k^l\partial_m\Phi_{<j}v_i^{\leq l}\partial_i v_m^{\leq l}).$$

We then obtain in a straightforward way using Proposition 2.5.15 and summing in l,

$$\|\Delta^{-1}Q_j^1\|_{W^{1,\infty}(\Omega)} \lesssim_A 2^{j(\frac{1}{2}-\varepsilon)}\|v\|^3_{C^{\frac{1}{2}+\varepsilon}(\Omega)} \lesssim_A 2^{j(\frac{1}{2}-\varepsilon)},$$

and from the $H^{-1}\to H^1$ estimate for the Dirichlet problem and Proposition 2.6.2,

$$\|\Delta^{-1}Q_j^2\|_{H^1(\Omega)} \lesssim_A 2^{-j(k-1-\varepsilon)}\|v\|_{H^{k-\varepsilon}(\Omega)}.$$

Finally, the last term in (2.7.7) can be handled by writing

$$\partial_i \partial_i v_j \partial_j p = \partial_i(\partial_i v_j \partial_j p) - \partial_i v_j \partial_i \partial_j p$$

and partitioning each term similarly to the first term in (2.7.7). Collecting all of the above partitions together completes the proof of the lemma. □

The following simple consequence of the above lemma will be useful for estimating $D_t a$ in pointwise norms.

Corollary 2.7.6. Given the assumptions of Lemma 2.7.5, there exists a sequence of partitions $D_t \nabla p = G_j^1 + G_j^2$ such that

$$\|G_j^1\|_{L^\infty(\Omega)} \lesssim_A 2^{j(\frac{1}{2}-\varepsilon)}$$
$$\|G_j^2\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)} \lesssim_A 2^{-j(k-\frac{3}{2}-2\varepsilon)}(\|v\|_{H^{k-\varepsilon}(\Omega)} + \|p\|_{H^{k+\frac{1}{2}-\varepsilon}(\Omega)} + \|D_t \nabla p\|_{H^{k-1-\varepsilon}(\Omega)}).$$

Proof. This follows from Lemma 2.7.5 by taking

$$G_j^1 = \Phi_{<j}(-\nabla \Phi_{<j} v \cdot \nabla p + \nabla F_j^1), \qquad G_j^2 = \Phi_{<j}(-\nabla \Phi_{\geq j} v \cdot \nabla p) + \Phi_{<j} \nabla F_j^2 + \Phi_{\geq j} D_t \nabla p.$$

□

L^2 based estimates for a and $D_t a$

Our next step will be to control $(a, D_t a)$ in $H^{k-1}(\Gamma) \times H^{k-\frac{3}{2}}(\Gamma)$ by the energy plus some lower order terms. Let us define for the rest of this section the lower order quantity

$$\Lambda_{k-\varepsilon} := \|\Gamma\|_{H^{k-\varepsilon}} + \|v\|_{H^{k-\varepsilon}(\Omega)} + \|p\|_{H^{k+\frac{1}{2}-\varepsilon}(\Omega)} + \|D_t \nabla p\|_{H^{k-1-\varepsilon}(\Omega)},$$

where $\varepsilon > 0$ is any small, but fixed, positive constant.

Lemma 2.7.7. We have

$$\|a\|_{H^{k-1}(\Gamma)} + \|D_t a\|_{H^{k-\frac{3}{2}}(\Gamma)} \lesssim_A (E^k)^{\frac{1}{2}} + \Lambda_{k-\varepsilon}.$$

Proof. To control a in $H^{k-1}(\Gamma)$, we use the ellipticity estimate for the Dirichlet-to-Neumann operator from Proposition 2.5.26 to obtain

$$\|a\|_{H^{k-1}(\Gamma)} \lesssim_A \|a\|_{L^2(\Gamma)} + \|\mathcal{N}^{k-1} a\|_{L^2(\Gamma)} + \|\Gamma\|_{H^{k-\varepsilon}} \|a\|_{C^\varepsilon(\Gamma)} \lesssim_A (E^k)^{\frac{1}{2}} + \Lambda_{k-\varepsilon}.$$

To estimate $D_t a$ in $H^{k-\frac{3}{2}}(\Gamma)$, we consider the partition $D_t\nabla p := G_j^1 + G_j^2$ from Corollary 2.7.6 and estimate using Proposition 2.5.26,

$$\|D_t a\|_{H^{k-\frac{3}{2}}(\Gamma)} \lesssim_A \|\mathcal{N}^{k-2} D_t a\|_{H^{\frac{1}{2}}(\Gamma)} + \|\Gamma\|_{H^{k-\varepsilon}} \sup_{j>0} 2^{-j(\frac{1}{2}-\varepsilon)} \|n_\Gamma \cdot G_j^1\|_{L^\infty(\Gamma)}$$
$$+ \sup_{j>0} 2^{j(k-2\varepsilon-\frac{3}{2})} \|n_\Gamma \cdot G_j^2\|_{H^\varepsilon(\Gamma)} + \Lambda_{k-\varepsilon}.$$

From the trace theorem,

$$\|\mathcal{N}^{k-2} D_t a\|_{H^{\frac{1}{2}}(\Gamma)} \lesssim_A \|\mathcal{H}\mathcal{N}^{k-2} D_t a\|_{H^1(\Omega)}.$$

Since $k \geq 3$ and

$$\int_\Gamma \mathcal{N}^{k-2} D_t a \, dS = \int_\Gamma n_\Gamma \cdot \nabla \mathcal{H}\mathcal{N}^{k-3} D_t a \, dS = 0,$$

we conclude by a Poincare type inequality that

$$\|\mathcal{H}\mathcal{N}^{k-2} D_t a\|_{H^1(\Omega)} \lesssim_A \|\nabla \mathcal{H}\mathcal{N}^{k-2} D_t a\|_{L^2(\Omega)} \lesssim_A (E^k)^{\frac{1}{2}}.$$

From Corollary 2.7.6, we have

$$\sup_{j>0} 2^{-j(\frac{1}{2}-\varepsilon)} \|n_\Gamma \cdot G_j^1\|_{L^\infty(\Gamma)} \lesssim_A 1.$$

On the other hand, from the trace theorem and Corollary 2.7.6,

$$2^{j(k-\frac{3}{2}-2\varepsilon)} \|n_\Gamma \cdot G_j^2\|_{H^\varepsilon(\Gamma)} \lesssim_A \Lambda_{k-\varepsilon},$$

which completes the proof. □

With our preliminary estimates in hand, let us proceed with the proof of the first (and harder) half of the coercivity estimate; namely,

$$\|(v,\Gamma)\|_{\mathbf{H}^k} \lesssim_A (E^k)^{\frac{1}{2}}.$$

Let us begin by proving the estimate

$$\|p\|_{H^{k+\frac{1}{2}}(\Omega)} + \|\Gamma\|_{H^k} \lesssim_A (E^k)^{\frac{1}{2}} + \Lambda_{k-\varepsilon}. \tag{2.7.10}$$

We start by recalling from Proposition 2.5.22 that we have

$$\|\Gamma\|_{H^k} + \|n_\Gamma\|_{H^{k-1}(\Gamma)} \lesssim_A 1 + \|\kappa\|_{H^{k-2}(\Gamma)},$$

where κ is the mean curvature of Γ. Therefore, to establish (2.7.10), it suffices to establish the same estimate except with $\|p\|_{H^{k+\frac{1}{2}}(\Omega)} + \|\kappa\|_{H^{k-2}(\Gamma)}$ on the left-hand side. To do this, we begin by relating the curvature to the pressure via the formula

$$\kappa = a^{-1}\Delta p - a^{-1}D^2p(n_\Gamma, n_\Gamma). \tag{2.7.11}$$

Here, we used the fact that $\Delta_\Gamma p = 0$ on Γ. We now estimate each term on the right-hand side of (2.7.11). For the first term, we use the Laplace equation for p and the bilinear frequency decomposition for $\Delta p = -\mathrm{tr}(\nabla v)^2$ as in Lemma 2.7.5 together with Proposition 2.5.9 to obtain

$$\begin{aligned}\|a^{-1}\Delta p\|_{H^{k-2}(\Gamma)} \lesssim_A\ & \|\mathrm{tr}(\nabla v)^2\|_{H^{k-2}(\Gamma)} \\ & + (\|a^{-1}\|_{H^{k-1-\varepsilon}(\Gamma)} + \|\Gamma\|_{H^{k-\varepsilon}}) \sup_{j>0} 2^{-j(1-\varepsilon)}\|\Phi_{<j}\partial_k(v_i^l\partial_i v_k^{\leq l})\|_{L^\infty(\Omega)} \\ & + \sup_{j>0} 2^{j(k-2-\varepsilon)}\|\Phi_{\geq j}\mathrm{tr}(\nabla v)^2\|_{L^2(\Gamma)}.\end{aligned}$$

Using the trace theorem, the product estimates Proposition 2.5.9 and Corollary 2.5.4, the latter two terms can be controlled by $C_A\Lambda_{k-\varepsilon}$ where C_A is a constant depending polynomially on A only. On the other hand, $\|\mathrm{tr}(\nabla v)^2\|_{H^{k-2}(\Gamma)}$ can be controlled using the balanced trace estimate Proposition 2.5.11 as well as Corollary 2.5.4 as follows:

$$\begin{aligned}\|\mathrm{tr}(\nabla v)^2\|_{H^{k-2}(\Gamma)} \lesssim_A\ & \|\mathrm{tr}(\nabla v)^2\|_{H^{k-\frac{3}{2}}(\Omega)} + \|\Gamma\|_{H^{k-\varepsilon}} \sup_{j>0} 2^{-j(1-\varepsilon)}\|\Phi_{<j}\partial_k(v_i^l\partial_i v_k^{\leq l})\|_{L^\infty(\Omega)} \\ & + \sup_{j>0} 2^{j(k-\frac{3}{2}-\varepsilon)}\|\Phi_{\geq j}\mathrm{tr}(\nabla v)^2\|_{L^2(\Omega)} \\ \lesssim_A\ & \Lambda_{k-\varepsilon}.\end{aligned}$$

To estimate $a^{-1}D^2p(n_\Gamma, n_\Gamma)$ in $H^{k-2}(\Gamma)$, we proceed similarly by starting with Proposition 2.5.9 and Lemma 2.7.5 to obtain

$$\begin{aligned}\|a^{-1}D^2p(n_\Gamma, n_\Gamma)\|_{H^{k-2}(\Gamma)} \lesssim_A\ & \|D^2p(n_\Gamma, n_\Gamma)\|_{H^{k-2}(\Gamma)} + \sup_{j>0} 2^{j(k-2-\varepsilon)}\|\Phi_{\geq j}D^2p\|_{L^2(\Gamma)} \\ & + (\|a^{-1}\|_{H^{k-1-\varepsilon}(\Gamma)} + \|\Gamma\|_{H^{k-\varepsilon}}) \sup_{j>0} 2^{-j(1-\varepsilon)}\|\Phi_{<j}D^2p\|_{L^\infty(\Omega)}.\end{aligned}$$

Similarly to the previous estimate, the latter two terms are controlled by $C_A\Lambda_{k-\varepsilon}$. For the term involving $D^2p(n_\Gamma, n_\Gamma)$, we use Proposition 2.5.9 again, combined with the estimates $\|n_\Gamma\|_{H^{k-1-\varepsilon}(\Gamma)} \lesssim_A \|\Gamma\|_{H^{k-\varepsilon}}$ and $\|n_\Gamma\|_{C^\varepsilon(\Gamma)} \lesssim_A 1$ to obtain (similarly to the above estimate but with a^{-1} replaced by n_Γ)

$$\|D^2p(n_\Gamma, n_\Gamma)\|_{H^{k-2}(\Gamma)} \lesssim_A \|D^2p\|_{H^{k-2}(\Gamma)} + \Lambda_{k-\varepsilon}.$$

Proposition 2.5.11 and the same partition of D^2p above then yields

$$\|D^2 p\|_{H^{k-2}(\Gamma)} \lesssim_A \|\nabla p\|_{H^{k-\frac{1}{2}}(\Omega)} + \Lambda_{k-\varepsilon}.$$

To complete the proof of (2.7.10), we now only need to control ∇p in $H^{k-\frac{1}{2}}$. For this, we use the div-curl estimate Proposition 2.5.27 for ∇p as well as Corollary 2.5.4, Proposition 2.5.9 and Proposition 2.5.31 to obtain

$$\begin{aligned} \|\nabla p\|_{H^{k-\frac{1}{2}}(\Omega)} &\lesssim_A \|\nabla p\|_{L^2(\Omega)} + \|\nabla^\top a\|_{H^{k-2}(\Gamma)} + \|\mathrm{tr}(\nabla v)^2\|_{H^{k-\frac{3}{2}}(\Omega)} + \|\Gamma\|_{H^{k-\varepsilon}} \|\nabla p\|_{C^\varepsilon(\Omega)} \\ &\lesssim_A (E^k)^{\frac{1}{2}} + \|a\|_{H^{k-1}(\Gamma)} + \Lambda_{k-\varepsilon} \\ &\lesssim_A (E^k)^{\frac{1}{2}} + \Lambda_{k-\varepsilon}, \end{aligned} \tag{2.7.12}$$

where we used Lemma 2.7.7 to go from the second to third line. From this, we finally obtain the estimate (2.7.10). To close the coercivity estimate, it remains to control v in $H^k(\Omega)$ and $D_t \nabla p$ in $H^{k-1}(\Omega)$ by the energy. We first reduce to the estimate

$$\|v\|_{H^k(\Omega)} \lesssim_A (E^k)^{\frac{1}{2}} + \|D_t \nabla p\|_{H^{k-1}(\Omega)} + \Lambda_{k-\varepsilon}.$$

For this, we start by relating the boundary term $\nabla^\top v \cdot n_\Gamma$ to $D_t \nabla p$. Indeed, we have

$$D_t \nabla p = \nabla D_t p - \nabla v \cdot \nabla p.$$

Since $\nabla p = -a n_\Gamma$ and $D_t p = 0$ on Γ, we obtain

$$\nabla^\top v \cdot n_\Gamma = a^{-1} (D_t \nabla p)^\top,$$

and so, since v is divergence free, we have from the div-curl estimate in Proposition 2.5.27,

$$\begin{aligned} \|v\|_{H^k(\Omega)} &\lesssim_A \|v\|_{L^2(\Omega)} + \|\omega\|_{H^{k-1}(\Omega)} + \|a^{-1}(D_t \nabla p)^\top\|_{H^{k-\frac{3}{2}}(\Gamma)} + \|\Gamma\|_{H^{k-\varepsilon}} \|v\|_{C^{\frac{1}{2}+\varepsilon}(\Omega)} \\ &\lesssim_A \|a^{-1}(D_t \nabla p)^\top\|_{H^{k-\frac{3}{2}}(\Gamma)} + (E^k)^{\frac{1}{2}} + \Lambda_{k-\varepsilon}. \end{aligned} \tag{2.7.13}$$

To estimate the first term on the right-hand side of (2.7.13), we use the decomposition $D_t \nabla p = G_j^1 + G_j^2$ from Corollary 2.7.6. By the balanced product and trace estimates Proposition 2.5.9 and Proposition 2.5.11 and a similar analysis to the estimate for $\|\kappa\|_{H^{k-2}(\Gamma)}$, we obtain

$$\begin{aligned} \|a^{-1}(D_t \nabla p)^\top\|_{H^{k-\frac{3}{2}}(\Gamma)} &\lesssim_A \|D_t \nabla p\|_{H^{k-1}(\Omega)} \\ &\quad + (\|a^{-1}\|_{H^{k-1-\varepsilon}(\Gamma)} + \|\Gamma\|_{H^{k-\varepsilon}}) \sup_{j>0} 2^{-j(\frac{1}{2}-\varepsilon)} \|G_j^1\|_{L^\infty(\Omega)} \\ &\quad + \sup_{j>0} 2^{j(k-\frac{3}{2}-2\varepsilon)} \|G_j^2\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)} \lesssim_A \|D_t \nabla p\|_{H^{k-1}(\Omega)} + \Lambda_{k-\varepsilon}. \end{aligned}$$

Finally, we need to show that

$$\|D_t\nabla p\|_{H^{k-1}(\Omega)} \lesssim_A (E^k)^{\frac{1}{2}} + \Lambda_{k-\varepsilon}.$$

For this, we will use the div-curl decomposition for $D_t\nabla p$. The divergence and curl are given by

$$\begin{cases} \nabla\cdot D_t\nabla p = 3\mathrm{tr}(\nabla^2 p\cdot\nabla v) + 2\mathrm{tr}(\nabla v)^3 \text{ in } \Omega, \\ \nabla\times D_t\nabla p = \nabla^2 p\cdot\nabla v - (\nabla v)^*\cdot\nabla^2 p \text{ in } \Omega. \end{cases}$$

Hence, using the div-curl estimate and the partition $D_t\nabla p = G_j^1 + G_j^2$ from Corollary 2.7.6 in conjunction with Corollary 2.5.4, we obtain

$$\begin{aligned}\|D_t\nabla p\|_{H^{k-1}(\Omega)} \lesssim_A\ & \|p\|_{H^{k+\frac{1}{2}}(\Omega)}\|v\|_{C^{\frac{1}{2}}(\Omega)} + \|p\|_{C^{1,\varepsilon}(\Omega)}\|v\|_{H^{k-\varepsilon}(\Omega)} + \|\mathrm{tr}(\nabla v)^3\|_{H^{k-2}(\Omega)} \\ & + \|\Gamma\|_{H^{k-\varepsilon}}\sup_{j>0} 2^{-j(\frac{1}{2}-\varepsilon)}\|G_j^1\|_{L^\infty(\Omega)} + \sup_{j>0} 2^{j(k-\frac{3}{2}-2\varepsilon)}\|G_j^2\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)} + \Lambda_{k-\varepsilon} \\ & + \|\nabla^\top(D_t\nabla p)\cdot n_\Gamma\|_{H^{k-\frac{5}{2}}(\Gamma)}.\end{aligned}$$

Estimating G_j^1 and G_j^2 as before and then using (2.7.12) gives

$$\begin{aligned}\|D_t\nabla p\|_{H^{k-1}(\Omega)} \lesssim_A\ & (E^k)^{\frac{1}{2}} + \|\nabla^\top(D_t\nabla p)\cdot n_\Gamma\|_{H^{k-\frac{5}{2}}(\Gamma)} + \|v\|_{H^{k-\varepsilon}(\Omega)} + \|\Gamma\|_{H^{k-\varepsilon}} \\ & + \|\mathrm{tr}(\nabla v)^3\|_{H^{k-2}(\Omega)} + \Lambda_{k-\varepsilon}.\end{aligned}$$

Using a trilinear frequency decomposition as in Lemma 2.7.5, we obtain easily

$$\|\mathrm{tr}(\nabla v)^3\|_{H^{k-2}(\Omega)} \lesssim_A \|v\|^2_{C^{\frac{1}{2}+\varepsilon}(\Omega)}\|v\|_{H^{k-\varepsilon}(\Omega)} \lesssim_A \Lambda_{k-\varepsilon}.$$

It remains to estimate the boundary term. We compute

$$\nabla^\top(D_t\nabla p)\cdot n_\Gamma = -\nabla^\top D_t a - D_t\nabla p\cdot\nabla^\top n_\Gamma. \tag{2.7.14}$$

By Proposition 2.5.9, Proposition 2.5.31 and using the decomposition $D_t\nabla p = G_j^1 + G_j^2$, the terms in (2.7.14) are controlled in a similar fashion to the above terms by

$$\|\nabla^\top(D_t\nabla p)\cdot n_\Gamma\|_{H^{k-\frac{5}{2}}(\Gamma)} \lesssim_A \|D_t a\|_{H^{k-\frac{3}{2}}(\Gamma)} + \Lambda_{k-\varepsilon} \lesssim_A (E^k)^{\frac{1}{2}} + \Lambda_{k-\varepsilon},$$

where we used Lemma 2.7.7 in the last inequality. Combining everything together, we have

$$\|D_t\nabla p\|_{H^{k-1}(\Omega)} + \|\Gamma\|_{H^k} + \|v\|_{H^k(\Omega)} + \|p\|_{H^{k+\frac{1}{2}}(\Omega)} \lesssim_A (E^k)^{\frac{1}{2}} + \Lambda_{k-\varepsilon}.$$

Using the definition of $\Lambda_{k-\varepsilon}$ and interpolating gives

$$\|D_t\nabla p\|_{H^{k-1}(\Omega)} + \|\Gamma\|_{H^k} + \|v\|_{H^k(\Omega)} + \|p\|_{H^{k+\frac{1}{2}}(\Omega)} \lesssim_A (E^k)^{\frac{1}{2}} + \|v\|_{L^2(\Omega)} + \|p\|_{H^1(\Omega)} + \|D_t\nabla p\|_{L^2(\Omega)}.$$

We can use the H^1 estimate for the Laplace equation for p to estimate

$$\|p\|_{H^1(\Omega)} \lesssim_A \|v\|_{H^1(\Omega)}.$$

Moreover, by writing $D_t\nabla p = \nabla D_t p - \nabla v \cdot \nabla p$, writing $D_t p$ in the form $\Delta^{-1}\nabla \cdot f$ as in the proof of Lemma 2.7.5 and using the $H^{-1} \to H^1$ estimate for Δ^{-1}, we have

$$\|D_t\nabla p\|_{L^2(\Omega)} \lesssim_A \|v\|_{H^1(\Omega)}.$$

Therefore, by interpolation we have

$$\|D_t\nabla p\|_{H^{k-1}(\Omega)} + \|\Gamma\|_{H^k} + \|v\|_{H^k(\Omega)} + \|p\|_{H^{k+\frac{1}{2}}(\Omega)} \lesssim_A (E^k)^{\frac{1}{2}}. \tag{2.7.15}$$

This finally establishes the desired estimate

$$\|(v,\Gamma)\|_{\mathbf{H}^k} \lesssim_A (E^k)^{\frac{1}{2}}.$$

Next, we show the easier part of the coercivity bound; namely,

$$(E^k)^{\frac{1}{2}} \lesssim_A 1 + \|(v,\Gamma)\|_{\mathbf{H}^k}.$$

Clearly, the only nontrivial part is to control the irrotational energy. More precisely, we have to show that

$$\|\nabla\mathcal{H}\mathcal{N}^{k-2}D_t a\|_{L^2(\Omega)} + \|a^{\frac{1}{2}}\mathcal{N}^{k-1}a\|_{L^2(\Gamma)} \lesssim_A 1 + \|(v,\Gamma)\|_{\mathbf{H}^k}. \tag{2.7.16}$$

To establish this, we will need the following L^2 based estimates for p and $D_t p$.

Lemma 2.7.8. The following estimate holds:

$$\|p\|_{H^{k+\frac{1}{2}}(\Omega)} + \|D_t p\|_{H^k(\Omega)} \lesssim_A \|(v,\Gamma)\|_{\mathbf{H}^k}.$$

Proof. First, from the balanced Dirichlet estimate in Proposition 2.5.19, as well as Corollary 2.5.4 and Lemma 2.7.5, we have

$$\|p\|_{H^{k+\frac{1}{2}}(\Omega)} \lesssim_A \|\mathrm{tr}(\nabla v)^2\|_{H^{k-\frac{3}{2}}(\Omega)} + \|\Gamma\|_{H^k}\|p\|_{W^{1,\infty}(\Omega)} \lesssim_A \|(v,\Gamma)\|_{\mathbf{H}^k}.$$

To estimate $D_t p$, recall that we can write $D_t p$ in the form $\Delta^{-1}\nabla \cdot f$. Indeed, similarly to Lemma 2.7.5, we can start by writing

$$D_t p = \Delta^{-1}\partial_i(\partial_i v_j \partial_j p) + 3\Delta^{-1}\partial_i(\partial_j p \partial_j v_i) + 2\Delta^{-1}\mathrm{tr}(\nabla v)^3 =: F_1 + F_2 + F_3. \tag{2.7.17}$$

We now will use Proposition 2.5.19 to estimate each term. We begin with F_1. We use the partition $F_1 = H_j^1 + H_j^2$ where $H_j^1 := \Delta^{-1}\partial_i(\partial_i\Phi_{\leq j}v_k\partial_k p)$ and Proposition 2.5.19 to obtain,

$$\|F_1\|_{H^k(\Omega)} \lesssim_A \|\nabla p\cdot\nabla v\|_{H^{k-1}(\Omega)} + \|\Gamma\|_{H^k}\sup_{j>0} 2^{-\frac{j}{2}}\|H_j^1\|_{W^{1,\infty}(\Omega)} + \sup_{j>0} 2^{j(k-1)}\|H_j^2\|_{H^1(\Omega)}.$$

Using Corollary 2.5.4 and the $H^{k+\frac{1}{2}}$ estimate for p above, we obtain

$$\|\nabla p\cdot\nabla v\|_{H^{k-1}(\Omega)} \lesssim_A \|v\|_{C^{\frac{1}{2}+\varepsilon}(\Omega)}\|p\|_{H^{k+\frac{1}{2}}(\Omega)} + \|p\|_{C^{1,\varepsilon}(\Omega)}\|v\|_{H^k(\Omega)} \lesssim_A \|(v,\Gamma)\|_{\mathbf{H}^k}.$$

We also have from Proposition 2.5.15 and the properties of $\Phi_{\leq j}$,

$$\sup_{j>0} 2^{-\frac{j}{2}}\|H_j^1\|_{W^{1,\infty}(\Omega)} \lesssim_A \|p\|_{C^{1,\varepsilon}(\Omega)}\|v\|_{C^{\frac{1}{2}+\varepsilon}(\Omega)} \lesssim_A 1,$$

and from the $H^{-1}\to H^1$ estimate for Δ^{-1} and Lemma 2.7.5, we have

$$\sup_{j>0} 2^{j(k-1)}\|H_j^2\|_{H^1(\Omega)} \lesssim_A \sup_{j>0} 2^{j(k-1)}\|\nabla p\|_{L^\infty(\Omega)}\|\nabla\Phi_{>j}v\|_{L^2(\Omega)} \lesssim_A \|v\|_{H^k(\Omega)}.$$

Hence,

$$\|F_1\|_{H^k(\Omega)} \lesssim_A \|(v,\Gamma)\|_{\mathbf{H}^k}. \tag{2.7.18}$$

By a very similar analysis, we obtain the same bound (2.7.18) for F_2. To estimate F_3, one uses the decomposition of $\mathrm{tr}(\nabla v)^3$ from (2.7.8) and (2.7.9) and then partitions one of the factors $\nabla v^{\leq l} = \nabla\Phi_{<j}v^{\leq l} + \nabla\Phi_{\geq j}v^{\leq l}$. After that, an estimate similar to F_1 yields the bound (2.7.18) for the term F_3. Therefore,

$$\|D_t p\|_{H^k(\Omega)} \lesssim_A \|(v,\Gamma)\|_{\mathbf{H}^k},$$

as desired. □

Now, returning to the proof of (2.7.16), for the term $\|a^{\frac{1}{2}}\mathcal{N}^{k-1}a\|_{L^2(\Gamma)}$, we have from Lemma 2.5.24 and Proposition 2.5.30,

$$\|a^{\frac{1}{2}}\mathcal{N}^{k-1}a\|_{L^2(\Gamma)} \lesssim_A \|a\|_{H^{k-1}(\Gamma)} + \|a\|_{L^\infty(\Gamma)}\|\Gamma\|_{H^k} \lesssim_A \|a\|_{H^{k-1}(\Gamma)} + \|\Gamma\|_{H^k}.$$

Then from Proposition 2.5.9, Proposition 2.5.11 and Lemma 2.7.8, we have

$$\|a\|_{H^{k-1}(\Gamma)} \lesssim_A \|p\|_{H^{k+\frac{1}{2}}(\Omega)} + \|\Gamma\|_{H^k} \lesssim_A \|(v,\Gamma)\|_{\mathbf{H}^k}.$$

To control the other part of the energy, we first note that by (2.5.20) we have

$$\|\nabla\mathcal{H}\mathcal{N}^{k-2}D_t a\|_{L^2(\Omega)} \lesssim_A \|\mathcal{N}^{k-2}D_t a\|_{H^{\frac{1}{2}}(\Gamma)}.$$

Then we apply Proposition 2.5.30, Proposition 2.5.11 and Proposition 2.5.9, in that order, to obtain

$$\begin{aligned}\|\mathcal{N}^{k-2}D_t a\|_{H^{\frac{1}{2}}(\Gamma)} &\lesssim_A \|D_t\nabla p\|_{H^{k-1}(\Omega)} + \|\Gamma\|_{H^k}\sup_{j>0} 2^{-\frac{j}{2}}\|G_j^1\|_{L^\infty(\Omega)} + \sup_{j>0} 2^{j(k-\frac{3}{2}-2\varepsilon)}\|G_j^2\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)} \\ &\lesssim_A \|D_t\nabla p\|_{H^{k-1}(\Omega)} + \|\Gamma\|_{H^k},\end{aligned}$$

where $D_t\nabla p = G_j^1 + G_j^2$ is the partition from Corollary 2.7.6. We then write $D_t\nabla p = -\nabla v\cdot\nabla p + \nabla D_t p$ and use Corollary 2.5.4 and Lemma 2.7.8 to obtain

$$\|D_t\nabla p\|_{H^{k-1}(\Omega)} \lesssim_A \|(v,\Gamma)\|_{\mathbf{H}^k}.$$

This completes the proof of (2.7.16) and thus the proof of part (i) of Theorem 2.7.1. Next, we turn to part (ii), which is the energy propagation bound.

L^∞ estimates for propagation

Now, we turn to the energy propagation bounds. As in the coercivity estimate, we will need certain L^∞ based estimates for p and $D_t p$, but in norms that have essentially $\frac{1}{2}$ more degrees of regularity compared to Lemma 2.7.5.

Lemma 2.7.9. Given the assumptions of Theorem 2.7.1, the following pointwise estimates for p and $D_t p$ hold.

(i) ($C^{1,\frac{1}{2}}$ estimate for p).

$$\|p\|_{C^{1,\frac{1}{2}}(\Omega)} \lesssim_A B.$$

(ii) ($W^{1,\infty}$ estimate for $D_t p$). Let $s\in\mathbb{R}$ with $s > \frac{d}{2}+1$. Then

$$\|D_t p\|_{W^{1,\infty}(\Omega)} \lesssim_A \log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})B.$$

Proof. We begin with the $C^{1,\frac{1}{2}}$ estimate. We have from Proposition 2.5.15, using the decomposition from (2.7.6) and a similar analysis to the $C^{1,\varepsilon}$ estimate for p,

$$\begin{aligned}\|p\|_{C^{1,\frac{1}{2}}(\Omega)} &\lesssim_A \|\Gamma\|_{C^{1,\frac{1}{2}}}(\|p\|_{C^{1,\varepsilon}(\Omega)} + \|v_i^l\partial_i v_j^{\leq l}\|_{C^\varepsilon(\Omega)}) + \|v_i^l\partial_i v_j^{\leq l}\|_{C^{\frac{1}{2}}(\Omega)} \\ &\lesssim_A \|\Gamma\|_{C^{1,\frac{1}{2}}} + \|v\|_{W^{1,\infty}(\Omega)} \\ &\lesssim_A B.\end{aligned}$$

Now, we turn to the more difficult $W^{1,\infty}$ estimate for $D_t p$. Again, we first recall from (2.7.7) that we have

$$D_t p = 4\Delta^{-1}(\partial_i\partial_j p \partial_i v_j) + 2\Delta^{-1}(\partial_j v_k \partial_k v_i \partial_i v_j) + \Delta^{-1}(\partial_i\partial_i v_j \partial_j p). \tag{2.7.19}$$

Using a very similar analysis to Lemma 2.7.5 (except without the partition of $D_t p$), we can estimate the second term in (2.7.19) in $W^{1,\infty}$ by

$$\|\Delta^{-1}(\partial_j v_k \partial_k v_i \partial_i v_j)\|_{W^{1,\infty}(\Omega)} \lesssim_A B.$$

For the first term in (2.7.19) we have the decomposition

$$\Delta^{-1}(\partial_i\partial_j p \partial_i v_j) = \Delta^{-1}(\partial_i\partial_j p^l \partial_i v_j^{\leq l}) + \Delta^{-1}(\partial_i\partial_j p^{\leq l} \partial_i v_j^l). \tag{2.7.20}$$

The first term in (2.7.20) can be estimated similarly using Proposition 2.5.15 by

$$\|\Delta^{-1}(\partial_i\partial_j p^l \partial_i v_j^{\leq l})\|_{W^{1,\infty}(\Omega)} = \|\Delta^{-1}\partial_j(\partial_i p^l \partial_i v_j^{\leq l})\|_{W^{1,\infty}(\Omega)} \lesssim_A \|p\|_{C^{1,\varepsilon}(\Omega)}\|v\|_{W^{1,\infty}(\Omega)} \lesssim_A B. \tag{2.7.21}$$

For the latter term in (2.7.20), we write

$$\Delta^{-1}(\partial_i\partial_j p^{\leq l} \partial_i v_j^l) = \Delta^{-1}\partial_i(\partial_i\partial_j p^{\leq l} v_j^l) - \Delta^{-1}\partial_j(\partial_i\partial_i p^{\leq l} v_j^l) \tag{2.7.22}$$

and use the fact that the pressure term is at low frequency compared to v and a similar analysis to the above to estimate

$$\|\Delta^{-1}(\partial_i\partial_j p^{\leq l} \partial_i v_j^l)\|_{W^{1,\infty}(\Omega)} \lesssim_A B. \tag{2.7.23}$$

We now focus on the last term in (2.7.19) which will be responsible for the logarithmic loss in the estimate. We begin by writing

$$\partial_i\partial_i v_j \partial_j p = \partial_i\partial_i v_j^l \partial_j p^{\leq l} + \partial_i\partial_i v_j^{\leq l} \partial_j p^l. \tag{2.7.24}$$

For the second term on the right-hand side of (2.7.24), we write

$$\partial_i\partial_i v_j^{\leq l} \partial_j p^l = \partial_j(\partial_i\partial_i v_j^{\leq l} p^l).$$

Again, similarly to the above, we have

$$\|\Delta^{-1}\partial_j(\partial_i\partial_i v_j^{\leq l} p^l)\|_{W^{1,\infty}(\Omega)} \lesssim_A B. \tag{2.7.25}$$

Now, for the first term on the right of (2.7.24) we have,

$$\partial_i\partial_i v_j^l \partial_j p^{\leq l} = \Delta(v_j^l \partial_j p^{\leq l}) + \partial_j(v_j^l \partial_i\partial_i p^{\leq l}) - 2\partial_i(v_j^l \partial_j \partial_i p^{\leq l}). \tag{2.7.26}$$

The latter two terms in (2.7.26) are estimated similarly to (2.7.25). We focus our attention on the first term, which corresponds to estimating $\Delta^{-1}\Delta(v_j^l\partial_j p^{\leq l})$ in $W^{1,\infty}$. We begin by writing

$$\Delta^{-1}\Delta(v_j^l\partial_j p^{\leq l}) = v_j^l \partial_j p^{\leq l} - \mathcal{H}(v_j^l \partial_j p^{\leq l}). \tag{2.7.27}$$

For the first term in (2.7.27) we note that

$$\nabla(v_j^l\partial_j p^{\leq l}) = v_j^l \partial_j \nabla p^{\leq l} + \nabla v_j^l \partial_j p^{\leq l}.$$

From the $C^{1,\varepsilon}$ bound for p from Lemma 2.7.5, we clearly have $\|v_j^l\partial_j \nabla p^{\leq l}\|_{L^\infty(\Omega)} \lesssim_A B$. On the other hand, we have the same estimate for $\nabla v_j^l \partial_j p^{\leq l}$ because

$$\nabla v_j^l \partial_j p^{\leq l} = \nabla v_j \partial_j p - \nabla v_j^{\leq l}\partial_j p^l.$$

This yields the estimate $\|v_j^l\partial_j p^{\leq l}\|_{W^{1,\infty}(\Omega)} \lesssim_A B$. It remains to estimate $\mathcal{H}(v_j^l\partial_j p^{\leq l})$, which is where we incur the logarithmic loss. By the maximum principle, it suffices to estimate $\|\nabla\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)}$. We begin by showing that for each $m \geq 0$

$$\|\Phi_m \nabla\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} \lesssim_A B, \tag{2.7.28}$$

with implicit constant independent of m. Indeed, we have

$$\|\Phi_m \nabla\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} \lesssim \|\Phi_m \nabla\mathcal{H}\Phi_{\leq m}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} + \|\Phi_m \nabla\mathcal{H}\Phi_{> m}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)}.$$

For the first term, we have from the regularization properties of Φ_m and the $C^{1,\varepsilon}$ estimate from Proposition 2.5.15,

$$\begin{aligned}\|\Phi_m \nabla\mathcal{H}\Phi_{\leq m}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} &\lesssim 2^{-\varepsilon m}\|\mathcal{H}\Phi_{\leq m}(v_j^l\partial_j p^{\leq l})\|_{C^{1,\varepsilon}(\Omega)} \lesssim_A 2^{-\varepsilon m}\|\Phi_{\leq m}(v_j^l\partial_j p^{\leq l})\|_{C^{1,\varepsilon}(\Omega)} \\ &\lesssim_A \|v_j^l\partial_j p^{\leq l}\|_{W^{1,\infty}(\Omega)}.\end{aligned}$$

Therefore, similarly to the estimate for $\nabla(v_j^l\partial_j p^{\leq l})$, we have

$$\|\Phi_m \nabla\mathcal{H}\Phi_{\leq m}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} \lesssim_A B.$$

For the other term, we have from the regularization properties of $\Phi_{\leq m}$ and $\Phi_{\geq m}$ and the maximum principle,

$$\begin{aligned}\|\Phi_m \nabla\mathcal{H}\Phi_{> m}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} &\lesssim_A 2^m\|\mathcal{H}\Phi_{> m}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} \leq 2^m\|\Phi_{> m}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} \\ &\lesssim_A \|v_j^l\partial_j p^{\leq l}\|_{W^{1,\infty}(\Omega)}.\end{aligned}$$

Combining everything gives (2.7.28). Now, to prove the full estimate, we fix an integer $m_0 > 0$ to be chosen later and estimate using (2.7.28),

$$\|\nabla\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} \lesssim_A m_0 B + \|\Phi_{\geq m_0}\nabla\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)}. \tag{2.7.29}$$

For the latter term, since $s > \frac{d}{2}+1$, we obtain by Sobolev embedding, the regularization properties of $\Phi_{\geq m_0}$ and the elliptic estimate for $\mathcal{H}$, the estimate

$$\|\Phi_{\geq m_0}\nabla\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} \lesssim_A 2^{-m_0\delta_0}\|\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{H^{s-\varepsilon}(\Omega)} \lesssim_A 2^{-m_0\delta_0}\|(v,\Gamma)\|_{\mathbf{H}^s}^r,$$

where $r \geq 1$ is some integer and $\delta_0 > 0$ is a constant depending on k. Taking $m_0 \approx r\delta_0^{-1}\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})$ and combining everything above with (2.7.29) then yields

$$\|\nabla\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)} \lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s}).$$

This completes the proof of the lemma. □

Remark 2.7.10. It is perhaps worth remarking that by using Proposition 2.5.15 and the maximum principle to estimate $\|\nabla\mathcal{H}(v_j^l\partial_j p^{\leq l})\|_{L^\infty(\Omega)}$ in the above proof in C^ε, we can also easily obtain the bound

$$\|D_t p\|_{W^{1,\infty}(\Omega)} \lesssim_A \|v\|_{C^{1,\varepsilon}(\Omega)}.$$

Of course, we do not want this in our energy estimates as it would force us to forfeit the scale invariant control parameter B.

Proof of energy propagation

Now, we turn to the second part of Theorem 2.7.1. Using (2.7.2) and the coercivity bound (2.7.1) it is straightforward to verify the following energy estimate for the rotational component of the energy:

$$\frac{d}{dt}E_r^k(v(t),\Gamma_t) \lesssim_A BE^k(v(t),\Gamma_t).$$

The main bulk of the work will be in establishing a propagation bound for the irrotational part of the energy. Namely, we want to show that

$$\frac{d}{dt}E_i^k(v(t),\Gamma_t) \lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})E^k(v(t),\Gamma_t).$$

To do this, we start by deriving a wave-type equation for a. The general procedure for deriving this equation is similar to [134]. However, we need to more precisely identify the source terms in order to obtain estimates with the required pointwise control parameters A and B.

We begin our derivation with the simple commutator identity

$$D_t \nabla p = -\nabla v \cdot \nabla p + \nabla D_t p.$$

Applying D_t and performing some elementary algebraic manipulations gives

$$\begin{aligned} D_t^2 \nabla p &= -\nabla D_t v \cdot \nabla p + D_t \nabla D_t p + \nabla v \cdot (\nabla v \cdot \nabla p) - \nabla v \cdot D_t \nabla p \\ &= \frac{1}{2}\nabla |\nabla p|^2 + \nabla D_t^2 p + 2\nabla v \cdot (\nabla v \cdot \nabla p) - 2\nabla v \cdot \nabla D_t p, \end{aligned}$$

where in the last line, we used the Euler equations to write $-\nabla D_t v \cdot \nabla p = \frac{1}{2}\nabla |\nabla p|^2$. As $\Delta p = -\mathrm{tr}(\nabla v)^2$ is lower order, it is natural to further split $\nabla |\nabla p|^2$ as

$$\frac{1}{2}\nabla |\nabla p|^2 = \frac{1}{2}\nabla \mathcal{H} |\nabla p|^2 + \frac{1}{2}\nabla \Delta^{-1}\Delta |\nabla p|^2.$$

From this, we obtain the equation

$$D_t^2 \nabla p - \frac{1}{2}\nabla \mathcal{H}|\nabla p|^2 = \frac{1}{2}\nabla \Delta^{-1}\Delta |\nabla p|^2 + \nabla D_t^2 p + 2\nabla v \cdot (\nabla v \cdot \nabla p) - 2\nabla v \cdot \nabla D_t p =: g. \quad (2.7.30)$$

It will be seen later that g can be thought of as a perturbative source term. In an effort to convert (2.7.30) into an equation for $D_t a$, we take the normal component of the trace on Γ_t to obtain

$$D_t^2 \nabla p \cdot n_{\Gamma_t} - \frac{1}{2}\mathcal{N}(a^2) = g \cdot n_{\Gamma_t}, \quad (2.7.31)$$

where we used the dynamic boundary condition $p|_{\Gamma_t} = 0$ to write $|\nabla p_{|\Gamma_t}|^2 = a^2$. Since D_t is tangent to Γ_t, we have

$$D_t^2 a = -D_t^2 \nabla p \cdot n_{\Gamma_t} - D_t \nabla p \cdot D_t n_{\Gamma_t} = -D_t^2 \nabla p \cdot n_{\Gamma_t} + a|D_t n_{\Gamma_t}|^2. \quad (2.7.32)$$

Note that for the latter equality in (2.7.32), we wrote $D_t \nabla p = -D_t(a n_{\Gamma_t})$ and used that $D_t n_{\Gamma_t}$ is tangent to Γ_t. Combining (2.7.31) and (2.7.32), we obtain the equation

$$D_t^2 a + \frac{1}{2}\mathcal{N}(a^2) = -g \cdot n_{\Gamma_t} + a|D_t n_{\Gamma_t}|^2,$$

which can be further reduced using the Leibniz type formula for $\mathcal{N}$ from (2.5.36) to the equation

$$D_t^2 a + a\mathcal{N} a = f, \quad (2.7.33)$$

where

$$f := -g \cdot n_{\Gamma_t} + a|D_t n_{\Gamma_t}|^2 + n_{\Gamma_t} \cdot \nabla\Delta^{-1}(|\nabla\mathcal{H}a|^2).$$

To propagate $(a, D_t a)$ in $H^{k-1}(\Gamma_t) \times H^{k-\frac{3}{2}}(\Gamma_t)$, one natural idea, in view of the ellipticity of $\mathcal{N}$, would be to use the spectral theorem to apply $\mathcal{N}^{k-\frac{3}{2}}$ to the above equation, and then read off the associated energy for the leading order wave-like equation. This is essentially the approach used in [134]. However, there is a much better choice for our purposes, which comes from instead applying $\nabla\mathcal{H}\mathcal{N}^{k-2}$ to the above equation. The benefit to this is twofold. The most important advantage is that we only have to work with integer powers of $\mathcal{N}$, which will allow us to make use of the balanced elliptic estimates from the previous sections. Secondly, this choice allows us to reinterpret the desired estimate for $(a, D_t a)$ in $H^{k-1}(\Gamma_t) \times H^{k-\frac{3}{2}}(\Gamma_t)$ as an L^2 type estimate for the linearized equation (2.2.8) with perturbative source terms. Indeed, by defining the variables

$$\begin{aligned} w &:= \nabla\mathcal{H}\mathcal{N}^{k-2} D_t a, \\ s &:= \mathcal{N}^{k-1} a, \\ q &:= \mathcal{H}(a\mathcal{N}^{k-1} a), \end{aligned}$$

we may interpret (w, s, q) to leading order as a solution to the linearized system (2.2.8). To verify this, note that we clearly have $\nabla \cdot w = 0$. Moreover, we observe that $q|_{\Gamma_t} = as$ and that $w|_{\Gamma_t} \cdot n_{\Gamma_t} = \mathcal{N}^{k-1} D_t a$. Hence,

$$D_t s - w|_{\Gamma_t} \cdot n_{\Gamma_t} = [D_t, \mathcal{N}^{k-1}]a =: \mathcal{R}.$$

We also note that in Ω_t, by using the equation (2.7.33) for a and the Leibniz formula for $\mathcal{N}$,

$$D_t w + \nabla q = \mathcal{Q},$$

where

$$\mathcal{Q} := -\nabla v \cdot w + \nabla[D_t, \mathcal{H}](\mathcal{N}^{k-2} D_t a) + \nabla\mathcal{H}[D_t, \mathcal{N}^{k-2}]D_t a + \nabla\mathcal{H}\mathcal{N}^{k-2} f - \nabla\mathcal{H}[\mathcal{N}^{k-2}, a]\mathcal{N}a. \tag{2.7.34}$$

To summarize the above in a compact form, we can write

$$\begin{cases} D_t w + \nabla q = \mathcal{Q} \text{ in } \Omega_t, \\ \nabla \cdot w = 0 \text{ in } \Omega_t, \\ D_t s - w \cdot n_{\Gamma_t} = \mathcal{R} \text{ on } \Gamma_t, \\ q = as \text{ on } \Gamma_t. \end{cases}$$

The linearized energy estimate from Proposition 2.2.2 combined with Cauchy-Schwarz and Lemma 2.7.9 immediately gives the preliminary bound

$$\frac{d}{dt}E_i^k \lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})E^k + (\|\mathcal{R}\|_{L^2(\Gamma_t)} + \|\mathcal{Q}\|_{L^2(\Omega_t)})(E^k)^{\frac{1}{2}}.$$

It remains to control the source terms $\mathcal{Q}$ and $\mathcal{R}$. This will be where the bulk of the work is situated. Our goal is to show that

$$\|\mathcal{Q}\|_{L^2(\Omega_t)} + \|\mathcal{R}\|_{L^2(\Gamma_t)} \lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.$$

We begin with the estimate for $\mathcal{Q}$. We proceed term by term. Clearly, we have

$$\|\nabla v\cdot w\|_{L^2(\Omega_t)} \lesssim B(E^k)^{\frac{1}{2}}.$$

To handle the second term in the definition of $\mathcal{Q}$, we begin by recalling the simple commutator identity from (2.5.42),

$$[D_t,\mathcal{H}]\psi = \Delta^{-1}\nabla\cdot\mathcal{B}(\nabla v,\nabla\mathcal{H}\psi),$$

where $\mathcal{B}$ is an $\mathbb{R}^d$-valued bilinear form. We then estimate using the $H^{-1}\to H^1$ bound for Δ^{-1} to obtain

$$\|\nabla[D_t,\mathcal{H}](\mathcal{N}^{k-2}D_ta)\|_{L^2(\Omega_t)} \lesssim_A B\|\nabla\mathcal{H}\mathcal{N}^{k-2}D_ta\|_{L^2(\Omega_t)} \lesssim_A B(E^k)^{\frac{1}{2}}.$$

For the third term in (2.7.34), we use the $H^{\frac{1}{2}}(\Gamma_t)\to H^1(\Omega_t)$ bound for $\mathcal{H}$ to obtain

$$\|\nabla\mathcal{H}([D_t,\mathcal{N}^{k-2}]D_ta)\|_{L^2(\Omega_t)} \lesssim_A \|[D_t,\mathcal{N}^{k-2}]D_ta\|_{H^{\frac{1}{2}}(\Gamma_t)}.$$

Then, from the commutator estimate Proposition 2.5.33 we obtain

$$\begin{aligned}\|[D_t,\mathcal{N}^{k-2}]D_ta\|_{H^{\frac{1}{2}}(\Gamma_t)} \lesssim_A\ & \|v\|_{H^k(\Omega_t)}\|D_ta\|_{L^\infty(\Gamma_t)} \\ &+ \|v\|_{W^{1,\infty}(\Omega_t)}\|D_ta\|_{H^{k-\frac{3}{2}}(\Gamma_t)} + \|D_ta\|_{L^\infty(\Gamma_t)}\|\Gamma\|_{H^k} \\ &+ \|v\|_{W^{1,\infty}(\Omega_t)}\|\Gamma\|_{H^k(\Omega_t)}\sup_{j>0}2^{-\frac{j}{2}}\|G_j^1\cdot n_{\Gamma_t}\|_{L^\infty(\Gamma_t)} \\ &+ \|v\|_{W^{1,\infty}(\Omega_t)}\sup_{j>0}2^{j(k-\frac{3}{2}-2\varepsilon)}\|G_j^2\cdot n_{\Gamma_t}\|_{H^\varepsilon(\Gamma_t)},\end{aligned}$$

where G_j^1 and G_j^2 are as in Corollary 2.7.6. Using Lemma 2.7.9, the energy coercivity, Lemma 2.7.7 and (2.7.15), we have

$$\|[D_t,\mathcal{N}^{k-2}]D_ta\|_{H^{\frac{1}{2}}(\Gamma_t)} \lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.$$

Next, we turn to the estimate for $\nabla\mathcal{H}\mathcal{N}^{k-2}f$, which involves the most work. We recall that

$$f := -g\cdot n_{\Gamma_t} + a|D_t n_{\Gamma_t}|^2 + \nabla_n\Delta^{-1}(|\nabla\mathcal{H}a|^2),$$

where g is defined as in (2.7.30). Using the identities $D_t n_{\Gamma_t} = -((Dv)^* n_{\Gamma_t})^\top = -(Dv)^* n_{\Gamma_t} + n_{\Gamma_t}(n_{\Gamma_t}\cdot(Dv)^* n_{\Gamma_t})$ and $|\nabla\mathcal{H}a|^2 = \frac{1}{2}\Delta|\mathcal{H}a|^2$, we may reorganize f into the expression

$$f = \frac{1}{2}\nabla_n\Delta^{-1}\Delta(\mathcal{H}a)^2 - \frac{1}{2}\nabla_n\Delta^{-1}\Delta|\nabla p|^2 - \nabla_n D_t^2 p + M_1 + M_2, \tag{2.7.35}$$

where M_1 is a multilinear expression in n_{Γ_t}, ∇p, ∇v with exactly two factors of ∇v (e.g., from (2.5.35), the term $a|D_t n_{\Gamma_t}|^2$), and M_2 is a multilinear expression in ∇p, ∇v, $\nabla D_t p$ and n_{Γ_t} with a single factor of each of $\nabla D_t p$ and ∇v (e.g., the term $n_{\Gamma_t}\cdot\nabla D_t p\cdot\nabla v$). We will abuse notation slightly and refer to terms of the first type as $M_1(\nabla v,\nabla v)$ and terms of the second type as $M_2(\nabla D_t p,\nabla v)$. Next, we estimate each term in $\nabla\mathcal{H}\mathcal{N}^{k-2}f$, with the expression (2.7.35) for f substituted in.

From Corollary 2.5.32, we have

$$\begin{aligned}\|\nabla\mathcal{H}\mathcal{N}^{k-2}\nabla_n\Delta^{-1}\Delta(\mathcal{H}a)^2\|_{L^2(\Omega_t)} &\lesssim_A \|\mathcal{N}^{k-2}\nabla_n\Delta^{-1}\Delta(\mathcal{H}a)^2\|_{H^{\frac{1}{2}}(\Gamma_t)} \\ &\lesssim_A \|\Gamma_t\|_{H^k}\|\Delta^{-1}\Delta(\mathcal{H}a)^2\|_{C^{\frac{1}{2}}(\Omega_t)} + \|\Delta^{-1}\Delta(\mathcal{H}a)^2\|_{H^k(\Omega_t)}.\end{aligned}$$

By writing $\Delta^{-1}\Delta(\mathcal{H}a)^2 = (\mathcal{H}a)^2 - \mathcal{H}(\mathcal{H}a)^2$ and using the $C^{\frac{1}{2}}$ estimate for $\mathcal{H}$ from Corollary 2.5.16 twice together with the maximum principle, we have

$$\|\Delta^{-1}\Delta(\mathcal{H}a)^2\|_{C^{\frac{1}{2}}(\Omega_t)} \lesssim_A \|\mathcal{H}a\|_{L^\infty(\Omega_t)}\|\mathcal{H}a\|_{C^{\frac{1}{2}}(\Omega_t)} \lesssim_A \|a\|_{C^{\frac{1}{2}}(\Gamma_t)} \lesssim_A B.$$

From Proposition 2.5.19, we obtain also

$$\|\Delta^{-1}\Delta(\mathcal{H}a)^2\|_{H^k(\Omega_t)} \lesssim_A B\|\Gamma_t\|_{H^k} + \|\Delta(\mathcal{H}a)^2\|_{H^{k-2}(\Omega_t)}.$$

Then using that $\Delta(\mathcal{H}a)^2 = 2|\nabla\mathcal{H}a|^2$, we obtain from Corollary 2.5.4,

$$\|\Delta(\mathcal{H}a)^2\|_{H^{k-2}(\Omega_t)} \lesssim \|\mathcal{H}a\|_{C^{\frac{1}{2}}(\Omega_t)}\|\mathcal{H}a\|_{H^{k-\frac{1}{2}}(\Omega_t)} \lesssim_A B\|\mathcal{H}a\|_{H^{k-\frac{1}{2}}(\Omega_t)}.$$

Then from Proposition 2.5.21, Lemma 2.7.7 and the energy coercivity bound (2.7.15), we obtain

$$\|\mathcal{H}a\|_{H^{k-\frac{1}{2}}(\Omega_t)} \lesssim_A \|a\|_{H^{k-1}(\Gamma_t)} + \|\Gamma\|_{H^k}\|a\|_{L^\infty(\Omega_t)} \lesssim_A (E^k)^{\frac{1}{2}}.$$

Therefore,

$$\|\Delta(\mathcal{H}a)^2\|_{H^{k-2}(\Omega_t)} \lesssim_A B(E^k)^{\frac{1}{2}}.$$

Next, we turn to the term $\nabla_n\Delta^{-1}\Delta|\nabla p|^2$ in (2.7.35). The procedure here is similar. Like with the previous estimate, we obtain

$$\|\nabla\mathcal{H}\mathcal{N}^{k-2}\nabla_n\Delta^{-1}\Delta|\nabla p|^2\|_{L^2(\Omega_t)} \lesssim_A \|\Gamma_t\|_{H^k}\|\Delta^{-1}\Delta(|\nabla p|^2)\|_{C^{\frac{1}{2}}(\Omega_t)} + \|\Delta(|\nabla p|^2)\|_{H^{k-2}(\Omega_t)} \tag{2.7.36}$$

and also

$$\|\Delta^{-1}\Delta(|\nabla p|^2)\|_{C^{\frac{1}{2}}(\Omega_t)} \lesssim_A B.$$

Moreover, by expanding $\Delta|\nabla p|^2$ (and some simple manipulations), we have

$$\|\Delta(|\nabla p|^2)\|_{H^{k-2}(\Omega_t)} \lesssim \||\nabla^2 p|^2\|_{H^{k-2}(\Omega_t)} + \||\Delta p|^2\|_{H^{k-2}(\Omega_t)} + \|\nabla p\Delta p\|_{H^{k-1}(\Omega_t)}.$$

Using Corollary 2.5.4 and Lemma 2.7.9, we have for the first two terms

$$\||\nabla^2 p|^2\|_{H^{k-2}(\Omega_t)} + \||\Delta p|^2\|_{H^{k-2}(\Omega_t)} \lesssim_A \|\nabla p\|_{C^{\frac{1}{2}}(\Omega_t)}\|p\|_{H^{k+\frac{1}{2}}(\Omega_t)} \lesssim_A B\|p\|_{H^{k+\frac{1}{2}}(\Omega_t)}.$$

To handle the other term, we use the Laplace equation for p to write

$$\|\nabla p\Delta p\|_{H^{k-1}(\Omega_t)} = \|\nabla p\partial_i v_j\partial_j v_i\|_{H^{k-1}(\Omega_t)}. \tag{2.7.37}$$

Then from (2.5.1), Corollary 2.5.4, Lemma 2.7.5 and Lemma 2.7.8, we have

$$\begin{aligned}\|\nabla p\partial_i v_j\partial_j v_i\|_{H^{k-1}(\Omega_t)} &\lesssim_A \|v\|_{W^{1,\infty}(\Omega_t)}\|\nabla p\partial_i v_j\|_{H^{k-1}(\Omega_t)} + \|\nabla p\partial_i v_j\|_{L^\infty(\Omega_t)}\|v\|_{H^k(\Omega_t)}\\ &\lesssim_A \|v\|_{W^{1,\infty}(\Omega_t)}(\|v\|_{C^{\frac{1}{2}}(\Omega_t)}\|p\|_{H^{k+\frac{1}{2}}(\Omega_t)} + \|p\|_{W^{1,\infty}(\Omega_t)}\|v\|_{H^k(\Omega_t)})\\ &\lesssim_A B\|(v,\Gamma)\|_{\mathbf{H}^k}.\end{aligned} \tag{2.7.38}$$

Combining the above with the energy coercivity (2.7.15), we obtain

$$\|\Delta(|\nabla p|^2)\|_{H^{k-2}(\Omega_t)} \lesssim_A B(E^k)^{\frac{1}{2}}.$$

Next, we turn to the estimate for M_1. We first write $M_1 = M_1'\mathcal{B}$ where M_1' is an $\mathbb{R}$-valued multilinear expression in n_{Γ_t} and ∇p and $\mathcal{B}$ is an $\mathbb{R}$-valued bilinear expression in ∇v. We use the bilinear frequency decomposition $\mathcal{B}(\nabla v,\nabla v) = \mathcal{B}(\nabla v^l, \nabla v^{\leq l}) + \mathcal{B}(\nabla v^{\leq l}, \nabla v^l)$ and consider the partition $\mathcal{B} = \mathcal{B}_j^1 + \mathcal{B}_j^2$ where $\mathcal{B}_j^1 := \mathcal{B}(\nabla\Phi_{<j}v^l, \nabla v^{\leq l}) + \mathcal{B}(\nabla v^{\leq l}, \nabla\Phi_{<j}v^l)$. Then using this partition, the trace inequality, energy coercivity and Proposition 2.5.30, we have

$$\begin{aligned}\|\nabla\mathcal{H}\mathcal{N}^{k-2}M_1\|_{L^2(\Omega_t)} &\lesssim_A \|M_1\|_{H^{k-\frac{3}{2}}(\Gamma_t)} + \|\Gamma_t\|_{H^k}\sup_{j>0} 2^{-\frac{j}{2}}\|\mathcal{B}_j^1\|_{L^\infty(\Omega_t)} \\ &+ \sup_{j>0} 2^{j(k-\frac{3}{2}-2\varepsilon)}\|\mathcal{B}_j^2\|_{H^{\frac{1}{2}+\varepsilon}(\Omega_t)} \\ &\lesssim_A \|M_1\|_{H^{k-\frac{3}{2}}(\Gamma_t)} + \|v\|_{W^{1,\infty}(\Omega_t)}\|v\|_{C^{\frac{1}{2}+\varepsilon}(\Omega_t)}\|\Gamma_t\|_{H^k} \\ &+ \|v\|_{W^{1,\infty}(\Omega_t)}\|v\|_{H^k(\Omega_t)} \\ &\lesssim_A \|M_1\|_{H^{k-\frac{3}{2}}(\Gamma_t)} + B(E^k)^{\frac{1}{2}}.\end{aligned} \tag{2.7.39}$$

Using the same partition as above and Proposition 2.5.9, Proposition 2.5.11 and Lemma 2.7.5, we have

$$\begin{aligned}\|M_1\|_{H^{k-\frac{3}{2}}(\Gamma_t)} &\lesssim_A \|\nabla v\|_{L^\infty(\Omega_t)}\|v\|_{H^k(\Omega_t)} \\ &+ (\|\Gamma_t\|_{H^k} + \|M_1'(\nabla p, n_{\Gamma_t})\|_{H^{k-1}(\Gamma_t)})\sup_{j>0} 2^{-\frac{j}{2}}\|\mathcal{B}_j^1\|_{L^\infty(\Omega_t)} \\ &+ \sup_{j>0} 2^{j(k-\frac{3}{2}-2\varepsilon)}\|\mathcal{B}_j^2\|_{\mathcal{H}^{\frac{1}{2}+\varepsilon}(\Omega_t)}.\end{aligned}$$

Estimating as in (2.7.39), this simplifies to

$$\|M_1\|_{H^{k-\frac{3}{2}}(\Gamma_t)} \lesssim_A B(E^k)^{\frac{1}{2}} + B\|M_1'(\nabla p, n_{\Gamma_t})\|_{H^{k-1}(\Gamma_t)}.$$

By Proposition 2.5.9, Proposition 2.5.11, Lemma 2.7.8 and the energy coercivity, we have also

$$\|M_1'(\nabla p, n_{\Gamma_t})\|_{H^{k-1}(\Gamma_t)} \lesssim_A (E^k)^{\frac{1}{2}},$$

from which we deduce

$$\|\nabla\mathcal{H}\mathcal{N}^{k-2}M_1\|_{L^2(\Omega_t)} \lesssim_A B(E^k)^{\frac{1}{2}}.$$

Next, we estimate M_2. This estimate is similar to M_1. One starts by writing $M_2 = M_2'\mathcal{B}$ where M_2' is multilinear in ∇p and n_{Γ_t} while $\mathcal{B}$ is bilinear in ∇v and $\nabla D_t p$. Using the partition $\mathcal{B} = \mathcal{B}_j^1 + \mathcal{B}_j^2$ with $\mathcal{B}_j^1 := \mathcal{B}(\nabla\Phi_{<j}v^l, \nabla(D_tp)^{\leq l}) + \mathcal{B}(\nabla v^{\leq l}, \nabla\Phi_{<j}(D_tp)^l)$ and a similar analysis to M_1, we have

$$\begin{aligned}\|\nabla\mathcal{H}\mathcal{N}^{k-2}M_2\|_{L^2(\Omega_t)} &\lesssim_A \|v\|_{W^{1,\infty}(\Omega_t)}\|D_tp\|_{H^k(\Omega_t)} \\ &+ \|D_tp\|_{W^{1,\infty}(\Omega_t)}(\|\Gamma_t\|_{H^k} + \|M_2'(\nabla p, n_{\Gamma_t})\|_{H^{k-1}(\Gamma_t)}) \\ &+ \|D_tp\|_{W^{1,\infty}(\Omega_t)}\|v\|_{H^k(\Omega_t)}.\end{aligned}$$

Then using the $W^{1,\infty}$ bound for $D_t p$ from Lemma 2.7.9 and the H^k bound for $D_t p$ from Lemma 2.7.8, we have

$$\|\nabla \mathcal{H}\mathcal{N}^{k-2} M_2\|_{L^2(\Omega_t)} \lesssim_A B \log(1 + \|(v, \Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.$$

Now we turn to the estimate for the term involving $D_t^2 p$. As usual, we first aim to write it in the form $\Delta^{-1}\nabla \cdot f$ but in such a way that f involves favorable frequency interactions. This presents some mild technical challenges as $D_t^2 p$ will have terms which are up to quadrilinear in ∇v. To deal with this, we have the following lemma.

Lemma 2.7.11. There exist bilinear, trilinear and quadrilinear expressions $\mathcal{B}$, $\mathcal{T}$ and $\mathcal{M}$ taking values in $\mathbb{R}^d$ such that

$$\Delta D_t^2 p = -2\Delta|\nabla p|^2 + \nabla \cdot \mathcal{B}(\nabla D_t p, \nabla v) + \nabla \cdot \mathcal{T}(\nabla p, \nabla v, \nabla v) + \nabla \cdot \mathcal{M}(v^m, \nabla v^{\leq m}, \nabla v^{\leq m}, \nabla v^{\leq m}).$$

Proof. First, using that v is divergence free, it is straightforward to verify

$$\Delta D_t^2 p = \partial_i(\partial_j D_t p \partial_j v_i) + \partial_i(\partial_i v_j \partial_j D_t p) + D_t \Delta D_t p = \nabla \cdot \mathcal{B} + D_t \Delta D_t p.$$

Next, we expand $D_t \Delta D_t p$. We start with the Laplace equation for $D_t p$ from (2.7.17),

$$\Delta D_t p = 3\partial_j(\partial_i p \partial_i v_j) + \partial_i(\partial_i v_j \partial_j p) + 2\partial_j v_k \partial_k v_i \partial_i v_j.$$

Using that v is divergence free, we have the commutator identity $[\partial_i, D_t]f = \partial_j(\partial_i v_j f)$. Combining this with the Euler equations, we obtain

$$\begin{aligned} D_t(3\partial_j(\partial_i p \partial_i v_j) + \partial_i(\partial_i v_j \partial_j p)) &= \nabla \cdot \mathcal{B} + \nabla \cdot \mathcal{T} - 4\partial_j(\partial_i p \partial_i \partial_j p) \\ &= \nabla \cdot \mathcal{B} + \nabla \cdot \mathcal{T} - 2\Delta|\nabla p|^2. \end{aligned}$$

It remains to expand $2D_t(\partial_j v_k \partial_k v_i \partial_i v_j)$. From the Euler equation and symmetry, we have

$$2D_t(\partial_j v_k \partial_k v_i \partial_i v_j) = 6D_t(\partial_j v_k)\partial_k v_i \partial_i v_j = -6\partial_j \partial_k p \partial_k v_i \partial_i v_j - 6\partial_j v_l \partial_l v_k \partial_k v_i \partial_i v_j.$$

We rearrange the first term as

$$\begin{aligned} -6\partial_j \partial_k p \partial_k v_i \partial_i v_j &= -6\partial_j(\partial_k p \partial_k v_i \partial_i v_j) + 6\partial_k p \partial_j \partial_k v_i \partial_i v_j \\ &= -6\partial_j(\partial_k p \partial_k v_i \partial_i v_j) + 3\partial_k p \partial_k(\partial_j v_i \partial_i v_j) \\ &= -6\partial_j(\partial_k p \partial_k v_i \partial_i v_j) + 3\partial_k(\partial_k p \partial_j v_i \partial_i v_j) - 3\partial_k \partial_k p \partial_j v_i \partial_i v_j \\ &= \nabla \cdot \mathcal{T} + 3|\Delta p|^2, \end{aligned} \tag{2.7.40}$$

where in the last line we used the Laplace equation for p. On the other hand, for the second term, by symmetry of the indices, we have the quadrilinear frequency decomposition,

$$\begin{aligned}-6\partial_j v_l \partial_l v_k \partial_k v_i \partial_i v_j &= -24\partial_j v_l^m \partial_l v_k^{\leq m} \partial_k v_i^{\leq m} \partial_i v_j^{\leq m} \\ &= \nabla\cdot\mathcal{M} + 24 v_l^m \partial_l \partial_j v_k^{\leq m} \partial_k v_i^{\leq m} \partial_i v_j^{\leq m} + 24 v_l^m \partial_l v_k^{\leq m} \partial_j \partial_k v_i^{\leq m} \partial_i v_j^{\leq m}.\end{aligned}$$

By symmetry and the fact that v is divergence free, the second term on the right-hand side can be rearranged as

$$24 v_l^m \partial_l \partial_j v_k^{\leq m} \partial_k v_i^{\leq m} \partial_i v_j^{\leq m} = 8 v_l^m \partial_l (\partial_j v_k^{\leq m} \partial_k v_i^{\leq m} \partial_i v_j^{\leq m}) = \nabla\cdot\mathcal{M}.$$

For the third term on the right-hand side, we have

$$\begin{aligned}24 v_l^m \partial_l v_k^{\leq m} \partial_j \partial_k v_i^{\leq m} \partial_i v_j^{\leq m} &= 12 v_l^m \partial_l v_k^{\leq m} \partial_k (\partial_j v_i^{\leq m} \partial_i v_j^{\leq m}) \\ &= \nabla\cdot\mathcal{M} - 12 \partial_k v_l^m \partial_l v_k^{\leq m} \partial_j v_i^{\leq m} \partial_i v_j^{\leq m} \\ &= \nabla\cdot\mathcal{M} - 3\partial_k v_l \partial_l v_k \partial_j v_i \partial_i v_j \\ &= \nabla\cdot\mathcal{M} - 3|\Delta p|^2,\end{aligned} \tag{2.7.41}$$

where we used the Laplace equation for p in the last line. Combining (2.7.40) and (2.7.41) to cancel the $3|\Delta p|^2$ terms then completes the proof of the lemma. □

Now, we return to the estimate for $\nabla\mathcal{HN}^{k-2}\nabla_n D_t^2 p$. We use Lemma 2.7.11 and estimate each term separately. The term $-2\nabla\mathcal{HN}^{k-2}\nabla_n \Delta^{-1}\Delta|\nabla p|^2$ can be estimated identically to (2.7.36). Let us then turn to the estimate for $\nabla\mathcal{HN}^{k-2}\nabla_n\Delta^{-1}(\nabla\cdot\mathcal{B})$. We use a partition $\mathcal{B} = \mathcal{B}_j^1 + \mathcal{B}_j^2$ where $\mathcal{B}_j^1$ is defined as follows: First, we perform the frequency decomposition,

$$\mathcal{B} = \mathcal{B}(\nabla(D_t p)^l, \nabla v^{\leq l}) + \mathcal{B}(\nabla(D_t p)^{\leq l}, \nabla v^l)$$

and then define

$$\mathcal{B}_1^j := \mathcal{B}(\nabla\Phi_{\leq j}(D_t p)^l, \nabla v^{\leq l}) + \mathcal{B}(\nabla(D_t p)^{\leq l}, \nabla\Phi_{\leq j} v^l).$$

Then Corollary 2.5.32 and Proposition 2.5.19 gives

$$\begin{aligned}\|\nabla\mathcal{HN}^{k-2}\nabla_n\Delta^{-1}(\nabla\cdot\mathcal{B})\|_{L^2(\Omega_t)} \lesssim_A \|\mathcal{B}\|_{H^{k-1}(\Omega_t)} + \|\Gamma_t\|_{H^k} \sup_{j>0} 2^{-\frac{j}{2}} \|\Delta^{-1}(\nabla\cdot\mathcal{B}_1^j)\|_{W^{1,\infty}(\Omega_t)} \\ + \sup_{j>0} 2^{j(k-1-\varepsilon)} \|\Delta^{-1}(\nabla\cdot\mathcal{B}_2^j)\|_{H^1(\Omega_t)}.\end{aligned}$$

From Sobolev product estimates and the H^k and L^∞ estimates for $D_t p$,

$$\begin{aligned}\|\mathcal{B}\|_{H^{k-1}(\Omega_t)} &\lesssim_A \|v\|_{W^{1,\infty}(\Omega_t)} \|D_t p\|_{H^k(\Omega_t)} + \|D_t p\|_{W^{1,\infty}(\Omega_t)} \|v\|_{H^k(\Omega_t)} \\ &\lesssim_A B\log(1 + \|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.\end{aligned}$$

Using Proposition 2.5.15, we also estimate

$$2^{-\frac{j}{2}}\|\Delta^{-1}(\nabla\cdot\mathcal{B}_1^j)\|_{C^{1,\varepsilon}(\Omega_t)}\lesssim_A \|D_tp\|_{W^{1,\infty}(\Omega_t)}\|v\|_{C^{\frac{1}{2}+\varepsilon}(\Omega_t)}\lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s}).$$

Finally, using the error bounds for $\Phi_{>j}$ and the L^∞ and H^k estimates for D_tp from Lemma 2.7.8 we see that

$$\begin{aligned}2^{j(k-1-\varepsilon)}\|\Delta^{-1}(\nabla\cdot\mathcal{B}_2^j)\|_{H^1(\Omega_t)}&\lesssim_A \|v\|_{W^{1,\infty}(\Omega_t)}\|D_tp\|_{H^k(\Omega_t)}+\|D_tp\|_{W^{1,\infty}(\Omega_t)}\|v\|_{H^k(\Omega_t)}\\&\lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.\end{aligned}$$

Hence,

$$\|\nabla\mathcal{H}\mathcal{N}^{k-2}\nabla_n\Delta^{-1}(\nabla\cdot\mathcal{B})\|_{L^2(\Omega_t)}\lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.$$

The estimates for $\nabla\mathcal{H}\mathcal{N}^{k-2}\nabla_n\Delta^{-1}(\nabla\cdot\mathcal{T})$ and $\nabla\mathcal{H}\mathcal{N}^{k-2}\nabla_n\Delta^{-1}(\nabla\cdot\mathcal{M})$ are very similar. The main difference is that we use the partition $\mathcal{T}=\mathcal{T}_1^j+\mathcal{T}_2^j$ with

$$\mathcal{T}_1^j=2\mathcal{T}(\nabla p,\nabla\Phi_{\leq j}v^l,\nabla v^{\leq l})$$

and the partition $\mathcal{M}=\mathcal{M}_1^j+\mathcal{M}_2^j$ with

$$\mathcal{M}_1^j:=\mathcal{M}(v^m,\nabla\Phi_{\leq j}v^{\leq m},\nabla v^{\leq m},\nabla v^{\leq m}).$$

Ultimately, we obtain

$$\|\nabla\mathcal{H}\mathcal{N}^{k-2}\nabla_nD_t^2p\|_{L^2(\Omega_t)}\lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}$$

which when combined with the previous analysis gives

$$\|\nabla\mathcal{H}\mathcal{N}^{k-2}f\|_{L^2(\Omega_t)}\lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}$$

as desired. The last term in the estimate for $\mathcal{Q}$ that we need to control is $\nabla\mathcal{H}[\mathcal{N}^{k-2},a]\mathcal{N}a$. For this, we have the following technical lemma.

Lemma 2.7.12. We have the following estimate:

$$\|\nabla\mathcal{H}[\mathcal{N}^{k-2},a]\mathcal{N}a\|_{L^2(\Omega_t)}\lesssim_A B\log(1+\|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.$$

Proof. Thanks to the $H^{\frac{1}{2}}(\Gamma_t)\to H^1(\Omega_t)$ bound for $\mathcal{H}$, it suffices to estimate $\|[\mathcal{N}^{k-2},a]\mathcal{N}a\|_{H^{\frac{1}{2}}(\Gamma_t)}$. We begin by using the Leibniz formula (2.5.36) to expand the commutator,

$$[\mathcal{N}^{k-2},a]\mathcal{N}a=\sum_{n+m=k-3}\mathcal{N}^n(\mathcal{N}a\mathcal{N}^{m+1}a)-2\mathcal{N}^n\nabla_n\Delta^{-1}(\nabla\mathcal{H}a\cdot\nabla\mathcal{H}\mathcal{N}^{m+1}a).\tag{2.7.42}$$

We focus on the latter term in (2.7.42) first as it is a bit more delicate to deal with. To simplify notation slightly, we write

$$a_j := \mathcal{H}\mathcal{N}^j a, \quad F := \nabla a_0 \cdot \nabla a_{m+1}, \quad \mathcal{N}_{<j} := n_{\Gamma_t} \cdot \nabla \Phi_{<j}\mathcal{H}, \quad \mathcal{N}_{\geq j} := n_{\Gamma_t} \cdot \nabla \Phi_{\geq j}\mathcal{H}.$$

Using Corollary 2.5.32 and then Proposition 2.5.19, we have

$$\begin{aligned}\|\mathcal{N}^n(\nabla_n \Delta^{-1} F)\|_{H^{\frac{1}{2}}(\Gamma_t)} \lesssim_A \|F\|_{H^n(\Omega_t)} + \|\Gamma\|_{H^k} \sup_{j>0} 2^{-j(m+\frac{3}{2})} \|\Delta^{-1} F_j^1\|_{W^{1,\infty}(\Omega_t)} \\ + \sup_{j>0} 2^{j(n+1-\varepsilon)} \|\Delta^{-1} F_j^2\|_{H^1(\Omega_t)},\end{aligned}$$

where $F = F_j^1 + F_j^2$ is a suitable partition of F to be chosen. To find a suitable partition, we start with a bilinear frequency decomposition similar to before. We define $a_j^l := \Phi_l a_j$ and $a_j^{\leq l} = \Phi_{\leq l} a_j$.

Remark 2.7.13. We note that the regularization operator $\Phi_{\leq l}$ does not preserve the harmonic property of a_j. However, using the definition of $\Phi_{\leq l}$ (see Section 2.6), the operator defined by $C_{\leq l} := [\Delta, \Phi_{\leq l}]$ is readily seen to satisfy the bounds,

$$\|C_{\leq l}\|_{C^\alpha \to L^\infty} \lesssim_A 2^{l(1-\alpha)} \quad \|C_{\leq l}\|_{H^\alpha \to L^2} \lesssim_A 2^{l(1-\alpha)}, \quad 0 \leq \alpha \leq 1 \tag{2.7.43}$$

for $\alpha, l \geq 0$. That is, $C_{\leq l}$ behaves like a differential operator of order 1 localized at dyadic scale $\lesssim 2^l$.

Now, using the same convention as before in this section (where repeated indices are summed over) we have

$$F = \nabla a_0^l \cdot \nabla a_{m+1}^{\leq l} + \nabla a_0^{\leq l} \cdot \nabla a_{m+1}^l =: F' + F''.$$

We can write F' and F'' to leading order as the divergence of some vector field. Using that a_0 and a_{m+1} are harmonic, we have

$$\begin{aligned} F' &= \nabla \cdot (a_0^l \nabla a_{m+1}^{\leq l}) - a_0^l C_{\leq l} a_{m+1} =: G' + H', \\ F'' &= \nabla \cdot (a_{m+1}^l \nabla a_0^{\leq l}) - a_{m+1}^l C_{\leq l} a_0 =: G'' + H''. \end{aligned} \tag{2.7.44}$$

We will focus on F' first. To choose a partition of F', we need to choose a suitable partition of G' and H'. We show the details for G' and remark later on the minor changes needed to deal with H'. We write $G' = (G')_j^1 + (G')_j^2$ with

$$(G')_j^1 = \nabla \cdot (a_0^l \nabla \Phi_{\leq l} a_{m+1,\leq j}), \quad a_{m+1,\leq j} := \Phi_{\leq j}(\mathcal{H}\mathcal{N}_{<j}^{m+1} a).$$

From Proposition 2.5.15, iterating the maximum principal and using the C^α bounds for $\mathcal{H}$ and the properties of $\Phi_{<j}$, we have

$$2^{-j(m+\frac{3}{2})}\|\Delta^{-1}(G')^1_j\|_{W^{1,\infty}(\Omega_t)} \lesssim_A \|a\|_{C^\varepsilon(\Gamma_t)}\|a\|_{C^{\frac{1}{2}}(\Gamma_t)} \lesssim_A B,$$

where we used Lemma 2.7.5 and Lemma 2.7.9 in the last inequality. For $(G')^2_j$, we can write

$$(G')^2_j = \nabla\cdot(a^l_0\nabla b^{\leq l}_{m+1,j}) + \sum_{0\leq i\leq m}\nabla\cdot(a^l_0\nabla b^{\leq l}_{i,j})$$

where

$$b^{\leq l}_{m+1,j} := \Phi_{\leq l}\Phi_{\geq j}a_{m+1}, \quad b^{\leq l}_{i,j} := \Phi_{\leq l}\Phi_{<j}\mathcal{H}\mathcal{N}^i_{<j}\mathcal{N}_{\geq j}\mathcal{N}^{m-i}a.$$

Using Corollary 2.5.16, the properties of the kernel Φ and the $H^{-1}\to H^1$ bound for Δ^{-1}, we obtain for each $0\leq i\leq m$,

$$\begin{aligned}2^{j(n+1-\varepsilon)}\|\Delta^{-1}\nabla\cdot(a^l_0\nabla b^{\leq l}_{i,j})\|_{H^1(\Omega_t)} &\lesssim_A 2^{j(n+1-\varepsilon)}\|a^l_0\|_{L^\infty(\Omega_t)}\|b^{\leq l}_{i,j}\|_{H^1(\Omega_t)}\\ &\lesssim_A 2^{j(n+1-\varepsilon)}\|a\|_{C^{\frac{1}{2}}(\Gamma_t)}\|\mathcal{H}\mathcal{N}^i_{<j}\mathcal{N}_{\geq j}\mathcal{N}^{m-i}a\|_{H^{\frac{1}{2}+\varepsilon}(\Omega_t)}.\end{aligned} \tag{2.7.45}$$

Repeatedly using the $H^\varepsilon\to H^{\frac{1}{2}+\varepsilon}$ estimate (2.5.20), the properties of Φ, the bound $\|n_{\Gamma_t}\|_{C^\varepsilon(\Gamma_t)}\lesssim_A 1$ and the trace inequality, we can estimate

$$\begin{aligned}2^{j(n+1-\varepsilon)}\|\mathcal{H}\mathcal{N}^i_{<j}\mathcal{N}_{\geq j}\mathcal{N}^{m-i}a\|_{H^{\frac{1}{2}+\varepsilon}(\Omega_t)} &\lesssim_A 2^{j(n+1+i-\varepsilon)}\|\nabla\Phi_{\geq j}\mathcal{H}\mathcal{N}^{m-i}a\|_{H^{\frac{1}{2}+\varepsilon}(\Omega_t)}\\ &\lesssim_A \|\mathcal{H}\mathcal{N}^{m-i}a\|_{H^{n+i+\frac{5}{2}}(\Omega_t)}.\end{aligned}$$

Using Proposition 2.5.30, Lemma 2.7.5, Lemma 2.7.7 and (2.7.15), we have

$$\|\mathcal{H}\mathcal{N}^{m-i}a\|_{H^{n+i+\frac{5}{2}}(\Gamma_t)} \lesssim_A \|a\|_{H^{k-1}(\Gamma_t)} + \|\Gamma\|_{H^k}\|a\|_{C^\varepsilon(\Gamma_t)} \lesssim_A (E^k)^{\frac{1}{2}}.$$

If $n\geq 1$, then doing a similar analysis for the term $\nabla\cdot(a^l_0\nabla b^{\leq l}_{m+1,j})$ and combining this with (2.7.45) and the bound $\|a\|_{C^{\frac{1}{2}}(\Gamma_t)}\lesssim_A B$, we obtain

$$2^{j(n+1-\varepsilon)}\|\Delta^{-1}(G')^2_j\|_{H^1(\Omega_t)} \lesssim_A B(E^k)^{\frac{1}{2}}.$$

If $n=0$, the term $\nabla\cdot(a^l_0\nabla b^{\leq l}_{m+1,j})$ is instead treated slightly differently. For this, we estimate similarly to before,

$$2^{j(1-\varepsilon)}\|\Delta^{-1}\nabla\cdot(a^l_0\nabla b^{\leq l}_{m+1,j})\|_{H^1(\Omega_t)} \lesssim_A \|a\|_{C^{\frac{1}{2}}(\Gamma_t)}\|\mathcal{H}\mathcal{N}^{m+1}a\|_{H^{\frac{3}{2}}(\Omega_t)}.$$

Then we use Proposition 2.5.18 to estimate the last term as

$$\|\mathcal{H}\mathcal{N}^{m+1}a\|_{H^{\frac{3}{2}}(\Omega_t)} \lesssim_A \|\mathcal{N}^{m+1}a\|_{H^1(\Gamma_t)},$$

and then estimate this term by $(E^k)^{\frac{1}{2}}$ similarly to the above. Next, one readily verifies analogous bounds for H', G'' and H'' by using the similar decompositions,

$$\begin{aligned}(H')^1_j &= -a^l_0 C_{\leq l}(a_{m+1,\leq j}), \quad (G'')^1_j = \nabla\cdot(\Phi_l(a_{m+1,\leq j})\nabla a_0^{\leq l}),\\ (H'')^1_j &= -C_{\leq l}a_0\Phi_l(a_{m+1,\leq j}).\end{aligned} \tag{2.7.46}$$

From these bounds, ultimately, we obtain

$$\|\mathcal{N}^n(\nabla_n\Delta^{-1}F)\|_{H^{\frac{1}{2}}(\Gamma_t)} \lesssim_A \|F\|_{H^n(\Omega_t)} + B(E^k)^{\frac{1}{2}}.$$

It remains to estimate F in H^n. We begin by looking at each summand in the bilinear frequency decomposition for F,

$$F_l := \nabla\Phi_l a_0\cdot\nabla\Phi_{\leq l}a_{m+1} + \nabla\Phi_{\leq l}a_0\cdot\nabla\Phi_l a_{m+1}.$$

For the latter term, we have

$$\|\nabla\Phi_{<l}a_0\cdot\nabla\Phi_l a_{m+1}\|_{H^n(\Omega_t)} \lesssim_A \|a\|_{C^{\frac{1}{2}}(\Gamma_t)}\|a_{m+1}\|_{H^{n+\frac{3}{2}}(\Omega_t)},$$

which when $n \geq 1$, we know from the above can be controlled by $B(E^k)^{\frac{1}{2}}$. For $n = 0$, we have the same bound by simply using Proposition 2.5.18. For the other term, we can further decompose

$$a_{m+1} = a^1_{m+1,l} + a^2_{m+1,l} \tag{2.7.47}$$

where $a^1_{m+1,l} = \mathcal{H}\mathcal{N}^{m+1}_{<l}a$. We then have from the properties of $\Phi_{\leq l}$ and the control of $\|\mathcal{H}a\|_{H^{n+m+\frac{5}{2}}(\Omega_t)}$ by the energy (as above),

$$\|\nabla\Phi_l a_0\cdot\nabla\Phi_{\leq l}a_{m+1}\|_{H^n(\Omega_t)} \lesssim_A \|a\|_{C^{\frac{1}{2}}(\Gamma_t)}(E^k)^{\frac{1}{2}}.$$

As ∇a_{m+1} is not at top order, we can easily verify using the decomposition above that we also have the following cruder bound for each l

$$\|F_l\|_{H^n(\Omega_t)} \lesssim_A 2^{-\delta l}\|(v,\Gamma)\|^r_{\mathbf{H}^s}(E^k)^{\frac{1}{2}}, \tag{2.7.48}$$

for some integer $r > 1$ and small constant $\delta > 0$. Arguing as in Lemma 2.7.9, we can combine the above two bounds to estimate

$$\|F\|_{H^n(\Omega_t)} \lesssim_A B\log(1 + \|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.$$

This handles the latter term in (2.7.42). Now, we turn to the first term. We have to estimate $\|\mathcal{N}^n(\mathcal{N}a\mathcal{N}^{m+1}a)\|_{H^{\frac{1}{2}}(\Gamma_t)}$ where $n, m \geq 0$ and $n+m = k-3$. Here, we only sketch the details as the procedure for this estimate is relatively similar to the previous term. We start by writing

$$\mathcal{N}a\mathcal{N}^{m+1}a = (\mathcal{H}n_{\Gamma_t} \cdot \nabla a_0)(\mathcal{H}n_{\Gamma_t} \cdot \nabla a_m) =: K_{|\Gamma_t}.$$

Then we apply Proposition 2.5.30 and Proposition 2.5.11 to estimate

$$\begin{aligned}\|\mathcal{N}^n K_{|\Gamma_t}\|_{H^{\frac{1}{2}}(\Gamma_t)} \lesssim_A \|K\|_{H^{n+1}(\Omega_t)} + \|\Gamma\|_{H^k} \sup_{j>0} 2^{-j(m+\frac{3}{2})}\|K_j^1\|_{L^\infty(\Omega_t)} \\ + \sup_{j>0} 2^{j(n+\frac{1}{2}-2\varepsilon)}\|K_j^2\|_{H^{\frac{1}{2}+\varepsilon}(\Omega_t)}\end{aligned}$$

where $K = K_j^1 + K_j^2$ and

$$K_j^1 := \Phi_{<j}((\mathcal{H}n_{\Gamma_t} \cdot \nabla\Phi_{<j}a_0)(\mathcal{H}n_{\Gamma_t} \cdot \nabla\Phi_{<j}\mathcal{H}\mathcal{N}_{<j}^m a)). \tag{2.7.49}$$

Similarly to the above, we can estimate

$$2^{-j(m+\frac{3}{2})}\|K_j^1\|_{L^\infty(\Omega_t)} \lesssim_A B.$$

We also have an estimate of the form

$$\begin{aligned}2^{j(n+\frac{1}{2}-2\varepsilon)}\|K_j^2\|_{H^{\frac{1}{2}+\varepsilon}(\Omega_t)} &\lesssim_A \|K\|_{H^{n+1}(\Omega_t)} + B(E^k)^{\frac{1}{2}} + 2^{j(n+1-\varepsilon)}\|\mathcal{B}(\nabla\Phi_{\geq j}a_0, \nabla a_m)\|_{L^2(\Omega_t)} \\ &\lesssim_A \|K\|_{H^{n+1}(\Omega_t)} + B(E^k)^{\frac{1}{2}} + \sup_{l>0} 2^{l(n+1-\varepsilon)}\|\mathcal{B}(\nabla\Phi_l a_0, \nabla a_m)\|_{L^2(\Omega_t)}\end{aligned}$$

for some bilinear expression $\mathcal{B}$. Using a decomposition of a_m similar to (2.7.47), we have

$$2^{l(n+1-\varepsilon)}\|\mathcal{B}(\nabla\Phi_l a_0, \nabla a_m)\|_{L^2(\Omega_t)} \lesssim_A B\log(1 + \|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}}.$$

Therefore, we have

$$\|\mathcal{N}^n K_{|\Gamma_t}\|_{H^{\frac{1}{2}}(\Gamma_t)} \lesssim_A B\log(1 + \|(v,\Gamma)\|_{\mathbf{H}^s})(E^k)^{\frac{1}{2}} + \|K\|_{H^{n+1}(\Omega_t)}.$$

To estimate K in $H^{n+1}(\Omega_t)$, the starting point is similar (but slightly more technical) than the estimate for F in H^n from above. The idea is to do a quadrilinear frequency decomposition for K and study each summand individually. The relevant terms correspond to terms essentially of the form $(\Phi_l\mathcal{H}n_{\Gamma_t} \cdot \nabla\Phi_{\leq l}a_0)(\Phi_{\leq l}\mathcal{H}n_{\Gamma_t} \cdot \nabla\Phi_{\leq l}a_m)$ and $(\Phi_{\leq l}\mathcal{H}n_{\Gamma_t} \cdot \nabla\Phi_l a_0)(\Phi_{\leq l}\mathcal{H}n_{\Gamma_t} \cdot \nabla\Phi_{\leq l}a_m)$ and $(\Phi_{\leq l}\mathcal{H}n_{\Gamma_t} \cdot \nabla\Phi_{\leq l}a_0)(\Phi_l\mathcal{H}n_{\Gamma_t} \cdot \nabla\Phi_{\leq l}a_m)$ and $(\Phi_{\leq l}\mathcal{H}n_{\Gamma_t} \cdot \nabla\Phi_{\leq l}a_0)(\Phi_{\leq l}\mathcal{H}n_{\Gamma_t} \cdot \nabla\Phi_l a_m)$. The second and fourth terms can be handled almost identically to the estimate for F in H^n (as by the maximum principle, one can dispense with the factors of $\mathcal{H}n_{\Gamma_t}$). The first and third terms are handled similarly by decomposing a_0 and a_m into low and high frequency parts as in (2.7.47) and using Proposition 2.5.21 when $\Phi_l \mathcal{H}n_{\Gamma_t}$ is at high frequency compared to the other factors. One then obtains the desired estimate similarly to the estimate for F in H^n above. We omit the remaining details. $\square$

We now turn to the estimate for the final source term, $\mathcal{R} = [D_t, \mathcal{N}^{k-1}]a$ in $L^2(\Gamma_t)$. To control this term, we first write

$$[D_t, \mathcal{N}^{k-1}]a = [D_t, \mathcal{N}]\mathcal{N}^{k-2}a + \mathcal{N}[D_t, \mathcal{N}^{k-2}]a.$$

For the latter term, we have by Lemma 2.5.24,

$$\|\mathcal{N}[D_t, \mathcal{N}^{k-2}]a\|_{L^2(\Gamma_t)} \lesssim_A \|[D_t, \mathcal{N}^{k-2}]a\|_{H^1(\Gamma_t)}.$$

Then using Proposition 2.5.33 and the coercivity bound, we estimate

$$\begin{aligned}\|[D_t, \mathcal{N}^{k-2}]a\|_{H^1(\Gamma_t)} &\lesssim_A \|v\|_{W^{1,\infty}(\Omega_t)}\|a\|_{H^{k-1}(\Gamma_t)} + \|a\|_{C^{\frac{1}{2}}(\Gamma_t)}(\|\Gamma\|_{H^k} + \|v\|_{H^k(\Omega_t)}) \\ &\quad + \|a\|_{L^\infty(\Gamma_t)}\|v\|_{W^{1,\infty}(\Omega_t)}\|\Gamma\|_{H^k} \\ &\lesssim_A B(E^k)^{\frac{1}{2}}.\end{aligned}$$

To conclude the proof of Theorem 2.7.1, it remains to estimate $[D_t, \mathcal{N}]\mathcal{N}^{k-2}a$ in $L^2(\Gamma)$. This term is rather delicate due to the lack of a trace estimate in $L^2(\Gamma)$. To deal with this term, we have the following proposition.

