UNIVERSITY OF CALIFORNIA

Los Angeles

Three Essays on Economic Theory

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Economics

by

Junrok Park

2022

ABSTRACT OF THE DISSERTATION

Three Essays on Economic Theory

by

Junrok Park
Doctor of Philosophy in Economics
University of California, Los Angeles, 2022
Professor Tomasz Marek Sadzik, Chair

This dissertation contributes to the field of mechanism design and, in particular, to the study of robust mechanisms, which concerns the construction of allocation mechanisms that perform well across a wide range of environments. It consists of two parts. The first part— Chapters 1 and 2—studies zero-sum games between a profit-maximizing mechanism designer and an information designer with the opposite goal in public good provision setups. They consider distinct yet related setups, yielding distinct economic implications. The second part—Chapter 3—concerns the construction of a double clock auction that implements various designer objectives other than exact efficiency in an exchange environment

Chapter 1 studies an informational remedy to a well-known example of market failure: a public good sold by a profit-seeking enterprise. A monopolist sells a public good to a group of buyers who privately observe a signal about their valuation. The public good is sold through an optimal mechanism subject to the ex-post incen-

tive constraints. This paper studies an information design problem, characterizing *the regulator's solution*, a signal distribution that maximizes the buyers' expected payoffs in an optimal mechanism. It limits the impact of free-riding while imposing indifference on the monopolist so that the public good is always sold with any number of agents. Such desirable properties are achieved through a carefully chosen negative correlation between signals and a lower bound on the total value of trade. Moreover, the monopolist earns a finite expected profit even with infinitely many buyers, thereby completely losing his market power.

Chapter 2 devises a robustly optimal mechanism for public good sales. A monopolist seller has a public good to sell to a group of buyers. He has limited knowledge of the buyers' private information: Only the maximal, minimal, and expected valuation of each buyer is known to the seller. We characterize a strong maximin mechanism of public good provision. The maximin mechanism qualitatively depends on the number of agents and the common expected valuation. Buyers report the quantity demanded to the monopolist. If the expected valuation is high relative to the number of buyers, then *the veto mechanism*, which supplies the public good by the maximal demand, is optimal. In other words, each buyer cannot enjoy more than their own demand report. Buyers are charged an exponential price in their quantity demand. Otherwise, *the capped proportional mechanism* is optimal. It supplies the public good by the sum of demand reports and each buyer can demand at most some portion of the public good.

Chapter 3[1] constructs a clock implementation of optimal mechanisms for exchange economies, which are a marketplace where traders enter an initial position

[1] Co-authored with Pavel Andreyanov (HSE) and Tomasz Sadzik (UCLA)

for some asset and may end up being a buyer or a seller after the trade. In our clock implementation, the auctioneer runs two clocks simultaneously to each side of the market and traders make a quantity report to each clock separately. Moreover, the auctioneer charges personalized prices, which are a sum of the common clock state and individual taxation, to implement mechanisms whose objective is other than exact efficiencies, such as profit maximization.

The dissertation of Junrok Park is approved.

Sushil Bikhchandani

Ichiro Obara

Moritz Meyer-ter-Vehn

Tomasz Marek Sadzik, Committee Chair

University of California, Los Angeles

2022

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

I am deeply indebted to Tomasz Sadzik, my primary advisor, for his support at every dissertation stage. I also would like to express my most profound appreciation to Moritz Meyer-ter-Vehn, Ichiro Obara, Simon Board, Jay Lu, and Sushil Bikhchandani, the participants of the Theory proseminars at UCLA, for valuable comments and suggestions. I enjoyed working with Pavel Andreyanov, who shared me with various insights. I also thank Younghoon Kim and Jay Yoo for their friendship.

VITA

2009–2015	B.E. (Economics), Sungkyunkwan University, Seoul, South Korea
2015–2016	M.A. (Economics), Sungkyunkwan University, Seoul, South Korea
2017	C.Phil (Economics), UCLA, Los Angeles, California
2017–2022	Teaching Assistant, Department of Economics, UCLA

CHAPTER 1

Informational Regulation of Public Good Monopolists

1.1 Introduction

Profit-oriented enterprises often supply a good that is non-excludable and non-rivalrous—a public good—to a group of consumers. A security service provides safety to all residents in a community. In some other cases, these enterprises make a collective decision, which resembles a public good, for a group of individuals. Survival of a financially troubled firm is also "a public good" to its creditors because, in spite of the fact that they privately have to bear the cost of lessening the debt, the benefit from survival is enjoyed by all creditors. Pollution claim settlement is another example, where a pollution emitting company tries to compensate local residents so that it can continue operation. In this case, pollution is a "public bad" to the residents. There are many other examples.

Efficiency in private provision of a public good is hampered by two well-known forces. First, profit-oriented companies wield market power to raise profit, while withholding the supply. Second, agents—buyers of a public good—have incentives to free-ride on contributions of others. Though previous studies, such as Güth and

Hellwig (1986) and Rob (1989) showed that the impact of the two forces is substantial, they considered a relatively specific setup, the independent private value model. Crémer and McLean (1988) stands on the other extreme, showing that it is possible to achieve efficient outcomes if private information is correlated. Then we have a question: How does the nature of private information determine the impact of market power and free-riding? Alternatively, we may study the same question from the policymaker's viewpoint. They could consider an indirect regulation that carefully designs private information of the buyers to tame the two forces to promote efficiency. This gives an alternative to policymakers, instead of direct regulations traditionally used, where governments take over private companies to operate them as a public enterprise, or place restrictions on their operation.[1]

To answer the above questions, this paper considers an information design problem, where we consider judicious information provision to the buyers of the public good, characterizing *the regulator's solution*. It is a signal distribution that maximizes the agents' welfare in an expected-profit maximizing mechanism of public good provision. It eliminates any efficiency loss due to under-provision of the public good via an optimal mechanism that always sells the public good at any signal profile realized on the regulator's solution, under the assumption that the public good is costless and the agents are not harmed by the public good. In addition, it limits the impact of free-riding by ensuring that the total value of sale is bounded below. We also construct *the posted-price distribution*, which is a signal distribution that maximizes the agents' welfare if the monopolistic seller is restricted to use simple mechanisms that

[1] To limit the impact of free-riding, countries across borders specify conditions where property rights of each agent could be overridden for a public project. (see Tirole (1988) for textbook treatment and Armstrong and Sappington (2007) for a recent survey)

post a fixed price to each buyer. The posted-price distribution has similar properties as the regulator's solution, but it ensures that each agent's signal is bounded below. We study the welfare properties with many agents, showing that each agent pays a vanishing amount and the monopolist earns a finite expected profit both in the regulator's solution and the posted-price distribution. In this sense, the monopolist loses his market power with many agents.

We obtain a couple of additional results. First, we characterize the set of surplus division among the monopolist seller and the buyers, which could be realized by some signal distribution. Notably, the regulator could implement any division of surplus among the seller and the buyers without causing under-production of the public good. Second, we construct *the maximin public good sales mechanism*, where the monopolist seller maximizes the worst-case expected profit subject to a limited knowledge on the true information structure.[2] The maximin mechanism is a scoring mechanism with two agents, which supplies the public good by the total score. On the other hand, with more than three agents, the maximin mechanism is no longer a scoring mechanism.

We consider the standard public good provision problem with private information, preceded by the *regulator's information design problem*. The regulator maximizes the agents' welfare by choosing a signal distribution, subject to the constraint that the signals are supported between 0 and 1, and are in expectation some $\mu \in (0, 1)$. Specifically, the sequential game between the regulator and the monopolist unfolds as follows. First, the regulator chooses a signal distribution, which lets the agents—buyers of the public good—to privately observe their expected valuation of the public

[2]We solve the maximin mechanism design problem as an intermediate step to solve the regulator's problem.

good. Next, the monopolist observes the regulator's choice and chooses an optimal mechanism. Then the agents play a dominant strategy equilibrium in the mechanism. We assume that the public good is costless so that it is always efficient to be supplied. This assumption allows us to focus on the interplay between the buyers' private information and the monopolist's profit-maximizing behavior.

We solve the regulator's problem by noticing the equivalence between the regulator's solution and the minimax solution, which is a signal distribution that minimizes the expected profit of an optimal mechanism, if there is an optimal mechanism that sells with probability 1 in the minimax solution. If the total welfare is fixed at the ex-ante efficient level, then maximizing the agents' payoff is equivalent to minimizing the monopolist's profit. We consider a zero-sum game between the monopolist and the 'profit-minimizer', whose objective is to minimize the expected profit of the monopolist. We then solve the minimax problem jointly with the maximin problem, where the monopolist maximizes the worst-case expected profit of a mechanism, subject to the same ex-post incentive constraints. We use the property that the minimax and maximin solution then constitutes a saddle point in the zero-sum game between the monopolist and the profit-minimizer to solve the minimax problem.

We identify a characterization for the minimax and maximin solutions, respectively, using the intuition that players make each other indifferent across equilibrium strategies in zero-sum games. First, the minimax solution is characterized by the monopolist's indifference, which is imposed by the zero-sum virtual values on the signal support, which puts a lower bound on the sum of signals. The monopolist is indifferent across any mechanism that does not sell the public good if the sum of reported signal is less than the lower bound, and sells the public good if all agents report the highest signal 1. Second, the maximin solution is characterized by the profit-minimizer's indifference, which is imposed by a mechanism whose total transfer is equal to the supporting hyperplane when the total transfer is positive. Here, the supporting hyperplane is characterized by two parameters, the slope[3] and intercept, whose ratio determines a lower bound on the total signals where a positive amount of the public good is provided. Since the profit-minimizer cannot change the expectation of each signal, she is indifferent across any signal distribution that is supported on signal profiles whose sum is bounded below.

Next, we characterize a saddle point of the minimax and maximin problems. We show that both minimax and maximin indifference conditions have a unique, well-defined solution, given the same set of parameters which define the supporting hyperplane in the maximin indifference and the lower bound on the sum of signals in the minimax indifference. We then pin down the parameters from the mutual best-response requirement in the zero-sum game. The maximin allocation rule must sell the full amount at the highest signal profile and the minimax signal distribution must satisfy the expectation constraints. Thus, the maximin and minimax solutions are mutual best-response to each other.

Then, we show that the minimax solution solves the regulator's problem since the minimax indifference condition implies that there is an optimal mechanism that always sells the public good in the minimax distribution. We then study how the regulator's solution achieves this indifference property. There are two channels: First, it puts a negative correlation between the signal of a single buyer and the sum of other buyers' signal. When one agent observes a high signal, the monopolist becomes pessimistic about signals of other agents, thereby weakening his incentive to discrim-

[3] Since the agents are symmetric, the hyperplane has the same slope in each signal.

inate across signal profiles. The optimality of negative correlation contrasts the usual exhibition of positive correlation in the context of correlated valuations, which arises often when studying allocation problems of common-valued assets. However, a positive correlation induces the monopolist to withhold the supply at lower signal profiles because he would be more pessimistic regarding a valuation of an agent if he knows that other agents have received lower signals. The regulator's solution thus illustrates that the buyers should learn about the value in a way that would induce different opinions among them. Second, it ensures that the total signal is bounded below at some threshold, which limits the impact of free-riding among the agents. Under the regulator's solution, there are always some agents who observe a high signal. Since both the monopolist and the regulator are concerned with the total value of trade, the regulator puts a lower bound on the sum of signals in her optimal solution so that any trade whose value is above the threshold is realized.

Though the regulator's solution induces the monopolist to always sell the public good, the optimal always-sell mechanism has an unappealing feature: It promises to sell the public good at the cheapest price when it is most highly valued since agents pay their pivot value, which decreases in signals of others. This feature makes the always-sell mechanism vulnerable to collusion among the agents and an imperfect commitment of the monopolist. However, the two issues are resolved if the monopolist posts a fixed price to each agent, so that the total transfer does not decrease. In addition, the monopolist could indirectly implement such a mechanism, without knowing the actual valuation of each buyer. We solve the regulator's problem when the monopolist is limited to choose such mechanisms, yielding the posted-price distribution, which is characterized by indifference among posted prices. The monopolist optimally posts the lowest price in the posted-price distribution. In addition, the lowest posted price is an optimal mechanism for the monopolist without the restriction to posted-price mechanisms. However, the posted-price distribution achieves its properties through distinct channels: First, in the posted-price distribution, individual signals are bounded below, while in the regulator's solution, the total signal is bounded below. Second, a signal is positively correlated with the sum of others' signals.

The regulator's solution limits the monopolist's market power in the large economy limit. The optimal always-sell mechanism may generate an infinite profit because the size of the ex-ante surplus is infinite with infinitely many agents. However, both under the regulator's solution and the posted-price distribution, the monopolist's market power is quickly dissipated when more consumers are introduced into the market due to the negative correlation. Under the posted-price distribution, the monopolist optimally charges the lowest price $\underline{s}$ to each agent, yielding the profit of $N\underline{s}$. Alternatively, it is also optimal to charge 1 to each agent, earning the same expected profit $N\underline{s}$, where $\underline{s}$ is the probability of the highest signal profile $(1, \cdots, 1)$ in the posted-price distribution. It is easier to have at least one agent observe a signal less than 1 when there are more agents. That is, the monopolist's share from the whole ex-ante surplus converges to zero, and the amount each buyer pays converges to zero as the number of agents grows to infinity. Since the posted-price distribution is not an optimal solution of the regulator's problem, we immediately see that the regulator's problem has the same large market properties.

Lastly, while the regulator's problem is the main focus of this paper, as an intermediate step toward solving the problem, we analyzed a zero-sum game between the monopolist and the profit-minimizing information designer. We call the resulting mechanism from the game as *the maximin mechanism*, a sales mechanism that solves the maximin problem where the monopolist maximizes its worst-cast expected profit through a sales mechanism. The maximin mechanism is valuable in other contexts as well, especially when the monopolist does not know the exact learning process of the buyers. It guarantees the monopolist to earn the worst-case expected profit, regardless of the actual information structure as long as the monopolist's partial knowledge of the information structure is correct. The maximin mechanism coincides with scoring mechanisms if there are two agents but if there are three or more, it qualitatively differs from the scoring mechanisms.

1.1.1 Literature Review

It is well known that inefficient outcomes are prevalent in the public good provision problems where the agents' private information is independently distributed as shown in Güth and Hellwig (1986), Rob (1989) and Mailath and Postlewaite (1990). In any mechanism, agents compare the probability of losing the public good against the gains from under-stating their value. When more agents are introduced into the game, it is less likely for each agent to be pivotal, and it becomes more difficult to induce truth-telling. Detragiache and Garella (1996) and Chillemi and Gui (1997) apply such insights to model debt restructuring of a financially troubled firm and human capital in teams. Since then, economists have sought for alternative setups where an efficient equilibrium exists as a reaction to the strong inefficiency result on the public good problems under the standard framework. We can broadly classify such responses into the following three categories.

First, property rights of individual agents can be compromised. Neeman (1999) and Schmitz (2002) show that efficiency can be achieved by "intermediate" property rights, giving the rights to the mechanism designer to build some amount of a public good or bad without the consent from the agents. In a similar line, Fryxell (2019) considers the "no-extortion constraint," which allows the designer to impose a negative payoff to each agent, yielding efficient outcomes in equilibrium.

Second, private information can be weakened. Pesendorfer (1998) studies the implications of a positively correlated signal distribution under limited liability constraint.[4] With many agents, the mechanism designer infers individual signals from the reports, which decreases the information rent of each agent. Li and Zhang (2021) considers a mechanism designer who is partially informed about the agents' private information, where the designer uses the information to limit the information rent of the agents.

Third, a law-of-large-number argument can be applied so that efficient outcomes arise asymptotically in equilibrium. Xi and Xie (2021) studies a mechanism that provides the public good if the reported total value exceeds the ex-ante surplus plus a wedge in the canonical IPV framework. Kocherlakota and Song (2019) considers ambiguity-averse agents, which could make the agents believe that they are a pivot agent, with sufficiently many agents. Bierbrauer and Winkelmann (2020) argues that bundling multiple public goods resolves inefficiency.

This paper provides an efficiency result from a different perspective from the three categories discussed above. Compared to the previous results, under the regulator's solution, there is no violation of the property rights of the agents since the monopolist is subject to a dominant strategy equilibrium. Interestingly, though correlation in the regulator's solution allows the monopolist to infer an individual signal, it only

[4] He excludes the use of Crémer and McLean (1988) style side-bet mechanism.

strengthens the bargaining position of the agents. That is, the regulator's solution weakens his incentive to withhold the supply of public good to raise the expected profit, which is an opposite implication from Pesendorfer (1998) that considers a positively correlated signal distribution. Finally, the efficiency result from this paper holds with any number of agents.

This paper is related to the information design literature. The influential work of Roesler and Szentes (2017) studies the buyer-optimal information in monopolist markets with one agent. Condorelli et al. (2020), and Armstrong and Zhou (2020) extends the basic model towards oligopoly markets. On the other hand, this paper extends Roesler and Szentes (2017) to a different direction: Here, we model a monopolist who supplies public good for many agents. Gentzkow and Kamenica (2016), Kolotilin (2018) and Dworczak and Martini (2019) consider the information design problem where the sender's preferences depend only on the expectation of the state of nature. All of the three papers, without loss of generality, consider a reduced-form information structure, which is the distribution of the expected state.

Finally, this paper is related to the literature on robust mechanism design problems. Robust mechanism design problems look for an optimal or efficient mechanism when the mechanism designer has limited knowledge of the true information structure. Carrasco et al. (2018) studies the maximin problem with one agent if the mechanism designer only knows up to the N-th moment of the state distribution. Brooks and Du (2019) shows that the "proportional auction" is a maximin mechanism in the common value auction problem when the agents play a Bayesian equilibrium. Che (2019), Suzdaltsev (2020) and Zhang (2021) are closely related to the maximin problem in this paper. They study the maximin problems in the auction context and in the bilateral trade context when the agents observe their valuation and are subject to the ex-post constraints.

Roadmap: In section 2, we introduce the model of the public good provision problem. In section 3, we discuss a simple example to illustrate the main results. In section 4, we characterize the regulator's solution via its conditional equivalence with the minimax solution. In section 5, we present the regulator's solution and discuss its economic properties. In section 6, we present the posted-price distribution and discuss its properties. In section 7, we present the maximin mechanism for the cases with two and three agents.

1.2 Model

In this section, we set up a model of public good provision with private information and specify who the players are as well as what actions they take around the provision problem.

1.2.1 Primitives

There are N *agents*, the potential buyers of the public good, indexed by $i = 1, \cdots, N$. Denote each agent i's valuation of the public good as $s_i \in [0, 1]$. The signal profile $s = (s_1, \cdots, s_N)$ is drawn from a commonly known joint probability distribution G on the signal space $[0, 1]^N$. In addition, we assume that the expectation of s_i is equal to μ for each agent i. Agent i's preferences are represented by the standard quasi-linear utility function of $s_i q - t_i$, where $q \in [0, 1]$ is the amount of the public good supplied to the agents, and t_i is the amount of money transfer paid by agent i to the monopolistic public good provider.

Then we have *a monopolistic seller* who produces the public good at a zero marginal cost. He chooses a sales mechanism (q, t) that receives messages from the agents and maps them to allocation and payments accordingly. The monopolist's preferences are represented by $\sum_{i=1}^{N} t_i$.

Lastly, we have *a regulator* who cannot directly control the monopolist and the agents but can choose an information structure to influence the outcomes or the other players' choices. The information structure π that she chooses determines learning process of the agents. The regulator's preferences are represented by $\sum_{i=1}^{N}(qs_i - t_i)$, which is tantamount of the sum of the agents' surplus.

1.2.2 Mechanism

Due to the revelation principle, it is without loss to consider direct mechanisms. A direct mechanism (M, q, t) consists of a message space $M_i = S_i = [0, 1]$, an allocation rule $q : M \to [0, 1]$ and a transfer rule $t : M \to \mathbb{R}^N$. We assume that the agents play a dominant strategy equilibrium in the public good provision game defined by a mechanism and an information structure.

Definition 1.2.1. *A direct mechanism (q, t) is dominant strategy incentive compatible (DSIC) if for each $i \in N$, $(s_i, s_{-i}) \in M$ and $(s_i', s_{-i}) \in M$,*

$$s_i q(s_i, s_{-i}) - t_i(s_i, s_{-i}) \geq s_i q(s_i', s_{-i}) - t_i(s_i', s_{-i}).$$

A mechanism (q, t) is ex-post individually rational (EPIR) if for each $i \in N$ and $s \in M$

$$s_i q(s_i, s_{-i}) - t_i(s_i, s_{-i}) \geq 0.$$

We state the characterization of DSIC mechanisms for completeness.

Proposition 1.2.1. *A mechanism (M, q, t) is DSIC if and only if q is weakly increasing in each s_i and the revenue equivalence formula holds at almost all $s \in M$.*

$$t_i(s) = s_i q(s_i, s_{-i}) - \int_0^{s_i} q(\nu, s_{-i}) d\nu - U_i(0, s_{-i}), \qquad \text{(Revenue Equivalence)}$$

where $U_i(s)$ is the equilibrium utility function of agent i at a signal profile (s_i, s_{-i}) which is defined by

$$U_i(s_i, s_{-i}) = \max_{s_i'} \left[s_i q(s_i', s_{-i}) - t_i(s_i', s_{-i}) \right].$$

In addition, a DSIC mechanism (M, q, t) is EPIR if and only if $U_i(0, s_{-i}) \geq 0$ for each i and $s_{-i} \in [0, 1]^{N-1}$.

Proof of Proposition 1.2.1. This is a standard result in mechanism design. See the textbook treatment of Börgers and Krahmer (2015) for proof. □

1.2.3 The Regulator's Problem

Here, we introduce the regulator's information design problem. The regulator governs the learning process of the agents before the monopolist moves, and she chooses a signal distribution G to maximize the agents' payoff against the monopolist. Observing the chosen signal distribution, the monopolist then chooses an optimal sales mechanism (q, t). This in turn defines a sequential game of complete information between the regulator and the monopolist.

Toward describing the regulator's problem, we do backward induction by first

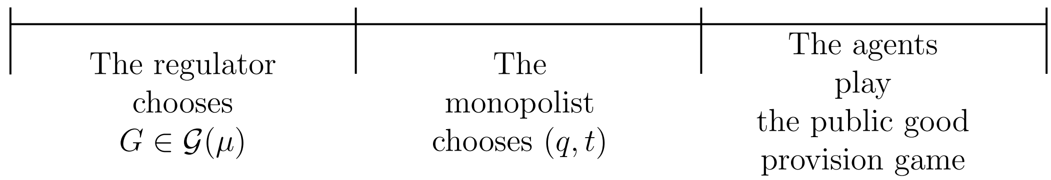

Figure 1.1: Timing of actions

discussing the monopolist's problem given some signal distribution chosen by the regulator. The monopolist chooses an optimal mechanism (q, t) subject to the DSIC and EPIR constraints to maximize the expected profit with respect to G. Since Proposition 1.2.1 finds the transfer rule that implements any monotone allocation rule, the problem reduces to choosing an allocation rule $q \in \mathcal{Q}$, where $\mathcal{Q}$ is the set of monotone allocation rules:

$$\max_{q \in \mathcal{Q}} \quad \int_s \sum_{i=1}^{N} \left(s_i q(s) - \int_0^{s_i} q(\tau, s_{-i}) d\tau \right) dG(ds). \qquad \text{(Monopolist's Problem)}$$

When stating the monopolist's problem, we take into account of the fact that an agent who observed a zero signal earns a zero payoff in equilibrium, i.e. $U_i(0, s_{-i}) = 0$.

Anticipating that the monopolist would best-respond to her choice of signal distribution G, the regulator maximizes the sum of the agents' expected payoff under an optimal sales mechanism. Each agent i's expected payoff from an optimal mechanism is determined from the revenue equivalence formula from Proposition 1.2.1, which is

$$\int_S U_i(s) dG(s) = \int_S (s_i q(s) - t_i(s)) dG(s) = \int_S \int_0^{s_i} q(\nu, s_{-i}) d\nu.$$

So the regulator chooses a feasible signal distribution $G \in \mathcal{G}(\mu)$ to maximize the sum of the agents' payoffs by solving

$$\max_{G \in \mathcal{G}(\mu)} \int_S \left(\sum_{i=1}^{N} \int_0^{s_i} q(\nu, s_{-i}) d\nu \right) dG(s) \tag{Regulator's Problem}$$

$$\text{s.t. } q \text{ solves the monopolist's problem given } G.$$

If the monopolist is indifferent between multiple allocation rules, we assume that he chooses an optimal mechanism that maximizes the regulator's payoff, or equivalently the largest allocation rule among optimal allocation rules. It is a standard assumption in the information design literature to break ties in the favor of the information designer[5].