Proposition 2.7.14. Let $s \in \mathbb{R}$ with $s > \frac{d}{2} + 1$. Then we have,

$$\|[\mathcal{N}, D_t]f\|_{L^2(\Gamma)} \lesssim_A B\log(1 + \|(v, \Gamma)\|_{\mathbf{H}^s})\|f\|_{H^1(\Gamma)}. \tag{2.7.50}$$

Our proof requires the following short lemma which is essentially a consequence of Proposition 2.5.18.

Lemma 2.7.15. For each $l = 1, ..., d$, we have

$$\|n_\Gamma \cdot (\nabla\Delta^{-1}\partial_l - e_l)\|_{H^{\frac{1}{2}}(\Omega) \to L^2(\Gamma)} \lesssim_A 1. \tag{2.7.51}$$

Proof. This will follow by interpolation if we can prove

$$\|n_\Gamma \cdot (\nabla\Delta^{-1}\partial_l - e_l)\|_{L^2(\Omega) \to H^{-\frac{1}{2}}(\Gamma)} + \|n_\Gamma \cdot (\nabla\Delta^{-1}\partial_l - e_l)\|_{H^{\frac{1}{2}+\delta}(\Omega) \to H^\delta(\Gamma)} \lesssim_A 1, \tag{2.7.52}$$

for some $0 < \delta < \varepsilon$. The $H^{\frac{1}{2}+\delta} \to H^{\delta}$ bound follows easily from the trace inequality, the bound $\|n_{\Gamma_t}\|_{C^{\varepsilon}(\Gamma_t)} \lesssim_A 1$ and Proposition 2.5.18. For the $L^2 \to H^{-\frac{1}{2}}$ bound we use duality. Indeed, let $f \in L^2(\Omega)$. Since $(\nabla\Delta^{-1}\partial_l - e_l)f$ is divergence free, we have

$$\int_{\Gamma} g n_{\Gamma} \cdot (\nabla\Delta^{-1}\partial_l - e_l) f \, dS = \int_{\Omega} \nabla\mathcal{H}g \cdot (\nabla\Delta^{-1}\partial_l - e_l) f \, dx \lesssim_A \|g\|_{H^{\frac{1}{2}}(\Gamma)} \|f\|_{L^2(\Omega)},$$

for every $g \in H^{\frac{1}{2}}(\Gamma)$. Therefore, we obtain (2.7.52) and thus also (2.7.51). □

Proof of Proposition 2.7.14. Now, returning to the proposition, we expand using (2.5.40),

$$[D_t, \mathcal{N}]f = D_t n_{\Gamma} \cdot \nabla\mathcal{H}f - n_{\Gamma} \cdot ((\nabla v)^*(\nabla\mathcal{H}f)) + n_{\Gamma} \cdot \nabla\Delta^{-1}\Delta(v \cdot \nabla\mathcal{H}f).$$

The first two terms on the right can easily be estimated in L^2 by the right-hand side of (2.7.50) by using (2.5.35) and Lemma 2.5.24. Now, we turn to the latter term. We write for simplicity $u := \mathcal{H}f$. We then split u as

$$u = \sum_{l \leq l_0} \Phi_l u + \Phi_{>l_0} u =: \sum_{l \leq l_0} u_l + u_{\geq l_0},$$

where l_0 is a parameter to be chosen. Note that u_l is not harmonic anymore, but it is to leading order. As usual, we also write the corresponding divergence free regularizations for v as $v_l := \Psi_l v$, $v_{<l} := \Psi_{<l} v$ and so forth.

The following lemma shows that we have a suitable estimate when u is replaced by a single dyadic regularization u_l.

Lemma 2.7.16. For each $l \in \mathbb{N}_0$, we have

$$\|\nabla_n \Delta^{-1}\Delta(v \cdot \nabla u_l)\|_{L^2(\Gamma)} \lesssim_A B\|f\|_{H^1(\Gamma)},$$

where the implicit constant does not depend on l.

Proof. We write

$$\nabla_n \Delta^{-1}\Delta(v \cdot \nabla u_l) = \nabla_n \Delta^{-1}\Delta(v_{<l} \cdot \nabla u_l) + \nabla_n \Delta^{-1}\Delta(v_{\geq l} \cdot \nabla u_l). \tag{2.7.53}$$

For the second term, where v is at high frequency, we use the identity $\Delta^{-1}\Delta = I - \mathcal{H}$ and the $H^1 \to L^2$ bound for $\mathcal{N}$ to estimate

$$\|\nabla_n \Delta^{-1}\Delta(v_{\geq l} \cdot \nabla u_l)\|_{L^2(\Gamma)} \lesssim_A \|\nabla(v_{\geq l} \cdot \nabla u_l)\|_{L^2(\Gamma)} + \|v_{\geq l} \cdot \nabla u_l\|_{H^1(\Gamma)}. \tag{2.7.54}$$

For the first term in (2.7.54), we distribute the derivative to obtain

$$\|\nabla(v_{\geq l}\cdot\nabla u_l)\|_{L^2(\Gamma)} \lesssim B\|\nabla u_l\|_{L^2(\Gamma)} + \|v_{\geq l}\cdot\nabla^2 u_l\|_{L^2(\Gamma)}. \tag{2.7.55}$$

For the first term in (2.7.55), we use the variant of the trace theorem leading to (2.4.8) and the fact that u_l is frequency localized to obtain

$$\|\nabla u_l\|_{L^2(\Gamma)} \lesssim \|\nabla u_l\|_{H^1(\Omega)}^{\frac{1}{2}}\|\nabla u_l\|_{L^2(\Omega)}^{\frac{1}{2}} \lesssim \|u\|_{H^{\frac{3}{2}}(\Omega)} \lesssim_A \|f\|_{H^1(\Gamma)}$$

where in the last estimate we used Proposition 2.5.18. For the second term in (2.7.55), we again use the trace theorem and the fact that $v_{\geq l}$ is higher frequency to obtain

$$\|v_{\geq l}\cdot\nabla^2 u_l\|_{L^2(\Gamma)} \lesssim \|v_{\geq l}\cdot\nabla^2 u_l\|_{L^2(\Omega)}^{\frac{1}{2}}\|v_{\geq l}\cdot\nabla^2 u_l\|_{H^1(\Omega)}^{\frac{1}{2}} \lesssim B\|u\|_{H^{\frac{3}{2}}(\Omega)} \lesssim B\|f\|_{H^1(\Gamma)}.$$

The term $\|v_{\geq l}\cdot\nabla u_l\|_{H^1(\Gamma)}$ in (2.7.54) is similarly estimated. For this, we only need to estimate $\|\nabla^\top(v_{\geq l}\cdot\nabla u_l)\|_{L^2(\Gamma)}$, and this is handled by an almost identical strategy to the above.

Now, to estimate the term in (2.7.53) where v is at low frequency, we distribute the Laplacian and use that $v_{<l}$ is divergence free to write $\nabla_n\Delta^{-1}\Delta(v_{<l}\cdot\nabla u_l)$ as a sum of terms of the form

$$\nabla_n\Delta^{-1}\partial_j(Dv_{<l}Du_l) + \nabla_n\Delta^{-1}\partial_j(v_{<l}C_l u),$$

where $C_l u := [\Delta, \Phi_l]u$. Using Lemma 2.7.15 we can then estimate

$$\|\nabla_n\Delta^{-1}\Delta(v_{<l}\cdot\nabla u_l)\|_{L^2(\Gamma)} \lesssim_A \|Dv_{<l}Du_l\|_{L^2(\Gamma)\cap H^{\frac{1}{2}}(\Omega)} + \|v_{<l}C_l u\|_{L^2(\Gamma)\cap H^{\frac{1}{2}}(\Omega)} =: J_1 + J_2.$$

Using that v is at low frequency, we can estimate similarly to the above,

$$J_1 \lesssim_A B\|f\|_{H^1(\Gamma)}.$$

For J_2, we note that C_l is an operator of order 1 and still retains essentially the frequency localization scale of 2^l. Therefore, we can estimate J_2 similarly. This completes the proof of the lemma. □

Returning to the proof of Proposition 2.7.14, we now estimate using Lemma 2.7.16,

$$\|\nabla_n\Delta^{-1}\Delta(v\cdot\nabla u)\|_{L^2(\Gamma)} \lesssim_A l_0 B\|f\|_{H^1(\Gamma)} + \|\nabla_n\Delta^{-1}\Delta(v\cdot\nabla u_{\geq l_0})\|_{L^2(\Gamma)}.$$

Again, using that v is divergence free, we can (as above) expand $\nabla_n \Delta^{-1}\Delta(v\cdot\nabla u_{\geq l_0})$ as a sum of terms of the form

$$\nabla_n \Delta^{-1}\partial_j(DvDu_{\geq l_0}) + \nabla_n\Delta^{-1}\partial_j(vC_{\leq l_0}u),$$

where $C_{\leq l_0}u = [\Delta, \Phi_{\leq l_0}]u$. For the latter term, we can simply estimate as above (since v is undifferentiated),

$$\|\nabla_n\Delta^{-1}\partial_j(vC_{\leq l_0}u)\|_{L^2(\Gamma)} \leq \sum_{l\leq l_0}\|\nabla_n\Delta^{-1}\partial_j(vC_l u)\|_{L^2(\Gamma)} \lesssim_A l_0 B\|f\|_{H^1(\Gamma)}.$$

For the other term, we use Lemma 2.7.15 to obtain

$$\|\nabla_n\Delta^{-1}\partial_j(DvDu_{\geq l_0})\|_{L^2(\Gamma)} \lesssim_A B\|Du_{\geq l_0}\|_{L^2(\Gamma)} + \|DvDu_{\geq l_0}\|_{H^{\frac{1}{2}}(\Omega)}.$$

Since u is harmonic we have

$$B\|Du_{\geq l_0}\|_{L^2(\Gamma)} \lesssim_A B\|f\|_{H^1(\Gamma)} + B\|Du_{<l_0}\|_{L^2(\Gamma)}.$$

Then expanding $u_{<l_0} = \sum_{l<l_0} u_l$ and using the trace theorem leading to (2.4.8) for each term as above, we get

$$B\|Du_{\geq l_0}\|_{L^2(\Gamma)} \lesssim_A Bl_0\|f\|_{H^1(\Gamma)}.$$

Finally, by product estimates and Sobolev embedding, it is easy to bound

$$\|DvDu_{\geq l_0}\|_{H^{\frac{1}{2}}(\Omega)} \lesssim_A B\|f\|_{H^1(\Gamma)} + \|Dv_{\geq l_0}\|_{H^{\frac{d}{2}+\varepsilon}(\Omega)}\|f\|_{H^1(\Gamma)} \lesssim_A (B + 2^{-l_0\delta}\|(v,\Gamma)\|_{\mathbf{H}^s})\|f\|_{H^1(\Gamma)}$$

for some $\delta > 0$. Then choosing $l_0 \approx_\delta \log(1 + \|(v,\Gamma)\|_{\mathbf{H}^s})$, we conclude the proof of the proposition. □

Finally, we conclude the proof of Theorem 2.7.1 by observing first from the above proposition that we have

$$\|[D_t,\mathcal{N}]\mathcal{N}^{k-2}a\|_{L^2(\Gamma)} \lesssim_A B\log(1 + \|(v,\Gamma)\|_{\mathbf{H}^s})\|\mathcal{N}^{k-2}a\|_{H^1(\Gamma)}.$$

Then, using Proposition 2.5.30, Lemma 2.7.5, Lemma 2.7.7 and (2.7.15), we have

$$\|\mathcal{N}^{k-2}a\|_{H^1(\Gamma)} \lesssim_A (E^k)^{\frac{1}{2}}.$$

This finally concludes the proof of Theorem 2.7.1.

2.8 Construction of regular solutions

In this section, we give a new, direct method for constructing solutions to the free boundary Euler equations in the high regularity regime. Solutions at low regularity will be obtained in the next section as unique limits of these regular solutions.

Previous approaches to constructing solutions to free boundary fluid equations include using Lagrangian coordinates, Nash Moser iteration or taking the zero surface tension limit in the capillary problem. A more recent approach in the case of a laterally infinite ocean with flat bottom can be found in [157]. The article [157] uses a paralinearization of the Dirichlet-to-Neumann operator and a complicated iteration scheme to construct solutions. In contrast, we propose a new, geometric approach, implemented fully within the Eulerian coordinates.

Our novel approach is roughly inspired by nonlinear semigroup theory, where one constructs an approximate solution by discretizing the problem in time. To execute this approach successfully, one needs to show that the energy bounds are uniformly preserved throughout the time steps. In our setting, a classical semigroup approach would require one to solve an elliptic free boundary problem with very precise estimates. However, on the other end of the spectrum, one could try to view our equation as an ODE and use an Euler type iteration. Of course, a naïve Euler method cannot work because it loses derivatives. A partial fix to this would be to combine the Euler method with a transport part, which would reduce but not eliminate the loss of derivatives.

Our goal is to retain the simplicity of the Euler plus transport method, while ameliorating the derivative loss by an initial regularization of each iterate in our discretization. In short, we will split the time step into two main pieces:

(i) Regularization.

(ii) Euler plus transport.

To ensure that the uniform energy bounds survive, the regularization step needs to be done carefully. For this, we will take a modular approach and try to decouple this process into two steps, where we regularize individually the domain and the velocity. We believe that this modular approach will serve as a recipe for a new and relatively simple method for constructing solutions to various free boundary problems.

The overarching scheme we employ in this section was carried out in the case of a compressible gas in [72]. While we follow the same rough roadmap here, we stress that the main difficulties in the incompressible liquid case are quite different than for the gas. One obvious reason for this is that the surface of a liquid carries a non-trivial energy. Also, we introduce another new idea here, which is to begin the iteration with a regularized version of the initial data, and then to partially propagate these regularized bounds through the iteration.

Basic setup and simplifications

We begin by fixing a smooth reference hypersurface Γ_* and a collar neighborhood $\Lambda_* := \Lambda(\Gamma_*, \varepsilon_0, \delta)$. Here, as usual, ε_0 and δ are some small but fixed positive constants. Given $k > \frac{d}{2} + 1$ sufficiently large and an initial state $(v_0, \Gamma_0) \in \mathbf{H}^k$, our aim is to construct a local solution $(v(t), \Gamma_t) \in \mathbf{H}^k$ whose lifespan depends only on the size of $\|(v_0, \Gamma_0)\|_{\mathbf{H}^k}$, the lower bound in the Taylor sign condition and the collar neighborhood Λ_*. We recall from Theorem 2.7.1 that we have the coercivity

$$1 + \|(v, \Gamma)\|_{\mathbf{H}^k}^2 \approx_A E^k(v, \Gamma)$$

for any state $(v, \Gamma) \in \mathbf{H}^k$. For technical convenience, we will work with the slightly modified energy,

$$\mathcal{E}^k(v, \Gamma) := \|\nabla \mathcal{H} \mathcal{N}^{k-2}(a^{-1} D_t a)\|_{L^2(\Omega)}^2 + \|a^{-\frac{1}{2}} \mathcal{N}^{k-1} a\|_{L^2(\Gamma)}^2 + \|\omega\|_{H^{k-1}(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 + 1. \quad (2.8.1)$$

This new energy is readily seen to be equivalent to the old one in the sense that

$$\mathcal{E}^k(v, \Gamma) \approx_A E^k(v, \Gamma). \quad (2.8.2)$$

The primary reason we modify the energy is that it will allow for cleaner cancellations in the energy when we later regularize the velocity.

Now, fix $M > 0$. Given a small time step $\varepsilon > 0$ and a suitable pair of initial data $(v_0, \Gamma_0) \in \mathbf{H}^k$ with $\|(v_0, \Gamma_0)\|_{\mathbf{H}^k} \leq M$, we aim to construct a sequence $(v_\varepsilon(j\varepsilon), \Gamma_\varepsilon(j\varepsilon)) \in \mathbf{H}^k$ satisfying the following properties:

(i) (Norm bound). There is a uniform constant $c_0 > 0$ depending only on Λ_*, M and the lower bound in the Taylor sign condition such that if j is an integer with $0 \leq j \leq c_0 \varepsilon^{-1}$,

then
$$\|(v_\varepsilon(j\varepsilon),\Gamma_\varepsilon(j\varepsilon))\|_{\mathbf{H}^k} \leq C(M),$$
where $C(M) > 0$ is some constant depending on M.

(ii) (Approximate solution).
$$\begin{cases} v_\varepsilon((j+1)\varepsilon) = v_\varepsilon(j\varepsilon) - \varepsilon(v_\varepsilon(j\varepsilon)\cdot\nabla v_\varepsilon(j\varepsilon) + \nabla p_\varepsilon(j\varepsilon) + ge_d) + \mathcal{O}_{C^1}(\varepsilon^2) \quad , \\ \nabla\cdot v_\varepsilon((j+1)\varepsilon) = 0 \quad \text{on } \Omega_\varepsilon((j+1)\varepsilon), \\ \Omega_\varepsilon((j+1)\varepsilon) = (I + \varepsilon v_\varepsilon(j\varepsilon))(\Omega_\varepsilon(j\varepsilon)) + \mathcal{O}_{C^1}(\varepsilon^2). \end{cases}$$
where the first equation holds on $\Omega_\varepsilon((j+1)\varepsilon)\cap\Omega_\varepsilon(j\varepsilon)$.

We will not have to concern ourselves too much with the Taylor sign condition in this section as we are working at high regularity and this is a pointwise property. In particular, we will suppress the lower bound in the Taylor sign condition from our notation. A nice feature about the above iteration scheme is that it suffices to only carry out a single step. For this, we have the following theorem.

Theorem 2.8.1. Let k be a sufficiently large even integer and $M > 0$. Consider an initial data $(v_0,\Gamma_0)\in\mathbf{H}^k$ so that $\|(v_0,\Gamma_0)\|_{\mathbf{H}^k} \leq M$ and v_0 and ω_0 satisfy the initial regularization bounds
$$\|v_0\|_{H^{k+1}(\Omega_0)} \leq K(M)\varepsilon^{-1}, \quad \|\omega_0\|_{H^{k+n}(\Omega_0)} \leq K'(M)\varepsilon^{-1-n}, \tag{2.8.3}$$
for $n = 0, 1$, where $K(M)$, $K'(M) > 0$ are constants, possibly much larger than M, such that $K'(M) \ll K(M)$. Then there exists a one step iterate $(v_0,\Gamma_0)\mapsto(v_1,\Gamma_1)$ with the following properties:

(i) (Energy monotonicity).
$$\mathcal{E}^k(v_1,\Gamma_1) \leq (1 + C(M)\varepsilon)\mathcal{E}^k(v_0,\Gamma_0). \tag{2.8.4}$$

(ii) (Good pointwise approximation).
$$\begin{cases} v_1 = v_0 - \varepsilon(v_0\cdot\nabla v_0 + \nabla p_0 + ge_d) + \mathcal{O}_{C^1}(\varepsilon^2) \quad \text{on } \Omega_1\cap\Omega_0, \\ \nabla\cdot v_1 = 0 \quad \text{on } \Omega_1, \\ \Omega_1 = (I+\varepsilon v_0)(\Omega_0) + \mathcal{O}_{C^1}(\varepsilon^2). \end{cases} \tag{2.8.5}$$

(iii) (Persistence of the regularization bounds). v_1 satisfies the regularization bounds

$$\|v_1\|_{H^{k+1}(\Omega_1)} \leq K(M)\varepsilon^{-1}, \quad \|\omega_1\|_{H^{k+n}(\Omega_1)} \leq (K'(M) + C(M)\varepsilon)\varepsilon^{-1-n}, \tag{2.8.6}$$

for $n = 0, 1$.

Remark 2.8.2. Property (2.8.6) ensures that v_1 retains the H^{k+1} regularization bound with the same constant compared to the first iterate, and ω_1 has a regularization bound which can only grow by an amount comparable to ε times the initial regularization bound, which is acceptable over $\approx_M \varepsilon^{-1}$ iterations. The energy monotonicity property, along with the energy coercivity bound from Theorem 2.7.1 will ensure that the resulting sequence $(v_\varepsilon(j\varepsilon), \Gamma_\varepsilon(j\varepsilon))$ of approximate solutions we construct remains uniformly bounded in $\mathbf{H}^k$ for $j \ll_M \varepsilon^{-1}$. The second property in Theorem 2.8.1 will ensure that $(v_\varepsilon(j\varepsilon), \Gamma_\varepsilon(j\varepsilon))$ converges in a weaker topology to a solution of the equation.

The assumption (2.8.3) for v_0 is for technical convenience. In the regularization step of the argument, it will allow us to decouple the process of regularizing the domain and regularizing the velocity into separate arguments (see Lemma 2.8.4 in the next section). The condition (2.8.6) ensures that (2.8.3) can be propagated from one iterate to the next. Assuming that the initial iterate satisfies (2.8.3) is harmless in practice. Indeed, by the regularization properties of $\Psi_{\leq \varepsilon^{-1}}$, we can replace the first iterate in the resulting sequence $(v_\varepsilon(j\varepsilon), \Gamma_\varepsilon(j\varepsilon))$ with a suitable ε^{-1} scale regularization so that the base case is satisfied. We note crucially that such a regularization is only done once - on the initial iterate - as we only know that this regularization is bounded on $\mathbf{H}^k$ (it does not necessarily satisfy the more delicate energy monotonicity). In contrast, we require the much stricter energy monotonicity bound (2.8.4) for all other iterations as in the above theorem. The condition on the vorticity in (2.8.3) can also be harmlessly assumed for the initial iterate. When we later regularize the velocity, we will not regularize the vorticity, but rather only the irrotational component. This is why, in contrast to the H^{k+1} bound for v_1, the constant for ω_1 in (2.8.6) gets slightly worse. Nonetheless, the careful tracking of its bound in (2.8.6) ensures that it only grows by an acceptable amount in each iteration. The heuristic reason why the regularization bound on ω_1 is expected is because the vorticity should be essentially transported by the flow, and therefore should not suffer the derivative loss of the full velocity in the iteration step.

Outline of the argument. We now give a brief overview of the section. The first step is selecting a suitable regularization scale. To motivate this, we recall that the evolution of the domain and the irrotational component of the velocity is essentially governed by the following approximate equation for a:

$$D_t^2 a \approx -a\mathcal{N}a. \tag{2.8.7}$$

Therefore, heuristically, D_t behaves roughly as a "spatial" derivative of order $\frac{1}{2}$. To control quadratic errors in the energy monotonicity bound in the Euler plus transport iteration later, it is therefore natural to attempt to regularize the domain and the irrotational part of the velocity on the ε^{-1} scale, as we do in Theorem 2.8.1. As the vorticity is essentially transported by the flow, we are able to leave the rotational part of the velocity alone, and instead track its growth as in (2.8.6).

With the above discussion in mind, we begin our analysis in earnest in Section 2.8 by regularizing the domain on the ε^{-1} scale. More specifically, given $(v_0, \Gamma_0) \in \mathbf{H}^k$ with v_0 satisfying (2.8.3), we construct for each $0 < \varepsilon \ll 1$ a domain $\Omega_\varepsilon \subseteq \Omega_0$ whose boundary is within $\mathcal{O}_{C^1}(\varepsilon^2)$ of Γ_0 and which satisfies the regularization bound $\|\Gamma_\varepsilon\|_{H^{k+\alpha}} \lesssim_{M,\alpha} \varepsilon^{-\alpha}$ for all $\alpha \geq 0$. This is achieved by performing a parabolic regularization of the graph parameterization η_0 on Γ_*, together with a slight contraction of the domain. We then define our new velocity $\tilde{v}_0 = \tilde{v}_0(\varepsilon)$ by restricting the old velocity v_0 to the new domain Ω_ε. As will be the case in every step of the argument, the main difficulty is to carefully track the effect of the regularization on the energy growth. The main point in this part of the argument is to show that the parabolic regularization of η_0 induces a corresponding parabolic gain in the surface component of the energy $\|a^{-\frac{1}{2}}\mathcal{N}^{k-1}a\|^2_{L^2(\Gamma)}$, allowing us to control all of the resulting errors.

With the domain now regularized, we move on to regularizing the velocity in Section 2.8, which is step 2 of the argument. In this step, we leave the domain and rotational part of the velocity alone, and regularize the irrotational part of the velocity on the ε^{-1} scale. The way we execute this is by using the functional calculus for the Dirichlet-to-Neumann operator. The main difficulty in this step of the argument is in tracking the effect of this regularization on the $\|\nabla\mathcal{H}\mathcal{N}^{k-2}(a^{-1}D_t a)\|^2_{L^2(\Omega)}$ portion of the energy, which at leading order controls the irrotational component of the velocity. An additional objective in this step of the argument is to improve the constant in (2.8.3) so that we can ultimately close the bootstrap in the upcoming Euler plus transport phase of the argument.

The final step in our construction is to use an Euler plus transport iteration to flow the regularized variables $(v_\varepsilon, \Gamma_\varepsilon)$ along a discrete version of the Euler evolution. It is in this step of the argument that we expect to observe a $\frac{1}{2}$ derivative loss (see the equation (2.8.7) for $D_t^2 a$, for instance), which is why the above regularization procedure is imperative. The Euler plus transport argument we employ is carried out in Section 2.8. Control of the resulting energy growth is shown by carefully relating the good variables a, $D_t a$ and ω for the new iterate to the corresponding good variables for the regularized data. Then, with the energy uniformly bounded and the variables appropriately iterated, in Section 2.8 we conclude that our scheme converges in a weaker topology, completing the construction of solutions.

Step 1: Domain regularization

We begin with the domain regularization step. For this, we have the following proposition.

Proposition 2.8.3. Given $(v_0, \Gamma_0) \in \mathbf{H}^k$ with v_0 satisfying (2.8.3), there exists a domain Ω_ε contained in Ω_0 with boundary $\Gamma_\varepsilon \in \Lambda_*$ such that the pair $(v_{0_{|\Omega_\varepsilon}}, \Gamma_\varepsilon)$ satisfies

(i) (Energy monotonicity).

$$\mathcal{E}^k(v_{0_{|\Omega_\varepsilon}}, \Gamma_\varepsilon) \leq (1 + C(M)\varepsilon)\mathcal{E}^k(v_0, \Gamma_0). \tag{2.8.8}$$

(ii) (Good pointwise approximation).

$$\eta_\varepsilon = \eta_0 + \mathcal{O}_{C^1}(\varepsilon^2) \;\; on \;\; \Gamma_*. \tag{2.8.9}$$

(iii) (Domain regularization bound). For every $\alpha \geq 0$, there holds,

$$\|\Gamma_\varepsilon\|_{H^{k+\alpha}} \lesssim_{M,\alpha} \varepsilon^{-\alpha}. \tag{2.8.10}$$

Proof. In the sequel, we will use $\tilde{v}_0$ as a shorthand for $v_{0_{|\Omega_\varepsilon}}$. To regularize Γ_0, we begin with the preliminary parabolic regularization of η_0 given by

$$\tilde{\eta}_\varepsilon = e^{\varepsilon^2 \Delta_{\Gamma_*}} \eta_0,$$

where Δ_{Γ_*} is the Laplace-Beltrami operator for Γ_*. The rationale for using the operator $e^{\varepsilon^2 \Delta_{\Gamma_*}}$ instead of, for instance, the operator $e^{-\varepsilon|D|}$ is to ensure that when k is large enough, we have $\|\partial_\varepsilon \tilde{\eta}_\varepsilon\|_{H^{k-2}(\Gamma_*)} \lesssim_M \varepsilon$. This ensures that the hypersurface parameterized by $\tilde{\eta}_\varepsilon$ in collar coordinates is at a distance on the order of no more than $\mathcal{O}_M(\varepsilon^2)$ from Γ_0 in the H^{k-2} topology (and thus the C^1 topology if k is large enough). We would also like to additionally guarantee that Ω_ε is contained in Ω_0, so that we can use the restriction of the velocity v_0 to Ω_ε as the velocity on the new domain. Therefore, we slightly correct the above parabolic regularization by defining our regularized hypersurface Γ_ε through the collar parameterization

$$\eta_\varepsilon = \tilde{\eta}_\varepsilon - C\varepsilon^2,$$

where C is some positive constant depending on M only, imposed to ensure that the domain Ω_ε associated to Γ_ε is contained in Ω_0. Clearly, η_ε satisfies (2.8.10) and the required pointwise approximation property in (2.8.9). The main bulk of the work in this step of the argument will therefore be in understanding how the above parabolic regularization of the surface (and also the restriction of the velocity to Ω_ε) affects the energy.

Given $(\tilde{v}_0, \Gamma_\varepsilon)$ as above, we define the associated quantities $\tilde{\omega}_0 := \nabla \times \tilde{v}_0$ and $\tilde{p}_0$, $D_t\tilde{p}_0$, $\tilde{a}_0$ and $D_t\tilde{a}_0$ on Ω_ε and Γ_ε by using the relevant Poisson equations, as in Section 2.7. We will use the notation $\mathcal{N}_\varepsilon$ to refer to the Dirichlet-to-Neumann operator for Γ_ε. Before proceeding to the proof of energy monotonicity, we note that the above construction gives rise to a flow velocity V_ε in the parameter ε for the family of hypersurfaces Γ_ε by composing $\partial_\varepsilon\eta_\varepsilon\nu$ with the inverse of the collar coordinate parameterization $x \mapsto x + \eta_\varepsilon(x)\nu(x)$. We may harmlessly assume that V_ε is defined on Ω_ε by harmonically extending it to Ω_ε. We use $D_\varepsilon := \partial_\varepsilon + V_\varepsilon \cdot \nabla$ to denote the associated material derivative, which will be tangent to the family of hypersurfaces Γ_ε.

We also importantly make note of the fact that for every $s \in \mathbb{R}$, we have

$$\|\tilde{\omega}_0\|_{H^s(\Omega_\varepsilon)} \leq \|\omega_0\|_{H^s(\Omega)}, \qquad \|\tilde{v}_0\|_{H^s(\Omega_\varepsilon)} \leq \|v_0\|_{H^s(\Omega)}. \tag{2.8.11}$$

Therefore, the bounds in (2.8.3) are retained from the initial data and, moreover, the rotational component of the energy does not increase.

Now we turn to the energy monotonicity bound (2.8.8). We will need the following two lemmas.

Lemma 2.8.4 (Material derivative bounds)**.** The following bound holds uniformly in ε:

$$\|D_\varepsilon \nabla \tilde{v}_0\|_{H^{k-1}(\Omega_\varepsilon)} \lesssim_M 1. \tag{2.8.12}$$

Lemma 2.8.5 (Variation of the surface energy)**.** Let k be a sufficiently large even integer. Then we have the following estimate for the $\tilde{a}_0$ component of the energy:

$$\frac{d}{d\varepsilon}\|\tilde{a}_0^{-\frac{1}{2}}\mathcal{N}_\varepsilon^{k-1}\tilde{a}_0\|^2_{L^2(\Gamma_\varepsilon)} \lesssim_M -\varepsilon\|\Gamma_\varepsilon\|^2_{H^{k+1}} + \mathcal{O}_M(1).$$

Lemma 2.8.4 will allow us to essentially ignore any contributions to the energy coming from the restriction $\tilde{v}_0$, while Lemma 2.8.5 will help in controlling the variation in ε of the irrotational components of the energy.

Before proving the above lemmas, let us see how they imply the energy monotonicity bound (2.8.8). Thanks to Lemma 2.8.5 and (2.8.11), we only need to study the $D_t a$ component of the energy. For this, we recall from the Laplace equation (2.7.4) that we have

$$\tilde{a}_0^{-1} D_t \tilde{a}_0 = \tilde{a}_0^{-1} n_{\Gamma_\varepsilon} \cdot \nabla \tilde{v}_0 \cdot \nabla \tilde{p}_0 - \tilde{a}_0^{-1} n_{\Gamma_\varepsilon} \cdot \nabla \Delta_{\Omega_\varepsilon}^{-1} (\Delta \tilde{v}_0 \cdot \nabla \tilde{p}_0 + 4\mathrm{tr}(\nabla^2 \tilde{p}_0 \cdot \nabla \tilde{v}_0) + 2\mathrm{tr}(\nabla \tilde{v}_0)^3) \quad \text{on } \Gamma_\varepsilon. \tag{2.8.13}$$

We apply $D_\varepsilon \nabla \mathcal{H}_\varepsilon \mathcal{N}_\varepsilon^{k-2}$ to (2.8.13) and distribute derivatives. We first dispense with the commutator. Using the standard $H^{\frac{1}{2}}(\Gamma_\varepsilon) \to H^1(\Omega_\varepsilon)$ bound for $\mathcal{H}_\varepsilon$, the $H^{k-\frac{3}{2}}(\Gamma_\varepsilon)$ to $H^{\frac{1}{2}}(\Gamma_\varepsilon)$ bound for $\mathcal{N}_\varepsilon^{k-2}$ from Proposition 2.5.30 and the $H^{\frac{1}{2}}(\Gamma_\varepsilon) \to H^1(\Omega_\varepsilon)$ bound for $[D_\varepsilon, \mathcal{H}_\varepsilon]$ from (2.5.39), we have

$$\|[D_\varepsilon, \nabla \mathcal{H}_\varepsilon \mathcal{N}_\varepsilon^{k-2}](\tilde{a}_0^{-1} D_t \tilde{a}_0)\|_{L^2(\Omega_\varepsilon)} \lesssim_M \|[D_\varepsilon, \mathcal{N}_\varepsilon^{k-2}](\tilde{a}_0^{-1} D_t \tilde{a}_0)\|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} + \|\tilde{a}_0^{-1} D_t \tilde{a}_0\|_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}.$$

Then, using the formula (2.5.43) and the elliptic estimates in Section 2.5 as well as the bound $\|V_\varepsilon\|_{H^{k-1}(\Gamma_\varepsilon)} \lesssim_M 1$, it is straightforward to verify the commutator bound

$$\|[D_\varepsilon, \mathcal{N}_\varepsilon^{k-2}]\|_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon) \to H^{\frac{1}{2}}(\Gamma_\varepsilon)} \lesssim_M 1.$$

By elliptic regularity, $\|\tilde{a}_0^{-1} D_t \tilde{a}_0\|_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}$ is $\mathcal{O}_M(1)$. Hence, we obtain

$$\|[D_\varepsilon, \nabla \mathcal{H}_\varepsilon \mathcal{N}_\varepsilon^{k-2}](\tilde{a}_0^{-1} D_t \tilde{a}_0)\|_{L^2(\Omega_\varepsilon)} \lesssim_M 1.$$

Using that

$$\|\nabla \mathcal{H}_\varepsilon \mathcal{N}_\varepsilon^{k-2} D_\varepsilon(\tilde{a}_0^{-1} D_t \tilde{a}_0)\|_{L^2(\Omega_\varepsilon)} \lesssim_M \|D_\varepsilon(\tilde{a}_0^{-1} D_t \tilde{a}_0)\|_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)},$$

it remains now to estimate $\|D_\varepsilon(\tilde{a}_0^{-1} D_t \tilde{a}_0)\|_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}$. For this, we distribute the operator D_ε onto the various terms in (2.8.13). To expedite this process, we collect a few useful bounds. First, using Lemma 2.8.4, the trace theorem ensures that we have the bound

$$\|D_\varepsilon \nabla \tilde{v}_0\|_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)} + \|D_\varepsilon \nabla \tilde{v}_0\|_{H^{k-1}(\Omega_\varepsilon)} \lesssim_M 1.$$

Using the identities for $[\Delta_{\Omega_\varepsilon}^{-1}, D_\varepsilon]$ and $D_\varepsilon n_{\Gamma_\varepsilon}$ in Section 2.5, the Laplace equation for p_ε, and the fact that V_ε is harmonic, we also readily verify the bounds

$$\|D_\varepsilon \tilde{p}_0\|_{H^{k+\frac{1}{2}}(\Omega_\varepsilon)} + \|D_\varepsilon n_{\Gamma_\varepsilon}\|_{H^{k-2}(\Gamma_\varepsilon)} + \|D_\varepsilon \tilde{a}_0\|_{H^{k-2}(\Gamma_\varepsilon)} \lesssim_M 1 \tag{2.8.14}$$

and

$$\|[D_\varepsilon,\nabla]\|_{H^k(\Omega_\varepsilon)\to H^{k-1}(\Omega_\varepsilon)}+\|D_\varepsilon n_{\Gamma_\varepsilon}\|_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}+\|D_\varepsilon\tilde{a}_0\|_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}\lesssim_M 1+\|V_\varepsilon\|_{H^{k-\frac{1}{2}}(\Gamma_\varepsilon)}.$$

From the above bounds and (2.8.13), we obtain the estimate

$$\|D_\varepsilon(\tilde{a}_0^{-1}D_t\tilde{a}_0)\|_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}\lesssim_M 1+\|V_\varepsilon\|_{H^{k-\frac{1}{2}}(\Gamma_\varepsilon)}.$$

The term $\|V_\varepsilon\|_{H^{k-\frac{1}{2}}(\Gamma_\varepsilon)}$ does not contribute an $\mathcal{O}_M(1)$ error, as it "loses" half a derivative. However, from the definition and regularization properties of V_ε, we have

$$\|V_\varepsilon\|_{H^{k-\frac{1}{2}}(\Gamma_\varepsilon)}\lesssim_M 1+\varepsilon^{\frac{1}{2}}\|\eta_\varepsilon\|_{H^{k+1}(\Gamma_*)}.$$

Hence, using Proposition 2.2.3 and Cauchy-Schwarz, we obtain

$$\frac{d}{d\varepsilon}\|\nabla\mathcal{H}_\varepsilon\mathcal{N}_\varepsilon^{k-2}(\tilde{a}_0^{-1}D_t\tilde{a}_0)\|_{L^2(\Omega_\varepsilon)}^2\lesssim_M 1+\delta_0\varepsilon\|\Gamma_\varepsilon\|_{H^{k+1}}^2,$$

where $\delta_0>0$ is some sufficiently small constant. Using the parabolic gain from Lemma 2.8.5, we notice that the latter term on the right-hand side is harmless as long as $\delta_0=\delta_0(M)$ is small enough.

It remains now to establish the two lemmas. We begin with Lemma 2.8.4, which is quite simple.

Proof. Since $\partial_\varepsilon\tilde{v}_0=0$, we have

$$D_\varepsilon\nabla\tilde{v}_0=V_\varepsilon\cdot\nabla\nabla\tilde{v}_0.$$

Then we use $\|V_\varepsilon\|_{H^{k-\frac{3}{2}}(\Omega_\varepsilon)}\lesssim_M\varepsilon$ and $\|V_\varepsilon\|_{H^{k-\frac{1}{2}}(\Omega_\varepsilon)}\lesssim_M 1$ together with the inductive bound for v_0 from (2.8.3); namely, $\|v_0\|_{H^{k+1}(\Omega_0)}\leq K(M)\varepsilon^{-1}$, to estimate

$$\|D_\varepsilon\nabla\tilde{v}_0\|_{H^{k-1}(\Omega_\varepsilon)}\lesssim_M\|V_\varepsilon\|_{H^{k-\frac{3}{2}}(\Omega_\varepsilon)}\|\tilde{v}_0\|_{H^{k+1}(\Omega_\varepsilon)}+\|V_\varepsilon\|_{H^{k-1}(\Omega_\varepsilon)}\|\tilde{v}_0\|_{H^k(\Omega_\varepsilon)}\lesssim_M 1.$$

This completes the proof of Lemma 2.8.4. □

Finally, we come to establishing Lemma 2.8.5, which is where the bulk of the work will be. We begin by establishing the following representation formula for the good variable $\mathcal{N}_\varepsilon^{k-1}\tilde{a}_0$:

$$\mathcal{N}_\varepsilon^{k-1}\tilde{a}_0=(-1)^m\tilde{a}_0\Delta_{\Gamma_\varepsilon}^m\kappa_\varepsilon+R_\varepsilon, \tag{2.8.15}$$

where κ_ε is the mean curvature for Γ_ε, $2m=k-2$ and R_ε is a remainder term satisfying the bounds

$$\|R_\varepsilon\|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}+\varepsilon^{\frac{1}{2}}\|R_\varepsilon\|_{H^1(\Gamma_\varepsilon)}+\|D_\varepsilon R_\varepsilon\|_{L^2(\Gamma_\varepsilon)}\lesssim_M 1. \tag{2.8.16}$$

The importance of (2.8.15) will be clear later. Roughly speaking, (2.8.15) states that to leading order $\mathcal{N}_\varepsilon^{k-1}\tilde{a}_0$ has a convenient local expression. Such an observation will facilitate the use of local formulas later on, consistent with our choice of domain regularization. Observe also that in (2.8.16), we have $D_\varepsilon R_\varepsilon = \mathcal{O}_{L^2(\Gamma_\varepsilon)}(1)$. This is stronger than the expected bound $D_\varepsilon R_\varepsilon = \mathcal{O}_{H^{-\frac{1}{2}}(\Gamma_\varepsilon)}(1)$. The reason for this improvement is the bound (2.8.12) for $D_\varepsilon \nabla \tilde{v}_0$; this term would have had to have been treated more carefully if we had attempted to regularize the velocity in this step of the argument.

Proof of (2.8.15). In the following analysis, R_ε will generically denote a remainder term satisfying (2.8.16) which is allowed to change from line to line. Likewise, $\tilde{R}_\varepsilon$ will denote an analogous remainder term but with

$$\tilde{R}_\varepsilon = \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}(1), \qquad \varepsilon^{\frac{1}{2}}\tilde{R}_\varepsilon = \mathcal{O}_{H^{k-1}(\Gamma_\varepsilon)}(1), \qquad D_\varepsilon \tilde{R}_\varepsilon = \mathcal{O}_{H^{k-2}(\Gamma_\varepsilon)}(1). \tag{2.8.17}$$

To establish (2.8.15), we begin by relating $\mathcal{N}_\varepsilon \tilde{a}_0$ to the mean curvature. Indeed, from $\Delta_{\Gamma_\varepsilon}\tilde{p}_0 = 0$ and the formula

$$\Delta \tilde{p}_0|_{\Gamma_\varepsilon} = \Delta_{\Gamma_\varepsilon}\tilde{p}_0 - \kappa_\varepsilon n_{\Gamma_\varepsilon}\cdot \nabla \tilde{p}_0 + D^2\tilde{p}_0(n_{\Gamma_\varepsilon}, n_{\Gamma_\varepsilon}),$$

we have

$$\begin{aligned} \tilde{a}_0\kappa_\varepsilon &= -n_i n_j \partial_i \partial_j \tilde{p}_0 + \Delta \tilde{p}_0 \\ &= -n_i n_j \partial_i \partial_j \tilde{p}_0 - \mathrm{tr}(\nabla \tilde{v}_0)^2 \\ &= -n_i n_j \partial_i \partial_j \tilde{p}_0 + \tilde{R}_\varepsilon, \end{aligned}$$

where in the last line, we used Lemma 2.8.4 to check the remainder property for $D_\varepsilon \tilde{R}_\varepsilon$ and the inductive assumption (2.8.3) and interpolation to control $\varepsilon^{\frac{1}{2}}\tilde{R}_\varepsilon$ in $H^{k-1}(\Gamma_\varepsilon)$. We now further expand using the Laplace equation for $\tilde{p}_0$,

$$\begin{aligned} -n_i n_j \partial_i \partial_j \tilde{p}_0 &= n_j \mathcal{N}_\varepsilon (n_j \tilde{a}_0) + n_j n_{\Gamma_\varepsilon}\cdot \nabla \Delta_{\Omega_\varepsilon}^{-1}\partial_j \mathrm{tr}(\nabla \tilde{v}_0)^2 \\ &= n_j \mathcal{N}_\varepsilon (n_j \tilde{a}_0) + \tilde{R}_\varepsilon. \end{aligned}$$

Next, we expand

$$\begin{aligned} n_j \mathcal{N}_\varepsilon (n_j \tilde{a}_0) &= \mathcal{N}_\varepsilon \tilde{a}_0 + \tilde{a}_0 n_j \mathcal{N}_\varepsilon n_j - 2 n_j n_{\Gamma_\varepsilon}\cdot \nabla \Delta_{\Omega_\varepsilon}^{-1}(\nabla \mathcal{H}_\varepsilon n_j \cdot \nabla \mathcal{H}_\varepsilon \tilde{a}_0) \\ &= \mathcal{N}_\varepsilon \tilde{a}_0 + \tilde{a}_0 n_j \mathcal{N}_\varepsilon n_j + \tilde{R}_\varepsilon \\ &= \mathcal{N}_\varepsilon \tilde{a}_0 + \tilde{R}_\varepsilon, \end{aligned}$$

where in the first equality, we used the Leibniz rule (2.5.36) for $\mathcal{N}_\varepsilon$. From the second to the third line, we used the Leibniz rule again, and that $\mathcal{N}_\varepsilon(n_j n_j) = 0$. In summary, what we have so far is the identity

$$\mathcal{N}_\varepsilon \tilde{a}_0 = \tilde{a}_0 \kappa_\varepsilon + \tilde{R}_\varepsilon. \tag{2.8.18}$$

The next step is to obtain the leading order identity,

$$\mathcal{N}_\varepsilon^{k-1} \tilde{a}_0 = \tilde{a}_0 \mathcal{N}_\varepsilon^{k-2}(\tilde{a}_0^{-1} \mathcal{N}_\varepsilon \tilde{a}_0) + R_\varepsilon \tag{2.8.19}$$

by applying $\mathcal{N}_\varepsilon^{k-2}$ to $\mathcal{N}_\varepsilon \tilde{a}_0$ and then commuting $\tilde{a}_0^{-1}$ with $\mathcal{N}_\varepsilon^{k-2}$. Here, R_ε can be seen to satisfy the required bounds through the use of the various commutator identities for D_ε listed in Section 2.5 as well as the Leibniz rule (2.5.36), the elliptic estimates in Section 2.5 for $\mathcal{N}_\varepsilon$ and the estimates in (2.8.14).

Before proceeding further, we recall the formula

$$\begin{aligned} -(\Delta_{\Gamma_\varepsilon} + \mathcal{N}_\varepsilon^2) f = \kappa_\varepsilon \mathcal{N}_\varepsilon f - 2 n_{\Gamma_\varepsilon} \cdot \nabla(-\Delta_{\Omega_\varepsilon})^{-1} (\nabla \mathcal{H}_\varepsilon n_{\Gamma_\varepsilon} \cdot \nabla^2 \mathcal{H}_\varepsilon f) \\ - \mathcal{N}_\varepsilon n_{\Gamma_\varepsilon} \cdot (\mathcal{N}_\varepsilon f n_{\Gamma_\varepsilon} + \nabla^\top f) \end{aligned} \tag{2.8.20}$$

from [140, Equation A.13]. Also, we recall from (4.23) of [140] the commutator estimate

$$\|[\Delta_{\Gamma_\varepsilon}, D_\varepsilon]\|_{H^s(\Gamma_\varepsilon) \to H^{s-2}(\Gamma_\varepsilon)} \lesssim_M \|V_\varepsilon\|_{H^{k-\frac{1}{2}}(\Omega_\varepsilon)} \lesssim_M 1, \quad 1 \le s \le k-1. \tag{2.8.21}$$

Then, given that $k - 2 = 2m$ is even, applying (2.8.18), (2.8.19) and iterating (2.8.20) m times, we have

$$\mathcal{N}_\varepsilon^{k-1} \tilde{a}_0 = \tilde{a}_0 \mathcal{N}_\varepsilon^{k-2}(\tilde{a}_0^{-1} \mathcal{N}_\varepsilon \tilde{a}_0) + R_\varepsilon = (-1)^m \tilde{a}_0 \Delta_{\Gamma_\varepsilon}^m (\tilde{a}_0^{-1} \mathcal{N}_\varepsilon \tilde{a}_0) + R_\varepsilon = (-1)^m \tilde{a}_0 \Delta_{\Gamma_\varepsilon}^m \kappa_\varepsilon + R_\varepsilon,$$

where by straightforward (but slightly tedious) computation we verify that the remainder term R_ε has the needed bounds through the use of the various commutator identities for D_ε listed in Section 2.5 as well as the above estimates (2.8.18)-(2.8.21), the relevant elliptic estimates in Section 2.5 and (2.8.14). □

Now, we are ready to establish the differential inequality in Lemma 2.8.5. For the sake of clarity, let us begin by assuming that the reference hypersurface is given by $\{x_d = 0\}$ and that Γ_ε is literally given by $x_d = \eta_\varepsilon(x_1, ..., x_{d-1})$. Then the mean curvature and Laplace-Beltrami operator take the form

$$\kappa_\varepsilon = -\frac{\Delta \eta_\varepsilon}{(1 + |\nabla \eta_\varepsilon|^2)^{\frac{1}{2}}} + \frac{\partial_i \eta_\varepsilon \partial_j \eta_\varepsilon \partial_i \partial_j \eta_\varepsilon}{(1 + |\nabla \eta_\varepsilon|^2)^{\frac{3}{2}}},$$

and

$$\Delta_{\Gamma_\varepsilon} f = \frac{1}{\sqrt{1+|\nabla\eta_\varepsilon|^2}}\partial_i(g_\varepsilon^{ij}\sqrt{1+|\nabla\eta_\varepsilon|^2}\partial_j f), \tag{2.8.22}$$

where $(g_\varepsilon^{ij}) = (\delta_{ij} + \partial_i\eta_\varepsilon\partial_j\eta_\varepsilon)^{-1}$. Observe that g_ε^{ij} and $\nabla\eta_\varepsilon$ are one derivative more regular than κ_ε. Therefore, by making use of the identity $\partial_\varepsilon\eta_\varepsilon = 2\varepsilon\Delta_{\Gamma_*}\eta_\varepsilon$ and the regularization bound (2.8.10), we can differentiate in ε and commute $2\varepsilon\Delta_{\Gamma_*}$ with these coefficients to obtain,

$$(D_\varepsilon(\mathcal{N}_\varepsilon^{k-1}\tilde{a}_0))_* = 2(-1)^m\varepsilon\Delta_{\Gamma_*}(\tilde{a}_0\Delta_{\Gamma_\varepsilon}^m\kappa_\varepsilon)_* + \mathcal{O}_{L^2(\Gamma_*)}(1), \tag{2.8.23}$$

where we define $f_*(x) := f(x+\eta_\varepsilon(x)\nu(x))$ for a function f defined on Γ_ε. Moreover, by an exercise in local coordinates, the reader may check that (2.8.23), as written, is valid for general reference hypersurfaces Γ_*. Now, using (2.5.41), the bounds for R_ε, and Cauchy-Schwarz, it follows that

$$\frac{d}{d\varepsilon}\|\tilde{a}_0^{-\frac{1}{2}}\mathcal{N}_\varepsilon^{k-1}\tilde{a}_0\|_{L^2(\Gamma_\varepsilon)}^2 \lesssim_M 1 - \varepsilon\||D|_{\Gamma_*}(\Delta_{\Gamma_\varepsilon}^m\kappa_\varepsilon)_*\|_{L^2(\Gamma_*)}^2,$$

where $|D|_{\Gamma_*} = (-\Delta_{\Gamma_*})^{\frac{1}{2}}$. To conclude, we now only need to show the coercivity type bound

$$\|\eta_\varepsilon\|_{H^{k+1}(\Gamma_*)} \lesssim_M 1 + \||D|_{\Gamma_*}(\Delta_{\Gamma_\varepsilon}^m\kappa_\varepsilon)_*\|_{L^2(\Gamma_*)}.$$

For this, we begin with Proposition 2.5.22 which yields

$$\|\eta_\varepsilon\|_{H^{k+1}(\Gamma_*)} \lesssim_M 1 + \|\kappa_\varepsilon\|_{H^{k-1}(\Gamma_\varepsilon)}.$$

Then, using (2.8.22) and the fact that $2m = k-2$ (this being relevant for ensuring domain dependent implicit constants are at most $\mathcal{O}_M(1)$ in size), one can easily verify the ellipticity bound

$$\|\kappa_\varepsilon\|_{H^{k-1}(\Gamma_\varepsilon)} \lesssim_M 1 + \|\Delta_{\Gamma_\varepsilon}^m\kappa_\varepsilon\|_{H^1(\Gamma_\varepsilon)} \lesssim_M 1 + \||D|_{\Gamma_*}(\Delta_{\Gamma_\varepsilon}^m\kappa)_*\|_{L^2(\Gamma_*)}.$$

This concludes the proof. □

Step 2: Velocity regularization

Now, we aim to regularize the velocity $\tilde{v}_0$ on the ε^{-1} scale, which will help us to improve the regularization constant in (2.8.3). This will be needed to compensate for the losses in this constant in the upcoming transport step of the argument. Thanks to the previous step, we are reduced to the situation of regularizing on a fixed domain which has boundary regularized at the ε^{-1} scale. To perform this step of the regularization, we decompose the velocity $\tilde{v}_0$ into a rotational component which is tangent to the boundary and an irrotational component. Roughly speaking, we will then regularize the irrotational component of $\tilde{v}_0$ and leave the rotational component alone. We will then reconstruct the regularized velocity using the regularized irrotational part and the original (not regularized) rotational part of $\tilde{v}_0$. The precise procedure for doing this will come with some slight technical subtleties due to the fact that the normal to the surface is half a derivative less regular than the trace of the velocity on the boundary. We will outline these nuances in more detail shortly. Heuristically, the reason it is unnecessary to regularize the rotational part of $\tilde{v}_0$ in this construction is because the vorticity will not lose derivatives in the transport step of our argument later. In other words, the vorticity bound in (2.8.3) is expected to only worsen by an $\mathcal{O}_M(1)$ error when measured in H^k and an $\mathcal{O}_M(\varepsilon^{-1})$ error when measured in H^{k+1}, which is acceptable.

Proposition 2.8.6. Given the pair $(\tilde{v}_0, \Gamma_\varepsilon)$ from the previous step, there exists a regularization $\tilde{v}_0 \mapsto v_\varepsilon$ defined on Ω_ε which satisfies:

(i) (Energy monotonicity).

$$\mathcal{E}^k(v_\varepsilon, \Gamma_\varepsilon) \leq (1 + C(M)\varepsilon)\mathcal{E}^k(\tilde{v}_0, \Gamma_\varepsilon).$$

(ii) (Good pointwise approximation).

$$\begin{cases} v_\varepsilon = \tilde{v}_0 + \mathcal{O}_{C^1}(\varepsilon^2), \\ \nabla \cdot v_\varepsilon = 0. \end{cases} \tag{2.8.24}$$

(iii) (Regularization bounds). For each $n = 1, 2$ and $K(M)$ large enough, there holds

$$\|v_\varepsilon\|_{H^{k+n}(\Omega_\varepsilon)} \leq \frac{1}{4} K(M)\varepsilon^{-n}. \tag{2.8.25}$$

Remark 2.8.7. The bound in (2.8.25) with $n = 1$ ensures that the constant in (2.8.3) is improved at this stage. The H^{k+2} bound will be needed to close the bootstrap in the final Euler plus transport step of the iteration in the next section because this step loses derivatives for the velocity.

Proof. We begin by recalling the rotational/irrotational decomposition of $\tilde{v}_0$ from Appendix A of [140]:

$$\tilde{v}_0 := \tilde{v}_0^{rot} + \tilde{v}_0^{ir},$$

where for a divergence free function v, we have $v^{ir} := \nabla\mathcal{H}_\varepsilon\mathcal{N}_\varepsilon^{-1}(v\cdot n_{\Gamma_\varepsilon})$. Naïvely, we would like to directly regularize the irrotational part of $\tilde{v}_0$. However, this does not quite work because the normal n_{Γ_ε} is half a derivative less regular than the trace of $\tilde{v}_0$ on Γ_ε. To get around this, we will regularize the irrotational part of a suitable high frequency component of $\tilde{v}_0$. More precisely, let us consider a subregularization v_- of $\tilde{v}_0$, defined by $v_- := \Psi_{\leq\varepsilon^{-\frac{1}{2}}}\tilde{v}_0$, which lives on an $\varepsilon^{\frac{1}{2}}$ enlargement of Ω_ε. We then define $w := \tilde{v}_0 - v_-$. Loosely speaking, we think of w as the portion of $\tilde{v}_0$ with frequency greater than $\varepsilon^{-\frac{1}{2}}$. In contrast to the full irrotational part of $\tilde{v}_0$, it is safe to regularize the irrotational part of w. The heuristic reason for this is that at leading order the term $w\cdot n_{\Gamma_\varepsilon}$ can be interpreted as a high-low paraproduct. That is, the contribution of the portion where n_{Γ_ε} is at comparable or higher frequency compared to w is lower order as there is still a nontrivial high frequency component of w to compensate for the $\frac{1}{2}$ derivative discrepancy between the trace of w and n_{Γ_ε}.

For the irrotational part of w, the regularization we choose has to respect the energy monotonicity bound. We will see below that the spectral multiplier $\mathcal{P}_{\leq\varepsilon^{-1}}(\mathcal{N}_\varepsilon) := 1_{[-\varepsilon^{-1},\varepsilon^{-1}]}(\mathcal{N}_\varepsilon)$ is very convenient for this purpose. We therefore define the irrotational component of our regularization v_ε of $\tilde{v}_0$ by removing the high frequency part of $w\cdot n_{\Gamma_\varepsilon}$ as follows:

$$\begin{aligned} v_\varepsilon^{ir} &:= \tilde{v}_0^{ir} - \nabla\mathcal{H}_\varepsilon\mathcal{N}_\varepsilon^{-1}\mathcal{P}_{>\varepsilon^{-1}}(w\cdot n_{\Gamma_\varepsilon}) \\ &= v_-^{ir} + \nabla\mathcal{H}_\varepsilon\mathcal{N}_\varepsilon^{-1}\mathcal{P}_{\leq\varepsilon^{-1}}(w\cdot n_{\Gamma_\varepsilon}). \end{aligned}$$

For simplicity, let us write

$$w_\varepsilon^{ir} := \nabla\mathcal{H}_\varepsilon\mathcal{N}_\varepsilon^{-1}\mathcal{P}_{\leq\varepsilon^{-1}}(w\cdot n_{\Gamma_\varepsilon}).$$

We define the full regularization v_ε of $\tilde{v}_0$ by

$$v_\varepsilon := \tilde{v}_0^{rot} + v_\varepsilon^{ir}.$$

If k is large enough, the combination of Sobolev embedding, ellipticity of $\mathcal{N}$ and spectral calculus allows us to easily establish the pointwise approximation property (2.8.24). Next, we establish the regularization bound (2.8.25) for v_ε. We begin by writing

$$v_\varepsilon = v_- + w_\varepsilon^{ir} + w^{rot},$$

where w^{rot} is the rotational part of w. We then estimate piece by piece. It is first of all clear that the corresponding bound holds for v_-. So, we turn to estimating w_ε^{ir}. For this, we note the following preliminary bound for $\mathcal{N}_\varepsilon^{-1}$ on the space $\dot{H}^s(\Gamma_\varepsilon) := \{f \in H^s(\Gamma_\varepsilon) : \int_{\Gamma_\varepsilon} f = 0\}$ from Proposition A.5 in [140]:

$$\|\mathcal{N}_\varepsilon^{-1} f\|_{\dot{H}^s(\Gamma_\varepsilon)} \lesssim_M \|f\|_{H^{s-1}(\Gamma_\varepsilon)}, \quad 0 \leq s \leq 1. \tag{2.8.26}$$

From this and the functional calculus for $\mathcal{N}_\varepsilon$, we deduce in particular the low regularity bound

$$\|\mathcal{P}_{\leq \varepsilon^{-1}} \mathcal{N}_\varepsilon^{-1}(w \cdot n_{\Gamma_\varepsilon})\|_{L^2(\Gamma_\varepsilon)} \lesssim_M \|w \cdot n_{\Gamma_\varepsilon}\|_{H^{-1}(\Gamma_\varepsilon)}. \tag{2.8.27}$$

This will be useful for handling the low frequency errors in the estimate for w_ε^{ir}. Next we check that (2.8.26) and (2.8.27), in conjunction with Proposition 2.5.9, Proposition 2.5.21, Proposition 2.5.26 and the regularization bounds for n_{Γ_ε} and w, yield

$$\|w_\varepsilon^{ir}\|_{H^{k+n}(\Omega_\varepsilon)} \lesssim_{M,n} \|\Gamma_\varepsilon\|_{H^{k+\frac{1}{2}+n}} \|w \cdot n_{\Gamma_\varepsilon}\|_{H^{k-2}(\Gamma_\varepsilon)} + \|\mathcal{P}_{\leq \varepsilon^{-1}}(w \cdot n_{\Gamma_\varepsilon})\|_{H^{k-\frac{1}{2}+n}(\Gamma_\varepsilon)} \lesssim_M \varepsilon^{-n},$$

where the implicit constant can be taken to be much smaller than $K(M)$ since $K(M) \gg M$. Note that in the above estimate, we used the paraproduct structure of $w \cdot n_{\Gamma_\varepsilon}$. More specifically, in the case when $k - \frac{1}{2}$ derivatives fall on n_{Γ_ε}, we compensated the half derivative loss by an $\varepsilon^{\frac{1}{2}}$ gain from w.

Finally, we move on to showing the regularization bound for w^{rot}. Here, we use Proposition 2.5.27 to obtain

$$\begin{aligned}
\|w^{rot}\|_{H^{k+n}(\Omega_\varepsilon)} &\lesssim_M \|w^{rot}\|_{L^2(\Omega_\varepsilon)} + \|\nabla \times w\|_{H^{k+n-1}(\Omega_\varepsilon)} + \|\Gamma_\varepsilon\|_{H^{k+n-\frac{1}{2}}} \\
&\quad + \|\nabla^\top w^{rot} \cdot n_{\Gamma_\varepsilon}\|_{H^{k+n-\frac{3}{2}}(\Gamma_\varepsilon)} \\
&\lesssim_{M,K'(M)} \varepsilon^{-n} + \|w^{rot}\|_{L^2(\Omega_\varepsilon)} + \|\nabla^\top w^{rot} \cdot n_{\Gamma_\varepsilon}\|_{H^{k+n-\frac{3}{2}}(\Gamma_\varepsilon)},
\end{aligned}$$

where we used (2.8.3) for $\tilde{\omega}_0$. Again, the implicit constant can be taken to be much smaller than $K(M)$ if $K'(M)$ in (2.8.3) is small enough compared to $K(M)$. To estimate $\|w^{rot}\|_{L^2(\Omega_\varepsilon)}$, we simply use (2.8.26), the identity $w^{rot} = w - w^{ir}$ and the $H^{\frac{1}{2}}(\Gamma_\varepsilon) \to H^1(\Omega_\varepsilon)$ bound for $\mathcal{H}_\varepsilon$ to crudely estimate

$$\|w^{rot}\|_{L^2(\Omega_\varepsilon)} \lesssim_M \|w\|_{H^1(\Omega_\varepsilon)}. \tag{2.8.28}$$

Then, using

$$\nabla^\top w^{rot} \cdot n_{\Gamma_\varepsilon} = -w^{rot} \cdot \nabla^\top n_{\Gamma_\varepsilon},$$

Proposition 2.5.9, Proposition 2.5.11 and the regularization bounds for Γ_ε, we have (if k is large enough)

$$\|w^{rot}\|_{H^{k+n}(\Omega_\varepsilon)} \lesssim_M \varepsilon^{-n} + \|w^{rot}\|_{H^{k-1+n}(\Omega_\varepsilon)} + \varepsilon^{-\frac{1}{2}-n}\|w^{rot}\|_{H^{k-2}(\Omega_\varepsilon)},$$

which implies by interpolation and (2.8.28) that

$$\|w^{rot}\|_{H^{k+n}(\Omega_\varepsilon)} \lesssim_{M,n} \varepsilon^{-n} + \varepsilon^{-\frac{1}{2}-n}\|w^{rot}\|_{H^{k-2}(\Omega_\varepsilon)}.$$

From Proposition 2.5.27, the inequality (2.8.28) and the fact that w is localized to frequency $\geq \varepsilon^{-\frac{1}{2}}$, we easily obtain

$$\|w^{rot}\|_{H^{k-2}(\Omega_\varepsilon)} \lesssim_M \|w\|_{H^{k-2}(\Omega_\varepsilon)} \lesssim_M \varepsilon.$$

Therefore, we have

$$\|w^{rot}\|_{H^{k+n}(\Omega_\varepsilon)} \lesssim_{M,n} \varepsilon^{-n},$$

with implicit constant much smaller than $K(M)$. This yields the desired regularization bounds for v_ε.

Next, we turn to the energy monotonicity. The domain is fixed in this step, so it is advantageous to compare the difference between $\mathcal{E}^k(v_\varepsilon,\Gamma_\varepsilon)$ and $\mathcal{E}^k(\tilde{v}_0,\Gamma_\varepsilon)$ directly. It will also be convenient to write the first term in $\mathcal{E}^k(v,\Gamma)$ as a surface integral:

$$\|\nabla\mathcal{H}\mathcal{N}^{k-2}(a^{-1}D_t a)\|^2_{L^2(\Omega)} = \|\mathcal{N}^{k-\frac{3}{2}}(a^{-1}D_t a)\|^2_{L^2(\Gamma)},$$

using integration by parts and the functional calculus for $\mathcal{N}$. Moreover, since the vorticity ω_ε is the same as $\tilde{\omega}_0$, we may restrict our attention to the two surface components of the energy in this step of the argument.

We begin with a simple algebraic identity for the a_ε component of the surface energy:

$$\begin{aligned}\int_{\Gamma_\varepsilon} a_\varepsilon^{-1}|\mathcal{N}_\varepsilon^{k-1}a_\varepsilon|^2\,dS &= \int_{\Gamma_\varepsilon}\tilde{a}_0^{-1}|\mathcal{N}_\varepsilon^{k-1}\tilde{a}_0|^2\,dS + 2\int_{\Gamma_\varepsilon} a_\varepsilon^{-1}\mathcal{N}_\varepsilon^{k-1}a_\varepsilon\mathcal{N}_\varepsilon^{k-1}(a_\varepsilon-\tilde{a}_0)\,dS\\ &\quad - \|a_\varepsilon^{-\frac{1}{2}}\mathcal{N}_\varepsilon^{k-1}(a_\varepsilon-\tilde{a}_0)\|^2_{L^2(\Gamma_\varepsilon)} + \mathcal{O}_M(\varepsilon).\end{aligned}$$

To derive an analogous relation for the other portion of the surface energy, we note that from the integer bounds for $\mathcal{N}$ in Section 2.5 and the identity $\|\mathcal{N}^{k-\frac{3}{2}}f\|_{L^2(\Gamma)} = \|\nabla\mathcal{H}\mathcal{N}^{k-2}f\|_{L^2(\Omega)}$, we have the estimate $\|\mathcal{N}_\varepsilon^{k-\frac{3}{2}}\|_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)\to L^2(\Gamma_\varepsilon)} \lesssim_M 1$. On the other hand, we have the elliptic regularity estimate

$$\|\tilde{a}_0 - a_\varepsilon\|_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)} \lesssim_M \|\tilde{p}_0 - p_\varepsilon\|_{H^k(\Omega_\varepsilon)} \lesssim_M \|\tilde{v}_0 - v_\varepsilon\|_{H^{k-1}(\Omega_\varepsilon)} \lesssim_M \varepsilon.$$

Together, these imply that

$$\begin{aligned}\int_{\Gamma_\varepsilon} |\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}D_t a_\varepsilon)|^2\, dS &= \int_{\Gamma_\varepsilon} |\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(\tilde{a}_0^{-1}D_t\tilde{a}_0)|^2\, dS \\ &\quad + 2\int_{\Gamma_\varepsilon} \mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}D_t a_\varepsilon)\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}(D_t a_\varepsilon - D_t\tilde{a}_0))\, dS \\ &\quad - \|\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}(D_t a_\varepsilon - D_t\tilde{a}_0))\|_{L^2(\Gamma_\varepsilon)}^2 + \mathcal{O}_M(\varepsilon).\end{aligned}$$

Motivated by the identities above, let us define the "energy" corresponding to $\tilde{v}_0 - v_\varepsilon$ by

$$\mathcal{E}^k(\tilde{v}_0 - v_\varepsilon) := \|\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}(D_t\tilde{a}_0 - D_t a_\varepsilon))\|_{L^2(\Gamma_\varepsilon)}^2 + \|a_\varepsilon^{-\frac{1}{2}}\mathcal{N}_\varepsilon^{k-1}(\tilde{a}_0 - a_\varepsilon)\|_{L^2(\Gamma_\varepsilon)}^2.$$

In light of the above identities, it suffices to show that

$$\begin{aligned}2\int_{\Gamma_\varepsilon} \mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}D_t a_\varepsilon)\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}(D_t a_\varepsilon - D_t\tilde{a}_0))\, dS + 2\int_{\Gamma_\varepsilon} a_\varepsilon^{-1}\mathcal{N}_\varepsilon^{k-1}a_\varepsilon\mathcal{N}_\varepsilon^{k-1}(a_\varepsilon - \tilde{a}_0)\, dS \\ \leq C(M)\varepsilon + \mathcal{E}^k(\tilde{v}_0 - v_\varepsilon).\end{aligned}$$

Our starting point is to observe the leading order relation given in the following lemma.

Lemma 2.8.8. We have the following relation between $D_t a_\varepsilon - D_t\tilde{a}_0$ and $(v_\varepsilon - \tilde{v}_0)\cdot n_{\Gamma_\varepsilon}$:

$$a_\varepsilon^{-1}(D_t a_\varepsilon - D_t\tilde{a}_0) = -\mathcal{N}_\varepsilon((v_\varepsilon - \tilde{v}_0)\cdot n_{\Gamma_\varepsilon}) + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}(\varepsilon). \tag{2.8.29}$$

Proof. We begin by noting the bound

$$\|\tilde{v}_0 - v_\varepsilon\|_{H^{k-1}(\Omega_\varepsilon)} \lesssim_M \varepsilon \tag{2.8.30}$$

and the elliptic regularity estimate

$$\|\tilde{p}_0 - p_\varepsilon\|_{H^k(\Omega_\varepsilon)} \lesssim_M \|\tilde{v}_0 - v_\varepsilon\|_{H^{k-1}(\Omega_\varepsilon)} \lesssim_M \varepsilon.$$

Using the equation for $D_t p$ from (2.7.4) we may therefore write

$$D_t a_\varepsilon - D_t\tilde{a}_0 = n_{\Gamma_\varepsilon}\cdot\nabla(v_\varepsilon - \tilde{v}_0)\cdot\nabla p_\varepsilon - n_{\Gamma_\varepsilon}\cdot\nabla\Delta^{-1}(\Delta(v_\varepsilon - \tilde{v}_0)\cdot\nabla p_\varepsilon) + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}(\varepsilon).$$

Then, using the standard identity $\mathcal{N}f_{|\Gamma} = n\cdot\nabla f - n\cdot\nabla\Delta^{-1}\Delta f$ and commuting $n_{\Gamma_\varepsilon}\cdot\nabla$ in the first term and Δ in the second term above, we can verify, from (2.8.30),

$$D_t a_\varepsilon - D_t\tilde{a}_0 = -\mathcal{N}_\varepsilon(a_\varepsilon(v_\varepsilon - \tilde{v}_0)\cdot n_{\Gamma_\varepsilon}) + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}(\varepsilon).$$

The conclusion then follows by commuting $\mathcal{N}_\varepsilon$ with a_ε using the Leibniz rule for $\mathcal{N}_\varepsilon$ and (2.8.30). In the case when everything falls on a_ε, we also compensate with the surface regularization bound (2.8.10) and the associated improvement in the bound for $\tilde{v}_0 - v_\varepsilon$ when measured in lower regularity Sobolev norms. □

We now turn to the a_ε component of the energy, which is straightforward. Indeed, by elliptic regularity,

$$2\int_{\Gamma_\varepsilon} a_\varepsilon^{-1}\mathcal{N}_\varepsilon^{k-1}a_\varepsilon\mathcal{N}_\varepsilon^{k-1}(a_\varepsilon-\tilde{a}_0)\,dS \lesssim_M \|a_\varepsilon-\tilde{a}_0\|_{H^{k-1}(\Gamma_\varepsilon)} \lesssim_M \|v_\varepsilon-\tilde{v}_0\|_{H^{k-\frac{1}{2}}(\Omega_\varepsilon)}.$$

To estimate $v_\varepsilon-\tilde{v}_0$, we observe the identity $v_\varepsilon-\tilde{v}_0=\nabla\mathcal{H}_\varepsilon\mathcal{N}_\varepsilon^{-1}\mathcal{P}_{>\varepsilon^{-1}}((v_\varepsilon-\tilde{v}_0)\cdot n_{\Gamma_\varepsilon})$, which follows from the idempotence $\mathcal{P}_{>\varepsilon^{-1}}=\mathcal{P}_{>\varepsilon^{-1}}^2$. Using this, Lemma 2.8.8 and ellipticity of $\mathcal{N}_\varepsilon$, we have

$$\|v_\varepsilon-\tilde{v}_0\|_{H^{k-\frac{1}{2}}(\Omega_\varepsilon)} \lesssim_M \varepsilon^{\frac{1}{2}}\|(v_\varepsilon-\tilde{v}_0)\cdot n_{\Gamma_\varepsilon}\|_{H^{k-\frac{1}{2}}(\Gamma_\varepsilon)} \lesssim_M \varepsilon^{\frac{1}{2}}(\mathcal{E}^k(\tilde{v}_0-v_\varepsilon))^{\frac{1}{2}}+C(M)\varepsilon,$$

which suffices by Cauchy-Schwarz.

Next, we move to the more difficult portion of the energy which involves D_ta_ε. We start by combining Lemma 2.8.8 with $(v_\varepsilon-\tilde{v}_0)\cdot n_{\Gamma_\varepsilon}=-\mathcal{P}_{>\varepsilon^{-1}}(w\cdot n_{\Gamma_\varepsilon})$ to obtain the relation

$$\begin{aligned}&\int_{\Gamma_\varepsilon}\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}D_ta_\varepsilon)\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}(D_ta_\varepsilon-D_t\tilde{a}_0))\,dS\\ &=\int_{\Gamma_\varepsilon}\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}D_ta_\varepsilon)\mathcal{N}_\varepsilon^{k-\frac{1}{2}}\mathcal{P}_{>\varepsilon^{-1}}(w\cdot n_{\Gamma_\varepsilon})\,dS+\mathcal{O}_M(\varepsilon).\end{aligned}$$

Define p_- and D_ta_- in the usual way using the relevant Laplace equations. We split the above integral into the two components,

$$\begin{aligned}\int_{\Gamma_\varepsilon}\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}D_ta_\varepsilon)\mathcal{N}_\varepsilon^{k-\frac{1}{2}}\mathcal{P}_{>\varepsilon^{-1}}(w\cdot n_{\Gamma_\varepsilon})\,dS&=\int_{\Gamma_\varepsilon}\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}D_ta_-)\mathcal{N}_\varepsilon^{k-\frac{1}{2}}\mathcal{P}_{>\varepsilon^{-1}}(w\cdot n_{\Gamma_\varepsilon})\,dS\\ +\int_{\Gamma_\varepsilon}\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}(D_ta_\varepsilon-D_ta_-))\mathcal{N}_\varepsilon^{k-\frac{1}{2}}\mathcal{P}_{>\varepsilon^{-1}}(w\cdot n_{\Gamma_\varepsilon})\,dS.&\end{aligned}\tag{2.8.31}$$

We begin by studying the first term in (2.8.31). By self-adjointness of $\mathcal{N}_\varepsilon$, we have

$$\begin{aligned}\int_{\Gamma_\varepsilon}\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}D_ta_-)\mathcal{N}_\varepsilon^{k-\frac{1}{2}}\mathcal{P}_{>\varepsilon^{-1}}(w\cdot n_{\Gamma_\varepsilon})\,dS&=\int_{\Gamma_\varepsilon}\mathcal{N}_\varepsilon^{k-\frac{1}{2}}(a_\varepsilon^{-1}D_ta_-)\mathcal{N}_\varepsilon^{k-\frac{3}{2}}\mathcal{P}_{>\varepsilon^{-1}}(w\cdot n_{\Gamma_\varepsilon})\,dS\\ \lesssim_M \varepsilon\|a_\varepsilon^{-1}D_ta_-\|_{H^{k-\frac{1}{2}}(\Gamma_\varepsilon)}\|\mathcal{P}_{>\varepsilon^{-1}}\mathcal{N}_\varepsilon^{k-\frac{1}{2}}(w\cdot n_{\Gamma_\varepsilon})\|_{L^2(\Gamma_\varepsilon)}+\mathcal{O}_M(\varepsilon),&\end{aligned}$$

where we used the multiplier $\mathcal{P}_{>\varepsilon^{-1}}$ to recover a power of $\mathcal{N}_\varepsilon$ in the high frequency term. Next, we show that $\|a_\varepsilon^{-1}D_ta_-\|_{H^{k-\frac{1}{2}}(\Gamma_\varepsilon)}\lesssim_M\varepsilon^{-\frac{1}{2}}$. By Sobolev product estimates and the fact that $\|a_\varepsilon^{-1}\|_{H^{k-\frac{1}{2}}(\Gamma_\varepsilon)}\lesssim_M\varepsilon^{-\frac{1}{2}}$, it suffices to show the same estimate for $\|D_ta_-\|_{H^{k-\frac{1}{2}}(\Gamma_\varepsilon)}$. To see this, recall that, by definition,

$$D_ta_-=n_{\Gamma_\varepsilon}\cdot\nabla v_-\cdot\nabla p_--n_{\Gamma_\varepsilon}\cdot\nabla D_tp_-.$$

Note then that by Proposition 2.5.11 we have the estimate $\|\nabla v_-\|_{H^{k-\frac{1}{2}}(\Gamma_\varepsilon)} \lesssim_M \varepsilon^{-\frac{1}{2}}$, since v_- is regularized at the $\varepsilon^{-\frac{1}{2}}$ scale. Moreover, as $n_{\Gamma_\varepsilon} = \mathcal{O}_{H^{k-1}}(1)$ and Γ_ε is regularized at the ε^{-1} scale, we have $\|n_{\Gamma_\varepsilon}\|_{H^{k-\frac{1}{2}}(\Gamma_\varepsilon)} \lesssim_M \varepsilon^{-\frac{1}{2}}$. By Proposition 2.5.11 and Proposition 2.5.19, we also have $\|\nabla p_-\|_{H^{k-\frac{1}{2}}(\Gamma_\varepsilon)} \lesssim_M \varepsilon^{-\frac{1}{2}}$. Therefore, by Proposition 2.5.9, we have $\|n_{\Gamma_\varepsilon} \cdot \nabla v_- \cdot \nabla p_-\|_{H^{k-\frac{1}{2}}(\Gamma_\varepsilon)} \lesssim_M \varepsilon^{-\frac{1}{2}}$.

Using Proposition 2.5.19 and the fact that the pressure terms in the Laplace equation for $D_t p_-$ always appear to one half derivative lower than top order, a similar analysis yields $\|n_{\Gamma_\varepsilon} \cdot \nabla D_t p_-\|_{H^{k-\frac{1}{2}}(\Gamma_\varepsilon)} \lesssim_M \varepsilon^{-\frac{1}{2}}$. Therefore, we obtain from Lemma 2.8.8 the bound,

$$\begin{aligned}\int_{\Gamma_\varepsilon} \mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1} D_t a_-)\mathcal{N}_\varepsilon^{k-\frac{1}{2}}\mathcal{P}_{>\varepsilon^{-1}}(w \cdot n_{\Gamma_\varepsilon})\, dS &\lesssim_M \varepsilon^{\frac{1}{2}}\|\mathcal{P}_{>\varepsilon^{-1}}\mathcal{N}_\varepsilon^{k-\frac{1}{2}}(w \cdot n_{\Gamma_\varepsilon})\|_{L^2(\Gamma_\varepsilon)} + \mathcal{O}_M(\varepsilon) \\ &\lesssim_M \varepsilon^{\frac{1}{2}}(\mathcal{E}^k(\tilde{v}_0 - v_\varepsilon))^{\frac{1}{2}} + \mathcal{O}_M(\varepsilon),\end{aligned}$$

as desired. It remains to deal with the other term in (2.8.31). For this, we need to expand $D_t a_\varepsilon - D_t a_-$. As a first reduction, we note that we can replace every appearance of p_- with p_ε in the definition of $D_t a_-$ if we allow for $\mathcal{O}_M(\varepsilon^{\frac{1}{2}})$ errors. This is because $\|p_\varepsilon - p_-\|_{H^k(\Omega_\varepsilon)} \lesssim_M \|v_\varepsilon - v_-\|_{H^{k-1}(\Omega_\varepsilon)} \lesssim_M \varepsilon^{\frac{1}{2}}$. Hence, we have

$$\begin{aligned}D_t a_- &= n_{\Gamma_\varepsilon} \cdot \nabla v_- \cdot \nabla p_\varepsilon - n_{\Gamma_\varepsilon} \cdot \nabla \Delta_{\Omega_\varepsilon}^{-1}(4\mathrm{tr}(\nabla^2 p_\varepsilon \cdot \nabla v_-) + 2\mathrm{tr}(\nabla v_-)^3 + \Delta v_- \cdot \nabla p_\varepsilon) \\ &\quad + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}(\varepsilon^{\frac{1}{2}}).\end{aligned}$$

We may also replace the lower order terms involving v_- by v_ε. Arguing similarly to Lemma 2.8.8, we then obtain the key identity

$$\begin{aligned}D_t a_- - D_t a_\varepsilon &= a_\varepsilon \mathcal{N}_\varepsilon((v_\varepsilon - v_-) \cdot n_{\Gamma_\varepsilon}) + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}(\varepsilon^{\frac{1}{2}}) \\ &= a_\varepsilon \mathcal{P}_{\leq \varepsilon^{-1}}\mathcal{N}_\varepsilon(w \cdot n_{\Gamma_\varepsilon}) + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}(\varepsilon^{\frac{1}{2}}).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\int_{\Gamma_\varepsilon} \mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}(D_t a_\varepsilon - D_t a_-))\mathcal{N}_\varepsilon^{k-\frac{1}{2}}\mathcal{P}_{>\varepsilon^{-1}}(w \cdot n_{\Gamma_\varepsilon})\, dS \\ &\lesssim_M -\int_{\Gamma_\varepsilon} \mathcal{N}_\varepsilon^{k-\frac{1}{2}}\mathcal{P}_{\leq \varepsilon^{-1}}(w \cdot n_{\Gamma_\varepsilon})\mathcal{N}_\varepsilon^{k-\frac{1}{2}}\mathcal{P}_{>\varepsilon^{-1}}(w \cdot n_{\Gamma_\varepsilon})\, dS + \varepsilon^{\frac{1}{2}}\|\mathcal{N}_\varepsilon^{k-\frac{1}{2}}\mathcal{P}_{>\varepsilon^{-1}}(w \cdot n_{\Gamma_\varepsilon})\|_{L^2(\Gamma_\varepsilon)}.\end{aligned}$$

The first term on the right-hand side above vanishes by orthogonality (this term is the reason we reweighted the energy in the first place) and the latter term is controlled by $\varepsilon^{\frac{1}{2}}(\mathcal{E}^k(\tilde{v}_0 - v_\varepsilon))^{\frac{1}{2}}$. Therefore, we obtain the desired bound for the $D_t a_\varepsilon$ portion of the energy. This completes the proof of Proposition 2.8.6. □

Step 3: Euler plus transport iteration

In this subsection, we construct the iterate (v_1, Γ_1) from the regularized data $(v_\varepsilon, \Gamma_\varepsilon)$. Intuitively, what remains to be done is to carry out something akin to the Euler iteration

$$v_1 := v_\varepsilon - \varepsilon(v_\varepsilon \cdot \nabla v_\varepsilon + \nabla p_\varepsilon + ge_d)$$

and then the domain transport

$$x_1(x) := x + \varepsilon v_\varepsilon(x).$$

Unfortunately, performed individually, these steps lose a full derivative in each iteration. Therefore, it is important that these two steps be carried out together. This will reduce the derivative loss and allow us to exploit a discrete version of the energy cancellation seen in the energy estimates. We will then use the regularization bounds from the previous subsections to control any remaining errors in the iteration. To carry out this process, we have the following proposition.

Proposition 2.8.9. Given $(v_\varepsilon, \Gamma_\varepsilon)$ as in the previous step, there exists an iteration $(v_\varepsilon, \Gamma_\varepsilon) \mapsto (v_1, \Gamma_1)$ such that the following properties hold:

(i) (Approximate solution).

$$\begin{cases} v_1 = v_\varepsilon - \varepsilon(v_\varepsilon \cdot \nabla v_\varepsilon + \nabla p_\varepsilon + ge_d) + \mathcal{O}_{C^1}(\varepsilon^2) \quad \text{on } \Omega_1 \cap \Omega_\varepsilon, \\ \nabla \cdot v_1 = 0 \quad \text{on } \Omega_1, \\ \Omega_1 = (I + \varepsilon v_\varepsilon)\Omega_\varepsilon. \end{cases}$$

(ii) (Energy monotonicity bound).

$$\mathcal{E}^k(v_1, \Gamma_1) \le (1 + C(M)\varepsilon)\mathcal{E}^k(v_\varepsilon, \Gamma_\varepsilon).$$

Moreover, v_1 and ω_1 satisfy the inductive bounds (2.8.6).

We define the change of coordinates $x_1(x) := x + \varepsilon v_\varepsilon(x)$ and the iterated domain Ω_1 by

$$\Omega_1 := (I + \varepsilon v_\varepsilon)\Omega_\varepsilon.$$

To define v_1, we proceed in two steps. First, we define

$$\tilde{v}_1(x_1) := v_\varepsilon - \varepsilon(\nabla p_\varepsilon + ge_d). \tag{2.8.32}$$

We note that $\tilde{v}_1$ is not divergence free, so we define the full iterate v_1 by correcting the divergence of $\tilde{v}_1$ by a gradient potential:

$$v_1 := \tilde{v}_1 - \nabla\Delta_{\Omega_1}^{-1}(\nabla \cdot \tilde{v}_1).$$

At this point, we can verify the inductive bound (2.8.6) for v_1 and ω_1. We start with v_1. We recall that we have to show that

$$\|v_1\|_{H^{k+1}(\Omega_1)} \leq K(M)\varepsilon^{-1}.$$

As a first step, using the regularization bound (2.8.25) for v_ε from the previous section, we have from the definition of $\tilde{v}_1$, the regularization bounds (2.8.10) for Γ_ε and the balanced elliptic estimate Proposition 2.5.19,

$$\|\tilde{v}_1\|_{H^{k+n}(\Omega_1)} \leq \frac{1}{3}K(M)\varepsilon^{-n}, \tag{2.8.33}$$

for $n = 0, 1, 2$. Next, we aim to control the error between v_1 and $\tilde{v}_1$ in $H^k(\Omega_1)$ and $H^{k+1}(\Omega_1)$ (but not $H^{k+2}(\Omega_1)$). We have for $n = 0, 1$ from the balanced elliptic estimate Proposition 2.5.19,

$$\begin{aligned}
\|v_1 - \tilde{v}_1\|_{H^{k+n}(\Omega_1)} &\lesssim_M \|\Gamma_1\|_{H^{k+\frac{1}{2}+n}} \|\nabla \cdot \tilde{v}_1\|_{H^{k-2}(\Omega_1)} + \|\nabla \cdot \tilde{v}_1\|_{H^{k-1+n}(\Omega_1)} \\
&\lesssim_M \varepsilon^{-\frac{1}{2}-n} \|\nabla \cdot \tilde{v}_1\|_{H^{k-2}(\Omega_1)} + \|\nabla \cdot \tilde{v}_1\|_{H^{k-1+n}(\Omega_1)}.
\end{aligned}$$

Above, we used the H^{k+1} and H^{k+2} (depending on if n is 0 or 1) regularization bounds for v_ε, Moser estimates, the bounds for Γ_ε and the relation $\Gamma_1 = (I + \varepsilon v_\varepsilon)(\Gamma_\varepsilon)$ to control $\|\Gamma_1\|_{H^{k+\frac{1}{2}+n}} \lesssim_M \varepsilon^{-\frac{1}{2}-n}$. By using the definition of $\tilde{v}_1$ and the regularization bounds for v_ε, it is straightforward to see that the divergence, $\nabla \cdot \tilde{v}_1$, contributes an error of size $\mathcal{O}_{H^{k-1+n}(\Omega_1)}(\varepsilon^{\frac{3}{2}-n})$ and also $\mathcal{O}_{H^{k-2}(\Omega_1)}(\varepsilon^2)$. Note that for this computation, one must use the cancellation between the velocity and the pressure in (2.8.32) in order to see the desired gain. Therefore, we have

$$\|\nabla\Delta_{\Omega_1}^{-1}(\nabla \cdot \tilde{v}_1)\|_{H^{k+n}(\Omega_1)} = \|v_1 - \tilde{v}_1\|_{H^{k+n}(\Omega_1)} \lesssim_M \varepsilon^{\frac{3}{2}-n}.$$

From this and (2.8.33), we conclude the inductive bound

$$\|v_1\|_{H^{k+1}(\Omega_1)} \leq K(M)\varepsilon^{-1},$$

and the leading order expansion for $v_1(x_1)$ in $H^k(\Omega_\varepsilon)$,

$$v_1(x_1) = v_\varepsilon - \varepsilon(\nabla p_\varepsilon + g e_d) + \mathcal{O}_{H^k(\Omega_\varepsilon)}(\varepsilon^{\frac{3}{2}}).$$

If k is large enough, then the leading order expansion (2.8) with $\mathcal{O}_{C^1}(\varepsilon^2)$ error can be seen by slightly modifying the above argument. Now, we verify the inductive bound $\|\omega_1\|_{H^{k+n}(\Omega_1)} \leq \varepsilon^{-1-n}(K'(M)+\varepsilon C(M))$ for $n=0,1$. It suffices to establish this for $\tilde{\omega}_1$ since v_1 and $\tilde{v}_1$ agree up to a gradient. Taking curl in the definition of $\tilde{v}_1$ and using that $\omega_\varepsilon = \tilde{\omega}_0$, we have

$$\|\nabla \times (\tilde{v}_1(x_1))\|_{H^{k+n}(\Omega_\varepsilon)} \leq \|\tilde{\omega}_0\|_{H^{k+n}(\Omega_\varepsilon)} \leq K'(M)\varepsilon^{-1-n}. \tag{2.8.34}$$

By chain rule, using (2.8.33) and the regularization bounds for v_ε, we have

$$\|\tilde{\omega}_1(x_1)\|_{H^{k+n}(\Omega_\varepsilon)} \leq \|\nabla \times (\tilde{v}_1(x_1))\|_{H^{k+n}(\Omega_\varepsilon)} + C(M)\varepsilon^{-n},$$

which by a change of variables and (2.8.34) yields

$$\|\tilde{\omega}_1\|_{H^{k+n}(\Omega_1)} \leq \varepsilon^{-1-n}(K'(M)+\varepsilon C(M)),$$

as desired. Note that in the above two lines, we treated $C(M)$ as an arbitrary constant, and relabelled it from line to line. Importantly, we did not do this for $K(M)$ and $K'(M)$.

Next, we work towards establishing the energy monotonicity bound for the transport part of the argument. As a first step, we aim to relate the good variables associated to the iterate v_1 to the good variables associated to v_ε at the regularity level of the energy. We have the following lemma.

Lemma 2.8.10 (Relations between the good variables)**.** The following relations hold:

(i) (Relation for ω_1).

$$\omega_1(x_1) = \omega_\varepsilon + \mathcal{O}_{H^{k-1}(\Omega_\varepsilon)}(\varepsilon).$$

(ii) (Relation for p_1).

$$p_1(x_1) - p_\varepsilon - \varepsilon D_t p_\varepsilon = \mathcal{O}_{H^{k+\frac{1}{2}}(\Omega_\varepsilon)}(\varepsilon). \tag{2.8.35}$$

(iii) (Relation for a_1).

$$a_1(x_1) = a_\varepsilon + \varepsilon D_t a_\varepsilon + \mathcal{O}_{H^{k-1}(\Gamma_\varepsilon)}(\varepsilon). \tag{2.8.36}$$

(iv) (Relation for $D_t a_1$).

$$D_t a_1(x_1) = D_t a_\varepsilon - \varepsilon a_\varepsilon \mathcal{N}_\varepsilon a_\varepsilon + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}(\varepsilon).$$

Proof. The relation for ω_1 is immediate. Next, we move to the relations for p_1 and a_1. By the chain rule and the Laplace equation (2.7.4) for $D_t p_\varepsilon$, we have

$$\begin{aligned}\Delta(p_1(x_1)) &= (\Delta p_1)(x_1) + \varepsilon\Delta v_\varepsilon\cdot(\nabla p_1)(x_1) + 2\varepsilon\nabla v_\varepsilon\cdot(\nabla^2 p_1)(x_1) + \mathcal{O}_{H^{k-\frac{3}{2}}(\Omega_\varepsilon)}(\varepsilon)\\ &= \Delta p_\varepsilon + \varepsilon\Delta D_t p_\varepsilon + \varepsilon\Delta v_\varepsilon\cdot((\nabla p_1)(x_1) - \nabla p_\varepsilon) + \mathcal{O}_{H^{k-\frac{3}{2}}(\Omega_\varepsilon)}(\varepsilon)\\ &= \Delta p_\varepsilon + \varepsilon\Delta D_t p_\varepsilon + \mathcal{O}_{H^{k-\frac{3}{2}}(\Omega_\varepsilon)}(\varepsilon),\end{aligned}$$

where in the last line, we controlled $\varepsilon\Delta v_\varepsilon\cdot((\nabla p_1)(x_1) - \nabla p_\varepsilon) = \mathcal{O}_{H^{k-\frac{3}{2}}(\Omega_\varepsilon)}(\varepsilon)$ by using the regularization bounds for v_ε as well as the error bound $(\nabla p_1)(x_1) - \nabla p_\varepsilon = \mathcal{O}_{L^\infty(\Omega_\varepsilon)}(\varepsilon)$, which is gotten by performing an $H^k(\Omega_\varepsilon)$ elliptic estimate in the second line, using the fact that $p_1(x_1) - p_\varepsilon$ vanishes on Γ_ε and that each of the source terms can be estimated directly in $H^{k-2}(\Omega_\varepsilon)$ (but not in $H^{k-\frac{3}{2}}(\Omega_\varepsilon)$). Therefore, since $p_1(x_1) - p_\varepsilon - \varepsilon D_t p_\varepsilon$ vanishes on Γ_ε, we may now do a $H^{k+\frac{1}{2}}(\Omega_\varepsilon)$ elliptic estimate to obtain the finer bound,

$$p_1(x_1) - p_\varepsilon - \varepsilon D_t p_\varepsilon = \mathcal{O}_{H^{k+\frac{1}{2}}(\Omega_\varepsilon)}(\varepsilon), \tag{2.8.37}$$

which gives (2.8.35). We also deduce from this that

$$\begin{aligned}(\nabla p_1)(x_1) &= \nabla p_\varepsilon + \varepsilon\nabla D_t p_\varepsilon - \varepsilon\nabla v_\varepsilon\cdot(\nabla p_1)(x_1) + \mathcal{O}_{H^{k-\frac{1}{2}}(\Omega_\varepsilon)}(\varepsilon)\\ &= \nabla p_\varepsilon + \varepsilon D_t\nabla p_\varepsilon + \mathcal{O}_{H^{k-\frac{1}{2}}(\Omega_\varepsilon)}(\varepsilon).\end{aligned}$$

From this we see that

$$\begin{aligned}a_1(x_1) &= a_\varepsilon + \varepsilon D_t a_\varepsilon - (n_{\Gamma_1}(x_1) - n_{\Gamma_\varepsilon})\cdot(\nabla p_1)(x_1) + \mathcal{O}_{H^{k-1}(\Gamma_\varepsilon)}(\varepsilon)\\ &= a_\varepsilon + \varepsilon D_t a_\varepsilon + \mathcal{O}_{H^{k-1}(\Gamma_\varepsilon)}(\varepsilon),\end{aligned}$$

where in the last line we used

$$\begin{aligned}(n_{\Gamma_1}(x_1) - n_{\Gamma_\varepsilon})\cdot(\nabla p_1)(x_1) &= -a_1(x_1)(n_{\Gamma_1}(x_1) - n_{\Gamma_\varepsilon})\cdot n_{\Gamma_1}(x_1) = -a_1(x_1)\frac{1}{2}|n_{\Gamma_1}(x_1) - n_{\Gamma_\varepsilon}|^2\\ &= \mathcal{O}_{H^{k-1}(\Gamma_\varepsilon)}(\varepsilon)\end{aligned}$$

This gives the relation (2.8.36).

Next, we prove the relation for $D_t a_1$. First, we see that

$$\begin{aligned}-(D_t\nabla p_1)(x_1) + D_t\nabla p_\varepsilon &= ((\nabla v_1\cdot\nabla p_1)(x_1) - \nabla v_\varepsilon\cdot\nabla p_\varepsilon) - ((\nabla D_t p_1)(x_1) - \nabla D_t p_\varepsilon)\\ &= ((\nabla v_1)(x_1) - \nabla v_\varepsilon)\cdot\nabla p_\varepsilon\\ &\quad - ((\nabla D_t p_1)(x_1) - \nabla D_t p_\varepsilon) + \mathcal{O}_{H^{k-1}(\Omega_\varepsilon)}(\varepsilon).\end{aligned} \tag{2.8.38}$$

To control the second term on the right-hand side above, we write out the Laplace equation for $D_t p_1(x_1)$:

$$\Delta(D_t p_1(x_1)) = (\Delta D_t p_1)(x_1) + \mathcal{O}_{H^{k-2}(\Omega_\varepsilon)}(\varepsilon).$$

By a similar analysis to the proof of (2.8.36) and the relation

$$(\Delta v_1)(x_1) = \Delta(v_1(x_1)) + \mathcal{O}_{H^{k-2}(\Omega_\varepsilon)}(\varepsilon) = \Delta v_\varepsilon - \varepsilon\nabla\Delta p_\varepsilon + \mathcal{O}_{H^{k-2}(\Omega_\varepsilon)}(\varepsilon) = \Delta v_\varepsilon + \mathcal{O}_{H^{k-2}(\Omega_\varepsilon)}(\varepsilon),$$

we obtain

$$\begin{aligned}(\Delta D_t p_1)(x_1) &= \Delta D_t p_\varepsilon + (\Delta v_1 \cdot \nabla p_1)(x_1) - \Delta v_\varepsilon \cdot \nabla p_\varepsilon + 4\mathrm{tr}(\nabla v_1 \cdot \nabla^2 p_1)(x_1) \\ &\quad - 4tr\nabla v_\varepsilon \cdot \nabla^2 p_\varepsilon + \mathcal{O}_{H^{k-2}(\Omega_\varepsilon)}(\varepsilon) \\ &= \Delta D_t p_\varepsilon + 4tr\left(\nabla v_\varepsilon \cdot ((\nabla^2 p_1)(x_1) - \nabla^2 p_\varepsilon)\right) + \mathcal{O}_{H^{k-2}(\Omega_\varepsilon)}(\varepsilon) \\ &= \Delta D_t p_\varepsilon + \mathcal{O}_{H^{k-2}(\Omega_\varepsilon)}(\varepsilon),\end{aligned}$$

where in the last line, we used (2.8.37) and that $\varepsilon D_t p_\varepsilon = \mathcal{O}_{H^k(\Omega_\varepsilon)}(\varepsilon)$. Combining the above with (2.8.38), one obtains by elliptic regularity,

$$-D_t\nabla p_1(x_1) + D_t\nabla p_\varepsilon = ((\nabla v_1)(x_1) - \nabla v_\varepsilon) \cdot \nabla p_\varepsilon + \mathcal{O}_{H^{k-1}(\Omega_\varepsilon)}(\varepsilon).$$

Then, noting from (2.8.36) that

$$(D_t\nabla p_1)(x_1) \cdot (n_{\Gamma_1}(x_1) - n_{\Gamma_\varepsilon}) = (D_t\nabla p_1)(x_1) \cdot (a_\varepsilon^{-1}\nabla p_\varepsilon - (a_1^{-1}\nabla p_1)(x_1)) = \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}(\varepsilon)$$

and using the fact that Δp_ε is lower order, we obtain

$$\begin{aligned}D_t a_1(x_1) - D_t a_\varepsilon &= -a_\varepsilon n_{\Gamma_\varepsilon} \cdot \nabla(v_1(x_1) - v_\varepsilon) \cdot n_{\Gamma_\varepsilon} - (D_t\nabla p_1)(x_1) \cdot (n_{\Gamma_1}(x_1) - n_{\Gamma_\varepsilon}) \\ &\quad + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}(\varepsilon) \\ &= \varepsilon a_\varepsilon n_{\Gamma_\varepsilon} \cdot \nabla\nabla p_\varepsilon \cdot n_{\Gamma_\varepsilon} + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}(\varepsilon) \\ &= \varepsilon a_\varepsilon \mathcal{N}_\varepsilon \nabla p_\varepsilon \cdot n_{\Gamma_\varepsilon} + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}(\varepsilon).\end{aligned}$$

Finally, noting that $\mathcal{N}_\varepsilon n_{\Gamma_\varepsilon} \cdot n_{\Gamma_\varepsilon}$ is lower order, we have, thanks to the Leibniz rule for $\mathcal{N}_\varepsilon$,

$$\varepsilon a_\varepsilon \mathcal{N}_\varepsilon \nabla p_\varepsilon \cdot n_{\Gamma_\varepsilon} = -\varepsilon a_\varepsilon \mathcal{N}_\varepsilon(n_{\Gamma_\varepsilon} a_\varepsilon) \cdot n_{\Gamma_\varepsilon} = -\varepsilon a_\varepsilon \mathcal{N}_\varepsilon a_\varepsilon + \mathcal{O}_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)}(\varepsilon).$$

Therefore, we have the desired relation for $D_t a_1$. This completes the proof of the lemma. $\square$

Energy monotonicity. To finish the proof of Proposition 2.8.9, it remains to establish energy monotonicity. The following lemma will allow us to more easily work with the relations in Lemma 2.8.10.

Lemma 2.8.11. Define the "pulled-back" energy $\mathcal{E}_*^k(v_1,\Gamma_1)$ by

$$\begin{aligned}\mathcal{E}_*^k(v_1,\Gamma_1) := 1 + \|\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_1^{-1}(x_1)D_t a_1(x_1))\|_{L^2(\Gamma_\varepsilon)}^2 + \|a_1^{-\frac{1}{2}}(x_1)\mathcal{N}_\varepsilon^{k-1}(a_1(x_1))\|_{L^2(\Gamma_\varepsilon)}^2 \\ + \|\omega_1(x_1)\|_{H^{k-1}(\Omega_\varepsilon)}^2 + \|v_1(x_1)\|_{L^2(\Omega_\varepsilon)}^2.\end{aligned}$$

Then we have the relation

$$\mathcal{E}^k(v_1,\Gamma_1) \leq \mathcal{E}_*^k(v_1,\Gamma_1) + \mathcal{O}_M(\varepsilon).$$

Before proving the above lemma, we show how it easily implies the desired energy monotonicity bound. In light of Lemma 2.8.11, it suffices to establish the bound

$$\mathcal{E}_*^k(v_1,\Gamma_1) \leq (1+C(M)\varepsilon)\mathcal{E}^k(v_\varepsilon,\Gamma_\varepsilon).$$

The monotonicity bound for the vorticity is immediate from Lemma 2.8.10. For the surface components of the energy, we first use Lemma 2.8.10, the fact that $\|\mathcal{N}_\varepsilon^{k-\frac{3}{2}}\|_{H^{k-\frac{3}{2}}(\Gamma_\varepsilon)\to L^2(\Gamma_\varepsilon)} \lesssim_M 1$ and the regularization bounds for Γ_ε and v_ε to obtain

$$\begin{aligned}&\int_{\Gamma_\varepsilon} |\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_1^{-1}(x_1)D_t a_1(x_1))|^2\,dS - \int_{\Gamma_\varepsilon} |\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}D_t a_\varepsilon)|^2\,dS \\ &= 2\int_{\Gamma_\varepsilon} \mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}D_t a_\varepsilon)\mathcal{N}_\varepsilon^{k-\frac{3}{2}}(a_\varepsilon^{-1}((D_t a_1)(x_1)-D_t a_\varepsilon))\,dS + \mathcal{O}_M(\varepsilon) \\ &= -2\varepsilon\int_{\Gamma_\varepsilon} a_\varepsilon^{-1}\mathcal{N}_\varepsilon^{k-1}D_t a_\varepsilon \mathcal{N}_\varepsilon^{k-1}a_\varepsilon\,dS + \mathcal{O}_M(\varepsilon),\end{aligned} \tag{2.8.39}$$

where in the last line, we used the commutator estimate $\|[\mathcal{N}_\varepsilon^{k-1},a_\varepsilon^{-1}]D_t a_\varepsilon\|_{L^2(\Gamma_\varepsilon)} \lesssim_M 1$ to shift a factor of $\mathcal{N}_\varepsilon^{\frac{1}{2}}$ onto $\mathcal{N}_\varepsilon^{k-\frac{3}{2}}D_t a_\varepsilon$. We similarly observe the leading order relation for the other component of the energy by using (2.8.36) to obtain,

$$\begin{aligned}\int_{\Gamma_\varepsilon} a_1^{-1}(x_1)|\mathcal{N}_\varepsilon^{k-1}(a_1(x_1))|^2\,dS - \int_{\Gamma_\varepsilon} a_\varepsilon^{-1}|\mathcal{N}_\varepsilon^{k-1}a_\varepsilon|^2\,dS = 2\varepsilon\int_{\Gamma_\varepsilon} a_\varepsilon^{-1}\mathcal{N}_\varepsilon^{k-1}D_t a_\varepsilon\mathcal{N}_\varepsilon^{k-1}a_\varepsilon\,dS \\ + \mathcal{O}_M(\varepsilon).\end{aligned}$$

The first term on the right-hand side of the above relation cancels the main term on the right-hand side of (2.8.39). Combining everything together then gives

$$\mathcal{E}^k(v_1,\Gamma_1) \leq (1+C(M)\varepsilon)\mathcal{E}^k(v_\varepsilon,\Gamma_\varepsilon),$$

as desired. It remains now to establish Lemma 2.8.11.

Proof of Lemma 2.8.11. By a simple change of variables, it is clear that the difference between $\|\omega_1(x_1)\|^2_{H^{k-1}(\Omega_\varepsilon)}$ and $\|\omega_1\|^2_{H^{k-1}(\Omega_1)}$ contributes only $\mathcal{O}_M(\varepsilon)$ errors. This is likewise true for the L^2 component of the velocity. The main difficulty is in dealing with the surface components of the energy. For this, we need the following proposition.

Proposition 2.8.12. *Let $-\frac{1}{2} \le s \le k-2$ and let $f \in H^{s+1}(\Gamma_1)$. Then we have the following bound on Γ_ε:*

$$\|(\mathcal{N}_1 f)(x_1) - \mathcal{N}_\varepsilon(f(x_1))\|_{H^s(\Gamma_\varepsilon)} \lesssim_M \varepsilon \|f\|_{H^{s+1}(\Gamma_1)}.$$

Proof. First, we handle the case $s = -\frac{1}{2}$. If $g \in C^\infty(\Gamma_\varepsilon)$, we write $h = g(x_1^{-1})\mathcal{H}_1 J$ where J is the Jacobian corresponding to the change of variables $y = x_1(x)$. Then we have by the divergence theorem,

$$\begin{aligned}\int_{\Gamma_\varepsilon} g((\mathcal{N}_1 f)(x_1) - \mathcal{N}_\varepsilon(f(x_1)))\, dS &= \int_{\Gamma_1} h\mathcal{N}_1 f\, dS - \int_{\Gamma_\varepsilon} g\mathcal{N}_\varepsilon(f(x_1))\, dS \\ &= \int_{\Omega_1} \nabla\mathcal{H}_1 h \cdot \nabla\mathcal{H}_1 f\, dx - \int_{\Omega_\varepsilon} \nabla\mathcal{H}_\varepsilon g \cdot \nabla\mathcal{H}_\varepsilon(f(x_1))\, dx.\end{aligned}$$

Using again the change of variables $x \mapsto x_1$ for the first term in the second line above, together with the estimates

$$\|\mathcal{H}_1 h\|_{H^1(\Omega_1)} \lesssim_M \|g\|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} \quad \text{and} \quad \|(\nabla\mathcal{H}_1 f)(x_1)\|_{L^2(\Omega_\varepsilon)} \lesssim_M \|f\|_{H^{\frac{1}{2}}(\Gamma_1)},$$

it is easy to verify

$$\begin{aligned}\int_{\Gamma_\varepsilon} g((\mathcal{N}_1 f)(x_1) - \mathcal{N}_\varepsilon(f(x_1)))\, dS \lesssim_M &\int_{\Omega_\varepsilon} \nabla((\mathcal{H}_1 h)(x_1) - \mathcal{H}_\varepsilon g) \cdot (\nabla\mathcal{H}_1 f)(x_1)\, dx \\ &+ \int_{\Omega_\varepsilon} \nabla H_\varepsilon g \cdot \nabla((\mathcal{H}_1 f)(x_1) - \mathcal{H}_\varepsilon(f(x_1)))\, dx \\ &+ \varepsilon\|g\|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}\|f\|_{H^{\frac{1}{2}}(\Gamma_1)}.\end{aligned} \tag{2.8.40}$$

We label the first and second terms on the right-hand side above by I_1 and I_2. For I_1, we use the fact that on Γ_ε we have

$$(\mathcal{H}_1 h)(x_1) - \mathcal{H}_\varepsilon g = (J(x_1) - I)g$$

to obtain the following simple elliptic estimate

$$I_1 \lesssim_M \varepsilon\|f\|_{H^{\frac{1}{2}}(\Gamma_1)}\|g\|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} + \|f\|_{H^{\frac{1}{2}}(\Gamma_1)}\|\Delta((\mathcal{H}_1 h)(x_1))\|_{H^{-1}(\Omega_\varepsilon)} \lesssim_M \varepsilon\|f\|_{H^{\frac{1}{2}}(\Gamma_1)}\|g\|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)},$$

where we used the chain rule and that $\mathcal{H}_1 h$ is harmonic to estimate $\Delta((\mathcal{H}_1 h)(x_1))$. A similar elliptic estimate yields the same bound for I_2. This establishes the case $s = -\frac{1}{2}$. By interpolation, we only need to handle the remaining cases when $\frac{1}{2} \leq s \leq k-2$. As a starting point, we have from some simple manipulations with the chain rule and the trace inequality,

$$\|(\mathcal{N}_1 f)(x_1) - \mathcal{N}_\varepsilon(f(x_1))\|_{H^s(\Gamma_\varepsilon)} \lesssim \varepsilon\|f\|_{H^{s+1}(\Gamma_1)} + \|(n_{\Gamma_1}(x_1) - n_{\Gamma_\varepsilon}) \cdot (\nabla\mathcal{H}_1 f)(x_1)\|_{H^s(\Gamma_\varepsilon)} + \|(\mathcal{H}_1 f)(x_1) - \mathcal{H}_\varepsilon(f(x_1))\|_{H^{s+\frac{3}{2}}(\Omega_\varepsilon)}.$$

By writing $n_{\Gamma_1}(x_1) - n_{\Gamma_\varepsilon} = a_\varepsilon^{-1}\nabla p_\varepsilon - a_1^{-1}(x_1)(\nabla p_1)(x_1)$ and using the relations in Lemma 2.8.10 and that $s \leq k-2$, the second term on the right is straightforward to control by $\varepsilon\|f\|_{H^{s+1}(\Gamma_\varepsilon)}$. For the third term, we do an elliptic estimate analogous to the $s = -\frac{1}{2}$ case (using that $(\mathcal{H}_1 f)(x_1) - \mathcal{H}_\varepsilon(f(x_1)) = 0$ on Γ_ε) to obtain

$$\|(\mathcal{H}_1 f)(x_1) - \mathcal{H}_\varepsilon(f(x_1))\|_{H^{s+\frac{3}{2}}(\Omega_\varepsilon)} \lesssim_M \|\Delta((\mathcal{H}_1 f)(x_1))\|_{H^{s-\frac{1}{2}}(\Omega_\varepsilon)} \lesssim_M \varepsilon\|f\|_{H^{s+1}(\Gamma_1)}.$$

This completes the proof. □

Now we return to the proof of Lemma 2.8.11. We note first that

$$\|(\mathcal{N}_1^{k-1} a_1)(x_1) - \mathcal{N}_\varepsilon^{k-1}(a_1(x_1))\|_{L^2(\Gamma_\varepsilon)} \lesssim \|\mathcal{N}_\varepsilon(\mathcal{N}_1^{k-2} a_1)(x_1) - \mathcal{N}_\varepsilon^{k-1}(a_1(x_1))\|_{L^2(\Gamma_\varepsilon)} + \|\mathcal{N}_\varepsilon(\mathcal{N}_1^{k-2} a_1)(x_1) - (\mathcal{N}_1^{k-1} a_1)(x_1)\|_{L^2(\Gamma_\varepsilon)}.$$

Applying Proposition 2.8.12 to the term in the second line and using the $H^1 \to L^2$ bound for $\mathcal{N}$, we have

$$\|(\mathcal{N}_1^{k-1} a_1)(x_1) - \mathcal{N}_\varepsilon^{k-1}(a_1(x_1))\|_{L^2(\Gamma_\varepsilon)} \lesssim_M \|(\mathcal{N}_1^{k-2} a_1)(x_1) - \mathcal{N}_\varepsilon^{k-2}(a_1(x_1))\|_{H^1(\Gamma_\varepsilon)} + \mathcal{O}_M(\varepsilon).$$

Iterating this procedure and applying Proposition 2.8.12 $k-2$ times, we see that we have

$$\|(\mathcal{N}_1^{k-1} a_1)(x_1) - \mathcal{N}_\varepsilon^{k-1}(a_1(x_1))\|_{L^2(\Gamma_\varepsilon)} \lesssim_M \varepsilon.$$

It follows from the above and a change of variables that we have

$$\|a_1^{-\frac{1}{2}}\mathcal{N}_1^{k-1} a_1\|_{L^2(\Gamma_1)}^2 \leq \|a_1^{-\frac{1}{2}}(x_1)\mathcal{N}_\varepsilon^{k-1}(a_1(x_1))\|_{L^2(\Gamma_\varepsilon)}^2 + \mathcal{O}_M(\varepsilon).$$

To conclude the proof of Lemma 2.8.11, we need to show that

$$\|\nabla\mathcal{H}_1(\mathcal{N}_1^{k-2}(a_1^{-1} D_t a_1))\|_{L^2(\Omega_1)}^2 \leq \|\nabla\mathcal{H}_\varepsilon\mathcal{N}_\varepsilon^{k-2}(a_1^{-1}(x_1) D_t a_1(x_1))\|_{L^2(\Omega_\varepsilon)}^2 + \mathcal{O}_M(\varepsilon).$$

From a change of variables, we see that

$$\|\nabla\mathcal{H}_1(\mathcal{N}_1^{k-2}(a_1^{-1} D_t a_1))\|_{L^2(\Omega_1)}^2 - \|\nabla\mathcal{H}_\varepsilon\mathcal{N}_\varepsilon^{k-2}(a_1^{-1}(x_1) D_t a_1(x_1))\|_{L^2(\Omega_\varepsilon)}^2 \lesssim_M \mathcal{J} + \mathcal{O}_M(\varepsilon),$$

where

$$\mathcal{J} := \|(\nabla \mathcal{H}_1 \mathcal{N}_1^{k-2}(a_1^{-1} D_t a_1))(x_1) - \nabla \mathcal{H}_\varepsilon \mathcal{N}_\varepsilon^{k-2}(a^{-1}(x_1) D_t a_1(x_1))\|_{L^2(\Omega_\varepsilon)}.$$

By elliptic regularity, it is easy to verify the bound

$$\mathcal{J} \lesssim_M \|(\mathcal{N}_1^{k-2}(a_1^{-1} D_t a_1))(x_1) - \mathcal{N}_\varepsilon^{k-2}(a_1^{-1}(x_1) D_t a_1(x_1))\|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} + \mathcal{O}_M(\varepsilon).$$

From here, we use Proposition 2.8.12 similarly to the other surface term in the energy to estimate

$$\|(\mathcal{N}_1^{k-2}(a_1^{-1} D_t a_1))(x_1) - \mathcal{N}_\varepsilon^{k-2}(a_1^{-1}(x_1) D_t a_1(x_1))\|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} \lesssim_M \varepsilon.$$

This completes the proof. □

Convergence of the iteration scheme

We have now arrived at the final step of the existence proof, where we use our one step iteration result in Theorem 2.8.1 in order to prove the existence of regular solutions. Precisely, we aim to establish the following theorem.

Theorem 2.8.13. Let k be a sufficiently large even integer and $M > 0$. Let $(v_0, \Gamma_0) \in \mathbf{H}^k$ be an initial data set so that $\|(v_0, \Gamma_0)\|_{\mathbf{H}^k} \leq M$. Then there exists $T = T(M)$ and a solution (v, Γ) to the free boundary incompressible Euler equations on $[0, T]$ with this initial data and the following regularity properties:

$$(v, \Gamma) \in L^\infty([0, T]; \mathbf{H}^k) \cap C([0, T]; \mathbf{H}^{k-1})$$

with the uniform bound

$$\|(v, \Gamma)(t)\|_{\mathbf{H}^k} \lesssim_M 1, \qquad t \in [0, T].$$

We remark that the solution we construct is unique by the result in Theorem 2.4.6. One missing piece here is the lack of continuity in $\mathbf{H}^k$, which does not follow from the proof below. However, this will be rectified in the next section. We now turn to the proof of the theorem.

Proof. Starting from the initial data $(v_0, \Gamma_0) \in \mathbf{H}^k$ with $\Gamma_0 \in \Lambda_* := \Lambda(\Gamma_*, \varepsilon_0, \delta)$, for each small time scale ε we construct a discrete approximate solution $(v_\varepsilon, \Gamma_\varepsilon)$ which is defined at discrete times $t = 0, \varepsilon, 2\varepsilon, \ldots$, as follows:

(i) We define $(v_\varepsilon(0), \Gamma_\varepsilon(0))$ by directly regularizing (v_0, Γ_0) at scale ε. Such a regularization is provided by Proposition 2.6.2 with $\varepsilon = 2^{-j}$. In view of the higher regularity bound there, these regularized data will satisfy the hypothesis of our one step Theorem 2.8.1, with M replaced by $\tilde{M} = C(A)M$.

(ii) We inductively define the approximate solutions $(v_\varepsilon(j\varepsilon), \Gamma_\varepsilon(j\varepsilon))$ by repeatedly applying the iteration step in Theorem 2.8.1.

To control the growth of the $\mathbf{H}^k$ norms of $(v_\varepsilon, \Gamma_\varepsilon)$ we rely on the energy monotonicity relation, together with the coercivity property in Theorem 2.7.1 (and also the relation (2.8.2)). We use the energy coercivity in both ways. At time $t = 0$ we have

$$\mathcal{E}^k(v_\varepsilon(0), \Gamma_\varepsilon(0)) \leq C_1(A)M.$$

We let our iteration continue for as long as

$$\begin{aligned} &\mathcal{E}^k(v_\varepsilon(j\varepsilon), \Gamma_\varepsilon(j\varepsilon)) \leq 2C_1(A)M, \\ &\Gamma_\varepsilon(j\varepsilon) \in 2\Lambda_* := \Lambda(\Gamma_*, \varepsilon_0, 2\delta). \end{aligned} \tag{2.8.41}$$

As long as this happens, using the coercivity in the other direction we get

$$\|(v_\varepsilon(j\varepsilon), \Gamma_\varepsilon(j\varepsilon))\|_{\mathbf{H}^k} \leq C_2(A)M.$$

Now by the energy monotonicity bound (2.8.4) we conclude that

$$\mathcal{E}^k(v_\varepsilon(j\varepsilon), \Gamma_\varepsilon(j\varepsilon)) \leq (1 + C(C_2(A)M)\varepsilon)^j \mathcal{E}^k(v_\varepsilon(0), \Gamma_\varepsilon(0)) \leq e^{C(C_2(A)M)\varepsilon j} \mathcal{E}^k(v_\varepsilon(0), \Gamma_\varepsilon(0)).$$

Hence we can reach the cutoff given by the first inequality in (2.8.41) no earlier than at time

$$t = \varepsilon j < T(M) := C(C_2(A)M)^{-1},$$

which is a bound that does not depend on ε. Similarly, for the second requirement in (2.8.41), the relations (2.8.5) ensure that at each step the boundary only moves by $\mathcal{O}(\varepsilon)$, so by step j it moves at most by $\mathcal{O}(j\varepsilon)$. This leads to a similar constraint as above on the number of steps. Analogous reasoning shows that the vorticity growth in (2.8.6) is also harmless on this time scale.

To summarize, we have proved that the discrete approximate solutions $(v_\varepsilon, \Gamma_\varepsilon)$ are all defined up to the above time $T(M)$, and satisfy the uniform bound

$$\|(v_\varepsilon, \Gamma_\varepsilon)\|_{\mathbf{H}^k} \lesssim_M 1 \qquad \text{in } [0, T],$$

with $\Gamma_\varepsilon \in 2\Lambda_*$. Since k is large enough, by Sobolev embeddings, this yields uniform bounds, say, in C^3,

$$\|v_\varepsilon\|_{C^3} + \|\eta_\varepsilon\|_{C^3} \lesssim_M 1 \qquad \text{in } [0,T], \tag{2.8.42}$$

where $\eta_\varepsilon := \eta_{\Gamma_\varepsilon}$ is the defining function for $\Gamma_\varepsilon \in 2\Lambda_*$.

The other piece of information we have about v_ε comes from (2.8.5). However, this only tells us what happens over a single time step of size ε, so we need to iterate it over multiple steps. We begin with the first relation for the velocity in (2.8.5), which implies that

$$|v_\varepsilon(t,x) - v_\varepsilon(s,y)| + |\nabla v_\varepsilon(t,x) - \nabla v_\varepsilon(s,y)| \lesssim_M |t-s| + |x-y|, \qquad t-s=\varepsilon.$$

Iterating this we arrive at

$$|v_\varepsilon(t,x) - v_\varepsilon(s,y)| + |\nabla v_\varepsilon(t,x) - \nabla v_\varepsilon(s,y)| \lesssim_M |t-s| + |x-y|, \qquad t,s \in \varepsilon\mathbb{N}\cap[0,T]. \tag{2.8.43}$$

A similar reasoning based on the last part of (2.8.5) yields

$$\|\eta_\varepsilon(t) - \eta_\varepsilon(s)\|_{C^1} \lesssim_M |t-s|, \qquad t,s \in \varepsilon\mathbb{N}\cap[0,T]. \tag{2.8.44}$$

Similarly, from (2.8.35) in Lemma 2.8.10 and the elliptic estimate $\|D_t p_\varepsilon\|_{H^k} \lesssim_M 1$ for each time, we also get a difference bound for the pressure; namely,

$$|\nabla p_\varepsilon(t,x) - \nabla p_\varepsilon(s,y)| \lesssim_M |t-s| + |x-y|, \qquad t,s \in \varepsilon\mathbb{N}\cap[0,T]. \tag{2.8.45}$$

Equipped with the last three Lipschitz bounds in time, we are now able to return to (2.8.5) and reiterate in order to obtain second order information. As above, we begin with the first relation in (2.8.5). Here we reiterate directly, using the bounds (2.8.43) and (2.8.45) in order to compare the expressions on the right at different times in the uniform norm. This yields

$$v_\varepsilon(t) = v_\varepsilon(s) - (t-s)(v_\varepsilon(s)\cdot\nabla v_\varepsilon(s) + \nabla p_\varepsilon(s) + ge_d) + \mathcal{O}((t-s)^2), \qquad t,s \in \varepsilon\mathbb{N}\cap[0,T]. \tag{2.8.46}$$

The same procedure applied to the last component of (2.8.5) yields

$$\Omega_\varepsilon(t) = (I + (t-s)v_\varepsilon(s))\Omega_\varepsilon(s) + \mathcal{O}((t-s)^2), \qquad t,s \in \varepsilon\mathbb{N}\cap[0,T]. \tag{2.8.47}$$

We now have enough information about our approximate solutions $(v_\varepsilon, \Gamma_\varepsilon)$, and we seek to obtain the desired solution (v, Γ) by taking the limit of $(v_\varepsilon, \Gamma_\varepsilon)$ on a subsequence as $\varepsilon \to 0$. For this it is convenient to take ε of the form $\varepsilon = 2^{-m}$, where we let $m \to \infty$. Then the time domains of the corresponding approximate solutions v_m are nested.

Starting from the Lipschitz bounds (2.8.43), (2.8.44) and (2.8.45), a careful application of the Arzela-Ascoli theorem yields uniformly convergent subsequences

$$\eta_m \to \eta, \qquad v_m \to v, \qquad \nabla v_m \to \nabla v, \qquad \nabla p_m \to \nabla p, \tag{2.8.48}$$

whose limits still satisfies the bounds (2.8.43), (2.8.44) and (2.8.45). It remains to show that (v, Γ) is the desired solution to the free boundary incompressible Euler equations, with Γ defined by η and p, where p is the associated pressure.

We begin by upgrading the spatial regularity of v and η. For this we observe that for $t \in 2^{-j}\mathbb{N} \cap [0, T]$ we can pass to the limit as $m \to \infty$ in (2.8.42) to obtain the uniform bound

$$\|v\|_{C^3} + \|\eta\|_{C^3} \lesssim_M 1.$$

Since both v and η are Lipschitz continuous in t, this extends easily to all $t \in [0, T]$. A similar argument applies to the $\mathbf{H}^k$ norm of (v, Γ).

Next we show that (v, Γ) solves the free boundary incompressible Euler equations, which we do in several steps:

i) The initial data. The fact that at the initial time we have $(v(0), \Gamma(0)) = (v_0, \Gamma_0)$ follows directly from the construction of $(v_\varepsilon(0), \Gamma_\varepsilon(0))$; namely, by Proposition 2.6.2.

ii) The pressure equation. To verify that p is the pressure associated to v and Γ we simply use the uniform convergence of ∇v_m, η_m and ∇p_m in order to pass to the limit in the pressure equation (2.1.5).

iii) The incompressible Euler equations. Here we directly use the uniform convergence (2.8.48) in order to pass to the limit in (2.8.46). This implies that v is differentiable in time, and that the incompressible Euler equations are verified.

iv) The kinematic boundary condition. Arguing as above, this time we directly use the uniform convergence (2.8.48) in order to pass to the limit in (2.8.47).

Finally, the $C(\mathbf{H}^{k-1})$ regularity of (v, Γ) follows directly from the incompressible Euler equations and the kinematic boundary condition. □

2.9 Rough solutions

In this section, we aim to construct solutions in the state space $\mathbf{H}^s$ as limits of regular solutions for $s > \frac{d}{2} + 1$. The general procedure for executing this construction will be as follows.

(i) We regularize the initial data.

(ii) We prove uniform bounds for the corresponding regularized solutions.

(iii) We show convergence of the regularized solutions in a weaker topology.

(iv) We combine the difference estimates and the uniform $\mathbf{H}^s$ bounds from step (ii) to obtain convergence in the $\mathbf{H}^s$ topology.

As will be seen below, this procedure carries with it various subtleties since it involves comparing functions defined on different domains. In addition, we must carefully address the fact that our control parameters in the difference and energy estimates are not entirely consistent.

Initial data regularization

Let $(v_0, \Gamma_0) \in \mathbf{H}^s$ be an initial data. The first step is to place Γ_0 within a suitable collar $\Lambda_* = \Lambda(\Gamma_*, \varepsilon, \delta)$ with $\delta \ll 1$. Since $\Gamma_0 \in H^s \subseteq C^{1,\varepsilon+}$, Γ_* is easily obtained by regularizing Γ_0 on a small enough spatial scale. We remark that the price to pay for a small enough regularization scale is that the higher Sobolev norms H^k of Γ_* will be large; but this is acceptable, as explained in Remark 2.3.4.

Let $M := \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}$ denote the data size measured relative to the collar Λ_*, and write c_0 for the lower bound on the Taylor term. We begin by constructing regularized data at each dyadic scale 2^j. For this, we define $\Gamma_{0,j}$ (along with $\Omega_{0,j}$) by regularizing the collar parameterization η_0. More specifically, we define $\eta_{0,j} := P_{\leq j}\eta_0$, where the meaning of $P_{\leq j}$ is as in Section 2.6. Then, we define the regularized velocity $v_{0,j} := \Psi_{\leq j} v_0$. Here, we recall that, as long as j is much larger than M, $v_{0,j}$ is defined on some 2^{-j} enlargement of both $\Omega_{0,j}$ and Ω_0. Indeed, by Sobolev embeddings, we have the distance bound

$$|\eta_{0,j} - \eta_0| \lesssim_M 2^{-\frac{3}{2}j}.$$

Moreover, for such j, we stay in the collar and have a uniform lower bound on the Taylor term.

Uniform bounds and lifespan of regular solutions

By Theorem 2.8.13, the regularized data $(v_{0,j}, \Gamma_{0,j})$ from the previous step generate corresponding smooth solutions (v_j, Γ_j). Our goal now is to establish uniform bounds for these regular solutions and, in particular, show that they have a lifespan which depends only on the size of the initial data (v_0, Γ_0) in $\mathbf{H}^s$, Taylor sign and the collar. To do this, we carry out a bootstrap argument with the $\mathbf{H}^s$ norm of (v_j, Γ_j).

In the argument below, we will be working with the enlarged control parameter $\tilde{B}_j(t) := \|v_j\|_{W^{1,\infty}(\Omega_j)} + \|\Gamma_j\|_{C^{1,\frac{1}{2}}} + \|D_t p_j\|_{W^{1,\infty}(\Omega_j)}$ for the corresponding solution (v_j, Γ_j). Note that the reason we work for now with $\tilde{B}_j$ instead of just $B_j(t) := \|v_j\|_{W^{1,\infty}(\Omega_j)} + \|\Gamma_j\|_{C^{1,\frac{1}{2}}}$ is because we will make use of the difference estimates which require control of $D_t p_j$. By elliptic regularity and Sobolev embeddings, it is easy to see that $\tilde{B}_j$ is controlled by some polynomial in $\|(v_j, \Gamma_j)\|_{\mathbf{H}^s}$.

Fix some large parameters A_0 and B_0 depending only on the numerical constants for the data (M, c_0 and so forth) such that $A_0 \ll B_0$. As alluded to above, we make the bootstrap assumption

$$\begin{aligned} &\|(v_j, \Gamma_j)(t)\|_{\mathbf{H}^s} \leq 2B_0, \quad A_j(t) \leq 2A_0, \quad a_j(t) \geq \frac{c_0}{2}, \quad \Gamma_j(t) \in 2\Lambda_*, \quad t \in [0,T], \\ &j(M) =: j_0 \leq j \leq j_1, \end{aligned}$$

with $j(M)$ sufficiently large depending on M, in a time interval $[0,T]$ where all the (v_j, Γ_j) are defined as smooth solutions with boundaries in the collar. Above, j_1 is some finite but arbitrarily large parameter, introduced for technical convenience to ensure that we run the bootstrap on only finitely many solutions at a time. Our aim will be to show that we can improve this bootstrap assumption as long as $T \leq T_0$ for some time $T_0 > 0$ which is independent of j_1.

For any large integer $k > s > \frac{d}{2} + 1$ as in Theorem 2.8.13, we may consider the solutions (v_j, Γ_j) as solutions in $\mathbf{H}^k$. In light of Theorems 2.7.1 and 2.8.13, for each $j \geq j_0$, the solution (v_j, Γ_j) can be continued past time T in $\mathbf{H}^k$ (and therefore $\mathbf{H}^s$) as long as the bootstrap is satisfied. Morally speaking, our choice for T_0 will be

$$T_0 \ll \frac{1}{P(B_0)},$$

for some fixed polynomial P, though this is not entirely accurate, as T_0 will also depend on the collar and c_0. Thanks to the energy bound in Theorem 2.7.1, if the bootstrap could be extended to such a T_0, it would guarantee uniform $\mathbf{H}^k$ bounds for (v_j, Γ_j) for any integer $k > \frac{d}{2} + 1$ in terms of its initial data in $\mathbf{H}^k$. The main difficulty we face is that, a priori, the $\mathbf{H}^s$ bounds for (v_j, Γ_j) do not necessarily propagate for noninteger s. The goal, therefore, is to establish $\mathbf{H}^s$ bounds for noninteger s. We will do this by working solely with the energy estimates for integer indices and the difference estimates.

We begin by letting c_j be the $\mathbf{H}^s$ admissible frequency envelope for the initial data (v_0, Γ_0) given by (2.6.3). We let $\alpha \geq 1$ be such that $k = s + \alpha$ is an integer. From Proposition 2.6.6 we know that the regularized data $(v_{0,j}, \Gamma_{0,j})$ satisfy the bounds

$$\|(v_{0,j}, \Gamma_{0,j})\|_{\mathbf{H}^{s+\alpha}} \lesssim_{A_0} 2^{\alpha j} c_j \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}. \tag{2.9.1}$$

From the energy bounds in Theorem 2.7.1 and the bootstrap hypothesis, we deduce from (2.9.1) and the definition of c_j that

$$\|(v_j, \Gamma_j)(t)\|_{\mathbf{H}^{s+\alpha}} \lesssim_{A_0} 2^{\alpha j} c_j (1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}), \qquad t \in [0, T], \tag{2.9.2}$$

as long as $T \leq T_0 \ll \frac{1}{P(B_0)}$. One may think of this as a high frequency bound, which roughly speaking allows us to control frequencies $\gtrsim 2^j$ in (v_j, Γ_j). Note that in (2.9.2) we suppressed the implicit dependence on the Taylor term and the collar. We will do this throughout the subsection except when these terms are of primary importance, as it will be clear that our argument can handle these minor technicalities.

To estimate low frequencies we use the difference estimates. Precisely, at the initial time we claim that we have the difference bound

$$D((v_{0,j}, \Gamma_{0,j}), (v_{0,j+1}, \Gamma_{0,j+1})) \lesssim_{A_0} 2^{-2js} c_j^2 \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}^2. \tag{2.9.3}$$

This bound is clear by Proposition 2.6.6 for the first term in (2.4.2). To see this for the surface integral, we use that on $\tilde{\Gamma}_{0,j} := \partial(\Omega_{0,j} \cap \Omega_{0,j+1})$, the pressure difference $p_{0,j} - p_{0,j+1}$ is proportional (with implicit constant depending on A_0) to the distance between $\Gamma_{0,j}$ and $\Gamma_{0,j+1}$, measured using the displacement function (2.4.1). Combining this with a change of variables, we have

$$\int_{\tilde{\Gamma}_{0,j}} |p_{0,j} - p_{0,j+1}|^2 \, dS \approx_{A_0} \|\eta_{0,j+1} - \eta_{0,j}\|_{L^2(\Gamma_*)}^2 \lesssim_{A_0} 2^{-2js} c_j^2 \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}^2,$$

from which (2.9.3) follows. By Theorem 2.4.2, we can propagate the difference bound (2.9.3) to obtain

$$D((v_j,\Gamma_j)(t),(v_{j+1},\Gamma_{j+1})(t)) \lesssim_{A_0} 2^{-2js}c_j^2\|(v_0,\Gamma_0)\|^2_{\mathbf{H}^s}, \qquad t\in[0,T], \tag{2.9.4}$$

as long as $T\le T_0 \ll \frac{1}{P(B_0)}$. In particular, this gives by a similar argument to the above,

$$\|v_{j+1}-v_j\|_{L^2(\Omega_j\cap\Omega_{j+1})}, \quad \|\eta_{j+1}-\eta_j\|_{L^2(\Gamma_*)} \lesssim_{A_0} 2^{-js}c_j\|(v_0,\Gamma_0)\|_{\mathbf{H}^s}. \tag{2.9.5}$$

Now, the goal is to combine the high frequency bound (2.9.2) and the L^2 difference bound (2.9.5) in order to obtain a uniform $\mathbf{H}^s$ bound of the form

$$\|(v_j,\Gamma_j)\|_{\mathbf{H}^s} \lesssim_{A_0} 1+\|(v_0,\Gamma_0)\|_{\mathbf{H}^s},$$

for $T\le T_0$. To establish such a bound for Γ_j, we consider the telescoping series on Γ_* given by

$$\eta_j=\eta_{j_0}+\sum_{j_0\le l\le j-1}(\eta_{l+1}-\eta_l). \tag{2.9.6}$$

From the higher energy bound (2.9.2), we have for each $j_0\le l\le j-1$,

$$\|\eta_{l+1}-\eta_l\|_{H^{s+\alpha}(\Gamma_*)} \lesssim_{A_0} 2^{l\alpha}c_l(1+\|(v_0,\Gamma_0)\|_{\mathbf{H}^s}). \tag{2.9.7}$$

Using the telescoping sum and interpolation, it is straightforward to verify from (2.9.5), (2.9.7) and an argument similar to Proposition 2.6.6 (see also [71]) that for each $k\ge 0$,

$$\|P_k\eta_j\|_{H^s(\Gamma_*)} \lesssim_{A_0} c_k(1+\|(v_0,\Gamma_0)\|_{\mathbf{H}^s}). \tag{2.9.8}$$

As a consequence, by almost orthogonality, we obtain the uniform bound

$$\|\Gamma_j\|_{H^s} \lesssim_{A_0} 1+\|(v_0,\Gamma_0)\|_{\mathbf{H}^s}. \tag{2.9.9}$$

Next, we turn to the bound for v_j. We first note that the analogous decomposition to (2.9.6) for v_j does not work because for each $l\le j-1$, v_l and v_{l+1} are defined on different domains. However, we can compare v_l and v_{l+1} by first regularizing each function $v_l\mapsto \Psi_{\le l}v_l$ which is defined on a 2^{-l} enlargement of Ω_l. For this comparison to work, we need to know that Γ_j and Γ_{j+1} are sufficiently close. By interpolating using (2.9.5) and (2.9.9) we have

$$\|\eta_{j+1}-\eta_j\|_{L^\infty(\Gamma_*)} \lesssim_{A_0} 2^{-\frac{3}{2}j}, \quad \|\eta_{j+1}-\eta_j\|_{C^{1,\frac{1}{2}}(\Gamma_*)} \lesssim_{A_0} 2^{-\delta j}, \tag{2.9.10}$$

for some $\delta > 0$. Now, we return to the uniform bound for v_j. Thanks to (2.9.10), we can safely consider the decomposition on Ω_j,

$$v_j = \Psi_{\leq j_0} v_{j_0} + \sum_{j_0 \leq l \leq j-1} \Psi_{\leq l+1} v_{l+1} - \Psi_{\leq l} v_l + (I - \Psi_{\leq j}) v_j. \tag{2.9.11}$$

The first term in the telescoping decomposition is trivial to bound. We therefore focus our attention on the remaining terms. First, define for $l \geq j_0$

$$\tilde{\Omega}_l = \bigcap_{k=l}^{j} \Omega_k.$$

Thanks again to (2.9.10), for j_0 large enough (independent of j and only depending on the data parameters), we can arrange for the regularization operator $\Psi_{\leq l}$ to be bounded from $H^s(\tilde{\Omega}_l)$ to $H^s(\tilde{\Omega}_l')$ where $\tilde{\Omega}_l'$ is some 2^{-l} enlargement of the union of all of the Ω_k for $k \geq l$. We will use this fact to establish the following lemma which will help us to estimate the intermediate terms in (2.9.11).

Lemma 2.9.1. Let $j_0 \leq l \leq j-1$, where j_0 is some universal parameter depending only on the numerical constants for the data. Then given the above decomposition for v_j, we have

$$\|\Psi_{\leq l+1} v_{l+1} - \Psi_{\leq l} v_l\|_{L^2(\Omega_j)} \lesssim_{A_0} 2^{-ls} c_l (1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}), \tag{2.9.12}$$

$$\|\Psi_{\leq l+1} v_{l+1} - \Psi_{\leq l} v_l\|_{H^{s+\alpha}(\Omega_j)} \lesssim_{A_0} 2^{l\alpha} c_l (1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}). \tag{2.9.13}$$

By Sobolev embedding, a corollary of this lemma is the following pointwise bound at the C^1 regularity.

Corollary 2.9.2. We have the estimate

$$\|\Psi_{\leq l+1} v_{l+1} - \Psi_{\leq l} v_l\|_{C^1(\Omega_j)} \lesssim_{A_0} 2^{-l\delta} (1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}), \quad \delta > 0.$$

Proof. The latter bound (2.9.13) is clear from the $H^{s+\alpha}$ boundedness of $\Psi_{\leq l}$ and (2.9.2). For the first bound, we split

$$\Psi_{\leq l+1} v_{l+1} - \Psi_{\leq l} v_l = (\Psi_{\leq l+1} - \Psi_{\leq l}) v_{l+1} + \Psi_{\leq l}(v_{l+1} - v_l).$$

Using Proposition 2.6.2 and (2.9.2), we have

$$\|(\Psi_{\leq l+1} - \Psi_{\leq l}) v_{l+1}\|_{L^2(\Omega_j)} \lesssim_{A_0} 2^{-ls} c_l (1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}).$$

For the remaining term, we use the difference bound and the L^2 boundedness of $\Psi_{\leq l}$ to obtain

$$\|\Psi_{\leq l}(v_{l+1}-v_l)\|_{L^2(\Omega_j)} \lesssim_{A_0} D((v_l,\Gamma_l),(v_{l+1},\Gamma_{l+1}))^{\frac{1}{2}} \lesssim_{A_0} 2^{-ls}c_l\|(v_0,\Gamma_0)\|_{\mathbf{H}^s}.$$

□

We also observe that the same bounds in Lemma 2.9.1 hold for the third term in (2.9.11) but with the parameter l replaced by j in the corresponding estimates. This is immediate for (2.9.13) and follows by telescopic summation from Proposition 2.6.6 in the case of (2.9.12).

We can use the above lemma (and the corresponding bounds for $(I-\Psi_{\leq j})v_j$) to estimate similarly to (2.9.8) that for each $k\geq 0$,

$$\|P_k v_j\|_{H^s(\mathbb{R}^d)} \lesssim_{A_0} c_k(1+\|(v_0,\Gamma_0)\|_{\mathbf{H}^s}),$$

where we carefully note here that for each $k\geq 0$, P_k should be interpreted as $P_kE_{\Omega_j}$ where E_{Ω_j} is the extension operator on Ω_j from Proposition 2.5.12. From this observation and almost orthogonality, we obtain the desired uniform bound,

$$\|(v_j,\Gamma_j)(t)\|_{\mathbf{H}^s} \lesssim_{A_0} 1+\|(v_0,\Gamma_0)\|_{\mathbf{H}^s},$$

for $t\in[0,T_0]$. In particular, if the constant B_0 is chosen to be sufficiently large relative to A_0 and the data size, this improves the bootstrap assumption for $\|(v_j,\Gamma_j)\|_{\mathbf{H}^s}$. It remains to improve the bootstrap assumption for A_j and at the same time the Taylor term and the collar neighborhood size. For this we rely on a computation similar to [134, 140] for the Lagrangian flow map $u_j(t,\cdot):\Omega_{0,j}\to\Omega_j(t)$, defined as the solution to the ODE

$$\partial_t u_j(t,y) = v_j(t,u_j(t,y)), \qquad y\in\Omega_{0,j}, \qquad u_j(0)=I.$$

Since $s>\frac{d}{2}+1$, if T_0 is small enough, then for any $0\leq t\leq T\leq T_0$ we have the bound

$$\begin{aligned}\|u_j(t,\cdot)-I\|_{H^s(\Omega_{0,j})} &\lesssim \int_0^t \|v_j(t',\cdot)\|_{H^s(\Omega_j(t'))}\|u_j(t',\cdot)\|^s_{H^s(\Omega_{0,j})}\,dt' \\ &\lesssim_{A_0} t\|(v_0,\Gamma_0)\|_{\mathbf{H}^s}.\end{aligned}$$

If A_0 is large enough relative to the data size, this easily implies simultaneously

$$\Gamma_j(t)\in\frac{3}{2}\Lambda_*, \qquad \|\Gamma_j(t)\|_{C^{1,\varepsilon}} \ll A_0,$$

as long as T_0 is small enough. Doing a similar computation with u_t in place of u and using the equation

$$\partial_t^2 u_j(t,y) = \partial_t(v_j(t,u_j(t,y))) = -(\nabla p_j + ge_d)(t,u_j(t,y))$$

together with the elliptic estimates for the pressure, we obtain also

$$\|v_j(t)\|_{C^{\frac{1}{2}+\varepsilon}(\Omega_j)} \ll A_0.$$

This improves the bootstrap assumption for A_j. Finally, a similar argument but instead with the pressure gradient and the H^s bound for $D_t p$ allows one to close the bootstrap for a_j as long as T_0 is sufficiently small depending on M and c_0.

The limiting solution

Here we show that for $T \leq T_0$,

$$(v,\Gamma) = \lim_{j\to\infty}(v_j,\Gamma_j) \quad \text{in} \quad C([0,T];\mathbf{H}^s).$$

First, we show domain convergence in H^s, which is more straightforward. Indeed, from (2.9.10) we see that the limiting domain Ω exists and has Lipschitz boundary Γ. Next, we let $j \geq j_0$ and consider the telescoping sum

$$\eta - \eta_j = \sum_{l=j}^{\infty} \eta_{l+1} - \eta_l.$$

An analysis similar to the previous subsection, using the difference bounds and the higher energy bounds, yields

$$\|\eta - \eta_j\|_{L^\infty(\Gamma_*)} \lesssim_{A_0} 2^{-\frac{3}{2}j} \tag{2.9.14}$$

and

$$\|\eta - \eta_j\|_{C([0,T];H^s(\Gamma_*))} \lesssim_{A_0} \|c_{\geq j}\|_{l^2}(1 + |(v_0,\Gamma_0)\|_{\mathbf{H}^s}),$$

which in particular shows convergence of $\Gamma_j \to \Gamma$ in $C([0,T];H^s(\Gamma_*))$. Next, we turn to showing the convergence $v_j \to v$ in $C([0,T];\mathbf{H}^s)$. We, formally, define v through the telescoping sum

$$v = \Psi_{\leq j_0} v_{j_0} + \sum_{l \geq j_0} \Psi_{\leq l+1} v_{l+1} - \Psi_{\leq l} v_l,$$

where, as usual, j_0 ensures that all the terms in the sum are defined on Ω. Thanks to (2.9.14), this is possible. We begin by showing that $\Psi_{\leq j} v_j \to v$ in $H^s(\Omega_t)$ uniformly in t (which is again unambiguous thanks to (2.9.14)). We have

$$v - \Psi_{\leq j} v_j = \sum_{l \geq j} \Psi_{\leq l+1} v_{l+1} - \Psi_{\leq l} v_l.$$

From this we see that

$$\|v - \Psi_{\leq j} v_j\|_{H^s(\Omega_t)} \lesssim_{A_0} \|c_{\geq j}\|_{l^2}(1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}),$$

which establishes the desired uniform convergence in $H^s(\Omega_t)$. To show convergence of v_j in the sense of Definition 2.3.5, we consider the regularization $\tilde{v} = \Psi_{\leq m} v_m$. We then have as above,

$$\|v - \Psi_{\leq m} v_m\|_{H^s(\Omega)} \lesssim_{A_0} \|c_{\geq m}\|_{l^2}(1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}),$$

which goes to 0 as $m \to \infty$. On the other hand, for $j > m$, we have

$$\begin{aligned}\|v_j - \Psi_{\leq m} v_m\|_{H^s(\Omega_j)} \lesssim_{A_0} &\|(1 - \Psi_{\leq j}) v_j\|_{H^s(\Omega_j)} + \|\Psi_{\leq j}(v_j - v)\|_{H^s(\Omega_j)} + \|\Psi_{\leq m}(v_m - v)\|_{H^s(\Omega_j)} \\ &+ \|\Psi_{\leq j} v - \Psi_{\leq m} v\|_{H^s(\Omega_j)}.\end{aligned}$$

Using (2.9.2) for the first term and the difference bounds for $D((v_j, \Gamma_j), (v, \Gamma))$, $D((v_m, \Gamma_m), (v, \Gamma))$ for the second and third terms, respectively, we obtain

$$\|v_j - \Psi_{\leq m} v_m\|_{H^s(\Omega_j)} \lesssim_{A_0} \|c_{\geq m}\|_{l^2}(1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}) + \|\Psi_{\leq j} v - \Psi_{\leq m} v\|_{H^s(\Omega_j)}.$$

To estimate the last term above, we have

$$\begin{aligned}\|\Psi_{\leq j} v - \Psi_{\leq m} v\|_{H^s(\Omega_j)} &\lesssim_{A_0} \|(\Psi_{\leq j} - \Psi_{\leq m})(v - \Psi_{\leq m} v_m)\|_{H^s(\Omega_j)} + \|(\Psi_{\leq j} - \Psi_{\leq m})\Psi_{\leq m} v_m\|_{H^s(\Omega_j)} \\ &\lesssim_{A_0} \|v - \Psi_{\leq m} v_m\|_{H^s(\Omega)} + 2^{-m\alpha}\|v_m\|_{H^{s+\alpha}(\Omega_m)} \\ &\lesssim_{A_0} \|c_{\geq m}\|_{l^2}(1 + \|(v_0, \Gamma_0)\|_{\mathbf{H}^s}),\end{aligned}$$

where we used (2.9.2) to estimate the second term in the last inequality. The combination of the above estimates establishes strong convergence in $\mathbf{H}^s$. A similar argument shows continuity of v with values in $\mathbf{H}^s$. Finally, one may also check that the limiting solution solves the free boundary Euler equations.

Continuous dependence

Given a sequence of initial data $(v_0^n, \Gamma_0^n) \in \mathbf{H}^s$ such that $(v_0^n, \Gamma_0^n) \to (v_0, \Gamma_0)$, we aim to show that we have the corresponding convergence of the solutions $(v^n, \Gamma^n) \to (v, \Gamma)$ in

$C([0,T];\mathbf{H}^s)$. First, we note that thanks to the data convergence, the corresponding solutions have a uniform in n lifespan in $\mathbf{H}^s$, and so, on some compact time interval $[0,T]$, we have $\|(v^n,\Gamma^n)\|_{\mathbf{H}^s}+\|(v,\Gamma)\|_{\mathbf{H}^s}\lesssim_M 1$. Let us denote by c_j^n and c_j the admissible frequency envelopes for the data (v_0^n,Γ_0^n) and (v_0,Γ_0), respectively. Now, let $\varepsilon>0$ and let $\delta=\delta(\varepsilon)>0$ be a small positive constant to be chosen. Moreover, let $n_0=n_0(\varepsilon)$ be some large integer to be chosen.

By definition of convergence in $\mathbf{H}^s$, there is a divergence free function $v_0^\delta\in H^s(\Omega_0^\delta)$ defined on some enlarged domain Ω_0^δ such that

$$\|v_0-v_0^\delta\|_{H^s(\Omega_0)}+\limsup_{n\to\infty}\|v_0^n-v_0^\delta\|_{H^s(\Omega_0^n)}<\delta.$$

Moreover, for n large enough, depending only on δ, v_0^δ is defined on a neighborhood of Ω_0 and Ω_0^n. Moreover, we may also assume that v_0^δ belongs to $H^s(\mathbb{R}^d)$. Indeed, for some $\delta'\ll\delta$, v_0^δ is defined on the domain Ω_0' defined by taking $\eta_0'=\eta_0+\delta'$. Then we can extend v_0^δ to $\mathbb{R}^d$ using Proposition 2.5.12. We note that v_0^δ is not necessarily divergence free on $\mathbb{R}^d$ but is on an enlargement of Ω_0 and Ω_0^n for n large enough. Now, let c_j^δ denote the admissible frequency envelope for (v_0^δ,Γ_0) (note that we are using the same domain Ω_0 as v_0 for the frequency envelope here; if δ is small enough, Taylor sign holds for this state) and denote by (v^δ,Γ^δ) the corresponding $\mathbf{H}^s$ solution (which we note has lifespan comparable to v and v^n for n large enough). We begin by choosing $j=j(\varepsilon)$ large enough so that

$$\|c_{\geq j}\|_{l^2}<\varepsilon. \tag{2.9.15}$$

We next observe that we can choose $\delta(\varepsilon)$ and then $n_0(\delta)$ so that

$$\|c_{\geq j}^n\|_{l^2}\lesssim_M\varepsilon+\|c_{\geq j}\|_{l^2}\lesssim_M\varepsilon, \tag{2.9.16}$$

for $n\geq n_0$. One can establish this by estimating the error when comparing terms in c_j^δ and c_j^n and then the error when comparing terms in c_j^δ and c_j by using (2.6.3) and square summing. The main error in the first comparison is essentially comprised of two terms. The first term to control involves the error between η_0^n and η_0. If δ is small enough and n is large enough, we have

$$\|\eta_0^n-\eta_0\|_{H^s(\Gamma_*)}<\delta<\varepsilon.$$

The second source of error comes from the extensions of the velocity functions,

$$\|E_{\Omega_0^n}v_0^n-E_{\Omega_0}v_0^\delta\|_{H^s(\mathbb{R}^d)}\leq\|E_{\Omega_0^n}v_0^\delta-E_{\Omega_0}v_0^\delta\|_{H^s(\mathbb{R}^d)}+\|E_{\Omega_0^n}(v_0^n-v_0^\delta)\|_{H^s(\mathbb{R}^d)}.$$

If $\delta \ll_M \varepsilon$, then the latter term is $\mathcal{O}(\varepsilon)$ by (uniform in n) boundedness of $E_{\Omega_0^n}$ and the definition of v_0^δ. The first term is $\mathcal{O}(\varepsilon)$ if n is large enough (relative to δ) thanks to the continuity property of the family $E_{\Omega_0^n}$ in Proposition 2.5.12. Then one establishes (2.9.16) by comparing c_j and c_j^δ which just involves controlling essentially the error term $\|E_{\Omega_0}(v_0^\delta - v_0)\|_{H^s(\mathbb{R}^d)}$.

Now that we have uniform smallness of the initial data frequency envelopes, the next step is to compare the corresponding solutions. First, thanks to the difference estimates, we observe that for large enough n, Γ^n and Γ^δ are within distance $\ll 2^{-j}$ as long as δ is chosen small enough relative to j (recall that j was chosen to ensure (2.9.15)). Indeed, by interpolating and using the uniform $\mathbf{H}^s$ bound, we have

$$\|\eta^n - \eta^\delta\|_{L^\infty(\Gamma_*)} \lesssim_M D((v^n,\Gamma^n),(v^\delta,\Gamma^\delta))^{\frac{3}{4s}} \lesssim_M \delta^{\frac{3}{2s}}.$$

This ensures that we may compare $\Psi_{\leq j}v^\delta$ to v^n. Denoting by (v_j^n,Γ_j^n) the regular solution corresponding to the regularized data $(v_{0,j}^n,\Gamma_{0,j}^n)$ (from the previous section), we have

$$\begin{aligned}
\|\Psi_{\leq j}v^\delta - v^n\|_{H^s(\Omega^n)} &\lesssim \|\Psi_{\leq j}(v^\delta - v^n)\|_{H^s(\Omega^n)} + \|\Psi_{\leq j}(v^n - v_j^n)\|_{H^s(\Omega^n)} + \|v^n - \Psi_{\leq j}v_j^n\|_{H^s(\Omega^n)}\\
&\lesssim_M \|c_{\geq j}^n\|_{l^2} + 2^{js}D((v^n,\Gamma^n),(v_j^n,\Gamma_j^n))^{\frac{1}{2}} + 2^{js}D((v^n,\Gamma^n),(v^\delta,\Gamma^\delta))^{\frac{1}{2}}\\
&\lesssim_M \|c_{\geq j}^n\|_{l^2} + 2^{js}D((v^n,\Gamma^n),(v^\delta,\Gamma^\delta))^{\frac{1}{2}},
\end{aligned}$$

which if δ is small enough gives

$$\|\Psi_{\leq j}v^\delta - v^n\|_{H^s(\Omega^n)} \lesssim_M \varepsilon.$$

Similarly, we may obtain

$$\|\eta^n - \eta\|_{H^s(\Gamma_*)} \lesssim_M \varepsilon$$

and

$$\|\Psi_{\leq j}v^\delta - v\|_{H^s(\Omega)} \lesssim_M \varepsilon.$$

This establishes continuous dependence.

Lifespan of rough solutions

Here, we finally establish the continuation criterion from Theorem 2.1.7 for $\mathbf{H}^s$ solutions. We consider initial data $(v_0,\Gamma_0) \in \mathbf{H}^s$ and the corresponding solution (v,Γ) in a time interval $[0,T)$ which has the property that

$$\mathcal{C} := \sup_{0\leq t<T} A(t) + \int_0^T B(t)\,dt < \infty, \quad a(t) \geq c_0 > 0, \quad t \in [0,T),$$

and whose domains Ω_t maintain a uniform thickness. Unlike with the construction of rough solutions, we now work with the weaker control parameter

$$B(t) = \|v\|_{W^{1,\infty}(\Omega_t)} + \|\Gamma_t\|_{C^{1,\frac{1}{2}}}.$$

One starting difficulty we face in this proof is that we do not a priori have a fixed reference collar neighborhood. However, the uniform bound on $A(t)$ guarantees that the free boundaries Γ_t are uniformly of class $C^{1,\varepsilon}$, and the uniform bound on v guarantees that they move at most with velocity $\mathcal{O}(1)$. This implies that the limiting boundary $\Gamma_T = \lim_{t\to T}\Gamma_t$ exists in the uniform topology, and also belongs to $C^{1,\varepsilon}$, with the corresponding domain Ω_T having positive thickness. Furthermore, by interpolation, it follows that

$$\lim_{t\to T}\Gamma_t = \Gamma_T \qquad \text{in } C^{1,\varepsilon_1}, \qquad 0 < \varepsilon_1 < \varepsilon.$$

This allows us define the reference boundary Γ_* as a regularization of Γ_T, so that $\Gamma_T \in \Lambda(\Gamma_*, \varepsilon/2, \delta/4)$ for an acceptable choice of δ ensuring that $\Lambda(\Gamma_*, \varepsilon/2, \delta/2)$ is also a well-defined collar (cf. Remark 2.3.4). Then the above convergence implies that $\Gamma_t \in \Lambda_* := \Lambda(\Gamma_*, \varepsilon/2, \delta/2)$ for t close to T.

Reinitializing the starting time close to T, we arrive at the case where we have the initial data $(v_0, \Gamma_0) \in \mathbf{H}^s$ and the corresponding solution (v, Γ) in a time interval $[0, T)$ with the property that

$$\Gamma_t \in \Lambda_*, \quad t \in [0, T).$$

From the local well-posedness theorem, it suffices to show that

$$\|(v, \Gamma)\|_{L^\infty([0,T);\mathbf{H}^s)} < \infty. \tag{2.9.17}$$

Similarly to the previous subsections, the strategy we would like to employ will involve showing that the control parameters for a suitable family of regularized solutions (v_j, Γ_j) can be controlled to leading order by the control parameters for (v, Γ). The main difficulty is that v_j and v are defined on different domains. As in the previous sections, as long as we can ensure that Γ_j and Γ are within distance $2^{-j(1+\delta)}$ of each other, we can compare v with $\Psi_{\leq j}v_j$. However, there is one added difficulty now. The difference bound, which ensured the closeness of domains in the previous sections, has a stronger control parameter involving the term $\|D_t p\|_{W^{1,\infty}(\Omega_t)}$ in addition to $B(t)$, which from Lemma 2.7.9 has size controlled by $B(t)$ and an additional logarithmic factor.

To overcome this, we will divide $[0,T)$ into two disjoint intervals $[0,\tilde{T}]$ and $[\tilde{T},T)$ where $0<\tilde{T}<T$ and $\tilde{T}$ has the property that

$$\int_{\tilde{T}}^{T} B(t)\,dt < \delta_0,$$

where δ_0 is some parameter to be chosen depending only on $\mathcal{C}$, c_0, the collar and the $\mathbf{H}^s$ norm of (v_0,Γ_0). Given such a $\tilde{T}$, we consider the regularized data $(v_{\tilde{T},j},\Gamma_{\tilde{T},j})$ of $(v(\tilde{T}),\Gamma_{\tilde{T}})$ and the corresponding solutions (v_j,Γ_j). We remark that $\tilde{T}$ and δ_0 need to be chosen carefully to not depend on j, but we postpone this choice for now. Their purpose is to guarantee that the stronger control parameter D_tp in the difference bounds as well as the logarithmic factor in the energy bounds does not cause the distance between Γ_j and Γ to grow larger than $2^{-j(1+\delta)}$ for times $t<T$ where (v_j,Γ_j) is defined.

From the continuous dependence result, the above regularized solutions converge to (v,Γ) in $[\tilde{T},T)$ and their lifespans T_j satisfy

$$\liminf_{j\to\infty} T_j \geq T-\tilde{T}.$$

However, a priori, we do not have a uniform L^1_T bound on their corresponding control parameters B_j, nor a uniform L^∞_T bound on A_j, nor a uniform lower bound on the corresponding Taylor terms a_j. Arguing similarly to the previous subsections, if such bounds could be established, one could hope to use them to establish a uniform $\mathbf{H}^s$ bound on the regularized solutions (v_j,Γ_j) and hence extend their time of existence by an amount uniform in j. To establish such uniform control on these pointwise parameters, we will run a relatively simple bootstrap argument. From here on, we write $M := \|(v_0,\Gamma_0)\|_{\mathbf{H}^s}$ and $M_{\tilde{T}} := \|(v(\tilde{T}),\Gamma_{\tilde{T}})\|_{\mathbf{H}^s}$. To set up the bootstrap, we begin by noting that at time $\tilde{T}$, we have by Sobolev embedding and interpolation, the bound

$$\|\eta_j(\tilde{T})-\eta(\tilde{T})\|_{C^{1,\varepsilon}(\Gamma_*)} \lesssim 2^{-\frac{j}{2}} M_{\tilde{T}}. \tag{2.9.18}$$

Moreover, by the properties of $\Psi_{\leq j}$, we have $\|v_j(\tilde{T})\|_{C^{\frac{1}{2}+\varepsilon}} \lesssim_{\mathcal{C}} 1$. Hence, initially we have

$$A_j(\tilde{T}) \leq P(\mathcal{C}) + 2^{-\frac{j}{2}} M_{\tilde{T}} \tag{2.9.19}$$

where $P>1$ is some sufficiently large positive polynomial. As long as the choice of $\tilde{T}$ we make later on depends only on $\mathcal{C}$ and c_0 (but not on j), we can arrange by taking j large enough, the initial bound

$$A_j(\tilde{T}) \leq 2P(\mathcal{C}). \tag{2.9.20}$$

Finally, if j is large enough, and $\tilde{T}$ is as above, we also initially have (for instance),

$$a_j(\tilde{T}) \geq \frac{2}{3}c_0.$$

Now, we make the bootstrap assumption that on a time interval $[\tilde{T}, T_0]$ with $\tilde{T} < T_0 < T$ we have the bounds

$$\int_{\tilde{T}}^{T_0} B_j(t)\, dt < 4C_1(A)\delta_0, \quad A_j(t) \leq 4P(\mathcal{C}), \quad a_j(t) \geq \frac{1}{2}c_0, \quad \Gamma_j(t) \in 2\Lambda_* \tag{2.9.21}$$

for $j \geq j_0(M, T_0)$ and some large universal constant $C_1 \gg 1$ depending only on $A := \sup_{t\in[0,T)} A(t)$. Our goal will be to show that the constant $4C_1\delta_0$ can be improved to $2C_1\delta_0$ and the constant $4P(\mathcal{C})$ can be improved to $2P(\mathcal{C})$, with similar improvements on the Taylor term and the collar. After we close this boostrap, we will give a separate argument which uses the uniform bounds on the control parameters to establish a uniform bound for (v_j, Γ_j) in $\mathbf{H}^s$, and hence permit us to continue the solution. To close the above bootstrap, we aim to establish the bounds

$$B_j \leq C_1(A)B + C_2 2^{-\delta j}, \quad A_j \leq P(\mathcal{C}) + C_2 2^{-\delta j}, \quad a_j \geq \frac{2}{3}c_0, \quad \Gamma_j(t) \in \frac{3}{2}\Lambda_*, \tag{2.9.22}$$

where $\delta > 0$ is some small positive constant and C_2 depends on the size of $M_{\tilde{T}}$ as well as the constant $\mathcal{C}$ above. The bootstrap can then be closed by choosing j_0 large enough to absorb the contribution of C_2.

As mentioned above, the main difficulty in comparing B_j with B and A_j with A is, as usual, the fact that the corresponding domains Ω_j and Ω are different. Our starting point is to select the parameter δ_0 and the time $\tilde{T}(\delta_0)$ to ensure that Ω_j and Ω are close enough. As mentioned above, in order for our argument not to be circular, we need to ensure that the choice of δ_0 depends only on c_0 and $\mathcal{C}$. Our first aim is to obtain some preliminary bounds for $\eta_j - \eta_{j+1}$ in L^∞ and $C^{1,\frac{1}{2}}$. We let k be the smallest integer larger than s. First, by the double exponential bound in Theorem 2.7.1 and the bootstrap hypothesis, we have for each j,

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^k}^2 \lesssim_A \exp\left(\exp(K\delta_0)\log(K(1 + 2^{2j(k-s)}\|(v(\tilde{T}), \Gamma_{\tilde{T}})\|_{\mathbf{H}^s}^2)\right).$$

Above, K is some (possibly large) constant depending on $\mathcal{C}$ and c_0 which we will let change from line to line. In the above estimate, if we take $K\delta_0 \ll 1$ (in particular, δ_0 does not depend on j), then we can arrange for

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^k}^2 \lesssim K 2^{2j(k-s)} M_{\tilde{T}}^2 (M_{\tilde{T}} 2^j)^\delta \tag{2.9.23}$$

for some small constant $\delta > 0$, where we assumed without loss of generality that $M_{\tilde{T}} \geq 1$ to simplify notation. Note here that there is a slight loss compared to (2.9.2) coming from the double exponential bound in the energy estimate. On the other hand, the difference estimates, Lemma 2.7.9 and the energy coercivity ensures that by Grönwall and the bootstrap assumption, we have

$$D((v_j, \Gamma_j), (v_{j+1}, \Gamma_{j+1})) \lesssim 2^{-2js} K M_{\tilde{T}}^2 \exp\left(K\delta_0 \mathcal{I}_j\right),$$

where $\mathcal{I}_j = \sup_{\tilde{T}<t\leq T_0}(\log(K + KE^k(v_j, \Gamma_j)) + \log(K + KE^k(v_{j+1}, \Gamma_{j+1})))$ and k is, again, the smallest integer larger than s. By the higher energy bound and the bootstrap assumption, we have

$$\mathcal{I}_j \lesssim K \log(1 + 2^{2jk} \|(v(\tilde{T}), \Gamma_{\tilde{T}})\|_{L^2}^2) \lesssim_k Kj,$$

where we used the higher energy bound for the regularized solution to propagate $\log(1 + E^k(v_j, \Gamma_j))$ and control $\log(1 + E^k(v_j, \Gamma_j))$ by $\log(1 + 2^{2kj}\|(v(\tilde{T}), \Gamma_{\tilde{T}})\|_{L^2}^2)$ as well as the fact that the volume of Ω_t is conserved and Hölder's inequality to estimate $\|(v(\tilde{T}), \Gamma_{\tilde{T}})\|_{L^2} \lesssim_A 1$. Again, we choose δ_0 small enough (and therefore $\tilde{T}$) depending only on $\mathcal{C}$ and c_0 so that

$$\exp(K\delta_0 \mathcal{I}_j) \leq 2^{j\delta},$$

for some sufficiently small $\delta > 0$ (depending only on s). Next, we pick j_0 depending on $M_{\tilde{T}}$, $\mathcal{C}$ and c_0 so that if $j \geq j_0$ (after possibly relabelling δ), we have

$$D((v_j, \Gamma_j), (v_{j+1}, \Gamma_{j+1})) \lesssim 2^{-2j(s-\delta)}, \quad \|(v_j, \Gamma_j)\|_{\mathbf{H}^k}^2 \lesssim 2^{2j(k-s)} 2^{j\delta}$$

with universal implicit constant. The key point to observe here is that there is now a slight loss in the difference estimates and energy estimates compared to the previous subsections because of the stronger control parameter in the difference bounds and the logarithmic factor in the energy estimates. However, by using these estimates, we still obtain by Sobolev embedding and interpolating, the bounds (after possibly relabelling δ)

$$\|\eta_j - \eta_{j+1}\|_{C^{1,\frac{1}{2}}(\Gamma_*)} \lesssim 2^{-\delta j}, \quad \|\eta_j - \eta_{j+1}\|_{C^{1,\varepsilon}(\Gamma_*)} \lesssim 2^{-\frac{1}{2}j}, \quad \|\eta_j - \eta_{j+1}\|_{L^\infty(\Gamma_*)} \lesssim 2^{-\frac{3}{2}j}, \tag{2.9.24}$$

all with universal implicit constant if j_0 is large enough. The first bound will give us control of $\|\Gamma_j\|_{C^{1,\frac{1}{2}}}$ in the first estimate in (2.9.22). The second bound above gives us control over $\|\Gamma_j\|_{C^{1,\varepsilon}}$ for the second estimate in (2.9.22) and also shows that $\Gamma_j \in \frac{3}{2}\Lambda_*$. The third bound ensures that Γ_j and Γ_{j+1} are sufficiently close. With this closeness established, we now work towards closing the bootstrap (2.9.22) for the $\|v_j\|_{W^{1,\infty}(\Omega_j)}$ component of B_j and the

$\|v_j\|_{C^{\frac{1}{2}+\varepsilon}(\Omega_j)}$ component of A_j. We show the details for $\|v_j\|_{W^{1,\infty}(\Omega_j)}$ as the other component is very similar. We estimate in three steps. First, we observe that from the bounds for $\Psi_{\leq j}$, we have

$$\|\Psi_{\leq j} v\|_{W^{1,\infty}} \lesssim_A B. \tag{2.9.25}$$

We can ensure that the implicit constant in this estimate is less than $C_1(A)$ if $C_1(A)$ is initially chosen large enough. Then we compare $\Psi_{\leq j} v$ and $\Psi_{\leq j} v_j$ which is justified thanks to (2.9.24). We have

$$\Psi_{\leq j} v - \Psi_{\leq j} v_j = \sum_{l \geq j} \Psi_{\leq j} v_{l+1} - \Psi_{\leq j} v_l.$$

By Sobolev embedding and a similar argument to the $C^{1,\frac{1}{2}}$ bound for $\eta_{j+1} - \eta_j$, we see that

$$\|\Psi_{\leq j} v_{l+1} - \Psi_{\leq j} v_l\|_{W^{1,\infty}} \leq C_2 2^{-l\delta},$$

which gives by summation

$$\|\Psi_{\leq j} v - \Psi_{\leq j} v_j\|_{W^{1,\infty}} \leq C_2 2^{-j\delta}. \tag{2.9.26}$$

Using the error bound for $I - \Psi_{\leq j}$, Sobolev embedding and the higher energy bounds, we also have

$$\|\Psi_{\leq j} v_j - v_j\|_{W^{1,\infty}} \leq C_2 2^{-j\delta}. \tag{2.9.27}$$

Combining (2.9.25), (2.9.26) and (2.9.27) shows that

$$\|v_j\|_{W^{1,\infty}(\Omega_j)} \leq C_1(A) B + C_2 2^{-j\delta}.$$

Doing a similar estimate for $\|v_j\|_{C^{\frac{1}{2}+\varepsilon}(\Omega_j)}$ and taking j large enough allows us to close the bootstrap for A_j.

It remains now to improve the bootstrap assumption for the Taylor term a_j. To do this, we need a suitable way of comparing the C^1 norms of the pressures p_j and p. We begin by defining the shrunken domain Ω' via $\eta' := \eta - 2^{-j_0}$. As Ω_j is within distance $\mathcal{O}(2^{-\frac{3}{2}j})$ of Ω for $j \geq j_0$, it follows that

$$\Omega' \subset \Omega \cap \bigcap_{j \geq j_0} \Omega_j.$$

We next note the following bound which holds on Ω' for any $0 < \delta < \frac{\varepsilon}{2}$,

$$\|v_j - v\|_{C^{\frac{1}{2}+\delta}(\Omega')} \leq C_2 2^{-j_0\delta}. \tag{2.9.28}$$

This follows by similar reasoning to the above. Now, we establish the following C^1 estimate for $p - p_j$:

$$\|p - p_j\|_{C^1(\Omega')} \leq C_2 2^{-j_0\delta}. \tag{2.9.29}$$

We begin by splitting $p - p_j$ into an inhomogeneous part plus a harmonic part on Ω',

$$p - p_j = \Delta^{-1}\Delta(p - p_j) + \mathcal{H}(p - p_j).$$

Using Proposition 2.5.15, the dynamic boundary condition and the fact that the boundary of Ω' is within distance 2^{-j_0} of the boundaries of Ω and Ω_j, we have

$$\|\mathcal{H}(p - p_j)\|_{C^{1,\delta}(\Omega')} \lesssim_{\mathcal{C}} 2^{-j_0\delta}(\|p\|_{C^{1,\varepsilon}(\Omega)} + \|p_j\|_{C^{1,\varepsilon}(\Omega_j)}).$$

By Lemma 2.7.5 and the bootstrap assumption on A_j, this gives

$$\|\mathcal{H}(p - p_j)\|_{C^{1,\delta}(\Omega')} \lesssim_{\mathcal{C}} 2^{-j_0\delta}.$$

To estimate the inhomogeneous part, we can argue similarly to the proof of Lemma 2.7.5 using a bilinear frequency decomposition for $\Delta(p_j - p)$, to obtain

$$\|\Delta^{-1}\Delta(p - p_j)\|_{C^1(\Omega')} \lesssim_{\mathcal{C}} \|v - v_j\|_{C^{\frac{1}{2}+\delta}(\Omega')} \leq C_2 2^{-j_0\delta},$$

where in the second inequality we used (2.9.28). Finally, to close the bootstrap on the Taylor term a_j, we can work in collar coordinates on Γ_* to estimate

$$\inf_{x\in\Gamma_j} |\nabla p_j(x)| \geq \inf_{x\in\Gamma} |\nabla p(x)| - \|p_j - p\|_{C^1(\Omega')} - 2^{-j_0\delta}(\|p_j\|_{C^{1,\varepsilon}(\Omega_j)} + \|p\|_{C^{1,\varepsilon}(\Omega)}).$$

In the above, we first estimate the error between $\nabla p_j(x + \eta_j(x)\nu(x))$ and $\nabla p_j(x + \eta'(x)\nu(x))$ (and also $\nabla p(x + \eta'(x)\nu(x))$ and $\nabla p(x + \eta(x)\nu(x))$) using the $C^{1,\varepsilon}$ Hölder regularity of p_j and p. Then, we estimate the difference between $\nabla p_j(x + \eta'(x)\nu(x))$ and $\nabla p(x + \eta'(x)\nu(x))$ on the common domain using our bounds for $\|p_j - p\|_{C^1(\Omega')}$.

Taking j_0 large enough and using (2.9.29) and Lemma 2.7.5, this gives

$$a_j \geq \frac{2}{3}c_0,$$

which closes the bootstrap for a_j.

From the above argument, we see that for $j \geq j_0$, the regular solutions (v_j, Γ_j) are defined on the interval $[\tilde{T}, T]$ and satisfy the assumptions (2.9.21). What we do not yet know is whether we have a uniform in j bound for the $\mathbf{H}^s$ norm of (v_j, Γ_j). Once we have this, (2.9.17) will follow from our continuous dependence result. From here on, we assume without loss of generality that $M_{\tilde{T}} \gg C(A)$. We let c_j denote the frequency envelope for the data at time $\tilde{T}$. Similarly to the above, on a time interval $[\tilde{T}, T_0]$, we make the bootstrap assumption that for finitely many $j \geq j_0$,

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^s} \leq M_{\tilde{T}}^2. \tag{2.9.30}$$

As in the previous subsection, we let $\alpha \geq 1$ be such that $s + \alpha$ is an integer. Then the higher energy bounds, (2.9.30) and (2.9.21) yield

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^{s+\alpha}} \lesssim 2^{j\alpha} c_j \exp(K\delta_0 \log(M_{\tilde{T}}^2)) M_{\tilde{T}}$$

where K is some constant depending on $\mathcal{C}$. As long as δ_0 is such that $K\delta_0 \ll 1$, we obtain

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^{s+\alpha}} \lesssim 2^{j\alpha} c_j M_{\tilde{T}}^{1+\delta} \tag{2.9.31}$$

for some positive constant $\delta \ll 1$. A similar argument with the difference bounds yields

$$D((v_j, \Gamma_j), (v_{j+1}, \Gamma_{j+1}))^{\frac{1}{2}} \lesssim 2^{-js} c_j M_{\tilde{T}}^{1+\delta}.$$

Arguing as in the local well-posedness result, we can use the above two bounds to estimate

$$\|(v_j, \Gamma_j)\|_{\mathbf{H}^s} \lesssim M_{\tilde{T}}^{1+\delta},$$

which improves the bootstrap. We are then able to finally conclude the bound (2.9.17) and thus the proof of Theorem 2.1.7.

Chapter 3

Ultrahyperbolic Schrödinger equations

3.1 Introduction

In this chapter, we consider the large data local well-posedness problem for general quasilinear ultrahyperbolic Schrödinger equations of the form

$$\begin{cases} i\partial_t u + g^{jk}(u, \overline{u}, \nabla u, \nabla \overline{u})\partial_j \partial_k u = F(u, \overline{u}, \nabla u, \nabla \overline{u}), \quad u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^m, \\ u(0, x) = u_0(x), \end{cases} \tag{3.1.1}$$

where g and F are assumed to be smooth functions of their arguments with g real, symmetric and uniformly non-degenerate and F vanishing at least quadratically at the origin.

In a recent series of articles [105, 107, 106], Marzuola, Metcalfe and Tataru have studied the well-posedness of the system (3.1.1) in low regularity Sobolev spaces. As a brief overview, the paper [107] considers the small data problem for cubic and higher nonlinearities in the Sobolev spaces $H^s(\mathbb{R}^d)$, $s > \frac{d+5}{2}$. The article [105], on the other hand, permits quadratic terms in the nonlinearity, but assumes that the data comes from the smaller space $l^1 H^s(\mathbb{R}^d)$, $s > \frac{d}{2} + 3$. Here, $l^1 H^s$ is an appropriate translation invariant Sobolev type space, imposing similar regularity requirements as H^s, but slightly stronger decay. To see that some additional decay is needed, it is instructive to look at the leading part of the linearized flow, which can be written schematically as

$$\begin{cases} i\partial_t v + \partial_j g^{jk} \partial_k v + b^j \partial_j v + \tilde{b}^j \partial_j \overline{v} = f, \\ v(0, x) = v_0(x). \end{cases} \tag{3.1.2}$$

Here, for the purposes of our heuristic discussion, we have written the principal operator in divergence form with $g^{jk} = g^{jk}(u, \overline{u})$ – we will elaborate further on this reduction later on. As is well-known, a necessary condition for L^2 well-posedness of a wide class of such linear systems is integrability of the first order coefficient $\mathrm{Re}(b^j)$ along the bicharacteristic (or Hamilton) flow of the principal differential operator $\partial_j g^{jk} \partial_k$. This is usually referred to as the Mizohata (or Takeuchi–Mizohata) condition. See, for instance, [70, 79, 104, 116, 117, 115, 145] for several manifestations of this ill-posedness mechanism. For cubic and higher nonlinearities, the integrability of $\mathrm{Re}(b^j)$ along the bicharacteristics is automatic for small H^s data, but for quadratic nonlinearities it is not. That being said, there are several natural ways to recover the above integrability condition. One common approach is to work in weighted Sobolev spaces. However, the alternative $l^1 H^s$ spaces also achieve this goal, but have the additional advantage of being translation invariant – they are also far less restrictive in terms of regularity and decay, as we will see below.

In contrast to the case of small data, the third paper in the series by Marzuola, Metcalfe and Tataru [106] considers the significantly more challenging large data problem. Here, the authors establish well-posedness in the same setting as their small data papers, but under two additional assumptions. The first assumption is that the initial metric $g(u_0)$ is nontrapping, meaning that all nontrivial bicharacteristics corresponding to the principal operator $\Delta_{g(u_0)}$ escape to spatial infinity at both ends. Such a condition is automatic in the small data regime (assuming sufficient regularity and asymptotic flatness of the metric) as in this setting the Hamilton trajectories are close to straight lines. For large data well-posedness, a nontrapping assumption is completely natural, in light of the Mizohata condition.

On the other hand, the methods in [106] also rely on the assumption of uniform ellipticity of the principal operator, i.e., the existence of a uniform constant $c > 0$ such that

$$c^{-1}|\xi|^2 \leq g^{jk}(x)\xi_j\xi_k \leq c|\xi|^2. \tag{3.1.3}$$

This assumption is critically used in the above article to effectively diagonalize the linearized equation (3.1.2) and remove the complex conjugate first order term. Roughly speaking, this diagonalization proceeds by considering the new variable

$$Sv := v + \mathcal{R}\overline{v},$$

where $\mathcal{R}$ is a pseudodifferential operator of order -1 with symbol which is essentially of the form

$$r(x, \xi) = \frac{i\tilde{b}^l \xi_l}{g^{jk}\xi_j\xi_k},$$

when $|\xi| \geq 1$. It is not difficult to see that, to leading order, Sv formally satisfies an equation like (3.1.2), but without the complex conjugate first order term. This diagonalization procedure is then used as a key ingredient in the proofs of the requisite local smoothing and $L_T^\infty L_x^2$ estimates for the linearized flow. A similar diagonalization is heavily relied upon in [24] and [87].

The primary objective of the current chapter is to generalize the main result of [106] to the full class of ultrahyperbolic quasilinear Schrödinger flows, while keeping the regularity and function spaces identical. That is, we shall relax the uniform ellipticity assumption

$$c^{-1}|\xi|^2 \leq g^{jk}\xi_j\xi_k \leq c|\xi|^2$$

to the much weaker uniform non-degeneracy condition

$$c^{-1}|\xi| \leq |g^{jk}\xi_k| \leq c|\xi|.$$

The lack of an ellipticity assumption on the metric in (3.1.1) causes significant difficulties, and is what prompted the development of the new well-posedness scheme that we present in this chapter (we were also inspired by the scheme in [81]). On the other hand, there are several physical sources of motivation for studying the general ultrahyperbolic problem. Some well-known examples arise naturally in the study of water waves [34] and others arise in the theory of completely integrable models [76, 137]. More recently, the Hall and electron magnetohydrodynamic equations without resistivity have been shown to behave at leading order like degenerate quasilinear Schrödinger systems of ultrahyperbolic type [80]. This dispersive character of the equations was used to great effect in [80, 81], leading to well-posedness in certain regimes and ill-posedness in others.

Although [106] requires ellipticity of the metric in order to achieve their low regularity results, significant progress has been made towards removing the ellipticity assumptions from the well-posedness theory of (3.1.1) in the high regularity regime. This is best illustrated by the pioneering series [84, 90, 86, 87] of Kenig, Ponce, Rolvung and Vega, which culminates in a proof of large data well-posedness under the nontrapping assumption for systems of the form (3.1.1) in high regularity weighted Sobolev spaces of the form $H^s \cap L^2(\langle x\rangle^N dx)$, where s and N are suitably large, dimension dependent parameters. In this fundamental series of papers, [87] studies the well-posedness problem assuming ellipticity of the principal operator $\partial_j g^{jk}\partial_k$, while [84, 90, 86] consider symmetric, non-degenerate metrics, first in the constant coefficient case and then later for variable coefficients. As should be evident from these articles, the ellipticity assumption on the metric is not easy to remove, even in the high regularity regime. The main objective of this thesis is to give a much simpler proof of well-posedness for the general system (3.1.1) that is also robust enough to work in low regularity spaces. To the best of our knowledge, this is the first low regularity well-posedness result that applies to the full class of ultrahyperbolic quasilinear Schrödinger flows.

The rough strategy used in [84] to prove well-posedness of the ultrahyperbolic flow (3.1.1) in high regularity weighted spaces is to first establish an estimate for the local energy type norm

$$\|v\|_{LE} := \|\langle x\rangle^{-\frac{N}{2}}\langle\nabla\rangle^{\frac{1}{2}}v\|_{L^2_T L^2_x}, \qquad N = N(d) \in \mathbb{N},$$

for the linearized equation (3.1.2) (assuming suitably strong asymptotic decay of the coefficients b^j, $\tilde{b}^j$ and $\nabla_x g^{jk}$) of the form

$$\|v\|_{LE} \lesssim \|v\|_{L^\infty_T L^2_x} + \|f\|_{LE^* + L^1_T L^2_x}. \tag{3.1.4}$$

Here, LE^* denotes the "dual" local energy space. The estimate (3.1.4) shows that the local energy norm of v remains under control, as long as v satisfies an a priori $L^\infty_T L^2_x$ bound. The preliminary estimate (3.1.4) follows, roughly speaking, from a suitable adaptation of Doi's construction in [38] to the ultrahyperbolic problem. The more significant technical obstruction in [84] is in establishing the a priori bound for the $L^\infty_T L^2_x$ norm. To understand the difficulties, we first note that when the real part of the coefficient b^j vanishes, it is a relatively straightforward exercise (in view of (3.1.4)) to obtain the bound

$$\|v\|_{L^\infty_T L^2_x} \lesssim \|v_0\|_{L^2_x} + \|f\|_{LE^* + L^1_T L^2_x}.$$

Indeed, this follows by a standard energy estimate, as one can integrate by parts to shift derivatives off of the first order terms and onto the coefficients b^j and $\tilde{b}^j$. Therefore, in the general case, one is motivated to try to conjugate away the badly behaved first order term $\mathrm{Re}(b^j)\partial_j v$. In [84], this conjugation is accomplished by constructing a (formally) zeroth order operator $\mathcal{O}$ which achieves the approximate cancellation

$$[\mathcal{O}, \partial_j g^{jk}\partial_k] + \mathcal{O}\,\mathrm{Re}(b^j)\partial_j \approx 0. \tag{3.1.5}$$

The idea here is very loosely akin to the method of integrating factors from ODE. On a formal level, the symbol for the operator $\mathcal{O}$ achieving (3.1.5) is given by

$$O(x,\xi) := \exp\left(-\int_{-\infty}^{0} \mathrm{Re}(b(x^t))\cdot\xi^t dt\right), \tag{3.1.6}$$

where (x^t, ξ^t) denotes the bicharacteristic flow

$$(\dot{x}^t, \dot{\xi}^t) = (\nabla_\xi a(x^t, \xi^t), -\nabla_x a(x^t, \xi^t)), \quad (x^0, \xi^0) = (x, \xi),$$

corresponding to the principal symbol $a(x, \xi) := -g^{jk}(x)\xi_j\xi_k$. Unfortunately, the symbol O does not belong to the standard symbol class S^0. Rather, (assuming that b has sufficient regularity and decay) it satisfies

$$|\partial_\xi^\alpha \partial_x^\beta O(x, \xi)| \lesssim_{\alpha,\beta} \langle \xi \rangle^{-|\alpha|} \langle x \rangle^{|\alpha|}. \tag{3.1.7}$$

In the case when the metric is positive-definite (i.e. Δ_g is elliptic), the mapping properties of the pseudodifferential operators associated with this class of symbols were intensively studied in the paper [33] of Craig, Kappeler and Strauss. In the case of a merely non-degenerate metric, Kenig, Ponce, Rolvung and Vega in [84] execute a systematic study of this symbol class as well as a very careful analysis of the bicharacteristic flow for $-g^{jk}(x)\xi_j\xi_k$ to establish suitable mapping properties for $\mathcal{O}$. In contrast, in the current chapter, to obtain the $L_T^\infty L_x^2$ estimate for (3.1.2) we will instead use a spatially truncated version of the above renormalization operator which achieves a suitable cancellation of the form (3.1.5), at least within a large compact set. The key advantage of this truncation is that the corresponding renormalization operator will be a classical pseudodifferential operator of order 0, which will dramatically simplify the analysis (perhaps at the cost of estimating some extra error terms). Moreover, it will allow us to considerably lower the regularity and decay assumptions on the coefficients in (3.1.2) compared to [84] when estimating the $L_T^\infty L_x^2$ norm of v. Of course, this idea comes with some technical caveats of its own, which will be discussed later.