1.3 A Binary Example

To fix ideas, we solve the regulator's problem with two agents under a simplifying restriction: only binary signal distributions are available to the regulator. Each agent observes either s^L or s^H, where $0 \leq s^L \leq \mu \leq s^H \leq 1$ in a binary signal distribution $G \in \mathcal{G}^B(\mu)$, where $\mathcal{G}^B(\mu)$ denotes the set of feasible binary signal distributions.

Given a binary signal distribution, an optimal mechanism is either of the following three: (1) charging s^L to each agent, (2) charging s^H to each agent, (3) charging s^L to one agent if the other agent reported s^H and charging s^H to one agent if the other

[5]For example, Kamenica and Gentzkow (2011) considers 'Sender-preferred subgame perfect equilibrium', where the receiver takes the sender's preferred action when indifferent among multiple actions.

agent reported s^L. In terms of a direct mechanism, they are expressed as below.

$$(q(s^L, s^L), q(s^H, s^L), q(s^L, s^H), q(s^H, s^H)) = \begin{cases} (1, 1, 1, 1), \\ (0, 1, 1, 1), \\ (0, 0, 0, 1). \end{cases}$$

The implementing transfer rule is identified by the revenue equivalence formula.

$$\begin{aligned}
t_i(s^L, s^L) &= s^L q(s^L, s^L), \\
t_i(s^L, s^H) &= s^L q(s^L, s^H), \\
t_i(s^H, s^L) &= s^H q(s^H, s^L) - (s^H - s^L) q(s^L, s^L), \\
t_i(s^H, s^H) &= s^H q(s^H, s^H) - (s^H - s^L) q(s^L, s^H),
\end{aligned}$$

where the first argument represents agent i's signal and the second argument represents the other agent's signal. The revenue equivalence formula then determines the ex-post equilibrium payoff of each agent. At each signal profile s, agent i earns $U_i(s)$ defined by

$$\begin{aligned}
U_i(s^L, s^L) &= 0, \\
U_i(s^L, s^H) &= 0, \\
U_i(s^H, s^L) &= (s^H - s^L) q(s^L, s^L), \\
U_i(s^H, s^H) &= (s^H - s^L) q(s^L, s^H).
\end{aligned}$$

Agents with the low signal s^L earn zero payoffs in equilibrium since an optimal mechanism charges at least s^L. On the other hand, agents with the high signal s^H earn some positive payoff as long as the monopolist charges s^L, selling the public good for the low signal agents. Thus, the regulator should induce the monopolist to sell if there is a low signal agent.

Now we solve the binary regulator's problem. First, we solve the regulator's problem with a restriction that s_1 and s_2 are perfectly correlated. This restriction essentially takes the problem back to the one-agent case, as studied by Roesler and Szentes (2017) and provides a benchmark that highlights the new channel that the regulator could use in the public good provision problem with multiple agents. In a perfectly correlated signal distribution $G \in \mathcal{G}^B(\mu)$, an optimal mechanism either charges s^L or s^H to both agents. The regulator earns a positive payoff only if there is an optimal mechanism that charges s^L to both agents. Thus, we state the binary regulator's problem with the perfect correlation restriction as below.

$$\begin{aligned} \max_{G \in \mathcal{G}^B(\mu)} \quad & 2(\mu - s^L), \\ \text{s.t. } & (s^H + s^H) g(s^H, s^H) \leq s^L + s^L, \\ & g(s^H, s^L) = 0, \quad g(s^L, s^H) = 0. \end{aligned}$$

The regulator maximizes $2(\mu - s^L)$, the total expected payoff of the two agents in the mechanism that charges s^L to each agent and she is subject to the constraint that *the inclusion constraint*, which ensures the monopolist optimally charge s^L to both agents and the perfect correlation constraint, which requires that $s_1 = s_2$ with probability 1. Equivalently, the regulator minimizes s^L while inducing the monopolist to charge the low price s^L.

It is easy to see that in an optimal solution of the regulator, the monopolist must be indifferent between charging the high price and the low price. Otherwise, the regulator could lower s^L without losing the sale. s^L is constrained by the expectation constraint and the upper bound on s^H, making the optimal solution of the regulator has $s^H = 1$. Thus, the optimal solution has $s^L = g(s^H, s^H)$, and s^L is pinned down from the expectation constraint. Observation 1.3.1 summarizes the binary regulator's solution with perfect correlation.

Observation 1.3.1. *The binary signal distribution $G^C \in \mathcal{G}^B(\mu)$ defined by*

$$\begin{aligned} s^L =& 1 - \sqrt{1-\mu}, \quad & g^C(s^L, s^L) =& \sqrt{1-\mu}, \\ s^H =& 1, & g^C(s^H, s^H) =& 1 - \sqrt{1-\mu}. \end{aligned}$$

solves the regulator's problem with the perfect correlation restriction.

Next, we lift the perfect correlation restriction and show that the regulator could improve her payoff using the environment, where a public good is sold to a group of agents. As it will be shown below, the regulator chooses a weaker correlation between signals while keeping the monopolist to sell the public good when there is a low signal agent. We describe how an optimal mechanism is chosen given a binary signal distribution to find the generalization of *the inclusion constraint*. The expected profit of an allocation rule is expressed in terms of the marginal revenue of each agent, which is also referred as the virtual value $\varphi_i(s)$ at each signal profile s.

$$R(q, G) = \sum_s q(s) \sum_{i=1}^{2} \varphi_i(s) g(s),$$

where the virtual value $\varphi_i(s)$ is defined by

$$\begin{aligned} \varphi_i(s^L, s_j) =& s^L - (s^H - s^L)\frac{g(s^H, s_j)}{g(s^L, s_j)} = s^L - (s^H - s^L)\frac{g(s^H|s_j)}{g(s^L|s_j)}, \\ \varphi_i(s^H, s_j) =& s^H. \end{aligned}$$

At each signal profile s, an optimal mechanism sells the public good if and only if the marginal revenue is positive, or the virtual values have a non-negative sum.

The regulator must induce the monopolist to sell the public good at signal profiles other than (s^H, s^H) to ensure some payoff to the agents. The inclusion constraint must induce trade at (s^L, s^H) and (s^H, s^L) so that the regulator earns some payoffs. However, it is not obvious if the regulator must induce sales at (s^L, s^L). Thus, we consider both cases to solve the binary regulator's problem.

First, suppose that the regulator induces the trade at (s^L, s^L). The total expected payoff of the agents is then equal to $2\mu - s^L - s^L$ since the sale always realizes and each agent pays s^L. Then the regulator's problem is expressed as

$$\max_{G \in \mathcal{G}^B(\mu)} \quad 2(\mu - s^L),$$
$$\text{s.t.} \sum_{i=1}^{2} \varphi_i(s^L, s^L) \geq 0, \quad \sum_{i=1}^{2} \varphi_i(s^L, s^H) \geq 0, \quad \sum_{i=1}^{2} \varphi_i(s^H, s^L) \geq 0.$$

The inclusion constraint induces the monopolist to charge s^L to both agents, selling at the lowest signal profile (s^L, s^L). The monopolist is made indifferent between selling and withholding the public good when he sells by the same reasoning. Otherwise, the regulator could lower the total virtual value without losing the sale. On top of the binding inclusion constraints, the regulator minimizes s^L, or equivalently, $g(s^H, s^H)$. Observation 1.3.2 summarizes the regulator's solution where the monopolist optimally charge s^L to both agents.

Observation 1.3.2. *The binary signal distribution $G^* \in \mathcal{G}^B(\mu)$ defined by*

$$
\begin{aligned}
s^L &= \frac{1}{2}\left(3-\mu-\sqrt{9-10\mu+\mu^2}\right), & g^*(s^L,s^L) &= 1-\mu,\\
& & g^*(s^H,s^L) &= \frac{1}{4}\left(-3+\sqrt{9-10\mu+\mu^2}+3\mu\right),\\
s^H &= 1, & g^*(s^L,s^H) &= \frac{1}{4}\left(-3+\sqrt{9-10\mu+\mu^2}+3\mu\right),\\
& & g^*(s^H,s^H) &= \frac{1}{2}\left(3-\sqrt{9-10\mu+\mu^2}-\mu\right).
\end{aligned}
$$

solves the binary regulator's problem.

Compared with G^C, the regulator's solution with the perfect correlation, the regulator's solution G^* puts a weaker correlation between s_1 and s_2 by having $g^*(s^L, s^H) > 0$. In addition, the regulator makes the monopolist indifferent at (s^L, s^H) and (s^H, s^L) by making it hard to induce truth-telling from the low signal agent. The high signal agent has the virtual value of 1, while the low signal agent has the virtual value of -1 at (s^L, s^H) and (s^H, s^L).

Next, suppose that the regulator does not induce the trade at (s^L, s^L). The total expected payoff of the agents is then equal to $2(s^H - s^L)g(s^H, s^H)$ because each agent earns a positive payoff only at (s^H, s^H). Then the regulator solves

$$
\begin{aligned}
&\max_{G\in\mathcal{G}^B(\mu)} \quad 2(s^H - s^L)g(s^H, s^H),\\
&\text{s.t.} \sum_{i=1}^{2}\varphi_i(s^L, s^L) \leq 0, \quad \sum_{i=1}^{2}\varphi_i(s^L, s^H) \geq 0, \quad \sum_{i=1}^{2}\varphi_i(s^H, s^L) \geq 0.
\end{aligned}
$$

If μ is small, then the regulator earns less payoff by giving up trade at (s^L, s^L). On the other hand, if μ is sufficiently high, the regulator could choose an even weaker correlation, or a higher probability of drawing disagreeing signals (s^L, s^H)

and (s^H, s^L). Observation 1.3.3 summarizes the solution where the public good is not sold at (s^L, s^L), for high μ.

Observation 1.3.3. *The binary signal distribution* $G^{**} \in \mathcal{G}^B(\mu)$ *defined by*

$$
\begin{aligned}
& & g^{**}(s^L, s^L) &= 0, \\
s^L &= \frac{1}{2}\left(1 + \mu - \sqrt{9 - 10\mu + \mu^2}\right), & g^{**}(s^H, s^L) &= \frac{1}{4}\left(-1 + \sqrt{9 - 10\mu + \mu^2} + \mu\right), \\
s^H &= 1, & g^{**}(s^L, s^H) &= \frac{1}{4}\left(-1 + \sqrt{9 - 10\mu + \mu^2} + \mu\right), \\
& & g^{**}(s^H, s^H) &= \frac{1}{2}\left(3 - \sqrt{9 - 10\mu + \mu^2} - \mu\right).
\end{aligned}
$$

solves the binary regulator's problem for $\mu \geq \frac{2}{3}$.

In this solution G^{**}, the regulator does not put any probability on (s^L, s^L) where there is no trade, which creates an weaker correlation than G^* does, as $g^{**}(s^H, s^L) > g^*(s^H, s^L)$.

From the binary regulator's problem, we saw that a weaker correlation plays an important role in the regulator's problem. The regulator takes advantage of the setup—a public good is sold to a group of buyers— by choosing an appropriate correlation between the signals. As Observation 1.3.3 illustrates, the regulator may choose not to draw some signal profiles to impose a stronger negative correlation. We then solve the regulator's problem without the binary signal restriction and investigate how the findings from the binary problem generalize.

1.4 The Regulator's Solution

We characterize the regulator's solution and discuss its properties in this section. As illustrated from the binary example, the inclusion constraint, which ensures that there is an optimal mechanism that sells the public good at a signal profile, plays an important role in the regulator's problem. The regulator maximizes the agents' payoff by minimizing the monopolist's profit while keeping him supplying the public good so that the agents can enjoy the public good at the smallest cost. Thus, the inclusion constraint binds on signal profiles that arise with a positive probability at the optimal solution, as illustrated from Observation 1.3.2 and 1.3.3. In other words, the monopolist is made indifferent at each signal profile on the signal support of the regulator's solution.

We define the binding inclusion constraint, or the monopolist's indifference condition in the general case. It is helpful to consider regular signal distributions so that an optimal mechanism is fully characterized by the sum of the virtual values, which is defined by[6]

$$\varphi_i(s_i|s_{-i}) = s_i - \frac{1 - G(s_i|s_{-i})}{g(s_i|s_{-i})}.$$

In a regular signal distribution G, an optimal mechanism conditions the supply on the sign of the sum of the virtual values, selling the public good if the sum of the virtual values is strictly positive, while withholding the public good if it is strictly negative. The monopolist is indifferent between any quantity if the sum of the virtual values is exactly zero.

It remains to determine the signal support where the binding inclusion constraint

[6]We omit the details of the monopolist's problem in a regular signal distribution because it is a standard exercise. See Börgers and Krahmer (2015) for a textbook treatment.

is imposed. We saw from the binary example that the regulator may drop a low signal profile in an optimal solution, if the expectation μ is high enough, which is a new feature available only in the public good problem. In general, without the binary support restriction, the regulator does not have choose a "square" signal support. Intuitively, the regulator would like to realize trades with higher values, or trades when the total signal is higher, while expanding the signal support while respecting the monopolist's indifference condition, so that the agents could pay a lower price. Thus, the signal support should be determined by the total value of the public good. We then formally define the monopolist's indifference condition below.

Definition 1.4.1. *A signal distribution G^* satisfies* the monopolist's indifference condition *if for each $s \in S$ such that $\sum_{i=1}^N s_i \geq \frac{X}{A}$ and $s \neq (1, \cdots, 1)$,*

$$0 = \sum_{i=1}^N \varphi_i^*(s_i|s_{-i}) = \sum_{i=1}^N \left(s_i - \frac{1 - G^*(s_i|s_{-i})}{g^*(s_i|s_{-i})} \right), \qquad \text{(Monopolist Indifference)}$$

for some $\frac{X}{A} > 0$ and $Supp(G^*) = \{s \in S | \sum_{i=1}^N s_i \geq \frac{X}{A}\}$.
A signal distribution is said to be the indifference distribution *if it satisfies the monopolist's indifference condition.*

In the indifference distribution G^*, the monopolist has many optimal mechanisms: any allocation rule q such that $q(1, \cdots, 1) = 1$ and $q(s) = 0$ for s such that $\sum_{i=1}^N s_i < \frac{X}{A}$. That is, any allocation rule that sells the public good at the highest signal profile $(1, \cdots, 1)$ and that does not sell the public good if the sum of signals is less than $\frac{X}{A}$.

We present the first main result of this paper, a characterization of the regulator's solution.

Theorem 1.4.1. *The indifference distribution $G^* \in \mathcal{G}(\mu)$ solves the regulator's problem.*

The indifference distribution, which is characterized by the monopolist's indifference or the binding inclusion constraint, minimizes the monopolist's profit without losing sale at any signal profile on the support. We prove Theorem 1.4.1 in Section 1.7 by exploiting a conditional equivalence between the regulator's solution G^* and the minimax solution, which is a signal distribution that minimizes the expected profit of an optimal mechanism.

1.4.1 Construction of the Regulator's Solution

Here we study the Monopolist Indifference condition to construct for the regulator's solution G^*. Since we cover technical steps here and will cover economic properties of the regulator's solution in the next subsection, readers may skip this subsection without loss.

First, we visually inspect the expression for the indifference condition to figure out technical properties of the indifference distribution. Then using the properties, we obtain a partial differential equation for the joint density function from the monopolist's indifference condition for the indifference distribution G^*. From the visual inspection of the monopolist's indifference condition, we can first observe that, under the indifference distribution G^*, the virtual value of each agent must be well-defined for $s_i < 1$. That is, for each i, $G^*(s_i|s_{-i})$ must have a well-defined conditional density $g^*(\cdot|s_{-i})$ for $s_i < 1$. However, the condition may allow for a signal distribution that has a discrete mass on $s_i = 1$, which makes the distribution to be discontinuous. Thus, the monopolist's indifference distribution may be a mixed probability distribution.

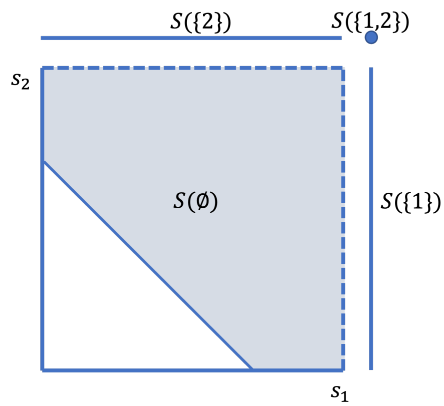

Figure 1.2: Illustration of the partition $\{S(H)|H \in \{\emptyset, \{1\}, \{2\}, \{1,2\}\}\}$ with two agents

To illustrate, we consider the case with two agents as in Figure 1.2. The monopolist's indifference condition requires that the indifference distribution is continuous on $S(\emptyset)$, where no agents observe the highest signal $s_i = 1$. The monopolist's indifference condition also requires that $g^*(s_1|1)$ and $g^*(s_2|1)$ to be well-defined along $S(\{2\})$ and $S(\{1\})$ respectively. Across the subsets $S(\emptyset)$, $S(\{1\})$, $S(\{2\})$ and $S(\{1,2\})$, the indifference distribution may be discontinuous.

In general, we can partition the signal space S with respect to the set of agents who observed the highest signal 1. Let $H \in 2^N$ be a generic subset of agents. Define $S(H)$ to be the subset of the signal space where $s_i = 1$ for $i \in H$. Define a partition on the signal space by the set of agents H who observed the highest signal.

$$S = \bigcup_{H \in 2^N} S(H) = \bigcup_{H \in 2^N} \{s \in S | s_i = 1 \text{ for each } i \in H\}. \tag{1.1}$$

For the virtual values to be well-defined, while it may not be continuous across different partition elements, the indifference distribution G^* needs to admit a well-defined density g^* on each partition element $S(H)$. For each measurable $A \subset S$, define $G^*(A)$ as the probability of observing the event A under G^* by

$$G^*(A) = \sum_{H \in 2^N} \int_{s \in S(H) \cap A} g^*(s) ds. \tag{1.2}$$

The minimax indifference condition then defines a partial differential equation for g^* since it is continuous on each partition element $S(H)$. For each $s \in S(H)$ such that $\sum_{i=1}^{N} s_i \geq \frac{X}{A}$ and $s \neq (1, \cdots, 1)$,

$$0 = \sum_{i=1}^{N} \varphi_i^*(s_i | s_{-i}) = \sum_{i=1}^{N} \left(s_i - \frac{\int_{s_i}^{1} g^*(t, s_{-i}) dt + g^*(1, s_{-i})}{g^*(s)} \right), \tag{1.3}$$

where we express the minimax indifference condition in terms of the joint density g^* instead of the conditional distributions on the right-hand side. We separately add the density g^* at $s_i = 1$ since this signal profile $(1, s_{-i})$ does not belong to $S(H)$, and G^* is not continuous across different partition elements. We solve the differential equation (1.3) to obtain the joint density g^* for each partition element $S(H)$.

Proposition 1.4.1. *For each partition element $S(H)$ and signal profile $s \in S(H)$ such that $\sum_{i=1}^{N} s_i \geq \frac{X}{A}$, the joint density g^* defined by (1.17) satisfies the minimax indifference condition (1.3).*

$$g^*(s) = \frac{c^*(\frac{X}{A})}{\left(\sum_{j \in N-H} s_j + H\right)^{N-H+1}} \frac{N+1}{\prod_{h=0}^{H}(N-h+1)}, \tag{1.4}$$

for some constant $c^*(\frac{X}{A})$ and threshold $\frac{X}{A}$.

We complete the construction of the indifference distribution G^* by finding the constant of integration $c(\frac{X}{A})$. For each $\frac{X}{A} > 0$, $c(\frac{X}{A})$ is pinned down by the constraint that G^* is a probability distribution,

$$1 = \int dG^*(s). \tag{1.5}$$

The profit-minimizer must choose the threshold $\left(\frac{X}{A}\right)^*$ of the indifference distribution G^* to satisfy the expectation constraint. We prove in Lemma 1.4.1 that there exists a well-defined $c(\frac{X}{A})$ and $\left(\frac{X}{A}\right)^*$ so that the indifference distribution G^* is a well-defined probability distribution.

Lemma 1.4.1. *For each $\mu \in (0,1)$, there exists a unique $\left(\frac{X}{A}\right)^* \in (0,N)$ that satisfies (1.6),*

$$\mu = \int s_i dG^*(s). \tag{1.6}$$

The indifference distribution G^* is fully described once we evaluate the integrals (1.5) and (1.6) to find an explicit solution for $c^*\left(\frac{X}{A}\right)$ and $\left(\frac{X}{A}\right)^*$. However, it is not easy to calculate the integrals with an arbitrary number of agents. There are N cases to consider with N agents. The integration region is the intersection of the $N-$dimensional unit cube and the hyperplane $\{s \in S | \sum_{i=1}^{N} s_i \geq \left(\frac{X}{A}\right)^*\}$, which is affected by the size of the threshold $\left(\frac{X}{A}\right)^*$. Thus, we find $c^*\left(\frac{X}{A}\right)$ and $\left(\frac{X}{A}\right)^*$ for two, three and four agents in the appendix instead of developing a general formula.

1.4.2 Properties of the Regulator's Solution

We discuss properties of the regulator's solution G^* previously constructed. Directly from its defining condition—the monopolist's indifference condition—the regulator's solution makes the monopolist indifferent across all mechanisms except for those that are clearly not optimal. For example, a mechanism that never sells the public good. Among the mechanisms that the monopolist becomes indifferent, we have the always-sell mechanism, a mechanism that sells the public good with probability 1 in G^*. We explain how the regulator's solution G^* achieves the indifference property by examining the individual virtual values φ_i and the conditional signal distributions of G^*. Lastly, we discuss how the signal support of G^* is determined to limit the impact of free-riding and to minimize the monopolist's profit.

First, we begin by examining the individual virtual value $\varphi_i^*(s_i|s_{-i})$ of each agent i at each signal profile s. We obtain the individual virtual values using the joint density g^* defined by (1.17).

Corollary 1.4.1. *For each signal profile $s \in S(H)$ such that $\sum_{i=1}^N s_i \geq d^*$, the regulator's solution G^* induces the individual virtual value of agent i as follows.*

$$\varphi_i^*(s_i|s_{-i}) = \begin{cases} s_i - \dfrac{\sum_{i=1}^N s_i}{N-H} & s_i < 1 \\ 1. & s_i = 1 \end{cases}$$

In the regulator's solution G^*, the virtual value of each agent is equal to the deviation from an adjusted mean signals, which is the total signal is divided by $N - H$, which is the number of agents whose signal is less than 1. The regulator's solution G^* diversifies the agents in terms of the virtual values around the adjusted mean signals so that at any signal profile, some agents have a positive virtual value while there are others who have a negative virtual value. When some agents observe the highest signal, there could be no information rent for them. In that case, the remaining agents are given more information rent to keep the indifference property.

The diversification of the virtual values imposes indifference on the monopolist on any signal profile, except for $s = (1, \cdots, 1)$.[7] Compared with the regulator's solution with a single agent studied in Roesler and Szentes (2017), diversification is a new feature that the regulator can use when faced with multiple agents who are trying to purchase a public good from a monopolist.

Next, we inspect the conditional signal distribution $G^*(\cdot|s_{-i})$ to study the shape of the information rent $\frac{\sum_{i=1}^N s_i}{N-H}$.

Corollary 1.4.2. *The conditional distribution of s_i given s_{-i} is found from the regulator's solution G^*.*

$$G^*(s_i|s_{-i}) = \begin{cases} 1 - \left(\dfrac{d^*}{\sum_{i=1}^N s_i}\right)^{N-H}, & s_i \geq \left(d^* - \sum_{j\neq i} s_j\right)^+ \\ 1, & s_i = 1 \geq \left(d^* - \sum_{j\neq i} s_j\right)^+ \end{cases}$$

where $s \in S(H)$.

On each partition element $S(H)$, the conditional distribution $G^*(\cdot|s_{-i})$ uniformly decreases in $\sum_{j\neq i} s_j$. Equivalently, $G^*(\cdot|s_{-i})$ is ordered in the sense of first-order stochastic dominance on each partition element $S(H)$ of the signal space by the sum of signals of other agents. The monopolist becomes pessimistic about s_i as other agents report higher signals.

We illustrate the negative correlation with two agents in Figure 1.3. It plots the conditional cumulative distribution $G^*(\cdot|s_2)$ across s_2 from 0 to 1. Yellower curves correspond to lower s_2, while redder curves correspond to higher s_2. Note that the

[7] The information rent of an agent at a signal profile corresponds to the amount of discounts that must be given to higher signals to maintain a truth-telling equilibrium.