We remark that the idea of using the above spatial truncation to close the energy estimate for (3.1.2) is inspired by the article [81] of Jeong and Oh, where they consider the well-posedness problem for the electron MHD equations near non-zero, constant magnetic fields, and perform an analogous truncation in their setting. As we shall see below, such a construction turns out to be tied heavily to the direction of propagation of the bicharacteristics of the principal part of the corresponding linear flow. For the electron MHD equations, the bicharacteristics have a distinguished direction of propagation. However, the bicharacteristics for the Schrödinger equations that we consider in this thesis do not exhibit this feature. Therefore, one key novelty of the present thesis is in dealing with the multi-directionality present in Schrödinger flows. Another important novelty is our ability to extend the truncation idea in order to give a new and very simple proof of the natural local smoothing type estimate for (3.1.2) in the local energy norms compatible with the translation invariant function spaces used in this thesis. The method that we present is very robust and requires only mild decay of the coefficients (e.g. uniform integrability along the Hamilton flow). A more detailed outline of the argument will be given in Section 3.3.

Statements of the results

We now state our main results more precisely. As in [106], our primary focus will be on the case of quadratic nonlinear interactions.

Let $d, m \geq 1$ and consider a system of equations of the form (3.1.1) where

$$g : \mathbb{C}^m \times \mathbb{C}^m \times (\mathbb{C}^m)^d \times (\mathbb{C}^m)^d \to \mathbb{R}^{d\times d} \;\text{ and }\; F : \mathbb{C}^m \times \mathbb{C}^m \times (\mathbb{C}^m)^d \times (\mathbb{C}^m)^d \to \mathbb{C}^m \tag{3.1.8}$$

are smooth functions. We assume that F vanishes at least quadratically at the origin, so that

$$|F(y,z)| \approx \mathcal{O}\left(|y|^2 + |z|^2\right) \text{ near } (y,z) = (0,0). \tag{3.1.9}$$

In [106], the authors assume that the metric g is uniformly elliptic and coincides with the identity matrix at the origin. That is, they assume that $g(0) = I_{d\times d}$ and that there is a fixed constant $c > 0$ so that

$$c^{-1}|\xi|^2 \leq g^{jk}(y,z)\xi_j\xi_k \leq c|\xi|^2, \;\forall \xi \in \mathbb{R}^d,\; y,z \in \mathbb{C}^m \times (\mathbb{C}^m)^d.$$

In this thesis, we only assume that g is symmetric and (uniformly) non-degenerate, in the sense that

$$c^{-1}|\xi| \leq |g(y,z)\xi| \leq c|\xi|, \;\forall \xi \in \mathbb{R}^d,\; y,z \in \mathbb{C}^m \times (\mathbb{C}^m)^d, \tag{3.1.10}$$

for some fixed constant $c > 0$.

As in [105, 107, 106], we also consider a second class of quasilinear Schrödinger equations of the form

$$\begin{cases} i\partial_t u + \partial_j g^{jk}(u,\overline{u})\partial_k u = F(u,\overline{u},\nabla u,\nabla\overline{u}), \quad u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}^m, \\ u(0,x) = u_0(x), \end{cases} \tag{3.1.11}$$

where F is as in (3.1.9), but where the metric g depends on u but not on ∇u. Such an equation arises by formally differentiating the system (3.1.1). Indeed, if u solves (3.1.1) then $(u, \nabla u)$ solves an equation of the form (3.1.11) with a nonlinearity F which depends at most quadratically on ∇u.

Remark 3.1.1. Note that the second order operator in (3.1.11) has a divergence structure, which can be achieved by commuting the first derivative with g and viewing the commutator as an additional term on the right-hand side. In contrast, the second order operator in (3.1.1) cannot be written in divergence form without possibly changing the type of the equations.

To state our main well-posedness theorem, we must recall the function spaces used in [105, 107, 106]. For now, we limit ourselves to an expository summary, giving more precise definitions in Section 3.2.

Consider a standard spatial Littlewood-Paley decomposition

$$1 = \sum_{j \in \mathbb{N}_0} S_j,$$

where S_j, $j \geq 1$, selects frequencies of size $\approx 2^j$ and S_0 selects all frequencies of size $\lesssim 1$. Corresponding to each dyadic frequency scale $2^j \geq 1$, we consider an associated partition $\mathcal{Q}_j$ of $\mathbb{R}^d$ into cubes of side length 2^j and an associated smooth partition of unity

$$1 = \sum_{Q \in \mathcal{Q}_j} \chi_Q.$$

We define the $l^1_j L^2$ norm by

$$\|u\|_{l^1_j L^2} = \sum_{Q \in \mathcal{Q}_j} \|\chi_Q u\|_{L^2}, \tag{3.1.12}$$

and the space $l^1 H^s$ via the norm

$$\|u\|^2_{l^1 H^s} = \sum_{j \geq 0} 2^{2sj} \|S_j u\|^2_{l^1_j L^2}. \tag{3.1.13}$$

Note that if one replaces the ℓ^1 sum by an ℓ^2 sum in (3.1.12) and defines $l^2 H^s$ analogously to (3.1.13), then $H^s = l^2 H^s$ with equivalent norms. The extra summability in the definition of the $l^1 H^s$ norm yields the decay necessary to circumvent Mizohata's ill-posedness mechanism. However, unlike the high regularity weighted Sobolev spaces used in previous works, the function spaces $l^1 H^s$ admit translation invariant equivalent norms and contain functions exhibiting weaker regularity and decay.

As mentioned above, in the large data problem, one has to contend with trapping. This is an obvious obstruction to well-posedness, so we will need to impose a nontrapping assumption on the initial metric $g(u_0)$ to prevent this. Then, as part of our well-posedness theorem, we will show that the nontrapping assumption propagates on a time interval whose length depends on the data size and the profile of the initial metric. Our definition of nontrapping is the same as [106].

Definition 3.1.2. We say that the metric $g(u_0)$ is *nontrapping* if all nontrivial bicharacteristics for $\Delta_{g(u_0)}$ escape to spatial infinity at both ends.

The above qualitative definition of nontrapping suffices in order to state our main results. However, as we shall see, the proofs require us to introduce a parameter L which gives a quantitative description of nontrapping. The precise way in which we define L is slightly different than [106], so as to better handle the case when Δ_g is not elliptic.

With the above discussion in mind, we may state our main well-posedness theorem as follows.

Theorem 3.1.3. Let $s > \frac{d}{2} + 3$ and suppose that the initial data $u_0 \in l^1 H^s$ makes $g(u_0)$ into a real, symmetric, uniformly non-degenerate, nontrapping metric. Then (3.1.1) with the quadratic nonlinearity (3.1.9) is locally well-posed in $l^1 H^s$. The same result holds if $s > \frac{d}{2} + 2$ for the equation (3.1.11).

Remark 3.1.4. We will prove the latter result in Theorem 3.1.3 as it will imply the former by differentiating the equation.

Remark 3.1.5. As in [106], the regularity and decay assumptions in the above results can be weakened if the metric and nonlinearity satisfy the stronger vanishing conditions

$$g(y,z) = g(0) + \mathcal{O}\left(|y|^2 + |z|^2\right), \quad |F(y,z)| \approx \mathcal{O}\left(|y|^3 + |z|^3\right) \text{ near } (y,z) = (0,0).$$

Namely, it can be shown that (3.1.1) is well-posed in the same sense as Theorem 3.1.3 when $u_0 \in H^s$ and $s > \frac{d+5}{2}$. An analogous result holds if $s > \frac{d+3}{2}$ for the equation (3.1.11). To prove this, one makes modifications to the quadratic case which are virtually identical to those made in [106]. In order to simplify our exposition, we omit the details for these relatively straightforward modifications and instead focus on the general case of quadratic nonlinearities.

Remark 3.1.6. In the above results, well-posedness is to be interpreted in the standard quasilinear fashion. More precisely, in the setting of Theorem 3.1.3 it includes the following key features.

- (Regular solutions). For large σ and nontrapping initial data $u_0 \in l^1 H^\sigma$ there is a unique solution $u \in C([0,T]; l^1 H^\sigma)$ which persists and remains nontrapping on some nontrivial maximal time interval $I = [0, T_*)$.

- (Rough solutions). For $s > \frac{d}{2} + 3$ and nontrapping data $u_0 \in l^1H^s$ there is a unique solution $u \in C([0,T]; l^1H^s) \cap l^1X^s([0,T])$ which persists and remains nontrapping on some nontrivial maximal time interval $I = [0, T_*)$. Here, the auxiliary space l^1X^s is a natural analogue of the local energy space LE described earlier. A precise definition of this space will be given in Section 3.2.

- (Continuous dependence). The maximal time $T_*(u_0)$ is a lower semicontinuous function of u_0 with respect to the l^1H^s topology and for each $T < T_*(u_0)$ the data-to-solution map $v_0 \mapsto v$ is continuous near u_0 from l^1H^s into $C([0,T]; l^1H^s) \cap l^1X^s([0,T])$.

Remark 3.1.7. As in [106, Remark 1.3.2], the maximal existence time $T_*(u_0)$ a priori depends on the full profile of the initial data u_0 rather than just its size in l^1H^s, due to the nontrapping condition on the metric.

Remark 3.1.8. As in [106], the arguments we use here are purely dispersive. This is in contrast to the viscosity methods used in earlier works, which are less tailored to the structure of the equations, and hence less suitable for low regularity analysis.

Organization of the chapter

The chapter is organized as follows. In Section 3.2 we recall the precise functional setting used in [106] as well as the standard Fourier-analytic, nonlinear, and pseudodifferential machinery that will be used throughout the chapter. In certain cases, we adapt this machinery in order to obtain refined estimates in the space-time function spaces where we aim to construct solutions to (3.1.1) and (3.1.11). In Section 3.3, we provide a detailed outline of the proof. Then, in Section 3.4, we analyze the bicharacteristic flow. The key objectives of this section are to quantify nontrapping, show that nontrapping is stable under small perturbations, and establish suitable asymptotic bounds for the bicharacteristics. In Section 3.5, we state our main well-posedness theorem for the linearized flow and reduce the main linear estimate to establishing a simplified bound for the corresponding inhomogeneous linear paradifferential flow in the l^1X^s spaces where we intend to construct solutions. Then, in Section 3.6, we aim to establish a suitable estimate for the $L^\infty_T L^2_x$ component of the l^1X^s norm by constructing a truncated version of the renormalization operator $\mathcal{O}$ in (3.1.6). Such an estimate will close on a short enough time interval, up to controlling a small factor of the local energy component of the l^1X^s norm. In Section 3.7, we control this remaining component of the l^1X^s norm for the linear paradifferential flow. Then, in Section 3.8, we deduce the full l^1X^s estimate for the paradifferential and linearized equations by combining the local energy estimate with the $L^\infty_T L^2_x$ estimate from Section 3.6. Finally, in Section 3.9 we use the linearized estimates from the previous sections along with a suitable paradifferential reduction of the full nonlinear equation to establish Theorem 3.1.3.

Acknowledgements

We thank Sung-Jin Oh for several enlightening discussions. We also thank both Sung-Jin Oh and In-Jee Jeong for kindly sharing with us the method of spatially truncating the renormalization operator in the energy bound for the linearized equation, which was an idea that they first implemented in [81] for the electron MHD equations. An important component of this argument, which we also use in our setting, is the high-frequency Calderon-Vaillancourt bound in Proposition 3.2.6.

3.2 Preliminaries

In this section, we recall some basic Fourier-analytic tools as well as the definitions and elementary properties of the function spaces that will be used in our analysis. We also recall some standard facts about pseudodifferential operators and establish some new estimates for these operators in our function spaces.

Littlewood-Paley decomposition

We begin by recalling the standard Littlewood-Paley decomposition. We let $\varphi : \mathbb{R}^d \to [0,1]$ be a smooth radial function supported in the ball of radius 2, $B_2 = B_2(0)$, which satisfies $\varphi = 1$ on B_1. We define Fourier multipliers S_0 and S_k by

$$\begin{aligned} \widehat{S}_k &:= \varphi(2^{-k}\xi) - \varphi(2^{-k+1}\xi), \qquad k \in \mathbb{N}, \\ \widehat{S}_0 &:= \varphi(\xi). \end{aligned}$$

We then define for each $k \in \mathbb{N}$,

$$S_{<k} := \sum_{0 \le j < k} S_j, \qquad S_{\ge k} := \sum_{k \le j < \infty} S_j.$$

With the above notation, we have the standard (inhomogeneous) Littlewood-Paley decomposition

$$1 = \sum_{k \ge 0} S_k.$$

In the sequel, we will often phrase our bilinear and nonlinear estimates in the language of paradifferential calculus. For a suitable pair of complex-valued functions f and g, we will write $T_g f$ to mean

$$T_g f := \sum_{k \geq 0} S_{<k-4} g S_k f. \tag{3.2.1}$$

In other words, $T_g f$ selects the portion of the product fg where f is at high frequency compared to g. With this notation, we have the so-called Bony decomposition or Littlewood-Paley trichotomy,

$$fg = T_f g + T_g f + \Pi(f, g).$$

We refer the reader to [14] and [109] for some basic properties of these operators. To compactify the above notation, we will sometimes write $f_{<k}$ as shorthand for $S_{<k} f$ and $f_{\geq k}$ as shorthand for $S_{\geq k} f$.

Function spaces

Next, we recall the definitions and basic properties of the function spaces that will be used in our analysis. Much of the material here is recalled from [105] and the large data paper [106]. For each frequency scale 2^k, we consider a partition of $\mathbb{R}^d$ into a set Q_k of disjoint cubes of side length 2^k along with a smooth partition of unity in physical space,

$$1 = \sum_{Q \in Q_k} \chi_Q.$$

For a translation-invariant Sobolev type space U, we define the spaces $l_k^p U$ by

$$\|u\|_{l_k^p U} := \left(\sum_{Q \in Q_k} \|\chi_Q u\|_U^p \right)^{\frac{1}{p}}, \ \ 1 \leq p < \infty, \quad \|u\|_{l_k^\infty U} := \sup_{Q \in Q_k} \|\chi_Q u\|_U.$$

As noted in [105], these spaces have a translation invariant equivalent norm, obtained by replacing the sum over cubes with an integral. Moreover, up to norm equivalence, the smooth partition by compactly supported cutoffs can be replaced by a partition consisting of cutoffs which are all localized to frequency zero.

We next recall the definition of the local energy type space X, which is defined for each $T > 0$ by

$$\|u\|_X := \sup_{l \in \mathbb{N}_0} \sup_{Q \in Q_l} 2^{-\frac{l}{2}} \|u\|_{L_T^2 L_x^2([0,T] \times Q)}.$$

Associated to X is the "dual" local energy type space $Y \subset L^2([0,T]\times\mathbb{R}^d)$ which satisfies the relation $X = Y^*$. See [105] for more details on the properties and construction of this space. For each non-negative integer k, we define

$$X_k := 2^{-\frac{k}{2}}X \cap L_T^\infty L_x^2, \quad \|u\|_{X_k} := 2^{\frac{k}{2}}\|u\|_X + \|u\|_{L_T^\infty L_x^2}$$

and

$$Y_k := 2^{\frac{k}{2}}Y + L_T^1 L_x^2, \quad \|u\|_{Y_k} := \inf\{2^{-\frac{k}{2}}\|u_1\|_Y + \|u_2\|_{L_T^1 L_x^2} : u = u_1 + u_2\}.$$

Loosely speaking, we will use X_k to measure solutions to the Schrödinger equation localized at frequency 2^k whereas Y_k will be used to measure inhomogeneous source terms localized at this frequency. Next, we define for each $s \in \mathbb{R}$,

$$\|u\|_{l^pX^s} := \left(\sum_{k\geq 0} 2^{2ks}\|S_k u\|_{l_k^p X_k}^2\right)^{\frac{1}{2}}, \quad \|u\|_{l^pY^s} := \left(\sum_{k\geq 0} 2^{2ks}\|S_k u\|_{l_k^p Y_k}^2\right)^{\frac{1}{2}},$$

for $1 \leq p < \infty$ (with the natural modification for $p = \infty$). We will also work with the corresponding spaces without the ℓ^p summability,

$$\|u\|_{X^s} := \left(\sum_{k\geq 0} 2^{2ks}\|S_k u\|_{X_k}^2\right)^{\frac{1}{2}}, \quad \|u\|_{Y^s} := \left(\sum_{k\geq 0} 2^{2ks}\|S_k u\|_{Y_k}^2\right)^{\frac{1}{2}}.$$

As already mentioned, throughout the chapter we will frequently make use of the standard tools of paradifferential calculus to estimate various multilinear and nonlinear expressions. A very nice bookkeeping device for efficiently tracking the frequency distribution of such terms is the language of frequency envelopes introduced by Tao in [147]. To define these, suppose that we are given a translation-invariant Sobolev type space U with the orthogonality relation,

$$\|u\|_U \approx \left(\sum_{k\geq 0}\|S_k u\|_U^2\right)^{\frac{1}{2}}.$$

An admissible frequency envelope for $u \in U$ is a positive sequence $(c_k) \subset \mathbb{N}_0$ such that for each $k \in \mathbb{N}_0$, we have

(i) (Boundedness and size).

$$\|S_k u\|_U \lesssim c_k\|u\|_U, \quad \|c_k\|_{l_k^2} \approx 1.$$

(ii) (Left-slowly varying).

$$c_j \geq 2^{\delta(j-k)} c_k, \quad j < k,$$

for some fixed parameter $\delta > 0$.

(iii) (Right-uniformly varying).

$$c_j \geq 2^{\sigma(k-j)} c_k, \quad j > k,$$

for some fixed parameter $\sigma > 0$.

For nonzero u, such a frequency envelope always exists. For instance, we may define

$$c_j = \|u\|_U^{-1} \left(\max_{k \geq j} 2^{-\delta|j-k|} \|S_k u\|_U + \max_{k \leq j} 2^{-\sigma|j-k|} \|S_k u\|_U \right).$$

In this chapter, the primary purpose of the above frequency envelopes will be to facilitate the proof of the continuity of the data-to-solution map for the quasilinear Schrödinger systems we consider.

Pseudodifferential calculus

Our objective in this subsection is to recall some basic properties of pseudodifferential operators and then establish some refined estimates for these operators in the local energy and "dual" local energy spaces defined above.

For $m \in \mathbb{R}$, we recall that the standard symbol class $S^m := S^m_{1,0}$ is defined by

$$S^m := \{a \in C^\infty(\mathbb{R}^{2d}) : |a|^{(j)}_{S^m} < \infty\,, \ \ j \in \mathbb{N}_0\},$$

where the corresponding seminorms $|a|^{(j)}_{S^m}$ are given by

$$|a|^{(j)}_{S^m} := \sup\{\|\langle\xi\rangle^{|\alpha|-m} \partial_x^\beta \partial_\xi^\alpha a(x,\xi)\|_{L^\infty(\mathbb{R}^{2d})} : \ \ |\alpha+\beta| \leq j\}.$$

To each symbol $a \in S^m$ we can associate the pseudodifferential operator $Op(a) \in OPS^m$, defined for $f \in \mathcal{S}(\mathbb{R}^d)$ by the quantization

$$Op(a)f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} a(x,\xi) e^{ix\cdot\xi} \widehat{f}(\xi) d\xi.$$

We now list some basic properties of pseudodifferential operators; proofs can be found in the standard reference [151]. We begin with an elementary result on Sobolev boundedness.

Proposition 3.2.1 (Sobolev boundedness)**.** Let $s, m \in \mathbb{R}$ and let $a \in S^m$. Then $Op(a)$ extends to a bounded linear operator from H^{s+m} to H^s and there exists j depending only on s, m and the dimension such that

$$\|Op(a)\|_{H^{s+m}\to H^s} \lesssim |a|_{S^m}^{(j)}.$$

We next recall the sharp Gårding inequality for symbols $a \in S^1$.

Proposition 3.2.2 (Sharp Gårding inequality)**.** Let $a \in S^1$ and let $R > 0$ be such that $\mathrm{Re}(a) \geq 0$ for $|\xi| \geq R$. Then $Op(a)$ is semi-positive. That is, there exists j depending on d such that for $f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\mathrm{Re}\langle Op(a)f, f\rangle \gtrsim_R -|a|_{S^1}^{(j)}\|f\|_{L^2}^2,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual L^2 inner product.

Proof. See, e.g., [68]. □

Remark 3.2.3. As shown in [68, 96], a variant of Proposition 3.2.2 also holds for $N \times N$ matrix-valued symbols. More specifically, if $a \in S^1$ is an $N \times N$ symbol satisfying $\mathrm{Re}(a) \geq 0$ then the associated pseudodifferential operator $Op(a)$ is semi-positive in the sense that $\mathrm{Re}\langle Op(a)f, f\rangle \geq -c\|f\|_{L^2}^2$ for all f in the Schwartz class.

Next, we recall a (weak) version of the Calderon-Vaillancourt theorem [19].

Proposition 3.2.4 (Calderon-Vaillancourt theorem)**.** Let $a \in S^0$. There exists an integer $j > 0$ depending on the dimension such that

$$\|Op(a)\|_{L^2\to L^2} \lesssim \sup_{|\alpha+\beta|\leq j} \|\partial_\xi^\alpha \partial_x^\beta a\|_{L^\infty(\mathbb{R}^{2d})},$$

where the implicit constant is universal.

Finally, we recall some elementary symbolic calculus which will allow us to perform basic manipulations with pseudodifferential operators.

Proposition 3.2.5 (Algebraic properties of pseudodifferential operators)**.** Let $m_1, m_2 \in \mathbb{R}$ and let $a_1 \in S^{m_1}$ and $a_2 \in S^{m_2}$. The following properties hold.

(i) (Composition property). There is $a \in S^{m_1+m_2-1}$ such that

$$Op(a_1)Op(a_2) = Op(a_1a_2) + Op(a)$$

and for every $j \in \mathbb{N}_0$, $|a|^{(j)}_{S^{m_1+m_2-1}}$ is controlled by $|a_1|^{(k)}_{S^{m_1}}|a_2|^{(k)}_{S^{m_2}}$ for some k depending on j and d.

(ii) (Adjoint). There is $a \in S^{m_1-1}$ such that

$$Op(a_1)^* = Op(\overline{a}_1) + Op(a)$$

and for every $j \in \mathbb{N}_0$, $|a|^{(j)}_{S^{m_1-1}}$ is controlled by $|a_1|^{(k)}_{S^{m_1}}$ for some k depending on j and d.

(iii) (Commutator). There is $a \in S^{m_1+m_2-2}$ such that

$$Op(a_1)Op(a_2) - Op(a_2)Op(a_1) = Op(-i\{a_1, a_2\}) + Op(a)$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket, which is defined by

$$\{a_1, a_2\} = \nabla_\xi a_1 \cdot \nabla_x a_2 - \nabla_\xi a_2 \cdot \nabla_x a_1.$$

Moreover, for every $j \in \mathbb{N}_0$, $|a|^{(j)}_{S^{m_1+m_2-2}}$ is controlled by $|a_1|^{(k)}_{S^{m_1}}|a_2|^{(k)}_{S^{m_2}}$ for some k depending on j and d.

Proof. See, e.g., [84, Theorem 2.1.2] for a precise statement and [55, 151] for proofs. □

In our construction, we will need the following refinement of the Calderon-Vaillancourt theorem for symbols $a \in S^0$, which ensures that the $L^2 \to L^2$ operator bound for $Op(a)$ depends only on the L^∞ norm of a when applied to functions localized at sufficiently high frequency. This refinement will be important later when we attempt to spatially localize the renormalization operator mentioned in the introduction. We remark that Proposition 3.2.6 is also used in the paper [81] to achieve a similar purpose. We include the simple proof for completeness.

Proposition 3.2.6 (Calderon-Vaillancourt theorem at high frequency)**.** Let $a \in S^0$. There is k_0 depending on a such that for $k \geq k_0$, $Op(a)$ satisfies the $L^2 \to L^2$ bound,

$$\|Op(a)S_{\geq k}\|_{L^2 \to L^2} \lesssim \|a\|_{L^\infty}.$$

That is, the $L^2 \to L^2$ bound for $Op(a)$ depends only on the L^∞ norm of the symbol a when applied to functions at sufficiently high frequency.

Proof. The proof is a simple scaling argument. The symbol for $S_{>k}$ is of the form $\psi_k(\xi) := 1-\varphi(2^{-k}\xi)$, where φ is a smooth bump function equal to one on the unit ball and supported in $B_2(0)$. Define the symbol $a_k := a\psi_k$. Let $\lambda > 0$ be some constant to be chosen, and define $a_{k,\lambda}(x,\xi) := a_k(\lambda^{-1}x, \lambda\xi)$, $v_\lambda(x) := v(\lambda x)$. We clearly have

$$Op(a_k)v = (2\pi)^{-d}\int_{\mathbb{R}^d} a_k(x,\lambda\xi)e^{i\lambda x\cdot\xi}\widehat{v_{\lambda^{-1}}}(\xi)d\xi.$$

Hence,

$$\|Op(a_k)v\|_{L^2} = \lambda^{-\frac{d}{2}}\|Op(a_{k,\lambda})v_{\lambda^{-1}}\|_{L^2}.$$

By Proposition 3.2.4, we have

$$\begin{aligned}\lambda^{-\frac{d}{2}}\|Op(a_{k,\lambda})v_{\lambda^{-1}}\|_{L^2} &\lesssim \lambda^{-\frac{d}{2}}\sup_{|\alpha|,|\beta|\le j(d)}\|\partial_x^\beta\partial_\xi^\alpha a_{k,\lambda}\|_{L^\infty}\|v_{\lambda^{-1}}\|_{L^2}\\ &= \sup_{|\alpha|,|\beta|\le j(d)}\|\partial_x^\beta\partial_\xi^\alpha a_{k,\lambda}\|_{L^\infty}\|v\|_{L^2},\end{aligned}$$

where $j(d)$ depends only on the dimension. To conclude, we therefore only need to show that for a suitable choice of λ, we have

$$\sup_{|\alpha|,|\beta|\le j(d)}\|\partial_x^\beta\partial_\xi^\alpha a_{k,\lambda}\|_{L^\infty} \lesssim \|a\|_{L^\infty}.$$

Taking $\lambda = 2^{\frac{k}{2}}$ and using that $a \in S^0$, we find

$$|\partial_x^\beta\partial_\xi^\alpha a_{k,\lambda}| \lesssim |a|_{S^0}^{(|\alpha|+|\beta|)}2^{-|\alpha|k}\lambda^{|\alpha|-|\beta|} \lesssim |a|_{S^0}^{(|\alpha|+|\beta|)}2^{-(|\alpha|+|\beta|)\frac{k}{2}}.$$

The proof is concluded by taking k sufficiently large (depending only on the symbol bounds for a). □

Next, we extend the above bounds to the X^0 and Y^0 spaces.

Proposition 3.2.7 (Operator bounds for X^0 and Y^0)**.** Let $a \in S^0$ be time-independent and let $T \lesssim 1$. Then there is $j = j(d)$ such that we have the operator bounds

$$\|Op(a)\|_{X^0\to X^0} + \|Op(a)\|_{Y^0\to Y^0} \lesssim 1 + |a|_{S^0}^{(j)}. \tag{3.2.2}$$

Moreover, there is $k_0 > 0$ depending only on a such that if $k \ge k_0$, we also have

$$\|Op(a)S_{\ge k}\|_{X^0\to X^0} + \|Op(a)S_{\ge k}\|_{Y^0\to Y^0} \lesssim 1 + \|a\|_{L^\infty}. \tag{3.2.3}$$

Remark 3.2.8. The inequality (3.2.3) can be thought of as the analogue of Proposition 3.2.6 for the X^0 and Y^0 spaces.

Proof. We prove (3.2.2) and remark on the very minor modifications required to prove (3.2.3) where necessary. For notational convenience, we let K_a denote the term on the right-hand side of (3.2.2). We begin with the $X^0 \to X^0$ bound. By definition, we have

$$\begin{aligned}\|Op(a)f\|_{X^0}^2 &= \sum_{k\geq 0}\|S_k Op(a)f\|_{X_k}^2 \\ &\lesssim \sum_{k\geq 0}\|S_k[Op(a),\tilde{S}_k]f\|_{X_k}^2 + \sum_{k\geq 0}\|S_k Op(a)\tilde{S}_k f\|_{X_k}^2,\end{aligned} \tag{3.2.4}$$

for some fattened Littlewood-Paley projection $\tilde{S}_k$. For the first term, we can crudely estimate using Hölder in T and dyadic summation,

$$\begin{aligned}\left(\sum_{k\geq 0}\|S_k[Op(a),\tilde{S}_k]f\|_{X_k}^2\right)^{\frac{1}{2}} &\lesssim \left(\sum_{k\geq 0}2^k\|S_k[Op(a),\tilde{S}_k]f\|_{L_T^\infty L_x^2}^2\right)^{\frac{1}{2}} \\ &\lesssim \sup_{k\geq 0}\|[Op(a),\tilde{S}_k]f\|_{L_T^\infty H_x^{\frac{1}{2}+\varepsilon}} \\ &\lesssim_\varepsilon K_a\|f\|_{L_T^\infty H_x^{-\frac{1}{2}+\varepsilon}} \\ &\lesssim K_a\|f\|_{X^0},\end{aligned}$$

where in the second to third line, we used Proposition 3.2.1 and that $[Op(a),\tilde{S}_k]\in OPS^{-1}$ (which has symbol bounds uniform in k, thanks to Proposition 3.2.5).

Remark 3.2.9. We remark briefly on one change needed here for the proof of (3.2.3). If f is replaced by $S_{>k}f$ for some sufficiently large k, then in the third line above, we can estimate

$$K_a\|S_{>k}f\|_{L_T^\infty H_x^{-\frac{1}{2}+\varepsilon}} \lesssim K_a 2^{-k(\frac{1}{2}-\varepsilon)}\|f\|_{X^0},$$

and so, the factor of K_a can be replaced by 1 in the above estimate by taking k large enough.

Now, we turn to the second term in (3.2.4). By square summing and Proposition 3.2.4 (or Proposition 3.2.6 when proving (3.2.3)), it suffices to estimate the $2^{-\frac{k}{2}}X$ component of the X_k norm. For this, we have

$$\|Op(a)\tilde{S}_k f\|_{2^{-\frac{k}{2}}X} = \sup_{l\in\mathbb{N}_0}\sup_{Q\in Q_l} 2^{\frac{k-l}{2}}\|\chi_Q Op(a)\tilde{S}_k f\|_{L_T^2 L_x^2}.$$

Using the $L^2 \to L^2$ bound for $Op(a)$ from Proposition 3.2.4 and that $[Op(a),\chi_Q]\in OPS^{-1}$ with bounds independent of l, we obtain for each $Q\in Q_l$,

$$\begin{aligned}2^{\frac{k-l}{2}}\|\chi_Q Op(a)\tilde{S}_k f\|_{L_T^2 L_x^2} &\lesssim K_a 2^{\frac{k-l}{2}}\|\chi_Q\tilde{S}_k f\|_{L_T^2 L_x^2} + 2^{\frac{k}{2}}K_a\|\tilde{S}_k f\|_{L_T^2 H_x^{-1}} \\ &\lesssim K_a\|\tilde{S}_k f\|_{X_k} + 2^{-\frac{k}{2}}K_a\|\tilde{S}_k f\|_{L_T^\infty L_x^2}.\end{aligned} \tag{3.2.5}$$

Therefore,

$$\left(\sum_{k\geq 0}\|Op(a)\tilde{S}_k f\|_{X_k}^2\right)^{\frac{1}{2}} \lesssim K_a\|f\|_{X^0},$$

which establishes the $X^0 \to X^0$ bound. The high-frequency variant (3.2.3) is proved by using instead Proposition 3.2.6 in place of Proposition 3.2.4 above and using the frequency gain in the latter term in the second line of (3.2.5) to absorb the factor of K_a.

Next, we turn to the $Y^0 \to Y^0$ bound. Again, by definition, we have

$$\begin{aligned}\|Op(a)f\|_{Y^0}^2 &= \sum_{k\geq 0}\|S_k Op(a)f\|_{Y_k}^2\\ &\lesssim \sum_{k\geq 0}\|S_k[Op(a),\tilde{S}_k]f\|_{Y_k}^2 + \sum_{k\geq 0}\|S_k Op(a)\tilde{S}_k f\|_{Y_k}^2.\end{aligned}$$

For the first term, we estimate using the embedding $L_T^1 L_x^2 \subset Y_k$ and that $[Op(a),\tilde{S}_k] \in OPS^{-1}$ to obtain

$$\left(\sum_{k\geq 0}\|S_k[Op(a),\tilde{S}_k]f\|_{Y_k}^2\right)^{\frac{1}{2}} \lesssim K_a\|f\|_{L_T^1 H_x^{-1+\varepsilon}} \lesssim K_a\|f\|_{Y^0},$$

where the last inequality follows from the fact that $Y^0 \subset L_T^1 H_x^{-\frac{1}{2}-\varepsilon}$. Similarly to before, for the bound (3.2.3) when f is replaced by $S_{>k}f$, we have

$$K_a\|S_{>k}f\|_{L_T^1 H_x^{-1+\varepsilon}} \lesssim K_a 2^{-k(\frac{1}{2}-2\varepsilon)}\|f\|_{Y^0},$$

and so, the factor of K_a can be replaced by 1 if k is large enough. For the second term, we use duality. Let $g \in X_k$ with $\|g\|_{X_k} \leq 1$. We have by Proposition 3.2.5 and similar embeddings as above,

$$\begin{aligned}|\langle S_k Op(a)\tilde{S}_k f, g\rangle| &\lesssim \|\tilde{S}_k f\|_{Y_k}\|\tilde{S}_k(Op(\overline{a}))S_k g\|_{X_k} + K_a\|\tilde{S}_k f\|_{L_T^1 H_x^{-1}}\|S_k g\|_{L_T^\infty L_x^2}\\ &\lesssim \|\tilde{S}_k f\|_{Y_k}\|\tilde{S}_k(Op(\overline{a}))S_k g\|_{X_k} + 2^{-k(\frac{1}{2}-\varepsilon)}K_a\|\tilde{S}_k f\|_{Y_k}.\end{aligned}$$

Again, if k is large enough, the K_a factor in the latter term can be discarded. Using the $X^0 \to X^0$ bound already established above, we also have

$$\|\tilde{S}_k(Op(\overline{a}))S_k g\|_{X_k} \lesssim \|Op(\overline{a})S_k g\|_{X^0} \lesssim K_a,$$

where K_a can be replaced by $1 + \|a\|_{L^\infty}$ if k is large enough. The proof of (3.2.2) is then concluded by dyadic summation. □

In our analysis later, we will sometimes need to estimate commutators of pseudodifferential and paradifferential operators. For this purpose, we recall the following Coifman-Meyer type estimate from (3.6.4) and (3.6.5) of [151].

Proposition 3.2.10 (Coifman-Meyer type bound)**.** For every $m, \sigma \in \mathbb{R}$ and $P \in OPS^m$, we have

$$\|[P, T_g]f\|_{H^\sigma} \leq C\|g\|_{W^{1,\infty}}\|f\|_{H^{\sigma+m-1}}, \tag{3.2.6}$$

where $C > 0$ is a constant depending on P and σ.

Multilinear and Moser estimates

Here we recall several of the multilinear and Moser-type estimates for the local energy and dual local energy spaces defined above.

Proposition 3.2.11 (Proposition 3.1 in [105])**.** Let $s > \frac{d}{2}$. Then for $u, v \in l^1X^s$ we have the algebra property

$$\|uv\|_{l^1X^s} \lesssim \|u\|_{l^1X^s}\|v\|_{l^1X^s}.$$

We also have the Moser-type estimate,

$$\|F(u)\|_{l^1X^s} \lesssim \|u\|_{l^1X^s}(1 + \|u\|_{l^1X^s})c(\|u\|_{L^\infty}),$$

for $s > \frac{d}{2}$ and any smooth function F with $F(0) = 0$.

We next recall some elementary bilinear estimates for the $l^1_kY_k$ spaces.

Proposition 3.2.12 (Bilinear estimates)**.** The following bilinear estimates hold for $l^1_kY_k$ spaces.

(i) (High-low interactions). If $j < k - 4$,

$$\|S_juS_kv\|_{l^1_kY_k} \lesssim 2^{j(\frac{d}{2}+1)}2^{-k}\|S_kv\|_{2^{-\frac{k}{2}}X}\|S_ju\|_{l^1_jL^\infty_TL^2_x}.$$

(ii) (Balanced interactions). If $|i - j| \leq 4$ and $i, j \geq k - 4$,

$$\|S_k(S_iuS_jv)\|_{l^1_kY_k} \lesssim 2^{\frac{jd}{2}}\|S_iu\|_{l^1_iL^2_TL^2_x}\|S_jv\|_{L^\infty_TL^2_x}.$$

Proof. This is a slight refinement of Lemma 4.3 in [106]. The proof is almost identical, so we omit the details. □

By dyadic summation, the following is a consequence of Proposition 3.2.12.

Proposition 3.2.13 (Paradifferential bilinear estimates). Let $s_0 > \frac{d}{2} + 2$. Then for every $\sigma \geq 0$, we have

$$\|T_u v\|_{l^1 Y^\sigma} + \|(T_v - v)u\|_{l^1 Y^\sigma} \lesssim \|u\|_{l^1 X^{s_0 - 1}} \|v\|_{X^{\sigma-1}},$$
$$\|T_u v\|_{l^1 Y^\sigma} + \|(T_v - v)u\|_{l^1 Y^\sigma} \lesssim \|u\|_{l^1 X^{s_0 - 2}} \|v\|_{X^{\sigma}}.$$

If $0 \leq \sigma \leq s_0$, we also have

$$\begin{aligned} \|(T_v - v)u\|_{l^1 Y^\sigma} &\lesssim \|u\|_{l^1 X^{\sigma - 1}} \|v\|_{X^{s_0-1}}, \\ \|(T_v - v)u\|_{l^1 Y^\sigma} &\lesssim \|u\|_{l^1 X^{\sigma - 2}} \|v\|_{X^{s_0}}. \end{aligned} \tag{3.2.7}$$

We next state a closely related commutator estimate, which is a slight refinement of the version in [105].

Proposition 3.2.14. Let $s > \frac{d}{2} + 2$ and let $A \in S^0$ be a Fourier multiplier. Then we have

$$\|\nabla[S_{<k-4} g, A(D)]\nabla S_k u\|_{l^1 Y^0} \lesssim_A \|g - g_\infty\|_{l^1 X^s} \|S_k u\|_{X^0},$$

where g_∞ is any constant matrix.

Proof. The proof of this is essentially identical to the proof of Proposition 3.2 in [105]. We omit the details. □

Remark 3.2.15. We note that if $A \in S^m$ is a Fourier multiplier for some real number $m \geq 0$ then we can write the commutator in the above proposition as

$$[S_{<k-4} g, A(D)]\nabla S_k u = 2^{mk} \tilde{S}_k [S_{<k-4} g, 2^{-mk} A(D) \tilde{S}_k] \nabla S_k u,$$

for some fattened projection $\tilde{S}_k$. Since $2^{-mk} A(D) \tilde{S}_k \in OPS^0$ with symbol bounds uniform in k, we have from Proposition 3.2.14,

$$\|[S_{<k-4} g, A(D)]\nabla S_k u\|_{l^1 Y^0} \lesssim 2^{(m-1)k} \|g - g_\infty\|_{l^1 X^s} \|S_k u\|_{X^0}.$$

The next bound will allow us to precisely estimate certain error terms in the dual local energy space Y^0 which involve commutators of pseudodifferential and paradifferential operators. This is essentially a variant of Proposition 3.2.10 but for the X and Y spaces.

Proposition 3.2.16 ($X^{-2} \to Y^0$ commutator estimate)**.** Let $T \lesssim 1$, $\mathcal{O} \in OPS^0$ be time-independent with symbol $O \in S^0$ and let $s_0 > \frac{d}{2}+2$. Moreover, let g be a function such that $g - g_\infty \in l^1 X^{s_0}$ for some constant g_∞. Then we have the estimate

$$\|[\mathcal{O}, T_g]f\|_{Y^0} \leq C\|g - g_\infty\|_{l^1 X^{s_0}} \|f\|_{X^{-2}},$$

where C depends only on $\mathcal{O}$.

Proof. Clearly, it suffices to prove the claim with $g_\infty = 0$. Moreover, it suffices to work with the principal part of the commutator since the remainder is bounded from $H_x^{-2} \to L_x^2$ uniformly in T with norm $\lesssim_{\mathcal{O}} \|g\|_{L_T^\infty C^{2,\varepsilon}}$ for some sufficiently small $\varepsilon > 0$, which by Sobolev embedding can be controlled by $\|g\|_{l^1 X^{s_0}}$. The principal symbol p for $[T_g, \mathcal{O}]$ is

$$\begin{aligned} p(x,\xi) &:= -i \sum_{k \geq 0} \{S_{<k-4} g(x) \widehat{S}_k(\xi), O\} \\ &= -i \sum_{k \geq 0} S_{<k-4} g(x) \nabla_\xi \widehat{S}_k(\xi) \cdot \nabla_x O + i \sum_{k \geq 0} S_{<k-4} \nabla_x g(x) \widehat{S}_k(\xi) \cdot \nabla_\xi O =: p_1 + p_2. \end{aligned}$$

First, we consider bounds for $P_1 := Op(p_1)$. Modulo an operator which is bounded from $H_x^{-2} \to L_x^2$ with norm $\lesssim_{\mathcal{O}} \|g\|_{L_T^\infty C^{2,\varepsilon}}$, we can write

$$P_1 = -i \sum_{k \geq 0} S_{<k-4} g(x) (\nabla_\xi \widehat{S}_k)(D) \cdot Op(\nabla_x O) \tilde{S}_k + \mathcal{O}_{L_T^\infty H_x^{-2} \to L_T^\infty L_x^2}(1),$$

where $\tilde{S}_k$ is a slightly fattened Littlewood-Paley projection. As $\nabla_\xi \widehat{S}_k$ is localized at frequency $\approx 2^k$, we can use Proposition 3.2.12 and dyadic summation to estimate

$$\|P_1 f\|_{Y^0} \lesssim \|g - g_\infty\|_{l^1 X^{s_0 - 1}} \|f\|_{X^{-2}}.$$

A similar argument for $P_2 := Op(p_2)$ gives

$$\|P_2 f\|_{Y^0} \lesssim \|\nabla_x g\|_{l^1 X^{s_0 - 1}} \|f\|_{X^{-2}},$$

which concludes the proof. □

Finally, we state versions of some of the above bilinear and Moser estimates which are phrased in terms of frequency envelopes. This will be convenient for establishing the finer properties of the solution map later on, such as the continuous dependence of the solution on the initial data. From Proposition 3.2 of [105], we have the following estimates.

Proposition 3.2.17 (Frequency localized estimates I)**.** Let $s > \frac{d}{2}$ and let $u, v \in l^1X^s$ with l^1X^s frequency envelopes given by a_k and b_k, respectively. Then for each $k \in \mathbb{N}_0$, we have

$$\|S_k(uv)\|_{l^1X^s} \lesssim (a_k + b_k)\|u\|_{l^1X^s}\|v\|_{l^1X^s}.$$

Moreover, if F is a smooth function with $F(0) = 0$, then we have

$$\|S_k(F(u))\|_{l^1X^s} \lesssim a_k\|u\|_{l^1X^s}(1 + \|u\|_{l^1X^s})c(\|u\|_{L^\infty}).$$

Proposition 3.2.18 (Frequency localized estimates II)**.** Let $s > \frac{d}{2} + 2$. The following estimates hold for $k \in \mathbb{N}_0$.

(i) Let $0 \leq \sigma \leq s$ and let $u \in l^1X^{\sigma-1}$ and $v \in l^1X^{s-1}$ with corresponding frequency envelopes a_k and b_k, respectively. We have

$$\|S_k(uv)\|_{l^1Y^\sigma} \lesssim (a_k + b_k)\|u\|_{l^1X^{\sigma-1}}\|v\|_{l^1X^{s-1}}.$$

(ii) Let $0 \leq \sigma \leq s - 1$ and let $u \in l^1X^\sigma$ and $v \in l^1X^{s-2}$ with corresponding frequency envelopes a_k and b_k, respectively. We have

$$\|S_k(uv)\|_{l^1Y^\sigma} \lesssim (a_k + b_k)\|u\|_{l^1X^\sigma}\|v\|_{l^1X^{s-2}}.$$

(iii) Let $0 \leq \sigma \leq s$ and let $u \in l^1X^\sigma$ and $v \in l^1X^{s-2}$ with corresponding frequency envelopes a_k and b_k, respectively. We have

$$\|S_k(vS_{\geq k-4}u)\|_{l^1Y^\sigma} \lesssim (a_k + b_k)\|u\|_{l^1X^\sigma}\|v\|_{l^1X^{s-2}}.$$

3.3 Overview of the proof

In this section, we give an overview of the key ideas that go into the proof of Theorem 3.1.3. We recall that our essential aim is to establish local well-posedness for the system

$$\begin{cases} i\partial_t u + \partial_j g^{jk}(u, \overline{u})\partial_k u = F(u, \overline{u}, \nabla u, \nabla \overline{u}), \quad u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^m, \\ u(0, x) = u_0(x), \end{cases} \tag{3.3.1}$$

in the l^1H^s scale, for $s > s_0 > \frac{d}{2} + 2$. As we shall see, our scheme builds on and complements the ideas from [81, 84, 90, 86, 87, 106], but also has several important novelties.

The linear and paradifferential ultrahyperbolic flows

The main component of our argument involves a careful analysis of the linear ultrahyperbolic flow

$$\begin{cases} i\partial_t v + \partial_j g^{jk}\partial_k v + b^j\partial_j v + \tilde{b}^j\partial_j \overline{v} = f, \\ v(0,x) = v_0(x), \end{cases} \tag{3.3.2}$$

which is naturally associated with the linearization of (3.3.1). Here, the metric g^{jk} is real, nontrapping, symmetric and non-degenerate, and the coefficients g, b, and $\tilde{b}$ satisfy the asymptotic flatness conditions

$$\|g - g_\infty\|_{l^1X^{s_0}} + \|\partial_t g\|_{l^1X^{s_0-2}} + \|(b,\tilde{b})\|_{l^1X^{s_0-1}} + \|\partial_t(b,\tilde{b})\|_{l^1X^{s_0-3}} \leq M, \tag{3.3.3}$$

where $M > 0$ is a fixed constant and g_∞ is a constant, non-degenerate, symmetric matrix. Note that in the special case where v corresponds to the linearization around a solution u to (3.3.1), the coefficients and inhomogeneous source term in (3.3.2) take the form

$$\begin{cases} b^j := \nabla_u g^{jk}\partial_k u - \nabla_{(\nabla u)_j} F, \quad \tilde{b}^j := \nabla_{\overline{u}} g^{jk}\partial_k u - \nabla_{(\nabla \overline{u})_j} F, \\ f := (\nabla_{\overline{u}} F - \partial_j(\nabla_{\overline{u}} g^{jk})\partial_k u)\overline{v} + (\nabla_u F - \partial_j(\nabla_u g^{jk})\partial_k u)v, \end{cases} \tag{3.3.4}$$

where we have suppressed the dependence on u in the coefficients for simplicity of notation. In our general analysis of (3.3.2), we will not require that b, $\tilde{b}$ and f arise from solutions to (3.3.1) via linearization.

An equation that is closely related to (3.3.2) is the associated linear paradifferential flow

$$\begin{cases} i\partial_t v + \partial_j T_{g^{jk}}\partial_k v + T_{b^j}\partial_j v + T_{\tilde{b}^j}\partial_j \overline{v} = f, \\ v(0,x) = v_0(x), \end{cases} \tag{3.3.5}$$

which extracts the leading part of the linear flow. Here, the paradifferential operator T_g is defined as in (3.2.1). Again, when v arises from the linearization around a solution u to (3.3.1), b^j and $\tilde{b}^j$ remain as in (3.3.4), but now

$$\begin{aligned} f :=& (\nabla_{\overline{u}} F - \partial_j(\nabla_{\overline{u}} g^{jk})\partial_k u)\overline{v} + (\nabla_u F - \partial_j(\nabla_u g^{jk})\partial_k u)v + (\partial_j T_{g^{jk}}\partial_k - \partial_j g^{jk}\partial_k)v \\ &+ (T_{b^j} - b^j)\partial_j v + (T_{\tilde{b}^j} - \tilde{b}^j)\partial_j \overline{v}. \end{aligned}$$

In this case, f can be thought of as being comprised of perturbative error terms when measured in the dual local energy space l^1Y^0. These terms either have a suitable algebraic balance of derivatives between the coefficients and v or have coefficient functions that are at high or comparable frequency relative to v.

Quantitative nontrapping and the bicharacteristic flow

As in [106], to adequately study the linear (and ultimately nonlinear) problem, we will need a quantitative measure of nontrapping. For our purposes, we will only need to define nontrapping for time-independent metrics g with regularity and decay given by

$$\|g - g_\infty\|_{l^1 H^{s_0}} \leq M, \quad \frac{d}{2} + 2 < s_0 < s, \tag{3.3.6}$$

where $M > 0$ is a fixed constant and g_∞ is a constant, non-degenerate, symmetric matrix. Note that the condition (3.3.6) guarantees that $g \in C^{2,\delta}$, which in particular ensures that the corresponding Hamilton flow,

$$(\dot{x}^t, \dot{\xi}^t) = (\nabla_\xi a(x^t, \xi^t), -\nabla_x a(x^t, \xi^t)), \quad a(x, \xi) = -g^{ij}(x)\xi_i\xi_j, \quad (x^0, \xi^0) = (x, \xi),$$

is locally well-posed. The first preliminary objective of Section 3.4 is to show that under the nontrapping assumption on g and the asymptotic flatness condition (3.3.6), the Hamilton flow is in fact globally defined. This is not automatic when Δ_g is not elliptic. Indeed, although $g^{ij}\xi_i\xi_j$ is conserved by the Hamilton flow, unlike in the elliptic case, it does not necessarily control the size of $|\xi^t|$ in our setting.

The second objective of Section 3.4 is to provide a quantitative measure of nontrapping. For this, we define a function $L : [0, \infty) \to [0, \infty)$ where $L(R)$ measures (roughly speaking) the maximal amount of time any initially unit speed bicharacteristic can intersect the ball $B_R(0)$. Our definition differs slightly from the definition in [106], as they define L in terms of the Hamilton flow projected onto the co-sphere bundle $|\xi^t| = 1$. This latter definition is natural in the elliptic case, in light of the conservation of $g^{ij}\xi_i\xi_j$, but is not quite suitable for our problem. Analogously to [106], we show that our nontrapping parameter L is stable under small perturbations of the metric, which will be important later on when we analyze the linear and nonlinear Schrödinger flows.

The main linear estimate

The crux of our argument centers around establishing the following key estimate for the linear paradifferential flow (3.3.5):

$$\|v\|_{l^1 X^\sigma} \leq C(M, L)(\|v_0\|_{l^1 H^\sigma} + \|f\|_{l^1 Y^\sigma}), \quad 0 \leq \sigma, \tag{3.3.7}$$

which as a simple consequence yields the following estimate for the linear flow (3.3.2):

$$\|v\|_{l^1 X^\sigma} \leq C(M, L)(\|v_0\|_{l^1 H^\sigma} + \|f\|_{l^1 Y^\sigma}), \quad 0 \leq \sigma \leq s_0.$$

Here, $C(M, L)$ is a constant depending on the coefficient size M in (3.3.3) and on the nontrapping parameter L for g within a fixed compact set whose size depends on the profile of the metric g and the rate of decay of the coefficients b^j and $\tilde{b}^j$. In Section 3.5, we reduce establishing the above two estimates to establishing the following simpler bound for the linear paradifferential flow:

$$\|v\|_{X^\sigma} \leq C(M, L)(\|v_0\|_{H^\sigma} + \|f\|_{Y^\sigma}), \quad \sigma \geq 0, \tag{3.3.8}$$

in the setting where $\widehat{v}$ is supported at frequencies $\gtrsim 2^{k_1}$, where k_1 is some sufficiently large parameter. This latter reduction follows in a straightforward manner as low-frequency errors can be controlled by taking T small enough depending on k_1. The reason we perform this reduction is so that we can make use of the more precise pseudodifferential mapping properties in Propositions 3.2.6 and 3.2.7, which provide high frequency operator bounds for pseudodifferential operators that depend only on the L^∞ norm of their symbols (as long as the symbol itself does not depend on k_1). This will be of critical importance in Sections 3.6 and 3.7, as we will explain below.

L^2 bounds for the linear flow

We next give an outline of Section 3.6, where we prove the first of the two main components of the bound (3.3.8). The main aim of Section 3.6 is to establish control of the $L^\infty_T H^\sigma_x$ norm of v. Throughout the discussion, we assume that $\widehat{v}$ is supported at frequencies larger than 2^{k_1}. Given $\varepsilon > 0$, our aim is to prove an estimate of the form

$$\|v\|_{L^\infty_T H^\sigma_x} \leq C(M, L)(\|v_0\|_{H^\sigma} + \|f\|_{Y^\sigma}) + \varepsilon\|v\|_{X^\sigma}, \quad \sigma \geq 0, \tag{3.3.9}$$

on a time interval $[0, T]$ whose length depends on M, L and ε. In order to clearly outline the main techniques, we focus on the case $\sigma = 0$. The key objective in this part of the proof is to construct a spatially truncated version of the renormalization operator in (3.1.6) which conjugates away the "main" portion of the term $\mathrm{Re}(b^j)\partial_j v$. As noted in the introduction, obtaining an $L^\infty_T L^2_x$ bound for (3.3.5) is straightforward in the absence of such a term. Unlike the symbol in (3.1.6), however, we want the symbol $O(x, \xi)$ of our renormalization operator $\mathcal{O}$ to be time-independent and to belong to S^0. In view of the first goal (and also to ensure that our symbol is smooth) we truncate in frequency and time, rewriting the paradifferential linear flow as

$$\begin{cases} i\partial_t v + \partial_j T_{g^{ij}} \partial_i v + b^j_{<k_0}(0)\partial_j v + \tilde{b}^j_{<k_0}(0)\partial_j \bar{v} = f + \mathcal{R}^1, \\ v(0) = v_0. \end{cases} \tag{3.3.10}$$

We then prove that if the frequency truncation parameter k_0 is large enough and T is small enough, the resulting error term $\mathcal{R}^1$ satisfies the perturbative bound

$$\|\mathcal{R}^1\|_{Y^0} \leq \varepsilon\|v\|_{X^0}.$$

In order to guarantee smoothness of our symbol O, we will only work with the Hamilton flow (x^t, ξ^t) for the truncated symbol $a := -g^{ij}_{<k_0}(0)\xi_i\xi_j$. By our stability results, the truncated metric $g^{ij}_{<k_0}(0)$ will be nontrapping with comparable parameters to g^{ij} if k_0 is sufficiently large and T is sufficiently small. The downside of working exclusively with these truncated quantities, however, is that we will need to obtain an estimate of the form

$$\|[\mathcal{O}, \partial_i(T_{g^{ij}} - g^{ij}_{<k_0}(0))\partial_j]\|_{X^0 \to Y^0} \leq \varepsilon,$$

when we commute the equation with $\mathcal{O}$. Establishing such a bound is not completely trivial, but can be handled expeditiously with the tools developed in Section 3.2. Knowing this, our construction of O proceeds as follows: First, we fix a large parameter R such that the coefficients in the equation are small outside of $B_R(0)$. That is,

$$\|(g - g_\infty)\chi_{>R}\|_{l^1 X^{s_0}} + \|(b, \tilde{b})\chi_{>R}\|_{l^1 X^{s_0-1}} \ll \varepsilon. \tag{3.3.11}$$

We then make the ansatz $O := e^{\psi_1+\psi_2}$, where $\psi_1, \psi_2 \in S^0$. The purpose of the symbol ψ_1 is to arrange for the leading order cancellation

$$\{a, \psi_1\} + \mathrm{Re}(b^j_{<k_0}(0))\xi_j \geq 0,$$

within the region $B_R(0)$ where the coefficient b^j is potentially large. Roughly speaking (but not exactly), we will take

$$\psi_1(x, \xi) := -\chi_{<2R}(x) \int_{-\infty}^{0} \mathrm{Re}((\chi_{<4R} b_{<k_0}(0))(x^t)) \cdot \xi^t dt.$$

The symbol ψ_2 will then be chosen to correct the error terms in the transition region $|x| \approx R$ which appear when derivatives are applied to the localization $\chi_{<2R}$. The resulting symbol O will turn out to be a classical time-independent S^0 symbol, allowing us to avoid the more exotic symbol class (3.1.7) used in [84].

We crucially note that the spatial localization in O comes with one significant caveat. Namely, it only conjugates away the bad first order term within the region $B_R(0)$. Therefore, we still have to estimate the residual error term $\chi_{>R}\,\mathrm{Re}(b^j_{<k_0}(0))\partial_j\mathcal{O}v$ in Y^0. Ideally, such

an estimate would follow easily from the smallness (3.3.11) of b^j outside of $B_R(0)$. However, the symbol bounds for O grow in the parameter R. Therefore, we have to somehow ensure that the $X^0 \to X^0$ bounds for $\mathcal{O}$ do not counteract the smallness coming from b^j. This is accomplished by using the observation that, unlike the higher order symbol bounds, the L^∞ norm of the symbol O is independent of the parameter R (as $R \to \infty$). Therefore, since $\widehat{v}$ is supported at frequencies $\gtrsim 2^{k_1}$, we can make use of the bounds in Proposition 3.2.6 and Proposition 3.2.7 to ensure that we have an estimate essentially of the form

$$\|\mathcal{O}v\|_{X^0} \lesssim \|O\|_{L^\infty}\|v\|_{X^0}.$$

This is what will ultimately allow us to break any potential circularity in our analysis. As mentioned earlier, an analogous construction was used to establish well-posedness for the electron MHD equations in [81].

Local energy bounds for the linear flow

In Section 3.7, we turn to the second of the two main components of the bound (3.3.8). Again, for simplicity of discussion, we take $\sigma = 0$. Here, the aim is to establish control of the local energy component of the X^0 norm of v in terms of the dual norm of f and the $L_T^\infty L_x^2$ norm of v. More precisely, we aim to prove an estimate of the form

$$\|v\|_{X^0} \leq C(M, L)(\|v\|_{L_T^\infty L_x^2} + \|f\|_{Y^0}). \tag{3.3.12}$$

Combining this bound with the $L_T^\infty L_x^2$ bound (3.3.9) (with ε sufficiently small), it is relatively straightforward to obtain the main bound (3.3.8). To obtain (3.3.12), we implement a novel approach based on the truncation idea used in the $L_T^\infty L_x^2$ bound. As before, we begin by fixing $R > 0$ so that we have the smallness (3.3.11) outside of $B_R(0)$.

Our first observation is that we can use the small data result from [105] (which holds for the general ultrahyperbolic Schrödinger flows that we consider here) to reduce having to control the entire local energy component of v to having to obtain the corresponding estimate within the compact set $B_R(0)$. Precisely, we can reduce matters to establishing the bound

$$\|\chi_{<R}v\|_{L_T^2 H_x^{\frac{1}{2}}} \leq C(M, L)(\|v\|_{L_T^\infty L_x^2} + \|f\|_{Y^0}) + \varepsilon\|v\|_{X^0}. \tag{3.3.13}$$

Note that in (3.3.13) we work with the stronger (but simpler) $L_T^2 H_x^{\frac{1}{2}}$ norm within the compact set $B_R(0)$. The starting point in the proof of this estimate is to rewrite (3.3.5) as a system for $\mathbf{v} := (v, \overline{v})$:

$$\partial_t \mathbf{v} + \mathbf{P}\mathbf{v} + \mathbf{B}^0_{k_0}\mathbf{v} = \mathbf{R},$$

where $\mathbf{P}$ is the corresponding principal operator. As in the $L^\infty_T L^2_x$ estimate, the operator $\mathbf{B}^0_{k_0}$ is a suitable time and frequency truncated version of the first order differential operator in the paralinearized Schrödinger equation and $\mathbf{R}$ is a source term which can be controlled by the right-hand side of (3.3.13) in Y^0. We write $\mathbf{P}^0_{k_0}$ as a shorthand for the associated time and frequency truncated principal operator.

The estimate (3.3.13) proceeds via a positive commutator argument. Our implementation can be thought of as a spatially truncated version of Doi's argument in [38]. Precisely, we aim to construct a real symbol $q \in S^0$ and a corresponding pseudodifferential operator $\mathbf{Q}$ such that the principal symbol for the commutator $[\mathbf{Q}, \mathbf{P}^0_{k_0}]$ is elliptic within $B_R(0)$ and controls the first order term $\mathbf{B}^0_{k_0}$ up to a small error. Like before, we work with the bicharacteristic flow (x^t, ξ^t) for the time and frequency truncated metric $g^{ij}_{<k_0}(0)$ to ensure that the symbol we construct is smooth and time-independent. To construct q, we first fix a secondary parameter $R' \gg R$ to be chosen. Similarly to before, we make the ansatz

$$q := e^{C(M)(p_1+p_2+p_3)},$$

where $C(M)$ is a suitably large constant and p_1, p_2, p_3 are S^0 symbols to be chosen. The choice of p_1 will simply ensure the ellipticity of $[\mathbf{Q}, \mathbf{P}^0_{k_0}]$ in $B_R(0)$. We can take

$$p_1(x,\xi) := -\chi_{<R'} \int_0^\infty \chi_{<R}(x^t, \xi^t)|\xi^t| dt.$$

A natural next step would be to correct this symbol in the transition region $|x| \approx R'$ and use the smallness of the coefficients $(b, \tilde{b})$ outside of $B_R(0)$ as in the $L^\infty_T L^2_x$ estimate. However, this will not work because the L^∞ bound for p_1 will not be uniform in R. Instead, we consider a second symbol p_2 whose purpose will be to ensure that the commutator $[\mathbf{Q}, \mathbf{P}^0_{k_0}]$ controls the first order term $\mathbf{B}^0_{k_0}$ within the much larger compact set $B_{R'}(0)$ but with an L^∞ bound which does not depend on the larger parameter R'. Roughly speaking, we will take p_2 to be

$$p_2(x,\xi) := -\chi_{<R'} \int_0^\infty \chi_{<R'}(x^t) \sqrt{|(b_{<k_0}(0))(x^t)|^2 + |(\tilde{b}_{<k_0}(0))(x^t)|^2 + L(R')^{-2}} \langle \xi^t \rangle dt,$$

which turns out to be a S^0 symbol with the desired properties. The symbol p_3 will then be chosen to correct the error in the transition region $|x| \approx R'$ similarly to the $L^\infty_T L^2_x$ bound. If $\mathbf{v}$ is localized at high enough frequency, the multiplier $\mathbf{Q}$ will then achieve the following key outcomes.

- $[\mathbf{Q}, \mathbf{P}^0_{k_0}]$ will have an essentially positive-definite principal symbol which is elliptic of order 1 within $B_R(0)$. This will permit the use of Gårding's inequality to control $\chi_{<R}\mathbf{v}$ in $L^2_T H^{\frac{1}{2}}_x$.

- $[\mathbf{Q}, \mathbf{P}^0_{k_0}]\mathbf{v}$ will control the first order term $\chi_{<R'}\mathbf{B}^0_{k_0}\mathbf{Q}\mathbf{v}$.

- If $\mathbf{v}$ is at high enough frequency 2^{k_1}, $\|\mathbf{Q}S_{>k_1-4}\|_{X^0\to X^0}$ will be independent of R'. This will allow us to control $\chi_{\geq R'}\mathbf{B}^0_{k_0}\mathbf{v}$ in Y^0 by a small factor of $\|v\|_{X^0}$ by taking R' sufficiently large and using the smallness of b^j and $\tilde{b}^j$ outside of $B_{R'}(0)$.

The above scheme turns out to be sufficient for closing the estimate (3.3.12). We remark that this method is very robust and works under extremely mild decay assumptions on the coefficients – we essentially only require integrability along the bicharacteristic flow (x^t, ξ^t). Such integrability is guaranteed by the asymptotic flatness condition (3.3.3) and the fact that the metric is nontrapping.

Well-posedness for the nonlinear equation

Finally, in Section 3.9 we will make use of the estimate (3.3.7) for both the linear and paradifferential flows as well as its various corollaries to establish well-posedness for the nonlinear flow. Having established our key linear estimate, the scheme for establishing well-posedness is virtually identical to the one implemented in Section 7 of [106]. We therefore only outline the minor changes, and refer to [106] for additional details. The interested reader may also consult [71] for an expository presentation of the overarching well-posedness scheme.

3.4 The bicharacteristic flow

In this section, we define our quantitative measure of nontrapping and establish basic properties of the bicharacteristic flow corresponding to the symbol $a(x,\xi) := -g^{ij}(x)\xi_i\xi_j$. We begin by fixing $s_0 > \frac{d}{2} + 2$ and letting g be a time-independent metric satisfying

$$\|g - g_\infty\|_{l^1 H^{s_0}} \leq M, \tag{3.4.1}$$

for some constant non-degenerate symmetric matrix g_∞. We moreover assume the non-degeneracy condition

$$c^{-1}|\xi| \leq |g^{ij}\xi_j| \leq c|\xi|, \quad \forall \xi \in \mathbb{R}^d, \tag{3.4.2}$$

for some constant $c > 0$. By Sobolev embedding, we have for some $\delta > 0$,

$$\|g\|_{C^{2,\delta}} \lesssim_{g_\infty} 1 + M.$$

As a consequence, for each $(x, \xi) \in \mathbb{R}^{2d}$, the bicharacteristic flow $(x^t, \xi^t) := (x^t_{(x,\xi)}, \xi^t_{(x,\xi)})$ given by

$$(\dot{x}^t, \dot{\xi}^t) = (\nabla_\xi a(x^t, \xi^t), -\nabla_x a(x^t, \xi^t)), \quad (x^0, \xi^0) = (x, \xi), \tag{3.4.3}$$

is well-defined in a neighborhood of $t = 0$ (whose size a priori depends on (x, ξ)).

In addition to the above decay and non-degeneracy assumptions, we will also impose the condition that the metric g be nontrapping. The meaning of this is given in a qualitative form by the following definition.

Definition 3.4.1 (Nontrapping metric)**.** A non-degenerate metric g is said to be *nontrapping* if for every $(x, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d - \{0\})$ and every compact set $K \subset \mathbb{R}^d$, the bicharacteristic x^t intersects K on a compact time interval.

As in [106], we will need a more quantitative description of the above definition. The quantitative parameter $L = L(R)$ we introduce should measure, in some sense, how long a given bicharacteristic can intersect $B_R(0)$. However, since the bicharacteristic flow satisfies the homogeneity law

$$\xi \mapsto \lambda\xi, \quad t \mapsto \lambda t, \tag{3.4.4}$$

such a parameter will not be uniform in the size of ξ. To deal with this, it is natural to restrict to data $\xi \in \mathbb{S}^{d-1}$. From the non-degeneracy of the metric, this restricts the initial speed of a given bicharacteristic to approximately unit size.

At this point, we fix a non-degenerate, nontrapping metric g satisfying (3.4.1) and (3.4.2). By a compactness argument, the function

$$L : [0, \infty) \to [0, \infty)$$

given formally by

$$L(R) := \inf\{s \geq 0 : |x^t| > R, \ \forall \ |t| \geq s, \ \forall (x, \xi) \in B_R(0) \times \mathbb{S}^{d-1}\} \tag{3.4.5}$$

is well-defined. We will use $L := L(R)$ as a quantitative measure of nontrapping.

Remark 3.4.2. In the case where Δ_g is elliptic, it is automatic that the bicharacteristic flow is globally well-defined because the quantity

$$g^{ij}\xi_i\xi_j \tag{3.4.6}$$

is preserved by the flow, which, by ellipticity, implies that $|\xi^t|$ remains bounded uniformly in t by $|\xi|$. The same is not immediate when the symbol $g^{ij}\xi_i\xi_j$ is not elliptic and therefore it still needs to be proved that the bicharacteristic flow is globally well-defined. We remark that our definition of the nontrapping parameter L is slightly different than the one used in [106]. In their article, they define L in terms of the maximal time any bicharacteristic for the projected flow onto $\{|\xi^t| = 1\}$ can intersect $B_R(0)$. In light of the above discussion, this is a natural definition in the case of an elliptic symbol, but is not so natural for our purposes because the bicharacteristic flow should not in general preserve any normalization of $|\xi|$ (even though (3.4.6) is still preserved by the flow). We therefore only restrict the initial ξ to the unit sphere in our quantitative measure of nontrapping.

Our next proposition addresses the problem of global existence and asymptotic bounds for the bicharacteristic flow when the metric is nontrapping and satisfies the decay condition $g - g_\infty \in l^1H^{s_0}$.

Proposition 3.4.3. Let $s_0 > \frac{d}{2} + 2$ and let g be a non-degenerate, nontrapping metric satisfying (3.4.1). Then

(i) For each $(x,\xi) \in \mathbb{R}^d \times (\mathbb{R}^d - \{0\})$, the bicharacteristic flow for $a(x,\xi) := -g^{ij}\xi_i\xi_j$ is globally defined.

(ii) For every $\varepsilon_0 > 0$ sufficiently small, there exists $R_0 > 0$ such that for any initially outgoing bicharacteristic (i.e. $\dot{x}^t(0)\cdot x \geq 0$) with data $(x,\xi) \in (\mathbb{R}^d - B_{R_0}(0)) \times (\mathbb{R}^d - \{0\})$, x^t is defined for $t \geq 0$ and is close to the flat flow in the sense that for all $t \geq 0$, we have

$$|x^t - x + 2tg_\infty^{ij}\xi_j| \leq t\varepsilon_0|\xi|, \quad |\xi^t - \xi| \leq \varepsilon_0|\xi|. \tag{3.4.7}$$

Proof. The proof of this is very similar to Lemma 5.1 in [106]. We include the short argument for completeness. We begin by choosing R_0 large enough so that g is sufficiently close to the flat metric g_∞ in $l^1H^{s_0}$ outside of $B_{\frac{R_0}{2}}(0)$. That is,

$$\|\chi_{>\frac{R_0}{2}}(g - g_\infty)\|_{l^1H^{s_0}} < \varepsilon, \tag{3.4.8}$$

where $0 < \varepsilon \ll \varepsilon_0 \ll 1$ is some sufficiently small constant relative to ε_0. We let (x^t, ξ^t) be any initially outgoing bicharacteristic with data $(x, \xi) \in (\mathbb{R}^d - B_{R_0}(0)) \times (\mathbb{R}^d - \{0\})$ and make the bootstrap assumption that the bicharacteristic (x^t, ξ^t) satisfies (3.4.7) on a time interval $t \in [0, T]$. Our goal will be to show that when $\varepsilon > 0$ is small enough, the factor of ε_0 in the bootstrap hypothesis can be improved to $\frac{\varepsilon_0}{2}$. Thanks to the nontrapping assumption on g, this will clearly suffice for establishing both (i) and (ii).

To close the bootstrap, we note that on $[0, T]$, thanks to (3.4.7) and the fact that x^t is initially outgoing, the bicharacteristic x^t remains outside $B_{\frac{3}{4}R_0}(0)$. Using this and the bootstrap hypothesis, we aim to prove the following simple lemma.

Lemma 3.4.4. The following estimate holds for every $t \in [0, T]$:

$$\int_0^t |\nabla_x g(x^s)| ds \lesssim \varepsilon |\xi|^{-1}.$$

Proof. We estimate

$$\begin{aligned}
\int_0^t |\nabla_x g(x^s)| ds &\lesssim \sum_{k \geq 0} \sum_{Q \in Q_k} \int_0^t |\chi_Q(x^s)(S_k(\chi_{> \frac{R_0}{2}} \nabla_x g))(x^s)| ds \\
&\lesssim |\xi|^{-1} \sum_{k \geq 0} \sum_{Q \in Q_k} 2^k \|\chi_Q S_k(\chi_{> \frac{R_0}{2}} \nabla_x g)\|_{L^\infty} \\
&\lesssim |\xi|^{-1} \|\chi_{> \frac{R_0}{2}} \nabla_x g\|_{l^1 H^{s_0 - 1}} \\
&\leq |\xi|^{-1} \varepsilon,
\end{aligned}$$

where in the second line we used (3.4.7) and the non-degeneracy of g_∞, which ensures that the bicharacteristic x^s intersects a cube of size 2^k for time at most $\lesssim 2^k |\xi|^{-1}$. In the third line, we used Bernstein's inequality and in the fourth line, we used (3.4.8). This concludes the proof. □

To close the bootstrap, we note that by (3.4.7) we have $|\xi^t| \leq (1 + \varepsilon_0)|\xi|$. Therefore, by using Lemma 3.4.4 and integrating in time the equation

$$\frac{d}{dt}(\xi^t - \xi) = \nabla_x g^{ij}(x^t) \xi_i^t \xi_j^t$$

we obtain

$$|\xi^t - \xi| \lesssim \varepsilon |\xi|.$$

Using this bound, integrating in time the equation

$$\frac{d}{dt}(x^t - x + 2t g_\infty^{ij} \xi_j) = 2(g_\infty^{ij} - g^{ij})(x^t) \xi_j^t - 2 g_\infty^{ij} (\xi_j^t - \xi_j)$$

and using that $|(g-g_\infty)(x^t)| \lesssim \varepsilon$, we also obtain

$$|x^t - x + 2tg_\infty^{ij}\xi_j| \lesssim t\varepsilon|\xi|,$$

which improves the bootstrap (3.4.7) if ε is small enough relative to ε_0. This concludes the proof of the proposition. □

The next proposition shows that the size of the nontrapping function L as well as the bicharacteristic bounds are stable under small perturbations of the metric.

Proposition 3.4.5. Let g_0 be a non-degenerate nontrapping metric satisfying (3.4.1). For every sufficiently small $\varepsilon_0 > 0$, there is a radius $R_0(\varepsilon_0) > 0$ and a constant $C_0 > 0$ depending only on M and the profile of g_0 such that if g_1 is another non-degenerate metric satisfying

$$\|g_0 - g_1\|_{l^1 H^{s_0}} < e^{-C_0 L(R_0)} \tag{3.4.9}$$

then the bicharacteristics corresponding to g_1 satisfy (ii) in Proposition 3.4.3 with comparable parameters R_0 and ε_0 and, moreover, g_1 is also nontrapping with comparable parameters L_1 and data size M_1.

Proof. Choosing $e^{-C_0 L(R_0)}$ so small that

$$\|g_0 - g_1\|_{l^1 H^{s_0}} \ll \varepsilon_0$$

ensures that the data size M_1 is comparable to M and also that the proof of part (ii) of Proposition 3.4.3 works equally well for the metric g_1. It therefore suffices to show that L_1 is comparable to L for $R \leq R_0$. To do this, we fix $(x,\xi) \in B_{R_0}(0) \times \mathbb{S}^{d-1}$. The desired conclusion will follow if we can show that the bicharacteristic flows corresponding to g_0 and g_1 are close within $B_{R_0}(0)$ in the sense that

$$|x_0^t - x_1^t|_{L_t^\infty} + |\xi_0^t - \xi_1^t|_{L_t^\infty} \lesssim e^{-C(M)L(R_0)} \tag{3.4.10}$$

for times in which x_0^t intersects $B_{R_0}(0)$. The proof of this is similar to the proof of Proposition 5.2 in [106] but since our nontrapping parameter L is slightly different than theirs, we include the short proof.

We implement a simple bootstrap. First, we can restrict to a time interval J such that $|J| \leq L(R_0)$. We will then assume the bound (3.4.10) on some smaller interval $I \subset J$ and establish the same bound with an improved constant. We begin by writing the equation for $\delta x^t := x_0^t - x_1^t$ and $\delta\xi^t := \xi_0^t - \xi_1^t$. Dropping the i, j indices, we obtain

$$\begin{cases} \frac{d}{dt}\delta x^t = 2(g_1 - g_0)(x_1^t)\xi_1^t + 2(g_0(x_1^t) - g_0(x_0^t))\xi_0^t - 2g_0(x_1^t)\delta\xi^t, \\ \frac{d}{dt}\delta\xi^t = -\xi_1^t\nabla(g_1 - g_0)(x_1^t)\xi_1^t - \xi_1^t(\nabla g_0(x_1^t) - \nabla g_0(x_0^t))\xi_1^t + (\xi_0^t\nabla g_0(x_0^t)\xi_0^t - \xi_1^t\nabla g_0(x_0^t)\xi_1^t), \\ (\delta x^0, \delta\xi^0) = (0, 0). \end{cases}$$

By definition, we have $|I| \leq L(R_0)$. Moreover, by a compactness argument, there is a constant $K_0 > 1$ depending on the profile of g_0 (but not on (x, ξ)) such that

$$|\xi_0^t| \lesssim K_0$$

for every $t \in J$. By the bootstrap hypothesis, this implies the same bound for ξ_1^t on I. From this, (3.4.9), the bootstrap hypothesis and the fact that $g_0 \in C^2$, we obtain the bound

$$\frac{d}{dt}[(\delta x^t)^2 + (\delta\xi^t)^2] \lesssim e^{-2C_0L(R_0)} + C(K_0)(1 + M)[(\delta x^t)^2 + (\delta\xi^t)^2].$$

By Grönwall's inequality and the bound $|I| \leq L(R_0)$, we obtain

$$(\delta x^t)^2 + (\delta\xi^t)^2 \lesssim e^{-2C_0L(R_0)}e^{C(K_0)(1+M)L(R_0)}$$

on I. Choosing C_0 large enough improves the bootstrap hypothesis and concludes the proof. □

By combining Proposition 3.4.3 with Proposition 3.4.5, we have the following immediate corollary which gives a precise quantitative bound for ξ^t.

Corollary 3.4.6. Let g_0 be as in Proposition 3.4.3. Then the corresponding bicharacteristic ξ_0^t (which is defined for all t) satisfies the bound

$$|\xi_0^t| \lesssim C_0|\xi|$$

for all $(x, \xi) \in \mathbb{R}^{2d}$ and some constant $C_0 > 1$ depending only on M and the profile of g_0. Moreover, if g_1 is any other metric satisfying the conditions of Proposition 3.4.5, then the corresponding bicharacteristic ξ_1^t is globally defined and satisfies the same bound with a similar constant.

Proof. For $|\xi| = 1$, this follows immediately from Proposition 3.4.3, Proposition 3.4.5, and the nontrapping assumption. The general case follows from this case and the homogeneity law (3.4.4). □

We next note the following bounds for the x and ξ derivatives of x^t and ξ^t.

Proposition 3.4.7 (Higher regularity bounds)**.** Let $(x,\xi) \in B_R(0) \times \mathbb{S}^{d-1}$. Let k be a positive integer. Assume that the metric satisfies $g \in C^{k+1}$ and write $M_k := \|g\|_{C^{k+1}}$. Then if $|x^t| \leq R$, there holds

$$|\partial_\xi^\alpha \partial_x^\beta x^t_{(x,\xi)}| + |\partial_\xi^\alpha \partial_x^\beta \xi^t_{(x,\xi)}| \leq e^{C(M_k)L(R)}, \quad |\alpha+\beta| \leq k.$$

Proof. The proof follows by differentiating (3.4.3) in the x and ξ variables, which leads to a differential inequality for

$$\frac{d}{dt}\left(|\partial_\xi^\alpha \partial_x^\beta x^t|^2 + |\partial_\xi^\alpha \partial_x^\beta \xi^t|^2\right).$$

One then concludes by inductively applying Grönwall's inequality. We omit the details which are straightforward. □

The final result of this section shows that functions in l^1H^s with $s > \frac{d}{2}+1$ are uniformly integrable along the bicharacteristic flow. This is what will allow us to recover the Mizohata condition.

Proposition 3.4.8. Let g be as in Proposition 3.4.3 and let $s > \frac{d}{2}+1$. Let $v \in l^1H^s$. Then v is integrable along the bicharacteristic flow and satisfies the bound

$$\sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{S}^{d-1}} \|v(x^t_{(x,\xi)})\|_{L^1_t(\mathbb{R})} \lesssim (1+L(R_0))\|v\|_{l^1H^s},$$

where R_0 is as in Proposition 3.4.3.

Proof. We abbreviate $x^t_{(x,\xi)}$ by x^t. Without loss of generality, we may assume that x^t intersects $B_{R_0}(0)$ only if $|t| < L(R_0)$. We then have

$$\int_{-L(R_0)}^{L(R_0)} |v(x^t)|dt \leq 2L(R_0)\|v\|_{L^\infty} \lesssim L(R_0)\|v\|_{l^1H^s}.$$

By homogeneity of the flow, it therefore suffices to show that

$$\int_{L(R_0)}^{\infty} |v(x^t)|dt \lesssim \|v\|_{l^1H^s}.$$

Without loss of generality, we may assume that $x^t(L(R_0))$ is outgoing. Using Proposition 3.4.3, we see that if $t \geq L(R_0)$, then for every cube $Q \subset \mathbb{R}^d$, x^t intersects the cube on a time interval I of size at most $|I| \lesssim |Q|^{\frac{1}{d}}$. Therefore, we have

$$\int_{L(R_0)}^{\infty} |v(x^t)|dt \lesssim \sum_{k\geq 0}\sum_{Q\in Q_k} 2^k\|\chi_Q S_k v\|_{L^\infty} \lesssim \|v\|_{l^1H^s},$$

where in the last step we used Bernstein's inequality and dyadic summation. Here, the strict inequality $s > \frac{d}{2} + 1$ was what allowed us to retain summability in k. This completes the proof. □

3.5 The linear ultrahyperbolic flow

Let $s_0 > \frac{d}{2} + 2$ and let $0 \le \sigma \le s_0$. Here we consider the $l^1 H^\sigma$ well-posedness of the linear ultrahyperbolic flow,

$$\begin{cases} i\partial_t v + \partial_j g^{jk}\partial_k v + b^j \partial_j v + \tilde{b}^j \partial_j \overline{v} = f, \\ v(0,x) = v_0(x), \end{cases} \tag{3.5.1}$$

as well as the corresponding linear paradifferential flow,

$$\begin{cases} i\partial_t v + \partial_j T_{g^{jk}}\partial_k v + T_{b^j} \partial_j v + T_{\tilde{b}^j} \partial_j \overline{v} = f, \\ v(0,x) = v_0(x). \end{cases} \tag{3.5.2}$$

We make the following basic assumptions on the metric g^{jk} and the coefficients b^j in the above equations:

(i) (Non-degeneracy). The metric g^{jk} is real, symmetric and non-degenerate. That is, there is $c > 0$ such that for all $\xi \in \mathbb{R}^d$ we have,

$$c^{-1}|\xi| \le |g^{jk}\xi_k| \le c|\xi|.$$

(ii) (Asymptotic flatness and size). There is a constant, symmetric, non-degenerate matrix g_∞ and a constant $M > 0$ such that

$$\|g - g_\infty\|_{l^1 X^{s_0}} + \|\partial_t g\|_{l^1 X^{s_0-2}} + \|(b, \tilde{b})\|_{l^1 X^{s_0-1}} + \|\partial_t (b, \tilde{b})\|_{l^1 X^{s_0-3}} \le M. \tag{3.5.3}$$

(iii) (Asymptotic smallness). For every $\varepsilon_0 > 0$, there is $R_0 > 0$ such that

$$\|(g^{jk} - g^{jk}_\infty)\chi_{>R_0}\|_{l^1 X^{s_0}} + \|(b^j, \tilde{b}^j)\chi_{>R_0}\|_{l^1 X^{s_0-1}} \le \varepsilon_0, \tag{3.5.4}$$

where $0 \le \chi_{>R_0} \le 1$ is a smooth cutoff which vanishes on $B_{R_0}(0)$ and is equal to 1 outside of $B_{2R_0}(0)$.

(iv) (Nontrapping). The metric is nontrapping with parameter L as defined in (3.4.5).

Note that condition (iii) follows from the asymptotic flatness condition (ii). However, we prefer to make statement (iii) explicit, as it will play a prominent role in the analysis.

In the sequel, we will write $C(L)$ to denote a constant which depends on the parameter L within some fixed compact set whose size depends on the profile of the metric g. The main result we aim to prove is the following.

Theorem 3.5.1. Let $s_0 > \frac{d}{2} + 2$ and $0 \leq \sigma \leq s_0$. Moreover, assume that $g^{jk}, b^j, \tilde{b}^j$ satisfy the above assumptions with parameters M and L. Then for every $f \in l^1 Y^\sigma$, the equation (3.5.1) is well-posed in $l^1 H^\sigma$. Furthermore, there is $T_0 > 0$ depending on the size of L within a compact set and on the data size M such that for every $0 \leq T \leq T_0$, we have

$$\|v\|_{l^1 X^\sigma} \leq C(M, L)(\|v_0\|_{l^1 H^\sigma} + \|f\|_{l^1 Y^\sigma}). \tag{3.5.5}$$

The same result holds for the paradifferential flow (3.5.2) for every $\sigma \geq 0$.

As the above result holds for the paradifferential flow for all $\sigma \geq 0$, it is a straightforward consequence to deduce the following frequency envelope variant using similar reasoning to Section 5 of [105] (see also [71]).

Corollary 3.5.2. Let $\sigma \geq 0$ and assume the other properties in the statement of Theorem 3.5.1. Let a_k be an admissible $l^1 H^\sigma$ frequency envelope for v_0 and let b_k be an admissible $l^1 Y^\sigma$ frequency envelope for f. Then the solution v to the paradifferential equation (3.5.2) satisfies the bound

$$\|S_k v\|_{l^1 X^\sigma} \leq C(M, L)\left(a_k \|v_0\|_{l^1 H^\sigma} + b_k \|f\|_{l^1 Y^\sigma}\right)$$

on a time interval $[0, T]$ whose length depends on the size of L within a compact set and on the data size M.

The main component of the proof of well-posedness for the equations (3.5.1) and (3.5.2) is the energy estimate (3.5.5). This is because the adjoint equation, which has essentially the same form, will also satisfy a similar energy estimate. Well-posedness then follows by a standard duality argument. Therefore, we focus our attention mainly on the bound (3.5.5).

Some simplifying reductions

We begin our analysis by making some straightforward but useful reductions which will allow us to simplify some of the steps in the proof of Theorem 3.5.1. Our first reduction shows that by restricting the time interval to be small enough, we may assume that $\widehat{v}$ is supported at high frequency. More precisely, we have the following lemma.

Lemma 3.5.3 (High frequency reduction)**.** Let $\varepsilon > 0$. Under the assumptions of Theorem 3.5.1, for every $k_1 > 0$ there is $T_0 > 0$ depending on k_1, ε, M and σ such that for $0 < T \leq T_0$, $v_{>k_1} := S_{>k_1} v$ satisfies the equation

$$\begin{cases} i\partial_t v_{>k_1} + \partial_j g^{ij} \partial_i v_{>k_1} + b^j \partial_j v_{>k_1} + \tilde{b}^j \partial_j \overline{v_{>k_1}} = h, \\ v_{>k_1}(0) := S_{>k_1} v_0, \end{cases}$$

where h and $S_{\leq k_1} v$ satisfy the estimate

$$\|S_{\leq k_1} v\|_{l^1 X^\sigma} + \|h\|_{l^1 Y^\sigma} \leq C(M, k_1, \sigma)(\|v_0\|_{l^1 H^\sigma} + \|f\|_{l^1 Y^\sigma}) + \varepsilon \|v\|_{l^1 X^\sigma}$$

for $0 \leq \sigma \leq s_0$. The analogous result holds for the paradifferential equation (3.5.2) for $\sigma \geq 0$.

Proof. We show the proof for the full linear equation. The proof for the paradifferential flow is similar. Using the notation of the lemma, we easily compute that

$$\begin{aligned} h = {} & S_{>k_1} f - (\partial_j g^{ij} \partial_i S_{\leq k_1} v + b^j \partial_j S_{\leq k_1} v + \tilde{b}^j \partial_j S_{\leq k_1} \overline{v}) \\ & + S_{\leq k_1}(\partial_j g^{ij} \partial_i v + b^j \partial_j v + \tilde{b}^j \partial_j \overline{v}). \end{aligned}$$

We clearly have

$$\|S_{>k_1} f\|_{l^1 Y^\sigma} \lesssim \|f\|_{l^1 Y^\sigma}.$$

For the remaining source terms, if $0 \leq \sigma \leq s_0 - 1$, we can estimate in $l^1 L^1_T H^\sigma_x \subset l^1 Y^\sigma$ in a naïve fashion using the frequency projection $S_{\leq k_1}$ to obtain

$$\|h\|_{l^1 Y^\sigma} \lesssim \|f\|_{l^1 Y^\sigma} + \varepsilon \|v\|_{l^1 L^\infty_T H^\sigma_x},$$

by applying Hölder's inequality in T and taking T small enough (depending on k_1). On the other hand, for $s_0 - 1 < \sigma \leq s_0$, we can instead use the bilinear estimates in Proposition 3.2.13 to obtain

$$\|h\|_{l^1 Y^\sigma} \lesssim \|f\|_{l^1 Y^\sigma} + C(M, k_1, \sigma) \|v\|_{l^1 L^\infty_T L^2_x}.$$

We can estimate the latter term on the right using the crude energy inequality

$$\|v\|_{l^1 L^\infty_T L^2_x} \lesssim_M \|v_0\|_{l^1 L^2_x} + \|v\|_{l^1 L^1_T H^1_x} + \|f\|_{l^1 Y^\sigma},$$

which follows from a direct energy estimate for (3.5.1) where the first order terms are estimated directly in $L^1_T L^2_x$. Since $\sigma > 1$, we may conclude by applying Hölder in T and taking $T \ll \varepsilon$ to control the second term on the right by $\varepsilon\|v\|_{l^1X^\sigma}$. It remains to estimate $\|S_{\leq k_1}v\|_{l^1X^\sigma}$. Using that $S_{\leq k_1}v$ is frequency localized, we easily have

$$\|S_{\leq k_1}v\|_{l^1X^\sigma} \lesssim 2^{k_1(\sigma+\frac{1}{2})}\|S_{\leq k_1}v\|_{l^1L^\infty_TL^2_x}.$$

We then note the naïve energy type estimate

$$\|S_{\leq k_1}v\|_{l^1L^\infty_TL^2_x} \lesssim_{M,k_1} \|v_0\|_{l^1L^2_x} + \|v\|_{l^1L^1_TL^2_x} + \|f\|_{l^1Y^0},$$

which follows from inspecting the equation for $S_{\leq k_1}v$ and using the fact that the first and second-order terms in the resulting equation are localized to frequencies $\lesssim k_1$. Then using Hölder in T and taking T small enough (depending on M, k_1 and ε) we can again control the second term on the right by $\varepsilon\|v\|_{l^1X^\sigma}$. This concludes the proof of the lemma for (3.5.1). A very similar argument works for the paradifferential analogue. We omit the details. □

Reduction to the paradifferential flow

As a second reduction, we reduce proving Theorem 3.5.1 to proving the corresponding estimate for the paradifferential equation. We begin by writing (3.5.1) in the paradifferential form

$$\begin{cases} i\partial_t v + \partial_j T_{g^{ij}}\partial_i v + T_{b^j}\partial_j v + T_{\tilde{b}^j}\partial_j\overline{v} = f + \mathcal{R}, \\ v(0) = v_0, \end{cases}$$

where $\mathcal{R}$ is a remainder term given by

$$\mathcal{R} = (T_{b^j}\partial_j v - b^j\partial_j v) + (T_{\tilde{b}^j}\partial_j\overline{v} - \tilde{b}^j\partial_j\overline{v}) + \partial_j(T_{g^{ij}}\partial_i v - g^{ij}\partial_i v). \tag{3.5.6}$$

Thanks to Lemma 3.5.3, we may harmlessly assume that v is localized to frequencies $\gtrsim 2^{k_1}$. Our next lemma shows that the error term $\mathcal{R}$ can be treated perturbatively if k_1 is large enough.

Lemma 3.5.4 (Paradifferential source terms)**.** Assume that the estimate in Theorem 3.5.1 holds for the paradifferential flow for each $\sigma \geq 0$. Let $\varepsilon > 0$ and assume that $\widehat{v}$ is supported at frequencies $|\xi| \gtrsim 2^{k_1}$. Then for k_1 large enough and T small enough depending on ε and k_1, the remainder term $\mathcal{R}$ satisfies the estimate

$$\|\mathcal{R}\|_{l^1Y^\sigma} \leq C(M,k_1,\sigma)(\|v_0\|_{l^1H^\sigma} + \|f\|_{l^1Y^\sigma}) + \varepsilon\|v\|_{l^1X^\sigma}.$$

Proof. We show the details for the first term as the estimates for the other two are similar. We split the analysis into two cases. First, assume that $\sigma \leq s_0 - \delta$ where $\delta > 0$ is such that $s_0 - 2\delta > \frac{d}{2} + 2$. Then since v is localized to frequencies $\gtrsim 2^{k_1}$, we may replace the coefficient b^j in $(T_{b^j}\partial_j v - b^j\partial_j v)$ with $S_{\geq k_1 - 5}b^j$. Therefore, by (3.2.7) in Proposition 3.2.13 and Bernstein's inequality, we have

$$\begin{aligned}\|(T_{b^j}\partial_j v - b^j\partial_j v)\|_{l^1Y^\sigma} &\lesssim \|S_{\geq k_1-5}b^j\|_{l^1X^{s_0-1-\delta}}\|v\|_{l^1X^\sigma}\\ &\lesssim_M 2^{-\delta k_1}\|v\|_{l^1X^\sigma}.\end{aligned}$$

Taking k_1 large enough, we therefore have

$$\|(T_{b^j}\partial_j v - b^j\partial_j v)\|_{l^1Y^\sigma} \leq \varepsilon\|v\|_{l^1X^\sigma}.$$

The other terms in (3.5.6) can be estimated similarly to obtain

$$\|\mathcal{R}\|_{l^1Y^\sigma} \leq \varepsilon\|v\|_{l^1X^\sigma}.$$

In the case $s_0 \geq \sigma \geq s_0 - \delta > \frac{d}{2} + 2$, we use instead the first estimate in Proposition 3.2.13 to obtain

$$\begin{aligned}\|(T_{b^j}\partial_j v - b^j\partial_j v)\|_{l^1Y^\sigma} &\lesssim \|b^j\|_{l^1X^{\sigma-1}}\|v\|_{l^1X^{s_0-2\delta}}\\ &\lesssim_M 2^{-k_1\delta}\|v\|_{l^1X^\sigma},\end{aligned}$$

where we used the fact that v is localized to frequencies greater than 2^{k_1}. Estimating the other terms in (3.5.6) in a similar fashion, and again taking k_1 large enough, we obtain

$$\|\mathcal{R}\|_{l^1Y^\sigma} \leq \varepsilon\|v\|_{l^1X^\sigma}.$$

This concludes the proof. □

Reduction to the X^σ estimate

To summarize what we have so far, it now suffices to establish (3.5.5) for the paradifferential flow under the assumption that v is localized to high frequency. As one final simplification, we reduce the proof of this estimate for the paradifferential flow to the corresponding X^σ estimate without the l^1 summability. For this, we will need the small data result from [105].