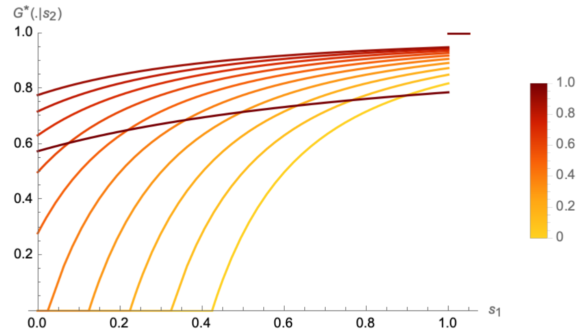

Figure 1.3: The conditional cumulative distribution $G^*(\cdot|s_2)$ with Two Agents

uniform ordering does not apply across different partition elements. In this case when $s_2 = 1$.

On top of the indifference property that ensures an optimal always-sell mechanism, the regulator's solution induces a signal support that bolsters the agents' payoffs. It requires that the sum of signals are bounded below by d^*. The lower bound on d^* limits the impact of free-riding behavior because, with probability one, the total value of trade is sufficiently high.

1.5 The Posted-price Distribution

This section presents the posted-price distribution G^P, which solves the regulator's problem if the monopolist is restricted to charges a fixed price to each agent. The monopolist is confined to post a fixed price if he is under additional strategic constraints, such as an imperfect commitment power and agents' collusion. The posted-price distribution is equivalently characterized by the zero-sum virtual value condition, but on a signal support where individual signals are bounded below. Thus, the posted-price distribution has similar features as the regulator's problem, exhibiting a similar negative correlation among signals. Moreover, we obtain the welfare properties of the regulator's solution in the large economy limit using the posted-price distribution.

1.5.1 Strategic Fragility of the Regulator's Solution

While the regulator's solution G^* maximizes the agents' payoff and realizes the trade with probability 1, it gives strategic strains on the monopolist and the agents. The total transfer is given by the sum of the pivot value of each agent, given the report from the other agent.

$$\sum_{i=1}^{N} t_i^{AS}(s) = q^{AS}(s) \sum_{i=1}^{N} \left(d^* - \sum_{j \neq i} s_j \right)^+$$

where $(\cdot)^+ := \max\{0, \cdot\}$. The total transfer decreases in each s_i since one agent's pivot value decreases as other agents report a higher signal. By choosing the always-sell mechanism, the monopolist promises to sell the public good at cheaper price as the agents value the public good highly. This feature of the always-sell mechanism implies two problems. First, it requires a strong commitment power of the monopolist.[8] Once a high signal profile realizes, the monopolist would be tempted to renege from the announced mechanism. Second, it is not robust to collusive behavior.[9] The agents could simply promise with each other to report $s_i = 1$ to pay less. The collusion is

[8]The second price auction has the same commitment power issue, but in the auction problem, the designer may use the clock implementation to resolve the problem. See, for example, Akbarpour and Li (2020). On the other hand, there is no alternative implementation for the public good problem that eases the designer's commitment burden to the best of my knowledge.

[9]In the auction problem, collusion between the bidders is constrained by mutual private information between the collusive bidders. Moreover, the timing of collusion and feasibility of side-payments also constrain the efficacy of collusion.

not constrained by private information nor by availability of side payments. Thus, the mechanism designer cannot easily make a quick adjustment to the always-sell mechanism to limit the impact of collusion.

Both strategic concerns related to the always-sell mechanism in G^*, which is a pivot mechanism, are resolved if the monopolist posts a fixed price to each agent, or if the monopolist chooses a posted-price mechanism. The commitment problem is resolved since posted-price mechanisms are indirectly implemented, where each agent is given the price, and they report whether they accept the price, keeping the monopolist ignorant of the exact signal of each agent, which prevents the monopolist from deviating from the chosen mechanism. The collusion problem is also resolved since the agents cannot alter the price anyone pays. In the public good provision problem, Proposition 1 of Bierbrauer and Hellwig (2016) shows that the monopolist cannot condition the payment of an agent on reports from others if he is subject to the ex-post coalition-proofness constraint. Thus, we solve the regulator's problem with a restriction on the monopolist that he must choose an optimal posted-price mechanism.

1.5.2 The Regulator's problem with a Posted-price Mechanism

Given a signal distribution G, suppose that the monopolist chooses an optimal posted-price p_i for each agent i. That is, the monopolist solves the following problem to maximize the expected profit of a posted-price mechanism, which charges a fixed price to each agent.

$$\max_{(p_1,\cdots,p_N)\in[0,1]^N} \quad \bar{G}(p)\sum_{i=1}^{N} p_i,$$

where $\bar{G}$ is the joint survival function of a signal distribution G.

Anticipating that the monopolist chooses an optimal posted-price mechanism, the regulator chooses a signal distribution to maximize the agents' payoff in an optimal posted-price mechanism.

$$\max_{G \in \mathcal{G}(\mu)} \int_{p_1}^{1} \cdots \int_{p_N}^{1} \sum_{i=1}^{N} (s_i - p_i)\, dG(s)$$
$$\text{s.t. } (p_1, \cdots, p_N) \text{ is an optimal posted-price in } G.$$

1.5.3 The Posted-Price Distribution

We construct the posted-price distribution G^P using a generalization of the unit-elastic demand condition of Roesler and Szentes (2017). Just as the monopolist is made indifferent across any price in the one-agent regulator's solution, the monopolist is made indifferent across any profile of posted prices on $[\underline{s}, 1]^N$ in the posted-price distribution G^P. We define the posted-price distribution as below.

Definition 1.5.1. *A signal distribution G^P is said to be the posted-price distribution if it satisfies* (1.7)

$$\bar{G}^P(s) \sum_{i=1}^{N} s_i = N\underline{s} \text{ for each } s \in [\underline{s}, 1]^N, \tag{1.7}$$

for some $\underline{s} > 0$, where $\bar{G}^P$ is the probability of sale of a posted price s.

The definition of the posted-price distribution imposes indifference between any posted-price mechanism on the monopolist, which plays the same role as the monopolist's indifference condition, which characterizes the regulator's solution. Indeed, the posted-price distribution solves the monopolist's problem if he is restricted to choose such mechanisms.

Proposition 1.5.1. *The posted-price distribution $G^P \in \mathcal{G}(\mu)$ solves the regulator's problem when the monopolist chooses an optimal posted-price mechanism.*

Using (1.7), we can easily recover the posted-price distribution G^P, which is a mixed probability distribution on the signal space. The signal space here is partitioned in the same way as the regulator's solution did, by the set of agents whose observed the highest signal 1. The posted-price distribution is continuous on each partition element but discrete across partition elements. We can fix a partition element $S(H) \subset S$ and partially differentiate $\bar{G}^P$ with respect to signal indices in $N-H$ to obtain the density g^P on $S(H)$.

$$g^P(s) = (N-H)! \frac{N\underline{s}}{\left(\sum_{i \in N-H} s_i + H\right)^{N-H+1}}$$

The probability of any measurable $A \subset S$ is then given as follows.

$$G^P(A) = \sum_{H \in 2^N} \int_{s \in S(H) \cap [\underline{s},1]^N} g^P(s) ds$$

There exists a unique $\underline{s} > 0$ that satisfies the expectation constraint. We obtain G_1^P, the marginal distribution of s_1 from (1.7) by taking $s_2 = \cdots = s_N = \underline{s}$.

$$G_1^P(s_1) = 1 - \frac{N\underline{s}}{s_1 + (N-1)\underline{s}}$$

The marginal distribution follows the Pareto distribution that is censored at 1. Then it is easy to find the expectation of s_1, which is

$$\mu = \int s_1 dG_1^P(s_1) = \underline{s} + N\underline{s} \log\left(\frac{1+(N-1)\underline{s}}{N\underline{s}}\right) \tag{1.8}$$

The expectation is increasing in $\underline{s}$. It is equal to one if $\underline{s} = 1$ and converges to zero as $\underline{s}$ goes to zero. Thus, for each $\mu \in (0, 1]$, there is a unique $\underline{s}$ that satisfies the expectation constraint.

The posted-price distribution makes the monopolist indifferent across mechanisms that charge each agent a fixed price. It is not obvious if the monopolist would continue to choose the lowest posted price if he could choose any mechanism that conditions one agent's transfer on reports from others. However, it turns out that the posted-price distribution G^P implies the zero-sum virtual values so that the lowest posted price is an optimal mechanism among any DSIC and EPIR mechanisms in the posted-price mechanism. Proposition 1.5.2 shows that the virtual values sum up to zero in G^P.

Proposition 1.5.2. *The virtual value φ_i^P of the posted-price distribution G^P sums up to zero at any signal profile $s \neq (1, \cdots, 1)$.*

Proof of Proposition 1.5.2. The virtual value of agent i is equivalently written in terms of the joint distribution. Consider a signal profile $s \in S(H)$ and an agent i such that $s_i < 1$.

$$\begin{aligned}\varphi_i^P(s) =& s_i - \frac{1 - G^P(s_i|s_{-i})}{g^P(s_i|s_{-i})} \\ =& s_i - \frac{\int_{s_i}^1 g^P(t, s_{-i})dt + g^P(1, s_{-i})}{g^P(s)} \\ =& s_i - \frac{1}{N - H}\left(\sum_{i=1}^N s_i\right)\end{aligned}$$

The virtual values of individual agents sum up to zero, except at the highest signal profile $s = (1, \cdots, 1)$. □

Proposition 1.5.2 implies that the posted-price distribution G^P is analogous to the regulator's solution G^* in that they both make the monopolist indifferent on each signal profile on the support. We construct the conditional cumulative distribution $G^P(\cdot|s_{-i})$ of the signals.

$$G^P(s_i|s_{-i}) = 1 - \left(\frac{\underline{s} + \sum_{j \neq i} s_j}{\sum_{i=1}^N s_i} \right)^{N-H}$$

for each signal profile $s \in S(H)$ and for each partition element.

However, the posted-price distribution exhibits a positive correlation among signals since $G^P(\cdot|s_{-i})$ uniformly increases in s_{-i}. Figure 1.4 illustrates $G^P(\cdot|s_{-i})$ with two agents. The conditional cumulative distribution $G^P(\cdot|s_2)$ increases in s_2. Yellower curves correspond to lower s_2, while redder curves correspond to higher s_2.

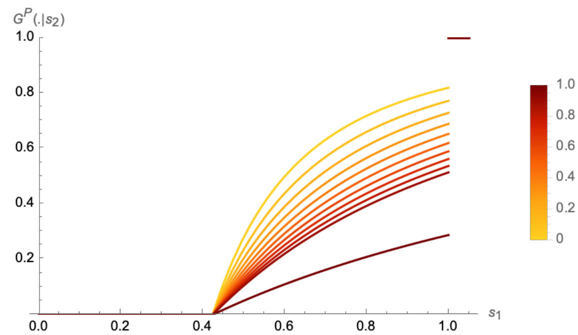

Figure 1.4: The conditional cumulative distribution $G^P(\cdot|s_2)$ with two agents

In addition, the posted-price distribution differs from the regulator's solution in the shape of the signal support. While the regulator's solution puts a lower bound on the sum of signals to ensure the total valuation of the public good is high enough, the posted-price distribution puts a lower bound on individual signals. With the bound on individual signals, the posted-price distribution implements trades when the total value of trade is lower at the cost of giving up trades when the total value of trade is higher. Figure 1.5 illustrates the supports of G^* and G^P with two agents. The orange region illustrates the support of G^P, while the blue region illustrates the support of G^*. In the posted-price distribution, the agents buy the public good when the total value is less than the lower bound d^* in the regulator's solution, while signal profiles on the left and lower region, where one agent's signal is less than $\underline{s}$ is not drawn in the posted-price distribution, while the total value of trade is high. Thus, the posted-price distribution is not optimal in the regulator's problem without the restriction on the monopolist.

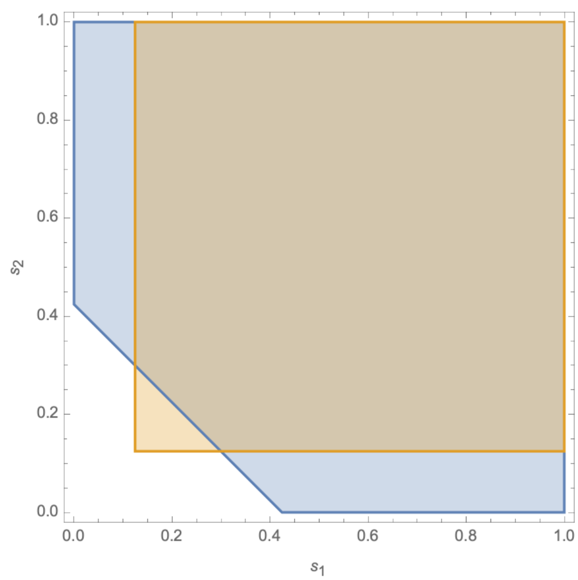

Figure 1.5: Comparison of the signal support of G^* and G^P

1.5.4 Welfare in the Large Economy

We now study the welfare properties of the regulator's solution G^* in the large economy limit. Since it is hard to compute the constant of integration $c^*(d)$ and the threshold d^* to fully describe the regulator's solution in general, we use the posted-price distribution G^P to obtain an upper bound on the monopolist's profit with any number of agents. The expected profit under the posted-price distribution is bounded in the large economy limit as illustrated in Figure 1.6. That is, the monopolist gets a negligible fraction of the total ex-ante surplus when there are many agents, i.e. the monopolist's market power gets dissipated as the number of agents increases. Proposition 1.5.3 formally proves the observation that the expected profit is bounded.

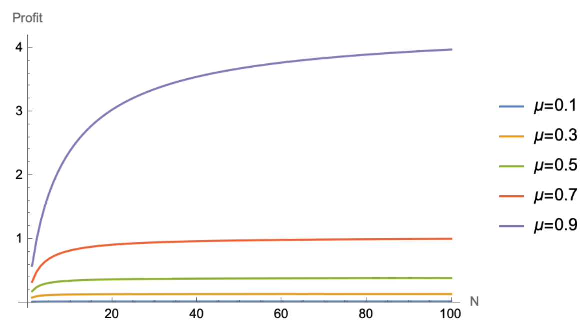

Figure 1.6: Expected profit under the posted-price distribution

Proposition 1.5.3. *Suppose that there are N symmetric agents with expected value $\mu \in [0, 1)$. Under the posted-price distribution $G^P \in \mathcal{G}(\mu)$, the monopolist's expected profit is bounded in N.*

The expected profit of an optimal mechanism in the posted-price distribution G^P is $N\underline{s}$, which is the expected profit of charging the lowest price $\underline{s}$ on the support, selling the public good with probability 1. Since in the posted-price distribution G^P, the monopolist is indifferent between any posted prices, the same expected profit could be obtained by charging 1 to each agent, selling the public good with probability of $g^P(1, \cdots, 1) = \underline{s}$. It is easier to have at least one agent observe a signal less than 1 and have a lower $\underline{s}$ with more agents. The proof of Proposition 1.5.3 shows that $\underline{s}$ is indeed proportional to $\frac{1}{N}$.

Proposition 1.5.3 provides a lower bound on the agents' payoff under the regulator's solution G^* as the posted-price distribution G^P is sub-optimal for the regulator's problem, and both the regulator's solution and the posted-price distribution ensure the always-sell mechanism to be optimal with any number of agents. The bounded expected profit in the large market then implies that, either under the posted-price distribution or the regulator's solution, each agent pays zero in the limit.

Corollary 1.5.1. *Under both the regulator's solution G^* and the posted-price distribution G^P, each agent takes the entire ex-ante surplus μ as N increases to infinity.*

It is intuitive that each agent pays a vanishing amount as the number of agents increases because the monopolist is selling a public good. On the other hand, the bounded profit in the large market in the regulator's solution contrasts with the large market results in the independent private value framework. For example, Güth and Hellwig (1986) shows that each agent pays a vanishing amount as the number of agents increases, assuming a symmetric regular signal distribution. However, they also find that the monopolist's profit is proportional to $\sqrt{N}$ as the number of agents increases.[10] In other words, the monopolist earns still an infinite amount of expected profit with many agents in the independent private value framework. Thus, the monopolist enjoys a substantial degree of market power with many agents, while the market power disappears under the regulator's solution.

Figure 1.7 illustrates the monopolist's expected profit (left) and expected profit per agent (right). The monopolist earns more expected profit with more agents or

[10] Proposition 4.5 of Güth and Hellwig (1986)

a higher expected value μ. However, the expected profit per agent decreases as the number of agents increase because the regulator is more flexible to choose a signal distribution with more agents. We provide explicit solutions to the regulator's problem only with a small number of agents, Corollary 1.5.1 finds the monopolist's expected profit in the large market limit to be flat at zero for $\mu < 1$.

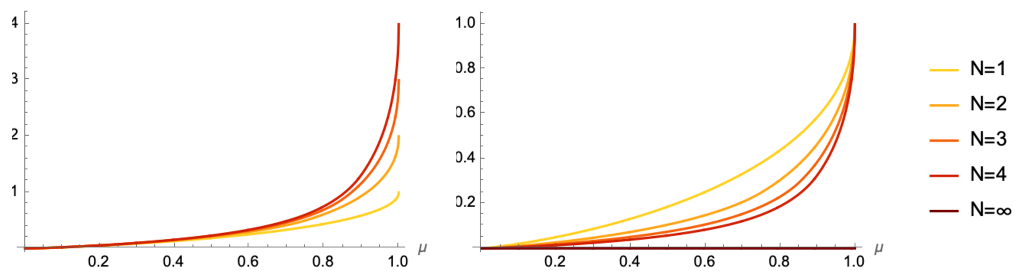

Figure 1.7: The monopolist's profit (left) / per-agent profit (right) under G^*

Figure 1.8 juxtaposes the regulator's payoff (left) and each agent's payoff (right). The regulator earns zero payoff at $\mu = 1$, where there is no uncertainty. Also, she also earns zero payoff at $\mu = 0$, where there is no value of trade. There is a trade-off for the regulator when μ increases because the increase in μ increases the total ex-ante value of trade while decreasing the regulator's ability to choose a signal distribution. The regulator's payoff is unbounded in the number of agents, while each agent pays zero in the large economy limit. Corollary 1.5.1 finds the agent's payoff in the limit to be a straight line for $\mu < 1$.

1.6 The Welfare Triangle

In this subsection, we show that the regulator may implement a different division of surplus between the monopolist and the agents from the regulator's solution G^*

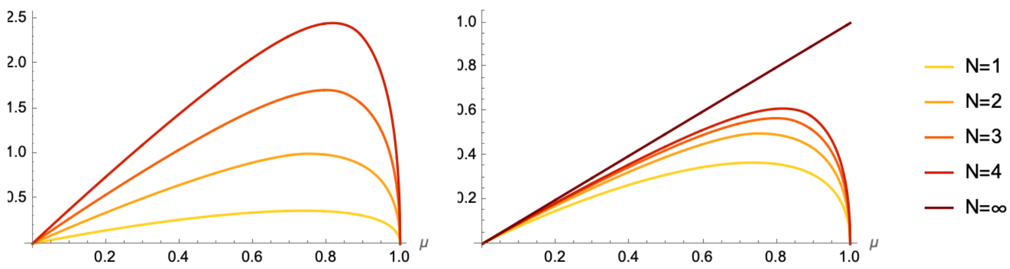

Figure 1.8: The regulator's payoff (left) / an agent's payoff (right) G^* and (q^{AS}, t^{AS})

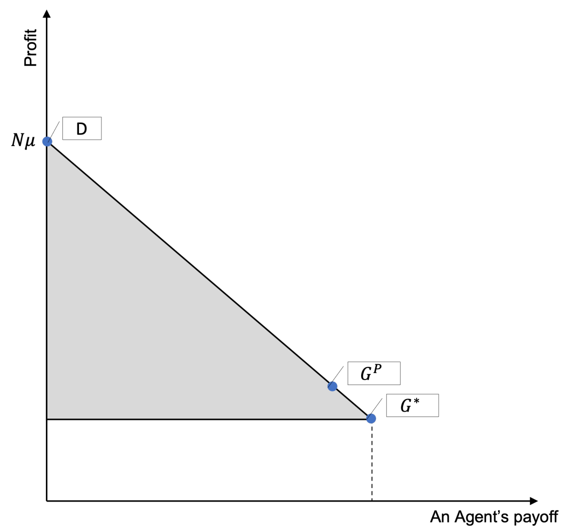

Figure 1.9: Outcome Triangle

without losing efficiency. Figure 1.9 describes the set of feasible welfare divisions between the monopolist and the agents. The regulator's solution G^* maximizes the expected payoff of the agents with the always-sell mechanism. The monopolist's earns the expected profit of $Ng^*(1, \cdots, 1) = \frac{c^*(d^*)}{N!}$ which is the expected profit of an optimal mechanism that sells the public good only if $s = (1, \cdots, 1)$. The opposite extreme **D**(egenerate distribution) maximizes the expected profit of the monopolist without losing efficiency by giving no information to the agents. That is, the monopolist takes the whole ex-ante value of trade $N\mu$. Other than G^* and $\mathbf{D}$, any intermediate division of surplus between the monopolist and the agents can be implemented without losing efficiency.

Proposition 1.6.1. *For each* $\pi \in \left[\frac{c^*(d^*)}{N!}, N\mu\right]$*, there exists a signal distribution* $\hat{G}$ *where there is an optimal always-sell mechanism in* $\hat{G}$ *that earns* π *for the monopolist.*

Any such divisions are implemented using a signal distribution that satisfies the indifference condition on the monopolist, and it does so on a narrow range of signals so that the always-sell mechanism could earn more profit. In this sense, the regulator's solution G^* puts the smallest lower bound on the sum of signals to make the monopolist to earn the smallest expected profit while always selling the public good.

1.7 Proof of Theorem 1.4.1

In this section, we characterize the regulator's solution using an indirect approach. We find *the minimax solution*, which is a signal distribution that minimizes the monopolist's expected profit. We say that the minimax solution solves the regulator's problem if there is an optimal mechanism that always sells the public good under the minimax solution. In such an optimal mechanism, the sum of the monopolist's profit and the agents's payoffs is equal to the ex-ante expected surplus. So minimizing the monopolist's profit is equivalent to maximizing the agents' payoffs since there is a fixed sum. We summarize this reasoning in Observation 1.7.1.

Observation 1.7.1. *The minimax solution* G^* *solves the regulator's problem if there is an optimal mechanism* (q^{AS}, t^{AS}) *with respect to* G^* *that sells the public good with probability* 1 *given* G^**.*

Then, to finish solving the regulator's problem, we need to solve the minimax problem and verify the existence of an optimal mechanism that always sells the public good.

1.7.1 The Minimax and Maximin Problem

Taking advantage of Observation 1.7.1, we solve the minimax problem by considering a zero-sum game between the monopolist and *the profit-minimizing information designer* (*the profit-minimizer*). The profit-minimizer's objective is to minimize the expected profit of the profit-maximizing monopolist, which is different from the regulator's objective we discussed earlier. In this zero-sum game, the profit-minimizer chooses a feasible signal distribution $G \in \mathcal{G}(\mu)$ and the monopolist chooses a DSIC/EPIR direct mechanism (q, t) at the same time. In this zero-sum game, the profit-minimizer solves the minimax problem, while the monopolist solves the maximin problem to obtain their optimal choices.

$$\inf_{G \in \mathcal{G}(\mu)} \sup_{q \in \mathcal{Q}} \int_{s \in [0,1]^N} \sum_{i=1}^{N} \left(s_i q(s) - \int_0^{s_i} q(\nu, s_{-i}) d\nu \right) dG(s), \qquad \text{(Minimax Problem)}$$

$$\sup_{p \in \mathcal{Q}} \inf_{H \in \mathcal{G}(\mu)} \int_{s \in [0,1]^N} \sum_{i=1}^{N} \left(s_i p(s) - \int_0^{s_i} p(\nu, s_{-i}) d\nu \right) dH(s). \qquad \text{(Maximin Problem)}$$

A signal distribution is said to be *the minimax solution* if it solves the minimax problem, and an allocation rule is said to be *the maximin solution* if it solves the maximin problem.

The minimax solution G^* and the maximin solution p^* constitutes a Nash equilibrium, or a saddle point in this zero-sum game. Let $R(p, G)$ denote the expected profit of an allocation rule p in a signal distribution G. A pair of an allocation rule and a signal distribution (p^*, G^*) constitutes *a saddle point* if

$$R(p^*, G) \geq R(p^*, G^*) \geq R(p, G^*),$$

for any $p \in \mathcal{Q}$ and $G \in \mathcal{G}(\mu)$. The definition of a saddle point requires (p^*, G^*) to be a mutual best response. The first inequality requires that G^* must be a worst-case signal distribution for p^*, which is the profit-minimizer's best response to p^*, while the second inequality requires that p^* is an optimal allocation rule in G^*, which is the monopolist's best response to G^*.