Theorem 3.5.5 (Small data well-posedness)**.** Let b^j, $\tilde{b}^j$, g^{ij} and M, σ be as above. Let $0 < T \leq 1$. For every $\sigma \geq 0$, there is $\delta > 0$ such that if $M \leq \delta$ then (3.5.2) is well-posed in both H^σ and l^1H^σ with the uniform bounds

$$\begin{aligned}\|v\|_{X^\sigma} &\lesssim \|v_0\|_{H^\sigma} + \|f\|_{Y^\sigma},\\ \|v\|_{l^1X^\sigma} &\lesssim \|v_0\|_{l^1H^\sigma} + \|f\|_{l^1Y^\sigma}.\end{aligned}$$

Remark 3.5.6. Strictly speaking, the small data result above is only explicitly stated in the case when g_∞ is the identity, but as remarked on page 1154 of [105], the result is also true when g_∞ is of the form we consider here, and the estimates above follow almost verbatim from the proof of Proposition 4.1 in their paper.

We may now phrase our final reduction as follows.

Lemma 3.5.7. Let b^j, $\tilde{b}^j$, g^{ij} and M, σ be as in Theorem 3.5.1. Assume that the paradifferential flow (3.5.2) admits the estimate

$$\|v\|_{X^\sigma} \leq C(M,L)(\|v_0\|_{H^\sigma} + \|f\|_{Y^\sigma}) \tag{3.5.7}$$

for each $\sigma \geq 0$. Then the corresponding estimate in Theorem 3.5.1 in l^1X^σ also holds for (3.5.2) for each $\sigma \geq 0$.

Proof. We can again harmlessly assume that v is localized to frequencies $\gtrsim 2^{k_1}$. Now let $\varepsilon > 0$ and let $R(\varepsilon)$ be such that (3.5.4) holds. Using Proposition 3.2.13 and Theorem 3.5.5, our first aim will be to reduce to estimating v in a compact set. More precisely, we aim to prove the estimate

$$\|\chi_{>2R}v\|_{l^1X^\sigma} \lesssim \|v_0\|_{l^1H^\sigma} + \|f\|_{l^1Y^\sigma} + \|\chi_{<4R}v\|_{l^1X^\sigma}. \tag{3.5.8}$$

This is a straightforward computation which follows by inspecting the equation for $v_{ext} := \chi_{>2R}v$. Indeed, if we define $g_{ext} := \chi_{>R}g + \chi_{\leq R}g_\infty$, $b_{ext} := \chi_{>R}b$ and $\tilde{b}_{ext} := \chi_{>R}\tilde{b}$, we obtain

$$\begin{cases} i\partial_t v_{ext} + \partial_i T_{g^{ij}_{ext}}\partial_j v_{ext} + T_{b^j_{ext}}\partial_j v_{ext} + T_{\tilde{b}^j_{ext}}\partial_j \overline{v}_{ext} = f_{ext}, \\ v_{ext}(0) = \chi_{>2R}v(0), \end{cases}$$

where

$$\begin{aligned} f_{ext} := \chi_{>2R}f &+ [\partial_i T_{g^{ij}}\partial_j + T_{b^j}\partial_j + T_{\tilde{b}^j}\partial_j, \chi_{>2R}]v + (\partial_i T_{g^{ij}_{ext}}\partial_j - \partial_i T_{g^{ij}}\partial_j)v_{ext} \\ &+ (T_{b^j_{ext}}\partial_j - T_{b^j}\partial_j)v_{ext} + (T_{\tilde{b}^j_{ext}}\partial_j - T_{\tilde{b}^j}\partial_j)\overline{v}_{ext}. \end{aligned}$$

Making use of Proposition 3.2.13 and paradifferential calculus, we can easily estimate

$$\begin{aligned} \|[\partial_i T_{g^{ij}}\partial_j + T_{b^j}\partial_j + T_{\tilde{b}^j}\partial_j, \chi_{>2R}]v\|_{l^1Y^\sigma} &\leq C(M,R)(\|\chi_{<4R}v\|_{l^1X^\sigma} + \|v\|_{l^1L^1_TH^\sigma_x}) \\ &\leq C(M,R)\|\chi_{<4R}v\|_{l^1X^\sigma} + \delta\|v\|_{l^1X^\sigma} \end{aligned}$$

for some small $\delta > 0$. We note that in the last inequality, we used Hölder's inequality in T and took T sufficiently small depending on R and M. Using the disjointness of the supports of $g_{ext} - g$ and v_{ext}, we obtain from the embedding $l^1 L^1_T H^\sigma_x \subset l^1 Y^\sigma$ and paradifferential calculus,

$$\|(\partial_i T_{g^{ij}_{ext}} \partial_j - \partial_i T_{g^{ij}} \partial_j) v_{ext}\|_{l^1 Y^\sigma} \lesssim_M \|v\|_{l^1 L^1_T H^\sigma_x} \lesssim \delta \|v\|_{l^1 X^\sigma}.$$

We can similarly estimate the last two terms in the definition of f_{ext}. In light of this and the small data result Theorem 3.5.5 which applies to the equation for v_{ext}, we obtain (3.5.8). We have therefore reduced the estimate for v in $l^1 X^\sigma$ to obtaining the bound

$$\|\chi_{<4R} v\|_{l^1 X^\sigma} \leq C(M, L)(\|v_0\|_{l^1 H^\sigma} + \|f\|_{l^1 Y^\sigma}).$$

However, this simply follows from (3.5.7) and the fact that the $l^1 X^\sigma$ and X^σ norms are equivalent within the set $B_{4R}(0)$ (with equivalence constant depending on R). □

3.6 The L^2 estimate for the linear flow

We begin our analysis by showing that we can close an estimate for the $L^\infty_T H^\sigma_x$ norm of a solution to the paradifferential linear equation (3.5.2) up to a small error term in X^σ as long as the time interval is small enough. Thanks to Lemma 3.5.3, we may from here on harmlessly assume that

$$\text{supp}(\widehat{v}) \subset \{|\xi| > 2^{k_1}\}$$

for some large parameter k_1 to be chosen. We will make this assumption for the rest of the section. The main estimate we wish to prove is the following.

Proposition 3.6.1 (L^2 estimate for the paradifferential linear flow)**.** Let s_0, g^{ij}, b^j and $\tilde{b}^j$ be as in Theorem 3.5.1 with parameters M and L. Let $\varepsilon > 0$. There is $T_0 = T_0(\varepsilon) > 0$ such that for $0 \leq T \leq T_0$, we have the a priori bound for v satisfying (3.5.2),

$$\|v\|_{L^\infty_T H^\sigma} \leq C(M, L)(\|v_0\|_{H^\sigma} + \|f\|_{Y^\sigma}) + \varepsilon \|v\|_{X^\sigma},$$

for every $\sigma \geq 0$.

As noted earlier, by $C(M, L)$ we mean a constant which depends on M and the trapping parameter L within some fixed compact set (which is allowed to depend on ε). The main obstruction to establishing Proposition 3.6.1 is essentially the presence of the real part of the first order term $T_{\text{Re}(b^j)} \partial_j v$. This is characterized somewhat by the following basic estimate for a truncated version of the linear flow in which the coefficient b^j is purely imaginary.

Lemma 3.6.2 (Basic energy estimate)**.** Let g^{ij} be smooth, real and symmetric and let b^j and $\tilde{b}^j$ be smooth functions. Assume that we have the size condition (3.5.3). Moreover, let $A(x, D) \in OPS^1$ be a time-independent pseudodifferential operator with symbol satisfying $Re(A) \geq 0$ and assume that v solves the equation

$$i\partial_t v + \partial_i T_{g^{ij}} \partial_j v + i \operatorname{Im}(b^j)\partial_j v + \tilde{b}^j \partial_j \overline{v} + iA(x, D)v = f. \tag{3.6.1}$$

Then for every $0 < \delta \ll 1$ there is $T_0 > 0$ depending on M, δ and A such that for $0 < T \leq T_0$, we have the L^2 estimate,

$$\|v\|^2_{L^\infty_T L^2_x} \lesssim \|v_0\|^2_{L^2_x} + \|v\|_{X^0} \|f\|_{Y^0} + \delta \|v\|^2_{X^0}.$$

In the above lemma, we allow for the extra first order term $iA(x, D)v$. This will afford us some flexibility when dealing with commutations of the principal operator $\partial_j T_{g^{ij}} \partial_i$ with various zeroth order Fourier multipliers and pseudodifferential operators later on when we deal with the full linear paradifferential flow.

Proof. We start with the basic energy identity:

$$\begin{aligned}\|v(t)\|^2_{L^2_x} + 2 \operatorname{Re}\langle A(x, D)v, v\rangle = \|v_0\|^2_{L^2_x} &+ 2 \operatorname{Re}\langle i\partial_i T_{g^{ij}} \partial_j v, v\rangle - 2 \operatorname{Re}\langle \operatorname{Im}(b^j)\partial_j v, v\rangle \\ &+ 2 \operatorname{Re}\langle i\tilde{b}^j \partial_j \overline{v}, v\rangle - 2 \operatorname{Re}\langle if, v\rangle,\end{aligned}$$

which holds for each $0 \leq t \leq T$. Here $\langle \cdot, \cdot\rangle$ denotes the inner product on $L^2_t L^2_x$. Unlike the operator $\partial_i g^{ij} \partial_j$, the paradifferential operator $\partial_i T_{g^{ij}} \partial_j$ is not quite self-adjoint. However, we do have the relation

$$\operatorname{Re}\langle i\partial_i T_{g^{ij}} \partial_j v, v\rangle = \operatorname{Re}\langle i\partial_i (T_{g^{ij}} - g^{ij})\partial_j v, v\rangle.$$

By standard paradifferential calculus and the fact that $\|g^{ij}\|_{L^\infty_T C^{2,\alpha}} \leq C(M)$ for some $\alpha > 0$, we have

$$\|\partial_i (T_{g^{ij}} - g^{ij})\partial_j v\|_{L^2_x} \lesssim_M \|v\|_{L^2_x}.$$

Hence, by Hölder in T and taking T sufficiently small, we have

$$2 \operatorname{Re}\langle i\partial_i T_{g^{ij}} \partial_j v, v\rangle \lesssim_M T\|v\|^2_{L^\infty_T L^2_x} \leq \delta \|v\|^2_{X^0}.$$

Now, we turn to the other terms in the energy estimate. Integrating by parts and making use of Sobolev embeddings, we obtain the bound

$$-2 \operatorname{Re}\langle \operatorname{Im}(b^j)\partial_j v, v\rangle + 2 \operatorname{Re}\langle i\tilde{b}^j \partial_j \overline{v}, v\rangle \lesssim MT\|v\|^2_{L^\infty_T L^2_x} \leq \delta \|v\|^2_{X^0},$$

if T is small enough. Moreover, by the $Y^* = X$ duality, we have

$$-2\,\mathrm{Re}\langle if, v\rangle \lesssim \|v\|_{X^0}\|f\|_{Y^0}.$$

Therefore, if T is small enough, we arrive at the bound

$$\|v\|^2_{L^\infty_T L^2_x} + \mathrm{Re}\langle A(x,D)v, v\rangle \lesssim \|v_0\|^2_{L^2_x} + \|v\|_{X^0}\|f\|_{Y^0} + \delta\|v\|^2_{X^0}.$$

Finally, by the sharp Gårding inequality Proposition 3.2.2 and Hölder in time, we have

$$\mathrm{Re}\langle A(x,D)v, v\rangle \gtrsim_A -T\|v\|^2_{L^\infty_T L^2_x}.$$

Taking T sufficiently small concludes the proof. □

The remainder of this section will be essentially devoted to transforming the equation (3.5.2) into an equation of the ideal form (3.6.1). Our primary means of doing this will be to construct a time-independent pseudodifferential renormalization operator $\mathcal{O} = Op(O) \in OPS^0$ which upon commuting $\mathcal{O}$ with the equation achieves this transformation within a compact ball $B_R(0)$. The hope is then to use the asymptotic smallness (3.5.4) to control the residual error terms outside $B_R(0)$. Quite a bit of care is needed here to avoid a circular argument because the higher order symbol bounds for O will grow in the parameter R, and so, at first glance, the operator bounds for $\mathcal{O}$ could counteract any smallness coming from the remaining error terms. Therefore, we will need to carefully track the dependence of the operator bounds for $\mathcal{O}$ on the parameters R and L. In our construction, it will turn out that the L^∞ norm of the symbol O will have a R independent bound (as $R \to \infty$). Therefore, for large enough k_1, the operator $\mathcal{O}S_{\geq k_1}$ will have R independent $L^2 \to L^2$, $X^0 \to X^0$ and $Y^0 \to Y^0$ bounds thanks to Proposition 3.2.6 and Proposition 3.2.7, respectively. This is how we will break the potential circularity.

First order truncations

Since we want the symbol for $\mathcal{O}$ to be time-independent and smooth, our first aim will be to show that the first order paradifferential coefficients in (3.5.2) can be replaced by smooth time-independent coefficients localized at a suitable frequency scale. To achieve this, let us fix another large parameter k_0 with $0 \ll k_0 \ll k_1$ to be chosen. We can rearrange the paradifferential equation as

$$\begin{cases} i\partial_t v + \partial_j T_{g^{ij}}\partial_i v + b^j_{<k_0}(0)\partial_j v + \tilde{b}^j_{<k_0}(0)\partial_j \overline{v} = f + \mathcal{R}^1, \\ v(0) = v_0, \end{cases} \tag{3.6.2}$$

where

$$\mathcal{R}^1 = (b^j_{<k_0}(0)\partial_j v - T_{b^j}\partial_j v) + (\tilde{b}^j_{<k_0}(0)\partial_j \overline{v} - T_{\tilde{b}^j}\partial_j \overline{v}). \tag{3.6.3}$$

We have the following short lemma which shows that for large enough k_0, k_1 and small enough T, the error term $\mathcal{R}_1$ can be treated perturbatively.

Lemma 3.6.3. For k_0 and k_1 sufficiently large and T sufficiently small, we have

$$\|\mathcal{R}^1\|_{Y^\sigma} \leq \varepsilon\|v\|_{X^\sigma}.$$

Proof. We estimate the first term in (3.6.3) as the other term is essentially identical. By Bernstein's inequality, averaging in T and the assumption (3.5.3), we have

$$\|b^j_{<k_0} - b^j_{<k_0}(0)\|_{l^1 X^{s_0-1}} \lesssim_M 2^{2k_0}T.$$

Therefore, by the assumption $k_1 \gg k_0$, Proposition 3.2.13 and taking T small enough (depending on k_0 and M), we have

$$\|(b^j_{<k_0} - b^j_{<k_0}(0))\partial_j v\|_{Y^\sigma} = \|T_{(b^j_{<k_0} - b^j_{<k_0}(0))}\partial_j v\|_{Y^\sigma} \leq \varepsilon\|v\|_{X^\sigma}.$$

Next, using $k_1 \gg k_0$, we can write

$$T_{b^j}\partial_j v - b^j_{<k_0}\partial_j v = T_{S_{\geq k_0} b^j}\partial_j v.$$

So, from Proposition 3.2.13, there is $\delta > 0$ depending only on s_0 such that

$$\begin{aligned}\|T_{b^j}\partial_j v - b^j_{<k_0}\partial_j v\|_{Y^\sigma} &\lesssim \|S_{\geq k_0} b^j\|_{l^1 X^{s_0-1-\delta}}\|v\|_{X^\sigma} \\ &\lesssim_M 2^{-k_0\delta}\|v\|_{X^\sigma}.\end{aligned}$$

The above term can be controlled by $\varepsilon\|v\|_{X^\sigma}$ by taking k_0 large enough. This completes the proof. □

Commuting with derivatives

The next step is to commute (3.5.2) with $\langle\nabla\rangle^\sigma$. This will essentially reduce matters to proving an L^2 estimate for the paradifferential flow and get us one step closer to a situation in which we can apply Lemma 3.6.2. This would typically be a completely straightforward matter since the equation is already in paradifferential form; however, the commutation of the principal operator $\mathcal{P}$ with $\langle\nabla\rangle^\sigma$ will generate a further first order term which cannot be treated perturbatively in the large data regime.

To proceed, we define $u := \langle\nabla\rangle^\sigma v$. We also compactify the notation for the principal and new first order terms by defining

$$\begin{aligned}
\mathcal{P} &:= \partial_j T_{g^{ij}}\partial_i,\\
\mathcal{B} &:= b^j_{<k_0}(0)\partial_j - [\mathcal{P}, \langle\nabla\rangle^\sigma]\langle\nabla\rangle^{-\sigma},\\
\tilde{\mathcal{B}} &:= \tilde{b}^j_{<k_0}(0)\partial_j.
\end{aligned}$$

By commuting (3.6.2) with $\langle\nabla\rangle^\sigma$, we obtain

$$i\partial_t u + \mathcal{P}u + \mathcal{B}u + \tilde{\mathcal{B}}\overline{u} = \langle\nabla\rangle^\sigma f + \mathcal{R}^1_\sigma + \mathcal{R}^2_\sigma,$$

where $\mathcal{R}^1_\sigma := \langle\nabla\rangle^\sigma\mathcal{R}^1$ and

$$\mathcal{R}^2_\sigma := -[\langle\nabla\rangle^\sigma, b^j_{<k_0}(0)]\partial_j v - [\langle\nabla\rangle^\sigma, \tilde{b}^j_{<k_0}(0)]\partial_j\overline{v}.$$

Thanks to Lemma 3.6.3, we have a suitable estimate for $\mathcal{R}^1_\sigma$ in Y^0 which allows us to treat this term perturbatively. The following lemma shows that $\mathcal{R}^2_\sigma$ can be estimated naïvely in $L^1_T L^2_x \subset Y^0$.

Lemma 3.6.4. For T small enough, the source term $\mathcal{R}^2_\sigma$ satisfies the bound

$$\|\mathcal{R}^2_\sigma\|_{L^1_T L^2_x} \le \varepsilon\|v\|_{X^\sigma}. \tag{3.6.4}$$

Proof. Since $k_0 \ll k_1$ and $\widehat{v}$ is supported at frequencies $\gtrsim 2^{k_1}$, we can write

$$[\langle\nabla\rangle^\sigma, b^j_{<k_0}(0)]\partial_j v = [\langle\nabla\rangle^\sigma, T_{b^j_{<k_0}(0)}]\partial_j v.$$

Hence, by Proposition 3.2.10, Sobolev embedding and the regularity assumptions on b^j, we have

$$\|[\langle\nabla\rangle^\sigma, b^j_{<k_0}(0)]\partial_j v\|_{L^1_T L^2_x} \lesssim_{M,k_0} \|v\|_{L^1_T H^\sigma_x} \lesssim_M T\|v\|_{L^\infty_T H^\sigma_x}.$$

The other term in $\mathcal{R}^2_\sigma$ can be estimated similarly. Hence, by taking T small enough, we obtain (3.6.4), as desired. □

Next, we further frequency and time truncate the commutator in the term $\mathcal{B}$. As we will see later, such truncations will ensure that our renormalization operator $\mathcal{O}$ belongs to OPS^0. Note that while we cannot directly truncate the principal operator $\mathcal{P}$ because it is second order, it is reasonable to expect that we can do this (as long as the truncation is sharp enough) for commutators involving $\mathcal{P}$, which are first order. We therefore define time and frequency truncated variants of $\mathcal{P}$, $\mathcal{B}$ and $\tilde{\mathcal{B}}$ (technically, this last term is unchanged) via

$$\begin{aligned}
\mathcal{P}^0_{k_0} &:= \partial_j g^{ij}_{<k_0}(0)\partial_i, \\
\mathcal{B}^0_{k_0} &:= b^j_{<k_0}(0)\partial_j - [\mathcal{P}^0_{k_0}, \langle\nabla\rangle^\sigma]\langle\nabla\rangle^{-\sigma}, \\
\tilde{\mathcal{B}}^0_{k_0} &:= \tilde{b}^j_{<k_0}(0)\partial_j,
\end{aligned}$$

and obtain the equation

$$i\partial_t u + \mathcal{P}u + \mathcal{B}^0_{k_0}u + \tilde{\mathcal{B}}^0_{k_0}\overline{u} = \langle\nabla\rangle^\sigma f + \mathcal{R}^1_\sigma + \mathcal{R}^2_\sigma + \mathcal{R}^3_\sigma, \tag{3.6.5}$$

where

$$\mathcal{R}^3_\sigma := (\mathcal{B}^0_{k_0} - \mathcal{B})u = -[\mathcal{P}^0_{k_0} - \mathcal{P}, \langle\nabla\rangle^\sigma]v.$$

The next lemma treats the new source term $\mathcal{R}^3_\sigma$.

Lemma 3.6.5. For k_0 and k_1 large enough and T small enough, we have

$$\|\mathcal{R}^3_\sigma\|_{Y^0} \le \varepsilon\|v\|_{X^\sigma}. \tag{3.6.6}$$

Proof. We begin by writing

$$\mathcal{P}^0_{k_0} - \mathcal{P} = (\partial_i g^{ij}_{<k_0}(0) - T_{\partial_i g^{ij}})\partial_j + (g^{ij}_{<k_0}(0) - T_{g^{ij}})\partial_i\partial_j.$$

As with the estimate for $\mathcal{R}^2_\sigma$, we have

$$\|[\langle\nabla\rangle^\sigma, (\partial_i g^{ij}_{<k_0}(0) - T_{\partial_i g^{ij}})]\partial_j v\|_{L^1_T L^2_x} \le \varepsilon\|v\|_{X^\sigma},$$

by taking T small enough. The term $[\langle\nabla\rangle^\sigma, (T_{g^{ij}} - g^{ij}_{<k_0}(0))]\partial_i\partial_j v$ is more difficult to deal with since it is like an operator of order $\sigma+1$ applied to v, and therefore cannot be estimated in $L^1_T L^2_x$ without losing derivatives. Consequently, we must estimate it in the weaker space Y^0. Since $k_1 \gg k_0$, we have the identity

$$[\langle\nabla\rangle^\sigma, (T_{g^{ij}} - g^{ij}_{<k_0}(0))]\partial_i\partial_j v = \sum_{k\ge 0} \tilde{S}_k[\langle\nabla\rangle^\sigma, S_{<k-4}(g^{ij} - g^{ij}_{<k_0}(0))]\partial_i\partial_j S_k v,$$

where $\tilde{S}_k$ is a fattened Littlewood-Paley projection. Therefore, by almost orthogonality, Proposition 3.2.14 and Remark 3.2.15 we have

$$\begin{aligned}
\|[\langle\nabla\rangle^\sigma, (T_{g^{ij}} - g^{ij}_{<k_0}(0))]\partial_i\partial_j v\|_{Y^0} &\lesssim \|g^{ij} - g^{ij}_{<k_0}(0)\|_{l^1 X^{s_0-\delta}} \left(\sum_{k\ge 0} 2^{2k(\sigma-1)}\|S_k\nabla v\|^2_{X^0}\right)^{\frac{1}{2}} \\
&\lesssim \|g^{ij} - g^{ij}_{<k_0}(0)\|_{l^1 X^{s_0-\delta}}\|v\|_{X^\sigma}
\end{aligned}$$

for some $\delta > 0$. By taking k_0 large enough and then T small enough, we can estimate using Bernstein type inequalities and the fundamental theorem of calculus,

$$\|g^{ij} - g^{ij}_{<k_0}(0)\|_{l^1 X^{s_0-\delta}} \leq \varepsilon.$$

Combining this with the above estimates concludes the proof of (3.6.6), as desired. □

To summarize what we have so far, $u := \langle \nabla \rangle^\sigma v$ solves the equation

$$i\partial_t u + \mathcal{P}u + \mathcal{B}^0_{k_0} u + \tilde{\mathcal{B}}^0_{k_0} \overline{u} = \mathcal{R}, \tag{3.6.7}$$

where the source term $\mathcal{R}$ can be estimated in Y^0 by

$$\|\mathcal{R}\|_{Y^0} \leq C\|f\|_{Y^\sigma} + \varepsilon \|v\|_{X^\sigma},$$

for some universal constant C.

Renormalization construction

Now we are ready to construct the renormalization operator $\mathcal{O}$ whose role will be to transform (3.6.7) into an equation essentially of the form (3.6.1). As alluded to earlier, the main enemy we have to deal with is the first order term $\mathrm{Re}(\mathcal{B}^0_{k_0})u$. The strategy will be to construct an operator with symbol in S^0 which conjugates away the "worst part" of this term. As noted in [84], conjugating the entire term away would give a symbol that does not belong to S^0. We opt therefore to conjugate away only a portion of the first-order term whose principal part is supported within some large compact set $B_R(0)$. The hope is that the remaining error term will contribute errors of size $\approx \varepsilon\|v\|_{X^\sigma}$ due to the smallness of the coefficients in (3.5.4) outside of $B_R(0)$. As mentioned earlier, this does not come for free. The trade-off is that we will also need to control the $X^0 \to X^0$, $Y^0 \to Y^0$, and $L^2 \to L^2$ norms of our renormalization operator to ensure that the smallness is retained when applying this operator (as the ξ derivatives of its symbol will not have uniform in R bounds).

The details of this construction will be given below. To set the stage, let us fix a large constant $R \gg 1$ to be chosen. We also define for each $\rho > 0$, the function $\chi_{<\rho}(x) := \chi(\rho^{-1}x)$ where χ is a radial cutoff function equal to 1 on the unit ball and vanishing outside $|x| > 2$. As a first constraint, we demand for R to be such that (3.5.4) holds with some $R_0 < \frac{R}{8}$ and $\varepsilon_0 \ll \varepsilon$. The bulk of the renormalization construction is given by the following proposition.

Proposition 3.6.6. Let u be as above. Let k_0 be large enough so that $g^{ij}_{<k_0}(0)$ is a non-trapping metric with comparable parameters to $g^{ij}(0)$ (the existence of which is guaranteed by Proposition 3.4.5). Define the truncated symbol $a(x,\xi) := -g^{ij}_{<k_0}(0)\xi_i\xi_j$, which is the principal symbol for $\mathcal{P}^0_{k_0}$. Write also $iB(x,\xi) := i\,\mathrm{Re}(b^j_{<k_0}(0))\xi_j + i\{a, \langle\xi\rangle^\sigma\}\langle\xi\rangle^{-\sigma}$ to denote the principal symbol of $\mathrm{Re}\,\mathcal{B}^0_{k_0}$ and

$$H_a := \nabla_\xi a \cdot \nabla_x - \nabla_x a \cdot \nabla_\xi$$

to denote the Hamiltonian vector field for a. Let the parameters R, M and L be as above. Then there exists a smooth, non-negative, real-valued, time-independent symbol $O \in S^0$ with the following properties.

(i) (Positive commutator with good error). There exists $r \in S^1$ such that if T is sufficiently small,

$$H_a O + \chi_{<2R} B(x,\xi) O(x,\xi) + r(x,\xi) O(x,\xi) \geq 0, \qquad \|Op(r)\|_{X^0 \to Y^0} \lesssim_M \varepsilon.$$

(ii) (Uniform L^2 bound at high frequency). For k_0, k_1 large enough and T small enough depending on R, M and L, $\mathcal{O} := Op(O)$ satisfies the estimates

$$\|\mathcal{O}u\|_{L^2} \approx \|u\|_{L^2}, \qquad \|\mathcal{O}u\|_{Y^0} \lesssim \|u\|_{Y^0}, \qquad \|\mathcal{O}u\|_{X^0} \lesssim \|u\|_{X^0}, \tag{3.6.8}$$

with implicit constants depending only on M and on L within a fixed compact set whose size is independent of R.

(iii) (Even in ξ within $B_{\frac{R}{8}}(0)$). The symbol $s := O(x,\xi) - O(x,-\xi)$ is supported in the region $|x| > \frac{R}{8}$ and for k_1 large enough, there holds

$$\|Op(s) S_{\geq k_1}\|_{Y^0 \to Y^0} \lesssim 1,$$

with implicit constants depending only on M and on L within a fixed compact set whose size is independent of R.

The first property will allow us to transform (3.6.7) into an equation of the type (3.6.1) up to an error term supported outside $B_R(0)$ (plus an acceptable remainder). The second property ensures that the $L^2 \to L^2$, $Y^0 \to Y^0$ and $X^0 \to X^0$ operator bounds for $\mathcal{O}$ do not depend on R, at least at high frequency. The third property ensures that $\mathcal{O} := Op(O)$ commutes with complex conjugation to leading order (i.e. within $B_{\frac{R}{8}}(0)$ where the coefficient $\tilde{b}^j$ can be large). The second and third properties will be important for avoiding the circularity mentioned earlier when trying to estimate the error terms supported outside $B_{\frac{R}{8}}(0)$.

We also emphasize that a is the principal symbol for the truncated operator $\mathcal{P}_{k_0}^0$ and not $\mathcal{P}$. This is to ensure that O will be a classical (time-independent) S^0 symbol with bounds not depending on higher derivatives of g^{ij} (however, they will depend on the frequency truncation scale 2^{k_0}). The trade-off is that when commuting the equation for u with $\mathcal{O}$, we will need to estimate an additional first order error term of the form

$$[\mathcal{P}-\mathcal{P}_{k_0}^0, \mathcal{O}]u$$

in Y^0. It will turn out that this can be made small by taking k_0, k_1 large enough and T small enough. We will discuss how to estimate this term later. For now, we start by proving Proposition 3.6.6.

Proof. We make the ansatz $O(x,\xi) = e^{\psi(x,\xi)}$ where ψ is some smooth real-valued function to be chosen. We begin by trying to enforce condition (i). For this, we recall that the vector field H_a corresponds to differentiation along the Hamilton flow of a, which is given by (3.4.3). That is,

$$(H_a\psi)(x,\xi) = \frac{d}{dt}\psi(x^t,\xi^t)_{|t=0},$$

where (x^t,ξ^t) are the bicharacteristics for a with initial data (x,ξ). We will perform our construction in two stages. That is, we will define two symbols ψ_1 and ψ_2 in S^0. The symbol ψ_1 will be chosen so that $H_a\psi_1$ cancels the bulk of the term $\chi_{<2R}B(x,\xi)$ but possibly with an additional error term which isn't small but has the redeeming feature that it is supported in the transition region $|x| \approx R$ where $g_{<k_0}^{ij}(0)$ is close to the corresponding flat metric. The second symbol ψ_2 will be chosen to correct ψ_1 so that the error term can be made sufficiently small. The full symbol ψ will then be defined by $\psi := \psi_1 + \psi_2$. Inspired by the previous works [33, 38, 67, 90], our starting point is to consider the ideal "symbol"

$$\psi_{ideal}(x,\xi) := -\frac{1}{2}\chi_{>1}(|\xi|)\int_{-\infty}^{0} B(x_{(x,\xi)}^t, \xi_{(x,\xi)}^t) + B(x_{(x,-\xi)}^t, \xi_{(x,-\xi)}^t)dt,$$

where $\chi_{>1}(|\xi|)$ is an increasing Fourier multiplier selecting frequencies ≥ 1. We note that since $g_{<k_0}^{ij}(0)$ is nontrapping and $b^j, \nabla_x g^{ij} \in l^1X^{s_0-1}$, the integral in ψ_{ideal} is well-defined. On a formal level, the commutator of the principal part of the equation with $Op(e^{\psi_{ideal}})$ conjugates away the leading part of the term $\operatorname{Re}\mathcal{B}_{k_0}^0 u$, but as mentioned above, the symbol ψ_{ideal} is not a classical S^0 symbol, so it is not ideal to work with such a construction directly.

In order to resolve this issue, we localize this symbol to the compact set $B_{2R}(0)$ by instead defining

$$\psi_1(x,\xi) := -\frac{1}{2}\chi_{>1}(|\xi|)\chi_{<2R}(x)\int_{-\infty}^{0}(\chi_{<4R}B)(x^t_{(x,\xi)},\xi^t_{(x,\xi)}) + (\chi_{<4R}B)(x^t_{(x,-\xi)},\xi^t_{(x,-\xi)})dt.$$

The corresponding pseudodifferential operator $Op(e^{\psi_1})$ will conjugate away the leading part of the first order term $\mathrm{Re}\,\mathcal{B}^0_{k_0}u$ within the ball $B_{2R}(0)$, which is the region where the $X^0 \to Y^0$ operator bounds for $\mathcal{B}^0_{k_0}$ are expected to be large. The difficulty is then shifted to controlling the remaining errors in the exterior region, but now we have the benefit of ψ_1 being a genuine S^0 symbol (this fact will be confirmed below). We remark that since $B(x,\xi)$ is real, ψ_1 is as well. Moreover, ψ_1 is even in ξ.

Since B is odd in ξ, it is straightforward to verify that we have the leading order cancellation,

$$H_a\psi_1 + \chi_{<2R}B(x,\xi) \geq -KR^{-1}|\chi'(\frac{1}{2}R^{-1}r)||\xi| - K\chi_{<2}(|\xi|), \tag{3.6.9}$$

where $K > 0$ is such that $K \gg_M \|\psi_{ideal}\|_{L^\infty}$. We remark that K is uniformly bounded in R because of Proposition 3.4.8. The term on the right-hand side of (3.6.9) is not quite suitable for defining a symbol r ensuring the bound in (i) (the corresponding operator need not have small $X^0 \to Y^0$ bound due to the insufficient spatial decay in the first term). For this reason, we seek to further correct ψ_1 by a symbol ψ_2 which is supported in the region $|x| \gtrsim R$. Precisely, our aim will be to construct ψ_2 so that

$$H_a\psi_2 - KR^{-1}|\chi'(\frac{1}{2}R^{-1}r)||\xi| - K\chi_{<2}(|\xi|) + r(x,\xi) \geq 0 \tag{3.6.10}$$

where $r \in S^1$ is a suitable remainder term satisfying the bound in (i). Before proceeding, to simplify the notation somewhat, for the remainder of the proof we will write $A := A(x)$ as a shorthand for $g^{ij}_{<k_0}(0)$ and A_∞ as a shorthand for g^{ij}_∞. We also define the functions $\theta(x,\xi) := \angle(x, A_\infty\xi)$, $\alpha(x,\xi) := \angle(x, A\xi)$, $\beta(\xi) := \angle(A\xi, A_\infty\xi)$ and $\gamma(x,\xi) := \frac{1}{2}(1+\cos(\theta))$.

Now, to proceed, we begin by recalling that the assumption (3.5.4) ensures that we have the bounds

$$|A - A_\infty| + |\nabla A| \ll \varepsilon, \quad |x| > \frac{R}{8}. \tag{3.6.11}$$

In particular, A is close to the flat metric in L^∞ when $|x| > \frac{R}{8}$. Now, let:

(i) ρ be a smooth, increasing function such that $\rho' \approx 1$ for $\frac{1}{7} \leq r \leq 2$, $\rho = 0$ for $r \leq \frac{1}{8}$ and $\rho = 1$ for $r \geq 3$. Define $\rho_R(x) = \rho(R^{-1}r)$ and $\rho_\theta(x,\xi) = \rho_R(x\gamma)$.

(ii) For $c \in [-1,1]$ and some fixed positive $\delta_0 \ll 1$, let $\varphi_{<c}$ be a decreasing smooth function which vanishes for $x > c+\delta_0$ and is identically one for $x \le c$. Define also $\varphi_{>c} := 1-\varphi_{<c}$.

We then define the symbol ψ_2 by

$$\psi_2(x,\xi) := K'\chi_{>1}(|\xi|)\left(\rho_R\varphi_{<-\frac{1}{2}}(\cos(\theta)) - \rho_\theta\varphi_{>-\frac{1}{2}}(\cos(\theta))\right) \tag{3.6.12}$$

where $K' \gg K$ is a constant to be chosen. We note that the weight ρ_R is increasing in the direction of the bicharacteristics in the regions of phase space where they are outgoing with respect to the flat metric. In such regions, this will give a good bound from below for the bulk of $H_a\psi_2$. The purpose of ρ_θ will be to accomplish the same task in the incoming region as well as the regions of phase space where $A_\infty\xi$ is nearly orthogonal to x. In such regions, a purely radially increasing cutoff (such as ρ_R) would be insufficient. The reason we use the average $\frac{1}{2}(1+\cos(\theta))$ in the definition of ρ_θ is to ensure that ρ_θ still vanishes for a suitable range of r on the support of $\varphi_{>-\frac{1}{2}}(\cos(\theta))$ ($r < \frac{R}{8}$, say). This, in particular, ensures that the pointwise error between A and A_∞ is small on the support of $\rho_\theta\varphi_{>-\frac{1}{2}}(\cos(\theta))$. To verify that ψ_2 has the required properties, we first make note of the following simple algebraic computation.

Lemma 3.6.7. For $r > \frac{R}{8}$, we have

$$A\xi\cdot\nabla_x\cos(\theta) = |A\xi|\left(\frac{\sin^2(\theta)}{r} + \delta(x,\xi)\right),$$

where $\delta(x,\xi)$ is an error term with $|\delta(x,\xi)| \ll \frac{1}{r}$.

Proof. This is a simple computation. We have

$$\begin{aligned} A\xi\cdot\nabla_x\cos(\theta) &= \frac{|A\xi|}{r}(\cos(\beta)-\cos(\alpha)\cos(\theta)) \\ &= \frac{|A\xi|}{r}\sin^2(\theta) + \frac{|A\xi|}{r}((\cos(\beta)-1)+\cos(\theta)(\cos(\theta)-\cos(\alpha))). \end{aligned} \tag{3.6.13}$$

By non-degeneracy of A and A_∞ and (3.6.11), we have

$$|\cos(\alpha)-\cos(\theta)| + |\cos(\beta)-1| \ll 1, \quad r \ge \frac{R}{8}.$$

Taking δ to be the coefficient of $|A\xi|$ in the second term in the second line of (3.6.13) concludes the proof. □

Now, we compute the Hamilton vector field applied to ψ_2. We define the remainder symbol $r \in S^1$ by

$$r(x,\xi) := -\xi_i\xi_j \nabla_\xi \psi_2 \cdot \nabla_x A^{ij} + K'' \chi_{<2}(|\xi|), \tag{3.6.14}$$

where $K'' \gg K'$ is some sufficiently large constant. We note that r essentially consists of the part of $H_a\psi_2$ in which ψ_2 is differentiated in ξ. This is expected to contribute a small $X^0 \to Y^0$ operator norm because its principal part includes a factor of $\nabla_x A$ which is small in $l^1 X^{s_0-1}$ when $|x| > \frac{R}{8}$. The subprincipal terms will contribute small $L^1_T L^2_x \to L^1_T L^2_x$ operator bounds by taking T to be sufficiently small. We then have

$$H_a\psi_2 + r(x,\xi) \geq -2A\xi \cdot \nabla_x \psi_2 + K''\chi_{<2}(|\xi|). \tag{3.6.15}$$

We now expand the first term on the right-hand side of (3.6.15) to obtain

$$\begin{aligned}
&- A\xi \cdot \nabla_x \psi_2 \\
&= -\frac{K'}{R}\chi_{>1}(|\xi|)|A\xi|\bigg(\cos(\alpha)\rho'(R^{-1}r)\varphi_{<-\frac{1}{2}}(\cos(\theta)) \\
&+ \frac{R}{r}\left(\sin^2(\theta) + r\delta\right)\rho(R^{-1}r)\varphi'_{<-\frac{1}{2}}(\cos(\theta))\bigg) \\
&+ \frac{K'}{R}\chi_{>1}(|\xi|)|A\xi|\bigg(\frac{1}{2}(\cos(\alpha)+\cos(\beta))\rho'(R^{-1}r\gamma)\varphi_{>-\frac{1}{2}}(\cos(\theta)) \\
&\qquad + \frac{R}{r}\big(\sin^2(\theta) + r\delta\big)\rho(R^{-1}r\gamma)\varphi'_{>-\frac{1}{2}}(\cos(\theta))\bigg),
\end{aligned}$$

where α and β are as in Lemma 3.6.7. If ε_0 is small enough in (3.5.4), we observe that on the support of $\rho'(R^{-1}r)\varphi_{<-\frac{1}{2}}(\cos(\theta))$, we have $\cos(\alpha) < -\frac{1}{3}$. Additionally, $(\sin^2(\theta) + r\delta)$ is non-negative on the support of $\rho(R^{-1}r)\varphi'_{<-\frac{1}{2}}(\cos(\theta))$ and $\rho(R^{-1}r\gamma)\varphi'_{>-\frac{1}{2}}(\cos(\theta))$. Moreover, on the support of $\rho'(R^{-1}r\gamma)\varphi_{>-\frac{1}{2}}(\cos(\theta))$, we have $(\cos(\alpha)+\cos(\beta)) > \frac{1}{3}$.

By non-degeneracy of A, we can choose K' depending only on g so that

$$K'|A\xi| \gg K|\xi|.$$

Combining the above, we can arrange for

$$-A\xi \cdot \nabla_x \psi_2 \geq KR^{-1}|\chi'(\frac{r}{2R})||\xi| - \frac{K''}{2}\chi_{<2}(|\xi|),$$

where K'' is as in (3.6.14). We then define the full symbol ψ by

$$\psi := \psi_1 + \psi_2.$$

It is left to verify the properties (i), (ii) and (iii) in Proposition 3.6.6. The positive commutator bound

$$H_a O + \chi_{<2R} B(x,\xi) O(x,\xi) + r(x,\xi) O(x,\xi) \geq 0$$

follows easily from the chain rule and the above construction if K' is large enough. Next, we verify that $r \in S^1$ and that r has the operator bound

$$\|Op(r)\|_{X^0 \to Y^0} \leq \varepsilon. \tag{3.6.16}$$

The fact that $r \in S^1$ is clear so we turn our attention to (3.6.16). Using the definition of r, we can write

$$Op(r) = (\chi_{>\frac{R}{8}} \nabla_x A^{ij}) \cdot Op(\xi_i \xi_j \nabla_\xi \psi_2) + K'' \chi_{<2}(|D|).$$

Using the embedding $L^1_T L^2_x \subset Y^0$ and that $\xi_i \xi_j \nabla_\xi \psi_2 \in S^1$, we can estimate using simple paradifferential calculus and Proposition 3.2.5,

$$\|(\chi_{>\frac{R}{8}} \nabla_x A^{ij} - T_{\chi_{>\frac{R}{8}} \nabla_x A^{ij}}) \cdot Op(\xi_i \xi_j \nabla_\xi \psi_2)\|_{L^1_T L^2_x \to Y^0} \lesssim_{M,k_0} 1.$$

Therefore, by Hölder's inequality in T, we have for T small enough (depending on M and k_0),

$$\|(\chi_{>\frac{R}{8}} \nabla_x A^{ij} - T_{\chi_{>\frac{R}{8}} \nabla_x A^{ij}}) \cdot Op(\xi_i \xi_j \nabla_\xi \psi_2)\|_{X^0 \to Y^0} \leq \varepsilon.$$

Hence, we now only need to show that the $X^0 \to Y^0$ norm for $T_{\chi_{>\frac{R}{8}} \nabla_x A^{ij}} \cdot Op(\xi_i \xi_j \nabla_\xi \psi_2)$ can be made small. For this, let us define $\tilde{r} \in S^0$ by

$$\tilde{r}(x,\xi) = \langle \xi \rangle^{-1} \xi_i \xi_j \nabla_\xi \psi_2.$$

By Proposition 3.2.5, one can verify that the operator $\langle \nabla \rangle Op(\tilde{r}) - Op(\xi_i \xi_j \nabla_\xi \psi_2)$ is bounded from $L^1_T L^2_x \to L^1_T L^2_x$ with norm depending only on M and k_0. Therefore, by taking T small, we can make the $X^0 \to Y^0$ bound of this operator smaller than ε. From Proposition 3.2.13 and the smallness assumption (3.5.4), we then have

$$\begin{aligned}
\|T_{\chi_{>\frac{R}{8}} \nabla_x A^{ij}} \cdot Op(\xi_i \xi_j \nabla_\xi \psi_2)\|_{X^0 \to Y^0} &\lesssim \|\chi_{>\frac{R}{8}} \nabla_x A^{ij}\|_{l^1 X^{s_0-1}} \|\langle \nabla \rangle\|_{X^0 \to X^{-1}} \|Op(\tilde{r})\|_{X^0 \to X^0} + \varepsilon \\
&\lesssim \varepsilon \|Op(\tilde{r})\|_{X^0 \to X^0} + \varepsilon \\
&\lesssim_M \varepsilon.
\end{aligned}$$

Clearly, the $X^0 \to Y^0$ bound for the remaining subprincipal term $K'' \chi_{<2}(|D|)$ can be made small by taking T small. This concludes the proof of (i). Now, we turn to (ii) and (iii). For this, we need the following lemma involving symbol bounds for O.

Lemma 3.6.8. The symbol O constructed above satisfies the following bounds.

(i) (R independent L^∞ bound). There is a constant C_0 depending only on the profile of $g(0)$ and on M but not on R or k_0 such that

$$\|O(x,\xi)\|_{L^\infty_{x,\xi}} \lesssim C_0.$$

(ii) (Higher order symbol bounds). For every $|\alpha+\beta| \geq 2$, there is a constant $C_{\alpha,\beta}$ depending on M, $L(R)$, R and k_0 such that

$$\|\langle\xi\rangle^{|\alpha|}\partial_\xi^\alpha \partial_x^\beta O(x,\xi)\|_{L^\infty_{x,\xi}} \leq C_{\alpha,\beta}.$$

If $|\alpha+\beta| = 1$, the constant can be taken to be uniform in k_0.

The crucial thing to note here is that only the higher order symbol bounds for O depend on R and k_0 while the L^∞ bound does not.

Proof. Clearly, it suffices to show each of the above symbol bounds for ψ_1 and ψ_2. Given the requisite bounds for ψ_1, the bounds for ψ_2 are clear. Therefore, we focus on ψ_1. We begin with the L^∞ bound. By homogeneity of the bicharacteristic flow, it further suffices to show that

$$\int_{\mathbb{R}} |B(x^t,\xi^t)|dt \lesssim_M C_0. \tag{3.6.17}$$

By homogeneity and a change of variables, we have

$$\int_{\mathbb{R}} |B(x^t,\xi^t)|dt = \int_{\mathbb{R}} |\xi|^{-1}|B(x^t_\omega, |\xi|\xi^t_\omega)|dt,$$

where $(x^t_\omega,\xi^t_\omega)$ denote the bicharacteristics with data $(x,\omega) := (x,\xi|\xi|^{-1})$. Then, we use Corollary 3.4.6 and the definition of the symbol B to obtain

$$|B(x^t_\omega, |\xi|\xi^t_\omega)| \leq C_0|\xi||(b^j_{<k_0}(0))(x^t_\omega)| + C_0|\xi||(\nabla_x A)(x^t_\omega)|.$$

The estimate (3.6.17) then follows (after possibly relabelling C_0) from Proposition 3.4.8, using the fact that $b^j_{<k_0}(0), \nabla_x A \in l^1H^{s_0-1}$ with norm $\lesssim_M 1$. This yields the L^∞ bound for ψ_1. The higher order symbol bounds follow immediately from Proposition 3.4.7 and repeated applications of the chain rule. $\square$

Now, we return to the proof of (3.6.8). From the above lemma and Proposition 3.2.6, we have the L^2 bound,

$$\|\mathcal{O}u\|_{L^2} \lesssim C_0\|u\|_{L^2} \tag{3.6.18}$$

for k_1 large enough, with universal implicit constant. We next aim to establish the bound

$$\|u\|_{L^2} \lesssim C_0 \|\mathcal{O}u\|_{L^2}. \tag{3.6.19}$$

Using Proposition 3.2.5, we see that $Op(e^{-\psi})$ is an approximate inverse for $Op(e^{\psi})$ in the sense that we have

$$Op(e^{-\psi})Op(e^{\psi}) = 1 + Op(q),$$

where $q \in S^{-1}$ with symbol bounds depending only on the symbol bounds for ψ. Therefore, we have

$$u = S_{\geq k_1-4}u = Op(e^{-\psi})S_{\geq k_1-4}\mathcal{O}u + Op(\tilde{q})u,$$

where $\tilde{q} \in S^{-1}$ with uniform in k_1 symbol bounds. Hence, from Proposition 3.2.1 we obtain

$$\|u\|_{L^2_x} \lesssim C_0 \|\mathcal{O}u\|_{L^2_x} + C_1 \|u\|_{H^{-1}_x},$$

where C_0 depends only on M and $g(0)$ and C_1 depends on a finite collection of semi-norms $|O|^{(j)}_{S^0}$. Since $\|u\|_{H^{-1}_x} \lesssim 2^{-k_1}\|u\|_{L^2_x}$, we can take k_1 large enough so that

$$\|u\|_{L^2_x} \lesssim C_0 \|\mathcal{O}u\|_{L^2}.$$

This gives (3.6.19). The $Y^0 \to Y^0$ and $X^0 \to X^0$ bounds for $\mathcal{O}$ follow from Proposition 3.2.7. This establishes property (ii) of Proposition 3.6.6. The proof of property (iii) follows almost identical reasoning to the proof of (ii), using the fact that ψ is even in ξ for $|x| < \frac{R}{8}$. This completes the proof of Proposition 3.6.6. □

Proof of Proposition 3.6.1

Now, we complete the proof of Proposition 3.6.1. We will slightly abuse notation from here on and write $\lesssim_M$ to mean that the implicit constant in the corresponding estimate depends on M and C_0 as above (but not on R). Moreover, we let $\mathcal{R}$ generically denote an error term such that

$$\|\mathcal{R}\|_{Y^0} \leq C(M, L)(\|f\|_{Y^\sigma} + \|v_0\|_{H^\sigma}) + \varepsilon \|v\|_{X^\sigma}. \tag{3.6.20}$$

We apply $\mathcal{O} := Op(O)$ from Proposition 3.6.6 to equation (3.6.5). Writing $w := \mathcal{O}u$, we obtain

$$i\partial_t w + \mathcal{P}w + \mathcal{O}\mathcal{B}^0_{k_0}u + [\mathcal{O}, \mathcal{P}]u + \mathcal{O}\tilde{\mathcal{B}}^0_{k_0}\overline{u} = \mathcal{R},$$

where by the $Y^0 \to Y^0$ bound for $\mathcal{O}$ (see (ii) in Proposition 3.6.6), $\mathcal{R}$ still satisfies the estimate (3.6.20) as long as k_1 is large enough. Performing similar frequency and time truncations as before and commuting $\mathcal{O}$ with the first order terms, we obtain

$$i\partial_t w + \mathcal{P}w + i\,\mathrm{Im}(\mathcal{B}^0_{k_0})w + \tilde{\mathcal{B}}^0_{k_0}\overline{w} + [\mathcal{O}, \mathcal{P}^0_{k_0}]u + \chi_{<2R}\,\mathrm{Re}(\mathcal{B}^0_{k_0})\mathcal{O}u = \tilde{\mathcal{R}},$$

where

$$\tilde{\mathcal{R}} = \mathcal{R} - [\mathcal{O}, \tilde{\mathcal{B}}^0_{k_0}]\overline{u} - [\mathcal{O}, \mathcal{B}^0_{k_0}]u - \chi_{\geq 2R}\,\mathrm{Re}(\mathcal{B}^0_{k_0})\mathcal{O}u + \tilde{B}^0_{k_0}(\overline{\mathcal{O}u} - \mathcal{O}\overline{u}) + [\mathcal{O}, (\mathcal{P}^0_{k_0} - \mathcal{P})]u. \tag{3.6.21}$$

We next estimate $\tilde{\mathcal{R}}$. To begin, note that the second and third terms in (3.6.21) are zeroth order and can be estimated in $L^1_T L^2_x \subset Y^0$, so that

$$\|[\mathcal{O}, \tilde{\mathcal{B}}^0_{k_0}]\overline{u} + [\mathcal{O}, \mathcal{B}^0_{k_0}]u\|_{Y^0} \lesssim_{M,L} T\|v\|_{X^\sigma},$$

which by taking T small can be controlled by $\varepsilon\|v\|_{X^\sigma}$. To get a suitable error estimate for the fourth term in (3.6.21), we first note that by property (ii) in Proposition 3.6.6, we have

$$\|\mathcal{O}u\|_{X^0} \lesssim C_0\|u\|_{X^0}$$

if k_1 is large enough. Here, we recall crucially that C_0 is a R independent constant. Therefore, it suffices to establish the bound

$$\|\chi_{\geq 2R}\,\mathrm{Re}(\mathcal{B}^0_{k_0})\|_{X^0 \to Y^0} \leq \varepsilon. \tag{3.6.22}$$

Clearly, it suffices to work with the principal part of $\chi_{\geq 2R}\,\mathrm{Re}(\mathcal{B}^0_{k_0})$ as the error term is bounded from $L^1_T L^2_x \to L^1_T L^2_x$. We can expand the principal part as

$$\chi_{\geq 2R}\,\mathrm{Re}(b_{<k_0}(0))m_1(D) + \chi_{\geq 2R}\nabla_x A m_2(D)$$

where $m_1, m_2 \in S^1$ are suitable (matrix-valued) Fourier multipliers with symbol bounds independent of M, L and R. We can replace the coefficients of m_1 and m_2 above with the paradifferential operators $T_{\chi_{\geq 2R}\,\mathrm{Re}(b_{<k_0}(0))}$ and $T_{\chi_{\geq 2R}\nabla A}$, as the error is an operator which maps $L^1_T L^2_x$ to $L^1_T L^2_x$ with norm depending only on M and k_0. Therefore, if T is small enough, such an error term can be discarded. Using Proposition 3.2.13 and the asymptotic smallness (3.5.4), the remaining term satisfies

$$\begin{aligned}\|T_{\chi_{\geq 2R}\,\mathrm{Re}(b_{<k_0}(0))}m_1(D) + T_{\chi_{\geq 2R}\nabla A}m_2(D)\|_{X^0 \to Y^0} &\lesssim_M \varepsilon(\|m_1(D)\|_{X^0 \to X^{-1}} + \|m_2(D)\|_{X^0 \to X^{-1}}) \\ &\lesssim \varepsilon.\end{aligned}$$

To deal with the fifth term in (3.6.21), we do a similar analysis. Using the definition of $\tilde{B}^0_{k_0}$ and property (iii) in Proposition 3.6.6, we can write

$$\tilde{B}^0_{k_0}(\overline{\mathcal{O}u} - \mathcal{O}\overline{u}) = \chi_{>\frac{R}{8}}\tilde{b}^j_{<k_0}(0)\partial_j(\overline{\mathcal{O}u} - \mathcal{O}\overline{u}).$$

If $\widehat{u}$ is supported at high enough frequency, we can estimate using (ii) in Proposition 3.6.6,

$$\|\partial_j(\overline{\mathcal{O}u} - \mathcal{O}\overline{u})\|_{X^{-1}} \lesssim_M \|u\|_{X^0}.$$

Combining this with the smallness

$$\|\chi_{>\frac{R}{8}}\tilde{b}_{<k_0}(0)\|_{l^1X^{s_0-1}} \leq \varepsilon,$$

we can argue as with the previous term to obtain

$$\|\tilde{B}^0_{k_0}(\overline{\mathcal{O}u} - \mathcal{O}\overline{u})\|_{Y^0} \lesssim_M \varepsilon\|v\|_{X^\sigma}.$$

Now, we turn to the most tricky part, which is estimating the last term in (3.6.21). For this, we have the following lemma.

Lemma 3.6.9 (Commutator bound)**.** For k_0 large enough and T sufficiently small, there holds

$$\|[\mathcal{O}, (\mathcal{P}^0_{k_0} - \mathcal{P})]u\|_{Y^0} \leq \varepsilon\|v\|_{X^\sigma}. \tag{3.6.23}$$

Proof. Clearly, we can write

$$\begin{aligned} [\mathcal{O}, (\mathcal{P} - \mathcal{P}^0_{k_0})] = [\mathcal{O}, (T_{\partial_i g^{ij}} - \partial_i g^{ij}_{<k_0}(0))\partial_j] + (T_{g^{ij}} - g^{ij}_{<k_0}(0))[\mathcal{O}, \partial_i\partial_j] \\ + [\mathcal{O}, (T_{g^{ij}} - g^{ij}_{<k_0}(0))]\partial_i\partial_j. \end{aligned} \tag{3.6.24}$$

The first term is zeroth order and is bounded from $L^1_TL^2_x \to L^1_TL^2_x$. Indeed, by taking T small enough, it is a straightforward consequence of Proposition 3.2.5 and Proposition 3.2.10 that

$$\|[\mathcal{O}, (T_{\partial_i g^{ij}} - \partial_i g^{ij}_{<k_0}(0))\partial_j]\|_{L^\infty_TL^2_x \to Y^0} \leq \varepsilon.$$

The second and third terms are first order, and, as usual, must be dealt with carefully to extract the necessary smallness. We start with the second term which is a bit easier. Since $(T_{g^{ij}_{<k_0}(0)} - g^{ij}_{<k_0}(0))[\mathcal{O}, \partial_i\partial_j]$ is bounded from $L^1_TL^2_x \to L^1_TL^2_x$, we can replace $T_{g^{ij}} - g^{ij}_{<k_0}(0)$ with $T_{g^{ij} - g^{ij}_{<k_0}(0)}$. Then by Proposition 3.2.13 and Bernstein inequalities, we have

$$\begin{aligned} &\|T_{g^{ij} - g^{ij}_{<k_0}(0)}[\mathcal{O}, \partial_i\partial_j]u\|_{Y^0} \\ &\lesssim (\|S_{\geq k_0}g^{ij}\|_{l^1X^{s_0-1-\delta}} + \|g^{ij}_{<k_0} - g^{ij}_{<k_0}(0)\|_{l^1X^{s_0-1-\delta}})\|\langle\nabla\rangle^{-1}[\mathcal{O}, \partial_i\partial_j]u\|_{X^0} \\ &\lesssim_M 2^{-(1+\delta)k_0}\|\langle\nabla\rangle^{-1}[\mathcal{O}, \partial_i\partial_j]u\|_{X^0}, \end{aligned}$$

for some $\delta > 0$. As $\langle\nabla\rangle^{-1}[\mathcal{O}, \partial_i\partial_j] \in OPS^0$, it suffices to consider its principal part when estimating the last term. This is because the subprincipal part is (crudely) bounded from $L^\infty_T H^{-\frac{1}{2}}_x$ to $L^\infty_T H^{\frac{1}{2}}_x \subset X^0$, so we can control such terms by using the fact that u is localized to frequencies $\gtrsim 2^{k_1}$ to gain a smallness factor $2^{-\frac{k_1}{2}}$, and then take k_1 sufficiently large. To estimate the principal symbol c_p for $\langle\nabla\rangle^{-1}[\mathcal{O}, \partial_i\partial_j]$, we can use that $\nabla g^{ij}(0), b^j \in C^{1,\delta}$ and Proposition 3.4.7 to obtain the bound

$$\|c_p\|_{L^\infty} \lesssim \|\nabla_x O\|_{L^\infty} \lesssim_{M,R,L} 1,$$

with implicit constant independent of k_0. Therefore, by taking k_0 and k_1 large enough and applying Proposition 3.2.7, we obtain

$$2^{-(1+\delta)k_0}\|Op(c_p)u\|_{X^0} \lesssim 2^{-(1+\delta)k_0}\|c_p\|_{L^\infty}\|u\|_{X^0} \leq \varepsilon\|v\|_{X^\sigma}.$$

Consequently, the second term in (3.6.24) can be estimated by

$$\|(T_{g^{ij}} - g^{ij}_{<k_0}(0))[\mathcal{O}, \partial_i\partial_j]u\|_{Y^0} \leq \varepsilon\|v\|_{X^\sigma}.$$

It remains to estimate the third term in (3.6.24) which is the most delicate because the commutator itself involves the metric g^{ij} at high frequencies. Our aim as above is to show that

$$\|[\mathcal{O}, (T_{g^{ij}} - g^{ij}_{<k_0}(0))]\partial_i\partial_j u\|_{Y^0} \leq \varepsilon\|v\|_{X^\sigma}.$$

Intuitively, this should be possible by taking k_0 large enough. There are, however, two complications in dealing with this. Firstly, the symbol bounds for O depend on k_0. Secondly, the coefficient in the paradifferential operator $T_{g^{ij}}$ has limited regularity, so the standard pseudodifferential calculus cannot be directly applied. Our strategy is to split this term into three parts to separate the issues. We write

$$\begin{aligned}[\mathcal{O}, (T_{g^{ij}} - g^{ij}_{<k_0}(0))]\partial_i\partial_j u = {} & [\mathcal{O}, (g^{ij}_{<m} - g^{ij}_{<k_0})]\partial_i\partial_j u \\ & + [\mathcal{O}, (T_{g^{ij}} - g^{ij}_{<m})]\partial_i\partial_j u \\ & + [\mathcal{O}, (g^{ij}_{<k_0} - g^{ij}_{<k_0}(0))]\partial_i\partial_j u,\end{aligned} \tag{3.6.25}$$

where m is some universal parameter with $k_0 \ll m \ll k_1$. For the first term, we do not need to worry about the presence of any functions of limited regularity, but we still need to worry about the dependence of O on k_0. For the second term, by taking m large enough, the k_0 dependence in O should be a non-issue, which puts us in a position to use Proposition 3.2.16. Control of the final term follows by taking $T \ll 2^{-2k_0}$ and averaging in T.

Let us begin by analyzing the first term. The principal symbol c_p for $[\mathcal{O},(g^{ij}_{<m}-g^{ij}_{<k_0})]\partial_i$ is given by

$$c_p = \{O,(g^{ij}_{<m}-g^{ij}_{<k_0})\}\xi_i.$$

Analogously to the principal part for the second term in (3.6.24), we have the bound

$$\|\langle\xi\rangle\nabla_\xi O\|_{L^\infty}\lesssim_{M,R,L} 1.$$

To estimate the full commutator, we then use Proposition 3.2.5 to write

$$[\mathcal{O},(g^{ij}_{<m}-g^{ij}_{<k_0})]\partial_i = \nabla_x(g^{ij}_{<m}-g^{ij}_{<k_0})\cdot Op(\nabla_\xi O\xi_i)+Op(r)$$

where $r\in S^0$ (with symbol bounds depending on m). Arguing as in the estimate for the second term in (3.6.24), it follows by using Proposition 3.2.13, then Proposition 3.2.7, then taking k_0 large enough and T small enough (depending on m, M, R and L) that

$$\|[\mathcal{O},(g^{ij}_{<m}-g^{ij}_{<k_0})]\partial_i\partial_j u\|_{Y^0}\leq\varepsilon\|v\|_{X^\sigma}.$$

This takes care of the first term in (3.6.25). For the second term, it suffices to estimate $[\mathcal{O},T_{g^{ij}-g^{ij}_{<m}}]\partial_i\partial_j u$, as the error will be bounded from $L^1_T L^2_x\to L^1_T L^2_x$. To estimate this term, we simply use Proposition 3.2.16 to obtain

$$\|[\mathcal{O},T_{g^{ij}-g^{ij}_{<m}}]\partial_i\partial_j u\|_{Y^0}\lesssim_{M,L,R,k_0}\|g^{ij}_{<m}-g^{ij}\|_{l^1X^{s_0-\delta}}\|u\|_{X^0}.$$

We then recall the smallness bound

$$\|g^{ij}_{<m}-g^{ij}\|_{l^1X^{s_0-\delta}}\lesssim_M 2^{-\delta m},$$

which tells us that if m is large enough relative to k_0, R, L and M then we have the estimate

$$\|[\mathcal{O},T_{g^{ij}-g^{ij}_{<m}}]\partial_i\partial_j u\|_{Y^0}\leq\varepsilon\|u\|_{X^0}\lesssim\varepsilon\|v\|_{X^\sigma}.$$

Finally, by averaging in T and arguing similarly to the above, the last term in (3.6.25) can be controlled by $\varepsilon\|v\|_{X^\sigma}$ by taking T small enough. This completes the proof of Lemma 3.6.9. □

Using the above lemma and Proposition 3.6.6, we now arrive at the following equation for w:

$$i\partial_t w+\mathcal{P}w+i\,\mathrm{Im}(\mathcal{B}^0_{k_0})w+\tilde{\mathcal{B}}^0_{k_0}\overline{w}+[\mathcal{O},\mathcal{P}^0_{k_0}]u+\chi_{<2R}\,\mathrm{Re}(\mathcal{B}^0_{k_0})\mathcal{O}u=\mathcal{R},$$

where $\mathcal{R}$ is as in (3.6.20). To conclude, we make one final reduction. From Proposition 3.2.5, $\mathcal{O}^{-1} := Op(e^{-\psi})$ is an approximate inverse for $\mathcal{O}$ in the sense that we have $Op(\mathcal{O})Op(\mathcal{O}^{-1}) = 1 + Op(q)$ for $q \in S^{-1}$. Therefore, by estimating the error term generated by $Op(q)$ in $L^1_T L^2_x$, we can write

$$i\partial_t w + \mathcal{P}w + i\,\mathrm{Im}(\mathcal{B}^0_{k_0})w + \tilde{\mathcal{B}}^0_{k_0}\overline{w} + iA(x,D)\mathcal{O}^{-1}w = \mathcal{R},$$

where

$$A(x,D) := -i[\mathcal{O}, \mathcal{P}^0_{k_0}] - i\chi_{<2R}\,\mathrm{Re}(\mathcal{B}^0_{k_0})\mathcal{O} + Op(r)\mathcal{O},$$

and $\mathcal{R}$ is again of the form (3.6.21) (as long as T is small enough). By construction, $A(x,D)\mathcal{O}^{-1}$ is a time-independent pseudodifferential operator of order 1 with non-negative principal symbol in S^1. Therefore, the above equation for w is now in the form (3.6.1) with a source term $\mathcal{R}$ satisfying (3.6.20). Hence, Proposition 3.6.1 easily follows by applying Lemma 3.6.2.

3.7 The local energy decay estimate

In this section, we complement the L^2 estimate in the previous section with an estimate for the local energy component of the norm $\|v\|_{X^\sigma}$ for a solution v to (3.5.2). For every $\sigma \geq 0$, we denote the local energy component of X^σ by

$$\|v\|_{\mathcal{X}^\sigma} = \left(\sum_{j\geq 0} 2^{2j(\sigma+\frac{1}{2})}\|S_j u\|_X^2\right)^{\frac{1}{2}}.$$

We remark that we have the obvious embedding $\|v\|_{\mathcal{X}^\sigma} \lesssim \|v\|_{L^2_T H^{\sigma+\frac{1}{2}}_x}$.

The local energy estimate

The local energy estimate we will need for (3.5.2) is given by the following proposition.

Proposition 3.7.1. Let $\sigma \geq 0$ and let s_0, g^{ij}, b^j and $\tilde{b}^j$ be as in Theorem 3.5.1 with parameters M and L. Suppose that v solves (3.5.2) and let $\varepsilon > 0$. There is $T_0 = T_0(\varepsilon) > 0$ such that for $0 \leq T \leq T_0$, we have the local energy bound

$$\|v\|_{\mathcal{X}^\sigma} \leq C(M,L)(\|v\|_{L^\infty_T H^\sigma_x} + \|f\|_{Y^\sigma}) + \varepsilon\|v\|_{X^\sigma},$$

where $C(M,L)$ depends on M and on the parameter L within some fixed compact set depending on ε.

Fix $\delta > 0$ to be some small parameter to be chosen. From (3.6.7) in the previous section, we can choose k_0 sufficiently large and T sufficiently small so that $u := \langle \nabla \rangle^\sigma v$ solves the equation

$$i\partial_t u + \mathcal{P}u + \mathcal{B}^0_{k_0} u + \tilde{\mathcal{B}}^0_{k_0}\overline{u} = \mathcal{R},$$

with the remainder estimate

$$\|\mathcal{R}\|_{Y^0} \lesssim \|f\|_{Y^\sigma} + \delta\|v\|_{X^\sigma}.$$

Also, as in the previous section, we may assume that u is localized to frequencies $\gtrsim 2^{k_1}$, where k_1 is some sufficiently large parameter to be chosen. Unlike with the L^2 estimate, however, we will not need the added energy structure coming from the complex-conjugate first order term. It is therefore convenient to write the equation as a system in u and $\overline{u}$. In doing this, we obtain the following compact form of the paradifferential linear equation:

$$\partial_t \mathbf{u} + \mathbf{P}\mathbf{u} + \mathbf{B}^0_{k_0}\mathbf{u} = \mathbf{R},$$

where

$$\mathbf{P} := i\begin{pmatrix} -\mathcal{P} & 0 \\ 0 & \mathcal{P} \end{pmatrix}, \quad \mathbf{B}^0_{k_0} := i\begin{pmatrix} -\mathcal{B}^0_{k_0} & -\tilde{\mathcal{B}}^0_{k_0} \\ \overline{\tilde{\mathcal{B}}^0_{k_0}} & \overline{\mathcal{B}^0_{k_0}} \end{pmatrix}, \quad \mathbf{u} := \begin{pmatrix} u \\ \overline{u} \end{pmatrix},$$

and $\mathbf{R}$ is a source term satisfying the bound

$$\|\mathbf{R}\|_{Y^0} \lesssim \|f\|_{Y^\sigma} + \delta\|v\|_{X^\sigma}. \tag{3.7.1}$$

We define analogously to before the truncated principal operator $\mathbf{P}^0_{k_0}$ by replacing the nonzero entries in $\mathbf{P}$ with $\mathcal{P}^0_{k_0}$ in the natural way.

By using Theorem 3.5.5 and arguing similarly to the proof of Lemma 3.5.7, for each $R > 0$ large enough and T small enough, there holds

$$\|\chi_{\geq R}\mathbf{u}\|_{\mathcal{X}^0} \leq C(M, R)(\|v\|_{L^\infty_T H^\sigma_x} + \|\mathbf{R}\|_{Y^0} + \|\chi_{<2R}\mathbf{u}\|_{L^2_T H^{\frac{1}{2}}_x}) + \varepsilon\|v\|_{X^\sigma}.$$

Therefore, to prove Proposition 3.7.2, it suffices to establish the bound

$$\|\chi_{<2R}\mathbf{u}\|_{L^2_T H^{\frac{1}{2}}_x} \leq C(M, L)(\|v\|_{L^\infty_T H^\sigma_x} + \|\mathbf{R}\|_{Y^0}) + \varepsilon\|v\|_{X^\sigma}. \tag{3.7.2}$$

This latter estimate is where we will concentrate the bulk of our efforts in this section.

Interior estimate

Now we turn to establishing the required interior estimate (3.7.2). The main construction we will need is given by the following result, which can very loosely be thought of as a spatially truncated version of Doi's construction in [38]. Our method will work under far less stringent decay assumptions, however. For similar reasons to the previous section, we will again work with the principal symbol for the truncated operator $\mathbf{P}^0_{k_0}$ in our analysis rather than $\mathbf{P}$ directly (at the cost of estimating a term with the same flavor as (3.6.23)). We will also write $|\mathbf{B}^0_{k_0}|$ to denote the maximum of the absolute values of the entries of the principal symbol for $\mathbf{B}^0_{k_0}$.

Proposition 3.7.2. Let $C(M) > 1$ be a constant depending on M to be chosen. Moreover, let k_0 be large enough so that $g^{ij}_{<k_0}(0)$ is nontrapping with comparable parameters to $g^{ij}(0)$ (which is possible by Proposition 3.4.5). Define $a := -g^{ij}_{<k_0}(0)\xi_i\xi_j$. Then for every $R' \gg R$ sufficiently large, there is a smooth, non-negative, time-independent S^0 symbol $q \geq 1$ with the following properties:

(i) (Positive commutator in $B_{R'}(0)$ with small error). There exists $r \in S^1$ such that if R' and k_1 are large enough and T is sufficiently small relative to R' and k_1, then we have

$$H_a q + C(M) r q \gtrsim C(M)\chi_{<R'}|\mathbf{B}^0_{k_0}|q, \quad \|Op(rq)S_{\geq k_1}\|_{X^0 \to Y^0} \lesssim \frac{\varepsilon}{C(M)}.$$

(ii) (Ellipticity in $B_{2R}(0)$).

$$H_a q + C(M) r q \geq C(M)\chi_{<2R}|\xi| q,$$

where r is as in (i).

(iii) (Zeroth order symbol bound). There is a constant $C_0(M, R)$ depending on M and R but not on R' such that

$$|q| \leq C_0.$$

(iv) (First order symbol bound). There is a constant $C_1(M, R, R')$ depending on M, R and R' such that

$$|\xi||\nabla_\xi q| + |\nabla_x q| \leq C_1.$$

(v) (Higher order symbol bounds). There is a constant $C_2(M, R, R', k_0)$ depending on M, R, R' and k_0 such that

$$\langle \xi \rangle^{|\alpha|}|\partial^\alpha_\xi \partial^\beta_x q| \lesssim_{\alpha,\beta} C_2, \quad |\alpha + \beta| \geq 2.$$

In the R and R' dependent constants above, we also allow for dependence on L within $B_R(0)$ and $B_{R'}(0)$, respectively. The first property will allow us to control the contribution of the first-order terms in the equation within the larger compact set $B_{R'}(0)$, up to a small error term, as long as $\mathbf{u}$ is localized at high enough frequency. The second property will give us the required control of $\chi_{<2R}\mathbf{u}$ in $L_T^2 H_x^{\frac{1}{2}}$ up to a suitable error term. We importantly remark that the zeroth order symbol bounds in (iii) for q depend only on M and R (more precisely, $L(R)$). This will ensure that the $Y^0 \to Y^0$ bound for $Op(q)$ depends only on M and R as long as $\mathbf{u}$ is at sufficiently high frequency, thanks to Proposition 3.2.7. As a consequence, we may argue similarly to the previous section and treat the first-order terms in the region outside of $B_{R'}(0)$ perturbatively as long as R' is large enough relative to R. We note that unlike in the construction in Proposition 3.6.6, this second parameter R' is needed because the uniform norm of the symbol q necessarily depends on the smaller radius R.

We emphasize that the first order symbol bounds in (iv) depend on M, R and R' but not on k_0. The purpose of this will be to control an error term that is similar to the commutator (3.6.23) from the previous section by taking k_0 large relative to M, R, and R'. The higher order symbol bounds in property (v) will ensure that q is a classical S^0 symbol and will allow us to estimate lower order error terms in $L_T^1 L_x^2$ by taking T small depending on M, R, R' and k_0, similarly to the previous section.

Proof. We begin by defining a smooth function that will be suitable for controlling the size of the first order coefficients within the larger compact set $B_{R'}(0)$. A reasonable choice is the following:

$$\eta_{R'} = \chi_{<2R'}\sqrt{|\tilde{b}_{<k_0}(0)|^2 + |b_{<k_0}(0)|^2 + |\nabla_x g^{ij}_{<k_0}(0)|^2 + L(2R')^{-2}}.$$

The term $L(2R')^{-2}$ is for technical convenience. It ensures that $\eta_{R'}$ is smooth and allows us to invoke Proposition 3.4.8 to obtain uniform integrability along the bicharacteristic flow for the truncated metric $g^{ij}_{<k_0}(0)$ with a bound independent of R'. Precisely, we have

$$\int_{\mathbb{R}} \eta_{R'}(x^t, \xi^t)|\xi^t| dt \leq C_0 \tag{3.7.3}$$

where C_0 is as above. Moreover, for $|x| \leq R'$, we clearly have $|\nabla_x g^{ij}_{<k_0}(0)||\xi| + |\mathbf{B}^0_{k_0}| \lesssim \eta_{R'}|\xi|$. Now, we move to constructing the symbol q. We start by defining a preliminary symbol p_1 via

$$p_1(x,\xi) := -\chi_{>1}(|\xi|)\chi_{<R'}\int_0^\infty (\chi_{<2R} + \eta_{R'})(x^t, \xi^t)|\xi^t| dt,$$

where similarly to the construction for O in Proposition 3.6.6, we localized the symbol in space to $B_{R'}(0)$ so that it will ultimately belong to S^0. As in Proposition 3.6.6, $H_a p_1$ will generate an error term coming from the localization $\chi_{<R'}$. To deal with this, we correct p_1 by another symbol p_2. To define p_2, we take our cue from the definition (3.6.12) in the previous section. Using the same notation as in (3.6.12) with the parameter R' replacing R in all instances, we define

$$p_2(x,\xi) := K'\chi_{>1}(|\xi|)\left(\rho_{R'}\varphi_{<-\frac{1}{2}}(\cos(\theta)) - \rho_\theta\varphi_{>-\frac{1}{2}}(\cos(\theta))\right),$$

where $K' := K'(R,M)$ is a constant such that

$$K' \gg \sup_{(x,\xi)\in\mathbb{R}^{2d}} \chi_{>1}(|\xi|)\int_0^\infty (\chi_{<2R} + \eta_{R'})(x^t,\xi^t)|\xi^t|dt. \tag{3.7.4}$$

We note that thanks to the nontrapping assumption and (3.7.3), K' can be chosen to depend only on R and M, but not on R'. We then define $p := p_1 + p_2$ and analogously to (3.6.14), we define the remainder symbol r by

$$r(x,\xi) := -\xi_i\xi_j\nabla_\xi p_2 \cdot \nabla_x g^{ij}_{<k_0}(0) + K''\chi_{<2}(|\xi|),$$

where $K'' \gg K'$ is some sufficiently large constant. We then define the required symbol q by

$$q := e^{C(M)p}, \tag{3.7.5}$$

for some sufficiently large constant $C(M) > 0$. Now, we turn to establishing each property in Proposition 3.7.2. First, arguing similarly to the proof of the first property in Proposition 3.6.6, we compute directly that

$$H_a q + C(M)rq \geq C(M)(\chi_{<2R} + \chi_{<R'}\eta_{R'})|\xi|q. \tag{3.7.6}$$

From this, we immediately obtain the positive commutator bounds in (i) and (ii) in Proposition 3.7.2. The $X^0 \to Y^0$ estimate for $Op(rq)S_{\geq k_1}$ follows from properties (iii)-(v) (to be established below), Proposition 3.2.7 and the fact that $\|\chi_{>R'}\nabla_x g^{ij}_{<k_0}(0)\|_{L^\infty} \to 0$ as $R' \to \infty$. Next, we verify the symbol bounds (iii)-(v). It clearly suffices to establish the analogous symbol bounds for the symbol p_1. To do this, we split

$$p_1 := -\chi_{>1}(|\xi|)\chi_{<R'}\left(\int_0^\infty \eta_{R'}(x^t,\xi^t)|\xi^t|dt + \int_0^\infty \chi_{<2R}(x^t,\xi^t)|\xi^t|dt\right) =: \chi_{>1}(|\xi|)\chi_{<R'}(a_1+a_2).$$

By a change of variables and homogeneity, we have for each $\xi \neq 0$,

$$a_i(x,\xi) = a_i\left(x, \frac{\xi}{|\xi|}\right), \quad i = 1,2.$$

By (3.7.3) and Corollary 3.4.6, we then easily verify property (iii) for $\chi_{>1}(|\xi|)\chi_{<R'}a_1$. By the nontrapping assumption, one may verify (iii) for the symbol $\chi_{>1}(|\xi|)\chi_{<R'}a_2$ as well. Properties (iv) and (v) for both $\chi_{>1}(|\xi|)\chi_{<R'}a_1$ and $\chi_{>1}(|\xi|)\chi_{<R'}a_2$ are a straightforward consequence of Proposition 3.4.7. This completes the proof of Proposition 3.7.2. □

Now, we turn to establishing the main estimate (3.7.2). We begin by defining the symbols $\mathbf{q}$ and $|\mathbf{q}|$:

$$\mathbf{q} := \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}, \qquad |\mathbf{q}| := q\mathbf{I}_{2\times 2}.$$

Define $\mathbf{Q} := \frac{1}{2}Op(\mathbf{q}) + \frac{1}{2}Op(\mathbf{q})^*$. Performing a similar calculation to Lemma 3.6.2, we note that $\mathbf{P}$ is skew-adjoint up to a $L^2_x \to L^2_x$ bounded error. Therefore, it is a straightforward algebraic manipulation to verify the inequality

$$\operatorname{Re}\langle \mathbf{QPu}, \mathbf{u}\rangle \geq \frac{1}{2}\operatorname{Re}\langle [\mathbf{Q}, \mathbf{P}]\mathbf{u}, \mathbf{u}\rangle - C_2\|v\|^2_{L^\infty_T H^\sigma_x},$$

where C_2 is as in Proposition 3.7.2. We then obtain the basic preliminary energy estimate,

$$\begin{aligned}
&\frac{1}{2}\operatorname{Re}\langle \mathbf{Qu}, \mathbf{u}\rangle(T) + \int_0^T \operatorname{Re}\langle (\frac{1}{2}[\mathbf{Q}, \mathbf{P}^0_{k_0}] + \mathbf{Q}\mathbf{B}^0_{k_0})\mathbf{u}, \mathbf{u}\rangle dt \\
\leq &\frac{1}{2}\operatorname{Re}\langle \mathbf{Qu}, \mathbf{u}\rangle(0) + \frac{1}{2}\int_0^T \operatorname{Re}\langle [\mathbf{Q}, (\mathbf{P}^0_{k_0} - \mathbf{P})]\mathbf{u}, \mathbf{u}\rangle dt + \int_0^T \operatorname{Re}\langle \mathbf{QR}, \mathbf{u}\rangle dt + C_2\|\mathbf{u}\|^2_{L^\infty_T L^2_x}.
\end{aligned} \tag{3.7.7}$$

Now, we estimate each term in (3.7.7). By Cauchy-Schwarz and Proposition 3.2.1, we have

$$|\operatorname{Re}\langle \mathbf{Qu}, \mathbf{u}\rangle(T)| + |\operatorname{Re}\langle \mathbf{Qu}, \mathbf{u}\rangle(0)| \leq C_2\|\mathbf{u}\|^2_{L^\infty_T L^2_x}.$$

Next, by Proposition 3.7.2, the principal symbol $\mathbf{c}(x,\xi)$ of $[\mathbf{Q}, \mathbf{P}^0_{k_0}]$ satisfies,

$$\mathbf{c}(x,\xi) + C(M)rq\mathbf{I}_{2\times 2} \geq \frac{1}{2}C(M)(\chi_{<2R}|\xi|q + \chi_{<R'}|\mathbf{B}^0_{k_0}|q)\mathbf{I}_{2\times 2}.$$

We can therefore choose $C(M)$ large enough so that

$$\mathbf{c}(x,\xi) + C(M)rq\mathbf{I}_{2\times 2} - \chi_{<2R}|\xi|q\mathbf{I}_{2\times 2} - \frac{1}{2}C(M)\chi_{<R'}q|\mathbf{B}^0_{k_0}|\mathbf{I}_{2\times 2} \geq \mathbf{0}.$$

Then, the classical Gårding inequality Proposition 3.2.2 along with its matrix version (see Remark 3.2.3) yields

$$\begin{aligned}
\int_0^T \operatorname{Re}\langle (\frac{1}{2}[\mathbf{Q}, \mathbf{P}^0_{k_0}] + \mathbf{Q}\mathbf{B}^0_{k_0})\mathbf{u}, \mathbf{u}\rangle dt \gtrsim &\|\chi_{<2R}\mathbf{u}\|^2_{L^2_T H^{\frac{1}{2}}_x} - C_2\|\mathbf{u}\|^2_{L^\infty_T L^2_x} \\
&- C(M)\|Op(rq)\mathbf{u}\|_{Y^0}\|\mathbf{u}\|_{X^0} - \|\mathbf{Q}\chi_{>R'}\mathbf{B}^0_{k_0}\mathbf{u}\|_{Y^0}\|\mathbf{u}\|_{X^0},
\end{aligned} \tag{3.7.8}$$

where we applied Hölder's inequality in T to control the lower order error term in Remark 3.2.3 by the $L^\infty_T L^2_x$ norm of $\mathbf{u}$ and the $Y^* = X$ duality to control the remaining first order terms. To control the first Y^0 error term on the right, we use property (i) from Proposition 3.7.2 to estimate

$$C(M)\|Op(rq)\mathbf{u}\|_{Y^0}\|\mathbf{u}\|_{X^0} \lesssim_M \varepsilon\|v\|^2_{X^\sigma},$$

which holds as long as $\mathbf{u}$ is localized at high enough frequency (i.e. k_1 is large enough). To control the latter Y^0 error term, we first note that by Proposition 3.2.5, the embedding $L^1_T L^2_x \subset Y^0$ and Hölder in T, we have

$$\|\mathbf{Q}\chi_{>R'}\mathbf{B}^0_{k_0}\mathbf{u}\|_{Y^0} \leq \|\chi_{>R'}\mathbf{B}^0_{k_0}\mathbf{Q}\mathbf{u}\|_{Y^0} + C_2\|\mathbf{u}\|_{L^\infty_T L^2_x}.$$

Then by using Proposition 3.2.13 and arguing as with the analogous terms in the previous section, we have

$$\begin{aligned}\|\chi_{>R'}\mathbf{B}^0_{k_0}\mathbf{Q}\mathbf{u}\|_{Y^0} \lesssim\ & \|\chi_{>R'}(\tilde{b}_{<k_0}(0), b_{<k_0}(0), \nabla_x g^{ij}_{<k_0}(0))\|_{l^1X^{s_0-1}}\|\mathbf{Q}S_{>k_1-4}\|_{X^0\to X^0}\|\mathbf{u}\|_{X^0} \\ &+ C_2\|\mathbf{u}\|_{L^\infty_T L^2_x}.\end{aligned}$$

Using the fact that the L^∞ norm of the symbol $\mathbf{q}$ depends only on R and M and not on R', we can take k_1 and R' large enough so that Proposition 3.2.7 and (3.5.4) ensure that

$$\|\chi_{>R'}(\tilde{b}_{<k_0}(0), b_{<k_0}(0), \nabla_x g^{ij}_{<k_0}(0))\|_{l^1X^{s_0-1}}\|\mathbf{Q}S_{>k_1-4}\|_{X^0\to X^0} \leq \varepsilon.$$

It then follows by Cauchy-Schwarz and (3.7.8) that we have

$$\int_0^T \mathrm{Re}\langle(\frac{1}{2}[\mathbf{Q},\mathbf{P}^0_{k_0}] + \mathbf{Q}\mathbf{B}^0_{k_0})\mathbf{u},\mathbf{u}\rangle dt \gtrsim \|\chi_{<2R}\mathbf{u}\|^2_{L^2_T H^{\frac{1}{2}}_x} - C_2\|\mathbf{u}\|^2_{L^\infty_T L^2_x} - \varepsilon\|v\|^2_{X^\sigma}.$$

Next, we estimate the contribution of the second term in the second line of (3.7.7). The procedure here is essentially identical to the estimate in (3.6.23). Using the symbol bounds for q in Proposition 3.7.2 (specifically, that the derivatives of q up to first order have uniform in k_0 bounds), we can estimate by taking k_0 large enough and T small enough as in the proof of Lemma 3.6.9 to obtain

$$\|[\mathbf{Q},(\mathbf{P}^0_{k_0} - \mathbf{P}_k)]\mathbf{u}\|_{Y^0} \leq \varepsilon\|v\|_{X^\sigma}.$$

To estimate the third term in the second line of (3.7.7), we use the $Y^* = X$ duality and Proposition 3.2.7 to obtain

$$\int_0^T \mathrm{Re}\langle\mathbf{Q}\mathbf{R},\mathbf{u}\rangle dt \leq C_0\|\mathbf{R}\|_{Y^0}\|v\|_{X^\sigma},$$

where the constant C_0 depends only on M and R if k_1 is large enough. Taking T small enough in (3.7.1) and using Cauchy-Schwarz, we have

$$\int_0^T \operatorname{Re}\langle \mathbf{QR}, \mathbf{u}\rangle dt \leq C_0 \|f\|_{Y^\sigma}^2 + \varepsilon^2 \|v\|_{X^\sigma}^2.$$

Putting the above estimates together, we obtain

$$\|\chi_{<2R}\mathbf{u}\|_{L_T^2 H_x^{\frac{1}{2}}} \leq C_2(\|v\|_{L_T^\infty H_x^\sigma} + \|f\|_{Y^\sigma}) + \varepsilon \|v\|_{X^\sigma}.$$

This establishes (3.7.2), which completes the proof of Proposition 3.7.1.

3.8 Proof of the main linear estimate

In this short section, we complete the proof of Theorem 3.5.1 by combining Proposition 3.6.1 and Proposition 3.7.1. First, note that by Lemma 3.5.3, Lemma 3.5.4 and Lemma 3.5.7, it suffices to establish for small enough T, the bound

$$\|v\|_{X^\sigma} \leq C(M, L)(\|v_0\|_{H^\sigma} + \|f\|_{Y^\sigma}), \quad \sigma \geq 0, \tag{3.8.1}$$

when v is a solution to (3.5.2) with $\widehat{v}$ supported at frequencies $\gtrsim 2^{k_1}$ for some arbitrarily large (but fixed) parameter k_1. Let $\varepsilon > 0$ be a small positive constant to be chosen. By Proposition 3.6.1 and Proposition 3.7.1, we have the initial estimate

$$\|v\|_{L_T^\infty H_x^\sigma} + \|v\|_{\mathcal{X}^\sigma} \leq C(M, L)(\|v_0\|_{H^\sigma} + \|f\|_{Y^\sigma}) + \varepsilon \|v\|_{X^\sigma}. \tag{3.8.2}$$

We would like to strengthen this bound by replacing the left-hand side of (3.8.2) with $\|v\|_{X^\sigma}$, which would suffice to complete the proof. For this, we require control of the slightly stronger (than the $L_T^\infty H_x^\sigma$) norm

$$\|v\|_{\mathcal{Z}^\sigma} := \left(\sum_{j \geq 0} 2^{2j\sigma} \|S_j v\|_{L_T^\infty L_x^2}^2 \right)^{\frac{1}{2}}.$$

Clearly, (3.8.1) will follow from (3.8.2) and the following lemma, for ε small enough (depending on M and L).

Lemma 3.8.1. Under the above assumptions, v satisfies the following estimate in the space $\mathcal{Z}^\sigma$:

$$\|v\|_{\mathcal{Z}^\sigma} \leq C(M, L)(\|v_0\|_{H^\sigma} + \|f\|_{Y^\sigma} + \|v\|_{\mathcal{X}^\sigma}) + \varepsilon \|v\|_{X^\sigma}.$$

Proof. We begin by defining $v_k := S_k v$ for each $k \geq 0$. We see that v_k satisfies the equation

$$\begin{cases} i\partial_t v_k + \partial_j T_{g^{ij}} \partial_i v_k + T_{b^j} \partial_j v_k + T_{\tilde{b}^j} \partial_j \overline{v}_k = S_k f + \mathcal{R}_k, \\ v_k(0) = S_k v_0, \end{cases} \tag{3.8.3}$$

where

$$\mathcal{R}_k := [T_{g^{ij}}, S_k]\partial_i \partial_j \tilde{S}_k v + [T_{\partial_j g^{ij}}, S_k]\partial_i \tilde{S}_k v + [T_{b^j}, S_k]\partial_j \tilde{S}_k v + [T_{\tilde{b}^j}, S_k]\partial_j \tilde{S}_k \overline{v}$$

and $\tilde{S}_k$ is a fattened version of the dyadic multiplier S_k. By dyadic summation and Proposition 3.6.1, the proof of the lemma will be concluded if we can show that

$$\|\mathcal{R}_k\|_{Y^\sigma} \leq C(M)\|\tilde{S}_k v\|_{\mathcal{X}^\sigma} + \varepsilon \|\tilde{S}_k v\|_{X^\sigma} \tag{3.8.4}$$

for some $\varepsilon > 0$ sufficiently small. This is an easy consequence of Proposition 3.2.10 for the latter three terms as we can estimate these in $L^1_T H^\sigma_x$ and take T small. To estimate the remaining term, we first observe that

$$[T_{g^{ij}}, S_k]\partial_i\partial_j \tilde{S}_k v = [T_{S_{<k} g^{ij}}, S_k]\partial_i\partial_j \tilde{S}_k v = [S_{<k} g^{ij}, S_k]\partial_i\partial_j \tilde{S}_k v + [T_{S_{<k} g^{ij}} - S_{<k} g^{ij}, S_k]\partial_i\partial_j \tilde{S}_k v.$$

The latter term above can be estimated easily in $L^1_T H^\sigma_x$ by the right-hand side of (3.8.4) by using paradifferential calculus and then by taking T small. For the remaining term, we use that

$$[S_{<k} g^{ij}, S_k]\partial_i\partial_j \tilde{S}_k v = 2^{-k} L(S_{<k}\nabla_x g^{ij}, \partial_i\partial_j \tilde{S}_k v),$$

where L is a translation invariant operator of the form

$$L(\phi_1, \phi_2)(x) = \int \phi_1(x+y)\phi_2(x+z)K(y,z)dydz, \quad \|K\|_{L^1} \lesssim 1.$$

See, for instance, [148]. As the spaces $l^1 X^{s_0}$ and $\mathcal{X}^\sigma$ are translation invariant (in that they admit translation invariant equivalent norms), it follows from Proposition 3.2.12 that we have

$$\|[S_{<k} g^{ij}, S_k]\partial_i\partial_j \tilde{S}_k v\|_{Y^\sigma} \leq C(M)\|\tilde{S}_k v\|_{\mathcal{X}^\sigma}.$$

This completes the proof of the lemma. □

3.9 Well-posedness for the nonlinear flow

Now, we proceed with the proof of Theorem 3.1.3. By differentiating (3.1.1), we obtain an equation for $(u, \nabla u)$ of the form (3.1.11). Therefore, it suffices to prove the second part of the theorem for (3.1.11). Given the key estimate and well-posedness in Theorem 3.5.1, the scheme for proving this follows a very similar path to [106, Section 7]. We only outline the main results and procedure here for the convenience of the reader, and refer to the corresponding parts of [106] where relevant. A fully detailed exposition of a simplified version of the scheme that we employ below can be found in [71].

The starting point is to rewrite the equation

$$\begin{cases} i\partial_t u + \partial_j g^{ij}(u,\overline{u})\partial_i u = F(u,\overline{u},\nabla u,\nabla\overline{u}), \\ u(0,x) = u_0(x), \end{cases} \tag{3.9.1}$$

in the paradifferential form

$$\begin{cases} i\partial_t u + \partial_j T_{g^{ij}}\partial_i u + T_{b^j}\partial_j u + T_{\tilde{b}^j}\partial_j \overline{u} = G(u,\overline{u},\nabla u,\nabla\overline{u}), \\ u(0,x) = u_0(x), \end{cases}$$

where

$$b := -\partial_{(\nabla u)}F, \quad \tilde{b} := -\partial_{(\nabla \overline{u})}F$$

and

$$G(u,\overline{u},\nabla u,\nabla\overline{u}) := (\partial_j T_{g^{ij}}\partial_i - \partial_j g^{ij}\partial_i)u + F(u,\overline{u},\nabla u,\nabla\overline{u}) + T_{b^j}\partial_j u + T_{\tilde{b}^j}\partial_j\overline{u}.$$

Existence of l^1X^s solutions to the nonlinear equation

Our first aim is to establish existence of l^1X^s solutions to the equation (3.1.11) for small time. This is given by the following proposition.

Proposition 3.9.1. Let $s > \frac{d}{2}+2$ and let $u_0 \in l^1H^s$ with $\|u_0\|_{l^1H^s} = M$. Suppose that $g(u_0)$ is a nontrapping, non-degenerate metric with parameters R_0 and L. Then there is $T_0 > 0$ depending on M, $L(R_0)$ and R_0 such that for every $T \le T_0$, there exists a solution $u \in l^1X^s$ to (3.1.11) such that

(i) (l^1X^s bound).

$$\|u\|_{l^1X^s} \le C(M,L)\|u_0\|_{l^1H^s}.$$

(ii) (Smallness outside B_{R_0}).

$$\|\chi_{>R_0}u\|_{l^1X^s} \le 2\varepsilon.$$

(iii) (Comparable nontrapping parameter).

$$L(u) \leq 2L(u_0).$$

As in Section 7 of [106], for each $n \geq 0$ we consider the following iteration scheme for the paradifferential form of the nonlinear equation:

$$\begin{cases} i\partial_t u^{n+1} + \partial_i T_{g^{ij}(u^n)}\partial_j u^{n+1} + T_{b^j(u^n)}\partial_j u^{n+1} + T_{\tilde{b}^j(u^n)}\partial_j \overline{u}^{n+1} = G(u^n), \\ u^{n+1}(0,x) = u_0(x), \end{cases} \tag{3.9.2}$$

with initialization $u^0 = 0$. Here, we are suppressing the dependence on derivatives of u^n and its complex conjugate in b^j, $\tilde{b}^j$ and G. It is clear that Proposition 3.9.1 will follow from our next proposition, which addresses the convergence and bounds for the iteration scheme.

Proposition 3.9.2. Let s, M, L, R_0, T_0 and u_0 be as in Proposition 3.9.1. Then there exists a constant $C(M,L)$ such that for every $n \geq 0$ there exists a solution u^n to (3.9.2) on $[0,T]$ such that

(i) (l^1X^s bound).
$$\|u^n\|_{l^1X^s} \leq C(M,L)\|u_0\|_{l^1H^s}.$$

(ii) (Smallness outside B_{R_0}).
$$\|\chi_{>R_0}u^n\|_{l^1X^s} \leq 2\varepsilon.$$

(iii) (Comparable nontrapping parameter).

$$L(u^n) \leq 2L(u_0).$$

Moreover, there is a function $u \in l^1X^s$ satisfying the same bounds as above such that u^n converges strongly to u in l^1X^σ for every $0 \leq \sigma < s$.

Remark 3.9.3. For simplicity of presentation, we have omitted the parameter $\frac{d}{2}+2 < s_0 < s$ used in [106, Section 7] from the statements of the results in this section. This parameter still needs to be taken into account in the (omitted) proofs to ensure that the bounds for the low-frequency coefficients g, b, and $\tilde{b}$ stay under control in each iteration.

The proof of the above proposition follows from a virtually identical line of reasoning as [106, Sections 7.1-7.3]. We simply use Proposition 3.4.5 and Theorem 3.5.1 in place of the analogues in their proof. We omit the details.

Uniqueness and the weak Lipschitz bound

In this subsection, we establish uniqueness of solutions in the class l^1X^s when $s > \frac{d}{2}+2$. In fact, our uniqueness result follows as a corollary of a weak Lipschitz type bound as noted in the following proposition.

Proposition 3.9.4. Let $s > \frac{d}{2}+2$ and let $u_0^1 \in l^1H^s$. Assume that $g(u_0^1)$ is a non-degenerate, nontrapping metric with parameters M, R_0 and L as above. Suppose that $u_0^2 \in l^1H^s$ is another initial datum satisfying

$$\|u_0^2\|_{l^1H^s} \lesssim M,$$

and suppose that u_0^2 is close to u_0^1 in the l^1L^2 topology in the sense that

$$\|u_0^1 - u_0^2\|_{l^1L^2} \ll_M e^{-C(M)L(R_0)}.$$

Then the following statements hold:

(i) $g(u_0^2)$ is nontrapping with comparable parameters to $g(u_0^1)$.

(ii) The solutions u^1 and u^2 generated by u_0^1 and u_0^2 exist on a time interval $[0,T]$ whose length depends only on the parameters M, R_0 and $L(R_0)$.

(iii) For $0 \leq \sigma < s_0 - 1$, we have the following weak Lipschitz type bound:

$$\|u^1 - u^2\|_{l^1X^\sigma} \leq C(M,L)\|u_0^1 - u_0^2\|_{l^1H^\sigma}.$$

Proof. The proof follows an identical line of reasoning as Section 7.4 in [106] except that we use Proposition 3.4.5 in place of Proposition 5.2 in [106] to prove (i). □

Frequency envelope bounds and continuous dependence

In this final subsection, our main objective is to establish continuous dependence for (3.9.1). More precisely, for $s > \frac{d}{2}+2$, we want to show that the data-to-solution map (given nontrapping data) $u_0 \mapsto u$ is continuous from l^1H^s to l^1X^s. As in [106], the main ingredient is the following frequency envelope bound for the solution $u \in l^1X^s$ in terms of the data.

Proposition 3.9.5. Let $u \in l^1X^s$ be a solution to (3.9.1) as in Proposition 3.9.1 with initial data $u_0 \in l^1H^s$. Let a_k be an admissible frequency envelope for u_0 in l^1H^s. Then the solution u satisfies the bound

$$\|S_ku\|_{l^1X^s} \leq a_kC(M,L)\|u_0\|_{l^1H^s}.$$

Proof. The proof follows identical reasoning as the proof of Proposition 7.5 in [106]. The only difference is that we use Corollary 3.5.2 in place of the analogous bound in their proof. □

Armed with Proposition 3.9.5, the proof of the continuity of the data-to-solution map in Section 7.6 of [106] now applies verbatim to establish the same property in our setting.

Chapter 4

Derivative nonlinear Schrödinger equations

4.1 Introduction

In this final chapter, we consider the *generalized derivative nonlinear Schrödinger equation*:

$$\begin{cases} i\partial_t u + \partial_x^2 u = i|u|^{2\sigma}\partial_x u, \\ u(0) = u_0, \end{cases} \tag{gDNLS}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ and $\sigma > 0$. We will be particularly interested in the case $\sigma < 1$, as this is where H^s local well-posedness is most difficult. We begin with a brief history of this family of equations, and some of its closely related analogues.

The (gDNLS) equations originate from the study of the so-called *derivative nonlinear Schrödinger equation*:

$$\begin{cases} i\partial_t u + \partial_x^2 u = i|u|^2\partial_x u, \\ u(0) = u_0, \end{cases} \tag{DNLS}$$

which corresponds to (gDNLS) with $\sigma = 1$. Physically, (DNLS) derives from the one-dimensional compressible magneto-hydrodynamic equation in the presence of the Hall effect, and the propagation of circular polarized nonlinear Alfvén waves in magnetized plasmas [113, 118, 125]. It also appears as a model for ultrashort optical pulses [1, 119], as well as in various other physical scenarios [22, 78, 135]. Mathematically, (DNLS) also has many interesting features. For example, like the 1D cubic NLS, it is completely integrable [83].

However, it scales like the 1D quintic NLS, which makes it L^2 critical. Moreover, although at first glance (DNLS) looks to be semilinear, it is known that uniform continuity of the solution map fails in H^s as long as $s < \frac{1}{2}$ (see [13, 143]). Therefore, this PDE has a clear quasilinear flavour.

In recent years, the (gDNLS) family of equations has seen increasing interest, stemming from the 2013 article of Liu, Simpson and Sulem [102]. One of the original motivations of [102] was to shed light on the global well-posedness of (DNLS) in the energy space H^1, which was an important open problem. However, in an interesting turn of events, Bahouri and Perelman [10] managed to prove global well-posedness for the (DNLS) equation *before* the global well-posedness of (gDNLS) could be established for any $\sigma \neq 1$. In this thesis we make progress towards resolving one half of the program of Liu, Simpson and Sulem by proving that (gDNLS) is globally well-posed in H^1 for $\sigma \in (\frac{\sqrt{3}}{2}, 1)$. Note that, although completed shortly after each other, our result for $\sigma < 1$ and the $\sigma = 1$ result of [10] are completely independent, and the methods used differ quite dramatically. Indeed, for $\sigma = 1$, local well-posedness in H^1 has been known for a long time [65], and can be established by employing a suitable gauge transformation, and standard Strichartz estimates. In fact, the smoothing properties of the equation are suitable to lower the well-posedness threshold to $H^{\frac{1}{2}}$ as in [144]. Global well-posedness, however, is considerably harder, as the problem is L^2 critical. For this reason, Bahouri and Perelman (as well as Harrop-Griffiths, Killip, Ntekoume and Vişan [59, 58, 91] in their subsequent work) crucially rely on the complete integrability of (DNLS). In the case $\sigma < 1$, the main difficulties are reversed. Establishing local well-posedness is difficult because of the lack of decay and roughness of the nonlinearity. On the other hand, one expects to be able to easily propagate any reasonable H^1 local well-posedness theory in time to obtain a global result. This is because when $\sigma < 1$ the problem becomes L^2 subcritical, and one expects to be able to use the conserved energy and mass of the problem to control the H^1 norm of a solution.

Another motivation for (gDNLS) is the rich family of soliton solutions, which is actually where the majority of [102]'s efforts were focused. Assuming a suitable H^1 well-posedness theory, the authors of [102] were able to use the abstract theory of Grillakis, Shatah and Strauss [49, 50] to investigate the orbital stability of the solitons. However, an H^1 well-posedness theory for $\sigma < 1$ had not been known until now.

When $\sigma < 1$, one can view (gDNLS) as a prototypical model for a quasilinear dispersive equation with a rough, low power nonlinearity (see [99] for a KdV analogue). Such nonlinearities in the context of semilinear NLS type equations are becoming increasingly well-understood [21, 154], and at modest regularity local well-posedness can usually be proven by a combination of regularization and perturbative arguments. However, the combination of derivative and low power coefficient in the nonlinearity of (gDNLS) causes many interesting technical issues, several of which are yet to be fully understood. One issue for low regularity well-posedness is that the coefficient $|u|^{2\sigma}$ in the nonlinearity is less than quadratic in order. Because of this, the smoothing properties of the linear part of the Schrödinger equation are seemingly not strong enough to directly compensate for the apparent derivative loss which occurs because of the u_x term in the nonlinearity. Another tool to avoid derivative loss - which has been successfully employed in the case $\sigma > 1$ in [57, 64] - is a gauge transformation. This technique allows one to re-normalize the equation to effectively remove the worst interactions in the derivative nonlinearity. However, again, it seems one can only directly apply this method when $\sigma \geq 1$ (i.e. $|u|^{2\sigma}$ is of quadratic order or higher), as in the case $\sigma < 1$ negative powers of $|u|$ eventually appear in the analysis. This is related to the roughness of the nonlinearity, and will be elaborated on further when we outline the proof of our results.

To contrast, the Benjamin-Ono equation,

$$\begin{cases} u_t + Hu_{xx} = uu_x, \\ u(0) = u_0, \end{cases} \tag{4.1.1}$$

has a similar low power derivative nonlinearity uu_x, and as with (gDNLS), the linear part of the equation does not have strong enough smoothing properties to directly compensate for the derivative loss in the nonlinearity. Nevertheless, H^1 well-posedness for this equation was established several years ago in [149]. One should note, however, that the Benjamin-Ono nonlinearity has a much nicer algebraic structure than that of (gDNLS) (it is smooth and multilinear), which makes the equation more amenable to normal form type techniques (such as cubic corrections or a gauge transformation). Moreover, Christ [26] showed that Schrödinger's equation with Benjamin-Ono's nonlinearity is ill-posed in any reasonable sense, so the analogies between these equations are at best heuristic. For (gDNLS), our solution to the above difficulties will be to introduce a family of *partial* gauge transformation adapted to each dyadic frequency scale and the corresponding paradifferential flow - which removes the portion of the nonlinearity which is large in a pointwise sense, on a scale which is balanced against the corresponding frequency localization scale of the nonlinearity. This will then be combined with smoothing and maximal function type arguments to attain the H^1_x

well-posedness threshold.

Another novel issue in the study of (gDNLS) is that the nonlinearity has only a finite degree of Hölder regularity, and so one does not expect to be able to construct smooth solutions from regular data. In our case, the nonlinearity is only $C^{1,2\sigma-1}$ Hölder continuous. We expect therefore to only be able to differentiate the equation with respect to some parameter "2σ times" to obtain estimates. To maximize the potential regularity of solutions, we note that the scaling of the Schrödinger equation suggests that we can convert L^2_x based estimates for one time derivative of a solution to estimates for two spatial derivatives. Therefore, it is advantageous to differentiate (gDNLS) in time rather than in space, and then convert time derivative estimates into estimates for spatial derivatives of a solution. After a single time differentiation, we are left with $2\sigma - 1$ degrees of regularity on the nonlinearity. By working with fractional space derivatives, one expects to be able to prove an energy estimate for the $H_x^{1+2\sigma}$ norm of a solution. However, working with fractional time derivatives (after suitably localizing in time), one expects to improve this further, and prove well-posedness in H^s_x up to $s = 2\cdot 1 + 2\cdot(2\sigma-1) = 4\sigma$. A similar heuristic argument applies to any dispersion generalized equation with rough nonlinearity, where one can convert time derivative estimates into estimates on a certain number of spatial derivatives, perhaps modulo some perturbative terms coming from the nonlinearity. In general, we expect this heuristic to give rather sharp results, but this is not even known for semilinear NLS equations with rough nonlinearities [21, 154], and is essentially unexplored in the quasilinear setting.

Finally, let u recall some basic symmetry properties of (gDNLS) as well as some conservation laws, which we will use to propagate our local well-posedness result to a global one. First, we have the scaling transformation

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda^{\frac{1}{2\sigma}} u(\lambda^2 t, \lambda x), \quad \lambda > 0,$$

which makes the critical Sobolev index $s_c = \frac{1}{2} - \frac{1}{2\sigma}$. In particular, the problem is L^2 subcritical when $\sigma < 1$. Moreover, (gDNLS) admits the following conserved quantities:

$$M(u) = \frac{1}{2}\int_{\mathbb{R}} |u|^2 dx, \tag{4.1.2}$$

$$P(u) = \frac{1}{2} Re \int_{\mathbb{R}} i\overline{u} u_x dx, \tag{4.1.3}$$

$$E(u) = \frac{1}{2}\int_{\mathbb{R}} |u_x|^2 dx + \frac{1}{2(\sigma+1)} Re \int_{\mathbb{R}} i|u|^{2\sigma}\overline{u} u_x dx, \tag{4.1.4}$$

which are the mass, momentum and energy, respectively. Unlike the standard NLS, (DNLS) doesn't enjoy the Galilean invariance nor the pseudo-conformal invariance symmetries, the latter being relevant for avoiding blowup. We also note that a simple change of variables allows us to change the sign of the nonlinearity in (gDNLS) and arrive at

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u = 0. \tag{4.1.5}$$

This latter equation is more common in the study of the solitary waves of (gDNLS).

Results

The main result of this chapter is global well-posedness of (gDNLS) in $H^s(\mathbb{R})$ when $\frac{\sqrt{3}}{2} < \sigma < 1$ and $s \in [1, 4\sigma)$. However, we divide this theorem into a "low-regularity" part and a "high-regularity" part, to maximize the range of σ. The high-regularity result is as follows:

Theorem 4.1.1. (High-Regularity) Let $\frac{1}{2} < \sigma < 1$ and let $2 - \sigma < s < 4\sigma$. Then (gDNLS) is locally well-posed in $H^s(\mathbb{R})$.

As mentioned, for a restricted range of σ, we can lower the well-posedness threshold down to H^1, where the conserved energy also gives global well-posedness:

Theorem 4.1.2. Let $\frac{\sqrt{3}}{2} < \sigma < 1$ and let $1 \leq s < 4\sigma$. Then (gDNLS) is globally well-posed in $H^s(\mathbb{R})$.

Remark 4.1.3. As a special case, Theorem 4.1.1 shows in particular that we have local well-posedness in H^s for $\frac{3}{2} \leq s \leq 2$. Therefore, we recover the only previously known local well-posedness results for (gDNLS) when $\sigma < 1$; namely, we recover the H^2 result of [64] and improve the result of [136], which used weighted Sobolev spaces.

Remark 4.1.4. In both Theorem 4.1.1 and Theorem 4.1.2, well-posedness is to be interpreted in the usual quasilinear fashion, including existence, uniqueness and continuous dependence on the data. More specifically, given an appropriate Sobolev index s and time $T > 0$, we first build a function space X_T^s that continuously embeds into $C([-T, T]; H_x^s)$. We then show that for each $u_0 \in H_x^s$ there exists a unique solution u to (gDNLS) that lies in X_T^s and satisfies $u(t = 0) = u_0$. Finally, we show that the data to solution map is continuous, even as a map from H_x^s to the stronger topology X_T^s.

Remark 4.1.5. Since (DNLS) is known to be globally well-posed in $H^{\frac{1}{2}}$, one may wonder why we only consider H^s well-posedness when $s \geq 1$. This is, in fact, not necessary. For each $\sigma \in (\frac{\sqrt{3}}{2}, 1)$, we expect that technical modifications of our proof should establish H^s well-posedness of (gDNLS) in a range $s \in [l(\sigma), 4\sigma)$ with $l(\sigma) < 1$ and $l(\sigma) \to \frac{1}{2}$ as $\sigma \to 1$. We avoid doing this for the sake of simplicity. It remains an open problem to prove well/ill-posedness in $H^{\frac{1}{2}}$ for any $\frac{1}{2} < \sigma < 1$, and to find the smallest $\sigma \in (0, 1)$ such that (gDNLS) is well-posed in H^1.

History on well-posedness and solitons

There is a vast literature devoted to the well-posedness of (DNLS), as it took several decades for the regularity to approach current thresholds, and for global results to emerge. We begin our review with the work of Tsutsumi and Fukuda [152, 153] who studied the well-posedness in $H^s(\mathbb{R})$ for $s > \frac{3}{2}$ by classical energy methods and parabolic regularization. The well-posedness in $H^1(\mathbb{R})$ was reached by Hayashi [65] by applying a gauge transformation to overcome the derivative loss, and Strichartz estimates to close a-priori estimates. The $H^1(\mathbb{R})$-solution was shown to be global by Hayashi and Ozawa [66], as long as the initial data satisfies $\|u_0\|_{L^2}^2 < 2\pi$. Later, Wu [162] improved this global result by relaxing the smallness condition to $\|u_0\|_{L^2}^2 < 4\pi$, which is natural in view of the soliton structure.

Below the energy space, there are also many results for (DNLS). Takaoka [144] proved local well-posedness in $H^s(\mathbb{R})$ when $s \geq \frac{1}{2}$ by the Fourier restriction method. This was complemented by a result of Biagioni and Linares [13] which notes that the solution map from $H^s(\mathbb{R})$ to $C([-T, T]; H^s(\mathbb{R}))$ cannot be locally uniformly continuous when $s < \frac{1}{2}$. By using the I-method, Colliander, Keel, Staffilani, Takaoka and Tao [31, 30] proved that the $H^s(\mathbb{R})$-solution is global if $s > \frac{1}{2}$ and $\|u_0\|_{L^2}^2 < 2\pi$. Guo and Wu [54] were later able to strengthen this result by proving that $H^{\frac{1}{2}}(\mathbb{R})$-solutions are global if $\|u_0\|_{L^2}^2 < 4\pi$. For an incomplete list of well-posedness results for (DNLS) on the torus, see [63, 121] and references therein.

There are also many works that use the complete integrability of the (DNLS) equation. The breakthrough result is [10], which establishes global well-posedness in $H^{\frac{1}{2}}(\mathbb{R})$. However, [10] was preceded by many results - see, e.g., [77, 126, 127] - highlights of which include a global well-posedness result in the weighted Sobolev space $H^{2,2}(\mathbb{R})$, and progress towards the soliton resolution conjecture. Moreover, although $H^{\frac{1}{2}}$ regularity is necessary for uniform continuity of the solution map, [59, 58, 91] are able to lower the global well-posedness threshold all the way to the critical Sobolev space L^2, definitively resolving the well-posedness theory for (DNLS) on the line. On the other hand, blowup for (DNLS) on non-standard domains (for example, the half-line with the Dirichlet boundary condition) is known to be possible [146, 161].

For (gDNLS), the literature on well-posedness is also quite large, though the results are far less definitive. As mentioned, (gDNLS) was popularized by [102], though well-posedness was not considered in that article. Possibly the first well-posedness result was by Hao, who in [57] was able to prove local well-posedness in $H^{\frac{1}{2}}(\mathbb{R})$ intersected with an appropriate Strichartz space for $\sigma \geq \frac{5}{2}$. Ambrose and Simpson [8] proved the existence and uniqueness of solutions $u \in C([0,T); H^2(\mathbb{T}))$ and the existence of solutions $u \in L^\infty([0,T), H^1(\mathbb{T}))$ for $\sigma \geq 1$. The uniqueness of $H^1(\mathbb{T})$-solutions was left unresolved, as the proof uses a compactness argument. Existence and uniqueness in $H^{\frac{1}{2}}(\mathbb{R})$ was proved by Santos in [136] for $\sigma > 1$, by utilizing global smoothing and maximal function estimates. A result in weighted Sobolev spaces was also proved in [136] for the case $\frac{1}{2} < \sigma < 1$, as adding weights helps compensate for the low power in the nonlinearity. In terms of $H^s(\mathbb{R})$ spaces, [64] proves local well-posedness in H^2 when $\sigma \geq \frac{1}{2}$, local well-posedness in H^1 when $\sigma \geq 1$, existence of weak solutions when $\sigma < 1$, and certain unconditional uniqueness results at high regularity. See [120] for more on unconditional uniqueness. The (gDNLS) equation with extremely rough nonlinearities $0 < \sigma < \frac{1}{2}$ is studied in [98, 100], but not in standard Sobolev spaces H^s.

We now turn to the history on stability of solitons. This is also a vast subject, and (gDNLS) is not the only generalization of (DNLS) whose solitons have been considered. For the sake of unification, therefore, let us consider the equation

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u + b|u|^{4\sigma}u = 0, \ x \in \mathbb{R}, \tag{4.1.6}$$

which is just a Schrödinger equation with a scale-invariant combination of derivative and power nonlinearities. Direct calculation verifies that the soliton solutions of (4.1.6) are given by

$$u_{\omega,c}(t,x) = e^{i\omega t}\phi_{\omega,c}(x - ct)$$

where

$$\phi_{\omega,c}(x) = \Phi_{\omega,c}(x)e^{i\theta_{\omega,c}(x)}, \qquad \theta_{\omega,c}(x) = \frac{c}{2}x - \frac{1}{2\sigma+2}\int_{-\infty}^{x} \Phi_{\omega,c}(y)^{2\sigma}dy,$$

and, using the notation $\gamma = 1 + \frac{(2\sigma+2)^2}{2\sigma+1}b$, the real-valued function $\Phi_{\omega,c}$ is given by

$$\Phi_{\omega,c}(x)^{2\sigma} = \begin{cases} \dfrac{(\sigma+1)(4\omega - c^2)}{\sqrt{c^2+\gamma(4\omega-c^2)}\cosh\left(\sigma\sqrt{4\omega-c^2}x\right) - c} & \gamma > 0, \quad -2\sqrt{\omega} < c < 2\sqrt{\omega}, \\ \dfrac{2(\sigma+1)c}{(\sigma c x)^2+\gamma} & \gamma > 0, \qquad c = 2\sqrt{\omega}, \\ \dfrac{(\sigma+1)(4\omega - c^2)}{\sqrt{c^2+\gamma(4\omega-c^2)}\cosh\left(\sigma\sqrt{4\omega-c^2}x\right) - c} & \gamma \leq 0, \\ & -2\sqrt{\omega} < c < -2\sqrt{-\gamma/(1-\gamma)}\sqrt{\omega}. \end{cases}$$

These solitons are, of course, related to the Hamiltonian structure of (4.1.6), as well as to the conservation of mass, energy and momentum, which we leave to the reader to compute.

As expected, the story on soliton stability for (4.1.6) begins with the (DNLS) equation. Indeed, in [51], Guo and Wu proved that the soliton solutions of (DNLS) are orbitally stable when $\omega > \frac{c^2}{4}$ and $c < 0$ by applying the abstract theory of Grillakis, Shatah, and Strauss [49, 50]. Colin and Ohta [29] removed the condition $c < 0$ and proved that $u_{\omega,c}$ is orbitally stable when $\omega > \frac{c^2}{4}$ by applying the variational characterization of solitons as in Shatah [138]. The endpoint case $c = 2\sqrt{\omega}$ is only partially resolved; progress was made by Kwon and Wu in [94], but with certain caveats, such as a non-standard definition of orbital stability. For the study of periodic travelling waves, we refer to [23, 56, 60, 63] and references therein.

For (gDNLS), the story on soliton stability is much richer. In [102] it was shown that the solitary waves $u_{\omega,c}$ are orbitally stable if $-2\sqrt{\omega} < c < 2z_0\sqrt{\omega}$, and orbitally unstable if $2z_0\sqrt{\omega} < c < 2\sqrt{\omega}$ when $1 < \sigma < 2$. Here the constant $z_0 = z_0(\sigma) \in (-1, 1)$ is the solution to

$$F_\sigma(z) := (\sigma-1)^2\left(\int_0^\infty (\cosh y - z)^{-\frac{1}{\sigma}}dy\right)^2 - \left(\int_0^\infty (\cosh y - z)^{-\frac{1}{\sigma}-1}(z\cosh y - 1)dy\right)^2 = 0.$$

Moreover, [102] proves that all solitary waves with $\omega > \frac{c^2}{4}$ are orbitally unstable when $\sigma \geq 2$ and orbitally stable when $0 < \sigma < 1$. As mentioned previously, these results are conditional on an appropriate well-posedness theory; there is also a minor numerical portion to the proof. In the borderline case when $c = 2z_0\sqrt{\omega}$ and $1 < \sigma < 2$, Fukaya ([44], see also [53]) proved orbital instability of the solitons. This completes the study of orbital stability of the solitons of (gDNLS), except in the case of the algebraic soliton, which requires special attention [52, 97].

In the case $\sigma = 1$, $b \neq 0$, there are also many works on soliton stability for (4.1.6), e.g. [29, 45, 63, 62, 61, 122, 123, 124]. On the other hand, there are no results in the case $\sigma \neq 1$, $b \neq 0$, as it seems the explicit formulas for the solitons were not previously known. We also mention that from the point of view of low regularity well-posedness, the additional term $b|u|^{4\sigma}u$ in (4.1.6) is both perturbative and maintains scaling, so in our usual range $\frac{\sqrt{3}}{2} < \sigma < 1$ our proof can easily be modified to establish global well-posedness in H^1, regardless of the size or sign of b. To contrast, recall that the known proof of global well-posedness in the case $\sigma = 1$, $b = 0$ is rather delicate; global well-posedness could, in principle, fail to persist once the effect of the focusing NLS is added. For state of the art global results when $\sigma = 1$, $b \neq 0$ we mention [63], which establishes global well-posedness below the soliton thresholds. In particular, (4.1.6) in the case $\sigma = 1$, $b \leq -\frac{3}{16}$ has been known to be globally well-posed for some time now, as at this point the energy becomes coercive, after a suitable gauge transformation.

Outline of the proofs

Here, we outline the key ideas in the proof of Theorem 4.1.1 and Theorem 4.1.2. We begin with a discussion of the low-regularity argument. Before describing the proof, however, it is instructive to discuss why the gauge transformation used in [64] combined with standard Strichartz estimates will not work. The following discussion is mostly heuristic and for the purpose of motivation only.

Firstly, by a standard energy estimate, one obtains for (regular enough) solutions to (gDNLS),

$$\|u\|_{L_T^\infty H_x^1} \lesssim \|u_0\|_{H_x^1} \exp\left(\int_0^T \|u\|_{L_x^\infty}^{2\sigma-1} \|u_x\|_{L_x^\infty}\right). \tag{4.1.7}$$

Therefore, one expects to be able to prove suitable H^1 bounds for solutions to (gDNLS) as long as one can estimate the Strichartz norm, $\|u_x\|_{L_T^1 L_x^\infty}$. However, applying Strichartz estimates directly to (gDNLS) leads to a loss of a derivative. Therefore, one might naïvely try to do some sort of gauge transformation to remove the $|u|^{2\sigma}u_x$ term in the equation, which is responsible for this loss. Indeed, if one (formally) defines

$$\Phi(t,x) = -\frac{1}{2}\int_{-\infty}^{x} |u|^{2\sigma} dy \tag{4.1.8}$$

and then

$$w = e^{i\Phi}u, \tag{4.1.9}$$

this leads to an equation for w of the form

$$iw_t + \partial_x^2 w = (-\partial_t\Phi + i\partial_x^2\Phi - (\partial_x\Phi)^2)w. \tag{4.1.10}$$

At first glance, it looks like one can prove Strichartz estimates for w_x without losing derivatives, to obtain the corresponding bound for $\|u_x\|_{L_T^1 L_x^\infty}$. Unfortunately, if we expand $\partial_t\Phi$, we get

$$\begin{aligned}\partial_t\Phi &= -\sigma\int_{-\infty}^x \mathrm{Re}(|u|^{2\sigma-2}\overline{u}u_t)dy\\ &= -\sigma\int_{-\infty}^x \mathrm{Re}(|u|^{2\sigma-2}\overline{u}i\partial_x^2 u)dy - \sigma\int_{-\infty}^x \mathrm{Re}(|u|^{4\sigma-2}\overline{u}u_x)dy.\end{aligned} \tag{4.1.11}$$

The first term above is problematic. To avoid losing derivatives, we are forced to integrate by parts off one derivative. However, since $|u|^{2\sigma-2}\overline{u}$ is not C^1 when $\sigma < 1$, this will inevitably introduce negative powers of u, so this approach will not work.

While the above calculations are not particularly useful for closing low-regularity estimates, they do clearly identify the main enemies in trying to close Strichartz estimates for the gauge transformed equation. That is, the portion of u which is small or vanishes will prevent us from closing Strichartz estimates for w. Therefore, it is natural to try to somehow perform a gauge transformation which only removes some portion of the derivative nonlinearity $|u|^{2\sigma}u_x$, which corresponds to a part of u for which u is bounded away from zero. Doing this is somewhat subtle. We can't simply fix a universal constant $\varepsilon > 0$, and remove the portion of the nonlinearity for which $|u| > \varepsilon$. This is because when the u_x factor in $|u|^{2\sigma}u_x$ is at very high frequency (compared to ε), we will still lose derivatives in the Strichartz estimate. To work around this issue, we perform a paradifferential expansion of the equation. That is, for each $j > 0$, we project onto frequencies of size $\sim 2^j$ and obtain

$$(i\partial_t + \partial_x^2)P_j u = iP_{<j-4}|u|^{2\sigma}P_j u_x + g_j \tag{4.1.12}$$

where g_j is a perturbative term. The idea now is to split the coefficient $P_{<j-4}|u|^{2\sigma} = P_{<j-4}|u_s|^{2\sigma} + P_{<j-4}|u_l|^{2\sigma}$, where u_l corresponds to the portion of u which is bounded away from zero (where the lower bound depends on the frequency parameter j), and u_s is the remaining portion of u which is bounded above by some small j dependent parameter. We then try to do a gauge transformation by defining

$$\Phi_j = -\frac{1}{2}\int_{-\infty}^x P_{<j-4}|u_l|^{2\sigma}dy \tag{4.1.13}$$

and

$$w_j = e^{i\Phi_j} P_j u. \tag{4.1.14}$$

This leads to an equation for w_j of the form,

$$(i\partial_t + \partial_x^2)w_j = (-\partial_t\Phi_j + i\partial_x^2\Phi_j - (\partial_x\Phi_j)^2)w_j + e^{i\Phi_j}g_j + ie^{i\Phi_j}P_{<j-4}|u_s|^{2\sigma}P_j u_x. \tag{4.1.15}$$

The point now is that the negative powers of u that arise in the $\partial_t\Phi_j$ term are bounded above by some parameter depending on the frequency scale 2^j. To avoid derivative loss, we would like this parameter to be as small as possible (i.e. u_l should be bounded below by a (j dependent) constant which is as large as possible). However, we still have to contend with the remainder of the original derivative nonlinearity, $ie^{i\Phi_j}P_{<j-4}|u_s|^{2\sigma}P_j u_x$, which is expected to cause derivative loss unless u_s is sufficiently small (depending on j). Therefore, we have to compromise between potential losses incurred by the $\partial_t\Phi_j$ term, and the remaining derivative nonlinearity. Unfortunately, by optimizing the appropriate splitting of u, it turns out that we will still lose $1-\sigma$ derivatives in estimating the Strichartz norm $\|u_x\|_{L^1_T L^\infty_x}$, and therefore, one only expects to be able to control $\|u_x\|_{L^1_T L^\infty_x}$ by $\|u\|_{L^\infty_T H^{2-\sigma}_x}$. As mentioned, while this is certainly an improvement over previous results [64, 136], this method is not quite robust enough to get well-posedness down to the energy space.

To get H^1 well-posedness, we combine this modified gauge transformation (and Strichartz estimates) with smoothing and maximal function type estimates, as in Propositions 4.2.3 and 4.2.4. However, we modify these Strichartz and maximal function norms (see the definition of Y^s_T below) to reflect the loss of $1-\sigma$ derivatives compared to the $L^\infty_T H^1_x$ norm, as mentioned above. That is, we build this deficiency into the function spaces where we construct solutions. In particular, the Strichartz ($L^1_T L^\infty_x$) component of the norm involves no more than σ derivatives. Therefore, the energy estimate (4.1.7) described above is no longer appropriate to close a priori estimates in H^1. Hence, the energy estimate has to be modified accordingly so that the control parameter (i.e. the Strichartz component) does not lead to a loss of derivatives (in excess of the H^1 norm) in the Strichartz/maximal function component of the estimate. It is actually this part of the argument that leads to the restriction on σ, which we will elaborate on later.

Next, we outline the proof of the high regularity well-posedness. As mentioned previously, the $C^{1,2\sigma-1}$ Hölder regularity of the function $z \mapsto |z|^{2\sigma}$ effectively limits the number of times one can differentiate the equation to obtain H^s estimates. A direct energy estimate, which involves differentiating the equation s times in the spatial variable (i.e. applying D_x^s to the equation) limits the range for which one can obtain estimates to $s \leq 2\sigma$. In [64], the authors managed to bypass this issue in the case $s = 2$ by instead obtaining an L_x^2 energy estimate for the time derivative $\partial_t u$. The point is that doing this only requires one to differentiate the nonlinearity a single time. Once an appropriate L_x^2 estimate is obtained, H_x^2 energy estimates for the solution can then be obtained by observing that up to an error of size $O(\|u\|_{L_T^\infty H_x^1}^{2\sigma+1})$, the equation gives,

$$\|(\partial_x^2 u)(t)\|_{L_x^2} \sim \|(\partial_t u)(t)\|_{L_x^2}. \tag{4.1.16}$$

In this thesis, we generalize this approach to derivatives of fractional order. It turns out that (after suitably localizing a solution in time), one can morally obtain an estimate (up to a suitable error term) essentially of the form

$$\|D_t^{\frac{s}{2}} u\|_{L_T^\infty L_x^2} \sim \|D_x^s u\|_{L_T^\infty L_x^2} \tag{4.1.17}$$

where $1 < s < 4\sigma$. The main idea for proving this estimate is a modulation type analysis. Namely, when the space-time Fourier transform of a solution u (after suitably localizing in time) is supported close to the characteristic hypersurface (or in the low modulation region), $\tau = -\xi^2$, one expects to be able to directly compare $D_t^{\frac{s}{2}} u$ and $D_x^s u$. On the other hand, when the space-time Fourier transform is supported far away from the hypersurface (or in the high modulation region), one expects to be able to control $D_t^{\frac{s}{2}} u$ and $D_x^s u$ in L_x^2 by a lower order error term stemming from the nonlinearity of the equation. This latter high modulation control can be loosely thought of as a space-time elliptic estimate.

With a method for suitably comparing space and time derivatives of a solution in hand, it then essentially suffices to obtain an energy estimate for $D_t^{\frac{s}{2}} u$ when u is localized near the characteristic hypersurface (which is precisely where one expects to be able to compare $D_t^{\frac{s}{2}} u$ to $D_x^s u$). Therefore, in light of the $C^{1,2\sigma-1}$ regularity of the nonlinearity, we should be able to obtain H_x^s estimates for a solution as long as $\frac{s}{2} < 2\sigma$. This explains the upper threshold of 4σ for our result. As hinted at earlier, the lower threshold of $2 - \sigma$ is explained by the fact that such an energy estimate closes as long as one can control $\|u_x\|_{L_T^1 L_x^\infty}$. Our low regularity estimates allow us to control this term by the $L_T^\infty H_x^s$ norm of u, as long as $s > 2 - \sigma$, where σ lies in the full range $(\frac{1}{2}, 1)$. This should be contrasted with the H^1 case where we employ a more complicated functional setting and only deal with a restricted range of σ. For clarity, we have chosen to present our high regularity results in the simplest possible functional setting, which is why the lower bound of $2 - \sigma$ appears in Theorem 4.1.1, as it comes naturally from our previous estimates. Since $2-\sigma<\frac{3}{2}$ when $\sigma>\frac{1}{2}$, this is a reasonable lower threshold for the high regularity result (as it encompasses the range for which $\|u_x\|_{L^1_T L^\infty_x}$ can be controlled by Sobolev embedding). Nonetheless, we emphasize that the main novelty in Theorem 4.1.1 is the upper threshold $s<4\sigma$.

4.2 Preliminaries

In this section we settle notation and recall some standard tools.

Littlewood-Paley decomposition

First, we recall the standard Littlewood-Paley decomposition. For this, let ϕ_0 be a radial function in $C_0^\infty(\mathbb{R})$ that satisfies

$$0 \leq \phi_0 \leq 1, \quad \phi_0(\xi)=1 \text{ for } |\xi|\leq 1, \quad \phi_0(\xi)=0 \text{ for } |\xi|\geq \frac{7}{6}.$$

Let $\phi(\xi):=\phi_0(\xi)-\phi_0(2\xi)$. For $j\in\mathbb{Z}$, define

$$\widehat{P_{\leq j} f}(\xi)=\phi_0(2^{-j}\xi)\widehat{f}(\xi),$$
$$\widehat{P_j f}(\xi)=\phi(2^{-j}\xi)\widehat{f}(\xi).$$

We will denote $P_{>j}=I-P_{\leq j}$, where I is the identity. Similarly, we define $P_{[a,b]}=\sum_{a\leq j\leq b}P_j$. We will also use the notation $\tilde{P}_j$, $\tilde{P}_{<j}$, $\tilde{P}_{>j}$ to denote a slightly enlarged or shrunken frequency localization. For example, we may denote $P_{[j-3,j+3]}$ by $\tilde{P}_j$.

Next, we recall a useful bookkeeping device. Following [73, 147], we denote by $L(\phi_1,\dots,\phi_n)$ a translation invariant expression of the form

$$L(\phi_1,\dots,\phi_n)(x)=\int K(y)\phi_1(x+y_1)\cdots\phi_n(x+y_n)dy,$$

where $K\in L^1$. Of interest is the following Leibniz type rule from [73, 147] which will make certain commutator expressions simpler to estimate:

Lemma 4.2.1. (Leibniz rule for P_j). We have the commutator identity

$$[P_j,f]g=L(\partial_x f,2^{-j}g). \tag{4.2.1}$$

Frequency envelopes

One way we will employ the Littlewood-Paley projections is to define frequency envelopes, which are another nice bookkeeping device introduced by Tao [147]. To define these, suppose we are given a Sobolev type space X such that

$$\|P_{\leq 0}u\|_X^2 + \sum_{j=1}^{\infty} \|P_j u\|_X^2 \sim \|u\|_X^2. \tag{4.2.2}$$

A frequency envelope for u in X is a positive sequence $(a_j)_{j\in\mathbb{N}_0}$ such that

$$\|P_{\leq 0}u\|_X \lesssim a_0\|u\|_X, \quad \|P_j u\|_X \lesssim a_j\|u\|_X, \quad \sum_{j=0}^{\infty} a_j^2 \lesssim 1. \tag{4.2.3}$$

We say that a frequency envelope is admissible if $a_0 \approx 1$ and it is slowly varying, meaning that

$$a_j \leq 2^{\delta|j-k|}a_k, \quad j,k \geq 0, \quad 0 < \delta \ll 1.$$

An admissible frequency envelope always exists, say by

$$a_j = 2^{-\delta j} + \|u\|_X^{-1} \max_{k\geq 0} 2^{-\delta|j-k|}\|P_k u\|_X. \tag{4.2.4}$$

In (4.2.4) - and in the definitions of the X_T^s and H_x^s frequency envelope formulas defined later - there is a slight notational conflict, and $P_0 u$ should really be interpreted as $P_{\leq 0}u$.

Remark 4.2.2. Frequency envelopes will be particularly convenient for expediting the proof of continuous dependence later on.

Strichartz and maximal function estimates

Next we recall some standard linear estimates for the Schrödinger equation on the line, which will play a key role in our analysis. We begin with the relevant maximal function and Strichartz estimates for the linear Schrödinger flow:

Proposition 4.2.3. (Homogeneous Strichartz and maximal function estimates) For $v \in \mathcal{S}(\mathbb{R})$, $\theta \in [0,1]$ and $T \in (0,1)$, we have for $j > 0$

$$\begin{aligned} \|e^{it\partial_x^2}v\|_{L_T^{\frac{4}{\theta}}L_x^{\frac{2}{1-\theta}}} &\lesssim \|v\|_{L^2}, \\ \|e^{it\partial_x^2}P_j v\|_{L_x^{\frac{2}{1-\theta}}L_T^{\frac{2}{\theta}}} &\lesssim 2^{j(\frac{1}{2}-\theta)}\|v\|_{L^2}. \end{aligned} \tag{4.2.5}$$

Proof. See [89, Lemma 3.1]. □

We will also need the inhomogeneous versions of these estimates. Here $D_x^s := |\partial_x|^s$, $\langle D_x\rangle^s := (1+|\partial_x|^2)^{\frac{s}{2}}$, and $|\partial_x| := H\partial_x$ where H is the Hilbert transform, $\widehat{Hu} = -i\text{sgn}(\xi)\widehat{u}$. We further note that both Propositions 4.2.3 and 4.2.4 hold for $j=0$, with the interpretation $P_0 = P_{\leq 0}$.

Proposition 4.2.4. (Inhomogeneous Strichartz and maximal function estimates) For $f \in \mathcal{S}(\mathbb{R}^2)$, $\theta \in [0,1]$ and $T \in (0,1)$, we have for $j > 0$

$$\begin{aligned}
\left\| \int_0^t e^{i(t-s)\partial_x^2} f(s)ds \right\|_{L_T^{\frac{4}{\theta}} L_x^{\frac{2}{1-\theta}}} &\lesssim \|f\|_{L_T^{(\frac{4}{\theta})'} L_x^{(\frac{2}{1-\theta})'}}, \\
\left\| \langle D_x\rangle^{\frac{\theta}{2}} \int_0^t e^{i(t-s)\partial_x^2} f(s)ds \right\|_{L_T^\infty L_x^2} &\lesssim \|f\|_{L_x^{p(\theta)} L_T^{q(\theta)}}, \\
\left\| D_x^{\frac{1+\theta}{2}} \int_0^t e^{i(t-s)\partial_x^2} f(s)ds \right\|_{L_x^\infty L_T^2} &\lesssim \|f\|_{L_x^{p(\theta)} L_T^{q(\theta)}}, \\
\left\| \langle D_x\rangle^{\frac{\theta}{2}} \int_0^t e^{i(t-s)\partial_x^2} P_j f(s)ds \right\|_{L_x^2 L_T^\infty} &\lesssim 2^{\frac{j}{2}} \|f\|_{L_x^{p(\theta)} L_T^{q(\theta)}}, \\
\left\| \int_0^t e^{i(t-s)\partial_x^2} P_j f(s)ds \right\|_{L_x^{\frac{2}{1-\theta}} L_T^{\frac{2}{\theta}}} &\lesssim 2^{j(\frac{1}{2}-\theta)} \|f\|_{L_T^1 L_x^2},
\end{aligned} \tag{4.2.6}$$

where

$$\frac{1}{p(\theta)} = \frac{3+\theta}{4}, \qquad \frac{1}{q(\theta)} = \frac{3-\theta}{4}. \tag{4.2.7}$$

Proof. See [89, Lemma 3.4 and Remark 3.7]. □

The following fractional Leibniz rules will also be useful for some of the following estimates:

Proposition 4.2.5. Let $\alpha \in (0,1)$, $\alpha_1, \alpha_2 \in [0,\alpha]$, $p, p_1, p_2, q, q_1, q_2 \in (1,\infty)$ satisfy $\alpha_1+\alpha_2 = \alpha$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then

$$\|D_x^\alpha(fg) - D_x^\alpha f g - f D_x^\alpha g\|_{L_x^p L_T^q} \lesssim \|D_x^{\alpha_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2} L_T^{q_2}}. \tag{4.2.8}$$

The endpoint cases $q_1 = \infty, \alpha_1 = 0$ as well as $(p,q) = (1,2)$ are also allowed.

Proof. See [85, Lemma 2.6] or [89, Lemma 3.8]. □

Another variant of the fractional Leibniz rule for L_x^p spaces is as follows:

Proposition 4.2.6. Let $\alpha \in (0,1)$, $\alpha_1, \alpha_2 \in (0, \alpha)$ and $p \in [1, \infty)$, $1 < p_1, p_2 < \infty$ satisfy $\alpha_1 + \alpha_2 = \alpha$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then

$$\|D_x^\alpha (fg) - D_x^\alpha f g - f D_x^\alpha g\|_{L_x^p} \lesssim \|D_x^{\alpha_1} f\|_{L_x^{p_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2}}. \tag{4.2.9}$$

The endpoint case $\alpha_2 = 0$, $1 < p_2 \leq \infty$ is also allowed if $p > 1$.

Proof. See [85, Lemma 2.6]. □

Next, we need a vector-valued Moser type estimate which will be convenient when derivatives fall on $|u|^{2\sigma}$.

Proposition 4.2.7. Let $F \in C^1(\mathbb{C})$. Let $\alpha \in (0,1)$, $p, q, p_1, p_2, q_2 \in (1, \infty)$ and $q_1 \in (1, \infty]$ with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}. \tag{4.2.10}$$

Then

$$\|D_x^\alpha F(u)\|_{L_x^p L_T^q} \lesssim \|F'(u)\|_{L_x^{p_1} L_T^{q_1}} \|D_x^\alpha u\|_{L_x^{p_2} L_T^{q_2}}. \tag{4.2.11}$$

Proof. See Theorem A.6 of [88]. □

We also recall the scalar version of the above estimate,

Proposition 4.2.8. Let $F \in C^1(\mathbb{C})$, $u \in L^\infty(\mathbb{R})$, $\alpha \in (0,1)$, $1 < p, q, r < \infty$, and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then

$$\|D_x^\alpha F(u)\|_{L^r} \lesssim \|F'(u)\|_{L^p} \|D_x^\alpha u\|_{L^q}. \tag{4.2.12}$$

Proof. See [25], Proposition 3.1. □

We will also make use of not only the standard Bernstein estimates (see, for example, [150, (A.2)-(A.6), page 333]) but the following vector-valued version:

Proposition 4.2.9. Let $1 \leq p, q \leq \infty$, $j > 0$ and $s \in \mathbb{R}$. Then we have

$$\|D_x^s P_j u\|_{L_x^p L_T^q} \sim 2^{js} \|P_j u\|_{L_x^p L_T^q}. \tag{4.2.13}$$

Proof. Let $\tilde{P}_j$ have corresponding multiplier $\tilde{\phi}_j$, where, as in the preliminaries on Littlewood-Paley theory, we have $\tilde{\phi}_j(\xi) = \tilde{\phi}(2^{-j}\xi)$. Notice that

$$D_x^s(\tilde{P}_j P_j u) = (D_x^s \mathcal{F}^{-1} \tilde{\phi}_j) * P_j u.$$

For each x, we have the inequality

$$\|D_x^s P_j u\|_{L_T^q} \leq |D_x^s \mathcal{F}^{-1}\tilde{\phi}_j| * \|P_j u\|_{L_T^q}.$$

Hence, applying L_x^p and Young's inequality, we have

$$\|D_x^s P_j u\|_{L_x^p L_T^q} \leq \|D_x^s \mathcal{F}^{-1}\tilde{\phi}_j\|_{L_x^1} \|P_j u\|_{L_x^p L_T^q} \lesssim 2^{js}\|P_j u\|_{L_x^p L_T^q}.$$

On the other hand,

$$2^{js}\|P_j u\|_{L_x^p L_T^q} = 2^{js}\|D_x^{-s} D_x^s P_j u\|_{L_x^p L_T^q} \lesssim \|D_x^s P_j u\|_{L_x^p L_T^q}.$$

□

A useful lemma

Finally, we need a Hölder estimate, which we will use to extract all of the $C^{1,2\sigma-1}$-regularity that our nonlinearity offers. We will use this lemma, e.g., when derivatives fall on $|u|^{2\sigma-2}\overline{u}$, or more generally on terms with regularity $C^{0,\alpha}$ for $0 < \alpha < 1$.

To set notation, for $\alpha \in (0,1]$ and $1 \leq p \leq \infty$ define the Hölder space $\dot{\Lambda}_\alpha^p(\mathbb{R})$ by

$$\|u\|_{\dot{\Lambda}_\alpha^p} := \sup_{|h|>0} \frac{\|u(\cdot + h) - u(\cdot)\|_{L^p}}{|h|^\alpha}. \tag{4.2.14}$$

This is just the usual homogeneous Hölder space $\dot{C}^{0,\alpha}$ when $p = \infty$.

Lemma 4.2.10. Suppose that $F \in \dot{C}^{0,\alpha}(\mathbb{C})$. Then for every $0 < \beta < \alpha < 1$, $p \in [1,\infty]$ with $\alpha p \geq 1$, we have

$$\|F(u)\|_{\dot{\Lambda}_\beta^p} \lesssim \|F\|_{\dot{C}^{0,\alpha}} \|u\|_{W^{\frac{\beta}{\alpha},p\alpha}}^\alpha. \tag{4.2.15}$$

Proof. We have

$$\begin{aligned} \frac{|F(u(x+h)) - F(u(x))|}{|h|^\beta} &= \frac{|F(u(x+h)) - F(u(x))|}{|u(x+h) - u(x)|^\alpha} \left(\frac{|u(x+h) - u(x)|}{|h|^{\frac{\beta}{\alpha}}} \right)^\alpha \\ &\leq \|F\|_{\dot{C}^{0,\alpha}} \left(\frac{|u(x+h) - u(x)|}{|h|^{\frac{\beta}{\alpha}}} \right)^\alpha. \end{aligned} \tag{4.2.16}$$

Hence,

$$\begin{aligned}\|F(u)\|_{\dot\Lambda^p_\beta} &\le \|F\|_{\dot C^{0,\alpha}} \sup_{|h|>0} \| \left(\frac{|u(x+h)-u(x)|}{|h|^{\frac{\beta}{\alpha}}} \right)^\alpha \|_{L^p} \\ &\le \|F\|_{\dot C^{0,\alpha}} \|u\|^\alpha_{\dot\Lambda^{p\alpha}_{\frac{\beta}{\alpha}}} \\ &\lesssim \|F\|_{\dot C^{0,\alpha}} \|u\|^\alpha_{W^{\frac{\beta}{\alpha},p\alpha}}\end{aligned} \tag{4.2.17}$$

where the last line follows from a standard embedding (c.f. [150, Exercise A.21]). □

We also have the following very useful corollary of the above lemma which we will use extensively.

Corollary 4.2.11. Suppose that $F \in \dot C^{0,\alpha}(\mathbb{C})$ with $F(0) = 0$. Then for every $0 < \beta < \alpha < 1$, $p \in [1,\infty]$ with $\alpha p \ge 1$ and $\varepsilon \in (0, \alpha - \beta)$, we have

$$\|F(u)\|_{W^{\beta,p}} \lesssim_\varepsilon \|F\|_{\dot C^{0,\alpha}} \|u\|^\alpha_{W^{\frac{\beta}{\alpha}+\varepsilon,p\alpha}}. \tag{4.2.18}$$

Proof. This follows from the embedding (c.f. [150, Exercise A.21]),

$$\|F(u)\|_{W^{\beta,p}} \lesssim_\varepsilon \|F(u)\|_{L^p} + \|F(u)\|_{\dot\Lambda^p_{\beta+\alpha\varepsilon}} \tag{4.2.19}$$

and Lemma 4.2.10 as well as the fact that

$$\|F(u)\|_{L^p} \lesssim \|F\|_{\dot C^{0,\alpha}} \|u\|^\alpha_{L^{p\alpha}}. \tag{4.2.20}$$

□

Remark 4.2.12. It is easy to see that $F(z) = \overline{z}|z|^{2\sigma-2}$ meets the hypothesis of the above corollary (c.f. [47, Lemma 2.4]). The price to pay when using Corollary 4.2.11 is that there is a sort of "loss of regularity" when derivatives fall on $F(u)$ in the sense that a derivative of order $0 < s < 2\sigma - 1$ will be amplified by a factor of $\frac{1}{2\sigma-1}$.

4.3 Low regularity estimates

Now, we proceed with the proof of Theorem 4.1.2. By the scaling symmetry $u_\lambda(t,x) := \lambda^{\frac{1}{2\sigma}} u(\lambda^2 t, \lambda x)$, we see that the L^2_x norm is subcritical with respect to scaling. Hence, we will assume without loss of generality throughout that for some small $0 < \varepsilon \ll 1$ the initial data satisfies $\|u_0\|_{H^s_x} \le \varepsilon$. We then will obtain local well-posedness on the time interval $[-T, T]$ where $T \lesssim 1$ is fixed.

Function spaces

We now define the spaces where we seek solutions. To begin, we define our baseline Strichartz type space Y_T^0 via

$$\|u\|_{Y_T^0} := \left(\sum_{j>0} \|P_j D_x^{\sigma-1} u\|_{L_T^4 L_x^\infty}^2\right)^{\frac{1}{2}} + \left(\sum_{j>0} \|P_j D_x^{\sigma-\frac{1}{2}} u\|_{L_x^\infty L_T^2}^2\right)^{\frac{1}{2}} + \left(\sum_{j>0} \|P_j D_x^{\sigma-\frac{3}{2}} u\|_{L_x^2 L_T^\infty}^2\right)^{\frac{1}{2}} + \|P_{\leq 0} u\|_{L_x^2 L_T^\infty}. \tag{4.3.1}$$

Then we define the space X_T^0 by:

$$\|u\|_{X_T^0} := \left(\sum_{j>0} \|P_j u\|_{L_T^\infty L_x^2}^2\right)^{\frac{1}{2}} + \|P_{\leq 0} u\|_{L_T^\infty L_x^2} + \|u\|_{Y_T^0}. \tag{4.3.2}$$

For higher Sobolev indices, $s \geq 0$, we define the spaces X_T^s and Y_T^s by

$$\|u\|_{Y_T^s} := \|\langle D_x\rangle^s u\|_{Y_T^0}, \quad \|u\|_{X_T^s} := \|\langle D_x\rangle^s u\|_{X_T^0}. \tag{4.3.3}$$

One should observe that we trivially have $\|u\|_{C([-T,T];H_x^s)} \leq \|u\|_{X_T^s}$.

Remark 4.3.1. One might wonder why the above Y_T^s space is not defined in a more standard way, where one replaces σ with 1. Indeed, one can see from the proof of the following estimates that by using this stronger norm, one will incur a loss of $1-\sigma$ derivatives in excess of the $L_T^\infty H_x^s$ norm. The function spaces defined above account for this loss.

Finally, it will be convenient to define the weaker norm S_T^s which just involves the purely Strichartz components of the X_T^s norm. Namely,

$$\|u\|_{S_T^s} = \|P_{\leq 0} u\|_{L_T^\infty L_x^2} + \left(\sum_{j>0} \|P_j \langle D_x\rangle^s u\|_{L_T^\infty L_x^2}^2\right)^{\frac{1}{2}} + \left(\sum_{j>0} \|P_j \langle D_x\rangle^{s-1+\sigma} u\|_{L_T^4 L_x^\infty}^2\right)^{\frac{1}{2}}. \tag{4.3.4}$$

The behavior of the S_T^1 norm will be relevant for continuing a local solution to a global one when $\sigma \in (\frac{\sqrt{3}}{2}, 1)$ in both the low and high regularity regimes.

X_T^s frequency envelopes

It is easy to see that for $s \geq 0$, we have

$$\|P_{\leq 0} u\|_{X_T^s}^2 + \sum_{j=1}^{\infty} \|P_j u\|_{X_T^s}^2 \sim \|u\|_{X_T^s}^2. \tag{4.3.5}$$

Hence, for $u \in X^s_T$, we use b_j to denote the X^s_T frequency envelope for u defined by

$$b_j = 2^{-\delta j} + \|u\|^{-1}_{X^s_T} \max_{k \geq 0} 2^{-\delta|j-k|} \|P_k u\|_{X^s_T} \tag{4.3.6}$$

where δ is some small, but fixed, positive parameter. Similarly, for $v \in H^s_x$, we use a_j to denote the H^s_x frequency envelope for v defined by

$$a_j = 2^{-\delta j} + \|v\|^{-1}_{H^s_x} \max_{k \geq 0} 2^{-\delta|j-k|} \|P_k v\|_{H^s_x}. \tag{4.3.7}$$

Unless otherwise stated, X^s_T and H^s_x frequency envelopes will always be defined by the above formulae.

Remark 4.3.2. In an identical fashion, one can also define S^s_T frequency envelopes.

Next, we state a technical lemma which will be useful for tracking the contributions of the rough part of the nonlinearity in (gDNLS) when derivatives fall on it.

Lemma 4.3.3. (Moser type estimate) Let $s \in [1, \frac{3}{2}]$, $\sigma \in (\frac{1}{2}, 1)$, $0 < T \lesssim 1$ and let b_j be a X^s_T frequency envelope for u. Write $\alpha = s - 1 + \sigma < 2\sigma$. For $j > 0$, we have the following Moser type estimate,

$$\|D^\alpha_x P_j |u|^{2\sigma}\|_{L^2_T L^\infty_x} \lesssim b_j \|u\|^{2\sigma-1}_{S^1_T} \|u\|_{X^s_T}. \tag{4.3.8}$$

Proof. There are two cases to consider. First assume $\alpha > 1$. We have

$$\begin{aligned} \|D^\alpha_x P_j(|u|^{2\sigma})\|_{L^2_T L^\infty_x} &\lesssim \|P_j D^{\alpha-1}_x (|u|^{2\sigma-2}\overline{u} u_x)\|_{L^2_T L^\infty_x} \\ &\lesssim \|P_j D^{\alpha-1}_x (P_{<j-4}(|u|^{2\sigma-2}\overline{u}) u_x)\|_{L^2_T L^\infty_x} \\ &+ \|P_j D^{\alpha-1}_x (P_{\geq j-4}(|u|^{2\sigma-2}\overline{u}) u_x)\|_{L^2_T L^\infty_x}. \end{aligned} \tag{4.3.9}$$

For the first term, we have by Bernstein,

$$\begin{aligned} \|P_j D^{\alpha-1}_x (P_{<j-4}(|u|^{2\sigma-2}\overline{u}) u_x)\|_{L^2_T L^\infty_x} &= \|P_j D^{\alpha-1}_x (P_{<j-4}(|u|^{2\sigma-2}\overline{u}) \tilde{P}_j u_x)\|_{L^2_T L^\infty_x} \\ &\lesssim 2^{j(\alpha-1)} \|u\|^{2\sigma-1}_{L^\infty_T L^\infty_x} \|\tilde{P}_j u_x\|_{L^2_T L^\infty_x} \\ &\lesssim \|u\|^{2\sigma-1}_{S^1_T} \|D^\alpha_x \tilde{P}_j u\|_{L^2_T L^\infty_x} \\ &\lesssim b_j \|u\|^{2\sigma-1}_{S^1_T} \|u\|_{X^s_T}. \end{aligned} \tag{4.3.10}$$

For the second term, we have for $\delta > 0$ small (under the additional assumption that $2^{-\delta j} \lesssim b_j$)

$$\begin{aligned} \|P_j D^{\alpha-1}_x (P_{\geq j-4}(|u|^{2\sigma-2}\overline{u}) u_x)\|_{L^2_T L^\infty_x} &\lesssim 2^{j(\alpha-1)} \|P_{\geq j-4}(|u|^{2\sigma-2}\overline{u}) u_x\|_{L^2_T L^\infty_x} \\ &\lesssim 2^{j(\alpha-1)} \|P_{\geq j-4}(|u|^{2\sigma-2}\overline{u})\|_{L^4_T L^\infty_x} \|u_x\|_{L^4_T L^\infty_x} \\ &\lesssim b_j \|D^{\alpha-1+\delta}_x (|u|^{2\sigma-2}\overline{u})\|_{L^4_T L^\infty_x} \|u_x\|_{L^4_T L^\infty_x} \\ &\lesssim b_j \|D^{\alpha-1+\delta}_x (|u|^{2\sigma-2}\overline{u})\|_{L^4_T L^\infty_x} \|u\|_{X^s_T}. \end{aligned} \tag{4.3.11}$$

It now suffices to show that

$$\|D_x^{\alpha-1+\delta}(|u|^{2\sigma-2}\overline{u})\|_{L_T^4 L_x^\infty} \lesssim \|u\|_{S_T^1}^{2\sigma-1}.$$

For this we fix $\varepsilon > 0$ small and invoke Corollary 4.2.11 and the fact that $2\sigma - 1 < 1$,

$$\begin{aligned} \|D_x^{\alpha-1+\delta}(|u|^{2\sigma-2}\overline{u})\|_{L_T^4 L_x^\infty} &\lesssim_T \|\langle D_x\rangle^{\frac{\alpha-1+\delta+\varepsilon}{2\sigma-1}} u\|_{L_T^4 L_x^\infty}^{2\sigma-1} \\ &\lesssim \|u\|_{S_T^1}^{2\sigma-1} \end{aligned} \tag{4.3.12}$$

where in the last line we take ε, δ small enough and used that $\frac{\alpha-1}{2\sigma-1} < \sigma$ when $s \in [1, \frac{3}{2}]$ and $\sigma \in (\frac{1}{2}, 1)$.

This handles the case $\alpha > 1$. Next, we assume $0 < \alpha \leq 1$. For this, we write

$$P_j|u|^{2\sigma} = P_j|P_{<j}u|^{2\sigma} + P_j(|u|^{2\sigma} - |P_{<j}u|^{2\sigma}). \tag{4.3.13}$$

We have for the first term,

$$\begin{aligned} \|D_x^\alpha P_j|P_{<j}u|^{2\sigma}\|_{L_x^\infty} &\lesssim 2^{j(\alpha-1)}\|P_j(|P_{<j}u|^{2\sigma-2}\overline{P_{<j}u}P_{<j}u_x)\|_{L_x^\infty} \\ &\lesssim 2^{j(\alpha-1)}\|P_j(P_{<j-4}(|P_{<j}u|^{2\sigma-2}\overline{P_{<j}u})\tilde{P}_j u_x)\|_{L_x^\infty} \\ &+ 2^{j(\alpha-1)}\|P_j(P_{\geq j-4}(|P_{<j}u|^{2\sigma-2}\overline{P_{<j}u})P_{<j}u_x)\|_{L_x^\infty} \\ &\lesssim \|u\|_{L_x^\infty}^{2\sigma-1}\|\tilde{P}_j D_x^\alpha u\|_{L_x^\infty} + 2^{-j\delta}\|D_x^{2\delta}(|P_{<j}u|^{2\sigma-2}\overline{P_{<j}u})\|_{L_x^\infty}\|D_x^{\alpha-1-\delta}u_x\|_{L_x^\infty}. \end{aligned} \tag{4.3.14}$$

Hence, by taking δ small enough, using Corollary 4.2.11, and the fact that $2^{-j\delta} \lesssim b_j$, we obtain

$$\|D_x^\alpha P_j|P_{<j}u|^{2\sigma}\|_{L_T^2 L_x^\infty} \lesssim_T b_j\|u\|_{S_T^1}^{2\sigma-1}\|u\|_{X_T^s}. \tag{4.3.15}$$

Next, we estimate

$$\begin{aligned} \|P_j D_x^\alpha(|u|^{2\sigma} - |P_{<j}u|^{2\sigma})\|_{L_T^2 L_x^\infty} &\lesssim 2^{j\alpha}\|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}\sum_{k\geq j}\|P_k u\|_{L_T^2 L_x^\infty} \\ &\lesssim \|u\|_{S_T^1}^{2\sigma-1}\|u\|_{X_T^s}\sum_{k\geq j}2^{-\alpha|k-j|}b_k \\ &\lesssim b_j\|u\|_{S_T^1}^{2\sigma-1}\|u\|_{X_T^s} \end{aligned} \tag{4.3.16}$$

where in the last line, we used the slowly varying property of b_j. This completes the proof. □

Remark 4.3.4. By repeating the proof almost verbatim, and taking b_j instead to be a S^s_T frequency envelope for u, we can modify the conclusion of the lemma to

$$\|D^\alpha_x P_j |u|^{2\sigma}\|_{L^2_T L^\infty_x} \lesssim b_j \|u\|^{2\sigma-1}_{S^1_T} \|u\|_{S^s_T}. \tag{4.3.17}$$

Remark 4.3.5. The $\|u\|^{2\sigma-1}_{S^1_T}$ coefficient in the estimate (4.3.8) could be optimized in terms of the parameters s and σ. We do not pursue this, for the sake of simplicity and also because it does not improve any of the later estimates in an important way.

Uniform bounds

In this subsection, we prove a priori estimates for solutions to (gDNLS). First, we prove uniform X^s_T bounds:

Proposition 4.3.6. Let $0 < \varepsilon \ll 1$, $s \in [1, \frac{3}{2}]$, $\sigma \in (\frac{\sqrt{3}}{2}, 1)$ and let $u_0 \in H^s_x$ with $\|u_0\|_{H^s_x} \le \varepsilon$. Let $T \lesssim 1$. Suppose $u \in X^s_T$ solves the equation,

$$\begin{cases} (i\partial_t + \partial_x^2)u = i|u|^{2\sigma}\partial_x u, \\ u(0) = u_0. \end{cases} \tag{4.3.18}$$

Furthermore, let a_j and b_j be a H^s_x and X^s_T frequency envelope for u_0 and u (on the time interval $[0, T]$), respectively, as defined in Section 4.3. Then we have the following X^s_T estimates for $j > 0$,

a) (Frequency localized X^s_T bound)

$$\|P_j u\|_{X^s_T} \lesssim_{\|u\|_{S^1_T}} a_j \|u_0\|_{H^s_x} + T^{\frac{1}{2}} b_j (1 + \|u\|^{4\sigma}_{S^1_T}) \|u\|_{X^s_T} + T^{\frac{1-\sigma}{2}} b_j \|u\|^{\sigma}_{X^1_T} \|u\|_{X^s_T}. \tag{4.3.19}$$

b) (Uniform X^s_T bound)

$$\|u\|_{X^s_T} \lesssim_{\|u\|_{X^1_T}} \|u_0\|_{H^s_x} \le \varepsilon. \tag{4.3.20}$$

We will also need the following result:

Proposition 4.3.7. Let $0 < \varepsilon \ll 1$ and σ, T and s be as in Proposition 4.3.6. Suppose $v \in X^0_T$ is a solution to the equation,

$$\begin{cases} (i\partial_t + \partial_x^2)v = i|w|^{2\sigma}\partial_x v + g\partial_x a v + \overline{g}\partial_x a \overline{v}, \\ v(0) = v_0, \end{cases} \tag{4.3.21}$$

for some $w \in X_T^1$ solving (gDNLS) (with possibly different initial data), $g \in Z := Z_T := L_x^{\frac{2}{2\sigma-1}} L_T^\infty \cap L_T^\infty W_x^{\frac{3}{4\sigma}-\frac{1}{2}+\varepsilon,\infty} \cap L_T^4 W_x^{\frac{3}{2}-\sigma+\varepsilon,\infty}$ and $a \in X_T^1$, all with sufficiently small norm $\ll 1$. Then v satisfies the bound

$$\|v\|_{X_T^0} \lesssim \|v_0\|_{L^2}. \tag{4.3.22}$$

Remark 4.3.8. In practice g will correspond to terms which are of similar regularity to the term $|u|^{2\sigma-1}$. For such terms to lie in Z (specifically the latter two components of this norm), we will need $\sigma > \frac{\sqrt{3}}{2}$. This will be elaborated on later in the proof.

Remark 4.3.9. Proposition 4.3.7 will be useful for establishing difference estimates for solutions in the weaker topology, X_T^0. This will allow us to show uniqueness for X_T^1 solutions, and to prove a weak Lipschitz type bound for the solution map.

We begin with the proof of Proposition 4.3.6. We divide the relevant estimates into two parts. First, we control the Y_T^s component of the norm. Then we do an energy type estimate to control the $L_T^\infty H_x^s$ component. For this purpose, we have the following lemmas:

Lemma 4.3.10. (Y_T^s estimate) Let $s \in [1, \frac{3}{2}]$, $\sigma \in (\frac{1}{2}, 1)$ and let u, T, a_j and b_j be as in Proposition 4.3.6. Then for $j > 0$ we have

$$\|P_j u\|_{Y_T^s} \lesssim_{\|u\|_{S_T^1}} a_j \|u_0\|_{H_x^s} + T^{\frac{1}{2}} b_j (1 + \|u\|_{S_T^1}^{4\sigma}) \|u\|_{X_T^s}. \tag{4.3.23}$$

Lemma 4.3.11. ($L_T^\infty H_x^s$ estimate) Let s, σ, T, a_j, b_j and u be as in *Proposition* 4.3.6. Then for $j > 0$ we have

$$\|P_j u\|_{L_T^\infty H_x^s} \lesssim a_j \|u_0\|_{H_x^s} + T^{\frac{1-\sigma}{2}} b_j \|u\|_{X_T^1}^\sigma \|u\|_{X_T^s}. \tag{4.3.24}$$

Proof. We begin with the proof of Lemma 4.3.10. For this purpose, let us apply P_j to (4.3.18) and write

$$(i\partial_t + \partial_x^2) u_j = i P_{<j-4} |u|^{2\sigma} \partial_x u_j + g_j \tag{4.3.25}$$

where

$$g_j = i P_j (P_{\geq j-4} |u|^{2\sigma} \partial_x u) + i [P_j, P_{<j-4} |u|^{2\sigma}] \partial_x u. \tag{4.3.26}$$

The term

$$i P_{<j-4} |u|^{2\sigma} \partial_x u_j \tag{4.3.27}$$

which corresponds to the worst interactions between $\partial_x u$ and $|u|^{2\sigma}$ is non-perturbative, and can lead to loss of derivatives in the Y_T^s estimates for u_j. It is desirable to remove as much of this bad interaction as possible. As mentioned earlier, one might try to remove it entirely with a gauge transformation, but this will not work, because the function $z \mapsto |z|^{2\sigma}$ is not smooth enough. Fortunately, in some sense, formally, the worst terms introduced by a gauge transformation are only poorly behaved when u is small (i.e. sufficiently close to 0). On the other hand, if u is sufficiently small (on a scale depending on j), then we expect to be able to treat the associated part of the term (4.3.27) perturbatively. One then expects to be able to remove the other part (in which u is bounded away from zero) with a gauge transformation, and gain some mileage.

With this strategy in mind, let φ be a smooth compactly supported function on $\mathbb{R}$ with $\varphi = 1$ on the unit interval and zero outside $(-2, 2)$. Likewise, define $\chi = 1 - \varphi$. We want to tailor these functions to a particular frequency, which we do by defining the rescaled functions $\varphi_j(x) = \varphi(2^j x)$ and $\chi_j(x) = \chi(2^j x)$. Next, we further rewrite (4.3.25) as the following equation,

$$(i\partial_t + \partial_x^2)u_j = iP_{<j-4}[\chi_j(|u|^2)|u|^{2\sigma}]\partial_x u_j + iP_{<j-4}[\varphi_j(|u|^2)|u|^{2\sigma}]\partial_x u_j + g_j. \tag{4.3.28}$$

Remark 4.3.12. One might wonder whether one can modify the 2^j scale in the definition of φ_j to $2^{j\alpha}$ for some $\alpha > 0$. It turns out that $\alpha = 1$ is the optimal choice, as one can ascertain from repeating the estimates below with this new parameter α. This optimization is obtained by balancing the contributions from the terms I_j^1 and I_j^3 in the below estimates.

Now, we do a partial gauge transformation to remove $iP_{<j-4}[\chi_j(|u|^2)|u|^{2\sigma}]\partial_x u_j$, which corresponds to the part of (4.3.27) for which the coefficient $|u|^{2\sigma}$ is bounded below by $2^{-j\sigma}$. Indeed, define

$$\Phi_j(t, x) := -\frac{1}{2}P_{<j-4}\partial_x^{-1}[\chi_j(|u|^2)|u|^{2\sigma}] \tag{4.3.29}$$

where

$$(\partial_x^{-1} f)(x) := \int_{-\infty}^{x} f(y)dy \tag{4.3.30}$$

and then define

$$w_j := u_j e^{i\Phi_j}. \tag{4.3.31}$$

Before proceeding, we need the following technical estimate which relates u_j to w_j.

Lemma 4.3.13. Let S refer to any of the four spaces, $L_T^\infty L_x^2$, $L_x^\infty L_T^2$, $L_x^2 L_T^\infty$, or $L_T^4 L_x^\infty$. Let $\beta \in (-1, 1)$ and $0 < \varepsilon \ll 1$. Then for $j > 0$, we have

$$\|\langle D_x\rangle^\beta u_j\|_S \lesssim_\varepsilon (1 + \|u\|_{S_T^1})^{2\sigma}(\|\langle D_x\rangle^\beta \tilde{P}_j w_j\|_S + \|\langle D_x\rangle^{\beta-\varepsilon} w_j\|_S). \tag{4.3.32}$$

Remark 4.3.14. As a brief remark, the range on β accounts for (more than) the greatest range of derivatives allowed in any component of the $X_T^{1-\sigma}$ norm, which will correspond to the situation in which we apply the estimate. Strictly speaking, this is overkill, but it lets us avoid dealing with several individual cases. Also, the $\beta - \varepsilon$ factor in the second term in the above estimate is to compensate for terms in which w_j is not frequency localized. In particular, later when applying Proposition 4.2.4, the ε will allow us to sum up the individual frequency dyadic contributions of w_j.

Proof. We have using the fact that u_j is frequency localized to frequency $\sim 2^j$,

$$\begin{aligned} \|\langle D_x\rangle^\beta u_j\|_S &= \|\langle D_x\rangle^\beta \tilde{P}_j(e^{-i\Phi_j}w_j)\|_S \\ &\lesssim \|D_x^\beta \tilde{P}_j(P_{<j-2}e^{-i\Phi_j}\tilde{P}_j w_j)\|_S + \|D_x^\beta \tilde{P}_j(P_{\geq j-2}e^{-i\Phi_j}w_j)\|_S. \end{aligned} \tag{4.3.33}$$

For the first term, we have by the (vector-valued) Bernstein's inequality

$$\|D_x^\beta \tilde{P}_j(P_{<j-2}e^{-i\Phi_j}\tilde{P}_j w_j)\|_S \lesssim \|D_x^\beta \tilde{P}_j w_j\|_S. \tag{4.3.34}$$

For the second term, we have from Bernstein's inequality (and since $j > 0$),

$$\begin{aligned} \|D_x^\beta \tilde{P}_j(P_{\geq j-2}e^{-i\Phi_j}w_j)\|_S &\lesssim 2^{j\beta}\|\tilde{P}_j(P_{\geq j-2}e^{-i\Phi_j}w_j)\|_S \\ &\lesssim 2^{j\beta}\|P_{\geq j-2}e^{-i\Phi_j}\|_{L_T^\infty L_x^\infty}\|P_{<j+2}w_j\|_S \\ &+ 2^{j\beta}\sum_{k\geq j}\|\tilde{P}_k e^{-i\Phi_j}\|_{L_T^\infty L_x^\infty}\|\tilde{P}_k w_j\|_S \\ &\lesssim_\varepsilon \|P_{\geq j-2}D_x^{|\beta|+2\varepsilon}e^{-i\Phi_j}\|_{L_T^\infty L_x^\infty}\|\langle D_x\rangle^{\beta-\varepsilon}w_j\|_S \end{aligned} \tag{4.3.35}$$

where $\varepsilon > 0$ is small enough so that for instance, $|\beta| + 2\varepsilon < 1$. Then we have by Bernstein,

$$\|P_{\geq j-2}D_x^{|\beta|+2\varepsilon}e^{-i\Phi_j}\|_{L_T^\infty L_x^\infty} \lesssim \|\partial_x P_{\geq j-2}e^{-i\Phi_j}\|_{L_T^\infty L_x^\infty} \lesssim \|u\|_{S_T^1}^{2\sigma}. \tag{4.3.36}$$

Combining the above estimates completes the proof. □

Given Lemma 4.3.13, we are in a position to convert estimates for w_j into estimates for u_j. A direct computation shows that w_j satisfies the following equation:

$$\begin{cases} (i\partial_t + \partial_x^2)w_j &= ie^{i\Phi_j}P_{<j-4}[\varphi_j(|u|^2)|u|^{2\sigma}]\partial_x u_j + (-\partial_t\Phi_j + i\partial_x^2\Phi_j - (\partial_x\Phi_j)^2)w_j + e^{i\Phi_j}g_j, \\ \quad w_j(0) &= e^{i\Phi_j}u_j(0). \end{cases} \tag{4.3.37}$$

The goal is to prove a priori estimates for w_j - and hence u_j - in Y_T^s. We observe a couple of useful facts. First, by Bernstein, we have $\|u_j\|_{Y_T^s} \lesssim 2^{j(\sigma+s-1)}\|u_j\|_{Y_T^{1-\sigma}}$. Secondly, we obviously have $\|gw_j\|_{L^1_T L^2_x} = \|gu_j\|_{L^1_T L^2_x}$ for measurable functions, g. Using these observations, Lemma 4.3.13, the maximal function estimates and the usual Strichartz estimates from Propositions 4.2.3 and 4.2.4 we have that

$$\begin{aligned}\frac{\|u_j\|_{Y^s_T}}{(1+\|u\|_{S^1_T})^{2\sigma}} &\lesssim \|u_j(0)\|_{H^s_x} + 2^{j(\sigma+s-1)}\|P_{<j-4}[\varphi_j(|u|^2)|u|^{2\sigma}]\partial_x u_j\|_{L^1_T L^2_x} + 2^{j(\sigma+s-1)}\|g_j\|_{L^1_T L^2_x} \\ &+ 2^{j(\sigma+s-1)}\|\partial_t \Phi_j u_j\|_{L^1_T L^2_x} + 2^{j(\sigma+s-1)}\|\partial_x^2 \Phi_j u_j\|_{L^1_T L^2_x} \\ &+ 2^{j(\sigma+s-1)}\|(\partial_x \Phi_j)^2 u_j\|_{L^1_T L^2_x} \\ &:= \|u_j(0)\|_{H^s_x} + I_1^j + I_2^j + I_3^j + I_4^j + I_5^j.\end{aligned} \tag{4.3.38}$$

We now estimate each of the above terms.

Estimate for I_1^j

By Bernstein and the fact that $|u| \lesssim 2^{-\frac{j}{2}}$ on the support of φ_j,

$$\begin{aligned}2^{j(\sigma+s-1)}\|P_{<j-4}[\varphi_j(|u|^2)|u|^{2\sigma}]\partial_x u_j\|_{L^1_T L^2_x} &\lesssim 2^{j(\sigma+s-1)}\|\varphi_j(|u|^2)|u|^{2\sigma}\|_{L^1_T L^\infty_x}\|\partial_x u_j\|_{L^\infty_T L^2_x} \\ &\lesssim T\|u_j\|_{L^\infty_T H^s_x} \\ &\lesssim T b_j \|u\|_{X^s_T}.\end{aligned} \tag{4.3.39}$$

Estimate for I_2^j

We have

$$g_j = iP_j(P_{\geq j-4}|u|^{2\sigma}\partial_x u) + i[P_j, P_{<j-4}|u|^{2\sigma}]\partial_x \tilde{P}_j u \tag{4.3.40}$$

where $\tilde{P}_j$ is a "fattened" projection to frequency $\sim 2^j$. By the standard Littlewood-Paley trichotomy, we write

$$\begin{aligned}P_j(P_{\geq j-4}|u|^{2\sigma}\partial_x u) &= P_j(\tilde{P}_j|u|^{2\sigma}\partial_x \tilde{P}_{<j} u) + P_j(\tilde{P}_j|u|^{2\sigma}\tilde{P}_j\partial_x u) \\ &+ \sum_{k>j} P_j(\tilde{P}_k|u|^{2\sigma}\tilde{P}_k\partial_x u).\end{aligned} \tag{4.3.41}$$

For the first term, we have by the Moser estimate (4.3.8) and Bernstein's inequality,

$$\begin{aligned}2^{j(\sigma+s-1)}\|P_j(\tilde{P}_j|u|^{2\sigma}\partial_x \tilde{P}_{<j} u)\|_{L^1_T L^2_x} &\lesssim 2^{j(\sigma+s-1)}\|\tilde{P}_j|u|^{2\sigma}\|_{L^1_T L^\infty_x}\|\partial_x u\|_{L^\infty_T L^2_x} \\ &\lesssim \|\tilde{P}_j D_x^{\sigma+s-1}|u|^{2\sigma}\|_{L^1_T L^\infty_x}\|\partial_x u\|_{L^\infty_T L^2_x} \\ &\lesssim T^{\frac{1}{2}} b_j \|u\|^{2\sigma}_{S^1_T}\|u\|_{X^s_T}.\end{aligned} \tag{4.3.42}$$

The second term is dealt with similarly. For the third term, we have by Bernstein's inequality

$$\begin{aligned}
&2^{j(\sigma+s-1)}\|\sum_{k>j} P_j(\tilde{P}_k|u|^{2\sigma}\tilde{P}_k\partial_x u)\|_{L^1_T L^2_x} \\
&\lesssim T^{\frac{3}{4}}\sum_{k>j}\|\tilde{P}_k u\|_{L^4_T L^\infty_x} 2^{j(\sigma+s-1)}2^k\|\tilde{P}_k|u|^{2\sigma}\|_{L^\infty_T L^2_x} \\
&\lesssim T^{\frac{3}{4}}\sum_{k>j} 2^{(j-k)(\sigma+s-1)}\|D_x^{\sigma+s-1}\tilde{P}_k u\|_{L^4_T L^\infty_x}\|\tilde{P}_k\partial_x|u|^{2\sigma}\|_{L^\infty_T L^2_x} \\
&\lesssim T^{\frac{3}{4}}\|u\|^{2\sigma}_{S^1_T}\|u\|_{X^s_T}\sum_{k>j} 2^{-(\sigma+s-1)|k-j|}b_k \\
&\lesssim T^{\frac{3}{4}} b_j\|u\|_{X^s_T}\|u\|^{2\sigma}_{S^1_T}\sum_{k>j} 2^{-(\sigma+s-1-\delta)|k-j|} \\
&\lesssim T^{\frac{3}{4}} b_j\|u\|_{X^s_T}\|u\|^{2\sigma}_{S^1_T}.
\end{aligned} \tag{4.3.43}$$

For the commutator term, we have by Lemma 4.2.1

$$2^{j(\sigma+s-1)}[P_j, P_{<j-4}|u|^{2\sigma}]\partial_x\tilde{P}_j u = 2^{j(\sigma+s-2)}L(\partial_x P_{<j-4}|u|^{2\sigma}, \tilde{P}_j\partial_x u) \tag{4.3.44}$$

for some appropriate translation invariant expression L.

This term is easily estimated by

$$\begin{aligned}
2^{j(\sigma+s-2)}\|L(\partial_x P_{<j-4}|u|^{2\sigma}, \tilde{P}_j\partial_x u)\|_{L^1_T L^2_x} &\lesssim 2^{j(\sigma+s-2)}\|\partial_x P_{<j-4}|u|^{2\sigma}\|_{L^\infty_T L^2_x}\|\tilde{P}_j\partial_x u\|_{L^1_T L^\infty_x} \\
&\lesssim \|u\|^{2\sigma-1}_{L^\infty_T L^\infty_x}\|\partial_x u\|_{L^\infty_T L^2_x}\|\tilde{P}_j D_x^{\sigma+s-1}u\|_{L^1_T L^\infty_x} \\
&\lesssim b_j T^{\frac{3}{4}}\|u\|^{2\sigma}_{S^1_T}\|u\|_{X^s_T}.
\end{aligned} \tag{4.3.45}$$

Hence, we have

$$I_2^j \lesssim T^{\frac{1}{2}} b_j\|u\|_{X^s_T}\|u\|^{2\sigma}_{S^1_T}. \tag{4.3.46}$$

Estimate for I_3^j

We expand

$$\partial_t\Phi_j = -\frac{1}{2}P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\partial_t|u|^2|u|^{2\sigma}] - \frac{1}{2}P_{<j-4}\partial_x^{-1}[\chi_j(|u|^2)\partial_t|u|^{2\sigma}] =: J_1 + J_2. \tag{4.3.47}$$

We have

$$\begin{aligned} J_1 &= -\frac{1}{2}P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\partial_t|u|^2|u|^{2\sigma}] \\ &= -P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\mathrm{Re}(\overline{u}u_t)|u|^{2\sigma}] \\ &= -P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\mathrm{Re}(i\overline{u}u_{xx})|u|^{2\sigma}] - P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\mathrm{Re}(\overline{u}|u|^{2\sigma}u_x)|u|^{2\sigma}] \\ &= -P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\partial_x\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}] - P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\mathrm{Re}(\overline{u}|u|^{2\sigma}u_x)|u|^{2\sigma}] \\ &:= K_1 + K_2. \end{aligned} \tag{4.3.48}$$

For the first term, K_1, in (4.3.48) we write

$$\begin{aligned} -P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\partial_x\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}] &= -P_{<j-4}[2^j\chi'(2^j|u|^2)\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}] \\ &\quad + P_{<j-4}\partial_x^{-1}[2^{2j}\chi''(2^j|u|^2)\partial_x|u|^2\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}] \\ &\quad + P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\mathrm{Re}(i\overline{u}u_x)\partial_x|u|^{2\sigma}]. \end{aligned} \tag{4.3.49}$$

We have for the first term in (4.3.49)

$$\begin{aligned} \|P_{<j-4}[2^j\chi'(2^j|u|^2)\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}]\|_{L^\infty_T L^2_x} &\lesssim 2^j\|\chi'(2^j|u|^2)\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}\|_{L^\infty_T L^2_x} \\ &\lesssim \|u\|^{2\sigma-1}_{L^\infty_T L^\infty_x}\|u_x\|_{L^\infty_T L^2_x} \end{aligned} \tag{4.3.50}$$

where we used the fact that

$$\|\chi'(2^j|u|^2)|u|^{2\sigma+1}\|_{L^\infty_T L^\infty_x} = \|\varphi'(2^j|u|^2)|u|^{2\sigma+1}\|_{L^\infty_T L^\infty_x} \lesssim 2^{-j}\|u\|^{2\sigma-1}_{L^\infty_T L^\infty_x}. \tag{4.3.51}$$

Now, for the second term in (4.3.49), we have

$$\begin{aligned} &2^{2j}\|P_{<j-4}\partial_x^{-1}[\chi''(2^j|u|^2)\partial_x|u|^2\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}]\|_{L^\infty_T L^\infty_x} \\ &\lesssim 2^{2j}\|\chi''(2^j|u|^2)\partial_x|u|^2\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}\|_{L^\infty_T L^1_x} \\ &\lesssim 2^{2j}\|\varphi''(2^j|u|^2)\mathrm{Re}(\overline{u}u_x)\mathrm{Re}(i\overline{u}u_x)|u|^{2\sigma}\|_{L^\infty_T L^1_x} \\ &\lesssim 2^{j(1-\sigma)}\|u_x\|^2_{L^\infty_T L^2_x}. \end{aligned} \tag{4.3.52}$$

The third term in (4.3.49) is estimated similarly to the second term.

Hence, we obtain

$$\begin{aligned} 2^{j(\sigma+s-1)}\|K_1u_j\|_{L^1_T L^2_x} &\lesssim 2^{j(\sigma+s-1)}2^{j(1-\sigma)}T\|u_x\|^2_{L^\infty_T L^2_x}\|u_j\|_{L^\infty_T L^2_x} \\ &\quad + 2^{j(\sigma+s-1)}\|u\|^{2\sigma-1}_{L^\infty_T L^\infty_x}T^{\frac{3}{4}}\|u_x\|_{L^\infty_T L^2_x}\|u_j\|_{L^4_T L^\infty_x} \\ &\lesssim T\|u_x\|^2_{L^\infty_T L^2_x}\|D^s_x u_j\|_{L^\infty_T L^2_x} \\ &\quad + T^{\frac{3}{4}}\|u\|^{2\sigma-1}_{L^\infty_T L^\infty_x}\|u_x\|_{L^\infty_T L^2_x}\|D^{\sigma+s-1}_x u_j\|_{L^4_T L^\infty_x}. \end{aligned} \tag{4.3.53}$$

Next, we estimate K_2. We have by Cauchy Schwarz, and Sobolev embedding,

$$\begin{aligned}\|P_{<j-4}\partial_x^{-1}[2^j\chi'(2^j|u|^2)\mathrm{Re}(\overline{u}|u|^{2\sigma}u_x)|u|^{2\sigma}]\|_{L_T^\infty L_x^\infty} &\lesssim 2^j\|\varphi'(2^j|u|^2)\mathrm{Re}(\overline{u}|u|^{2\sigma}u_x)|u|^{2\sigma}\|_{L_T^\infty L_x^1}\\ &\lesssim 2^{j(\frac{1}{2}-\sigma)}\|u\|_{L_T^\infty L_x^{4\sigma}}^{2\sigma}\|u_x\|_{L_T^\infty L_x^2}\\ &\lesssim 2^{j(\frac{1}{2}-\sigma)}\|u\|_{S_T^1}^{2\sigma}\|u_x\|_{L_T^\infty L_x^2}\\ &\lesssim \|u\|_{S_T^1}^{2\sigma}\|u_x\|_{L_T^\infty L_x^2}\end{aligned} \tag{4.3.54}$$

where we used the fact that $\sigma \geq \frac{1}{2}$.

Hence, we finally obtain the estimate,

$$\begin{aligned}2^{j(\sigma+s-1)}\|u_jJ_1\|_{L_T^1L_x^2} &\lesssim T\|u_x\|_{L_T^\infty L_x^2}^2\|D_x^s u_j\|_{L_T^\infty L_x^2} + T^{\frac{3}{4}}\|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}\|u_x\|_{L_T^\infty L_x^2}\|D_x^{\sigma+s-1}u_j\|_{L_T^4L_x^\infty}\\ &\quad + T\|u\|_{S_T^1}^{2\sigma}\|u_x\|_{L_T^\infty L_x^2}\|D_x^{\sigma+s-1}u_j\|_{L_T^\infty L_x^2}\\ &\lesssim T^{\frac{3}{4}}(1+\|u\|_{S_T^1}^{4\sigma})\|u_j\|_{X_T^s}.\end{aligned} \tag{4.3.55}$$

Next, we turn to the estimate for J_2. We have

$$\begin{aligned}J_2 &= -\frac{1}{2}P_{<j-4}\partial_x^{-1}[\chi_j(|u|^2)\partial_t|u|^{2\sigma}]\\ &= -\sigma P_{<j-4}\partial_x^{-1}[\chi_j(|u|^2)|u|^{2\sigma-2}\mathrm{Re}(\overline{u}u_t)]\\ &= -\sigma P_{<j-4}\partial_x^{-1}[\chi_j(|u|^2)|u|^{2\sigma-2}\mathrm{Re}(i\overline{u}u_{xx})] - \sigma P_{<j-4}\partial_x^{-1}[\chi_j(|u|^2)|u|^{2\sigma-2}\mathrm{Re}(\overline{u}|u|^{2\sigma}u_x)]\\ &:= K_3 + K_4.\end{aligned} \tag{4.3.56}$$

For the first term, we have

$$\begin{aligned}K_3 &= -\sigma P_{<j-4}[\chi_j(|u|^2)|u|^{2\sigma-2}\mathrm{Re}(i\overline{u}u_x)] + \sigma P_{<j-4}\partial_x^{-1}[\chi_j(|u|^2)\partial_x|u|^{2\sigma-2}\mathrm{Re}(i\overline{u}u_x)]\\ &\quad - 2^j\sigma P_{<j-4}\partial_x^{-1}[\varphi'(2^j|u|^2)\partial_x|u|^2|u|^{2\sigma-2}\mathrm{Re}(i\overline{u}u_x)]\\ &= K_{3,1}+K_{3,2}+K_{3,3}.\end{aligned} \tag{4.3.57}$$

We now must estimate each of the above terms. For the first two terms, we have

$$\|K_{3,1}\|_{L_T^\infty L_x^2} \lesssim \|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}\|u_x\|_{L_T^\infty L_x^2} \tag{4.3.58}$$

and

$$\begin{aligned}\|K_{3,2}\|_{L_T^\infty L_x^\infty} &\lesssim \|\chi_j(|u|^2)\partial_x|u|^{2\sigma-2}\mathrm{Re}(i\overline{u}u_x)\|_{L_T^\infty L_x^1}\\ &\lesssim \|\chi_j(|u|^2)|u|^{2\sigma-4}\mathrm{Re}(\overline{u}u_x)\mathrm{Re}(i\overline{u}u_x)\|_{L_T^\infty L_x^1}\\ &\lesssim 2^{j(1-\sigma)}\|u_x\|_{L_T^\infty L_x^2}^2\end{aligned} \tag{4.3.59}$$

where we used the fact that

$$\chi_j(|u|^2)|u|^{2\sigma-2} \lesssim 2^{j(1-\sigma)}. \tag{4.3.60}$$

Remark 4.3.15. It should be emphasized that the main point of the partial gauge transformation is to be able to estimate the term $K_{3,2}$ above, which involves negative powers of $|u|$.