We solve the minimax problem to obtain the regulator's solution and the maximin problem is jointly solved with the minimax problem as an intermediate step. However, the maximin problem has a separate economic interest. The maximin mechanism earns at least the worst-case expected profit as long as the actual signal distribution is drawn from $\mathcal{G}(\mu)$. In this way, the maximin solution robustly maximizes the expected profit of the monopolist given limited knowledge about the information structure.

1.7.2 The Indifference Conditions

Now we introduce the indifference conditions we impose on the the minimax and maximin problems. Under these conditions, the profit-minimizer and the monopolist make each other indifferent across all mechanisms and signal distributions, except for the ones that can never be optimal. Then we use these conditions to characterize the minimax and maximin solutions.

In an equilibrium of the zero-sum game, players make it harder for the opponent to best-respond by giving them a number of optimal choices.[11] Borrowing this intuition from the zero-sum game equilibrium, the profit-minimizer ensures that the monopolist has many optimal mechanisms that he is indifferent, while the monopolist ensures that the profit-minimizer has many worst-case signal distributions that she is indifferent in equilibrium.

The profit-minimizer imposes indifference on the monopolist by choosing the indifference distribution G^*, where any mechanism such that $q(1, \cdots, 1) = 1$ and $q(s) = 0$ for signal profiles whose sum is less than $\frac{X}{A}$.

It takes some work to figure out how the monopolist could impose indifference on the profit-minimizer. Then we consider the maximin problem, where the monopolist first chooses an allocation rule and is followed by the profit-minimizer who chooses a worst-case signal distribution. To implement the indifference intuition, we need to know properties of a worst-case distribution to make the profit-minimizer indifferent across many signal distributions. Thus, we consider the profit-minimizer's problem of finding a worst-case distribution $G \in \mathcal{G}(\mu)$ given a mechanism.

$$\min_G \int \sum_i t_i(s) dG(s) \tag{Profit-Minimizer}$$

$$\text{s.t.} \int_{v \in [0,1]^N} s_i dG(s) = \mu \text{ for each } i = 1, \cdots, N \tag{PM-1}$$

$$\int_{s \in [0,1]^N} dG(s) = 1 \tag{PM-2}$$

$$G(A) \geq 0 \text{ for each measurable subset } A \subset [0,1]^N \tag{PM-3}$$

[11]For example, the rock-scissor-paper game has a mixed strategy equilibrium where players are indifferent between rock, scissor and paper.

The profit-minimizer's problem is a linear programming problem. PM-1 requires that the expectation of each s_i must be μ. PM-2 and PM-3 requires G to be a probability distribution. Thus, its dual program and the complementary slackness condition characterize the solutions. The dual of the profit minimizer's problem is to find two real numbers A and X.

$$\max_{\phi,\lambda} \quad -X + NA\mu \qquad \text{(Dual Profit-minimizer)}$$

$$\text{s.t. } \sum_i t_i(s) \geq -X + \sum_{i=1}^{N} As_i \text{ for each } s \in [0,1]^N \qquad \text{(D-PM-1)}$$

The dual variables A and X are multipliers of PM-1 and PM-2. Lemma 1.7.1 provides a sufficient optimality condition for Profit-Minimizer and Dual Profit-minimizer.

Lemma 1.7.1. *There is no duality gap between the profit-minimizer's problem and its dual problem. Moreover, an optimal solution of the profit-minimizer's problem and its dual problem is characterized by the complementary slackness condition.*

$$\int \left(\sum_{i=1}^{N} t_i(s) + X - \sum_{i=1}^{N} As_i \right) dG(s) = 0 \qquad \text{(C)}$$

The complementary slackness (C) characterizes worst-case signal distributions given a mechanism, which requires that a worst-case distribution must be supported on signal profiles where the total transfer of the given mechanism coincides with the supporting hyperplane $A\sum s_i - X$. Then, the monopolist could impose indifference on the profit-minimizer by choosing a mechanism whose total transfer coincides with its supporting hyperplane. We formally define the maximin indifference below.

Definition 1.7.1. *An allocation rule* p^* *satisfies* the maximin indifference condition *if for each* s,

$$\left(A\sum_{i=1}^{N} s_i - X\right)^+ = \sum_{i=1}^{N}\left(s_i p^*(s) - \int_{(X/A-\sum_{j\neq i} s_j)^+}^{s_i} p^*(\nu, s_{-i})d\nu\right), \tag{Maximin Indifference}$$

for some $A > 0$ *and* $X > 0$*. An allocation rule that satisfies the maximin indifference condition is said to be* the indifference allocation rule*.*

The indifference allocation rule p^* implements the indifference intuition by making the profit-minimizer indifferent across any signal distribution G that is supported on signal profiles whose sum exceeds the threshold $\frac{X}{A}$. The indifference condition implies that $\sum_{i=1}^{N} t_i(s) = 0$ and $p^*(s) = 0$ for signal profile s whose sum is less than $\frac{X}{A}$. Then, the complementary slackness condition implies that any signal distribution whose support is not a subset of $\{s|\sum_{i=1}^{N} s_i \geq \frac{X}{A}\}$ cannot be a worst-case distribution since the total transfer is strictly larger than the supporting hyperplane at signal profiles whose sum is less than $\frac{X}{A}$.

1.7.3 The Indifference Allocation Rule

The Maximin Indifference condition defines a partial integral equation for an allocation rule. We expect to see a piecewise-defined solution from the maximin indifference condition because the lower bound of the integration interval is truncated at 0. With an arbitrary number of agents, it is generally difficult to explicitly solve for the indifference allocation rule as there are rapidly increasing number of cases we need to consider. Still, we show that the maximin indifference condition implies a well-defined allocation rule and explicitly construct the indifference allocation rule with two and three agents.

Proposition 1.7.1. *Given $A > 0$ and $X > 0$, there exists the indifference allocation rule p^*. In addition, p^* is non-negative and strictly increasing in s.*

We present a constructive proof for Proposition 1.7.1 at the appendix, devising a procedure that solves the maximin indifference condition for the indifference allocation rule. Namely, the procedure differentiates the indifference condition equation, which is a partial integral equation, with respect to the signals. It then yields an ordinary integral equation of a cross-partial derivative of the allocation rule that we can easily solve. Finally, We integrate the cross-partial derivatives up to recover the allocation rule. Applying such procedure to the case with two or three agents, we fully solve for the indifference allocation rule to understand the economics behind Proposition 1.7.1.

We apply the procedure developed in Proposition 1.7.1 to solve Maximin Indifference for the maximin allocation rule p^* with two and three agents to study properties of the maximin mechanism. It is qualitatively different when there are two agents and when there are three or more agents. With two agents, the maximin mechanism uses a scoring rule to determine supply. On the other hand, with three or more agents, the allocation rule is not a scoring rule.

First, suppose that there are two agents. We obtain the maximin allocation rule in Corollary 1.7.1.

Corollary 1.7.1.

$$p^*(s) = \begin{cases} 0 & s_1 + s_2 < \frac{X}{A} \\ \frac{A^2}{X}(s_1 + s_2) - A & s_1 + s_2 \geq \frac{X}{A} \quad s_1 < \frac{X}{A} \quad s_2 < \frac{X}{A} \\ A \log\left(\frac{A}{X}s_1\right) + \frac{A^2}{X}s_2 & s_1 + s_2 \geq \frac{X}{A} \quad s_1 > \frac{X}{A} \quad s_2 < \frac{X}{A} \\ A \log\left(\frac{A}{X}s_1\right) + A \log\left(\frac{A}{X}s_2\right) + A & s_1 + s_2 \geq \frac{X}{A} \quad s_1 > \frac{X}{A} \quad s_2 > \frac{X}{A} \end{cases} \tag{1.9}$$

With two agents, the maximin allocation rule allows a simple interpretation since it transforms reports of each agent into ϕ, a scoring rule, which is defined as below.

$$\phi^*(s) = \begin{cases} \frac{A^2}{X}s - \frac{A}{2} & s < \frac{X}{A} \\ A \log\left(\frac{A}{X}s\right) + \frac{A}{2}. & s \geq \frac{X}{A} \end{cases}$$

The scoring rule depends on the size of signal, which is linear when the report is small while being a log function when the report is large.

$$p^*(s) = \max\{0, \phi^*(s_1) + \phi^*(s_2)\}$$

The allocation rule p^* is then the total score if it is non-negative. The maximin allocation rule provides incentive to agents with low signals by allowing them to make a negative contribution to the supply. Reports from one agent determines the level of public good supply and money transfer from the other agents, but they are independent on the margin.

Then we consider the three agents case. Again, we obtain the maximin allocation rule by solving the maximin indifference condition. However, with three agents, it takes too much space to fully describe the allocation rule. Thus, we relegate the complete description of the maximin allocation rule to the appendix and proceed with the marginal allocation rule $p_1^*(s)$, which is the partial derivative of p^* with respect to s_1 in Corollary 1.7.2.

Corollary 1.7.2.

$$p_1^*(s) := \frac{\partial p^*}{\partial s_1}(s) = \begin{cases} \frac{A^3}{X^2}\left(s_1 + \min\{s_2, \frac{X}{A} - s_1\} + \min\{s_3, \frac{X}{A} - s_1\}\right) & s_1 < \frac{X}{A} \\ \frac{A}{s_1} & s_1 \geq \frac{X}{A} \end{cases} \tag{1.10}$$

The maximin allocation is qualitatively different with three agents from the two agents case. Notably, the maximin allocation rule is no longer a scoring rule because p^* is not additively separable in signals. Figure 1.10 illustrates the marginal allocation rule q_1, assuming without loss of generality that $s_2 < s_3$. When s_1 is small, the agents complement each other in the mechanism on the margin. That is, increasing s_1 allows the mechanism to sell the public good at lower signals of other agents. On the other hand, as s_1 increases, the marginal allocation plateaus and then decreases because the agents are no longer complements on the margin.

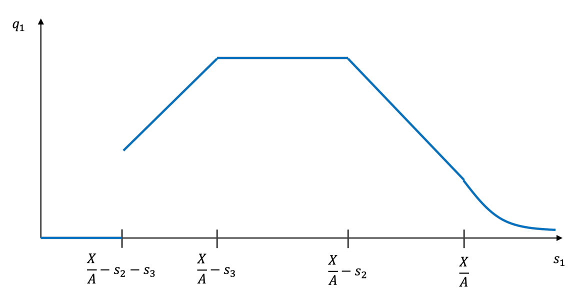

Figure 1.10: q_1 with three agents

1.7.4 Characterization of A Saddle Point

So far, we constructed the indifference distribution G^* and verified that the maximin indifference condition implies a well-defined allocation rule p^* for some parameter A and X. Now, we have all the ingredients to construct a saddle point in the zero-sum game between the monopolist and the profit-minimizer.

Proposition 1.7.2. *The pair of the indifference allocation rule p^* and the indifference distribution G^* with parameters A^* and X^* that satisfy*

$$\mu = \int_S s_i dG^*(s), \tag{1.11}$$

$$1 = p^*(1, \cdots, 1) \tag{1.12}$$

is a saddle point.

Proof of Proposition 1.7.2. We first argue that there exists a unique A^* and X^* that satisfy (1.11) and (1.12). First, Lemma 1.4.1 shows that there exists a unique ratio between A^* and X^* so that G^* is a feasible signal distribution since the indifference distribution is fully determined by the ratio $\frac{X^*}{A^*}$. The parameters A^* and X^* are individually pinned down by (1.12), which requires that the indifference allocation sells the full amount at the highest signal profile.

Then we show that the indifference allocation and distribution constitute a saddle point, by proving that they mutually best-response to each other. We first show that p^* is an optimal mechanism in the indifference distribution G^*. From the minimax indifference condition, we know that an optimal mechanism in G^* must sell the full amount of the public good at $s = (1, \cdots, 1)$, while it sells no public good if the sum of reported signals is less than $\frac{X^*}{A^*}$. (1.12) requires the first requirement for optimality, and the maximin indifference implies $p^*(s) = 0$ for signal profiles where $\sum_{i=1}^{N} s_i \leq \frac{X^*}{A^*}$. Thus, the indifference allocation p^* is an optimal mechanism in the indifference distribution G^*.

Next, we show that the indifference distribution G^* is the worst-case signal distribution for the indifference allocation p^*. We use Lemma 1.7.1, which characterizes worst-case signal distributions given any mechanism to prove the point. Given the indifference allocation p^*, the complementary slackness (C) requires that a worst-case distribution to be supported on signal profiles such that $\sum s_i \geq \frac{X^*}{A^*}$, which coincides with the support of the indifference distribution G^*.

Thus, we verified that the pair of the indifference allocation p^* and the indifference distribution G^* constitutes a saddle point, when the parameters A^* and X^* are chosen by (1.11) and (1.12). □

Proposition 1.7.2 jointly characterizes both the minimax and maximin solution. However, for the purpose of solving the regulator's problem, we only need the minimax solution. It is characterized independently from the maximin solution, only by the minimax indifference and the expectation constraint since G^* depends on the ratio between A^* and X^*, which is pinned down by the expectation constraint.

Corollary 1.7.3. *The minimax distribution G^* is characterized by the Monopolist Indifference condition and the expectation constraint* (1.11)*.*

1.7.5 Characterization of the Regulator's Solution

Cororllary 1.7.3 above shows that the minimax solution G^* is characterized by the Monopolist Indifference and the expectation constriant only. The minimax indifference implies that the always-sell mechanism (q^{AS}, t^{AS}), which sells the public good at any signal profile that realizes under G^*, is an optimal mechanism in G^*.

$$\begin{aligned} q^{AS}(s) =& \mathbb{1}\left\{\sum_{i=1}^{N} s_i \geq \left(\frac{X}{A}\right)^*\right\}. \\ t_i^{AS}(s) =& q^{AS}(s) \min\left\{\hat{s}_i | \hat{s}_i + \sum_{j \neq i} s_j \geq \left(\frac{X}{A}\right)^*, \quad \hat{s}_i \geq 0\right\}, \\ =& q^{AS}(s) \left(\left(\frac{X}{A}\right)^* - \sum_{j \neq i} s_j\right)^+. \end{aligned} \tag{1.13}$$

Then we leverage on Observation 1.7.1 to prove that the minimax solution G^* also solves the regulator's problem.

Corollary 1.7.4. *The minimax solution $G^* \in \mathcal{G}(\mu)$ solves the Regulator's Problem.*

1.8 Discussion and Concluding Remarks

1.8.1 Discussion

We would like to discuss and interpret the assumptions on the regulator's problem relative to the standard approach in the information design literature. The regulator is restricted by the expectation of each signal and the range of signal to be between $[0, 1]$ for each coordinate. This modeling approach contrasts with the standard information design literature, where there is a state of nature, and an information structure is formally defined as a joint distribution over signals and the state of nature. In our context, the true valuation of the public good is the state of nature, and the agents are the receivers.

Recent papers such as Gentzkow and Kamenica (2016), Roesler and Szentes (2017) and Dworczak and Martini (2019) consider signal distributions instead of full information structures because sender and receiver have preferences that only depend on the expectation of the state. They identify the set of feasible signal distributions, which could arise from some full information structure to solve the information design problem. Feasibility requires that the signal distribution is a mean-preserving contraction of the distribution of the state. The three-way characterization of the mean-preserving spread from Rothschild and Stiglitz (1971) plays an important role in characterizing the set of such signal distributions, especially for the comparison between the integrated cumulative distribution functions.

However, in our case with multi-dimensional state of nature, there is no simple characterization of the mean-preserving contraction relationship as we cannot employ the comparison of integrated cumulative distribution functions anymore. However, the modeling approach we take here does not give an excessive power to the regulator relative to the standard assumptions because, even with the standard assumptions, the regulator's solution is characterized by the minimax and maximin indifference conditions as before. The regulator's solution will still be characterized by the zero-sum virtual values condition, but on a narrower signal support under the standard assumptions.

1.8.2 Concluding Remarks

In this paper, we studied a problem of signal distribution choice that would maximize the buyers' payoffs in an optimal public good sales mechanism. The optimal signal distribution must tame the monopolist's market power and the agents' free-riding incentives at the same time so that the agents could buy the public good at the cheapest price possible. We characterized the regulator's solution and the signal support. The solution achieves the goal through the indifference condition on the monopolist so that he always sells the public good under the chosen signal distribution. The signal support exhibits the property that the total signal is bounded below and in turn limits the impact of free-riding of the agents.

We solved the regulator's problem using the conditional equivalence with the minimax solution, which is the signal distribution that minimizes the expected profit of an optimal mechanism. Toward solving the minimax problem, we solved the maximin problem where the monopolist chooses a mechanism that maximizes the worst-case expected profit. The maximin mechanism depends on the number of agents, which is a scoring mechanism with two agents, while it becomes qualitatively different from scoring mechanism with three or more agents.

1.9 Appendix

Proof of Proposition 1.4.1. We first show that G^* is well-defined. The constant of integration $c(d)$ is given by

$$c(d) = \frac{1}{\sum_{H \in 2^N} \int_{S \in S(H), \sum_{i=1}^{N} s_i \geq d} \frac{1}{\left(\sum_{j \in N-H} s_j + H\right)^{N-H+1}} \frac{N+1}{\prod_{h=0}^{H}(N-h+1)} ds}$$

ensures that G^* is a probability distribution. $c(d)$ increases in d since the integral on the denominator decreases in d. In addition, c converges to 0 as d goes to 0 since the integral on the denominator diverges and $c(N) = N * N!$.

The expectation of s_i increases in d, for each i since an increase in d drops lower s_i from the expectation and increases the weight $c(d)$ on higher s_i.

$$\int s_i dG^*(s) = \sum_{H \in 2^N} \int_{S \in S(H), \sum_{i=1}^N s_i \geq d} s_i \frac{c(d)}{\left(\sum_{j \in N-H} s_j + H\right)^{N-H+1}} \frac{N+1}{\prod_{h=0}^{H}(N-h+1)} ds$$

Suppose that $d = N$, then G^* is degenerate at $s = (1, \cdots, 1)$ and the expectation of each s_i is 1. Next, suppose that d approaches to 0 from above. Then the expectation of s_i converges to 0 as $c(d)$ vanishes as d gets smaller. Thus, there exists a unique d^* that satisfies the expectation constraint by the intermediate value theorem.

We next show that the density g^* satisfies the minimax indifference condition on each partition element $S(H)$, for each $H \in 2^N$. Fix a partition element $S(H)$. For each s such that $\sum_{i=1}^N s_i \geq d^*$ and $s_i < 1$, the density g^* implies the following.

$$\frac{1 - G^*(s_i|s_{-i})}{g^*(s_i|s_{-i})} = \frac{\int_{s_i}^1 g^*(t, s_{-i})dt + g^*(1, s_{-i})}{g^*(s)} = \frac{1}{N-H}\left(\sum_{j \in N-H} s_j + H\right)$$

Then we obtain the individual virtual value for agent i. For each signal profile $s \in S(H)$ such that $\sum_{i=1}^N s_i \geq d$,

$$\varphi_i^*(s_i|s_{-i}) = \begin{cases} s_i - \frac{1}{N-H}\left(\sum_{j \in N-H} s_j + H\right) & i \in N-H \\ 1 & i \in H \end{cases}$$

It is clear that if there is a joint density g^* that induces the virtual value as above, it must satisfy the minimax indifference condition.

□

$c^*(d)$ and d^* for Two, Three and Four Agents

- Two agents

$$c^*(d) = \begin{cases} d & 0 < d \leq 1 \\ d^2 & 1 < d \leq 2 \end{cases} \quad \mu = \begin{cases} \frac{1}{4}d^*(3 - 2\log d^*) & 0 < d^* \leq 1 \\ d^* - \frac{1}{4}(d^*)^2 & 1 < d^* \leq 2 \end{cases}$$

- Three agents

$$c^*(d) = \begin{cases} 2d & 0 < d \leq 1 \\ \frac{2d^3}{2d-1} & 1 < d \leq 2 \\ \frac{2d^3}{3} & 2 < d \leq 3 \end{cases} \quad \mu = \begin{cases} \frac{1}{18}d^*\left(11 + 6\log\left(\frac{1}{d^*}\right)\right) & 0 < d^* \leq 1 \\ \frac{-4(d^*)^3+24(d^*)^2-9d^*}{36d^*-18} & 1 < d^* \leq 2 \\ \frac{d^*}{2} - \frac{(d^*)^3}{54} & 2 < d^* \leq 3 \end{cases}$$

- Four agents

$$c^*(d) = \begin{cases} 6d & 0 < d \leq 1 \\ \frac{6d^4}{3d^2-3d+1} & 1 < d \leq 2 \\ \frac{6d^4}{9d-11} & 2 < d \leq 3 \\ \frac{3d^4}{8} & 3 < d \leq 4 \end{cases} \quad \mu = \begin{cases} \frac{25d^*}{48} - \frac{1}{4}d^*\log(d^*) & 0 < d^* \leq 1 \\ \frac{d^*(16-9d^*((d^*-8)d^*+6))}{48(3(d^*-1)d^*+1)} & 1 < d^* \leq 2 \\ \frac{d^*\left(-3(d^*)^3+162d^*-176\right)}{48(9d^*-11)} & 2 < d^* \leq 3 \\ \frac{1}{768}(d^*)^4\left(\frac{256}{(d^*)^3} - 1\right) & 3 < d^* \leq 4 \end{cases}$$

Proof of Proposition 1.5.1. We first show that for any symmetric signal distribution $G \in \mathcal{G}(\mu)$, there exists another symmetric signal distribution G^{PP} where the monopolist earns the same expected profit and the sale realizes with probability 1.

Fix a symmetric $G \in \mathcal{G}(\mu)$ and let p^* denote an optimal price in G^*, which satisfies

$$\pi^* = \bar{G}(p^*) \sum_{i=1}^{N} p_i^* \geq \bar{G}(p) \sum_{i=1}^{N} p_i,$$

for each $(p_1, \cdots, p_N) \in [0,1]^N$. Equivalently,

$$\frac{\pi^*}{\sum_{i=1}^{N} p_i} \geq \bar{G}(p), \tag{1.14}$$

for each $(p_1, \cdots, p_N) \in [0,1]^N$.

Define a joint survival function $\bar{G}^{PP}$ by

$$\bar{G}^{PP}(s) = \frac{\pi^*}{\sum_{i=1}^{N} s_i},$$

for $s_i \in [\pi^*/N, 1]$ for each i.

(1.14) implies that $\bar{G}^{PP}$ is larger than $\bar{G}$ in the upper orthant order, which implies that the expected signal in $\bar{G}^{PP}$ is larger than in $\bar{G}$ by Theorem 6.G.1. of Shaked and Shanthikumar (2007),

$$\int_{\pi^*/N}^{1} \cdots \int_{\pi^*/N}^{1} s_i dG^{PP}(s) \geq \mu = \int_{0}^{1} \cdots \int_{0}^{1} s_i dG(s) \geq \pi^*/N,$$

for each i.

For each i, define the function $f_i(\bar{s})$ by

$$f_i(\bar{s}) = \int_{\pi^*/N}^{\bar{s}} \cdots \int_{\pi^*/N}^{\bar{s}} s_i dG^{PP}(s) - \mu$$

Each f_i is continuous, $f_i(\pi^*/N) < 0$ and $f_i(1) > 0$. Thus, by the intermediate value theorem, there exists $\bar{s}^* \in [\pi^*/N, 1]$ where $f_i(\bar{s}^*) = 0$ for each i and $G^{PP} \in \mathcal{G}(\mu)$.

It is optimal for the monopolist to charge π^*/N to each agent in G^{PP}, which earns the expected profit π^* and sells the public good with probability 1. It is without loss to consider G^{PP} with $\bar{s} \leq 1$ to solve the regulator's problem when the monopolist chooses an optimal posted-price. Thus, the posted-price distribution G^P solves the regulator's problem when the monopolist chooses an optimal posted-price.

□

Proof of Proposition 1.6.1. We consider a variant of the indifference distribution, where signal profiles are drawn from a smaller range, where each signal is at most $m \leq 1$, for each agent. The signal space is partitioned by the set of agents who observed a signal higher than m. Define $\hat{S}(H)$ to be the subset of the signal space where $H \in 2^N$, a subset of agents observed a signal higher than m. Define a partition on the signal space by the set of agents H who observed a signal higher than m.

$$S = \cup_{H \in 2^N} \hat{S}(H) = \cup_{H \in 2^N} \{s \in S | s_i \geq m \text{ for each } i \in H\}. \tag{1.15}$$

For each measurable $A \subset S$, define $\hat{G}(A)$, the probability of observing the event A under $\hat{G}$ by

$$\hat{G}(A) = \sum_{H \in 2^N} \int_{s \in \hat{S}(H) \cap A} \hat{g}(s) ds. \tag{1.16}$$

The joint density $\hat{g}$ is chosen so that the always-sell mechanism is optimal under $\hat{G}$. For $s \in \hat{S}(H)$ such that $\sum_{i=1}^N s_i \geq d$ and $s_i \leq m$ for each i,

$$\hat{g}(s) = \frac{\hat{c}(d)}{\left(\sum_{i=1}^N s_i\right)^{N-H+1}} \frac{N+1}{\prod_{h=0}^{H}(N-h+1)}, \tag{1.17}$$

for some constant $\hat{c}(d)$ and the threshold d which ensures that $\hat{G}$ is a probability distribution and satisfies the expectation constraint.