Now, we turn to the estimate for $K_{3,3}$. We have

$$\begin{aligned} \|K_{3,3}\|_{L_T^\infty L_x^\infty} &\lesssim 2^j \|\varphi'(2^j|u|^2)|u|^{2\sigma-2}\mathrm{Re}(\overline{u}u_x)\mathrm{Re}(i\overline{u}u_x)\|_{L_T^\infty L_x^1} \\ &\lesssim 2^{j(1-\sigma)}\|u_x\|_{L_T^\infty L_x^2}^2. \end{aligned} \tag{4.3.61}$$

Hence, we have

$$2^{j(\sigma+s-1)}\|K_3 u_j\|_{L_T^1 L_x^2} \lesssim T^{\frac{3}{4}}\|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}\|u_x\|_{L_T^\infty L_x^2}\|D_x^{\sigma+s-1}u_j\|_{L_T^4 L_x^\infty} + T\|u_x\|_{L_T^\infty L_x^2}^2\|D_x^s u_j\|_{L_T^\infty L_x^2}. \tag{4.3.62}$$

Finally, we estimate K_4. We have

$$\begin{aligned} \|K_4\|_{L_T^\infty L_x^\infty} &\lesssim \|P_{<j-4}\partial_x^{-1}[\chi_j(|u|^2)|u|^{2\sigma-2}\mathrm{Re}(\overline{u}|u|^{2\sigma}u_x)]\|_{L_T^\infty L_x^\infty} \\ &\lesssim \|\chi_j(|u|^2)|u|^{2\sigma-2}\mathrm{Re}(\overline{u}|u|^{2\sigma}u_x)\|_{L_T^\infty L_x^1} \\ &\lesssim \|u\|_{L_T^\infty L_x^\infty}^{4\sigma-2}\|u_x\|_{L_T^\infty L_x^2}\|u\|_{L_T^\infty L_x^2}. \end{aligned} \tag{4.3.63}$$

Hence, combining with the estimate for K_3, we obtain

$$\begin{aligned} 2^{j(\sigma+s-1)}\|u_j J_2\|_{L_T^1 L_x^2} &\lesssim T^{\frac{3}{4}}\|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}\|u_x\|_{L_T^\infty L_x^2}\|D_x^{\sigma+s-1}u_j\|_{L_T^4 L_x^\infty} + T\|u_x\|_{L_T^\infty L_x^2}^2\|D_x^s u_j\|_{L_T^\infty L_x^2} \\ &\quad + T\|u\|_{L_T^\infty L_x^\infty}^{4\sigma-2}\|u_x\|_{L_T^\infty L_x^2}\|u\|_{L_T^\infty L_x^2}\|D_x^{\sigma+s-1}u_j\|_{L_T^\infty L_x^2} \\ &\lesssim_T T^{\frac{3}{4}}(1+\|u\|_{S_T^1}^{4\sigma})\|u_j\|_{X_T^s}. \end{aligned} \tag{4.3.64}$$

Now combining this with the estimate for J_1 finally yields the desired estimate for I_3^j. Namely, we have

$$\begin{aligned} I_3^j &\lesssim T^{\frac{3}{4}}(1+\|u\|_{S_T^1}^{4\sigma})\|u_j\|_{X_T^s} \\ &\lesssim T^{\frac{3}{4}}b_j(1+\|u\|_{S_T^1}^{4\sigma})\|u\|_{X_T^s}. \end{aligned} \tag{4.3.65}$$

Estimate for I_4^j

This term is straightforward to deal with. Indeed, after expanding $\partial_x^2\Phi_j$ we have

$$\begin{aligned} \|\partial_x^2\Phi_j\|_{L_T^\infty L_x^2} &\lesssim 2^j\|\varphi'(2^j|u|^2)\mathrm{Re}(\overline{u}u_x)|u|^{2\sigma}\|_{L_T^\infty L_x^2} + \|\chi_j(|u|^2)\mathrm{Re}(|u|^{2\sigma-2}\overline{u}u_x)\|_{L_T^\infty L_x^2} \\ &\lesssim \|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}\|u_x\|_{L_T^\infty L_x^2}. \end{aligned} \tag{4.3.66}$$

Hence,

$$\begin{aligned} I_4^j &\lesssim T^{\frac{3}{4}}\|u\|_{L_T^\infty L_x^\infty}^{2\sigma-1}\|u_x\|_{L_T^\infty L_x^2}\|D_x^{\sigma+s-1}u_j\|_{L_T^4 L_x^\infty} \\ &\lesssim T^{\frac{3}{4}}\|u\|_{S_T^1}^{2\sigma}\|u_j\|_{X_T^s} \\ &\lesssim T^{\frac{3}{4}}b_j\|u\|_{S_T^1}^{2\sigma}\|u\|_{X_T^s}. \end{aligned} \tag{4.3.67}$$

Estimate for I_5^j

The estimate for I_5^j is also straightforward as it doesn't involve any differentiated terms. Indeed, we have

$$\|\partial_x\Phi_j\|_{L_T^\infty L_x^\infty}^2 \lesssim \|u\|_{L_T^\infty L_x^\infty}^{4\sigma}. \tag{4.3.68}$$

Hence, by Sobolev embedding,

$$\begin{aligned} I_5^j &\lesssim T\|u\|_{L_T^\infty L_x^\infty}^{4\sigma}\|D_x^{\sigma+s-1}u_j\|_{L_T^\infty L_x^2} \\ &\lesssim T\|u\|_{S_T^1}^{4\sigma}\|u_j\|_{X_T^s} \\ &\lesssim Tb_j\|u\|_{S_T^1}^{4\sigma}\|u\|_{X_T^s}. \end{aligned} \tag{4.3.69}$$

Now, combining all the estimates above completes the proof of Lemma 4.3.10.

Remark 4.3.16. By taking b_j to instead be a S_T^s frequency envelope for u, and repeating the proof almost verbatim with Remark 4.3.4 in place of (4.3.8), we instead obtain

$$\|P_j u\|_{Y_T^s} \lesssim_{\|u\|_{S_T^1}} a_j\|u_0\|_{H_x^s} + T^{\frac{1}{2}}b_j(1+\|u\|_{S_T^1}^{4\sigma})\|u\|_{S_T^s}. \tag{4.3.70}$$

This will be relevant for when we later establish local well-posedness in the high regularity regime $2-\sigma < s < 4\sigma$ for the full range of $\frac{1}{2} < \sigma < 1$. Specifically, this will be important for establishing a priori bounds in the range $2-\sigma < s \leq \frac{3}{2}$ when Sobolev embedding is not suitable for controlling the term $\|u_x\|_{L_T^4 L_x^\infty}$. The reason the proof of (4.3.70) is almost identical to the current proof is that we have not yet used the maximal function part of the norm of X_T^s; we will begin using this part of the norm in the proof of Lemma 4.3.11.

Remark 4.3.17. As a second important remark, the estimate (4.3.70) also holds for $T \lesssim 1$ if the nonlinearity $i|u|^{2\sigma}u_x$ is replaced by the spatially regularized and time-truncated nonlinearity $i\eta P_{<k}|u|^{2\sigma}u_x$, where $k \in \mathbb{N}$ and $\eta = \eta(t)$ is a time-dependent cutoff function supported in $(-2,2)$ and equal to 1 on $[-1,1]$. This fact won't be relevant for the low regularity construction, but will be important for the high regularity construction in Sections 5 and 6 where the cutoff η is needed for estimating (fractional order) time derivatives of a

solution u to (gDNLS). Since the proof of this estimate is nearly identical to Lemma 4.3.10, we omit the details. Nevertheless, for the sake of completeness, we state this observation in the following lemma.

Lemma 4.3.18. Let $k \in \mathbb{N}$, $\sigma \in (\frac{1}{2}, 1)$, $s \in [1, \frac{3}{2}]$, and $T \lesssim 1$. Let η be a time-dependent cutoff function supported in $(-2, 2)$ with $\eta = 1$ on $[-1, 1]$. Let $v, w \in S^s_T$ with $\|v\|_{S^s_T}$, $\|w\|_{S^s_T} \lesssim 1$. Assume that $u, v \in S^s_T$ solve the equations

$$\begin{cases} (i\partial_t + \partial_x^2)u = i\eta P_{<k}|v|^{2\sigma}\partial_x u, \\ u(0) = u_0, \end{cases} \tag{4.3.71}$$

and

$$\begin{cases} (i\partial_t + \partial_x^2)v = i\eta P_{<k}|w|^{2\sigma}\partial_x v, \\ v(0) = u_0, \end{cases} \tag{4.3.72}$$

respectively. Then u satisfies the estimate

$$\|u\|_{Y^s_T} \lesssim \|u_0\|_{H^s_x} + T^{\frac{1}{2}}\|u\|_{S^s_T}. \tag{4.3.73}$$

As mentioned, the proof of Lemma 4.3.18 proceeds in a nearly identical fashion to Lemma 4.3.10, so we omit the details. The main difference is that Φ_j is replaced by

$$\Phi_j = -\frac{1}{2}\eta(t)P_{<j-4}P_{<k}\partial_x^{-1}[\chi_j(|v|^2)|v|^{2\sigma}]. \tag{4.3.74}$$

The requirement (4.3.72) that v solves an additional (gDNLS) type equation is merely relevant for the I_3^j estimate when time derivatives fall on Φ_j, and hence on v. In practice, Lemma 4.3.18 will be used in the construction of solutions at high regularity in Sections 5, 6 and 7.

Next, we turn to proving Lemma 4.3.11.

Proof. Again, we begin by writing the equation in a paradifferential fashion,

$$i\partial_t u_j + \partial_x^2 u_j = iP_{<j-4}|u|^{2\sigma}\partial_x u_j + iP_j(P_{\geq j-4}|u|^{2\sigma}\partial_x u) + i[P_j, P_{<j-4}|u|^{2\sigma}]\partial_x u. \tag{4.3.75}$$

A simple energy estimate (i.e. multiplying by $-i2^{2js}\overline{u_j}$, taking real part and integrating), and Bernstein's inequality gives

$$\begin{aligned}\|u_j\|^2_{L^\infty_T H^s_x} \lesssim \|u_j(0)\|^2_{H^s_x} &+ 2^{2js}\int_0^T \left|\int_{\mathbb{R}} P_{<j-4}|u|^{2\sigma}\partial_x|u_j|^2\right| + 2^{2js}\int_0^T\left|\int_{\mathbb{R}}\overline{u_j}P_j(P_{\geq j-4}|u|^{2\sigma}\partial_x u)\right| \\ &+ 2^{2js}\int_0^T\left|\int_{\mathbb{R}}\overline{u_j}[P_j, P_{<j-4}|u|^{2\sigma}]\partial_x u\right| \\ &:= \|u_j(0)\|^2_{H^s_x} + I_1^j + I_2^j + I_3^j.\end{aligned} \tag{4.3.76}$$

Estimate for I_1^j

For the first term, we integrate by parts and estimate using standard interpolation inequalities, Bernstein's inequality, Hölder's inequality and Proposition 4.2.7

$$\begin{aligned}&2^{2js}\int_0^T\left|\int_{\mathbb{R}}|u_j|^2 P_{<j-4}\partial_x|u|^{2\sigma}\right| \\ &\lesssim 2^{2js}\|P_{<j-4}\partial_x|u|^{2\sigma}|u_j|^{2(1-\sigma)}\|_{L^1_x L_T^{\frac{1}{1-\sigma}}}\|u_j\|^{2\sigma}_{L^\infty_x L^2_T} \\ &\lesssim 2^{2js}\|P_{<j-4}\partial_x|u|^{2\sigma}\|_{L_x^{\frac{1}{\sigma}} L_T^{\frac{1}{\varepsilon(1-\sigma)}}}\|u_j\|^{2(1-\sigma)}_{L^2_x L_T^{\frac{2}{1-\varepsilon}}}\|u_j\|^{2\sigma}_{L^\infty_x L^2_T} \\ &\lesssim \|P_{<j-4}(D_x^{\sigma-\frac{1}{2}}|u|^{2\sigma})\|_{L_x^{\frac{1}{\sigma}} L_T^{\frac{1}{\varepsilon(1-\sigma)}}}\|D_x^{s-c_1\varepsilon}u_j\|^{2(1-\sigma)}_{L^2_x L_T^{\frac{2}{1-\varepsilon}}}\|D_x^{s+\frac{3}{4\sigma}-\frac{1}{2}+c_2\varepsilon}u_j\|^{2\sigma}_{L^\infty_x L^2_T} \\ &\lesssim T^{(1-\sigma)(1-\varepsilon)}\|P_{<j-4}(D_x^{\sigma-\frac{1}{2}-\varepsilon}|u|^{2\sigma})\|_{L_x^{\frac{1}{\sigma}} L_T^{\frac{1}{\varepsilon(1-\sigma)}}}\|u_j\|^2_{X^s_T} \\ &\lesssim T^{1-\sigma}\|u\|^{2\sigma-1}_{L^2_x L^\infty_T}\|D_x^{\sigma-\frac{1}{2}-\varepsilon}u\|_{L^2_x L^\infty_T}\|u_j\|^2_{X^s_T} \\ &\lesssim T^{1-\sigma}\|u\|^{2\sigma}_{Y^1_T}\|u_j\|^2_{X^s_T} \\ &\lesssim T^{1-\sigma}b_j^2\|u\|^{2\sigma}_{Y^1_T}\|u\|^2_{X^s_T},\end{aligned} \tag{4.3.77}$$

where c_1, c_2 are fixed positive constants, and $\varepsilon > 0$ is sufficiently small. Observe that going from line 3 to line 4 uses the fact that $\sigma > \frac{\sqrt{3}}{2}$ since $s+\frac{3}{4\sigma}-\frac{1}{2} < s+\sigma-\frac{1}{2}$ precisely when $\sigma > \frac{\sqrt{3}}{2}$.

Estimate for I_2^j

We have by the Littlewood-Paley trichotomy

$$2^{2js}\int_{\mathbb{R}} P_j(P_{\geq j-4}|u|^{2\sigma}\partial_x u)\overline{u_j} = 2^{2js}\int_{\mathbb{R}}\tilde{P}_j(|u|^{2\sigma})\tilde{P}_{<j}\partial_x u\overline{u}_j + 2^{2js}\sum_{k>j}\int_{\mathbb{R}}\overline{u_j}P_j(\tilde{P}_k(|u|^{2\sigma})\tilde{P}_k\partial_x u) \tag{4.3.78}$$

for appropriate "fattened" Littlewood-Paley projections $\tilde{P}_j$. For the first term, using Bernstein's inequality and Hölder's inequality, and that $2^{-\delta j} \lesssim b_j$ we have,

$$\begin{aligned} &2^{2js} \int_0^T \left| \int_{\mathbb{R}} \tilde{P}_j(|u|^{2\sigma}) \tilde{P}_{<j} \partial_x u \overline{u}_j \right| \\ &\lesssim 2^{j(\frac{5}{2}-\sigma+s)} \|\tilde{P}_j |u|^{2\sigma}\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2} \|\tilde{P}_j u\|_{L_x^\infty L_T^2}^{2\sigma-1} \|\tilde{P}_j u\|_{L_x^2 L_T^2}^{2(1-\sigma)} \|\tilde{P}_{<j} D_x^{\sigma+s-\frac{3}{2}} u\|_{L_x^2 L_T^\infty} \\ &\lesssim T^{1-\sigma} \|D_x^{2+\sigma-2\sigma^2+\delta} \tilde{P}_j(|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2} \|\tilde{P}_j u\|_{X_T^s} \|u\|_{X_T^s} \\ &\lesssim T^{1-\sigma} b_j \|D_x^{2+\sigma-2\sigma^2+2\delta} \tilde{P}_j(|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2} \|\tilde{P}_j u\|_{X_T^s} \|u\|_{X_T^s}. \end{aligned} \tag{4.3.79}$$

Note that the first line follows since $s \in [1, \frac{3}{2}]$. Now, we estimate $\|D_x^{2+\sigma-2\sigma^2+2\delta} \tilde{P}_j(|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2}$. For notational convenience, write $2 + \sigma - 2\sigma^2 + 2\delta = \alpha$. We employ the Littlewood-Paley trichotomy and then Hölder's and Bernstein's inequality to obtain

$$\begin{aligned} \|D_x^\alpha \tilde{P}_j(|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2} &\lesssim \|D_x^{\alpha-1} \tilde{P}_j(|u|^{2\sigma-2} \overline{u} u_x)\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2} \\ &\lesssim \|D_x^{\alpha-1} \tilde{P}_j(\tilde{P}_{<j}(|u|^{2\sigma-2}\overline{u}) \tilde{P}_j u_x)\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2} \\ &+ \|D_x^{\alpha-1} \tilde{P}_j(\tilde{P}_{>j}(|u|^{2\sigma-2}\overline{u}) u_x)\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2} \\ &\lesssim \|u\|_{L_x^2 L_T^\infty}^{2\sigma-1} \|D_x^\alpha \tilde{P}_j u\|_{L_x^\infty L_T^2} + \|D_x^{\alpha-1}(|u|^{2\sigma-2}\overline{u})\|_{L_T^\infty L_x^\infty} \|u_x\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2}. \end{aligned} \tag{4.3.80}$$

Observe that $\|D_x^\alpha u\|_{L_x^\infty L_T^2} \lesssim \|u\|_{Y_T^1}$ since $\alpha < \sigma + \frac{1}{2}$ when $\sigma > \frac{\sqrt{3}}{2}$. Furthermore, by Corollary 4.2.11 and Sobolev embedding, we have

$$\|D_x^{\alpha-1}(|u|^{2\sigma-2}\overline{u})\|_{L_T^\infty L_x^\infty} \lesssim \|\langle D_x \rangle^{\frac{\alpha-1+\varepsilon}{2\sigma-1}} u\|_{L_T^\infty L_x^\infty}^{2\sigma-1} \lesssim \|u\|_{S_T^1}^{2\sigma-1} \tag{4.3.81}$$

where the last inequality again follows because $\sigma > \frac{\sqrt{3}}{2}$. Furthermore, by interpolating $\|u_x\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2}$ between $L_x^2 L_T^2$ and $L_x^\infty L_T^2$, we see that $\|u_x\|_{L_x^{\frac{2}{2\sigma-1}} L_T^2} \lesssim \|u\|_{X_T^1}$. Hence, we can control (4.3.79) by

$$T^{1-\sigma} b_j^2 \|u\|_{X_T^1}^{2\sigma} \|u\|_{X_T^s}^2. \tag{4.3.82}$$

For the other term in (4.3.78), we have

$$\begin{aligned}
&2^{2js}\int_0^T \left|\sum_{k>j}\int_{\mathbb{R}} \overline{u_j} P_j(\tilde{P}_k(|u|^{2\sigma})\tilde{P}_k\partial_x u)\right| \\
&\lesssim 2^{js}T^{(1-\sigma)}\|D_x^s u_j\|^{2(1-\sigma)}_{L^\infty_T L^2_x}\|D_x^s u_j\|^{2\sigma-1}_{L^\infty_x L^2_T}\sum_{k>j} 2^k \|\tilde{P}_k(|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}}L^2_T}\|\tilde{P}_k u\|_{L^2_x L^\infty_T} \\
&\lesssim 2^{j(s-\frac{1}{2}(1-2\sigma)^2)}T^{(1-\sigma)}\|u_j\|_{X^s_T}\sum_{k>j} 2^k \|\tilde{P}_k(|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}}L^2_T}\|\tilde{P}_k u\|_{L^2_x L^\infty_T} \\
&\lesssim 2^{j(s-\frac{1}{2}(1-2\sigma)^2)}T^{(1-\sigma)}\|u_j\|_{X^s_T}\sum_{k>j} 2^{k(\frac{3}{2}-\sigma-s+1)} \|\tilde{P}_k(|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}}L^2_T}\|\tilde{P}_k D_x^{s+\sigma-\frac{3}{2}} u\|_{L^2_x L^\infty_T} \\
&\lesssim T^{(1-\sigma)}\|u_j\|_{X^s_T}\sum_{k>j} 2^{(j-k)(s-\frac{1}{2}(1-2\sigma)^2)} \|\tilde{P}_k(D_x^{2+\sigma-2\sigma^2}|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}}L^2_T}\|\tilde{P}_k D_x^{s+\sigma-\frac{3}{2}} u\|_{L^2_x L^\infty_T} \\
&\lesssim T^{(1-\sigma)} b_j^2 \|u\|^2_{X^s_T}\|u\|^{2\sigma}_{X^1_T}\sum_{k>j} 2^{(j-k)((s-\frac{1}{2}(1-2\sigma)^2)-\delta)} \\
&\lesssim T^{(1-\sigma)} b_j^2 \|u\|^2_{X^s_T}\|u\|^{2\sigma}_{X^1_T}
\end{aligned} \tag{4.3.83}$$

where we estimated $\|\tilde{P}_k(D_x^{2+\sigma-2\sigma^2}|u|^{2\sigma})\|_{L_x^{\frac{2}{2\sigma-1}}L^2_T}$ in essentially the same way as we did with the previous term.

Estimate for I_3^j

We have

$$\begin{aligned}
[P_j, P_{<j-4}|u|^{2\sigma}]\partial_x u &= [P_j, P_{<j-4}|u|^{2\sigma}]\partial_x \tilde{P}_j u \\
&= 2^{-j}\int_{\mathbb{R}^2} K(y)\partial_x P_{<j-4}|u|^{2\sigma}(x+y_1)\partial_x\tilde{P}_j u(x+y_2)dy
\end{aligned} \tag{4.3.84}$$

for some kernel $K \in L^1$ with $\|K\|_{L^1} \lesssim 1$ (with a bound independent of j), see Lemma 4.2.1. Hence,

$$\int_0^T \left|\int_{\mathbb{R}} \overline{u_j}[P_j, P_{<j-4}|u|^{2\sigma}]\partial_x\tilde{P}_j u\right| \lesssim 2^{-j}\sup_{y\in\mathbb{R}^2}\int_0^T\int_{\mathbb{R}} |\partial_x P_{<j-4}|u|^{2\sigma}(x+y_1)||\partial_x\tilde{P}_j u(x+y_2)||u_j|. \tag{4.3.85}$$

This is estimated analogously to I_j^1. Indeed, we obtain by Cauchy Schwarz, Bernstein's inequality and Proposition 4.2.7,

$$\begin{aligned}
&2^{2js}\int_0^T \left|\int_{\mathbb{R}} \overline{u_j}[P_j, P_{<j-4}|u|^{2\sigma}]\partial_x \tilde{P}_j u\right| \\
&\lesssim 2^{2js}\|\tilde{P}_j D_x^{\frac{3}{4\sigma}-\frac{1}{2}+c_1\varepsilon} u\|^{2\sigma}_{L^\infty_x L^2_T}\|\tilde{P}_j u\|^{2(1-\sigma)}_{L^2_T L^2_x}\|u\|^{2\sigma-1}_{L^2_x L^\infty_T}\|D_x^{\sigma-\frac{1}{2}-c_2\varepsilon}u\|_{L^2_x L^\infty_T} \\
&\lesssim T^{(1-\sigma)}\|u\|^{2\sigma}_{Y^1_T}\|\tilde{P}_j u\|^2_{X^s_T} \\
&\lesssim T^{(1-\sigma)} b_j^2 \|u\|^{2\sigma}_{Y^1_T}\|u\|^2_{X^s_T},
\end{aligned} \tag{4.3.86}$$

where c_1, c_2 are positive constants depending on σ, s. The second line follows from the fact that $\frac{3}{4\sigma}-\frac{1}{2} < \sigma - \frac{1}{2}$ as long as $\sigma > \frac{\sqrt{3}}{2}$.

Hence, we obtain

$$\|P_j u\|_{L^\infty_T H^s_x} \lesssim a_j\|u_0\|_{H^s_x} + T^{\frac{1-\sigma}{2}} b_j \|u\|^{\sigma}_{X^1_T}\|u\|_{X^s_T}, \tag{4.3.87}$$

thus completing the proof of the $L^\infty_T H^s_x$ estimate. □

Proof of Proposition 4.3.6

We combine the energy estimate and the Y^s estimate to obtain

$$\|P_j u\|_{X^s_T} \lesssim_{\|u\|_{S^1_T}} a_j\|u_0\|_{H^s_x} + T^{\frac{1}{2}} b_j (1+\|u\|^{4\sigma}_{S^1_T})\|u\|_{X^s_T} + T^{\frac{1-\sigma}{2}} b_j\|u\|^{\sigma}_{X^1_T}\|u\|_{X^s_T}. \tag{4.3.88}$$

This proves part a) of Proposition 4.3.6.

Now we move to part b). Let us first assume $T \ll 1$ (but independent of ε). There are two components to consider. For high frequency, square summing over $j>0$ shows

$$\|P_{>0} u\|_{X^s_T} \lesssim_{\|u\|_{S^1_T}} \|u_0\|_{H^s_x} + T^{\frac{1}{2}} (1+\|u\|^{4\sigma}_{S^1_T})\|u\|_{X^s_T} + T^{\frac{1-\sigma}{2}} \|u\|^{\sigma}_{X^1_T}\|u\|_{X^s_T}. \tag{4.3.89}$$

On the other hand, directly applying the maximal function/Strichartz estimates in Proposition 4.2.3 and Proposition 4.2.4 and Bernstein's inequality to $P_{\le 0}u$, we easily obtain

$$\|P_{\le 0} u\|_{X^s_T} \lesssim \|u_0\|_{L^2_x} + \|P_{\le 0}(|u|^{2\sigma}u_x)\|_{L^1_T L^2_x} \lesssim \|u_0\|_{L^2_x} + T\|u\|^{2\sigma+1}_{S^1_T}. \tag{4.3.90}$$

From the above bounds, we see that the X^s_T norm of u converges to the H^1_x norm of the initial data as $T \to 0^+$. Let us now make the bootstrap assumption $\|u\|_{X^1_T} \le \varepsilon^{\frac{1}{2}}$. We then obtain from the above estimates,

$$\|u\|_{X^s_T} \lesssim_{\|u\|_{X^1_T}} \|u_0\|_{H^s_x} \le \varepsilon \tag{4.3.91}$$

where $T \ll 1$ (but independent of ε) and $1 \leq s \leq \frac{3}{2}$. To obtain the estimate for $T \sim 1$, we iterate the above procedure $O(T^{-1})$ many times (after suitable translating the initial data). This proves part b) of Proposition 4.3.6.

Next, we turn to the proof of Proposition 4.3.7. We proceed in a similar manner to Proposition 4.3.6, and prove separate estimates for the Y_T^0 and $L_T^\infty L_x^2$ components of the X_T^0 norm. For this purpose, we have the following two lemmas:

Lemma 4.3.19. (Y_T^0 estimate) Let v, σ, T, w, g and a be as in Proposition 4.3.7. Then we have the Y_T^0 estimate,

$$\|v\|_{Y_T^0} \lesssim \|v_0\|_{L_x^2} + T^{\frac{1}{2}}(1 + \|w\|_{X_T^1}^{4\sigma})\|v\|_{X_T^0} + T^{1-\sigma}\|g\|_Z\|a\|_{X_T^1}\|v\|_{X_T^0}. \tag{4.3.92}$$

Lemma 4.3.20. ($L_T^\infty L_x^2$ estimate) Let v, σ, T, w, g and a be as in Proposition 4.3.7. Then we have the estimate,

$$\|P_j v\|_{l_j^2 L_T^\infty L_x^2}^2 \lesssim \|v_0\|_{L_x^2}^2 + T^{1-\sigma}\|g\|_Z\|a\|_{X_T^1}\|v\|_{X_T^0}^2 + T^{1-\sigma}\|w\|_{X_T^1}^{2\sigma}\|v\|_{X_T^0}^2. \tag{4.3.93}$$

We begin with Lemma 4.3.19. The proof is almost the same as Lemma 4.3.10 with a couple of small differences. As in (4.3.28), we consider a similar paradifferential truncation of (4.3.21),

$$(i\partial_t + \partial_x^2)v_j = iP_{<j-4}(\chi_j(|w|^2)|w|^{2\sigma})\partial_x v_j + iP_{<j-4}(\varphi_j(|w|^2)|w|^{2\sigma})\partial_x v_j + f_j + g_j \tag{4.3.94}$$

where φ_j and χ_j are as in (4.3.28) and

$$f_j := iP_j(P_{\geq j-4}|w|^{2\sigma}\partial_x v) + i[P_j, P_{<j-4}|w|^{2\sigma}]\partial_x v, \tag{4.3.95}$$

$$g_j := 2P_j(\partial_x a \mathrm{Re}(gv)). \tag{4.3.96}$$

Analogously to the proof of Proposition 4.3.6, we define

$$\Psi_j(x) = -\frac{1}{2}P_{<j-4}\partial_x^{-1}[\chi_j(|w|^2)|w|^{2\sigma}] \tag{4.3.97}$$

and consider the new variable

$$\tilde{v}_j := v_j e^{i\Psi_j}. \tag{4.3.98}$$

By direct computation, $\tilde{v}_j$ solves the equation,

$$\begin{cases} (i\partial_t + \partial_x^2)\tilde{v}_j = ie^{i\Psi_j}P_{<j-4}[\varphi_j(|w|^2)|w|^{2\sigma}]\partial_x v_j + (-\partial_t\Psi_j + i\partial_x^2\Psi_j - (\partial_x\Psi_j)^2)\tilde{v}_j \\ \qquad\qquad +2e^{i\Psi_j}P_j(\partial_x a\mathrm{Re}(gv)) + e^{i\Psi_j}f_j, \\ \tilde{v}_j(0) = e^{i\Psi_j}v_j(0). \end{cases} \tag{4.3.99}$$

Now, Proposition 4.2.3, Proposition 4.2.4 and a similar argument to Proposition 4.3.6 yields the estimate

$$\begin{aligned}\|v\|_{Y^0_T} &\lesssim_T \|v_0\|_{L^2_x} + T^{\frac{1}{2}}[1+\|w\|_{X^1_T}]^{4\sigma}\|v\|_{X^0_T} \\ &\quad + \left(\sum_{j>0}\|\langle D_x\rangle^{\sigma-1}P_j(g\partial_x av)\|^2_{L^1_TL^2_x}\right)^{\frac{1}{2}}.\end{aligned} \tag{4.3.100}$$

It remains to control the last term. Indeed, we have by Bernstein and Sobolev embedding,

$$\begin{aligned}\|\langle D_x\rangle^{\sigma-1}P_j(g\partial_x av)\|_{L^1_TL^2_x} &\lesssim 2^{j(\sigma-1)}\|P_j(g\partial_x av)\|_{L^1_TL^2_x} \\ &\lesssim 2^{j(\sigma-1)}\|P_{<j-4}(\partial_x ag)\tilde{P}_jv\|_{L^1_TL^2_x} + \|P_j(P_{\geq j-4}(\partial_x ag)v)\|_{L^1_TL_x^{\frac{2}{3-2\sigma}}}.\end{aligned} \tag{4.3.101}$$

For the first term, we have by Bernstein's inequality,

$$\begin{aligned}2^{j(\sigma-1)}\|P_{<j-4}(\partial_x ag)\tilde{P}_jv\|_{L^1_TL^2_x} &\lesssim T^{\frac{3}{4}}\|\partial_x a\|_{L^\infty_TL^2_x}\|g\|_{L^\infty_TL^\infty_x}\|\tilde{P}_jD_x^{\sigma-1}v\|_{L^4_TL^\infty_x} \\ &\lesssim T^{\frac{3}{4}}\|a\|_{X^1_T}\|g\|_Z\|\tilde{P}_jD_x^{\sigma-1}v\|_{L^4_TL^\infty_x}.\end{aligned} \tag{4.3.102}$$

For the second term, we have by the usual Littlewood-Paley trichotomy,

$$\begin{aligned}\|P_j(P_{\geq j-4}(\partial_x ag)v)\|_{L^1_TL_x^{\frac{2}{3-2\sigma}}} &\lesssim \|P_j(\tilde{P}_j(\partial_x ag)P_{<j}v)\|_{L^1_TL_x^{\frac{2}{3-2\sigma}}} + \sum_{k\geq j}\|P_j(\tilde{P}_k(\partial_x ag)\tilde{P}_kv)\|_{L^1_TL_x^{\frac{2}{3-2\sigma}}} \\ &:= K_1^j + K_2^j.\end{aligned} \tag{4.3.103}$$

To estimate K_1^j, we have

$$\begin{aligned}\|P_j(\tilde{P}_j(\partial_x ag)P_{<j}v)\|_{L^1_TL_x^{\frac{2}{3-2\sigma}}} &\lesssim \|\tilde{P}_j(\partial_x ag)\|_{L^2_TL^2_x}\|P_{<j}v\|_{L^2_TL_x^{\frac{1}{1-\sigma}}} \\ &\lesssim \|D_x^{(1-\sigma+\varepsilon)(2\sigma-1)}\tilde{P}_j(g\partial_x a)\|_{L^2_TL^2_x}\|P_{<j}v\|^{2(1-\sigma)}_{L^2_TL^2_x}\|P_{<j}D_x^{\sigma-1-\varepsilon}v\|^{2\sigma-1}_{L^2_TL^\infty_x} \\ &\lesssim T^{1-\sigma}\|D_x^{(1-\sigma+\varepsilon)(2\sigma-1)}\tilde{P}_j(g\partial_x a)\|_{L^2_TL^2_x}\|v\|_{X^0_T}\end{aligned} \tag{4.3.104}$$

where in the last line we used the fact that by Sobolev embedding,

$$\|P_{<j}D_x^{\sigma-1-\varepsilon}v\|_{L^2_TL^\infty_x} \lesssim \|v\|_{L^\infty_TL^2_x} + \|P_{>0}v\|_{X^0_T} \lesssim \|v\|_{X^0_T} \tag{4.3.105}$$

as well as $\|P_{<j}v\|_{L^2_TL^2_x} \lesssim T^{\frac{1}{2}}\|v\|_{X^0_T}$. Now, setting $\alpha = (1-\sigma+\varepsilon)(2\sigma-1)$, we have by Bernstein's inequality, and a simple application of the Littlewood-Paley trichotomy,

$$\begin{aligned}\|D_x^\alpha\tilde{P}_j(g\partial_x a)\|_{L^2_TL^2_x} &\lesssim 2^{-j\varepsilon}\|D_x^{\alpha+\varepsilon}\tilde{P}_j(g\partial_x a)\|_{L^2_TL^2_x} \\ &\lesssim 2^{-j\varepsilon}\|D_x^{\alpha+\varepsilon}\partial_x a\|_{L_x^{\frac{1}{1-\sigma}}L^2_T}\|g\|_{L_x^{\frac{2}{2\sigma-1}}L^\infty_T} + 2^{-j\varepsilon}\|\partial_x a\|_{L^\infty_TL^2_x}\|D_x^{\alpha+\varepsilon}g\|_{L^2_TL^\infty_x}.\end{aligned} \tag{4.3.106}$$

Next, by interpolating $\|D_x^{\alpha+\varepsilon}\partial_x a\|_{L_x^{\frac{1}{1-\sigma}}L_T^2}$ between $D_x^{\frac{\alpha+2\varepsilon}{2\sigma-1}}a$ in $L_x^\infty L_T^2$ and $\partial_x a$ in $L_x^2L_T^2$, we see that for ε small enough, $\|D_x^{\alpha+\varepsilon}\partial_x a\|_{L_x^{\frac{1}{1-\sigma}}L_T^2} \lesssim \|a\|_{X_T^1}$ as long as $\sigma > \frac{3}{4}$ (because this corresponds to when $\frac{\alpha}{2\sigma-1} < \sigma - \frac{1}{2}$). Furthermore, clearly $\|D_x^{\alpha+\varepsilon}g\|_{L_T^2L_x^\infty} \lesssim \|g\|_Z$. Hence,

$$\|D_x^\alpha \tilde{P}_j(g\partial_x a)\|_{L_T^2L_x^2} \lesssim 2^{-j\varepsilon}\|g\|_Z\|a\|_{X_T^1}. \tag{4.3.107}$$

It is easy to see that a similar analysis works for K_2^j. Hence, we ultimately deduce that

$$K_1^j + K_2^j \lesssim 2^{-j\varepsilon}T^{1-\sigma}\|g\|_Z\|a\|_{X_T^1}\|v\|_{X_T^0}. \tag{4.3.108}$$

Square summing now gives

$$\left(\sum_{j>0}\|\langle D_x\rangle^{\sigma-1}P_j(g\partial_x av)\|_{L_T^1L_x^2}^2\right)^{\frac{1}{2}} \lesssim T^{1-\sigma}\|g\|_Z\|a\|_{X_T^1}\|v\|_{X_T^0}. \tag{4.3.109}$$

□

Next, we turn to the energy type $L_T^\infty L_x^2$ estimate in Lemma 4.3.20. First, it is straightforward to verify by a simple energy estimate that $P_{\leq 0}v$ is controlled in $L_T^\infty L_x^2$ by the right hand side of (4.3.93). Hence, let us restrict to controlling $P_{>0}v$.

Proof. Let $j > 0$. Projecting (4.3.21) onto frequency 2^j, multiplying by $-i\overline{P_jv}$, taking real part and integrating from 0 to T gives

$$\begin{aligned}\|P_jv\|_{L_T^\infty L_x^2}^2 &\lesssim \|P_jv_0\|_{L_x^2}^2 + \int_0^T\left|\int_{\mathbb{R}}P_j(g\partial_x av)\overline{v_j} + P_j(\overline{g}\partial_x a\overline{v})\overline{v_j}\right| + \int_0^T\left|\int_{\mathbb{R}}P_j(|w|^{2\sigma}\partial_x v)\overline{v_j}\right| \\ &:= \|P_jv_0\|_{L_x^2}^2 + I_1^j + I_2^j.\end{aligned} \tag{4.3.110}$$

Estimate for I_1^j

For simplicity, we show how to deal with the first term,

$$\int_{\mathbb{R}}P_j(g\partial_x av)\overline{v}_j \tag{4.3.111}$$

as the other term (involving the complex conjugate of gv) is essentially identical.

We have by the Littlewood-Paley trichotomy,

$$\int_{\mathbb{R}}P_j(g\partial_x av)\overline{v_j} = \int_{\mathbb{R}}P_j(P_{\geq j-4}(g\partial_x a)v)\overline{v_j} + \int_{\mathbb{R}}\tilde{P}_{<j}(g\partial_x a)\tilde{P}_jv\overline{\tilde{P}_jv}. \tag{4.3.112}$$

We expand the first term as

$$P_j(P_{\geq j-4}(g\partial_x a)v) = P_j(\tilde{P}_j(g\partial_x a)\tilde{P}_{<j}v) + \sum_{k\geq j} P_j(\tilde{P}_k(g\partial_x a)\tilde{P}_k v). \tag{4.3.113}$$

We obtain by Bernstein's inequality, Hölder and a simple application of the Littlewood-Paley trichotomy,

$$\begin{aligned}
&\int_0^T \left|\int_{\mathbb{R}} P_j(\tilde{P}_j(g\partial_x a)\tilde{P}_{<j}v)\overline{v_j}\right| \\
&\lesssim \|\tilde{P}_j D_x^{\frac{3}{4\sigma}-\frac{1}{2}+\varepsilon}(g\partial_x a)\|_{L_x^{\frac{2}{2\sigma-1}}L_T^2} \|\tilde{P}_j D_x^{\frac{3}{4\sigma}-\frac{1}{2}} v\|_{L_x^\infty L_T^2}^{2\sigma-1} \|\tilde{P}_j v\|_{L_x^2 L_T^2}^{2(1-\sigma)} \|\tilde{P}_{<j}\langle D_x\rangle^{\sigma-\frac{3}{2}-\varepsilon} v\|_{L_x^2 L_T^\infty} \\
&\lesssim 2^{-j\varepsilon} T^{1-\sigma} \|\tilde{P}_j D_x^{\frac{3}{4\sigma}-\frac{1}{2}+2\varepsilon}(g\partial_x a)\|_{L_x^{\frac{2}{2\sigma-1}}L_T^2} \|v\|_{X_T^0}^2 \\
&\lesssim 2^{-j\varepsilon} T^{1-\sigma} (\|D_x^{\frac{3}{4\sigma}-\frac{1}{2}+3\varepsilon} g\|_{L_x^\infty L_T^\infty} \|\partial_x a\|_{L_x^{\frac{2}{2\sigma-1}}L_T^2} + \|g\|_{L_x^{\frac{2}{2\sigma-1}}L_T^\infty} \|D_x^{\frac{3}{4\sigma}-\frac{1}{2}+2\varepsilon}\partial_x a\|_{L_x^\infty L_T^2}) \|v\|_{X_T^0}^2 \\
&\lesssim 2^{-j\varepsilon} T^{1-\sigma} \|g\|_Z \|a\|_{X_T^1} \|v\|_{X_T^0}^2
\end{aligned} \tag{4.3.114}$$

where in the last line, we used the assumption $\sigma > \frac{\sqrt{3}}{2}$. The second term in (4.3.113) is similarly estimated by $2^{-j\varepsilon}T^{1-\sigma}\|g\|_Z\|a\|_{X_T^1}\|v\|_{X_T^0}^2$. Hence,

$$\|P_j(P_{\geq j-4}(g\partial_x a)v)\overline{v_j}\|_{L_T^1 L_x^1} \lesssim 2^{-j\varepsilon} T^{1-\sigma} \|g\|_Z \|a\|_{X_T^1} \|v\|_{X_T^0}^2. \tag{4.3.115}$$

For the remaining term, we have

$$\begin{aligned}
-g\partial_x a &= gD_x Ha \\
&= D_x^{\frac{3}{2}-\sigma+\varepsilon}(gD_x^{\sigma-\frac{1}{2}-\varepsilon}Ha) - D_x^{\frac{3}{2}-\sigma+\varepsilon} g D_x^{\sigma-\frac{1}{2}-\varepsilon} Ha \\
&\quad - D_x^{\frac{3}{2}-\sigma+\varepsilon}(gD_x^{\sigma-\frac{1}{2}-\varepsilon}Ha) + D_x^{\frac{3}{2}-\sigma+\varepsilon} g D_x^{\sigma-\frac{1}{2}-\varepsilon} Ha + gD_x Ha.
\end{aligned} \tag{4.3.116}$$

Now, we estimate each term, thinking of the second line as a single term for which we will apply fractional Leibniz. For the first term in (4.3.116), we have by Hölder and Bernstein inequalities,

$$\begin{aligned}
&\int_0^T \left|\int_{\mathbb{R}} \tilde{P}_{<j} D_x^{\frac{3}{2}-\sigma+\varepsilon}(gD_x^{\sigma-\frac{1}{2}-\varepsilon}Ha)\tilde{P}_j\overline{v}\tilde{P}_j v\right| \\
&\lesssim \|\tilde{P}_{<j} D_x^{\frac{3}{2}-\sigma+\varepsilon}(gD_x^{\sigma-\frac{1}{2}-\varepsilon}Ha)|\tilde{P}_j v|^{2(1-\sigma)}\|_{L_x^1 L_T^{\frac{1}{1-\sigma}}} \|\tilde{P}_j v\|_{L_x^\infty L_T^2}^{2\sigma} \\
&\lesssim \|\tilde{P}_{<j} D_x^{\frac{3}{2}-\sigma+\varepsilon}(gD_x^{\sigma-\frac{1}{2}-\varepsilon}Ha)\|_{L_x^{\frac{1}{\sigma}} L_T^\infty} \|\tilde{P}_j v\|_{L_x^2 L_T^2}^{2(1-\sigma)} \|\tilde{P}_j v\|_{L_x^\infty L_T^2}^{2\sigma} \\
&\lesssim T^{1-\sigma} \|\tilde{P}_{<j}(gD_x^{\sigma-\frac{1}{2}-\varepsilon}Ha)\|_{L_x^{\frac{1}{\sigma}} L_T^\infty} \|\tilde{P}_j v\|_{X_T^0}^2 \\
&\lesssim T^{1-\sigma} \|g\|_{L_x^{\frac{2}{2\sigma-1}} L_T^\infty} \|D_x^{\sigma-\frac{1}{2}-\varepsilon} Ha\|_{L_x^2 L_T^\infty} \|\tilde{P}_j v\|_{X_T^0}^2,
\end{aligned} \tag{4.3.117}$$

where going from the second to the third line uses the fact that $\sigma > \frac{\sqrt{3}}{2}$.

Next, we estimate the second term in (4.3.116),

$$\begin{aligned}\int_0^T \left| \int_{\mathbb{R}} \tilde{P}_{<j}(D_x^{\frac{3}{2}-\sigma+\varepsilon} g D_x^{\sigma-\frac{1}{2}-\varepsilon} Ha)\tilde{P}_j \overline{v} \tilde{P}_j v \right| &\lesssim \|\tilde{P}_j v\|_{L_T^\infty L_x^2}^2 \|D_x^{\frac{3}{2}-\sigma+\varepsilon} g\|_{L_T^2 L_x^\infty} \|D_x^{\sigma-\frac{1}{2}-\varepsilon} Ha\|_{L_T^2 L_x^\infty} \\ &\lesssim T^{\frac{1}{2}} \|g\|_Z \|a\|_{X_T^1} \|\tilde{P}_j v\|_{L_T^\infty L_x^2}^2.\end{aligned} \tag{4.3.118}$$

Using Sobolev embedding and the fractional Leibniz rule, the third term is estimated analogously to the second term.

Combining the estimates and square summing then shows

$$\|I_1^j\|_{l_j^1(\mathbb{N})} \lesssim T^{1-\sigma} \|g\|_Z \|a\|_{X_T^1} \|v\|_{X_T^0}^2. \tag{4.3.119}$$

Estimate for I_2^j. A similar argument to Lemma 4.3.11 shows that

$$\|I_2^j\|_{l_j^1(\mathbb{N})} \lesssim T^{1-\sigma} \||w|^{2\sigma-1}\|_Z \|w\|_{X_T^1} \|v\|_{X_T^0}^2. \tag{4.3.120}$$

We now use the fact that for $\sigma > \frac{\sqrt{3}}{2}$, we have

$$\||w|^{2\sigma-1}\|_Z \lesssim \|w\|_{X_T^1}^{2\sigma-1}. \tag{4.3.121}$$

To see (4.3.121), first note that the $L_x^{\frac{2}{2\sigma-1}} L_T^\infty$ component is controlled by

$$\||w|^{2\sigma-1}\|_{L_x^{\frac{2}{2\sigma-1}} L_T^\infty} \lesssim \|w\|_{L_T^2 L_x^\infty}^{2\sigma-1} \lesssim \|w\|_{X_T^1}^{2\sigma-1}. \tag{4.3.122}$$

For the $L_T^\infty W_x^{\frac{3}{4\sigma}-\frac{1}{2}+\varepsilon,\infty}$ component, we have by Corollary 4.2.11, Sobolev embedding, and the fact that $\frac{(\frac{3}{4\sigma}-\frac{1}{2})}{2\sigma-1} < \frac{1}{2}$,

$$\|D_x^{\frac{3}{4\sigma}-\frac{1}{2}+\varepsilon} |w|^{2\sigma-1}\|_{L_T^\infty L_x^\infty} \lesssim \|w\|_{L_T^\infty H_x^1}^{2\sigma-1} \lesssim \|w\|_{X_T^1}^{2\sigma-1}. \tag{4.3.123}$$

This easily gives

$$\||w|^{2\sigma-1}\|_{L_T^\infty W_x^{\frac{3}{4\sigma}-\frac{1}{2}+\varepsilon,\infty}} \lesssim \|w\|_{X_T^1}^{2\sigma-1}. \tag{4.3.124}$$

Finally, for the $L_T^4 W_x^{\frac{3}{2}-\sigma+\varepsilon,\infty}$ component, we have by Corollary 4.2.11 and the fact that $\frac{\frac{3}{2}-\sigma}{2\sigma-1} < \sigma$,

$$\|D_x^{\frac{3}{2}-\sigma+\varepsilon} |w|^{2\sigma-1}\|_{L_T^4 L_x^\infty} \lesssim \|w\|_{L_T^4 W_x^{\sigma,\infty}}^{2\sigma-1} \lesssim \|w\|_{X_T^1}^{2\sigma-1} \tag{4.3.125}$$

which clearly gives

$$\||w|^{2\sigma-1}\|_{L_T^4 W_x^{\frac{3}{2}-\sigma+\varepsilon}} \lesssim \|w\|_{X_T^1}^{2\sigma-1}. \tag{4.3.126}$$

Combining the above three estimates gives (4.3.121).

Combining (4.3.121) and (4.3.120) gives

$$\|I_2^j\|_{l_j^1(\mathbb{N})} \lesssim T^{1-\sigma}\|w\|_{X_T^1}^{2\sigma}\|v\|_{X_T^0}^2. \tag{4.3.127}$$

Combining the above estimates for I_j^1 and I_j^2 completes the proof of Lemma 4.3.20.

Proof of Proposition 4.3.7. Now we complete the proof of Proposition 4.3.7.

Proof. Combining Lemma 4.3.19 and Lemma 4.3.20 with an argument similar to what was done in Proposition 4.3.6 gives for $T \sim 1$ and $\|g\|_Z, \|w\|_{X_T^1}, \|a\|_{X_T^1} \ll 1$,

$$\|v\|_{X_T^0} \lesssim \|v_0\|_{L_x^2}. \tag{4.3.128}$$

□

4.4 Well-posedness at low regularity

In this section, we aim to prove local well-posedness in H_x^s for $s \in [1, \frac{3}{2}]$ and $\sigma > \frac{\sqrt{3}}{2}$ assuming the conclusion of Theorem 4.1.2 when $\frac{3}{2} < s < 4\sigma$, which will be justified in a later section when we prove high-regularity estimates. Given the estimates established in the previous section, the scheme to prove well-posedness is relatively standard. We essentially follow the approach of [105]. See also the recent preprint [71] for a more detailed overview.

Frequency envelope bounds

Proposition 4.4.1. Let $\frac{\sqrt{3}}{2} < \sigma < 1$ and let u be as in Proposition 4.3.6. If a_j is an admissible frequency envelope for u_0 in H_x^s, then a_j is an admissible frequency envelope for u in X_T^s.

Indeed, let b_j be a X_T^s frequency envelope for the solution u. Obviously $b_0 \lesssim a_0$, so let us consider $j > 0$. By Proposition 4.3.6 a), we have

$$\|P_j u\|_{X_T^s} \lesssim_T a_j\|u_0\|_{H_x^s} + T^{\frac{1}{2}} b_j(1 + \|u\|_{S_T^1}^{4\sigma})\|u\|_{X_T^s} + T^{\frac{1-\sigma}{2}} b_j\|u\|_{X_T^1}^{\sigma}\|u\|_{X_T^s}. \tag{4.4.1}$$

Hence, by definition we have

$$b_j \lesssim a_j(1 + \|u_0\|_{H_x^s}\|u\|_{X_T^s}^{-1}) + T^{\frac{1-\sigma}{2}} b_j \|u\|_{X_T^1}^{\sigma} + T^{\frac{1}{2}} b_j (1 + \|u\|_{X_T^1}^{4\sigma}). \tag{4.4.2}$$

For T small enough, it follows from Proposition 4.3.6 that

$$b_j \lesssim a_j. \tag{4.4.3}$$

Iterating this procedure $O(T^{-1})$ many times shows that this is true for $T \lesssim 1$. This completes the proof.

Existence of H^s solutions

Now, we construct local H^s solutions to (gDNLS) for $1 \leq s \leq \frac{3}{2}$ as limits of more regular solutions.

Indeed, let $u_0 \in H^s$. Let $u^{(n)}$ be the globally well-posed $C_{loc}(\mathbb{R}; H_x^3)$ solution (to be constructed in a later section) to the equation,

$$\begin{cases} (i\partial_t + \partial_x^2)u^{(n)} = i|u^{(n)}|^{2\sigma}\partial_x u^{(n)}, \\ u_0^{(n)} = P_{<n}u_0. \end{cases} \tag{4.4.4}$$

Let $n > m$. We see that $v^{(m,n)} := u^{(n)} - u^{(m)}$ satisfies the equation

$$\begin{cases} (i\partial_t + \partial_x^2)v^{(m,n)} = i|u^{(n)}|^{2\sigma}\partial_x v^{(m,n)} + iG^{(n,m)}\partial_x u^{(m)} v^{(m,n)}, \\ v^{(m,n)}(0) = P_{m\leq \cdot < n}u_0, \end{cases} \tag{4.4.5}$$

where

$$G^{(n,m)} := \frac{(|u^{(n)}|^{2\sigma} - |u^{(m)}|^{2\sigma})}{u^{(n)} - u^{(m)}}. \tag{4.4.6}$$

Using Corollary 4.2.11, Sobolev embedding, the fact that $\sigma > \frac{\sqrt{3}}{2}$ and Proposition 4.3.6, one easily verifies that $G^{(n,m)}$ satisfies the conditions of Proposition 4.3.7 with $\|G^{(n,m)}\|_Z \lesssim_{\|u_0\|_{H_x^s}} 1$ (with the implicit constant independent of n and m). One likewise checks using Proposition 4.3.6 that $u^{(n)}$ satisfies $\|u^{(n)}\|_{X_T^1} \lesssim_{\|u_0\|_{H_x^s}} 1$ uniformly in n. Hence, by Proposition 4.3.7, we obtain for T small enough (depending on the size of the H_x^s norm of u_0),

$$\|v^{(m,n)}\|_{X_T^0} \lesssim \|P_{m\leq \cdot < n}u_0\|_{L_x^2}. \tag{4.4.7}$$

Hence, $u^{(n)}$ is Cauchy in X^0_T and thus converges to some $u \in X^0_T$. We show that in fact $u^{(n)} \to u$ in X^s_T.

To see this, we let a^n_j and a_j be admissable frequency envelopes for $P_{<n}u_0$ and u_0 respectively, in H^s_x. Clearly $(a^n_j) \to (a_j)$ in $l^2_j(\mathbb{N}_0)$ as $n \to \infty$. Now let $\varepsilon > 0$. Then thanks to Proposition 4.4.1, we have

$$\|P_{>j}u^{(n)}\|_{X^s_T} \lesssim \|(a^n_j)_{N>j}\|_{l^2_N(\mathbb{N})}\|u_0\|_{H^s_x}. \tag{4.4.8}$$

Hence, for $n \geq n_0(\varepsilon)$ large enough, we obtain the bound,

$$\|P_{>j}u^{(n)}\|_{X^s_T} \lesssim (\varepsilon + \|(a_j)_{N>j}\|_{l^2_N(\mathbb{N})})\|u_0\|_{H^s_x} \tag{4.4.9}$$

where the implicit constant is independent of j and n. Hence, there is $j = j(\varepsilon)$ such that for every $n > n_0$, we have

$$\|P_{>j}u^{(n)}\|_{X^s_T} \lesssim \varepsilon. \tag{4.4.10}$$

On the other hand, since $u^{(n)}$ converges in X^0_T, it follows that for $m, n > n_0$ large enough that

$$\|u^{(n)} - u^{(m)}\|_{X^s_T} \lesssim 2^{js}\|u^{(n)} - u^{(m)}\|_{X^0_T} + \|P_{\geq j}u^{(n)}\|_{X^s_T} + \|P_{\geq j}u^{(m)}\|_{X^s_T} \lesssim \varepsilon. \tag{4.4.11}$$

Hence, $u^{(n)}$ is Cauchy in X^s_T and thus converges to u. It is clear at this regularity that u solves the equation (gDNLS) in the sense of distributions. This shows existence.

Uniqueness and Lipschitz dependence in X^0

Here, we aim to show that solutions in X^1_T (and thus, also in X^s_T for $s > 1$) are unique and that they satisfy a weak Lipschitz type bound in X^0_T. For this, consider the difference of two solutions u^1 and u^2, $v := u^1 - u^2$. We see that v solves the equation,

$$\begin{cases} (i\partial_t + \partial_x^2)v = i|u^1|^{2\sigma}\partial_x v + iG\partial_x u^2 v, \\ v(0) = u^1(0) - u^2(0), \end{cases} \tag{4.4.12}$$

where

$$G = \frac{|u^1|^{2\sigma} - |u^2|^{2\sigma}}{u^1 - u^2}. \tag{4.4.13}$$

We see that Proposition 4.3.7 applies, and we obtain the weak Lipschitz bound

$$\|u^1 - u^2\|_{X^0_T} \lesssim \|u^1(0) - u^2(0)\|_{L^2_x}. \tag{4.4.14}$$

In particular, this shows uniqueness.

Continuous dependence in H^s

Here, we aim to show that the solution map is continuous in H^s. Specifically, we show that for each $R > 0$, there is $T = T(R) > 0$ such that the solution map from $\{u_0 : \|u_0\|_{H^s} < R\}$ to the corresponding X^s_T space is continuous. By rescaling the data and restricting to small enough time, we may assume without loss of generality that the conditions of Proposition 4.4.1 are satisfied.

Now, let $u_0^{(n)}$ be a sequence in H^s_x converging to u_0 in H^s_x. Let a_j and $a_j^{(n)}$ be the associated frequency envelopes for u_0 and $u_0^{(n)}$ given by (4.3.7). We have $(a_j^{(n)}) \to (a_j)$ in l^2. Now, let $\varepsilon > 0$. Let $N = N(\varepsilon)$ be such that $\|a^{(n)}_{j>N}\|_{l^2_j} \lesssim \varepsilon$. Using Proposition 4.4.1, we have $\|P_{>N} u^{(n)}\|_{X^s_T} \lesssim \varepsilon$ for all n. On the other hand, using the Lipschitz dependence at low frequency, we have

$$\|P_{<N}(u^{(n)} - u)\|_{X^s_T} \lesssim 2^{sN} \|u_0^{(n)} - u_0\|_{L^2}. \tag{4.4.15}$$

Now, for $n(N)$ large enough, we have

$$\|P_{<N} u^{(n)} - P_{<N} u\|_{X^s_T} \lesssim \varepsilon. \tag{4.4.16}$$

Hence, for such n, we have

$$\|u^{(n)} - u\|_{X^s_T} \lesssim \|P_{<N}(u^{(n)} - u)\|_{X^s_T} + \|P_{\geq N} u^{(n)}\|_{X^s_T} + \|P_{\geq N} u\|_{X^s_T} \lesssim \varepsilon. \tag{4.4.17}$$

It follows that

$$\limsup_{n\to\infty} \|u^{(n)} - u\|_{X^s_T} \lesssim \varepsilon. \tag{4.4.18}$$

Taking $\varepsilon \to 0$ then yields

$$\lim_{n\to\infty} \|u^{(n)} - u\|_{X^s_T} = 0 \tag{4.4.19}$$

as desired. This completes the proof of continuous dependence and also concludes the local well-posedness portion of the proof of Theorem 4.1.2 when $s \leq \frac{3}{2}$. □

Further discussion of the proofs

We now provide a brief discussion on how one can, in principle, go below the H^1_x well-posedness threshold, as well as justify some of the choices made in the proof.

It is instructive to discuss a version of this gauge transformation method which was successfully implemented in Tao's article [149] which established local well-posedness of the

Benjamin-Ono equation,

$$\begin{cases} u_t + Hu_{xx} = uu_x, \\ u(0) = u_0, \end{cases} \tag{4.4.20}$$

in H_x^1. The idea in Tao's paper was to do a type of gauge transformation by defining essentially,

$$w = P_{+hi}(e^{-iF}) \tag{4.4.21}$$

where $F(t,x)$ is a suitable spatial primitive of $u(t,x)$ and P_{+hi} is a projection onto large positive frequencies. Then one proves a priori H_x^2 estimates for w (which can be translated into H_x^1 estimates for u). While the coefficient u in the nonlinearity in Benjamin-Ono is only of linear order (and so one might at first naïvely suspect that this equation behaves similarly to (gDNLS) when $\sigma = \frac{1}{2}$), the spatial primitive F still essentially solves a linear Schrödinger equation (up to a perturbative error). A refinement of this gauge transformation idea appeared in [73] in which L_x^2 well-posedness (among other results) for Benjamin-Ono was proven. Loosely speaking, in this latter paper, the authors performed a gauge transformation on each frequency scale to remove the leading order paradifferential part of the nonlinearity and then performed a quadratic normal form correction to remove the milder terms in the nonlinearity. Our so-called partial gauge transformation is more analogous to what was done in that paper. Specifically, the analogue of F in our proof is essentially the family of functions Φ_j as defined in (4.3.29), which in addition to the frequency localization scale, takes into account the pointwise size of u relative to the frequency scale. However, in our case, there is no obvious cancellation arising in the term $(i\partial_t\Phi_j + \partial_x^2\Phi_j)$, which forces us to estimate each term $\partial_t\Phi_j$ and $\partial_x^2\Phi_j$ separately. This is one of the major sources for the losses in our low regularity estimates.

This issue actually also adds technical difficulty when trying to lower the local well-posedness threshold below H_x^1. For instance, when estimating $\partial_t\Phi_j$ in Proposition 4.3.6, there are expressions essentially of the form

$$P_{<j}\partial_x^{-1}(gv_1v_2) \tag{4.4.22}$$

that we bound in $L_T^1 L_x^\infty$, where g is some bounded function and v_1 and v_2 are linear expressions in u_x or $\overline{u_x}$. Unfortunately, in these expressions, it doesn't seem that typically the output frequency of the product gv_1v_2 is comparable to the frequencies of the individual terms v_1 and v_2, and so the ∂_x^{-1} can't be "distributed" amongst these factors to obtain expressions with lower order derivatives in place of u_x. One workaround to this issue could be to place any factors of u_x arising in such an expression in an appropriate maximal function/smoothing space as in Proposition 4.2.4. Proceeding this way will likely lead to losses worse than the $1-\sigma$ derivatives already observed in the current low regularity estimates. However, this should work in principle to lower the well-posedness threshold below H^1_x when σ is close to 1. We decided not to do this for the sake of simplicity, as our preliminary calculations suggested that the dependence of the well-posedness threshold on σ would be rather complicated when $s<1$, at least without introducing some new tools.

4.5 High regularity estimates

In this section, we aim to prove a priori H^{2s}_x-type bounds for a global solution u to a family of regularizations of (gDNLS),

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\eta P_{<k}|v|^{2\sigma}u_x, \\ u(0) = P_{<k}u_0, \end{cases} \tag{4.5.1}$$

where $k\in\mathbb{N}$, $v\in C^2(\mathbb{R};H^\infty_x)$, $2s$ is in the range $2-\sigma<2s<4\sigma$, η is a suitable time-dependent cutoff function which is equal to 1 on the unit time interval $[-1,1]$ and supported within $(-2,2)$, and $u_0\in H^{2s}_x$ has sufficiently small norm. The key difficulty here is to obtain estimates independent of the regularization parameter k. As mentioned earlier, this is somewhat subtle because the nonlinearity is too rough to directly obtain an energy estimate by simply applying D^{2s}_x to the equation. Our overarching idea, morally, is to instead obtain suitable estimates for time derivatives, $D^s_t u$, of order $s<2\sigma$ for solutions to (4.5.1). This is one of the key technical reasons for truncating the nonlinearity with the time-dependent cutoff η and working with global in-time solutions to (4.5.1). For small enough data, one expects to be able to construct a solution u to this equation on the time interval $[-2,2]$, and then extend it to a global solution using the fact that u should solve the linear Schrödinger equation for $|t|>2$. The idea of truncating the nonlinearity with a time-dependent cutoff in order to obtain global in time solutions (to facilitate use of Fourier analysis in the time variable) is not a new idea. See for instance, [15] and [16].

Before outlining our strategy in more detail, we give an overview of the functional setting and relevant notation for this problem.

Function spaces and notation

Here, we fix some basic notation and describe the function spaces used in our construction of solutions at high regularity.

We will use S_k, $S_{<k}$ and $S_{\geq k}$ to denote the temporal variants of the spatial Littlewood-Paley projections $P_k, P_{<k}$ and $P_{\geq k}$ as defined in Section 4.2. We write $\phi(2^{-j}\xi)$ to denote the spatial Fourier multiplier for P_j and $\psi(2^{-k}\tau)$ to denote the temporal Fourier multiplier for S_k.

We will also need to sometimes distinguish between a compact time interval and the whole space in our estimates. For this purpose, let us denote for a Banach space X, $L^p_t X := L^p(\mathbb{R}; X)$ (that is, we use a lowercase t to emphasize when the underlying time interval is $\mathbb{R}$). For $T > 0$, we use $L^p_T X := L^p([-T,T]; X)$ when we want to emphasize that the time interval is compact.

Next, for the range of $2s \in (2-\sigma, 4\sigma)$ we are considering, the smoothing and maximal function type norms from the low regularity estimates are not needed. We modify our function spaces accordingly and only use standard L^2_x based Sobolev spaces and standard Strichartz spaces (see below). Since both spatial and temporal regularity will be relevant in our analysis, we make the convention from here on that a real number s will correspond to the Sobolev regularity of a function in the time variable. In light of the scaling of the linear Schrödinger equation, it is natural to use $2s$ to denote the corresponding spatial regularity. With this in mind, for $s \geq 0$ and $T > 0$, we denote the relevant Strichartz type space by $\mathcal{S}^{2s}_T := L^4_T W^{2s,\infty}_x \cap L^\infty_T H^{2s}_x$. We also define the energy type space $\mathcal{X}^{2s}_T$ by the norm,

$$\|u\|_{\mathcal{X}^{2s}_T} := \|P_{\leq 0}u\|_{L^\infty_T H^{2s}_x} + \left(\sum_{j>0} \|P_j u\|^2_{L^\infty_T H^{2s}_x}\right)^{\frac{1}{2}}. \tag{4.5.2}$$

Clearly this controls the $C([-T,T]; H^{2s}_x)$ norm. The reason we opt for this slightly stronger norm (as opposed to just $\|u\|_{L^\infty_T H^{2s}_x}$) is because it will be slightly more convenient for proving frequency envelope bounds. Furthermore, we have the trivial embedding

$$X^{2s}_T \subseteq \mathcal{X}^{2s}_T. \tag{4.5.3}$$

Finally, since estimates for time derivatives will play a key role in our analysis, it will also be convenient to introduce the auxiliary norm

$$\|u\|_{Z^s_{p,q}} := \|\langle D_t\rangle^s u\|_{L^p_t L^q_x} + \|\langle D_x\rangle^{2s} u\|_{L^p_t L^q_x}. \tag{4.5.4}$$

When $q = 2$, we will simply abbreviate this by Z^s_p.

The reader should keep in mind that although we will often time-localize u (or the nonlinearity) to be compactly supported in time, some mild care must be taken in the estimates when nonlocal operators such as D^s_t are involved. This is especially relevant when comparing $L_t X$ and $L_T X$ type norms.

A frequency localized H^{2s}_x bound

The key result for this section is the following frequency localized H^{2s}_x a priori bound for (4.5.1).

Proposition 4.5.1. Let $2-\sigma < 2s < 4\sigma$, $T = 2$ and $u_0 \in H^{2s}_x$. Suppose that $u \in C^2(\mathbb{R}; H^\infty_x)$ solves (4.5.1). Furthermore, let a_j be a H^{2s}_x frequency envelope for u_0 and let b^1_j and b^2_j be $\mathcal{X}^{2s}_T$ frequency envelopes for u and v, respectively. Let $b_j := \max\{b^1_j, b^2_j\}$. Furthermore, let $0 < \varepsilon \ll 1$ and assume that for each $0 < \delta \ll 1$

$$\|v\|_{\mathcal{S}^{1+\delta}_T} + \|(i\partial_t + \partial^2_x)v\|_{Z^{s-1+\delta}_\infty \cap \mathcal{S}^\delta_T} \lesssim_\delta \varepsilon. \tag{4.5.5}$$

Then $P_j u$ satisfies the estimate,

$$\begin{aligned}\|P_j u\|^2_{\mathcal{X}^{2s}_T} \lesssim a^2_j \|u_0\|^2_{H^{2s}_x} + b^2_j \varepsilon^{2\sigma}(\|u\|^2_{\mathcal{X}^{2s}_T} + \|u\|^2_{\mathcal{S}^1_T}) + b^2_j \varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}^1_T}\|u\|_{\mathcal{X}^{2s}_T}\|v\|_{\mathcal{X}^{2s}_T} \\ + b^2_j \varepsilon^{4\sigma-2}\|u\|^2_{\mathcal{S}^1_T}\|v\|^2_{\mathcal{X}^{2s}_T}.\end{aligned} \tag{4.5.6}$$

Furthermore, by square summing, we also have

$$\|u\|^2_{\mathcal{X}^{2s}_T} \lesssim \|u_0\|^2_{H^{2s}_x} + \varepsilon^{2\sigma}(\|u\|^2_{\mathcal{X}^{2s}_T} + \|u\|^2_{\mathcal{S}^1_T}) + \varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}^1_T}\|u\|_{\mathcal{X}^{2s}_T}\|v\|_{\mathcal{X}^{2s}_T} + \varepsilon^{4\sigma-2}\|u\|^2_{\mathcal{S}^1_T}\|v\|^2_{\mathcal{X}^{2s}_T}. \tag{4.5.7}$$

Remark 4.5.2. Crucially, it should be noted that the implied constant in the bound above does not depend on the regularization parameter k.

Remark 4.5.3. The reader should carefully observe the restriction $T = 2$ and not $T \leq 2$ in Proposition 4.5.1. This is because η is localized in time to a unit scale. More work is required to show that we have suitable bounds for $T \leq 2$. This will be studied further in Section 4.6.

Next, we give a brief outline for how we will obtain such an estimate. As mentioned above, to minimize the number of derivatives which fall on the rough part of the inhomogeneous term, $|v|^{2\sigma}$, we will prove what is essentially an energy type estimate for $D^s_t u$ instead of $D_x^{2s}u$ and use the bounds for $D_t^s u$ to estimate $D_x^{2s}u$. This is consistent with the scaling symmetry of (gDNLS). There is one technical caveat however. Namely, one expects to be able to convert estimates for $D_t^s u$ to estimates for $D_x^{2s}u$ when the time frequency τ of a solution u to (4.5.1) is close to $-\xi^2$ where ξ is the spatial frequency (i.e. in the so-called low modulation region). However, this is not guaranteed due to the presence of the inhomogeneous term in the equation. Therefore, we need a suitable way of controlling $D_x^{2s}u$ for the portion of u which has space-time Fourier support far away from the characteristic hypersurface $\tau = -\xi^2$. In other words, we also need an estimate for u in the so-called high modulation region.

With this in mind, we split our analysis into two parts. First, we prove an elliptic type estimate in the high modulation region for solutions to (4.5.1) which will allow us to suitably control $D_x^{2s}u$ in terms of the portion of $D_x^{2s}u$ localized near the characteristic hypersurface, as well as a lower order term stemming from the nonlinearity. To control $D_x^{2s}u$ in the low modulation region, we essentially obtain an energy type estimate for $D_t^s u$ (the benefit being that we only have to differentiate the nonlinearity s times in the time variable as opposed to $2s$ times in the spatial variable). When u is localized near the characteristic hypersurface, this is precisely the regime in which we expect to be able to suitably control $D_x^{2s}u$ by $D_t^s u$. Proposition 4.5.1 will then follow from combining the low and high modulation analysis.

The high modulation estimate

We begin with the high modulation estimate, Lemma 4.5.4. This will be useful for estimating the portion of a (time-localized) solution to (4.5.1) which has space-time Fourier support away from the characteristic hypersurface. This can also be thought of as an elliptic space-time estimate.

Lemma 4.5.4. Let $u_0 \in H_x^\infty$ and suppose $u \in C^1(\mathbb{R}; H_x^\infty)$ solves the equation,

$$\begin{cases} (i\partial_t + \partial_x^2)u = f, \\ u(0) = u_0. \end{cases} \tag{4.5.8}$$

Let $0 \leq s \leq 1$, $j, k > 0$, $p, q \in [1, \infty]$ and suppose $|k - 2j| > 4$. Then $P_j S_k u$ satisfies the estimate,

$$\|P_j S_k \langle D_x \rangle^{2s} u\|_{L_t^p L_x^q} + \|P_j S_k \langle D_t \rangle^s u\|_{L_t^p L_x^q} \lesssim \|\tilde{P}_j \tilde{S}_k \langle D_t \rangle^{s-1} f\|_{L_t^p L_x^q}. \tag{4.5.9}$$

The result also holds for $k = 0$, when S_0 is replaced by $S_{\leq 0}$.

Proof. We prove the estimate for $\langle D_x\rangle^{2s}u$. The estimate for $\langle D_t\rangle^s u$ is similar. Notice that

$$\begin{aligned}[\mathcal{F}_{t,x}(\langle D_x\rangle^{2s}S_kP_ju)](\tau,\xi) &= \langle\xi\rangle^{2s}\psi(2^{-k}\tau)\phi(2^{-j}\xi)[\mathcal{F}_{t,x}(\tilde{S}_k\tilde{P}_ju)](\tau,\xi)\\ &= -\frac{\langle\xi\rangle^{2s}}{\tau+\xi^2}\psi(2^{-k}\tau)\phi(2^{-j}\xi)[\mathcal{F}_{t,x}\tilde{S}_k\tilde{P}_j(i\partial_t+\partial_x^2)u](\tau,\xi).\end{aligned} \tag{4.5.10}$$

Hence, by Young's inequality and (4.5.8), we have (using that $\psi(2^{-k}\tau)\phi(2^{-j}\xi)$ is supported away from $\tau+\xi^2=0$),

$$\begin{aligned}\|\langle D_x\rangle^{2s}S_kP_ju\|_{L_t^pL_x^q} &\lesssim \|\mathcal{F}_{t,x}^{-1}[\frac{\langle\xi\rangle^{2s}}{\tau+\xi^2}\psi(2^{-k}\tau)\phi(2^{-j}\xi)]\|_{L_t^1L_x^1}\|(i\partial_t+\partial_x^2)\tilde{S}_k\tilde{P}_ju\|_{L_t^pL_x^q}\\ &\lesssim \|\mathcal{F}_{t,x}^{-1}[\frac{\langle\xi\rangle^{2s}}{\tau+\xi^2}\psi(2^{-k}\tau)\phi(2^{-j}\xi)]\|_{L_t^1L_x^1}\|\tilde{S}_k\tilde{P}_jf\|_{L_t^pL_x^q}.\end{aligned} \tag{4.5.11}$$

It remains then to show that

$$\|\mathcal{F}_{t,x}^{-1}[\frac{\langle\xi\rangle^{2s}}{\tau+\xi^2}\psi(2^{-k}\tau)\phi(2^{-j}\xi)]\|_{L_t^1L_x^1} \lesssim 2^{-k(1-s)}. \tag{4.5.12}$$

A simple change of variables shows that

$$\|\mathcal{F}_{t,x}^{-1}[\frac{\langle\xi\rangle^{2s}}{\tau+\xi^2}\psi(2^{-k}\tau)\phi(2^{-j}\xi)]\|_{L_t^1L_x^1} = \|\mathcal{F}_{t,x}^{-1}[\frac{\langle 2^j\xi\rangle^{2s}}{2^k\tau+2^{2j}\xi^2}\psi(\tau)\phi(\xi)]\|_{L_t^1L_x^1}. \tag{4.5.13}$$

Then we have

$$\frac{\langle 2^j\xi\rangle^{2s}}{2^k\tau+2^{2j}\xi^2}\psi(\tau)\phi(\xi) = 2^{k(s-1)}\frac{(2^{-k}+2^{2j-k}\xi^2)^s}{\tau+2^{2j-k}\xi^2}\psi(\tau)\phi(\xi) := 2^{k(s-1)}F_{j,k}(\tau,\xi). \tag{4.5.14}$$

It is easy to see that for multi-indices $0\le|\alpha|\le 3$,

$$|\partial_{\tau,\xi}^{\alpha}F_{j,k}| \lesssim 1 \tag{4.5.15}$$

so that (since $\phi\psi$ is supported on $[-2,2]\times[-2,2]$)

$$\|\partial_{\tau,\xi}^{\alpha}F_{j,k}\|_{L_{\tau,\xi}^1} \lesssim 1 \tag{4.5.16}$$

with bound independent of j and k. It follows that

$$\|\mathcal{F}_{t,x}^{-1}[\frac{\langle 2^j\xi\rangle^{2s}}{2^k\tau+2^{2j}\xi^2}\psi(\tau)\phi(\xi)]\|_{L_t^1L_x^1} \lesssim 2^{k(s-1)}\|(1+|x|+|t|)^{-3}\|_{L_t^1L_x^1} \lesssim 2^{k(s-1)} \tag{4.5.17}$$

which is what we wanted to show. The case for $\langle D_t\rangle^s u$ is similar. □

From this lemma, we obtain a very useful corollary which will allow us to control derivatives of u in the high modulation region with convenience and reduce matters to proving a suitable low modulation bound.

Corollary 4.5.5. Let $u \in C^2(\mathbb{R}; H_x^\infty)$, and let the notation be as in Lemma 4.5.4. Then for every $\delta > 0$ and $j > 0$, we have

a) If $0 \leq s < 1$,

$$\|P_j \langle D_x \rangle^{2s} u\|_{L_t^p L_x^q} + \|P_j \langle D_t \rangle^s u\|_{L_t^p L_x^q} \lesssim_\delta \|\tilde{S}_{2j} P_j \langle D_x \rangle^{2s} u\|_{L_t^p L_x^q} + \|\tilde{P}_j \langle D_t \rangle^{s-1+\delta} f\|_{L_t^p L_x^q} \quad (4.5.18)$$

and

b) If $1 \leq s < 2\sigma$,

$$\|P_j \langle D_x \rangle^{2s} u\|_{L_t^p L_x^2} + \|P_j \partial_t \langle D_t \rangle^{s-1} u\|_{L_t^p L_x^2} \lesssim_\delta \|\tilde{S}_{2j} P_j \langle D_x \rangle^{2s} u\|_{L_t^p L_x^2} + \|\tilde{P}_j f\|_{Z_{p,2}^{s-1+\delta}} \quad (4.5.19)$$

where $\tilde{S}_{2j} = S_{[2j-4, 2j+4]}$.

Proof. For a), this follows from the Bernstein type estimate $\|D_x^{2s} \tilde{S}_{2j} P_j u\|_{L_t^p L_x^q} \sim \|D_t^s \tilde{S}_{2j} P_j u\|_{L_t^p L_x^q}$ and from Lemma 4.5.4 by summing over $k > 0, |k - 2j| > 4$ (which is where the requirement of having $\delta > 0$ comes in to play). Then b) follows from part a) with u replaced by $\partial_t u$ and s replaced by $s - 1$, and then by expanding $\partial_t D_x^{2s-2} P_j u = i\partial_x^2 D_x^{2s-2} P_j u - i D_x^{2s-2} P_j f$. □

Remark 4.5.6. We remark that in part b), if f takes the form of $f = i\eta P_{<k} |u|^{2\sigma} u_x$ as in (4.5.1) then if δ is sufficiently small, we expect to be able to control the last term on the right as long as $2s - 2 < 2\sigma$ which is satisfied automatically, because $2s < 4\sigma < 2\sigma + 2$ in the range $\frac{1}{2} < \sigma < 1$. If we were looking at the case $\sigma > 1$, this would present a new limiting threshold for which we expect to obtain estimates for u, c.f. [154].

In light of the above remark, one should observe at this point that the high modulation estimate above essentially reduces proving Proposition 4.5.1 to obtaining an estimate for the $L_T^\infty H_x^{2s}$ norm of a solution u to (4.5.1) in the low modulation region, as well as controlling an essentially perturbative source term stemming from the nonlinearity in (4.5.1). With this in mind, we now turn to the low modulation estimate, which is essentially the heart of the matter.

Low modulation estimates

Next we prove suitable bounds for the $L_T^\infty H_x^{2s}$ norm of a solution u to (4.5.1) in the low modulation region. Specifically, we prove the following energy type bound to control the portion of u which is localized near the characteristic hypersurface.

Lemma 4.5.7. Let $u_0 \in H_x^{2s}$ and suppose that $u \in C^2(\mathbb{R}; H_x^\infty)$ solves (4.5.1). Let $T = 2$, $2-\sigma < 2s < 4\sigma$, a_j be an admissible H_x^{2s} frequency envelope for u_0, and b_j^1, b_j^2 be $\mathcal{X}_T^{2s}$ frequency envelopes for u and v, respectively. Take $b_j := \max\{b_j^1, b_j^2\}$. Let $0 < \varepsilon \ll 1$ and suppose v satisfies the estimates,

$$\|v\|_{\mathcal{S}_T^{1+\delta}} + \|(i\partial_t + \partial_x^2)v\|_{Z_\infty^{s-1+\delta} \cap \mathcal{S}_T^\delta} \lesssim_\delta \varepsilon \tag{4.5.20}$$

for each $0 < \delta \ll 1$. Then for every $j \geq 0$, we have

$$\begin{aligned}\|\tilde{S}_{2j} P_j D_x^{2s} u\|_{L_T^\infty L_x^2}^2 \lesssim_\delta a_j^2 \|u_0\|_{H_x^{2s}}^2 + b_j^2 \varepsilon^{2\sigma}(\|u\|_{\mathcal{X}_T^{2s}}^2 + \|u\|_{\mathcal{S}_T^1}^2) + b_j^2 \varepsilon^{2\sigma-1} \|u\|_{\mathcal{S}_T^1} \|u\|_{\mathcal{X}_T^{2s}} \|v\|_{\mathcal{X}_T^{2s}} \\ + b_j^2 \varepsilon^{4\sigma-2} \|u\|_{\mathcal{S}_T^1}^2 \|v\|_{\mathcal{X}_T^{2s}}^2.\end{aligned} \tag{4.5.21}$$

Remark 4.5.8. As a brief but important remark, it should be noted that for $\alpha \geq 0$ there is no need to distinguish between $\|u\|_{L_t^\infty H_x^\alpha}$ and $\|u\|_{L_T^\infty H_x^\alpha}$. This is because outside of $[-2, 2]$, u solves a linear Schrödinger equation, and so the H_x^α norms are constant on both $(-\infty, -2]$ and $[2, \infty)$.

It will also be convenient to introduce the notation $\tilde{v} := \tilde{\eta} v$ where $\tilde{\eta}$ is a time-dependent cutoff supported in $(-2, 2)$ which is equal to 1 on the support of η. For notational convenience, we also write $|v|_{<k}^{2\sigma}$ to denote $P_{<k}|v|^{2\sigma}$. Now, we begin with the proof of the energy type bound in Lemma 4.5.7.

Proof. Note that we can write $\eta |v|_{<k}^{2\sigma} = \eta |\tilde{v}|_{<k}^{2\sigma}$. Next, we apply $\tilde{S}_{2j} P_j := S_{[2j-4, 2j+4]} P_j$ to the equation and see that $\tilde{S}_{2j} P_j u$ solves the equation,

$$(i\partial_t + \partial_x^2)\tilde{S}_{2j} P_j u = i\tilde{S}_{2j} P_j (\eta |\tilde{v}|_{<k}^{2\sigma} u_x), \tag{4.5.22}$$

with initial data $(\tilde{S}_{2j} P_j u)(0)$. Next, we do a paradifferential expansion of the "nonlinear" term $i\tilde{S}_{2j} P_j (\eta |\tilde{v}|_{<k}^{2\sigma} u_x)$, in both the space and time variable, which splits this term into five interactions. Indeed, first by commuting the spatial projection P_j, we have

$$\begin{aligned}\tilde{S}_{2j} P_j (i\eta |\tilde{v}|_{<k}^{2\sigma} u_x) = \tilde{S}_{2j}(i\eta P_{<j-4} |\tilde{v}|_{<k}^{2\sigma} \partial_x P_j u) + \tilde{S}_{2j}(i\eta [P_j, P_{<j-4} |\tilde{v}|_{<k}^{2\sigma}] \partial_x u) \\ + \tilde{S}_{2j} P_j (i\eta P_{\geq j-4} |\tilde{v}|_{<k}^{2\sigma} \partial_x u).\end{aligned} \tag{4.5.23}$$

Then by commuting the temporal projection $\tilde{S}_{2j}$ in the first term, we obtain

$$\begin{aligned}\tilde{S}_{2j}P_j(i\eta|\tilde{v}|^{2\sigma}_{<k}u_x) &= S_{<2j-8}(i\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})\partial_x P_j\tilde{S}_{2j}u + [\tilde{S}_{2j}, S_{<2j-8}(i\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})]\partial_x P_j u \\ &+ \tilde{S}_{2j}(S_{\geq 2j-8}(i\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})\partial_x P_j u) + \tilde{S}_{2j}(i\eta[P_j, P_{<j-4}|\tilde{v}|^{2\sigma}_{<k}]\partial_x u) \\ &+ i\tilde{S}_{2j}P_j(\eta P_{\geq j-4}|\tilde{v}|^{2\sigma}_{<k}\partial_x u).\end{aligned} \tag{4.5.24}$$

We label these terms in the order they appear above as $A_1, \ldots, A_5$.

We make a brief remark about each of the above interactions before proceeding with the estimates. The first term, A_1, which corresponds to the low-high interaction (in spatial frequency) between the coefficient $i\eta|\tilde{v}|^{2\sigma}_{<k}$ and $\partial_x u$ reacts well to a standard energy type estimate for $P_j\tilde{S}_{2j}u$ since the single spatial derivative ∂_x on $P_j\tilde{S}_{2j}u$ can be integrated by parts onto the coefficient $i\eta|\tilde{v}|^{2\sigma}_{<k}$. The terms A_2, A_3 and A_4 are expected to be treated perturbatively. These in a very loose sense correspond to more balanced frequency interactions for which (space or time) derivatives can be distributed somewhat evenly between the terms $\partial_x u$ and $i\eta|\tilde{v}|^{2\sigma}_{<k}$. The most serious issue comes from A_5, which is the situation in which the coefficient $i\eta|\tilde{v}|^{2\sigma}_{<k}$ is at high spatial frequency compared to $\partial_x u$. Some care must be taken here to ensure that this term is not "differentiated" $2s$ times in the spatial variable, but instead "differentiated" at most only s times in the time variable.

Now, we continue with the proof. We begin with a standard energy type estimate. Indeed, multiplying (4.5.22) by $-i2^{4js}\overline{\tilde{S}_{2j}P_j u}$, taking real part and integrating over $\mathbb{R}$ in the spatial variable and from 0 to $\overline{T}$ with $|\overline{T}| \leq 2$ gives,

$$\|D_x^{2s}\tilde{S}_{2j}P_j u\|^2_{L^\infty_T L^2_x} \lesssim 2^{4js}\|(\tilde{S}_{2j}P_j u)(0)\|^2_{L^2_x} + \sum_{k=1}^{5} I_j^k \tag{4.5.25}$$

where

$$I_j^k := 2^{4js}\int_{-T}^{T}\big|\mathrm{Re}\int_{\mathbb{R}} -iA_k\overline{\tilde{S}_{2j}P_j u}\big|. \tag{4.5.26}$$

Now, we estimate each term. We need to deal with both the initial data term $2^{4js}\|(\tilde{S}_{2j}P_j u)(0)\|^2_{L^2_x}$ and the I_j^k terms for $k = 1, ..., 5$. First we deal with the latter terms.

Estimate for I_j^1

We integrate by parts and use Bernstein's inequality to obtain

$$\begin{aligned} I_j^1 &= 2^{4js}\int_{-T}^{T}\left|\mathrm{Re}\int_{\mathbb{R}} S_{<2j-8}(\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})\partial_x\tilde{S}_{2j}P_ju\overline{\tilde{S}_{2j}P_ju}\right| \\ &\lesssim 2^{4js}\int_{-T}^{T}\left|\mathrm{Re}\int_{\mathbb{R}} S_{<2j-8}\partial_x(\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})|\tilde{S}_{2j}P_ju|^2\right| \\ &\lesssim 2^{4js}\|v\|^{2\sigma-1}_{L^\infty_T L^\infty_x}\|v_x\|_{L^1_T L^\infty_x}\|P_ju\|^2_{L^\infty_T L^2_x} \\ &\lesssim \|v\|^{2\sigma}_{\mathcal{S}^1_T}\|D^{2s}_xP_ju\|^2_{L^\infty_T L^2_x} \\ &\lesssim b_j^2\|v\|^{2\sigma}_{\mathcal{S}^1_T}\|u\|^2_{\mathcal{X}^{2s}_T} \\ &\lesssim b_j^2\varepsilon^{2\sigma}\|u\|^2_{\mathcal{X}^{2s}_T}. \end{aligned} \tag{4.5.27}$$

Estimate for I_j^2

As mentioned above, this term can be treated perturbatively. For simplicity, we denote $g := i\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k}$. Then Lemma 4.2.1 gives

$$[\tilde{S}_{2j}, S_{<2j-8}(i\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})]\partial_xP_ju = 2^{-2j}\int_{\mathbb{R}^2} K(s)[\partial_tS_{<2j-8}g](t+s_1,x)[\partial_xP_ju](t+s_2,x)ds \tag{4.5.28}$$

for some $K\in L^1(\mathbb{R}^2)$. Hölder's inequality, Minkowski's inequality, Bernstein's inequality and the fact that $\|P_ju\|_{L^\infty_t L^2_x} = \|P_ju\|_{L^\infty_T L^2_x}$ then gives

$$\begin{aligned} I_j^2 &\lesssim 2^{-2j}2^{4js}\int_{\mathbb{R}^2}|K(s)|\int_{-T}^{T}\int_{\mathbb{R}}|[\partial_tS_{<2j-8}g](t+s_1,x)[\partial_xP_ju](t+s_2,x)||(\tilde{S}_{2j}P_ju)(t,x)|dxdtds \\ &\lesssim 2^{-j}2^{4js}\|\partial_tS_{<2j-8}g\|_{L^2_tL^\infty_x}\|P_ju\|^2_{L^\infty_TL^2_x} \\ &\lesssim 2^{(\varepsilon_0-1)j}2^{4js}\|\partial_tS_{<2j-8}g\|_{L^2_tL^{\frac{1}{\varepsilon_0}}_x}\|P_ju\|^2_{L^\infty_TL^2_x} \\ &\lesssim \|D_t^{\frac{1}{2}+\frac{\varepsilon_0}{2}}(\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})\|_{L^2_tL^{\frac{1}{\varepsilon_0}}_x}\|P_jD^{2s}_xu\|^2_{L^\infty_TL^2_x}, \end{aligned} \tag{4.5.29}$$

where $\varepsilon_0<\delta$ is some small positive constant. From the fractional Leibniz rule and then the vector valued Moser bound Proposition 4.2.7, Sobolev embedding and then Corollary 4.5.5, we obtain

$$\begin{aligned} \|D_t^{\frac{1}{2}+\frac{\varepsilon_0}{2}}(\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})\|_{L^2_tL^{\frac{1}{\varepsilon_0}}_x} &\lesssim \|\tilde{v}\|^{2\sigma}_{\mathcal{S}^1_T} + \|\tilde{v}\|^{2\sigma-1}_{L^\infty_tL^\infty_x}\|D_t^{\frac{1}{2}+\frac{\varepsilon_0}{2}}\tilde{v}\|_{L^4_tL^{\frac{1}{\varepsilon_0}}_x} \\ &\lesssim \varepsilon^{2\sigma}. \end{aligned} \tag{4.5.30}$$

Hence,

$$I_j^2 \lesssim b_j^2 \varepsilon^{2\sigma} \|u\|^2_{\mathcal{X}_T^{2s}}. \tag{4.5.31}$$

Estimate for I_j^3

This term can also be dealt with perturbatively. Indeed, we can use Hölder and then Bernstein's inequality to shift a factor of $D_t^{\frac{1}{2}}$ onto the rough part of the nonlinearity,

$$\begin{aligned} I_j^3 &\lesssim 2^{4js} \|\tilde{S}_{2j}(S_{\geq 2j-8}(\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})\partial_x P_j u)\|_{L_t^2 L_x^2} \|P_j u\|_{L_T^2 L_x^2} \\ &\lesssim 2^j \|S_{\geq 2j-8}(\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})\|_{L_t^2 L_x^\infty} \|P_j D_x^{2s} u\|^2_{L_T^\infty L_x^2} \\ &\lesssim \|S_{\geq 2j-8} D_t^{\frac{1}{2}} (\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})\|_{L_t^2 L_x^\infty} \|P_j D_x^{2s} u\|^2_{L_T^\infty L_x^2} \\ &\lesssim b_j^2 \|D_t^{\frac{1}{2}+\frac{\varepsilon_0}{2}} (\eta P_{<j-4}|\tilde{v}|^{2\sigma}_{<k})\|_{L_t^2 L_x^{\frac{1}{\varepsilon_0}}} \|u\|^2_{\mathcal{X}_T^{2s}}. \end{aligned} \tag{4.5.32}$$

By a similar argument to the estimate for I_j^2, we then obtain,

$$I_j^3 \lesssim b_j^2 \varepsilon^{2\sigma} \|u\|^2_{\mathcal{X}_T^{2s}}. \tag{4.5.33}$$

Estimate for I_j^4

This term is also straightforward to deal with directly. The estimate is somewhat analogous to I_j^2. We have by Lemma 4.2.1,

$$[P_j, P_{<j-4}|\tilde{v}|^{2\sigma}_{<k}]\partial_x u = 2^{-j} \int_{\mathbb{R}^2} K(y)[P_{<j-4}\partial_x |\tilde{v}|^{2\sigma}_{<k}](x+y_1)[\tilde{P}_j \partial_x u](x+y_2)dy \tag{4.5.34}$$

for some integrable kernel $K \in L^1(\mathbb{R}^2)$. Hence, by Minkowski's inequality, Hölder's inequality and Bernstein's inequality,

$$\begin{aligned} I_j^4 &\lesssim \|\partial_x |\tilde{v}|^{2\sigma}_{<k}\|_{L_T^1 L_x^\infty} \|D_x^{2s} \tilde{P}_j u\|^2_{L_T^\infty L_x^2} \\ &\lesssim b_j^2 \varepsilon^{2\sigma} \|u\|^2_{\mathcal{X}_T^{2s}}. \end{aligned} \tag{4.5.35}$$

Estimate for I_j^5

As remarked on earlier, this is the most troublesome term to deal with since the rough coefficient $|\tilde{v}|^{2\sigma}_{<k}$ is at high spatial frequency. To deal with this, first write $w = \eta u$. We expand using the Littlewood-Paley trichotomy,

$$\tilde{S}_{2j} P_j(\eta P_{\geq j-4}|\tilde{v}|^{2\sigma}_{<k}\partial_x u) = \sum_{m \geq j} \tilde{S}_{2j} P_j(\tilde{P}_m |\tilde{v}|^{2\sigma}_{<k} \partial_x \tilde{P}_m w) + \tilde{S}_{2j} P_j(\tilde{P}_j |\tilde{v}|^{2\sigma}_{<k} \partial_x \tilde{P}_{<j} w). \tag{4.5.36}$$

The first term above, where the frequency interactions between $\partial_x w$ and $|\tilde{v}|^{2\sigma}_{<k}$ are balanced, is relatively straightforward to estimate. Indeed,

$$\begin{aligned}
&2^{4js}\int_{-T}^{T}|\int_{\mathbb{R}}\overline{\tilde{S}_{2j}P_j u}\sum_{m\geq j}\tilde{S}_{2j}P_j(\tilde{P}_m|\tilde{v}|^{2\sigma}_{<k}\partial_x\tilde{P}_m w)| \\
&\lesssim \|D_x^{2s}\tilde{S}_{2j}P_j u\|_{L^\infty_T L^2_x}2^{2js}\|\sum_{m\geq j}\tilde{S}_{2j}P_j(\tilde{P}_m|\tilde{v}|^{2\sigma}_{<k}\partial_x\tilde{P}_m w)\|_{L^1_T L^2_x} \\
&\lesssim \|D_x^{2s}\tilde{S}_{2j}P_j u\|_{L^\infty_T L^2_x}\sum_{m\geq j}2^{2js}\|\tilde{P}_m|\tilde{v}|^{2\sigma}_{<k}\partial_x\tilde{P}_m w\|_{L^1_t L^2_x} \\
&\lesssim b_j\|u\|_{\mathcal{X}^{2s}_T}\sum_{m\geq j}2^{2js}\|\tilde{P}_m|\tilde{v}|^{2\sigma}_{<k}\partial_x\tilde{P}_m w\|_{L^1_t L^2_x} \\
&\lesssim b_j\|\partial_x|\tilde{v}|^{2\sigma}_{<k}\|_{L^1_T L^\infty_x}\|u\|_{\mathcal{X}^{2s}_T}\sum_{m\geq j}2^{2(j-m)s}\|\tilde{P}_m D_x^{2s}u\|_{L^\infty_T L^2_x} \\
&\lesssim b_j\varepsilon^{2\sigma}\|u\|^2_{\mathcal{X}^{2s}_T}\sum_{m\geq j}2^{2(j-m)s}b_m \\
&\lesssim b_j^2\varepsilon^{2\sigma}\|u\|^2_{\mathcal{X}^{2s}_T}
\end{aligned} \tag{4.5.37}$$

where in the last line we used the slowly varying property of b_j.