The signal distribution $\hat{G}$ coincides with the regulator's solution if $m = 1$. □

Proof of Proposition 1.5.3. With N symmetric agents, the lowest signal $\underline{s}$ is chosen so that the expected signal is equal to μ.

$$\mu = \underline{s} + N\underline{s}\log\left(\frac{1 + (N-1)\underline{s}}{N\underline{s}}\right)$$

An optimal mechanism earns $\pi(N) = N\underline{s}$ under the posted-price distribution. Rearrange the expectation constraint to obtain a sequence of implicit functions of $\pi(N)$.

$$\Phi(\pi(N), N) = \pi(N)\left[\frac{1}{N} + \log\left(\frac{1}{\pi(N)} + \frac{N-1}{N}\right)\right]$$

For each N, $\Phi(\pi(N), N)$ is continuous and increasing in $\pi(N)$ for $\pi(N) < N$. Moreover, $\Phi(N, N) = 1$ and $\lim_{\pi \to 0} \Phi(\pi, N) = 0$ for each N. Thus, there exists a unique $\pi(N) \in [0, N]$ for each N.

Denote the limit expected profit by $\pi(\infty) = \lim_{N\to\infty} \pi(N)$. Take the limit on Φ to obtain the limit expected profit.

$$\mu = \lim_{N\to\infty} \Phi(\pi(N), N) = \pi(\infty)\log\left(\frac{1}{\pi(\infty)} + 1\right)$$

The right hand side is 0 if $\pi(\infty) = 0$, is 1 if $\pi(\infty) = \infty$ and is increasing in $\pi(\infty)$. Thus, for each $\mu \in [0, 1)$, there exists a finite $\pi(\infty)$ that satisfies the expectation constraint. □

Proof of Lemma 1.7.1. The value of the profit-minimizer's problem is bounded above by $N\mu$. The degenerate distribution at the value $v = \mu$ is an interior point of the set of feasible primal variables. Thus, there is no duality gap between the primal and dual profit-minimzier's problem by Theorem 3.12 of Anderson and Nash (1987)

Next, we show that the complementary slackness condition characterizes an optimal solution. Suppose that G and (A, X) are feasible for the primal and the dual, respectively and they satisfy the complementary slackness condition. From the complementary slackness condition,

$$\int \sum_{i=1}^{N} t_i(s) dG(s) = -X + NA\mu$$

The integrals on the right hand side were simplified using the primal constraints. Next, we show that G is an optimal solution. Consider a feasible primal variable $\tilde{G}$. Integrate the dual constraint with respect to $\tilde{G}$ to obtain

$$\int \sum_{i=1}^{N} t_i(s) d\tilde{G}(s) \geq -X + NA\mu$$

Again, we simplified the integrals on the right hand side using the primal constraints. Therefore, for any feasible $\tilde{G}$, G is an optimal solution for the primal problem. Using a mirror argument, (A, X) is an optimal solution for the dual problem given the assumptions.

Next, suppose that A, X and G are optimal solutions to the dual and primal problem. Integrating the dual constraint yields

$$\int \sum_{i} t_i(s) dG \geq -X + NA\mu$$

The left-hand side is the primal value and the right-hand side is the dual value, which are equal. Thus, we obtain the complementary slackness condition. □

Proof of Proposition 1.7.1. We apply the same strategy to solve Maximin Indifference with any number of agents. Suppose that the monopolist has N symmetric agents. We start from finding the marginal allocation rule $p_1(s) := \frac{\partial p}{\partial s_1}(s)$ for special cases. Differentiate Maximin Indifference with respect to s_1. For each s such that $A\sum_{i=1}^{N} s_i - X \geq 0$,

$$A = \sum_{i=1}^{N} s_i p_1(s) - \sum_{i=2}^{N} \int_{(X/A-\sum_{j\neq i} s_j)^+}^{s_i} p_1(t, s_{-i})dt \tag{1.18}$$

where we used $p(X/A - \sum_{j\neq 1} s_j, s_{-1}) = 0$ since the signal sums up to X/A. p_1^* is again found for some special cases, exploiting symmetry.

$$p_1^*(s) = \begin{cases} \frac{A^N}{X^{N-1}} \left(\sum_{i=1}^{N} s_i\right)^{N-2} & s_1 < \frac{X}{A}, \quad \sum_{j\neq i} s_j \geq \frac{X}{A} \text{ for each } i \neq 1, \\ \frac{A}{s_1} & s_1 \geq \frac{X}{A} \end{cases} \tag{1.19}$$

We solve for the rest of p_1^* using the same strategy. Let $p_{12}^*(s) := \frac{\partial^2 p^*}{\partial s_1 \partial s_2}(s)$ denote the cross-marginal allocation rule with respect to s_1 and s_2. Differentiate (1.18) with respect to s_2 yields an integral equation for p_{12}^*.

$$\sum_{i=3}^{N} \mathbb{1}\left\{\frac{X}{A} - \sum_{j\neq i} s_j \geq 0\right\} p_1^*\left(\frac{X}{A} - \sum_{j\neq i} s_j, s_{-j}\right) = \sum_{i=1}^{N} s_i p_{12}^*(s) - \sum_{i=3}^{N} \int_{(X/A-\sum_{j\neq i} s_j)^+}^{s_i} p_{12}^*(\tau, s_{-j})d\tau \tag{1.20}$$

The left-hand side of (1.20) is either 0 or a positive constant. For each i such that $X/A - \sum_{j\neq i} s_j \geq 0$, $p_1^*(X/A - \sum_{j\neq i} s_j, s_{-j}) = \frac{A^2}{X}$ from (1.19) since the signal sums up to $\frac{X}{A}$. Thus, the left-hand side of the above equation is a non-negative constant.

$$\sum_{i=3}^{N} \mathbb{1}\left\{\frac{X}{A} - \sum_{j\neq i} s_j \geq 0\right\} p_1^* \left(\frac{X}{A} - \sum_{j\neq i} s_j, s_{-j}\right) = \sum_{i=3}^{N} \mathbb{1}\left\{\frac{X}{A} - \sum_{j\neq i} s_j \geq 0\right\} \frac{A^2}{X}$$

Each differentiation of (1.18) yields the cross partial derivative of p^* evaluated at $(X/A - \sum_{j\neq i} s_j, s_{-j})$ if $X/A - \sum_{j\neq i} s_j \geq 0$. Since the signal sums up to $\frac{X}{A}$, we see that the cross partial derivatives are all constant from (1.19). Thus, the constant term disappears after another differentiation.

After differentiating the equation with respect to $s_1, \cdots, s_{N-1}$, we obtain an ordinary differential equation for $p^*_{-N} := \frac{\partial^{N-1}}{\partial s_1 \cdots \partial s_{N-1}} p^*$. Again, the left-hand side is a constant.

$$\mathbb{1}\left\{\frac{X}{A} - \sum_{j\neq N} s_j \geq 0\right\} \frac{A^{N-1}}{X^{N-2}} \frac{1}{(N-2)!} = \sum_{i=1}^{N} s_i p^*_{-N}(s) - \int_{(X/A - \sum_{j\neq N} s_j)^+}^{s_N} p^*_{-N}(\tau, s_{-j}) d\tau$$

Thus, p^*_{-N} is a non-negative constant. Then the allocation rule is recovered from integrating the cross partial derivatives over signals.

During the procedure, we found that any cross partial derivative of p^* is non-negative at any signal profile. Thus, we have found the maximin allocation rule p^* must be monotone in signals. Summarizing the discussion above, we obtain Proposition 1.7.1. □

Proof of Corollary 1.7.1. Suppose that the monopolist has two symmetric agents. We solve Maximin Indifference by finding the marginal allocation rule $p_1 := \frac{\partial p}{\partial s_1}$ and then integrating it to recover the allocation rule. Differentiate Maximin Indifference with respect to s_1 to yield an ordinary integral equation.

$$A = (s_1 + s_2)p_1(s) - \int_{(X/A - s_1)^+}^{s_2} p_1(s_1, \tau)d\tau \tag{1.21}$$

The condition that $p^*(s_1, X/A - s_1) = 0$ was used to simplify the expression. Consider (1.21) as an ordinary integral equation for $p_1^*(s_1, \cdot)$, holding s_1 fixed to obtain p_1^*.

$$p_1^*(s) = \begin{cases} \frac{A^2}{X} & s_1 < \frac{X}{A} \\ \frac{A}{s_1} & s_1 \geq \frac{X}{A} \end{cases}$$

It remains to integrate p_1^* to recover the allocation rule p^*. First, suppose that $s_1 < X/A$ and $s_2 < X/A$. Then Maximin Indifference implies $p^*(X/A - s_2, s_2) = 0$. Next, suppose that $s_1 > X/A$. Then the boundary condition needs $p^*(X/A, s_2)$, which is recovered from the first case. Next, suppose that $s_1 < X/A$ and $s_2 \geq X/A$. The boundary condition $p^*(0, s_2)$ is recovered from the second case by permuting the agents. Thus, we obtain the allocation rule as below. Consider $s_1 \geq s_2$ since the agents are symmetric. □

Proof of Corollary 1.7.2. Next, suppose that there are three agents. We take the same approach by differentiating Maximin Indifference to simplify the partial integral equation into an ordinary integral equation. Differentiate Maximin Indifference with respect to s_1 to obtain the equation below.

$$A = \sum_{i=1}^{3} s_i p_1(s) - \sum_{i=2}^{3} \int_{(X/A - \sum_{j \neq i} s_j)^+}^{s_i} p_1(t, s_{-i})dt \tag{1.22}$$

Though (1.22) is a partial integral equation, p_1^* could be found on some special cases where it has a symmetric shape.

$$p_1^*(s) = \begin{cases} \frac{A^3}{X^2} \sum_{i=1}^3 s_i & \sum_{i=1}^3 s_i \geq \frac{X}{A}, \frac{X}{A} - \sum_{j \neq 2} s_j > 0, \frac{X}{A} - \sum_{j \neq 3} s_j > 0 \\ \frac{A}{s_1} & s_1 \geq \frac{X}{A} \end{cases} \tag{1.23}$$

We recover the rest of p_1^* from finding $p_{12}^* := \frac{\partial^2 p^*}{\partial s_1 \partial s_2}$, differentiate (1.22) with respect to s_2 to obtain an ordinary integral equation for p_{12}^*.

$$p_1^* \left(s_1, s_2, \frac{X}{A} - \sum_{j \neq 3} s_j \right) \mathbb{1} \left\{ \frac{X}{A} - \sum_{j \neq 3} s_j > 0 \right\} = \sum_{i=1}^{3} s_i p_{12}^*(s) - \int_{(\frac{X}{A} - s_1 - s_2)^+}^{s_3} p_{12}^*(t, s_{-3}) dt \tag{1.24}$$

From (1.23), we see that the left-hand side of (1.24) is either 0 or a positive constant. Suppose $X/A - s_1 - s_2 > 0$, then $p_1^*(s_1, s_2, X/A - s_1 - s_2) = A^3/X^2$. Otherwise, the left-hand side is zero. Thus, we obtain an ordinary integral equation for $p_{12}^*(s)$.

$$p_{12}^*(s) = \begin{cases} 0 & X/A - s_1 - s_2 < 0 \\ \frac{A^3}{X^2} & X/A - s_1 - s_2 \geq 0 \end{cases}$$

Following similar steps, p_{13}^* is also identified. Then, p_1^* is found from integrating p_{12}^* and p_{13}^* along s_2 and s_3. □

We state the maximin allocation rule with three agents. Integrate (1.10) with respect to s_1 to obtain p^*. Without loss of generality, we consider signal profiles where $s_1 \leq s_2 \leq s_3$.

1. $s_1 \leq s_2 \leq s_3 \leq X/A$

(a) $X/A - s_1 - s_2 > 0,\ X/A - s_1 - s_3 > 0,\ X/A - s_2 - s_3 > 0$

$$p^*(s) = \frac{A^3 s_1^2}{2X^2} + \frac{A^3 s_1 s_2}{X^2} + \frac{A^3 s_1 s_3}{X^2} + \frac{A^3 s_2^2}{2X^2} + \frac{A^3 s_2 s_3}{X^2} + \frac{A^3 s_3^2}{2X^2} - \frac{A}{2} \quad (1.25)$$

(b) $X/A - s_1 - s_2 > 0,\ X/A - s_1 - s_3 > 0,\ X/A - s_2 - s_3 < 0$

$$p^*(s) = \frac{A^3 s_1^2}{2X^2} + \frac{A^3 s_1 s_2}{X^2} + \frac{A^3 s_1 s_3}{X^2} + \frac{A^2 s_2}{X} + \frac{A^2 s_3}{X} - A \quad (1.26)$$

(c) $X/A - s_1 - s_2 > 0,\ X/A - s_1 - s_3 < 0,\ X/A - s_2 - s_3 < 0$

$$p^*(s) = \frac{A^3 s_1 s_2}{X^2} - \frac{A^3 s_3^2}{2X^2} + \frac{A^2 s_1}{X} + \frac{A^2 s_2}{X} + \frac{2A^2 s_3}{X} - \frac{3A}{2} \quad (1.27)$$

(d) $X/A - s_1 - s_2 < 0,\ X/A - s_1 - s_3 < 0,\ X/A - s_2 - s_3 < 0$

$$p^*(s) = -\frac{A^3 s_1^2}{2X^2} - \frac{A^3 s_2^2}{2X^2} - \frac{A^3 s_3^2}{2X^2} + \frac{2A^2 s_1}{X} + \frac{2A^2 s_2}{X} + \frac{2A^2 s_3}{X} - 2A \quad (1.28)$$

2. $s_1 \leq s_2 \leq X/A \leq s_3$

(a) $X/A - s_1 - s_2 > 0,\ X/A - s_1 - s_3 < 0,\ X/A - s_2 - s_3 < 0$

$$p^*(s) = \frac{A^3 s_1 s_2}{X^2} + \frac{A^2 s_1}{X} + \frac{A^2 s_2}{X} + A \log\left(\frac{A s_3}{X}\right) \quad (1.29)$$

(b) $X/A - s_1 - s_2 < 0,\ X/A - s_1 - s_3 < 0,\ X/A - s_2 - s_3 < 0$

$$p^*(s) = -\frac{A^3 s_1^2}{2X^2} - \frac{A^3 s_2^2}{2X^2} + \frac{2A^2 s_1}{X} + \frac{2A^2 s_2}{X} + A \log\left(\frac{A s_3}{X}\right) - \frac{A}{2} \quad (1.30)$$

3. $s_1 \leq X/A \leq s_2 \leq s_3$

(a) $X/A - s_1 - s_2 < 0$, $X/A - s_1 - s_3 < 0$, $X/A - s_2 - s_3 < 0$

$$p^*(s) = -\frac{A^3 s_1^2}{2X^2} + \frac{2A^2 s_1}{X} + A\log\left(\frac{As_2}{X}\right) + A\log\left(\frac{As_3}{X}\right) + A \qquad (1.31)$$

4. $X/A \leq s_1 \leq s_2 \leq s_3$

(a) $X/A - s_1 - s_2 < 0$, $X/A - s_1 - s_3 < 0$, $X/A - s_2 - s_3 < 0$

$$p^*(s) = A\log\left(\frac{As_1}{X}\right) + A\log\left(\frac{As_2}{X}\right) + A\log\left(\frac{As_3}{X}\right) + \frac{5A}{2} \qquad (1.32)$$

CHAPTER 2

Informationally Robust Public Good Provision

2.1 Introduction

How should a monopolist sell a pure public good, which is non-excludable and non-rivalrous to a group of buyers once it is produced? One may resort to the classical Bayesian optimal mechanism as in Güth and Hellwig (1986), where the monopolist seller maximizes the expected profit from the sale. However, the Bayesian approach requires the seller to know the details of the buyers' information structure to optimize. Otherwise, the Bayesian recipe is not very useful from the seller's perspective. For example, consider a seller of an innovation.[1] He may only have a proposal of the final product to show at the time of sale, which makes it hard for him to have detailed information about its appeal to the buyers. Even if the seller has a complete product, he does not want to let the buyers examine the product before purchase, which may leak the knowledge and eliminate the seller's bargaining position. For another example, consider a seller who needs a boilerplate contract for public good sales for repeated use. It is desirable for the template to depend on a few parameters and perform reasonably well across various information structures.

[1] Though the patent system helps innovators to keep their works exclusive, its coverage is far from perfect. Patent laws require innovations to satisfy several properties to be patentable.

Then, how should the monopolist sell a public good when he does not have a clear belief on the buyers' information structure? This paper considers a minimalist approach, constructing a strong maximin solution due to Brooks and Du (2020) when the monopolist knows N, the number of buyers and μ, the expected value each buyer puts on the public good and the range of valuations, but any other details of the true information structure is not available to him.

This paper explicitly constructs a robustly optimal mechanism of public good sale that does not require detailed knowledge about the true information structure and is simple enough to be practical. It maximizes the profit guarantee, the worst-case expected profit from sales across any information structure that is consistent with the monopolist's knowledge and equilibrium played in the mechanism.

In the optimal mechanism, each buyer reports her quantity demanded to the monopolist, which is aggregated into the amount of the public good provided. The optimal allocation rule qualitatively depends on the two parameters, N and μ, which represents the intensity of free-riding and the monopolist's confidence in his product, respectively. First, if the monopolist is sufficiently confident, (μ is sufficiently high given N) the monopolist fully shuts down the possibility to free-ride by having the smallest demand determine the supply of public good. In other words, each buyer cannot consume more than she demands for herself. Second, if the monopolist is not sufficiently confident, (μ is not sufficiently high given N) then he must condone the possibility of free-riding since there might be one buyer whose demand is lower than that of others. Each buyer is allowed to demand a fraction of the full amount, and the public good is provided by the total demand reports. In both cases, the maximin allocation rule is implemented via a transfer rule that satisfies an indifference condition, which gives a number of worst-case information structures to the profit-minimizing information designer. The condition admits a number of different transfer rules.

This paper considers the standard public good provision setup where buyers have private information about their willingness to pay for the public good. They have quasi-linear preferences for the public good and a numeraire. The buyers observe a private signal about the value of the public good according to an information structure consistent with the monopolist's knowledge. Compared with the setup of Chapter 1, we allow value interdependence, on which buyers learn from signals of other buyers, in Chapter 2. The monopolist chooses a sales mechanism to maximize profit, but has limited knowledge about the buyers' information structure. He only knows that each buyer's expected valuation of the public good is μ and the actual valuation is between 0 and 1. The monopolist evaluates a mechanism by its worst-case expected profit, across any information structure that is consistent with his knowledge and the Bayes-Nash equilibrium played in the public good provision game defined by the mechanism and the worst-case information structure.

This paper solves the max2min problem of public good provision, where the monopolist evaluates a mechanism by its profit guarantee, the worst-case expected profit across any information structure that is consistent with his knowledge and any Bayes-Nash equilibrium played on the public good provision game defined by the pair of a mechanism and an information structure. Following Brooks and Du (2020), we leverage on the restricted max2min and min2max problems to obtain a necessary condition for a strong maximin solution. The restricted problems are obtained by relaxing the second mover's problem in the max2min and the min2max problems. Thus, the restricted problems provide an upper and a lower bound for the max2min and the min2max problems. They provide a necessary condition for a strong maximin solution, but it is hard to explicitly find an intuitive solution.

We consider an equivalent representation of the restricted minimax problem, which is conducive to discovery of essential features of the maximin allocation rule. The restricted minimax problem finds an information structure, which is a joint distribution over signals and values, that minimizes expected profit of an optimal Bayesian mechanism. Since the choice of an information structure is only subject to the monopolist's knowledge about the true information structure, which is concerned with the expected value of each buyer, it is equivalent to choosing an interim value function, which is the buyer's expected valuation of the public good given a signal profile, where the ex-ante expectation coincides with the given expected value μ and the interim values are between 0 and 1 at any signal profile.

The restricted min2max problem of choosing an interim value yields its dual problem, where an allocation rule and the shadow value on the constraint that the interim value is at most 1. We show that this dual problem implies the max2min allocation rule to be either *the veto* or *the capped proportional* rule. In addition, the equivalent dual problem is reducible to choosing a couple of parameters. The complementary slackness condition ensures that the min2max interim value function exists and is pinned down by the same set of parameters. Thus, we do not need to explicitly find the min2max information structure to certify optimality of the maximin mechanism.

This paper is closely related to a recent literature on the strong maximin mechanisms, pioneered by Brooks and Du (2020). They show that the proportional auction is a maximin mechanism for common-value private good sales, in which each bidder submits quantity demanded and wins the good proportional to own demand report relative to the total reports. A follow up paper Brooks and Du (2021a) studies a general structure of the strong maximin problem using an approximation argument.

It also suggests the restricted problems, which this paper exploits to obtain its main results. Next, Brooks and Du (2021b) shows that the proportional auction continues to be a strong maximin solution if the auctioneer only knows the expected valuation of each bidder.

This paper contributes to this series of papers by providing a set of strong maximin solutions in the public good provision situation, providing intuitive solutions. Unlike the strong maximin solution for private good sales, we find that the maximin mechanism qualitatively depends on the common expected value. In our setup, the proportional mechanism is no longer optimal, instead it must put *a cap* on individual demand to curb free-riding. The veto mechanism provides a novel way of selling public goods when sellers are confident about quality. Additionally, this paper considers the equivalent representation of the restricted min2max problem, deducing the max2min allocation rule. This consideration may help the search of a strong maximin solution in other problems.

The rest of this paper consists of the following sections. We first introduce the setup where there is a monopolist seller of a public good and a number of buyers. We then discuss the construction of the restricted maximin problem and a characterization of its solutions. Then we use the characterization for the restricted maximin solution to obtain a strong maximin solution.

2.2 Model

Consider the sale of a public good by a profit-maximizing monopolist seller, who designs a sales mechanism. The public good is divisible and could be produced up to one unit at no cost. There are N buyers who could non-exclusively and nonrivalrously enjoy the public good. Each buyer either finds the public good useful, $v_i = 1$ or useless, $v_i = 0$. Let $V = \{0, 1\}^N$ denote the valuation space. The designer knows that each buyer's value of the public good is in expectation $E[v_i]$ is $\mu \in (0, 1)$, but other details of the true information structure are not available to the designer. The buyers have preferences over the amount of public good provided q and money transfer t_i. Their preferences are quasi-linear with respect to money transfer, represented by a utility function $v_i q - t_i$. The designer wants to maximize profit from trade. His preferences is represented by $\sum_{i=1}^{N} t_i$.

An information structure $\mathcal{I} = (S, w, g)$ describes the distribution of buyers' signals and values, where $S = \prod_{i=1}^{N} S_i$ is a measurable signal space, $w : S \to [0, 1]^N$ is an interim expected value of agents, which determines the buyers' expected valuation of the public good given a profile of signals and G is the signal distribution. We assume that the information structure must be consistent with the designer's knowledge. That is, for each agent i,

$$\int_S w_i(s) G(ds) = \mu$$

Equivalently, an information structure could be defined by a tuple (S, π), where $\pi \in \Delta(V \times S)$ whose marginal over S is G and

$$\int_{s \in B} \sum_{v} v_i \pi(ds, v) = \mu$$

for any measurable subset B of S and each agent i. Denote the set of information structures which are consistent with the designer's knowledge by $\mathbf{I}$

The mechanism designer chooses a mechanism $\mathcal{M} = (\prod_{i=1}^{N} M_i, q, t)$, where M_i

is a measurable message space, $q : \prod_{i=1}^{N} M_i \to [0,1]$ is an allocation rule, and $t : \prod_{i=1}^{N} M_i \to \mathbb{R}^N$ is a transfer rule. Assume that the allocation and transfer rule are measurable functions. In addition, the mechanism must have an opt-out message m_i^0 for each agent i where $t_i(m_i^0, m_{-i}) = 0$ for any m_{-i}. Denote the set of feasible mechanisms by $\mathbf{M}$

A pair of an information structure and a mechanism $(\mathcal{M}, \mathcal{I})$ defines a game of incomplete information. Agent i's strategy is a measurable function $\beta_i : S_i \to \Delta(M_i)$. A profile of strategies β is a function $\beta : S \to \Delta(M)$, where $\beta(s)$ is the product measure $\prod_{i=1}^{N} \beta_i(s_i)$. Let $U_i(\beta)$ denote agent i's expected payoff under the strategy profile β, where

$$U_i(\beta) = \int_{s \in S} \int_{m \in M} (w_i(s)q(s) - t_i(s))\, \beta(dm|s) G(ds).$$

A strategy profile β is a Bayes-Nash equilibrium if $U_i(\beta) \geq U_i(\beta_i', \beta_{-i})$ for any other strategy β_i' and for any agent i. Denote the set of BNEs by $\mathcal{B}(\mathcal{M}, \mathcal{I})$.

Let $\Pi(\mathcal{M}, \mathcal{I}, \beta)$ denote the expected profit of a mechanism $\mathcal{M}$ under an information structure $\mathcal{I}$ and a strategy profile β,

$$\Pi(\mathcal{M}, \mathcal{I}, \beta) = \int_{s \in S} \int_{m \in M} \sum_{i=1}^{N} t_i(m) \beta(dm|s) G(ds).$$

Following Brooks and Du (2020), we construct a strong maximin solution to identify a robustly profit-maximizing mechanism for the mechanism designer. A strong maximin solution consists of a mechanism, an information structure and an equilibrium $(\mathcal{M}^*, \mathcal{I}^*, \beta^*)$ that satisfies the following three conditions.

1. For all $\mathcal{M}' \in \mathbf{M}$ and $\beta' \in \mathcal{B}(\mathcal{M}', \mathcal{I}^*)$, $\Pi(\mathcal{M}^*, \mathcal{I}^*, \beta^*) \geq \Pi(\mathcal{M}', \mathcal{I}^*, \beta')$,

2. For all $\mathcal{I}' \in \mathbf{I}$ and $\beta' \in \mathcal{B}(\mathcal{M}^*, \mathcal{I}')$, $\Pi(\mathcal{M}^*, \mathcal{I}^*, \beta^*) \leq \Pi(\mathcal{M}^*, \mathcal{I}', \beta')$,

3. $\beta \in \mathcal{B}(\mathcal{M}^*, \mathcal{I}^*)$.

Condition 1 requires that the minimax information structure $\mathcal{I}^*$ limits expected profit of the designer across any choice of mechanism $\mathcal{M}' \in \mathbf{M}$ and any Bayes-Nash equilibrium $\beta' \in \mathcal{B}(\mathcal{M}', \mathcal{I}^*)$. Equivalently, the maximin mechanism $\mathcal{M}^*$ is an expected-profit maximizing mechanism under the minimax information structure $\mathcal{I}^*$. On the other hand, Condition 2 requires that the maximin mechanism $\mathcal{M}^*$ earns at least expected profit $\Pi(\mathcal{M}^*, \mathcal{I}^*, \beta^*)$ across any information structure $\mathcal{I}' \in \mathbf{I}$ and any Bayes-Nash equilibrium $\beta' \in \mathcal{B}(\mathcal{M}^*, \mathcal{I}')$. Likewise, the minimax information structure $\mathcal{I}^*$ is the expected-profit minimizing information structure given the maximin mechanism $\mathcal{M}^*$. Finally, it requires that the game of the maximin mechanism $\mathcal{M}^*$ and the minimax information structure $\mathcal{I}^*$ has a Bayes-Nash equilibrium $\beta^* \in \mathcal{B}(\mathcal{M}^*, \mathcal{I}^*)$.

If $(\mathcal{M}^*, \mathcal{I}^*, \beta^*)$ is a strong maximin solution, the maximin mechanism $\mathcal{M}^*$ solves the following max2min problem,

$$\sup_{\mathcal{M} \in \mathbf{M}} \inf_{\mathcal{I} \in \mathbf{I}} \inf_{\beta \in \mathcal{B}(\mathcal{M}, \mathcal{I})} \Pi(\mathcal{M}, \mathcal{I}, \beta). \tag{max2min}$$

The minimax information structure $\mathcal{I}^*$ solves the following min2max problem,

$$\inf_{\mathcal{I} \in \mathbf{I}} \sup_{\mathcal{M} \in \mathbf{M}} \sup_{\beta \in \mathcal{B}(\mathcal{M}, \mathcal{I})} \Pi(\mathcal{M}, \mathcal{I}, \beta). \tag{min2max}$$

In addition, the max2min and min2max problems have the same value if there is a strong maximin solution.

2.3 Restricted Maximin Problem

This section studies the restricted problems, where we obtain an educated guess for a strong maximin solution. Since the max2min problem is a complicated non-linear optimization problem, we consider the restricted max2min and min2max problems, where the second mover is subject to a weaker set of constraints, providing an upper and lower bound on the value of max2min and min2max problems. In addition, the restricted problems are a linear programming problem, which is easier to solve. Brooks and Du (2020) showed that the restricted problems are 'almost' dual to each other, yielding a necessary condition for a strong maximin solution. However, the necessary condition itself does not provide detailed information about the solution. Thus, we consider an equivalent representation of the restricted minimax problem, of which dual problem is conducive to more information about the maximin allocation rule and the minimax interim value. Namely, it implies a specific form for the maximin allocation and characterizes the minimax interim value up to a few parameters.

The *double revelation principle* (Brooks and Du (2021b)) provides a convenient normalization on the message and signal space to characterize a strong maximin solution. It consists of the following three parts.

1. The maximin mechanism is a direct mechanism on the minimax information structure.
2. The minimax information structure is a direct recommendation information structure on the maximin mechanism.
3. Buyers play a truth-telling equilibrium in the game of incomplete information defined by the maximin mechanism and the minimax information structure.

This allows us to use the same message and signal space. Furthermore, we suppose that each buyer i observes and reports a signal $s_i \in [0, \infty)$. That is, $M_i = S_i = \mathbb{R}_+ = [0, \infty)$ for each buyer i and assume that the signal follows the independent unitary exponential distribution. Buyer i's signal s_i is drawn from the cumulative distribution function $G(s_i) = 1 - e^{-s_i}$.

Even after the normalization on the message and signal spaces, it is hard to directly solve the max2min problem since the max2min is a non-linear optimization problem. Instead, as Brooks and Du (2020) suggested, we consider the restricted maximin and minimax problem to obtain an educated guess for a strong maximin solution, which is later evaluated against the three requirements. In the restricted problems, the second mover is subject to a smaller set of constraints. Simply put, all non-local constraints of the second mover are dropped in the restricted problems, providing an upper and lower bound for the max2min and min2max values.

We proceed with a discrete setup without dealing with mathematical subtleties of infinite dimensional optimization problems.[2] Later, the discreteness restriction will be relaxed. For each natural number k, define the signal and message space to be $X_i(k)$, which is defined as below.

$$X_i(k) = \left\{ \frac{l}{k} \mid 0 \leq l \leq k^2, l \in \mathbb{Z} \right\},$$

where $\mathbb{Z}$ is the set of integers, and let $X(k) = \prod_{i=1}^{N} X_i(k)$.

Let (q, t) denote a mechanism whose message space is $X(k)$, where each buyer reports a message $x_i \in X_i(k)$. $q : X(k) \to \mathbb{R}_+$ is an allocation rule and $t : X(k) \to \mathbb{R}$

[2]We follow the notations and definitions from Brooks and Du (2020).

is a transfer rule. Let $\pi \in \Delta(X(k) \times V)$ denote an information structure, where buyer i observes a signal $x_i \in X_i(k)$.

Given a function $f : X(k) \to \mathbb{R}^N$, let the discrete upward partial derivative $\nabla_i^+ f(x)$ defined by

$$\nabla_i^+ f(x) = \mathbb{1}_{x_i < k^2}(k-1)(f(x_i + 1/k, x_{-i}) - f(x))$$

Let $\nabla^+ f(x)$ denote the discrete upward gradient of f, $(\nabla_1^+ f(x), \cdots, \nabla_N^+ f(x))$. The discrete upward divergence is denoted by $\nabla^+ \cdot f(x) = \sum_{i=1}^N \nabla_i^+ f(x)$. Also, let

$$g_i(x_i) = \left(1 - \frac{1}{k}\right)^{kx_i} \frac{1}{k^{\mathbb{1}_{x_i < k^2}}}$$

denote the censored geometric distribution on $X_i(k)$ with arrival rate $\frac{1}{k}$ and let $g(x)$ denote the joint distribution on $X(k)$.

First, we describe $\underline{\Pi}^{Max-2Min}(k)$, the restricted maximin problem. Given a mechanism (q, t), the second mover's problem finds an expected profit minimizing Bayes correlated equilibrium (BCE), where an information structure π is chosen subject to the consistency and obedience constraint. The consistency constraint dictates that information structures must be consistent with the monopolist's knowledge, inducing the expected valuation of μ for each buyer. The obedience constraint requires that it is each buyer's best interest to obey the signal they observed. In the restricted problem, we keep only locally upward obedience constraints and drop every others.

Since the BCE problem is a linear programming, it is then jointly solved with its dual problem, where the dual variables are the shadow values for the consistency and obedience constraint of the primal problem. In addition, we restrict the dual variable on the local obedience constraint to be 1. Then the restriction yields a maximization problem where a mechanism and a dual variable for the consistency constraints are chosen, subject to the dual constraint of the dual BCE problem. We obtain the restricted max2min problem below.

$$\begin{aligned}\underline{\Pi}^{Max-2Min}(k) = \max_{q,t,\lambda(1),\lambda(0)} \quad & \sum_{i=1}^{N} (\lambda(1)\mu + \lambda(0)(1-\mu)) \\ \text{s.t. } & q(m) \leq 1 \\ & t_i(0, m_{-i}) = 0 \\ & \sum_{i=1}^{N} \lambda(v_i) \leq \sum t(x) + v \cdot \nabla^{+} q(x) - \nabla^{+} \cdot t(x)\end{aligned} \tag{2.1}$$

Second, we describe $\bar{\Pi}^{Min-2Max}(k)$, the restricted minimax problem. Given an information structure σ, the second mover's problem finds a Bayesian optimal mechanism. The revelation principle ensures that it is without loss to choose an optimal direct mechanism, subject to the incentive compatibility and individual rationality constraints. In the restricted problem, we keep only locally downward incentive compatibility constraints and the individual rationality constraints of the buyers with the smallest signal. Again, the Bayesian mechanism design problem is a linear programming, which is jointly solved with the dual problem. We normalize the shadow value on the incentive constraints to be 1 to obtain the restricted min2max problem below.

$$
\begin{aligned}
\bar{\Pi}^{Min-2Max}(k) = \min_{\gamma,\sigma,w} \quad & \sum_{x \in X(k)} \gamma(x) \\
\text{s.t. } \mu = & \sum_{x \in X(k), v_{-i}} \sigma((1, v_{-i}), x) \\
& g(x) = \sum_{v} \sigma(v, x) \\
& w_i(x) = \frac{1}{g(x)} \sum_{v} v_i \sigma(v, x) \\
& \gamma(x) \geq g(x) \sum_{i=1}^{N} \left(w_i(x) - \nabla_i^{+} w_i(x) \right) \\
& \gamma(x) \geq 0 \quad \sigma(v, x) \geq 0
\end{aligned}
$$

Theorem 1[3] of Brooks and Du (2020) shows that the restricted min2max and max2min problems are 'almost dual' to each other, in the sense that they have the same value as k grows to infinity and weak duality holds for any k.

$$
\bar{\Pi}^{Min-2Max}(k) \geq \underline{\Pi}^{Max-2Min}(k)
$$

$$
\lim_{k \to \infty} \underline{\Pi}^{Max-2Min}(k) = \lim_{k \to \infty} \bar{\Pi}^{Min-2Max}(k) = \Pi^*
$$

In addition, the theorem proves that Π^* is equal to the value of the unrestricted max2min and min2max problems.

The following complementary slackness condition characterizes solutions of the restricted max2min and min2max problems in the limit, which provides us with an

[3]They consider the single unit auction setup, but only slight modification takes the theorem to the current setup.

educated guess for a strong maximin solution. Both restricted problems consider local obedience and incentive compatibility constraints on the opposite direction, which becomes immaterial in the limit as k grows.

$$\begin{aligned} 0 &= q(m)\left(\gamma(m) - g(m)\sum_{i=1}^{N}\left(w_i(m) - \frac{\partial w_i}{\partial m_i}(m)\right)\right) \\ 0 &= \gamma(m)(1-q(m)) \\ 0 &= \int_M \sigma(v,m)\left[\sum_i\left(\frac{\partial t_i}{\partial m_i}(m) - t_i(m)\right) + \sum_i \lambda_i(v_i) - \sum_{i=1}^{N} v_i \frac{\partial q}{\partial m_i}(m)\right] dm \end{aligned} \tag{2.2}$$

The complementary slackness condition could be understood through the indifference intuition from zero-sum games. Namely, the first mover in the restricted max2min and min2max problems make the second mover indifferent across their choice. In the restricted minimax problem, the information designer gives a number of optimal mechanisms to the mechanism designer, which is illutrated by the first and second line of (2.2). On the other hand, in the restricted maximin problem, the mechanism designer, who is the first mover, gives a number of worst-case information structure to the information designer by choosing a mechanism that satisfies the second and third line of (2.2).

However, the complementary slackness condition has multiple solutions, which means that it is easy to verify whether a mechanism and an information structure satisfy the condition, but it is hard to deduce a solution from the condition. Instead of making a guess out of blue, we consider an equivalent representation of the restricted min2max problem, where the dual program provides more information about the maximin and minimax solutions.

A consistent information structure σ implies an interim value function w that satisfies the expectation constraint and has value between 0 and 1. Given an interim value function w that satisfies the expectation constraint, one can find a consistent information structure σ. Thus, we may consider the restricted min2max by choosing an interim value function, instead of an information structure σ. Consider the equivalent representation of the restricted min2max problem below.

$$
\begin{aligned}
\bar{\Pi}^{Min-2max*}(k) = \min_{\gamma, w} \quad & \sum_{x \in X(k)} \gamma(x) \\
\text{s.t. } & \gamma(x) \geq g(x) \sum_{i=1}^{N} \left[w_i(x) - \nabla_i^+ w_i(x) \right] && (q(x)) \\
& 1 \geq w_i(x) && (\tau_i(x)) \\
& \mu = \sum_{s} w_i(x) g(x) && (\eta) \\
& w_i(x) \geq 0
\end{aligned}
$$

Note that the expectation constraint will still bind if $\mu \leq \sum_s w_i(x)g(x)$ since we may lower the expected profit by choosing an interim value function with lower expectations. Thus, we suppose that the dual variable η must be non-negative in the dual problem. Again, consider the dual of the equivalent representation of the restricted min2max problem.

$$\bar{\Pi}_D^{Min-2max*}(k) = \max_{q,\tau,\eta} \quad N\mu\eta - \sum_{i=1}^{N} \sum_{x \in X(k)} \tau_i(x) g(x)$$

$$\text{s.t. } 1 \geq q(x)$$

$$\nabla_i^- q(x) \geq \eta - \tau_i(x)$$

$$\nabla_i^- q(x) = \begin{cases} Kq(x) & x_i = 0 \\ K(q(x) - q(x_i - 1/k, x_{-i})) & 0 < x_i < k \\ q(x) - q(x_i - 1/k, x_{-i}) & x_i = K \end{cases}$$

$$q(x) \geq 0 \quad \tau_i(x) \geq 0 \quad \eta \geq 0$$

The dual of the equivalent min2max problem $\bar{\Pi}_D^{Min-2max*}(k)$ chooses an allocation rule q, which is the multiplier on the designer's indifference constraint. Though the restricted minimax problem is not exactly dual to the restricted maximin problem, the dual variable q must be a well-defined allocation rule, as found from the dual problem. The dual problem also chooses τ_i, which is the multiplier on the upper bound of the interim value and η, which is the multiplier on the consistency constraint.

As will be shown in Proposition 2.3.1, the dual of the equivalent min2max problem implies an intuitive allocation rule, which is determined by only two parameters η and m^*. First, define the *veto allocation rule* as below.

$$q(x) = \min\{1, \eta \min\{x_i, m^*\}\}. \tag{2.3}$$

In this allocation rule, the monopolist fully shuts down any free-riding by supplying the public good at the smallest demand. Each buyer can enjoy the public good at most the amount they demanded themselves. The optimal parameters m^* and η maximizes

$$\begin{aligned} &\max_{\eta, m^*} \quad \eta \left(N\mu - \sum_{i=1}^{N} \sum_{x_{-i}} (1 - G(\min\{x_{-i}, m^*\}))g(x_{-i}) \right), \\ &\text{s.t. } 1 = m^*\eta. \end{aligned} \tag{2.4}$$

Next, define the *capped proportional allocation rule* as below.

$$q(x) = \min\{1, \eta \sum_{i=1}^{N} \min\{x_i, m^*\}\}. \tag{2.5}$$

In the capped proportional allocation rule, buyers can demand up to some portion m^* of the public good and the public good is supplied by the total demand. It balances between free-riding prevention and hedging against a low demanded buyer from whom the veto allocation rule is substantially affected. The optimal parameters m^* and η maximizes

$$\max_{\eta, m^*} \quad \eta \left(N\mu - \sum_{i=1}^{N} \sum_{x_{-i}} (1 - G(\min\{m^*, 1/\eta - \sum_{j \neq i} \min\{x_j, m^*\}\}))g(x_{-i}) \right). \tag{2.6}$$

We prove that the dual equivalent min2max problem $\bar{\Pi}_D^{Min-2max*}(k)$ is solved either by the veto allocation rule or the capped proportional allocation rule in Proposition 2.3.1 below.

Proposition 2.3.1. *The dual of the equivalent min2max problem* $\bar{\Pi}_D^{Min-2max*}(k)$ *is solved by either*

1. The veto allocation rule (2.3) *with* (η, m^*) *that solves* (2.4) *and*

$$\tau_i(x) = \eta \mathbb{1}\{x_i \geq \min\{m^*, \min x_{-i}\}\}$$

2. The capped proportional allocation rule (2.5) *with* (η, m^*) *that solves* (2.6) *and*

$$\tau_i(x) = \eta \mathbb{1}\left\{x_i \geq \min\left\{m^*, 1/\eta - \sum_{j \neq i} \min\{m^*, x_j\}\right\}\right\}$$

Proof of Proposition 2.3.1. We start with identifying the set of binding inequality constraints in the dual problem $\bar{\Pi}_D^{Min-2max*}(k)$. In an optimal solution, the inequality constraint

$$\nabla_i q(x) \geq \eta - \tau_i(x)$$

must bind with an equality for any $x \in X(k)$ for each i. Suppose otherwise that the constraint does not bind at some $x' \in X(k)$ for some i. Then we may lower $\tau_i(x)$ (if it is positive), increasing the objective function without violating the constraints. Otherwise, we may raise η and lower $\nabla_i q(x)$ without violating the supply constraint. Thus, for each i and $x \in X(k)$ the inequality constraint binds at an optimal solution,

$$\nabla_i q(x) + \tau_i(x) = \eta.$$

Since we consider a censored geometric distribution for the signal distribution, it is less costly to have τ_i at higher signal profiles than smaller profiles. Moreover, since it is a linear programming, at an optimal solution a variable is either 0 or at its maximum level that satisfies the constraints. Thus, for each i, there exists a cutoff

$f(x_{-i})$ that determines an optimal dual solution.

$$\tau_i(x) = \eta \mathbb{1}\{x_i > f(x_{-i})\}$$
$$\nabla_i q(x) = \eta \mathbb{1}\{x_i \leq f(x_{-i})\}$$

Existence of the cutoff f and piecewise constant slope of q dictate a particular shape of an optimal allocation rule q. It takes either of the two forms, parameterized by m^*, which is the highest message that affects the allocation rule.

$$q(x) = \begin{cases} \min\{1, \eta \min\{x_i, m^*\}\} \\ \min\{1, \eta \sum_{i=1}^N \min\{x_i, m^*\}\} \end{cases}$$

□

Thus, the dual equivalent restricted min2max problem implies that the maximin mechanism must have either the veto allocation rule or the capped proportional allocation rule. Next, strong duality between the equivalent restricted min2max problem and its dual provides a sharper characterization of the minimax information structure, as shown in Proposition 2.3.2.

Proposition 2.3.2. *The minimax interim value w^* is parameterized by m^* and η.*

Proof of Proposition 2.3.2. The solution of the equivalent restricted minimax problem are characterized by the complementary slackness condition and the constraints.

$$
\begin{aligned}
0 &= \gamma(x)(1 - q(x)) \\
0 &= w_i(x)(\nabla_i^- q(x) - \lambda + \tau_i(x)) \\
0 &= q(x)(\gamma(x) - g(x) \sum_{i=1}^{N} \big[w_i(x) - \nabla_i^+ w_i(x) \big]) \\
0 &= \tau_i(x)(1 - w_i(x)) \\
\mu &= \sum_{x \in X(k)} w_i(x) g(x) \\
w_i(x) \geq 0, &\quad q(x) \geq 0, \quad \gamma(x) \geq 0, \quad \tau_i(x) \geq 0
\end{aligned}
$$

Proposition 2.3.1 pins down the primary variables γ and w_i as below. Given the veto allocation rule,

$$
\begin{aligned}
\gamma(x) &= N \mathbb{1}\{\min x \geq m^*\} g(x), \\
\gamma(x) &= g(x) \sum_{i=1}^{N} \big[w_i(x) - \nabla_i^+ w_i(x) \big]).
\end{aligned}
$$

Given the capped proportional allocation rule,

$$
\begin{aligned}
\gamma(x) &= N \mathbb{1} \left\{ \eta \sum_{i=1}^{N} \min\{m^*, x_i\} \geq 1 \right\} g(x), \\
\gamma(x) &= g(x) \sum_{i=1}^{N} \big[w_i(x) - \nabla_i^+ w_i(x) \big]).
\end{aligned}
$$

Thus, the minimax interim value w^* is parameterized by m^* and η. □

In the next section, we finish constructing an educated guess for a strong maximin solution using the complementary slackness conditions and Proposition 2.3.1 and 2.3.2. We then check if the pair satisfies the three requirements of the strong maximin solution to finish the characterization of a strong maximin solution.

In the next section, we characterize a strong maximin solution for the public good provision problem. The maximin mechanism qualitatively depends on the expected value μ and the number of buyers N, which are related to the mechanism designer's confidence on his product and the intensity of free-riding incentives, respectively. Lemma 2.4.1 shows that there is a threshold $\bar{\mu}(N)$ that determines the shape of the maximin mechanism.

2.4 Veto Mechanism

This subsection presents the veto mechanism and the veto information structure. Suppose that $\mu \geq \bar{\mu}(N)$, where the mechanism designer is sufficiently confident in the quality of the public good. Knowing this, the designer could fully shut down any possibility of free-riding by supplying the public good by the buyer with the lowest demand. In other words, no buyer can enjoy the amount of the public good she did not demand for herself. This allocation rule is sensitively affected by a buyer with a low demand. However, the monopolist is willing to take the risk since he is sufficiently confident with the public good.

2.4.1 Minimax Information Structure

We start with the veto information structure $\mathcal{I}^* = (S^*, G^*, w^*)$. In the veto information structure, each buyer privately observes a positive real numbered signal, which is independently distributed by the exponential distribution with arrival rate 1. However, any individual signal above a threshold m^* does not affect the interim value. It exhibits an interdependent valuation, where all buyers except for the smallest signal buyer have the interim valuation of 1 about the public good, but the smallest signal buyer's interim valuation is less than 1 and depends on the difference between her own signal and the minimal signal among others.

$$
\begin{aligned}
S_i^* &= [0, \infty), \\
G^*(s_i) &= 1 - e^{-s_i}, \\
w_i^*(s) &= \min\{1, 1 + N(e^{s_i - \min\{m^*, \min_{j \neq i} s_j\}} - 1)\},
\end{aligned}
$$

where m^* is determined by the expectation constraint $\mu = \int_S w_i(s) dG(s)$. In other words, m^* must satisfy (2.7).

$$
\mu = 1 - \frac{1}{N} + N(m^* + 1)e^{-m^* N} - \frac{N-1}{N} e^{-m^* N}(1 + N + Nm^*). \tag{2.7}
$$

Under the veto information structure, the mechanism designer has a variety of Bayesian optimal mechanism since it ensures that its virtual values add up to zero, at any signal profile except at the highest profile $(m^*, \cdots, m^*)$. Since an optimal Bayesian mechanism sells the full amount if and only if the total virtual value is positive, any mechanism (q, t) whose allocation rule q sells the full amount at $(m^*, \cdots, m^*)$ is optimal, earning $Ne^{-m^* N}$.

$$
\varphi_i^*(s) = w_i^*(s) - \frac{1 - G^*(s_i)}{g^*(s_i)} \frac{\partial w_i^*}{\partial s_i}(s) = \begin{cases} -(N-1), & s_i < \min_{j \neq i} s_j \\ 1. & s_i \geq \min_{j \neq i} s_j \end{cases}
$$

Specifically, all buyer except for the smallest signal buyer has the expected value 1, which gives them the virtual value of 1. On the other hand, the smallest signal buyer has the virtual value of $-(N-1)$, offsetting the total virtual values of everyone else. Thus, the veto information structure satisfies (2.2), the complementary slackness condition of the restricted maximin problem.