For the second term in (4.5.36), we distribute the temporal projection to obtain

$$\tilde{S}_{2j}P_j(\tilde{P}_j|\tilde{v}|^{2\sigma}_{<k}\partial_x\tilde{P}_{<j}w) = \tilde{S}_{2j}P_j(\tilde{P}_j|\tilde{v}|^{2\sigma}_{<k}\partial_x\tilde{P}_{<j}S_{\geq 2j-8}w) + \tilde{S}_{2j}P_j(\tilde{P}_j\tilde{S}_{2j}|\tilde{v}|^{2\sigma}_{<k}\partial_x P_{<j}S_{<2j-8}w). \tag{4.5.38}$$

For the first term in (4.5.38), we use Bernstein's inequality and then Corollary 4.5.5, which yields

$$\begin{aligned}
2^{2js}\|\tilde{S}_{2j}P_j(\tilde{P}_j|\tilde{v}|^{2\sigma}_{<k}\partial_x\tilde{P}_{<j}S_{\geq 2j-8}w)\|_{L^1_T L^2_x} &\lesssim 2^{-j\varepsilon_0}\|D_x^{1+\varepsilon_0}|v|^{2\sigma}\|_{L^1_T L^\infty_x}\|S_{\geq 2j-8}D_t^s w\|_{L^\infty_t L^2_x} \\
&\lesssim 2^{-j\varepsilon_0}\|D_x^{1+\varepsilon_0}|v|^{2\sigma}\|_{L^1_T L^\infty_x}\|P_{\leq 0}S_{\geq 2j-8}D_t^s w\|_{L^\infty_t L^2_x} \\
&+ 2^{-j\varepsilon_0}\|D_x^{1+\varepsilon_0}|v|^{2\sigma}\|_{L^1_T L^\infty_x}\left(\sum_{m>0}\|P_m D_t^s w\|^2_{L^\infty_t L^2_x}\right)^{\frac{1}{2}} \\
&\lesssim 2^{-j\varepsilon_0}\|D_x^{1+\varepsilon_0}|v|^{2\sigma}\|_{L^1_T L^\infty_x}(\|u\|_{\mathcal{X}^{2s}_T} + \|g\|_{Z^{s-1+\delta}_\infty})
\end{aligned} \tag{4.5.39}$$

where $g := (i\partial_t + \partial_x^2)w$ and $0 < \varepsilon_0 \ll \delta$ is some small positive constant. If ε_0 is small enough, then Corollary 4.2.11 gives $\|D_x^{1+\varepsilon_0}|v|^{2\sigma}\|_{L^1_T L^\infty_x} \lesssim \|v\|^{2\sigma}_{\mathcal{S}^{1+\delta}_T} \lesssim \varepsilon^{2\sigma}$. Then finally by taking $2^{-j\varepsilon_0} \lesssim b_j$, it follows that

$$2^{2js}\|\tilde{S}_{2j}P_j(\tilde{P}_j|\tilde{v}|^{2\sigma}_{<k}\partial_x\tilde{P}_{<j}S_{\geq 2j-8}w)\|_{L^1_TL^2_x} \lesssim b_j\varepsilon^{2\sigma}(\|u\|_{\mathcal{X}^{2s}_T} + \|g\|_{Z^{s-1+\delta}_\infty}). \tag{4.5.40}$$

Now we look at controlling the second term in (4.5.38). We use Bernstein's inequality and the fact that $w = \eta u$ is time-localized to obtain

$$\begin{aligned} 2^{2js}\|\tilde{S}_{2j}P_j(\tilde{P}_j\tilde{S}_{2j}|\tilde{v}|^{2\sigma}_{<k}\partial_xP_{<j}S_{<2j-8}w)\|_{L^1_TL^2_x} &\lesssim 2^{2js}\|\tilde{P}_j\tilde{S}_{2j}|\tilde{v}|^{2\sigma}_{<k}\partial_xP_{<j}S_{<2j-8}w\|_{L^1_tL^2_x} \\ &\lesssim \|u\|_{\mathcal{S}^1_T}\|D^s_t\tilde{P}_j\tilde{S}_{2j}|\tilde{v}|^{2\sigma}\|_{L^2_tL^2_x}. \end{aligned} \tag{4.5.41}$$

Here we crucially ensured that the time derivative D^s_t, rather than the spatial derivative D^{2s}_x fell on the rough part of the nonlinearity.

To control $\|D^s_t\tilde{P}_j\tilde{S}_{2j}|\tilde{v}|^{2\sigma}\|_{L^2_tL^2_x}$ we will need the following low modulation Moser type estimate.

Lemma 4.5.9. Given the conditions of Lemma 4.5.7, the following estimate holds:

$$\|\tilde{P}_j\tilde{S}_{2j}D^s_t|\tilde{v}|^{2\sigma}\|_{L^2_tL^2_x} \lesssim b_j\varepsilon^{2\sigma-1}(\varepsilon + \|v\|_{\mathcal{X}^{2s}_T}). \tag{4.5.42}$$

We will postpone the proof of this technical lemma until the end of the section.

Combining Lemma 4.5.9 and the estimate (4.5.37) allows us to estimate I^5_j by

$$I^5_j \lesssim b^2_j\varepsilon^{2\sigma}(\|u\|^2_{\mathcal{X}^{2s}_T} + \|u\|^2_{\mathcal{S}^1_T} + \|g\|^2_{Z^{s-1+\delta}_\infty}) + b^2_j\varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}^1_T}\|u\|_{\mathcal{X}^{2s}_T}\|v\|_{\mathcal{X}^{2s}_T}. \tag{4.5.43}$$

Finally, combining the estimates for $I^1_j, \ldots, I^5_j$ now yields

$$\begin{aligned} \|D^{2s}_x\tilde{S}_{2j}P_ju\|^2_{L^\infty_TL^2_x} \lesssim{}& 2^{4js}\|(\tilde{S}_{2j}P_ju)(0)\|^2_{L^2_x} + b^2_j\varepsilon^{2\sigma}(\|u\|^2_{\mathcal{X}^{2s}_T} + \|u\|^2_{S^1_T} + \|g\|^2_{Z^{s-1+\delta}_\infty}) \\ &+ b^2_j\varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}^1_T}\|u\|_{\mathcal{X}^{2s}_T}\|v\|_{\mathcal{X}^{2s}_T}. \end{aligned} \tag{4.5.44}$$

Next, we need to control $(\tilde{S}_{2j}P_ju)(0)$ in terms of P_ju_0. To accomplish this, we use the high modulation estimate Lemma 4.5.4. Namely,

$$\begin{aligned} 2^{2js}\|(\tilde{S}_{2j}P_ju)(0)\|_{L^2_x} &\lesssim \|D^{2s}_xP_ju_0\|_{L^2_x} + \|(1-\tilde{S}_{2j})P_jD^{2s}_xu\|_{L^\infty_tL^2_x} \\ &\lesssim \|D^{2s}_xP_ju_0\|_{L^2_x} + \|S_{\leq 0}P_jD^{2s}_xu\|_{L^\infty_tL^2_x} + \sum_{m>0,|m-2j|>4}\|P_jS_mD^{2s}_xu\|_{L^\infty_tL^2_x} \\ &\lesssim \|D^{2s}_xP_ju_0\|_{L^2_x} + \|\langle D_t\rangle^{s-1+\delta}\tilde{P}_j(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{L^\infty_tL^2_x}. \end{aligned} \tag{4.5.45}$$

In light of (4.5.44) and (4.5.45), to complete the proof of Lemma 4.5.7 it remains to estimate the latter term on the right hand side of (4.5.45) as well as $\|g\|_{Z_\infty^{s-1+\delta}}$. This is done in the following lemma.

Lemma 4.5.10. Let $s, \sigma, T, u_0, u, a_j$ and b_j be as in Proposition 4.5.1. Let v also be as in Proposition 4.5.1, but with (4.5.5) replaced by the weaker assumption that for all $0 < \varepsilon \ll 1$ and $0 < \delta \ll 1$,

$$\|v\|_{\mathcal{S}_T^{1+\delta}} + \|(i\partial_t + \partial_x^2)v\|_{Z_\infty^{s-\frac{3}{2}+\delta}} \lesssim_\delta \varepsilon. \tag{4.5.46}$$

Then we have

$$\|\tilde{P}_j(\eta|\tilde{v}|_{<k}^{2\sigma}u_x)\|_{Z_\infty^{s-1+\delta}} \lesssim b_j\varepsilon^{2\sigma}(\|u\|_{\mathcal{S}_T^1} + \|u\|_{\mathcal{X}_T^{2s-1+c\delta}}) + b_j\varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}_T^1}\|v\|_{\mathcal{X}_T^{2s-1+c\delta}} \tag{4.5.47}$$

and

$$\|(i\partial_t + \partial_x^2)w\|_{Z_\infty^{s-1+\delta}} := \|g\|_{Z_\infty^{s-1+\delta}} \lesssim \|u\|_{\mathcal{X}_T^{2s-1+c\delta}} + \|u\|_{\mathcal{S}_T^1} + \|u\|_{\mathcal{S}_T^1}\|v\|_{\mathcal{X}_T^{2s-1+c\delta}}, \tag{4.5.48}$$

for some constant $c > 0$.

Remark 4.5.11. The reader may wonder why we estimate the full $Z_\infty^{s-1+\delta}$ norm in the above lemma. Although the argument up until this point only requires us to estimate the component of the $Z_\infty^{s-1+\delta}$ norm involving the time derivative, we will need to also estimate the component involving spatial derivatives in the next section when we establish well-posedness for the full equation in $\mathcal{X}_T^{2s}$.

Proof. We begin with (4.5.47). For the purpose of not having to track all the factors of δ that appear throughout the proof, we will denote by $c > 0$ some positive constant which is allowed to grow from line to line. First we study the component of the $Z_\infty^{s-1+\delta}$ norm which involves the time derivative. By considering separately temporal frequencies larger than 2^{2j} and smaller than 2^{2j}, we obtain (using the vector valued Bernstein inequality),

$$\begin{aligned}\|\tilde{P}_j\langle D_t\rangle^{s-1+\delta}(\eta|\tilde{v}|_{<k}^{2\sigma}u_x)\|_{L_t^\infty L_x^2} \lesssim\ & 2^{-2j\delta}\|\tilde{P}_j\langle D_x\rangle^{2s-2+c\delta}(\eta|\tilde{v}|_{<k}^{2\sigma}u_x)\|_{L_t^\infty L_x^2} \\ &+ 2^{-2j\delta}\|\tilde{P}_jS_{>2j}\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|_{<k}^{2\sigma}u_x)\|_{L_t^\infty L_x^2}.\end{aligned} \tag{4.5.49}$$

Hence,

$$\begin{aligned}\|\tilde{P}_j(\eta|\tilde{v}|_{<k}^{2\sigma}u_x)\|_{Z_\infty^{s-1+\delta}} \lesssim\ & 2^{-2j\delta}\|\tilde{P}_j\langle D_x\rangle^{2s-2+c\delta}(\eta|\tilde{v}|_{<k}^{2\sigma}u_x)\|_{L_t^\infty L_x^2} \\ &+ 2^{-2j\delta}\|\tilde{P}_jS_{>2j}\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|_{<k}^{2\sigma}u_x)\|_{L_t^\infty L_x^2}.\end{aligned} \tag{4.5.50}$$

We now look at the first term in (4.5.50). The bound

$$2^{-2j\delta}\|\tilde{P}_j\langle D_x\rangle^{2s-2+c\delta}(\eta|\tilde{v}|_{<k}^{2\sigma}u_x)\|_{L_t^\infty L_x^2} \lesssim b_j\varepsilon^{2\sigma}\|u\|_{\mathcal{X}_T^{2s-1+c\delta}} + b_j\varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}_T^1}\|v\|_{\mathcal{X}_T^{2s-1+c\delta}} \tag{4.5.51}$$

is a straightforward consequence of $2^{-2j\delta} \lesssim b_j$ and the fractional Leibniz rule if $2s-2<1$. If $2s-2 \geq 1$, then for the homogeneous component, we have

$$\begin{aligned}\|D_x^{2s-2+c\delta}(i\eta P_{<k}|v|^{2\sigma}u_x)\|_{L_t^\infty L_x^2} \lesssim \|D_x^{2s-3+c\delta}&(i\eta P_{<k}|v|^{2\sigma}u_{xx})\|_{L_t^\infty L_x^2}\\ &+\|\eta D_x^{2s-3+c\delta}(\mathrm{Re}P_{<k}(|v|^{2\sigma-2}\overline{v}v_x)u_x)\|_{L_t^\infty L_x^2}.\end{aligned} \tag{4.5.52}$$

By the fractional Leibniz rule and Sobolev embedding, clearly the first term above can be controlled by $\varepsilon^{2\sigma}\|u\|_{\mathcal{X}_T^{2s-1+c\delta}}$. Using the fact that $2s-3<2\sigma-1$ and applying the fractional Leibniz rule, Corollary 4.2.11 (when $D_x^{2s-3+c\delta}$ falls on $|v|^{2\sigma-2}\overline{v}$) and interpolation, we can control the second term by

$$\varepsilon^{2\sigma}\|u\|_{\mathcal{X}_T^{2s-1+c\delta}}+\varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}_T^1}\|v\|_{\mathcal{X}_T^{2s-1+c\delta}} \tag{4.5.53}$$

to obtain the desired bound (4.5.51).

Now, to estimate the second term on the right hand side of (4.5.50), we use that $2^{-2j\delta} \lesssim b_j$ and estimate

$$\begin{aligned}&2^{-2j\delta}\|\tilde{P}_jS_{>2j}\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|_{<k}^{2\sigma}u_x)\|_{L_t^\infty L_x^2}\\ &\lesssim b_j\|\tilde{P}_jS_{>2j}\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|_{<k}^{2\sigma}u_x)\|_{L_t^\infty L_x^2}\\ &\lesssim b_j\sum_{m\geq 2j}\|\tilde{P}_jS_m\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|_{<k}^{2\sigma}u_x)\|_{L_t^\infty L_x^2}\\ &\lesssim b_j\sum_{m\geq 2j}\|\tilde{P}_jS_m\langle D_t\rangle^{s-1+c\delta}(S_{<m-4}(\eta|\tilde{v}|_{<k}^{2\sigma})\tilde{S}_mu_x)\|_{L_t^\infty L_x^2}\\ &+b_j\sum_{m\geq 2j}\|\tilde{P}_jS_m\langle D_t\rangle^{s-1+c\delta}(S_{\geq m-4}(\eta|\tilde{v}|_{<k}^{2\sigma})u_x)\|_{L_t^\infty L_x^2}.\end{aligned} \tag{4.5.54}$$

For the first term in (4.5.54), we have by Bernstein's inequality,

$$\begin{aligned}&b_j\sum_{m\geq 2j}\|\tilde{P}_jS_m\langle D_t\rangle^{s-1+c\delta}(S_{<m-4}(\eta|\tilde{v}|_{<k}^{2\sigma})\tilde{S}_mu_x)\|_{L_t^\infty L_x^2}\\ &\lesssim b_j\sum_{m\geq 2j}\|\tilde{P}_jS_m\langle D_t\rangle^{s-\frac{1}{2}+c\delta}(S_{<m-4}(\eta|\tilde{v}|_{<k}^{2\sigma})\tilde{S}_mu)\|_{L_t^\infty L_x^2}\\ &+b_j\sum_{m\geq 2j}\|\tilde{P}_jS_m\langle D_t\rangle^{s-1+c\delta}(S_{<m-4}(\eta\partial_x|\tilde{v}|_{<k}^{2\sigma})\tilde{S}_mu)\|_{L_t^\infty L_x^2}.\end{aligned} \tag{4.5.55}$$

Using Bernstein's inequality and Corollary 4.5.5, we may control the first term by

$$\begin{aligned}
&b_j \sum_{m\geq 2j} \|\tilde{P}_j S_m \langle D_t\rangle^{s-\frac{1}{2}+c\delta}(S_{<m-4}(\eta|\tilde{v}|^{2\sigma}_{<k})\tilde{S}_m u)\|_{L^\infty_t L^2_x} \\
&\lesssim b_j \|v\|^{2\sigma}_{\mathcal{S}^1_T}\|\langle D_t\rangle^{s-\frac{1}{2}+c\delta}u\|_{L^\infty_t L^2_x} \\
&\lesssim b_j \varepsilon^{2\sigma}\|\langle D_t\rangle^{s-\frac{1}{2}+c\delta}u\|_{L^\infty_t L^2_x} \\
&\lesssim b_j \varepsilon^{2\sigma}(\|u\|_{\mathcal{X}^{2s-1+c\delta}_T} + \|\eta|v|^{2\sigma}_{<k}u_x\|_{Z^{s-\frac{3}{2}+c\delta}_\infty}).
\end{aligned} \tag{4.5.56}$$

For the second term in (4.5.55), we obtain also

$$\begin{aligned}
&b_j \sum_{m\geq 2j} \|\tilde{P}_j S_m \langle D_t\rangle^{s-1+c\delta}(S_{<m-4}(\eta\partial_x|\tilde{v}|^{2\sigma}_{<k})\tilde{S}_m u)\|_{L^\infty_t L^2_x} \\
&\lesssim b_j \||\tilde{v}|^{2\sigma}\|_{L^\infty_t H^1_x}\|\langle D_t\rangle^{s-1+c\delta}u\|_{L^\infty_t L^\infty_x} \\
&\lesssim b_j \||\tilde{v}|^{2\sigma}\|_{L^\infty_t H^1_x}\|\langle D_t\rangle^{s-1+c\delta}\langle D_x\rangle^{\frac{1}{2}+\delta}u\|_{L^\infty_t L^2_x} \\
&\lesssim b_j \varepsilon^{2\sigma}(\|\langle D_t\rangle^{s-\frac{1}{2}+c\delta}u\|_{L^\infty_t L^2_x} + \|\langle D_x\rangle^{s-\frac{1}{2}+c\delta}u\|_{L^\infty_t L^2_x}) \\
&\lesssim b_j \varepsilon^{2\sigma}(\|u\|_{\mathcal{X}^{2s-1+c\delta}_T} + \|\eta|v|^{2\sigma}_{<k}u_x\|_{Z^{s-\frac{3}{2}+c\delta}_\infty}).
\end{aligned} \tag{4.5.57}$$

For the second term in (4.5.54), we obtain

$$\begin{aligned}
b_j \sum_{m\geq 2j} \|\tilde{P}_j S_m \langle D_t\rangle^{s-1+c\delta}(S_{\geq m-4}(\eta|\tilde{v}|^{2\sigma}_{<k})u_x)\|_{L^\infty_t L^2_x} &\lesssim b_j \|u_x\|_{L^\infty_T L^2_x}\|\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|^{2\sigma}_{<k})\|_{L^\infty_t L^\infty_x} \\
&\lesssim b_j \|u\|_{\mathcal{S}^1_T}\|\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|^{2\sigma}_{<k})\|_{L^\infty_t L^\infty_x}.
\end{aligned} \tag{4.5.58}$$

We have by Sobolev embedding, the fractional Leibniz rule and the Moser bound Proposition 4.2.8,

$$\begin{aligned}
&\|\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|^{2\sigma}_{<k})\|_{L^\infty_t L^\infty_x} \\
&\lesssim \|\langle D_t\rangle^{s-1+c\delta}(\eta|\tilde{v}|^{2\sigma}_{<k})\|_{L^\infty_x L^{\frac{1}{\delta}}_t} \\
&\lesssim \||\tilde{v}|^{2\sigma}\|_{L^\infty_x L^{\frac{2}{\delta}}_t} + \|\langle D_t\rangle^{s-1+c\delta}|\tilde{v}|^{2\sigma}_{<k}\|_{L^\infty_x L^{\frac{2}{\delta}}_t} \\
&\lesssim \||\tilde{v}|^{2\sigma}\|_{L^{\frac{2}{\delta}}_t L^\infty_x} + \||\tilde{v}|^{2\sigma-1}\|_{L^\infty_x L^{\frac{4}{\delta}}_t}\|\langle D_t\rangle^{s-1+c\delta}\tilde{v}\|_{L^\infty_x L^{\frac{4}{\delta}}_t} \\
&\lesssim \varepsilon^{2\sigma} + \||\tilde{v}|^{2\sigma-1}\|_{L^{\frac{4}{\delta}}_t L^\infty_x}\|\langle D_t\rangle^{s-1+c\delta}\tilde{v}\|_{L^{\frac{4}{\delta}}_t L^\infty_x} \\
&\lesssim \varepsilon^{2\sigma} + \|v\|^{2\sigma-1}_{\mathcal{S}^1_T}\|\langle D_t\rangle^{s-1+c\delta}\tilde{v}\|_{L^{\frac{4}{\delta}}_t L^\infty_x}.
\end{aligned} \tag{4.5.59}$$

Now, notice that by Corollary 4.5.5,

$$\begin{aligned}
&\|\langle D_t\rangle^{s-1+c\delta}\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^\infty} \\
&\lesssim \sum_{j\geq 0}\|\langle D_t\rangle^{s-1+c\delta}P_j\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^\infty} \\
&\lesssim \sum_{j\geq 0}\|\langle D_t\rangle^{s-1+c\delta}\langle D_x\rangle^{\frac{1}{2}}P_j\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^2} \\
&\lesssim \sum_{j\geq 0}\|\langle D_x\rangle^{s-\frac{1}{2}+c\delta}P_j\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^2} + \sum_{j\geq 0}\|\langle D_t\rangle^{s-\frac{1}{2}+c\delta}P_j\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^2} \\
&\lesssim \|\tilde{v}\|_{\mathcal{X}_T^{2s-1+c\delta}} + \sum_{j\geq 0}\left(\|\langle D_x\rangle^{2s-1+c\delta}P_j\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^2} + \|\tilde{P}_j\langle D_t\rangle^{s-\frac{3}{2}+c\delta}(i\partial_t+\partial_x^2)\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^2}\right) \\
&\lesssim \|v\|_{\mathcal{X}_T^{2s-1+c\delta}} + \sum_{j\geq 0}\|\tilde{P}_j\langle D_t\rangle^{s-\frac{3}{2}+c\delta}(i\partial_t+\partial_x^2)\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^2}.
\end{aligned} \tag{4.5.60}$$

To control the latter term above, there are two cases. If $s-\frac{3}{2}<0$, then this term can be easily controlled by ε by commuting $(i\partial_t+\partial_x^2)$ with $\tilde{\eta}$ and applying Hölder's inequality. If $s-\frac{3}{2}\geq 0$, then we have (after possibly enlarging $c\delta$)

$$\begin{aligned}
&\sum_{j\geq 0}\|\tilde{P}_j\langle D_t\rangle^{s-\frac{3}{2}+c\delta}(i\partial_t+\partial_x^2)\tilde{v}\|_{L_t^{\frac{4}{\delta}}L_x^2} \\
&\lesssim \|\langle D_t\rangle^{s-\frac{3}{2}+c\delta}(\eta(i\partial_t+\partial_x^2)v)\|_{L_t^{\frac{4}{\delta}}L_x^2} + \|\langle D_t\rangle^{s-\frac{3}{2}+c\delta}(\partial_t\tilde{\eta}v)\|_{L_t^{\frac{4}{\delta}}L_x^2} \\
&\lesssim \sum_{k\geq 0}\|\langle D_t\rangle^{s-\frac{3}{2}+c\delta}S_k(\eta(i\partial_t+\partial_x^2)v)\|_{L_t^{\frac{4}{\delta}}L_x^2} + \|\langle D_t\rangle^{s-\frac{3}{2}+c\delta}S_k(\partial_t\tilde{\eta}v)\|_{L_t^{\frac{4}{\delta}}L_x^2}.
\end{aligned} \tag{4.5.61}$$

By doing a paraproduct expansion of $S_k(\eta(i\partial_t+\partial_x^2)v) = S_k(S_{<k-4}\eta(i\partial_t+\partial_x^2)\tilde{S}_kv)+S_k(S_{\geq k-4}\eta(i\partial_t+\partial_x^2)v)$, using Bernstein and Hölder's inequality, summing over k, and possibly enlarging the factor of $c\delta$, we obtain

$$\sum_{k\geq 0}\|\langle D_t\rangle^{s-\frac{3}{2}+c\delta}S_k(\eta(i\partial_t+\partial_x^2)v)\|_{L_t^{\frac{4}{\delta}}L_x^2} \lesssim \|(i\partial_t+\partial_x^2)v\|_{Z_\infty^{s-\frac{3}{2}+c\delta}} \lesssim \varepsilon. \tag{4.5.62}$$

A similar argument involving a paraproduct expansion of $S_k(\partial_t\eta v)$ can be used to show

$$\|\langle D_t\rangle^{s-\frac{3}{2}+c\delta}S_k(\partial_t\eta v)\|_{L_t^{\frac{4}{\delta}}L_x^2} \lesssim \varepsilon. \tag{4.5.63}$$

Therefore, the second term in (4.5.54) can be controlled by

$$b_j\varepsilon^{2\sigma}\|u\|_{\mathcal{S}_T^1} + b_j\varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}_T^1}\|v\|_{\mathcal{X}_T^{2s-1+c\delta}}. \tag{4.5.64}$$

Combining this and (4.5.55) with (4.5.51) yields the estimate,

$$\begin{aligned}\|\tilde{P}_j(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{Z^{s-1+\delta}_\infty} \lesssim b_j\varepsilon^{2\sigma}(\|u\|_{\mathcal{X}^{2s-1+c\delta}_T} + \|u\|_{\mathcal{S}^1_T} + \|\eta|v|^{2\sigma}_{<k}u_x\|_{Z^{s-\frac{3}{2}+c\delta}_\infty}) \\ + b_j\varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}^1_T}\|v\|_{\mathcal{X}^{2s-1+c\delta}_T}.\end{aligned} \tag{4.5.65}$$

By square summing (4.5.65) and applying (4.5.65) with $s-1$ replaced by $s-\frac{3}{2}$, we obtain

$$\begin{aligned}\|\eta|v|^{2\sigma}_{<k}u_x\|_{Z^{s-\frac{3}{2}+c\delta}_\infty} \lesssim \varepsilon^{2\sigma}(\|u\|_{\mathcal{X}^{2s-1+c\delta}_T} + \|u\|_{\mathcal{S}^1_T} + \|\eta|v|^{2\sigma}_{<k}u_x\|_{Z^{s-2+c\delta}_\infty}) \\ + \varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}^1_T}\|v\|_{\mathcal{X}^{2s-1+c\delta}_T}\end{aligned} \tag{4.5.66}$$

and since $s<2$, it follows that if δ is small enough, then

$$\|\eta|v|^{2\sigma}_{<k}u_x\|_{Z^{s-2+c\delta}_\infty} \lesssim \varepsilon^{2\sigma}\|u\|_{\mathcal{X}^{2s-1+c\delta}_T}. \tag{4.5.67}$$

Therefore, the bound

$$\|\tilde{P}_j(\eta|\tilde{v}|^{2\sigma}_{<k}u_x)\|_{Z^{s-1+\delta}_\infty} \lesssim b_j\varepsilon^{2\sigma}(\|u\|_{\mathcal{X}^{2s-1+c\delta}_T} + \|u\|_{\mathcal{S}^1_T}) + b_j\varepsilon^{2\sigma-1}\|u\|_{\mathcal{S}^1_T}\|v\|_{\mathcal{X}^{2s-1+c\delta}_T} \tag{4.5.68}$$

follows.

For the estimate (4.5.48), we have

$$\|g\|_{Z^{s-1+\delta}_\infty} \lesssim \|\partial_t\eta u\|_{Z^{s-1+\delta}_\infty} + \|\eta P_{<k}|v|^{2\sigma}u_x\|_{Z^{s-1+\delta}_\infty}. \tag{4.5.69}$$

The first term above is controlled using Corollary 4.5.5 by

$$\begin{aligned}\|\partial_t\eta u\|_{Z^{s-1+\delta}_\infty} &\lesssim \|u\|_{\mathcal{X}^{2s-1+c\delta}_T} + \|\langle D_t\rangle^{s-1+\delta}(\partial_t\eta u)\|_{L^\infty_t L^2_x} \\ &\lesssim \|u\|_{\mathcal{X}^{2s-1+c\delta}_T} + \|\partial_t^2\eta u\|_{Z^{s-2+c\delta}_\infty} + \|\partial_t\eta(\eta P_{<k}|v|^{2\sigma}u_x)\|_{Z^{s-2+c\delta}_\infty} \\ &\lesssim \|u\|_{\mathcal{X}^{2s-1+c\delta}_T}\end{aligned} \tag{4.5.70}$$

where in the last line, we used that $s<2$. The second term in (4.5.69) can be estimated by square summing (4.5.68). This completes the proof of Lemma 4.5.10. □

Finally, we complete the proof of Lemma 4.5.7. This simply follows by combining Lemma 4.5.10 with the estimates (4.5.44) and (4.5.45). □

Proof of Proposition 4.5.1

Finally, we prove the main estimate of the section, Proposition 4.5.1.

Proof. Let $0 < \delta \ll 1$. From Corollary 4.5.5, we have

$$\|P_j u\|^2_{L^\infty_T H^{2s}_x} \lesssim_\delta \|\tilde{S}_{2j} P_j u\|^2_{L^\infty_T H^{2s}_x} + \|\tilde{P}_j(\eta |\tilde{v}|^{2\sigma}_{<k} u_x)\|^2_{Z^{s-1+\delta}_\infty}. \tag{4.5.71}$$

By Lemma 4.5.7, we have

$$\begin{aligned}\|\tilde{S}_{2j} P_j u\|^2_{L^\infty_T H^{2s}_x} \lesssim_\delta\ & a_j^2 \|u_0\|^2_{H^{2s}_x} + b_j^2 \varepsilon^{2\sigma}(\|u\|^2_{\mathcal{X}^{2s}_T} + \|u\|^2_{\mathcal{S}^1_T}) + b_j^2 \varepsilon^{2\sigma-1} \|u\|_{\mathcal{S}^1_T} \|u\|_{\mathcal{X}^{2s}_T} \|v\|_{\mathcal{X}^{2s}_T} \\ & + b_j^2 \varepsilon^{4\sigma-2} \|u\|^2_{\mathcal{S}^1_T} \|v\|^2_{\mathcal{X}^{2s}_T}.\end{aligned} \tag{4.5.72}$$

Furthermore, by Lemma 4.5.10, we have

$$\begin{aligned}\|\tilde{P}_j(\eta |\tilde{v}|^{2\sigma}_{<k} u_x)\|^2_{Z^{s-1+\delta}_\infty} &\lesssim b_j^2 \varepsilon^{4\sigma}(\|u\|^2_{\mathcal{X}^{2s}_T} + \|u\|^2_{\mathcal{S}^1_T}) + b_j^2 \varepsilon^{4\sigma-2} \|u\|^2_{\mathcal{S}^1_T} \|v\|^2_{\mathcal{X}^{2s}_T} \\ &\lesssim b_j^2 \varepsilon^{2\sigma}(\|u\|^2_{\mathcal{X}^{2s}_T} + \|u\|^2_{\mathcal{S}^1_T}) + b_j^2 \varepsilon^{4\sigma-2} \|u\|^2_{\mathcal{S}^1_T} \|v\|^2_{\mathcal{X}^{2s}_T}.\end{aligned} \tag{4.5.73}$$

This completes the proof. □

Proof of Lemma 4.5.9

It remains to prove the technical estimate Lemma 4.5.9. This will follow from the slightly more general estimate:

Lemma 4.5.12. Let $T = 2$, $\frac{1}{2} < \sigma < 1$ and u be a $C^2(\mathbb{R}; H^\infty_x)$ solution to the inhomogeneous Schrödinger equation,

$$(i\partial_t + \partial_x^2)u = f \tag{4.5.74}$$

supported in the time interval $[-2, 2]$. Furthermore, let b_j be an admissible $\mathcal{X}^{2s}_T$ frequency envelope for u (here we don't assume that the formula is necessarily given explicitly by (4.2.4)). Then for $j > 0$ we have,

a) If $0 < s < 1$, then

$$\|\tilde{P}_j \tilde{S}_{2j} D^s_t(|u|^{2\sigma})\|_{L^2_t L^2_x} \lesssim b_j \|u\|^{2\sigma-1}_{\mathcal{S}^1_T}(\|u\|_{\mathcal{X}^{2s}_T} + \|f\|_{\mathcal{S}^0_T}). \tag{4.5.75}$$

b) If $1 \le s < 2\sigma$ and $0 < \delta \ll 1$, then

$$\|\tilde{P}_j \tilde{S}_{2j} D^s_t(|u|^{2\sigma})\|_{L^2_t L^2_x} \lesssim_\delta b_j \Lambda(\|u\|_{\mathcal{X}^{2s}_T} + \|f\|_{\mathcal{S}^0_T} + \|f\|_{Z^{s-1+\delta}_\infty}) \tag{4.5.76}$$

where

$$\Lambda := (\|u\|_{\mathcal{S}_T^{1+\delta}} + \|f\|_{\mathcal{S}_T^{\delta}})^{2\sigma-1}\Lambda_0 \tag{4.5.77}$$

and Λ_0 is some polynomial in $\|u\|_{\mathcal{S}_T^{1+\delta}} + \|f\|_{\mathcal{S}_T^{\delta}}$.

Remark 4.5.13. We only prove the above estimate for $\tilde{P}_j\tilde{S}_{2j}D_t^s(|u|^{2\sigma})$ in $L_t^2L_x^2$. Although the estimate is almost certainly true for a suitable range of $p \geq 1$ in $L_t^pL_x^2$, we do not pursue this here, so as not to further complicate the argument (specifically, the proof of b)).

Remark 4.5.14. We do not claim that the factors of $\|f\|_{\mathcal{S}_T^0}$, $\|f\|_{Z_\infty^{s-1+\delta}}$ and Λ that appear in the estimate are in any way optimal (in fact, in many instances in the below estimates, they arise in relatively crude ways). We opted not to carefully optimize the inequality because it will not affect the range of s for which Lemma 4.5.7 holds, and also because the current form of Lemma 4.5.12 can be more easily applied to establish Proposition 4.5.1.

Proof. a) For notational convenience, we will sometimes write $F(z) = |z|^{2\sigma-2}\overline{z}$ and $P_{<j}u = u_{<j}$. Now, for each $j > 0$, we write

$$\begin{aligned} D_t^s\tilde{S}_{2j}P_j|u|^{2\sigma} &= D_t^s\tilde{S}_{2j}P_j|P_{<j}u|^{2\sigma} - D_t^s\tilde{S}_{2j}P_j(|P_{<j}u|^{2\sigma} - |u|^{2\sigma}) \\ &= D_t^s\tilde{S}_{2j}P_j|P_{<j}u|^{2\sigma} + 2\sigma\mathrm{Re}\int_0^1 P_j\tilde{S}_{2j}D_t^s(F(y(\theta))P_{\geq j}u)d\theta \end{aligned} \tag{4.5.78}$$

where

$$y(\theta) := \theta u + (1-\theta)P_{<j}u. \tag{4.5.79}$$

For the first term, interpolating gives

$$\|D_t^s\tilde{S}_{2j}P_j|P_{<j}u|^{2\sigma}\|_{L_t^2L_x^2} \lesssim \|\tilde{S}_{2j}P_j|P_{<j}u|^{2\sigma}\|_{L_t^2L_x^2}^{1-s}\|\tilde{S}_{2j}P_j(F(u_{<j})P_{<j}u_t)\|_{L_t^2L_x^2}^{s}. \tag{4.5.80}$$

By expanding u_t in the second factor, we obtain

$$\begin{aligned} \|\tilde{S}_{2j}P_j(F(u_{<j})P_{<j}u_t)\|_{L_t^2L_x^2} &\lesssim \|\tilde{S}_{2j}P_j(F(u_{<j})P_{<j}u_{xx})\|_{L_t^2L_x^2} \\ &\quad + \|\tilde{S}_{2j}P_j(F(u_{<j})P_{<j}f)\|_{L_t^2L_x^2}. \end{aligned} \tag{4.5.81}$$

We expand the first term in (4.5.81) using the Littlewood-Paley trichotomy. Then Bernstein's inequality and Corollary 4.2.11 yields

$$\begin{aligned} &\|\tilde{S}_{2j}P_j(F(u_{<j})P_{<j}u_{xx})\|_{L_t^2L_x^2} \\ &\lesssim \|P_{<j-4}F(u_{<j})\tilde{P}_ju_{xx}\|_{L_t^2L_x^2} + \|P_{\geq j-4}F(u_{<j})P_{<j}u_{xx}\|_{L_t^2L_x^2} \\ &\lesssim b_j2^{2j(1-s)}\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|u\|_{\mathcal{X}_T^{2s}} + 2^{2j(1-s)}2^{-\delta j}\|D_x^\delta F(u_{<j})\|_{L_t^\infty L_x^\infty}\|D_x^{2s}u\|_{L_t^\infty L_x^2} \\ &\lesssim b_j2^{2j(1-s)}\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|u\|_{\mathcal{X}_T^{2s}}. \end{aligned} \tag{4.5.82}$$

For the second term in (4.5.81), we obtain (by taking $2^{2j(s-1)} \lesssim b_j$)

$$\|\tilde{S}_{2j}P_j(F(u_{<j})P_{<j}f)\|_{L_t^2L_x^2} \lesssim b_j 2^{2j(1-s)}\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|f\|_{L_t^\infty L_x^2} \tag{4.5.83}$$

and so by Bernstein, the estimate (4.5.80) becomes

$$\begin{aligned} &\|D_t^s\tilde{S}_{2j}P_j|P_{<j}u|^{2\sigma}\|_{L_t^2L_x^2} \\ &\lesssim 2^{2js(1-s)}[b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|u\|_{\mathcal{X}_T^{2s}} + b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|f\|_{L_t^\infty L_x^2}]^s\|\tilde{S}_{2j}P_j|P_{<j}u|^{2\sigma}\|_{L_t^2L_x^2}^{1-s} \\ &\lesssim [b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|u\|_{\mathcal{X}_T^{2s}} + b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|f\|_{L_t^\infty L_x^2}]^s\|D_t^s\tilde{S}_{2j}P_j|P_{<j}u|^{2\sigma}\|_{L_t^2L_x^2}^{1-s}. \end{aligned} \tag{4.5.84}$$

Hence,

$$\|D_t^s\tilde{S}_{2j}P_j|P_{<j}u|^{2\sigma}\|_{L_t^2L_x^2} \lesssim b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|u\|_{\mathcal{X}_T^{2s}} + b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|f\|_{L_t^\infty L_x^2}. \tag{4.5.85}$$

For the second term in (4.5.78), using that $2^{2js}\|P_{\geq j}u\|_{L_t^2L_x^2} \lesssim \|D_x^{2s}P_{\geq j}u\|_{L_t^2L_x^2}$ and Corollary 4.2.11 leads to the estimate,

$$\begin{aligned} &\|P_j\tilde{S}_{2j}D_t^s(F(y(\theta))P_{\geq j}u)\|_{L_t^2L_x^2} \\ &\lesssim 2^{2js}\|P_j(F(y(\theta))P_{\geq j}u)\|_{L_t^2L_x^2} \\ &\lesssim b_j\|\langle D_x\rangle^\delta F(y(\theta))\|_{L_t^\infty L_x^\infty}\|u\|_{\mathcal{X}_T^{2s}} \\ &\lesssim b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|u\|_{\mathcal{X}_T^{2s}}. \end{aligned} \tag{4.5.86}$$

Hence, by Minkowski's inequality,

$$2\sigma\|\mathrm{Re}\int_0^1 P_j\tilde{S}_{2j}D_t^s(F(y(\theta))P_{\geq j}u)d\theta\|_{L_t^2L_x^2} \lesssim b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|u\|_{\mathcal{X}_T^{2s}}. \tag{4.5.87}$$

Combining everything shows that

$$\|D_t^s\tilde{S}_{2j}P_j|u|^{2\sigma}\|_{L_t^2L_x^2} \lesssim b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|u\|_{\mathcal{X}_T^{2s}} + b_j\|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|f\|_{L_t^\infty L_x^2}. \tag{4.5.88}$$

This proves part a).

Next, we prove part b). By commuting through the temporal projection, we obtain

$$\begin{aligned} \|\tilde{S}_{2j}P_jD_t^s(|u|^{2\sigma})\|_{L_t^2L_x^2} &\lesssim \|\tilde{S}_{2j}D_t^{s-1}(\tilde{S}_{<2j}(|u|^{2\sigma-2}\overline{u})\partial_t\tilde{S}_{2j}u)\|_{L_t^2L_x^2} \\ &\quad + \|\tilde{S}_{2j}D_t^{s-1}(\tilde{S}_{\geq 2j}(|u|^{2\sigma-2}\overline{u})\partial_t u)\|_{L_t^2L_x^2}. \end{aligned} \tag{4.5.89}$$

The first term in (4.5.89) can be estimated by Bernstein's inequality to obtain

$$\|\tilde{S}_{2j}D_t^{s-1}(\tilde{S}_{<2j}(|u|^{2\sigma-2}\overline{u})\partial_t\tilde{S}_{2j}u)\|_{L_t^2L_x^2} \lesssim \|u\|_{\mathcal{S}_T^1}^{2\sigma-1}\|D_t^{s-1}\partial_t\tilde{S}_{2j}u\|_{L_t^2L_x^2}. \tag{4.5.90}$$

Then writing

$$\|D_t^{s-1}\partial_t\tilde{S}_{2j}u\|_{L_t^2L_x^2} \sim \|D_t^{s-1}\partial_t\tilde{S}_{2j}P_{\le 0}u\|_{L_t^2L_x^2} + \left(\sum_{k>0}\|D_t^{s-1}\partial_t\tilde{S}_{2j}P_k u\|_{L_t^2L_x^2}^2\right)^{\frac{1}{2}} \tag{4.5.91}$$

and requiring $b_j \ge 2^{-j\delta}$, applying Lemma 4.5.4 and then square summing over k yields

$$\begin{aligned}\|D_t^{s-1}\partial_t\tilde{S}_{2j}u\|_{L_t^2L_x^2} &\lesssim \|D_x^{2s}\tilde{S}_{2j}P_j u\|_{L_t^2L_x^2} + 2^{-j\delta}\|\langle D_t\rangle^{s-1+\delta}\tilde{S}_{2j}f\|_{L_t^2L_x^2} \\ &\lesssim b_j\|u\|_{\mathcal{X}_T^{2s}} + b_j\|f\|_{Z_\infty^{s-1+\delta}}.\end{aligned} \tag{4.5.92}$$

To estimate the second term in (4.5.89), we have two cases:

If $1 \le s \le \sigma + \frac{1}{2}$, we obtain from the equation,

$$\begin{aligned}&\|\tilde{S}_{2j}D_t^{s-1}(\tilde{S}_{\ge 2j}(|u|^{2\sigma-2}\overline{u})\partial_t u)\|_{L_t^2L_x^2} \\ &\lesssim \|\tilde{S}_{2j}D_t^{s-1}(\tilde{S}_{\ge 2j}(|u|^{2\sigma-2}\overline{u})\partial_x^2 u)\|_{L_t^2L_x^2} + \|\tilde{S}_{2j}D_t^{s-1}(\tilde{S}_{\ge 2j}(|u|^{2\sigma-2}\overline{u})f)\|_{L_t^2L_x^2}.\end{aligned} \tag{4.5.93}$$

By Hölder and Bernstein's inequality, Sobolev embedding and Corollary 4.2.11, the first term can be estimated by

$$\begin{aligned}\|\tilde{S}_{2j}D_t^{s-1}(\tilde{S}_{\ge 2j}(|u|^{2\sigma-2}\overline{u})\partial_x^2 u)\|_{L_t^2L_x^2} &\lesssim 2^{-j\delta}\|D_t^{s-1+\frac{\delta}{2}}(|u|^{2\sigma-2}\overline{u})\|_{L_t^{\frac{1}{s-1}}L_x^{\frac{1}{s-1}}}\|\partial_x^2 u\|_{L_t^\infty L_x^{\frac{2}{3-2s}}} \\ &\lesssim 2^{-j\delta}\|\langle D_t\rangle^{\frac{s-1+\delta}{2\sigma-1}}u\|^{2\sigma-1}_{L_t^{\frac{2\sigma-1}{s-1}}L_x^{\frac{2\sigma-1}{s-1}}}\|\partial_x^2 u\|_{L_t^\infty L_x^{\frac{2}{3-2s}}} \\ &\lesssim 2^{-j\delta}\|\langle D_t\rangle^{\frac{s-1+\delta}{2\sigma-1}}u\|^{2\sigma-1}_{L_t^{\frac{2\sigma-1}{s-1}}L_x^{\frac{2\sigma-1}{s-1}}}\|u\|_{L_t^\infty H_x^{s+1}}.\end{aligned} \tag{4.5.94}$$

Applying Corollary 4.5.5 gives

$$\|\langle D_t\rangle^{\frac{s-1+\delta}{2\sigma-1}}u\|^{2\sigma-1}_{L_t^{\frac{2\sigma-1}{s-1}}L_x^{\frac{2\sigma-1}{s-1}}} \lesssim \|\langle D_x\rangle^{\frac{2s-2+4\delta}{2\sigma-1}}u\|^{2\sigma-1}_{L_t^{\frac{2\sigma-1}{s-1}}L_x^{\frac{2\sigma-1}{s-1}}} + \|\langle D_t\rangle^{\frac{s-1+2\delta}{2\sigma-1}-1}f\|^{2\sigma-1}_{L_t^{\frac{2\sigma-1}{s-1}}L_x^{\frac{2\sigma-1}{s-1}}}. \tag{4.5.95}$$

Since $\frac{2s-2}{2\sigma-1} \le 1 - (\frac{1}{2} - \frac{s-1}{2\sigma-1})$, when $s \le \sigma + \frac{1}{2}$, we have by Sobolev embedding in the spatial variable,

$$\|\langle D_x\rangle^{\frac{2s-2+4\delta}{2\sigma-1}}u\|^{2\sigma-1}_{L_t^{\frac{2\sigma-1}{s-1}}L_x^{\frac{2\sigma-1}{s-1}}}\|u\|_{L_t^\infty H_x^{s+1}} \lesssim \|u\|^{2\sigma-1}_{\mathcal{S}_T^{1+c\delta}}\|u\|_{L_t^\infty H_x^{2s}} \tag{4.5.96}$$

for some fixed constant $c > 0$.

Next, applying Sobolev embedding in the time variable, and using the inequality $\|g\|_{L^p_x L^q_t} \lesssim \|g\|_{L^q_t L^p_x}$ when $p \geq q$, we also obtain

$$\begin{aligned} \|\langle D_t\rangle^{\frac{s-1+2\delta}{2\sigma-1}-1} f\|_{L_t^{\frac{2\sigma-1}{s-1}} L_x^{\frac{2\sigma-1}{s-1}}} &\lesssim \|f\|_{L_x^{\frac{2\sigma-1}{s-1}} L_t^2} \lesssim \|f\|_{L_t^2 L_x^{\frac{2\sigma-1}{s-1}}} \\ &\lesssim \|f\|_{\mathcal{S}^0_T} \end{aligned} \tag{4.5.97}$$

and so, the first term in (4.5.93) can be controlled by (after possibly relabelling δ),

$$2^{-j\delta}(\|u\|_{\mathcal{S}^{1+c\delta}_T} + \|f\|_{\mathcal{S}^0_T})^{2\sigma-1}\|u\|_{L^\infty_t H^{2s}_x} \lesssim b_j\Lambda\|u\|_{L^\infty_t H^{2s}_x} \lesssim b_j\Lambda\|u\|_{\mathcal{X}^{2s}_T}. \tag{4.5.98}$$

For the second term in (4.5.93), we simply have by Bernstein, and Corollary 4.2.11 and Corollary 4.5.5,

$$\begin{aligned} \|\tilde{S}_{2j} D_t^{s-1}(\tilde{S}_{\geq 2j}(|u|^{2\sigma-2}\overline{u})f)\|_{L^2_t L^2_x} &\lesssim 2^{-j\delta}\|D_t^{s-1+\frac{\delta}{2}}(|u|^{2\sigma-2}\overline{u})\|_{L_t^{\frac{4}{2\sigma-1}} L_x^{\frac{4}{2\sigma-1}}} \|f\|_{L^4_t L_x^{\frac{4}{3-2\sigma}}} \\ &\lesssim 2^{-j\delta}\|\langle D_t\rangle^{\frac{1}{2}+\frac{\delta}{2}} u\|^{2\sigma-1}_{L^4_t L^4_x}\|f\|_{\mathcal{S}^0_T} \\ &\lesssim 2^{-j\delta}(\|u\|^{2\sigma-1}_{\mathcal{S}^{1+\delta}_T} + \|f\|^{2\sigma-1}_{\mathcal{S}^0_T})\|f\|_{\mathcal{S}^0_T} \\ &\lesssim b_j\Lambda\|f\|_{\mathcal{S}^0_T}. \end{aligned} \tag{4.5.99}$$

This handles the case $1 \leq s \leq \sigma + \frac{1}{2}$.

Next, suppose $\sigma + \frac{1}{2} < s < 2\sigma$. By Bernstein's inequality,

$$\begin{aligned} \|\tilde{S}_{2j} D_t^{s-1}(\tilde{S}_{\geq 2j}(|u|^{2\sigma-2}\overline{u})\partial_t u)\|_{L^2_t L^2_x} &\lesssim 2^{-j\delta}\|D_t^{s-1+\frac{\delta}{2}}(|u|^{2\sigma-2}\overline{u})\|_{L_t^{\frac{2}{2\sigma-1}} L_x^{\frac{2}{2\sigma-1}}} \|\partial_t u\|_{L_t^{\frac{1}{1-\sigma}} L_x^{\frac{1}{1-\sigma}}} \\ &\lesssim b_j\|D_t^{s-1+\frac{\delta}{2}}(|u|^{2\sigma-2}\overline{u})\|_{L_t^{\frac{2}{2\sigma-1}} L_x^{\frac{2}{2\sigma-1}}} \|\partial_t u\|_{L_t^{\frac{1}{1-\sigma}} L_x^{\frac{1}{1-\sigma}}}. \end{aligned} \tag{4.5.100}$$

Using Corollary 4.2.11 and then Corollary 4.5.5, we estimate,

$$\begin{aligned} \|D_t^{s-1+\frac{\delta}{2}}(|u|^{2\sigma-2}\overline{u})\|_{L_t^{\frac{2}{2\sigma-1}} L_x^{\frac{2}{2\sigma-1}}} &\lesssim \|\langle D_t\rangle^{\frac{s-1+\delta}{2\sigma-1}} u\|^{2\sigma-1}_{L^2_t L^2_x} \\ &\lesssim \|P_{\leq 0}\langle D_t\rangle^{\frac{s-1+\delta}{2\sigma-1}} u\|^{2\sigma-1}_{L^2_t L^2_x} + \left(\sum_{j>0}\|\langle D_t\rangle^{\frac{s-1+\delta}{2\sigma-1}} P_j u\|^2_{L^2_t L^2_x}\right)^{\frac{1}{2}(2\sigma-1)} \\ &\lesssim \|\langle D_x\rangle^{\frac{2s-2+2\delta}{2\sigma-1}} u\|^{2\sigma-1}_{L^2_t L^2_x} + \|f\|^{2\sigma-1}_{\mathcal{S}^0_T}. \end{aligned} \tag{4.5.101}$$

Furthermore, we have by Sobolev embedding and the equation,

$$\|\partial_t u\|_{L_t^{\frac{1}{1-\sigma}} L_x^{\frac{1}{1-\sigma}}} \lesssim \|\langle D_x\rangle^{\sigma+\frac{3}{2}} u\|_{L_t^\infty L_x^2} + \|\langle D_x\rangle^{\sigma-\frac{1}{2}} f\|_{L_t^\infty L_x^2}. \tag{4.5.102}$$

Hence, we obtain

$$\begin{aligned} &b_j\|D_t^{s-1+\frac{\delta}{2}}(|u|^{2\sigma-2}\overline{u})\|_{L_t^{\frac{2}{2\sigma-1}} L_x^{\frac{2}{2\sigma-1}}}\|\partial_t u\|_{L_t^{\frac{1}{1-\sigma}} L_x^{\frac{1}{1-\sigma}}} \\ &\lesssim b_j(\|\langle D_x\rangle^{\sigma+\frac{3}{2}} u\|_{L_t^\infty L_x^2} + \|\langle D_x\rangle^{\sigma-\frac{1}{2}} f\|_{L_t^\infty L_x^2})\|\langle D_x\rangle^{\frac{2s-2+2\delta}{2\sigma-1}} u\|_{L_t^2 L_x^2}^{2\sigma-1} \\ &+ \Lambda b_j(\|u\|_{L_t^\infty H_x^{2s}} + \|f\|_{Z_\infty^{s-1+\delta}}). \end{aligned} \tag{4.5.103}$$

To control the first term, interpolating each factor between $L_t^\infty H_x^{2s}$ and $L_t^\infty H_x^1$ shows that

$$\|\langle D_x\rangle^{\frac{2s-2+2\delta}{2\sigma-1}} u\|_{L_t^2 L_x^2}^{2\sigma-1}\|\langle D_x\rangle^{\sigma+\frac{3}{2}} u\|_{L_t^\infty L_x^2} \lesssim \|u\|_{\mathcal{S}_T^{1+\delta}}^{2\sigma-1}\|u\|_{L_t^\infty H_x^{2s}}. \tag{4.5.104}$$

For the second term, interpolating the $\langle D_x\rangle^{\frac{2s-2+2\delta}{2\sigma-1}} u$ factor between $L_t^\infty H_x^1$ and $L_t^\infty H_x^{2s}$ and the $\langle D_x\rangle^{\sigma-\frac{1}{2}} f$ factor between $L_t^\infty L_x^2$ and $L_t^\infty H_x^{2s-2+\delta}$ and using that $s > \sigma + \frac{1}{2}$ leads to

$$\|\langle D_x\rangle^{\frac{2s-2+2\delta}{2\sigma-1}} u\|_{L_t^2 L_x^2}^{2\sigma-1}\|\langle D_x\rangle^{\sigma-\frac{1}{2}} f\|_{L_t^\infty L_x^2} \lesssim \Lambda(\|u\|_{L_t^\infty H_x^{2s}} + \|f\|_{Z_\infty^{s-1+\delta}}). \tag{4.5.105}$$

Now, collecting all of the estimates and using that $\|u\|_{L_t^\infty H_x^{2s}} \lesssim \|u\|_{\mathcal{X}_T^{2s}}$ completes the proof. □

Finally, we use Lemma 4.5.12 to establish Lemma 4.5.9.

Proof. First, it is straightforward to verify that b_j^2 is a $\mathcal{X}_T^{2s}$ frequency envelope for $\tilde{v}$ in the sense that b_j^2 satisfies property (4.2.3) and is slowly varying. Next, we expand

$$(i\partial_t + \partial_x^2)\tilde{v} = i\partial_t\tilde{\eta}v + \tilde{\eta}(i\partial_t + \partial_x^2)v := f. \tag{4.5.106}$$

Using an argument similar to what was done to estimate (4.5.61) and applying Corollary 4.5.5, it is straightforward to verify $\|f\|_{\mathcal{S}_T^\delta} + \|f\|_{Z_\infty^{s-1+\delta}} \lesssim \varepsilon + \|v\|_{\mathcal{X}_T^{2s}}$, and so the conclusion immediately follows from Lemma 4.5.12. □

4.6 Well-posedness at high regularity

In this section, we aim to prove Theorem 4.1.1. We begin by studying a suitable regularized equation.

Well-posedness of a regularized equation

Since there is an apparent limit to the possible regularity of solutions to (gDNLS), we construct H_x^{2s} solutions as limits of smooth solutions to an appropriate regularized approximate equation. Like in the previous section η will denote a time-dependent cutoff with $\eta = 1$ on $[-1, 1]$ with support in $(-2, 2)$. To construct the requisite solutions, we need the following lemma:

Lemma 4.6.1. Let $2 - \sigma < 2s < 4\sigma$. Let $2s \geq \alpha > \max\{2 - \sigma, 2s - 1\}$. Then there is an $\varepsilon > 0$ such that for every $u_0 \in H_x^{2s}$ with $\|u_0\|_{H_x^\alpha} \leq \varepsilon$ and for all $j > 0$, the regularized equation

$$\begin{cases} (i\partial_t + \partial_x^2)u = i\eta P_{<j}|u|^{2\sigma}\partial_x u, \\ u(0) = P_{<j}u_0, \end{cases} \tag{4.6.1}$$

admits a global solution $u \in C^2(\mathbb{R}; H_x^\infty)$. Moreover, we have the following bounds for $T = 2$,

$$\begin{aligned} &\|u\|_{\mathcal{X}_T^\alpha \cap \mathcal{S}_T^{1+\delta}} \lesssim \varepsilon, \\ &\|(i\partial_t + \partial_x^2)u\|_{\mathcal{S}_T^\delta \cap Z_\infty^{s-1+\delta}} \lesssim \varepsilon, \end{aligned} \tag{4.6.2}$$

where the implicit constant in the above inequality is independent of the parameter j and where $0 < \delta \ll 1$ is any small positive constant.

Remark 4.6.2. The smallness assumption on the H_x^α norm of u_0 will turn out to be inconsequential (by L_x^2 subcriticality for (gDNLS)). This assumption is made for convenience to guarantee (4.6.2).

Let us now construct solutions to (4.6.1). The first step is to construct solutions to an appropriate linear equation. For this, we have the following lemma.

Lemma 4.6.3. Let $\eta = \eta(t)$ be a smooth time-dependent cutoff with $\eta = 1$ on $[-1, 1]$ and with support in $(-2, 2)$. Let $T > 0$ and $v \in L_T^{2\sigma} L_x^\infty$. Let $u_0 \in H_x^{2s}$. Then for each $j > 0$, there exists a unique solution $w \in C([-T, T]; H_x^\infty)$ solving the equation

$$\begin{cases} \partial_t w = i\partial_x^2 w + \eta P_{<j}|v|^{2\sigma}\partial_x w, \\ w(0) = P_{<j}u_0. \end{cases} \tag{4.6.3}$$

Proof. First, observe that for each $n > j$ a simple (iterated) application of the contraction mapping theorem in the closed subspace of $C([-T, T]; L_x^2)$ consisting of functions whose spatial Fourier transform is supported on $[-2^n, 2^n]$ gives rise to a solution $w^{(n)} \in C([-T, T]; H_x^\infty)$

to the following regularized linear equation,

$$\begin{cases} \partial_t w^{(n)} = i\partial_x^2 w^{(n)} + \eta P_{\leq n}(P_{<j}|v|^{2\sigma}\partial_x w^{(n)}), \\ w^{(n)}(0) = P_{<j}u_0. \end{cases} \tag{4.6.4}$$

We show that the sequence $w^{(n)}$ converges as $n \to \infty$ to some $w \in C([-T,T];H_x^\infty)$ which solves (4.6.3). This follows in two stages, but is standard. First, for each integer $k \geq 0$, a standard energy estimate and Bernstein's inequality shows that $w^{(n)}$ satisfies the bound

$$\|w^{(n)}\|_{C([-T,T];H_x^k)} \lesssim \exp(2^{j(k+1)}\|v\|^{2\sigma}_{L_T^{2\sigma}L_x^\infty})\|P_{<j}u_0\|_{H_x^k} \tag{4.6.5}$$

where importantly, the bound is independent of n (but can depend on j). Furthermore, a simple energy estimate in L_x^2 for differences of solutions $w^{(n)} - w^{(m)}$ to (4.6.4) shows that the sequence $w^{(n)}$ is Cauchy in $C([-T,T];L_x^2)$ and thus converges to some $w \in C([-T,T];L_x^2)$. Interpolating against (4.6.5) shows that in fact $w^{(n)}$ converges to some w in $C([-T,T];H_x^\infty)$ and that w solves (4.6.3) in the sense of distributions, and furthermore that w satisfies the bound (4.6.5) for each $k \geq 0$. □

The next step in the proof of Lemma 4.6.1 is to construct the corresponding $C^2(\mathbb{R};H_x^\infty)$ solution to (4.6.1). For this purpose, consider the following iteration scheme,

$$\begin{cases} (i\partial_t + \partial_x^2)u^{(n+1)} = i\eta P_{<j}|u^{(n)}|^{2\sigma}\partial_x u^{(n+1)}, \\ u^{(n+1)}(0) = P_{<j}u_0, \end{cases} \tag{4.6.6}$$

with the initialization $u^{(0)} = 0$. Thanks to Lemma 4.6.3 it follows that for each n, there is a solution $u^{(n+1)} \in C([-2,2];H_x^\infty)$ to the above equation. In particular, $u^{(n+1)}$ can be extended globally in time because for $|t| > 2$, $u^{(n+1)}$ solves the linear Schrödinger equation.

Next, we have the following lemma concerning the convergence of this iteration scheme, from which Lemma 4.6.1 is immediate.

Lemma 4.6.4. Let $2-\sigma < 2s < 4\sigma$. Let $2s \geq \alpha > \max\{2-\sigma, 2s-1\}$. Let $u_0 \in H_x^{2s}$ and let $u^{(n+1)}$ be the corresponding $C(\mathbb{R};H_x^\infty)$ solution to (4.6.6). Then there is $\varepsilon > 0$ independent of j such that if $\|u_0\|_{H_x^\alpha} \leq \varepsilon$, then $u^{(n)}$ converges to some $u \in C(\mathbb{R};H_x^\infty)$ solving (4.6.1). Furthermore, we have $u \in C^2(\mathbb{R};H_x^\infty)$ and the bounds

$$\begin{aligned} &\|u\|_{\mathcal{X}_T^\alpha \cap \mathcal{S}_T^{1+\delta}} \lesssim \varepsilon, \\ &\|(i\partial_t + \partial_x^2)u\|_{\mathcal{S}_T^\delta \cap Z_\infty^{s-1+\delta}} \lesssim \varepsilon. \end{aligned} \tag{4.6.7}$$

Proof. We begin by showing that $u^{(n+1)}$ satisfies the bounds

$$\begin{aligned} &\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha \cap \mathcal{S}_T^{1+\delta}} \lesssim \varepsilon, \\ &\|(i\partial_t + \partial_x^2)u^{(n+1)}\|_{\mathcal{S}_T^\delta \cap Z_\infty^{s-1+\delta}} \lesssim \varepsilon, \end{aligned} \tag{4.6.8}$$

for $T = 2$ uniformly in n. Given the initialization $u^{(0)} = 0$, we may make the inductive hypothesis that (4.6.8) holds with $n+1$ replaced by n. Now, we prove the above two bounds for $u^{(n+1)}$.

We begin by showing $\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha \cap \mathcal{S}_T^{1+\delta}} \lesssim \varepsilon$. Indeed, it follows from the modification of the low regularity bounds outlined in Lemma 4.3.18 that for $2s \geq \alpha > 2 - \sigma$,

$$\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha \cap \mathcal{S}_T^{1+\delta}} \lesssim \|u^{(n+1)}\|_{\mathcal{X}_T^\alpha}. \tag{4.6.9}$$

Then Proposition 4.5.1 and the inductive hypothesis gives

$$\begin{aligned} \|u^{(n+1)}\|_{\mathcal{X}_T^\alpha}^2 \lesssim \|u_0\|_{H_x^\alpha}^2 + \varepsilon^{2\sigma}(\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha}^2 + \|u^{(n+1)}\|_{\mathcal{S}_T^1}^2) + \varepsilon^{2\sigma-1}\|u^{(n+1)}\|_{\mathcal{S}_T^1}\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha}\|u^{(n)}\|_{\mathcal{X}_T^\alpha} \\ + \varepsilon^{4\sigma-2}\|u^{(n+1)}\|_{\mathcal{S}_T^1}^2\|u^{(n)}\|_{\mathcal{X}_T^\alpha}^2, \end{aligned} \tag{4.6.10}$$

and so,

$$\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha}^2 \lesssim \|u_0\|_{H_x^\alpha}^2 + \varepsilon^{2\sigma}\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha}^2. \tag{4.6.11}$$

From this, we deduce

$$\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha} \lesssim \varepsilon. \tag{4.6.12}$$

Next, we aim to verify the bound,

$$\|(i\partial_t + \partial_x^2)u^{(n+1)}\|_{\mathcal{S}_T^\delta \cap Z_\infty^{s-1+\delta}} \lesssim \varepsilon. \tag{4.6.13}$$

For this, we use the equation,

$$(i\partial_t + \partial_x^2)u^{(n+1)} = i\eta P_{<j}|u^{(n)}|^{2\sigma}\partial_x u^{(n+1)}. \tag{4.6.14}$$

From Lemma 4.5.10 and (4.6.9), we have

$$\|i\eta P_{<j}|u^{(n)}|^{2\sigma}\partial_x u^{(n+1)}\|_{\mathcal{S}_T^\delta \cap Z_\infty^{s-1+\delta}} \lesssim \varepsilon^{2\sigma}\|u^{(n+1)}\|_{\mathcal{X}_T^\alpha} \lesssim \varepsilon. \tag{4.6.15}$$

This verifies the uniform in n bound (4.6.8).

Next, we show that that $u^{(n)}$ converges to $u \in C(\mathbb{R}; L_x^2)$. Clearly it suffices to show (by the localization properties of η) that $u^{(n)}$ converges to $u \in C([-2,2]; L_x^2)$.

We begin by estimating the L_x^2 norm of $u^{(n+1)}(t) - u^{(n)}(t)$ for $|t| \leq 2$. Indeed, we see that $u^{(n+1)} - u^{(n)}$ satisfies the equation,

$$\begin{cases} (i\partial_t + \partial_x^2)(u^{(n+1)} - u^{(n)}) = i\eta P_{<j}|u^{(n)}|^{2\sigma}\partial_x(u^{(n+1)} - u^{(n)}) \\ +i\eta P_{<j}(|u^{(n)}|^{2\sigma} - |u^{(n-1)}|^{2\sigma})\partial_x u^{(n)}, \\ (u^{(n+1)} - u^{(n)})(0) = 0. \end{cases} \tag{4.6.16}$$

A simple energy estimate shows that for each $-2 \leq T \leq 2$

$$\begin{aligned} &\|u^{(n+1)} - u^{(n)}\|^2_{L_T^\infty L_x^2} \\ &\lesssim \|u^{(n)}\|_{S_T^1}\|u^{(n)}\|^{2\sigma-1}_{L_T^\infty L_x^\infty}\|u^{(n+1)} - u^{(n)}\|^2_{L_T^\infty L_x^2} \\ &+ \|u^{(n)}\|_{S_T^1}(\|u^{(n)}\|^{2\sigma-1}_{L_T^\infty L_x^\infty} + \|u^{(n-1)}\|^{2\sigma-1}_{L_T^\infty L_x^\infty})\|u^{(n)} - u^{(n-1)}\|_{L_T^\infty L_x^2}\|u^{(n+1)} - u^{(n)}\|_{L_T^\infty L_x^2} \end{aligned} \tag{4.6.17}$$

where all the implicit constants are independent of j. Using (4.6.8) and Cauchy Schwarz, we obtain

$$\|u^{(n+1)} - u^{(n)}\|^2_{L_T^\infty L_x^2} \leq \frac{1}{4}\|u^{(n+1)} - u^{(n)}\|^2_{L_T^\infty L_x^2} + \frac{1}{4}\|u^{(n)} - u^{(n-1)}\|^2_{L_T^\infty L_x^2}. \tag{4.6.18}$$

From this, one obtains

$$\|u^{(n+1)} - u^{(n)}\|^2_{L_T^\infty L_x^2} \leq \frac{1}{2}\|u^{(n)} - u^{(n-1)}\|^2_{L_T^\infty L_x^2}. \tag{4.6.19}$$

Hence, we see that $u^{(n)}$ converges to u in $C([-2,2]; L_x^2)$. By a simple energy estimate, and Bernstein's inequality, it is straightforward to verify that for each integer $k \geq 0$, we have the uniform (in n) bound

$$\|u^{(n+1)}\|_{C([-2,2];H_x^k)} \lesssim \exp(2^{j(k+1)}\|u^{(n)}\|^{2\sigma}_{L_T^{2\sigma} L_x^\infty})\|P_{<j}u_0\|_{H_x^k} \lesssim_j \|u_0\|_{H_x^{2s}}. \tag{4.6.20}$$

Hence, by interpolating against (4.6.20), we see that $u^{(n)}$ converges to u in $C([-2,2]; H_x^\infty)$. By differentiating the equation in time, we find $u \in C^2([-2,2]; H_x^\infty)$.

It remains to show (4.6.7). Since $u^{(n)} \to u$ in $C([-2,2]; H_x^\infty)$, the $\mathcal{X}_T^\alpha \cap \mathcal{S}_T^{1+\delta}$ bound follows immediately from (4.6.8). For the remaining estimate, we may clearly control

$$(i\partial_t + \partial_x^2)u = i\eta P_{<j}|u|^{2\sigma}\partial_x u \tag{4.6.21}$$

in $\mathcal{S}_T^\delta \cap Z_\infty^{s-1+\delta}$ by (after possibly slightly enlarging δ)

$$\|u\|^{2\sigma}_{\mathcal{X}_T^\alpha\cap\mathcal{S}_T^{1+\delta}} + \|i\eta P_{<j}|u|^{2\sigma}\partial_x u\|_{Z_\infty^{s-1+\delta}} \lesssim \varepsilon + \|iP_{<j}\eta|u|^{2\sigma}\partial_x u\|_{Z_\infty^{s-1+\delta}}. \tag{4.6.22}$$

From Lemma 4.5.10, we have

$$\|i\eta P_{<j}|u|^{2\sigma}\partial_x u\|_{Z_\infty^{s-\frac{3}{2}+\delta}} \lesssim \varepsilon. \tag{4.6.23}$$

Then applying Lemma 4.5.10 again, using (4.6.23) then gives

$$\|i\eta P_{<j}|u|^{2\sigma}\partial_x u\|_{Z_\infty^{s-1+\delta}} \lesssim \varepsilon. \tag{4.6.24}$$

This completes the proof. □

Remark 4.6.5. Note that at this point, we haven't said anything about the behavior of (4.6.1) as $j \to \infty$. For this, we will again need the uniform bounds from Proposition 4.5.1.

Well-posedness for the full equation

In this section, we prove the local well-posedness of (gDNLS) in H_x^{2s} for $2-\sigma < 2s < 4\sigma$.

Indeed, let $u_0 \in H_x^{2s}$ and let $2-\sigma < \alpha \le 2s$. By rescaling (recalling the problem is L_x^2 subcritical), we may assume without loss of generality that $\|u_0\|_{H_x^\alpha} \le \varepsilon$ for some $\varepsilon > 0$ sufficiently small, and construct the corresponding H_x^{2s} solution on the time interval $[-1,1]$. For $2-\sigma < 2s \le \frac{3}{2}$, we construct the solution in the Strichartz type space $\mathcal{X}_T^{2s}\cap\mathcal{S}_T^{1+\delta}$, where $0<\delta\ll 1$ is any sufficiently small positive constant. When $s > \frac{3}{2}$, the extra $\mathcal{S}_T^{1+\delta}$ component is, of course, redundant, thanks to Sobolev embedding.

We will realize H_x^{2s} well-posed solutions as (restrictions to the interval $[-1,1]$ of) limits of smooth solutions to the regularized equation (4.6.1). To establish this, we have the following lemma.

Lemma 4.6.6. Let $2-\sigma < 2s < 4\sigma$. Let $2s \ge \alpha > \max\{2-\sigma, 2s-1\}$. Then there is an $\varepsilon > 0$ such that for every $u_0 \in H_x^{2s}$ with $\|u_0\|_{H_x^\alpha} \le \varepsilon$, the time-truncated equation,

$$\begin{cases} (i\partial_t + \partial_x^2)u = i\eta|u|^{2\sigma}\partial_x u, \\ u(0) = u_0, \end{cases} \tag{4.6.25}$$

admits a global solution $u \in C^2(\mathbb{R}; H_x^\infty)$. Moreover, we have the following bounds for $T = 2$,

$$\begin{aligned} &\|u\|_{\mathcal{X}_T^\alpha \cap \mathcal{S}_T^{1+\delta}} \lesssim \varepsilon, \\ &\|(i\partial_t + \partial_x^2)u\|_{\mathcal{S}_T^\delta \cap Z_\infty^{s-1+\delta}} \lesssim \varepsilon, \end{aligned} \tag{4.6.26}$$

and also

$$\|u\|_{\mathcal{X}_T^{2s}}^2 \lesssim \frac{1}{1 - C\varepsilon^{2\sigma}} \|u_0\|_{H_x^{2s}}^2, \tag{4.6.27}$$

where $C > 0$ is some universal constant.

Proof. If ε is small enough, thanks to Lemma 4.6.1, for each $j > 0$, there is a smooth solution $u^{(j)} \in C^2(\mathbb{R}; H_x^\infty)$ to the equation,

$$\begin{cases} (i\partial_t + \partial_x^2)u^{(j)} = i\eta P_{<j}|u^{(j)}|^{2\sigma} u_x^{(j)}, \\ u^{(j)}(0) = P_{<j}u_0, \end{cases} \tag{4.6.28}$$

satisfying

$$\|u^{(j)}\|_{\mathcal{X}_T^\alpha \cap \mathcal{S}_T^{1+\delta}} + \|(i\partial_t + \partial_x^2)u^{(j)}\|_{\mathcal{S}_T^\delta \cap Z_\infty^{s-1+\delta}} \lesssim \varepsilon \tag{4.6.29}$$

uniformly in j. Now, define for $k > j$, $v^{(k,j)} := u^{(k)} - u^{(j)}$. Then $v^{(k,j)}$ satisfies the equation,

$$\begin{cases} (i\partial_t + \partial_x^2)v^{(k,j)} = i\eta P_{<k}|u^{(k)}|^{2\sigma}\partial_x v^{(k,j)} + i\eta P_{<k}(|u^{(k)}|^{2\sigma} - |u^{(j)}|^{2\sigma})\partial_x u^{(j)} \\ +i\eta P_{j\le \cdot < k}|u^{(j)}|^{2\sigma}\partial_x u^{(j)}, \\ v^{(k,j)}(0) = P_{j\le\cdot<k}u_0. \end{cases} \tag{4.6.30}$$

Multiplying by $-i\overline{v^{(k,j)}}$ taking real part and integrating over $\mathbb{R}$ and from 0 to t with $|t| \le T$ leads to the simple energy estimate

$$\begin{aligned} \|v^{(k,j)}\|_{L_T^\infty L_x^2}^2 \lesssim \|P_{j\le\cdot<k}u_0\|_{L_x^2}^2 + (\|u^{(j)}\|_{\mathcal{S}_T^1}^{2\sigma-1} + \|u^{(k)}\|_{\mathcal{S}_T^1}^{2\sigma-1})\|u^{(j)}\|_{\mathcal{S}_T^1}\|v^{(k,j)}\|_{L_T^\infty L_x^2}^2 \\ + \|u^{(k)}\|_{\mathcal{S}_T^1}^{2\sigma}\|v^{(k,j)}\|_{L_T^\infty L_x^2}^2 + \|P_{j\le\cdot<k}|u^{(j)}|^{2\sigma}\|_{L_T^\infty L_x^2}\|u^{(j)}\|_{\mathcal{S}_T^1}\|v^{(k,j)}\|_{L_T^\infty L_x^2}. \end{aligned} \tag{4.6.31}$$

Using the uniform in j bound

$$\|u^{(j)}\|_{\mathcal{S}_T^{1+\delta}} \lesssim \varepsilon \tag{4.6.32}$$

from Lemma 4.6.1 and Cauchy Schwarz gives

$$\|v^{(k,j)}\|_{L_T^\infty L_x^2}^2 \lesssim \|P_{j\le\cdot<k}u_0\|_{L_x^2}^2 + \|P_{j\le\cdot<k}|u^{(j)}|^{2\sigma}\|_{L_T^\infty L_x^2}^2\|u^{(j)}\|_{\mathcal{S}_T^1}^2. \tag{4.6.33}$$

Furthermore,

$$\|P_{j\le\cdot<k}|u^{(j)}|^{2\sigma}\|_{L_T^\infty L_x^2} \lesssim 2^{-j}\|u^{(j)}\|_{\mathcal{S}_T^1}^{2\sigma}. \tag{4.6.34}$$

Hence, the right hand side of (4.6.33) goes to zero as $j, k \to \infty$. Therefore, $u^{(j)}$ converges to some u in $C([-2,2]; L^2_x)$. On the other hand, thanks to the uniform (in k) bounds from the energy estimate Proposition 4.5.1, we obtain

$$\|P_j u^{(k)}\|^2_{\mathcal{X}^{2s}_T} \lesssim a_j^2 \|u_0\|^2_{H^{2s}_x} + [b_j^{(k)}]^2 \varepsilon^{2\sigma} \|u^{(k)}\|^2_{\mathcal{X}^{2s}_T}, \tag{4.6.35}$$

where $b_j^{(k)}$ is a $\mathcal{X}^{2s}_T$ frequency envelope for $u^{(k)}$. Using that $\|u^{(k)}\|_{S^{1+\delta}_T} \lesssim \varepsilon$, an argument similar to the low regularity well-posedness shows that for ε small enough, a_j is a $\mathcal{X}^{2s}_T$ frequency envelope for $u^{(k)}$. Analogously to the low regularity argument, this can be used to show that $u^{(k)} \to u$ in $\mathcal{X}^{2s}_T$ and that a_j is a $\mathcal{X}^{2s}_T$ frequency envelope for u and that u solves the time truncated equation,

$$\begin{cases} i\partial_t u + u_{xx} = i\eta |u|^{2\sigma} u_x, \\ u(0) = u_0, \end{cases} \tag{4.6.36}$$

in the sense of distributions. Moreover, by square summing over j and passing to the limit in (4.6.35), we obtain the uniform bound

$$\|u\|^2_{\mathcal{X}^{2s}_T} \lesssim \frac{1}{1 - C\varepsilon^{2\sigma}} \|u_0\|^2_{H^{2s}_x}. \tag{4.6.37}$$

□

Next, we establish local well-posedness for the full equation (gDNLS).

For existence, we may rescale (using the L^2_x subcriticality of the equation) to assume $u_0 \in H^{2s}_x$ has sufficiently small data. Then we may construct a $\mathcal{X}^{2s}_T$ solution to (gDNLS) on the time interval $[-1, 1]$ by applying Lemma 4.6.6 and restricting to $|t| \leq 1$.

For uniqueness, we consider the difference of two H^{2s}_x solutions u_1, u_2 to (gDNLS) and obtain, by a standard energy estimate, the weak Lipschitz bound,

$$\|u_1 - u_2\|_{L^\infty_T L^2_x} \lesssim_{\|u_1\|_{\mathcal{S}^1_T}, \|u_2\|_{\mathcal{S}^1_T}} \|u_1(0) - u_2(0)\|_{L^2_x}. \tag{4.6.38}$$

for $T > 0$. Among other things, this shows uniqueness in $C([-1,1]; H^{2s}_x) \cap \mathcal{S}^1_T$.

For continuous dependence, again assume without loss of generality that u_0 has sufficiently small H^{2s}_x norm. To show continuous dependence for the full equation (gDNLS), it clearly suffices (by restricting to $T \leq 1$) to show that the data to solution map $u_0 \in H^{2s}_x \mapsto u \in \mathcal{X}^{2s}_{T=2} \cap \mathcal{S}^{1+\delta}_{T=2}$ for the time-truncated equation (4.6.36) is continuous. For this, let $u_0^n \in H^{2s}_x$

be a sequence of initial data converging to some u_0 in H^{2s}_x. Let u^n and u denote the corresponding $\mathcal{X}^{2s}_{T=2} \cap \mathcal{S}^{1+\delta}_{T=2}$ solutions to the time-truncated equation (4.6.36), respectively. From the frequency envelope bound (4.6.35) and an argument almost identical to the proof of continuous dependence at low regularity, it follows that

$$\lim_{n\to\infty} \|u^n - u\|_{\mathcal{X}^{2s}_{T=2}\cap\mathcal{S}^{1+\delta}_{T=2}} = 0. \tag{4.6.39}$$

We omit the details. This finally completes the proof of Theorem 4.1.1.

4.7 Global well-posedness

Here, we complete the proof of Theorem 4.1.2. That is, we show that for $\frac{\sqrt{3}}{2} < \sigma < 1$ and $1 \le 2s < 4\sigma$, (gDNLS) is globally well-posed in H^{2s}_x. The proof of local well-posedness in H^{2s}_x for $1 \le 2s \le \frac{3}{2}$ and $\sigma > \frac{\sqrt{3}}{2}$ established in Section 4.4 relied on having global well-posedness when $\frac{3}{2} < 2s < 4\sigma$, so we establish this first. Ultimately, global well-posedness will follow from the conservation laws, which we use in the next lemma to establish uniform control of the H^1_x norm of solutions:

Lemma 4.7.1. (H^1_x norm remains bounded) Let $u_0 \in H^{2s}_x$, $1 \le 2s < 4\sigma$ and $\frac{\sqrt{3}}{2} < \sigma < 1$. Let $T > 0$ be sufficiently small. If $2s \le \frac{3}{2}$, suppose that there is a corresponding well-posed solution $u \in X^{2s}_T$ to (gDNLS). Likewise, if $4\sigma > 2s > \frac{3}{2}$, let $u \in \mathcal{X}^{2s}_T$ be the corresponding well-posed solution to (gDNLS). Then for $0 \le |t| \le T$, we have

$$\|u(t)\|_{H^1_x} \lesssim_{\|u_0\|_{H^1_x}} 1 \tag{4.7.1}$$

where the implied constant depends only on the size of $\|u_0\|_{H^1_x}$. In particular, the H^1_x norm of u cannot blow up in finite time.

Remark 4.7.2. There is one small technical caveat to be aware of. Namely, in Lemma 4.7.1, it is assumed for $1 \le 2s \le \frac{3}{2}$ that the equation (gDNLS) is locally well-posed X^{2s}_T. As mentioned above, this will follow from the results proven in Section 4.4 once we have established global well-posedness in the range $\frac{3}{2} < 2s < 4\sigma$ (where we already have local well-posedness from Section 6).

Proof. Recall that we have the conserved mass and energy, respectively

$$M(u) := \frac{1}{2}\int_{\mathbb{R}} |u|^2 dx = M(u_0), \tag{4.7.2}$$

$$E(u) := \frac{1}{2}\int_{\mathbb{R}} |u_x|^2 dx + \frac{1}{2(1+\sigma)}\mathrm{Re}\int_{\mathbb{R}} i|u|^{2\sigma}\overline{u}u_x dx = E(u_0). \tag{4.7.3}$$

It is also straightforward to verify that any well-posed solution in $\mathcal{X}_T^{2s}$ (when $\frac{3}{2} < 2s < 4\sigma$) or X_T^{2s} (when $1 \leq 2s \leq \frac{3}{2}$) satisfies these conservation laws. By interpolation, we have the following lower bound for the energy (where C is some constant that may change from line to line)

$$\begin{aligned} E(u) &\geq \frac{1}{2}\|u_x\|_{L_x^2}^2 - C\|u\|_{L_x^{4\sigma+2}}^{2\sigma+1}\|u_x\|_{L_x^2} \\ &\geq \frac{1}{4}\|u_x\|_{L_x^2}^2 - C\|u\|_{L_x^2}^{\frac{1+\sigma}{1-\sigma}} \\ &\geq \frac{1}{4}\|u_x\|_{L_x^2}^2 - CM(u)^{\frac{1+\sigma}{2(1-\sigma)}}. \end{aligned} \tag{4.7.4}$$

Hence, for $0 \leq |t| \leq T$, we have

$$\|u(t)\|_{H_x^1}^2 \lesssim E(u_0) + M(u_0) + M(u_0)^{\frac{1+\sigma}{2(1-\sigma)}} \lesssim_{\|u_0\|_{H_x^1}} 1. \tag{4.7.5}$$

□

Corollary 4.7.3. Let $u_0 \in H_x^{2s}$, $0 < T^* < \infty$, $\frac{3}{2} < 2s < 4\sigma$ and $\frac{\sqrt{3}}{2} < \sigma < 1$. Suppose that for each $T < T^*$, there is a corresponding well-posed solution $u \in \mathcal{X}_T^{2s}$ with initial data u_0. Then for each $0 < \delta \ll 1$, we have

$$\limsup_{T \nearrow T^*} \|u\|_{\mathcal{S}_T^{1+\delta} \cap X_T^{2-\sigma+2\delta}} < \infty. \tag{4.7.6}$$

In particular, the $\mathcal{S}_T^{1+\delta} \cap X_T^{2-\sigma+2\delta}$ norm of a solution cannot blow up in finite time.

Proof. Lemma 4.7.1 shows that for all $0 < T < T^*$, the norm $\|u\|_{L_T^\infty H_x^1}$ is bounded by a constant depending on the initial data $\|u_0\|_{H_x^1}$. Therefore, iterating (after appropriately translating and rescaling the initial data) Proposition 4.3.6 shows that

$$\limsup_{T \nearrow T^*} \|u\|_{X_T^1} \lesssim_{\|u_0\|_{H_x^1}} 1. \tag{4.7.7}$$

By virtue of (4.7.7) and iterating Proposition 4.3.6, we find that

$$\limsup_{T \nearrow T^*} \|u\|_{X_T^{2-\sigma+2\delta}} < \infty. \tag{4.7.8}$$

It follows that

$$\limsup_{T \nearrow T^*} \|u\|_{\mathcal{S}_T^{1+\delta}} \leq \limsup_{T \nearrow T^*} \|u\|_{X_T^{2-\sigma+2\delta}} < \infty. \tag{4.7.9}$$

□

Next, we use Corollary 4.7.3 and Lemma 4.6.6 to establish global well-posedness in the high regularity regime $\frac{3}{2} < 2s < 4\sigma$. Indeed, for $u_0 \in H^{2s}_x$ let $T^* > 0$ be the maximal time for which there is a corresponding well-posed solution $u \in \mathcal{X}^{2s}_T$ for each $T < T^*$. If $T^* = \infty$, then we are done. We can therefore assume for the sake of contradiction that $T^* < \infty$. Then we have

$$\limsup_{T \nearrow T^*} \|u\|_{\mathcal{X}^{2s}_T} = \infty. \tag{4.7.10}$$

We show that this is impossible. By rescaling and translation, we may without loss of generality take $T^* = 1$.

We begin with the case $\frac{3}{2} < 2s < 2$. Set $\alpha = 2 - \sigma + 2\delta$ where δ is some small positive constant.

Let $0 < \varepsilon \ll 1$. Define now the rescaled solution $u_\lambda(t,x) = \lambda^{\frac{1}{2\sigma}} u(\lambda^2 t, \lambda x)$ to (gDNLS), where λ satisfies $k := \lambda^{-2} \in \mathbb{N}$ and where λ is small enough so that for each $T < \lambda^{-2}$,

$$\|u_\lambda\|_{L^\infty_{T<\lambda^{-2}} H^\alpha_x} \lesssim \lambda^{\frac{1}{2\sigma} - \frac{1}{2}} \|u\|_{L^\infty_{T<1} H^\alpha_x} \lesssim \varepsilon. \tag{4.7.11}$$

By assumption u_λ is a $\mathcal{X}^{2s}_T$ solution to (gDNLS) for $T < \lambda^{-2}$ with

$$\limsup_{T \nearrow \lambda^{-2}} \|u_\lambda\|_{\mathcal{X}^{2s}_T} = \infty. \tag{4.7.12}$$

Now, we iterate Lemma 4.6.6. We consider the initial value problem for each natural number $n < k$,

$$\begin{cases} (i\partial_t + \partial_x^2) w_n = i\eta |w_n|^{2\sigma} \partial_x w_n, \\ w_n(0) = u_\lambda(n). \end{cases} \tag{4.7.13}$$

By Lemma 4.6.6 by taking $\alpha = 2-\sigma+2\delta$, and (4.7.11) there is a global solution $w \in C(\mathbb{R}; H^{2s}_x)$ to the above equation satisfying

$$\|w_n\|^2_{\mathcal{X}^{2s}_{T=2}} \lesssim \frac{1}{1 - C\varepsilon^{2\sigma}} \|u_\lambda(n)\|^2_{H^{2s}_x} \tag{4.7.14}$$

from which we deduce (by restricting w to times in $[-1,1]$),

$$\|u_\lambda(n + \cdot)\|^2_{\mathcal{X}^{2s}_{T=1}} \lesssim \frac{1}{1 - C\varepsilon^{2\sigma}} \|u_\lambda(n)\|^2_{H^{2s}_x}. \tag{4.7.15}$$

Iterating this k times gives the bound

$$\|u_\lambda\|^2_{\mathcal{X}^{2s}_{T<\lambda^{-2}}} \lesssim \left(\frac{1}{1 - C\varepsilon^{2\sigma}} \right)^k \|u_\lambda(0)\|^2_{H^{2s}_x}. \tag{4.7.16}$$

This contradicts (4.7.12). Therefore $T^* = \infty$ and the $\mathcal{X}_T^{2s}$ norm cannot blow up in finite time when $\frac{3}{2} < 2s < 2$.

Next, we proceed with the case $2 \leq 2s < 4\sigma$. If $2 \leq 2s < 3$, then if we assume a maximal time of existence $T^* < \infty$ for a $\mathcal{X}_T^{2s}$ solution, then the previous case shows that for $\delta > 0$ sufficiently small,

$$\limsup_{T \nearrow T*} \|u\|_{\mathcal{X}_T^{2s-1+\delta}} < \infty. \tag{4.7.17}$$

Replacing α in the previous case with $\max\{2s - 1 + \delta, 2 - \sigma + 2\delta\}$ and repeating the proof verbatim shows once again that $T^* = \infty$. Iterating once more shows that in the case $3 \leq 2s < 4\sigma$, we also have the same conclusion. Thus, (gDNLS) is globally well-posed in H_x^{2s} when $\frac{3}{2} < 2s < 4\sigma$.

We finally turn to the last case. Namely, we show that (gDNLS) is globally well-posed when $1 \leq 2s \leq \frac{3}{2}$.

Indeed, at this point, we know from Section 4 and the previous two cases that we have a locally well-posed X_T^{2s} solution. Iterating the low regularity bounds Proposition 4.3.6 and using Lemma 4.7.1 shows that such a solution can be continued for all time. This finally completes the proof of Theorem 4.1.2.

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