Note that any signal higher than m^* obtains the same interim value as m^* in the veto information structure. Thus, we obtain an equivalent representation of the veto information structure where the signal space is a finite closed interval and the signal distribution follows the exponential distribution censored at m^* as below.

$$S_i^* = [0, m^*],$$
$$G^*(s_i) = \begin{cases} 1 - e^{-s_i} & s_i < m^*, \\ 1 & s_i = m^*. \end{cases}$$

Later, we use the second representation as it frees us from the mathematical subtlety of dealing with an infinite signal space. In the second representation, m^* determines the upper end of the signal space.

For the veto information structure to be well-defined, its expected valuation w^* must be between 0 and 1 at any signal profile on S^*. Lemma 2.4.1 identifies a cutoff $\bar{\mu}(N)$ on the common expected value.

Lemma 2.4.1. *For each number of buyers N, the veto information structure is well defined only if $\mu \geq \bar{\mu}(N)$, where the cutoff $\bar{\mu}(N)$ is defined by (2.8),*

$$\bar{\mu}(N) = \frac{1}{N}\left[N + \left(\frac{N-1}{N}\right)^N \left(N \log\left(\frac{N}{N-1}\right) + 1\right) - 1\right]. \tag{2.8}$$

Proof of Lemma 2.4.1. Given μ, the expectation constraint pins down the upper end of the signal space m^*. At the same time, m^* must keep the value function w_i non-negative at any signal profile. The value function w_i is minimized at a signal profile $s_i = 0$ and $\min_{j \neq i} s_j = m^*$. Thus, we obtain an upper bound on m^* from such signal profiles.

$$w_i(0, m^*, \cdots, m^*) = 1 + N(e^{-m^*} - 1) \geq 0 \quad \rightarrow \quad m^* \leq \log\left(\frac{N}{N-1}\right).$$

Since the right hand side of the expectation constraint decreases in m^*, the expected value μ has to be at least $\bar{\mu}(N)$ to satisfy the expected value constraint,

$$\bar{\mu}(N) = \frac{1}{N}\left[N + \left(\frac{N}{N-1}\right)^{-N}\left(N \log\left(\frac{N}{N-1}\right) + 1\right) - 1\right].$$

Thus, $\mu \geq \bar{\mu}(N)$ ensures a well-defined value function in the veto information structure. □

Figure 2.1 illustrates the cutoff $\bar{\mu}(N)$ across N. The cutoff $\bar{\mu}(N)$ increases and converges to 1 as N grows to infinity. In other words, the veto information structure requires a higher common expected valuation μ with more buyers. To maintain the designer's indifference, the smallest signal buyer must have a steep increase in the interim valuation as her signal increases so that her negative virtual value compensates the total virtual value of others, which is $N-1$. Thus, with more buyers, the veto information structure must have a smaller m^*, which requires a larger μ.

Next, we construct the veto mechanism using the veto information structure and the complementary slackness conditions.

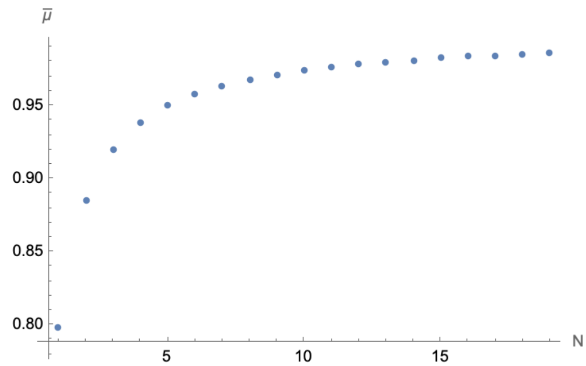

Figure 2.1: The cutoff $\bar{\mu}$ from $N = 2$ to $N = 20$

2.4.2 Maximin Mechanism

Now we construct the veto mechanism, leveraging on the properties of the veto information structure and the complementary slackness conditions. Let the message space $M_i^* = S_i^* = [0, m^*]$ coincide with the signal space S_i^* of the veto information structure. The mechanism designer is confident in the quality of his product and is willing to fully shut down room for free-riding behavior, which is implemented by the allocation rule that supplies the public good by the minimal reported message.

$$q^*(m) = \eta^* \min_{i \in N} \{m_i\},$$

where $\eta^* m^* = 1$. In this allocation rule, each buyer can enjoy the public good by the amount she demanded for herself, or equivalently, each buyer essentially refuses to buy any amount exceeding her own demand.

It remains to construct the veto transfer rule t^* that satisfies the complementary slackness condition (2.2). Given the veto allocation rule defined above, the transfer rule t_i^* must satisfy

$$\sum_i \left(\frac{\partial t_i^*}{\partial m_i}(m) - t_i^*(m) \right) = \Xi(m) = \begin{cases} \eta - N\lambda^*(1) & \min m < m^* \\ -N\lambda^*(1) & \min m = m^* \end{cases}, \tag{2.9}$$

for each $m \in M^*$. (2.9) places an aggregate requirement on the veto transfer rule t_i^*. However, we need to specify how individual excess growth (relative to the exponential level) of transfer, $\frac{\partial t_i^*}{\partial m_i}(m) - t_i^*(m)$ is distributed at each signal profile m. There are multiple transfer rules that satisfy the complementary slackness. We discuss two of such transfer rules below.

We identify an optimal $\lambda^*(1)$ and $\lambda^*(0)$ for the restricted maximin problem. At any message profile m, the primal inequality constraint binds both for $v_i = 0$ and $v_i = 1$ if buyer i's message is smallest among all buyers since buyer i's interim value is strictly less than 1 in the veto information structure. Thus, we obtain the difference between λ^* as

$$\lambda^*(0) = \lambda^*(1) - \eta^*.$$

$\lambda^*(1)$ is found from the strong duality in the dual and primal restricted maximin problem. Any optimal mechanism to the veto information structure earns Ne^{-Nm^*} since the virtual value sums up to N only when all buyers observe the highest signal m^* and to zero otherwise. Thus,

$$\lambda^*(1) = e^{-Nm^*} + \eta^* - \mu\eta^*.$$

Transfer rule 1: One natural transfer rule would charge only the buyer with the smallest demand report, whose report determines the amount of the public good provided in the veto allocation rule. Let $\underline{m} = \min m_i$ denote the smallest message of a message profile m and $\#(\underline{m})$ denote the number of buyers with the smallest message, in case there is a tie. Buyer i is charged

$$t_i^*(m) = \begin{cases} \frac{1}{\#(\underline{m})} T(\underline{m}) & m_i = \underline{m}, \\ 0 & m_i > \underline{m}, \end{cases} \tag{2.10}$$

where $T(\underline{m})$ is defined to be

$$T(\underline{m}) = \begin{cases} (\eta - N\lambda^*(1))(e^{\underline{m}} - 1) & \underline{m} < m^*, \\ N\lambda^*(1) & \underline{m} = m^*. \end{cases} \tag{2.11}$$

It is easy to verify that the above transfer rule satisfies the complementary slackness condition with the veto allocation rule.

Transfer Rule 2: The canonical transfer rule due to Brooks and Du (2021a) also satisfies the complementary slackness condition. Let Z denote the set of permutations on $\{1, \cdots, N\}$ with a typical element ζ. Denote the subset of agents whose order is less than k in ζ by

$$[\zeta \leq k] = \{j | \zeta(j) \leq k\}.$$

and analogously define $[\zeta > k]$. Let

$$\tau_{\zeta,k}(m) = \int_{[0,m^*]^{N-k}} \Xi(m_{[\zeta \leq k]}, x_{[\zeta > k]}) dG(x_{[\zeta > k]})$$

The function τ simplifies under the veto allocation rule. Namely, $\tau_{\zeta,k}$ is determined whether the smallest signal in $\min m_{[\zeta\leq k]} \geq m^*$ or $\min m_{[\zeta\leq k]} < m^*$.

$$\tau_{\zeta,k}(m) = \begin{cases} \eta^* - N\lambda^*(1) & \min m_{[\zeta\leq k]} < m^* \\ \eta^*(1 - e^{-m^*(N-k)}) - N\lambda^*(1) & \min m_{[\zeta\leq k]} \geq m^* \end{cases}$$

Then define the individual excess growth in transfer $\xi_i(m)$ as below.

$$\xi_i(m) = \frac{1}{N!} \sum_{\zeta\in Z} \left[\tau_{\zeta,\zeta(i)}(m) - \tau_{\zeta,\zeta(i)-1}(m)\right].$$

The excess growth ξ_i only depends on the number of buyers whose demand report exceeds m^* since

$$\tau_{\zeta,\zeta(i)}(m) - \tau_{\zeta,\zeta(i)-1}(m) = \begin{cases} 0 & m_{[\zeta\leq\zeta(i)-1]} < m^*, m_{[\zeta\leq\zeta(i)]} < m^* \\ e^{-m^*(N-\zeta(i)+1)} & m_{[\zeta\leq\zeta(i)-1]} \geq m^*, m_{[\zeta\leq\zeta(i)]} < m^* \\ \eta e^{-m^*(N-\zeta(i))}(e^{-m^*} - 1) & m_{[\zeta\leq\zeta(i)-1]} \geq m^*, m_{[\zeta\leq\zeta(i)]} \geq m^* \end{cases}$$

Due to the definition of ξ_i, the individual excess growth in transfer,

$$t_i(m) = e^{m_i} \int_0^{m_i} \xi_i(x_i, m_{-i}) e^{-x_i} dx_i.$$

This transfer rule satisfies the complementary slackness condition.

2.4.3 Strong Maximin Solution

Finally, an equilibrium $\beta^* \in \mathcal{B}(\mathcal{M}^*, \mathcal{I}^*)$ completes the statement of a strong maximin solution. On the veto mechanism and the veto information structure, agent i uses a truth-telling strategy,

$$\beta_i^*(s_i) = s_i.$$

Theorem 2.4.1. *Suppose that $\mu \geq \bar{\mu}(N)$. The triple $(\mathcal{M}^*, \mathcal{I}^*, \beta^*)$ that consists of the veto mechanism, the veto information structure and the truth-telling equilibrium is a strong maximin solution. Moreover, the profit guarantee of $\mathcal{M}^*$ is Π^*,*

$$\Pi^* = \Pi(\mathcal{M}^*, \mathcal{I}^*, \beta^*) = Ne^{-Nm^*},$$

where m^* is defined by (2.7).

Proof of Theorem 2.4.1. Theorem 2.4.1 follows from Proposition 2.4.1, 2.4.2 and 2.4.3. Each proposition checks that the triple $(\mathcal{M}^*, \mathcal{I}^*, \beta^*)$ satisfies the three requirements of the strong maximin solution. □

2.4.4 Proof of Theorem 2.4.1

Proposition 2.4.1. *For all $\mathcal{M}'$ and $\beta' \in \mathcal{B}(\mathcal{M}', \mathcal{I}^*)$, $\bar{\Pi} \geq \Pi(\mathcal{M}', \mathcal{I}^*, \beta')$.*

Proof of Proposition 2.4.1. We solve the Bayesian optimal mechanism design problem under the veto information structure to obtain an upper bound of $\Pi(\mathcal{M}', \mathcal{I}^*, \beta')$ across any mechanism $\mathcal{M}'$ and equilibrium $\beta' \in \mathcal{B}(\mathcal{M}', \mathcal{I}^*)$. Since the veto information structure is given, the revelation principle allows us to focus on direct mechanisms. Let (q, t) denote a direct mechanism under the veto information structure.

The Bayesian optimal mechanism design problem maximizes expected revenue

$$\int_S \sum_{i=1}^N t_i(s) dG(s)$$

subject to the Bayesian incentive compatibility and individual rationality constraints. The revenue equivalence formula is applicable under the veto information structure since the signal distribution is independent and the expected value function w_i is monotone in s_i.

$$\int_{S_{-i}} t_i(s_i, s_{-i})dG(s_{-i}) = \int_{S_{-i}} \left(w_i(s_i, s_{-i})q(s_i, s_{-i}) - \int_0^{s_i} \frac{\partial w_i}{\partial s_i}(\nu, s_{-i})q(\nu, s_{-i})d\nu \right) dG(s_{-i})$$

Following the standard procedure, expected revenue is expressed in terms of an allocation rule and the virtual value.

$$\int_S \sum_{i=1}^{N} t_i(s)dG(s) = \int_S \sum_{i=1}^{N} \left(w_i(s_i, s_{-i}) - \frac{\partial w_i}{\partial s_i}(s_i, s_{-i}) \right) q(s)dG(s)$$

Under the veto information structure, the virtual value of agents adds up to 0 if $\min s_i < m^*$ and N otherwise. Thus, the optimal direct mechanism earns

$$N \int_{\{s \in S | s_i > m^*, \forall i \in N\}} dG(s) = Ne^{-m^* N}$$

which is the revenue guarantee $\Pi(\mathcal{M}^*, \mathcal{I}^*, \beta^*)$ of the veto mechanism. $\square$

Proposition 2.4.2. *Suppose that $\mu \geq \bar{\mu}(N)$. For all $\mathcal{I}'$ and $\beta' \in \mathcal{B}(\mathcal{M}^*, \mathcal{I}')$, $\bar{\Pi} \leq \Pi(\mathcal{M}^*, \mathcal{I}', \beta')$. That is, the veto mechanism $\mathcal{M}^*$ earns at least Π as its expected profit across any information structure and equilibrium.*

Before proving Proposition 2.4.2, we prove a lemma, which is equivalent to local optimality of an equilibrium β.

Lemma 2.4.2. *Let $\mathcal{M}^*$ be the veto mechanism. Then for any information structure*

$\mathcal{I}$ and an equilibrium $\beta \in \mathcal{B}(\mathcal{M}^*, \mathcal{I})$,

$$\int_S \int_M \sum_{i=1}^{N} \left[w_i(s) \frac{\partial q}{\partial m_i}(m) - \frac{\partial t_i}{\partial m_i}(m) \right] \beta(dm|s) G(ds) \leq 0 \tag{2.12}$$

Proof of Lemma 2.4.2. Fix an information structure $\mathcal{I}$ and an equilibrium $\beta \in \mathcal{B}(\mathcal{M}^*, \mathcal{I})$. For each $\Delta > 0$,

$$\int_S \int_M \sum_{i=1}^{N} \left[w_i(s) \frac{q(m_i + \Delta, m_{-i}) - q(m)}{\Delta} - \frac{t_i(m_i + \Delta, m_{-i}) - t_i(m)}{\Delta} \right] \beta(dm|s) G(ds) \leq 0$$

since β is an equilibrium. Since $\frac{\partial q}{\partial m_i}$ and $\frac{\partial t_i}{\partial m_i}$ are bounded, the integrand is bounded for any Δ. Take the limit on $\delta \to 0$ and invoke the dominated convergence theorem to obtain the conclusion. □

Proof of Proposition 2.4.2. First, we show that

$$\sum_{i=1}^{N} \lambda(v_i) \leq \sum_{i=1}^{N} v_i \frac{\partial q}{\partial m_i}(m) - \Xi(m) \tag{2.13}$$

for each $v \in \{0, 1\}^N$ and $m \in M$. From the definition of Ξ, the right-hand side is equal to

$$\sum_{i=1}^{N} \left((v_i - 1) \frac{\partial q}{\partial m_i}(m) + \lambda(1) \right)$$

For each i, we have

$$(v_i - 1) \frac{\partial q}{\partial m_i}(m) + \lambda(1) = \begin{cases} \lambda(1) & v_i = 1 \\ -\eta \mathbb{1}\{m_i = \min_{j \in N} m_j < m^*\} + \lambda(1) & v_i = 0 \end{cases}$$

Thus, (2.13) holds for any v and m as desired.

Now, fix an information structure $\mathcal{I}'$ and an equilibrium $\beta' \in \mathcal{B}(\mathcal{M}^*, \mathcal{I}')$. By taking the expectation with respect to v, (2.13) implies

$$\sum_{i=1}^{N} (w_i(s)\lambda(1) + (1 - w_i(s))\lambda(0)) \leq \sum_{i=1}^{N} w_i(s) \frac{\partial q}{\partial m_i}(m) - \Xi(m)$$

Moreover, (2.12) also holds. Thus,

$$\begin{aligned}
\int_S \int_M \sum_{i=1}^{N} t_i(m)\beta(dm|s)G(ds) &\geq \int_S \int_M \sum_{i=1}^{N} \left(t_i(m) + w_i(s)\frac{\partial q}{\partial m_i}(m) - \frac{\partial t_i}{\partial m_i}(m) \right) \beta(dm|s)G(ds) \\
&= \int_S \int_M \sum_{i=1}^{N} \left(w_i(s)\frac{\partial q}{\partial m_i}(m) - \Xi(m) \right) \beta(dm|s)G(ds) \\
&\geq \int_S \int_M \sum_{i=1}^{N} (w_i(s)\lambda(1) + (1 - w_i(s))\lambda(0))\, \beta(dm|s)G(ds) \\
&\geq \sum_{i=1}^{N} (\mu\lambda(1) + (1-\mu)\lambda(0)) \\
&= \bar{\Pi}
\end{aligned}$$

Thus, the veto mechanism earns at least $\bar{\Pi}$. □

Proposition 2.4.3. $\beta^* \in \mathcal{B}(\mathcal{M}^*, \mathcal{I}^*)$. *That is, the truth-telling strategy profile* β^* *is an equilibrium of the veto mechanism* $\mathcal{M}^*$ *and the veto information structure* $\mathcal{I}^*$*.*

Proof of Proposition 2.4.3. We need to show that truth-telling is a best response for any $s_i, s_i' \in [0, m^*]$. Let $U_i(s_i, s_i')$ denote agent i's expected payoff under the truth-telling equilibrium if she observed s_i and reported s_i'.

$$U_i(s_i, s_i') = \int_{S_{-i}} w_i(s_i, s_{-i}) q(s_i', s_{-i}) dG(s_{-i}) - \int_{S_{-i}} t_i(s_i', s_{-i}) dG(s_{-i})$$

Truth-telling is a best response if $U_i(s_i, s_i) \geq U_i(s_i, s_i')$ for any s_i and s_i'. Equivalently, truth-telling is a best response if

$$\frac{\partial U_i}{\partial s_i'}(s_i, s_i')|_{s_i'=s_i} = 0 \quad \text{and} \quad \frac{\partial^2 U_i}{\partial s_i'^2}(s_i, s_i')|_{s_i'=s_i} \leq 0$$

Start from the derivative of the first term.

$$\frac{\partial}{\partial s_i'} \int_{S_{-i}} w_i(s_i, s_{-i}) q(s_i', s_{-i}) dG(s_{-i})|_{s_i'=s_i} = \int_{S_{-i}} w_i(s) \eta \mathbb{1}\{s_i < \min\{s_{-i}\}\} dG(s_{-i})$$

Since q and w_i depends on s_i and the minimum of s_{-i},

$$\begin{aligned}
\frac{\partial}{\partial s_i'} \int_{S_{-i}} w_i(s_i, s_{-i}) q(s_i', s_{-i}) dG(s_{-i})|_{s_i'=s_i} =& \int_0^{m^*} w_i(s_i, y) \eta \mathbb{1}\{s_i < y\}\} dG_{min}^{N-1}(y) \\
=& \int_{s_i}^{m^*} \left(1 + N(e^{s_i - y} - 1)\right) \eta dG_{min}^{N-1}(y) \\
=& \eta \int_{s_i}^{m^*} \left[-(N-1)^2 e^{-y(N-1)} + N(N-1) e^{s_i - yN}\right] dy \\
& + \eta(1 + N(e^{s_i - m^*} - 1)) e^{-m^*(N-1)} \\
=& \eta e^{s_i - m^* N}
\end{aligned} \tag{2.14}$$

where G_{min}^{N-1} is the distribution of the minimum of $N-1$ independent exponential distributions censored at m^*. G_{min}^{N-1} is defined by

$$G_{min}^{N-1}(y) = \begin{cases} 1 - e^{-y(N-1)} & y < m^* \\ 1 & y = m^* \end{cases}$$

Before working on the derivative of the second term, we need to obtain an expression for the interim transfer $t_i(s_i)$.

$$\begin{aligned} t_i(s_i') &= \int_{S_{-i}} t_i(s_i', s_{-i}) dG(s_{-i}) \\ &= e^{s_i'} \int_0^{s_i'} \int_{S_{-i}} \xi_i(x, s_{-i}) e^{-x} dG(s_{-i}) dx \\ &= e^{s_i'} \int_0^{s_i'} \int_{S_{-i}} \sum_{i=1}^{N} \xi_i(x, s_{-i}) e^{-x} dG(s_{-i}) dx \\ &= e^{s_i'} \int_0^{s_i'} \int_{S_{-i}} \sum_{i=1}^{N} \left(\frac{\partial q}{\partial m_i}(x, s_{-i}) - \lambda(1) \right) e^{-x} dG(s_{-i}) dx \end{aligned}$$

From the second to the third line, we used the fact that

$$0 = \int_0^{m^*} \xi_i(x, s_{-i}) dG(x)$$

for each $j \neq i$. Then,

$$\begin{aligned} t_i(s_i) =& e^{s_i} \int_0^{s_i} \int_{S_{-i}} \left[\eta \mathbb{1}\{x < \min\{s_{-i}\}\} + \sum_{j \neq i} \eta \mathbb{1}\{s_j < \min\{x, s_{-\{i,j\}}\}\} \right] dG(s_{-i}) e^{-x} dx \\ & - N\lambda(1)(e^{s_i} - 1) \end{aligned}$$

Now we evaluate the integral on S_{-i}, holding x fixed.

$$\begin{aligned}\int_{S_{-i}} \eta \mathbb{1}\{x < \min\{s_{-i}\}\} dG(s_{-i}) &= \int_0^{m^*} \eta \mathbb{1}\{x < y\}\} dG_{min}^{N-1}(y) \\ &= \eta(1 - G_{min}^{N-1}(x)) \\ &= \eta e^{-x(N-1)}\end{aligned}$$

For each $j \neq i$,

$$\begin{aligned}\int_{S_{-i}} \eta \mathbb{1}\{s_j < \min\{x, s_{-\{i,j\}}\}\} dG(s_{-i}) &= \int_0^x \int_{s_j}^{m^*} \eta e^{-s_j} dG_{min}^{N-2}(y) ds_j \\ &= \eta \int_0^x e^{-s_j}(1 - G_{min}^{N-2}(s_j)) ds_j \\ &= \eta \int_0^x e^{-s_j(N-1)} ds_j \\ &= \frac{\eta}{N-1}(1 - e^{-x(N-1)})\end{aligned}$$

Finally, from the definition of $\lambda(1)$,

$$\begin{aligned}N\lambda(1) &= \int_S \sum_{i=1}^N \frac{\partial q}{\partial m_i}(s) \mathbb{1}_{m_i < m^*} dG(s) \\ &= \eta(1 - e^{-m^* N})\end{aligned}$$

Thus, we obtain an expression for $t_i(s_i)$ as below.

$$\begin{aligned}t_i(s_i) &= \eta e^{s_i} \int_0^{s_i} \left(e^{-x(N-1)} + (N-1)\frac{1}{N-1}(1 - e^{-x(N-1)}) \right) e^{-x} dx - \eta(1 - e^{-m^* N})(e^{s_i} - 1) \\ &= \eta(e^{s_i} - 1)e^{-m^* N}\end{aligned}$$

It is easy to find the derivative of t_i.

$$t_i'(s_i) = \eta e^{s_i} e^{-m^* N} \tag{2.15}$$

Then we obtain the first-order condition from (2.14) and (2.15).

$$\frac{\partial U_i}{\partial s_i'}(s_i, s_i')|_{s_i'=s_i} = 0$$

The second order condition follows since $\frac{\partial^2 q}{\partial m_i^2}(m) = 0$ for any m and $t_i''(s_i) > 0$. Thus, truth-telling is an equilibrium of the veto mechanism and the veto information structure. □

2.5 Capped Proportional Mechanism

Suppose that $\mu \leq \bar{\mu}(N)$. The monopolist is no longer confident enough in the public good's quality. The veto mechanism then is no longer a strong maximin solution to the monopolist. Instead, it becomes important to balance between free-riding prevention and hedging against a low demanded buyer. The capped proportional mechanism, where the public good is supplied according to the total demand reported from the buyers, achieves this goal. However, to balance the insurance motive while curbing the free-riding incentives, each buyer is given a cap on the maximal quantity she can ask.

2.5.1 Minimax Information Structure

Proposition 2.3.1 and 2.3.2 ensures that the proportional information structure exists and is pinned down by the parameters (m^*, η) and the indifference condition on the message and signal space $X(k)$, for any natural number k. Here, we illustrate numerical solutions of the restricted minimax problem to probe other properties of the minimax information structure.

We present a numerical solution of the restricted minimax problem.[4] Figure 2.2 illustrates the interim value w^* from the restricted min2max problem. The interim value w_i^* is monotone and continuous in own signal. Figure 2.3 illustrates the virtual value φ_i^* and Figure 2.4 illustrates the total virtual values $\sum \varphi_i^*$ induced by the proportional information structure.

The shape of the numerical solutions suggest that the interim value is characterized by three parameters $\hat{m}$, m^* and η. Specifically, w^* behaves similarly with the veto interim value when either signal is smaller than $\hat{m}$. Buyers with higher signal has the interim value and the virtual value at 1 and buyers with lower signal has the opposite. However, when either signal is higher than $\hat{m}$, the virtual value strictly increases in own signal. Visual examination of the numerical solution suggests a functional form for the interim value. Figure 2.5 illustrates how w^* is parameterized.

The individual virtual value determines the interim value w^* on some signal profiles.

[4]The solution is obtained with an additional constraint that the individual virtual value is monotone in own signal, which does not affect the restricted min2max value.

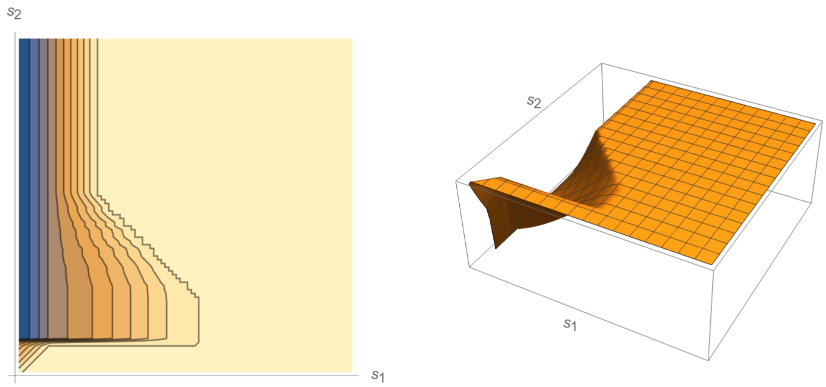

Figure 2.2: Proportional Interim Value w_i^*

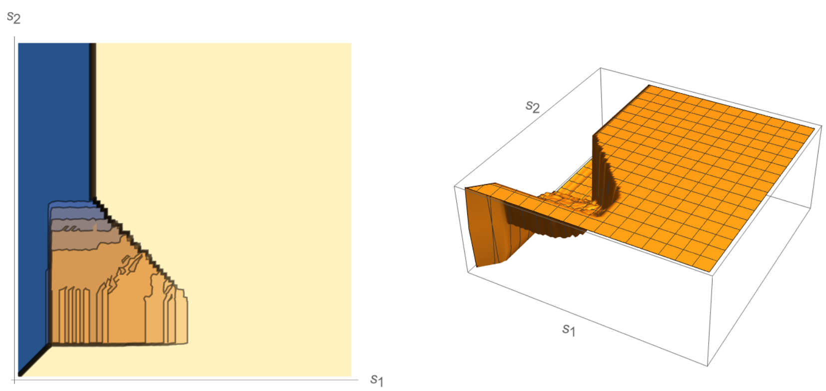

Figure 2.3: Virtual Value φ_i^*

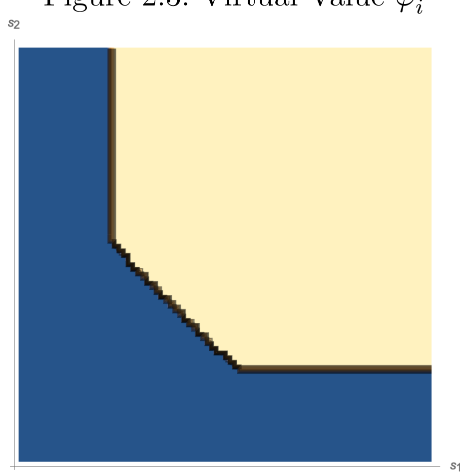

Figure 2.4: Total Virtual Values $\sum \varphi_i$

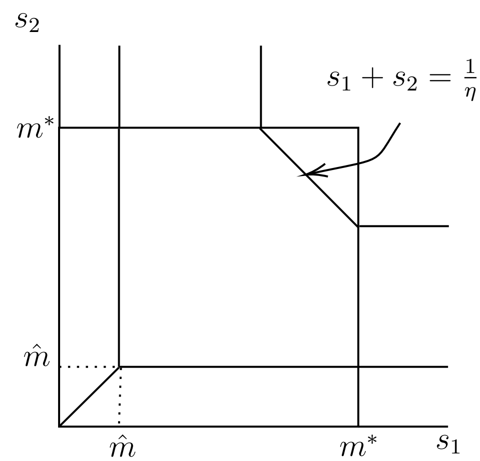

Figure 2.5: $N = 2$ Proportional Information Structure

- $s_j = m^*$

$$w_i^*(s) = \begin{cases} -1 + e^{s_i} & \text{for } s_i \leq \frac{1}{\eta} - m^* \\ 1 & \text{for } s_i \geq \frac{1}{\eta} - m^* \end{cases}$$

- $\hat{m} \leq s_j \leq m^*$

$$w_i^*(s) = \begin{cases} -1 + e^{s_i} & \text{for } s_i \leq \hat{m} \\ \omega_i^*(s) & \text{for } \hat{m} \leq s_i \leq \min\{\frac{1}{\eta} - s_j, m^*\} \\ 1 & \text{for } \min\{\frac{1}{\eta} - s_j, m^*\} \leq s_i \end{cases}$$

- $0 \leq s_j \leq \hat{m}$

$$w_i^*(s) = \begin{cases} -1 + 2e^{s_i - s_j} & \text{for } s_i \leq s_j \\ 1 & \text{for } s_i \geq s_j \end{cases}$$

where the parameters η, $\hat{m}$ and m^* and the function ω^* are determined so that w^* is well-defined and induces individual virtual values that add up to zero.

The aggregate virtual value $\sum \varphi_i^*(s)$ of the proportional information structure is identified as below.

$$\sum \varphi_i^*(s) = Ng(s)\mathbb{1}\{\eta \sum_i \min\{m^*, s_i\} \geq 1\}. \tag{2.16}$$

As we approximate the discrete space $X(k)$ to the real line, the optimal parameter problem (2.6) converges to the optimization problem below, which finds an optimal parameter and the minimax value $\Pi^*(N)$ in the continuous model. The optimal parameter (η, m^*) solves the following problem. For each $\mu \leq \bar{\mu}(N)$,

$$\Pi^*(N) = \max_{\eta, m^*} \quad \eta N \left(\mu - \int_{[0,\infty)^{N-1}} e^{-\min\left\{m^*, \frac{1}{\eta} - \sum_{j\neq 1} \min\{m^*, s_j\}\right\} - \sum_{j \neq 1} s_j} ds_{-1} \right). \tag{2.17}$$

2.5.2 Maximin Mechanism

Proposition 2.3.1 implies that the maximin mechanism must be capped proportional as the discreteness restriction relaxes. The capped proportional mechanism $\mathcal{M}^* = (M^*, q^*, t^*)$ is defined by

$$M_i = [0, m^*], \quad q(m) = \min\left\{1, \eta^* \sum_{i=1}^{N} m_i \right\}.$$

In the capped proportional mechanism, each buyer can demand up to $\eta^* m^*$ of the public good. It is the maximal individual demand, or the individual quota that the monopolist uses to balance between free-riding prevention and hedging against low demanded buyers. As the monopolist's confidence μ increases, he may allow a smaller quota, closer to $\frac{1}{N}$ to curb free-riding. In a similar vein, $\eta m^* N$, which is the total quota, measures how much free-riding the monopolist allows. Interestingly, the monopolist optimally uses a similar level of the total quota when he faces more buyers. Figure 2.6 illustrates the individual and total quota on intermediate μ with different number of buyers. Note that the quota is plotted for $\mu \leq \bar{\mu}(N)$, where the cutoff $\bar{\mu}(N)$ is defined in (2.8).

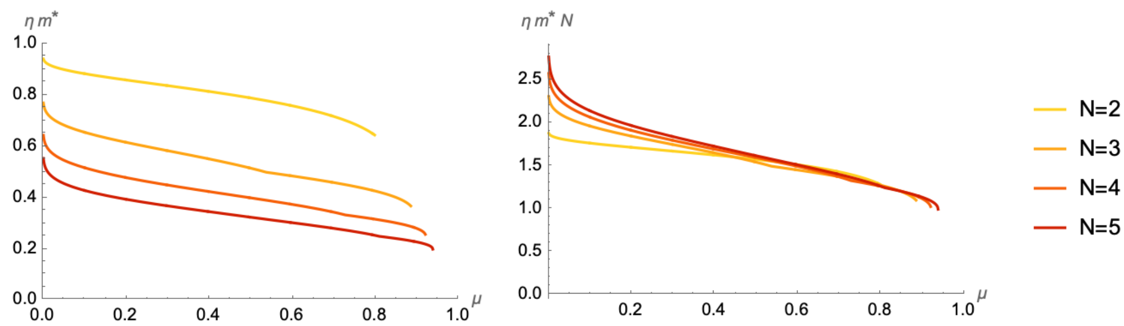

Figure 2.6: The individual quota ηm^* (left) and the total quota $\eta m^* N$ (right) of the optimal capped proportional mechanism

The proportional mechanism must satisfy the complementary slackness (2.2). Thus, the maximin transfer must satisfy

$$\sum_i \left(\frac{\partial t_i^*}{\partial m_i}(m) - t_i^*(m) \right) = \sum_i \eta^* \mathbb{1}_{\{m_i < \min\{m^*, 1/\eta^* - \sum_{j \neq i} m_j\}\}} - N\lambda^*(1) \tag{2.18}$$

for each $m \in M^*$.

We identify optimal $\lambda^*(1)$ and $\lambda^*(0)$ for the restricted maximin problem, using the same idea as before.

$$\lambda^*(0) = \lambda^*(1) - \eta^*.$$

$\lambda^*(1)$ is found from the strong duality in the dual and primal restricted maximin problem. Any optimal mechanism to the proportional information structure earns

$\Pi^*(N)$. Since the value of the restricted maximin problem is equal to that of its dual, we find λ^* as below.

$$\lambda^*(1) = \frac{\Pi^*(N)}{N} + \eta^*(1-\mu)$$

To construct a maximin transfer rule, we need to determine the individual $\frac{\partial t_i^*}{\partial m_i}(m) - t_i^*(m)$ as before. Let $\Xi(m)$ denote the right-hand side of (2.18).

$$\Xi(m) = \sum_i \eta^* \mathbb{1}_{\{m_i < \min\{m^*, 1/\eta^* - \sum_{j \neq i} m_j\}\}} - N\lambda^*(1)$$

and let Z denote the set of permutations on $\{1, \cdots, N\}$ with a typical element ζ. Denote the subset of agents whose order is less than k in ζ by

$$[\zeta \leq k] = \{j | \zeta(j) \leq k\}$$

and analogously define $[\zeta > k]$. Let

$$\tau_{\zeta,k}(m) = \int_{[0,m^*]^{N-k}} \Xi(m_{[\zeta \leq k]}, x_{[\zeta > k]}; q) dG(x_{[\zeta > k]})$$

and

$$\xi_i(m) = \frac{1}{N!} \sum_{\zeta \in Z} \left[\tau_{\zeta,\zeta(i)}(m) - \tau_{\zeta,\zeta(i)-1}(m)\right]$$

The proportional transfer rule is obtained from the definition of ξ_i.

$$t_i(m) = exp(m_i) \int_0^{m_i} \xi_i(x_i, m_{-i}) exp(-x_i) dx_i$$

2.6 Conclusion

This paper constructed a strong maximin solution in the public good provision setup. The monopolist seller needs to know a minimal amount of knowledge about the buyers' information structure to implement the strong maximin mechanism constructed in this paper. It guarantees some expected profit as long as the buyers' actual information structure is consistent with the monopolist's knowledge. In addition, it only asks the buyers to report a quantity demanded, which makes the maximin mechanism simple and practical.

The maximin mechanism qualitatively depends on the number of buyers N and the common expected valuation μ. If μ is high relative to N, then the veto mechanism, which fully shuts down free-riding, is robustly optimal. In the veto mechanism, the buyer with the smallest demand report is charged an exponential price, while everyone else is charged nothing. On the other hand, if μ is not high enough, the capped proportional mechanism is robustly optimal, where the public good is supplied by the total demand, but each buyer can only demand some portion of the public good.

CHAPTER 3

Optimal Robust Double Clock Auctions

3.1 Introduction

Exchange economies, a fundamental model of microeconomics, well describe a variety of essential market institutions. For example, consider centralized stock, bond, and commodity markets. They are populated with traders who submit their demand or supply to the marketplace. Their after-trade position may be positive or negative relative to their initial holdings, which is determined by the market price. Owners of such exchanges may have various objectives, including efficiency and maximizing profit from market making. Studies of mechanism design prescribe exchange owners a direct mechanism that maximizes the owners' objectives.

On the other hand, clock implementations are widely used in practice, where traders gradually reveal their private information to the exchange and other traders. In this sense, clock implementations are considered an open format. Most notably, Ausubel (2004) constructs a clock implementation of the Vickery mechanism with multiple units of goods, achieving efficient allocation in an open fashion. However, for two reasons, his clock implementation does not smoothly carry over to the exchange situation. First, it is confined to the implementation of the Vickery mechanism, which aims for efficiency. However, the exchange can't achieve efficient allocation without suffering from budget deficits due to two-sided private information. Second, the exchange cannot ex-ante partition traders into buyers and sellers. The two points call for a novel clock implementation for the exchange problem.

When values are private, we offer a dynamic (clock) multi-unit double auction format for homogeneous goods that is robust, meaning that all the incentive and market clearing constraints are satisfied ex-post, and at the same time achieves a variety of objectives of the auctioneer. Simply put, we extend the Ausubel (2004) clock auction into two dimensions. First, our clock auction endogenously determines buyers and sellers. Second, our clock auction achieves goals other than exact efficiency.

At the core of our double auction, are two Ausubel (clinching) auctions run simultaneously, one for buyers and one for sellers, such that supply continuously matches with demand, until the two clocks meet. The payments consist of two parts. The first part is the standard per-unit payments associated with the quantities clinched on either side of the market, the second part is a special per-unit tax, that may also depend on the clock. Our first main result is that this auction has an ex-post equilibrium, which is also dynamically consistent. Our second main result is that this auction is capable of implementing a variety of mechanisms, including efficient, nearly efficient and, most importantly, optimal (profit maximizing) ones.

This paper belongs to the literature on clock auctions. Ausubel (2004) constructs a clock implementation to sell multiple units of homogeneous goods. The same author's follow-up paper Ausubel (2006) extends the clock implementation to sell multiple units of heterogeneous goods, at the cost of additional complexity. This paper further extends the clock implementation towards two other direction: two-sided private information and various designer objective. Loertscher and Marx (2020a) and

Loertscher and Mezzetti (2021) constructs a clock implementation where traders are ex-ante partitioned to be buyers and sellers.

This paper is also related to the study of optimal or efficient mechanism design problems in exchange settings, where the mechanism designer endogenously determine whether a trader end up being a buyer or a seller. Lu and Robert (2001) and Loertscher and Marx (2020b) consider the optimal mechanism design problem where traders have a constant marginal utility. Loertscher and Marx (2020b) also discusses a clock implementation of the optimal mechanism, but it is limited to unit supply and demand. Andreyanov and Sadzik (2021) constructs a robust mechanism that implements an asymptotically efficient outcome without making a budget deficit, which converges to full ex-post efficiency as the number of traders grows. Our clock auction can be used to implement this mechanism in an open fashion.

3.2 Model

The designer, or the owner of a trading platform, wishes to reallocate a single, homogeneous, and perfectly divisible good among N agents. Each agent i has a quasi-linear utility function $U_i(q_i) - t_i$, where q_i,is the quantity of the good the agent gets. We assume that the asset is not continuously divisible: $q_i \in Q_K := \{\cdots, -2/K, -1/K, 0, 1/K, 2/K, \cdots\}$ for some positive integer K. $t_i \in Q_K$ is the payment he makes to the trading platform. In particular, the utility does not depend on the information of other agents (*private values*). We assume that the marginal utility functions $MU_i : Q_K \to M_K := \{1/K, 2/K, \cdots, (MK-1)/K, M\}$ are bounded, weakly decreasing. The utility function of agent i is then defined by the cumulative sum of her marginal utility.

$$U_i(q_i) = \begin{cases} \sum_{0<x\leq q_i, x\in Q_K} MU_i(x) & q_i > 0 \\ 0 & q_i = 0 \\ \sum_{0>x\geq q_i, x\in Q_K} MU_i(x) & q_i < 0 \end{cases}$$

Agent i enters the exchange with an initial position where she has no asset.

The environment is an example of the classic exchange economy, with quasi-linear utilities with respect to an asset and a numeraire. For any weakly increasing menu of (per-unit) prices $p(\cdot)$ we define a *demand correspondence* of agent i as

$$Q_i(p) = \underset{q\in Q_K}{argmax} \left\{ U_i(q) - \sum_{0\leq x\leq q, x\in Q_K} p(x) \right\}.$$

The demand correspondence reduces to the textbook Walrasian demand correspondence once we assume that the per-unit price is constant. We consider *non-linear* per-unit prices to facilitate the implementation of various designer objectives.

Note that each agent could potentially be either a buyer or a seller, and so the demand correspondence can take on both positive and negative values. Decreasing marginal utilities guarantee that the demand correspondences are weakly decreasing in price menus, and determined by the local conditions. In particular, it implies the existence of the efficiency benchmark, the *Walrasian equilibrium*: a constant per-unit price p^* together with $q_i^* \in Q_i(p^*)$, $i \leq N$, such that market clears, $\sum_{i=1}^N q_i^* = 0$.

3.3 Double-Clock Auctions

In this section we define a *double-clock auction* with individualized prices. The auction has three characteristic features, distinguishing it from a classical single-good clock auction. First, at any time, each player faces separate buyer and seller prices. Second, prices are not anonymous. At any point during the auction, each agent i faces individual prices, which may differ across the agents. Third, prices for agent i depend also on how many units the agent already bought or sold. Effectively, a player faces two menus of prices, with per-unit prices as a function of quantity. As we shall see, both features enhance the scope for price discrimination, and at the same time do not compromise any of the strategic virtues of the auction.

Taxation Functions. An auction is parametrized by a *system of taxation functions* $(\tau_1, ..., \tau_N)$, with $\tau_i : M_K \times Q_K \to M_K$, for every $i \in N$. The first argument λ of any τ_i is interpreted as a clock state of the auction. Its dynamics will be determined shortly. The second argument q is the quantity that the agent desires to purchase (or sell, when $q < 0$). Thus $\tau_i(\lambda, q)$ is the marginal, or per-unit 'taxation' on top of the base price for the q-th unit for agent i, if the auction is at state λ. Put differently, at a clock state λ agent i faces a demand-supply schedule, or a menu of per-unit prices $\lambda + \tau_i(\lambda, \cdot)$.

We require that i) for each player i, $\tau_i(0, \cdot) = 0$ and $\tau_i(M, \cdot) = 0$; ii) prices $\tau_i(\cdot, q)$ are weakly increasing in q, for every λ; iii) prices $\lambda + \tau_i(\lambda, q)$ are weakly increasing in λ, for every q; *iv*) at each λ at most one player's pricing schedule changes by more than one unit in λ.

Clock Dynamics. An auction is parameterized by two clock processes, $(\lambda^b(t), \lambda^s(t))_{t \le MK}$. The two clocks of the auction are initiated at $\lambda^b(0) = 0$ and $\lambda^s(0) = M$; $\lambda^b(t)$ (weakly) increases over time, and $\lambda^s(t)$ (weakly) decreases over time, and both functions are continuous. We also require that there are no "lulls", with either clock changing at any time. Thus, the clock auction lasts at most MK periods.

The exact details of how the clocks adjust are open, and are a parameter (along with the system of prices, and information structure) of an auction, and we propose several options below in the paper. In particular, the clocks will typically adjust in response to the submitted demands at both clock states.

Strategies, Clinching, and Unloading. At any point in period t, given the clock states $\lambda^b(t)$ and $\lambda^s(t)$, each player i faces two schedules of prices, $\lambda^b(t) + \tau_i(\lambda^b(t), \cdot)$ and $\lambda^s(t) + \tau_i(\lambda^s(t), \cdot)$. In the auction, his strategy is to provide two demands (or supplies) at any point in period t.

Specifically, consider any specification of private histories—that is, histories of play prior to period t that are observable to a player i at t, for each $t = 1, \cdots, MK$ and $i \leq N$. There is a leeway in what is observable by the players, such as the history of prices, or the exact history of submitted demands. We assume that a player at time t knows, besides own past actions, at least the buyer-clock and seller-clock states $\lambda^b(t)$ and $\lambda^s(t)$. A *strategy* $(x_i^b(t), x_i^s(t))_{t \leq MK}$ of a player i specifies a submitted demand and a submitted supply, each in $\mathbb{R}$, for every period t and every private history.[1]

In the process or running an auction, players secure purchases and sales of the good; when a player secures a sale we will say that he *unloads* the good and, following Ausubel (2004), we say that he *clinches* it when he secures its purchase. The quantity

[1]We suppress the dependence of submitted demands and supplies on private histories, as those play no role in the results. Note also that we define a strategy as a plan of action for a given, realized utility function. Later on, in the incomplete information setting, we will talk explicitly about "mappings from utility functions to strategies", avoiding the term "incomplete information strategy", to avoid confusion.

of the good that player i clinched at time t, $C_i(t)$, and the quantity that he unloaded, D_i, are defined as

$$\begin{aligned} C_i(t) &= max\{0, -\sum_{j\neq i} x_j^b(t)\}, \\ D_i(t) &= min\{0, -\sum_{j\neq i} x_j^s(t)\}. \end{aligned} \tag{3.1}$$

A player starts clinching goods when the total demand from other players—the *residual demand*—at the lower, buyer-prices drops below zero. Then, he clinches additional units when the residual demand drops further. Similarly, he starts unloading the good when the total supply from other players—the *residual supply*—at the seller-prices reaches above zero, and he unloads more as the residual supply goes up.

We make the following assumptions on strategies. We require that $i)$ before the clocks meet, submitted demand on the buyer-clock is greater than that on the seller-clock, $x_i^b(t) \geq x_i^s(t)$, for every $t \geq 0$; $ii)$ demand on the buyer-clock $x_i^b(t)$ is decreasing in t and demand on the seller-clock $x_i^s(t)$ is increasing in t; and $iii)$ player can't submit two different demands at a given price scheme,

$$x_i^*(t) = x_i^{\#}(t'), \text{ if } \lambda^*(t) = \lambda^{\#}(t'), \text{ for } t' \leq t \text{ and } *, \# \in \{b, s\}. \tag{3.2}$$

The last assumption eliminates the "redundancy" of submitting demands twice, at different points of time, for a given price scheme. It is only needed to guarantee unique weak rationalizability of sincere bidding below (see Ausubel (2004)). In particular, it implies that a strategy needs to be defined only until the clocks meet at some end period T. $iv)$ players must demand at least the amount she clinched/unloaded,

$x_i^b(t) \geq C_i(t)$ and $x_i^s(t) \leq D_i(t)$.

Allocations and Payments. Finally, we define the allocation and the payments resulting from running an auction. Each agent is allocated the quantity that he clinched or unloaded by the time respective side of the market clears. The price paid for each unit clinched (or received for each one unloaded) is the price for that unit at the clock when it was clinched. This definition of the payments conforms with the guiding idea behind the standard single-unit English auction, or the dynamic multi-unit auction in Ausubel (2004).

Let T^b be the first time such that in the next period there is excess supply on the buyer-clock, and T^s be the first time such that in the next period there is excess demand on the seller-clock,

$$T^b = \min\{\tau : \sum_i x_i^b(t) \leq 0, \forall t > \tau\},$$
$$T^s = \min\{\tau : \sum_i x_i^s(t) \geq 0, \forall t > \tau\}.$$

Both payments and allocations in an auction are determined when T^b and T^s are crossed, at which point the auction may end.

The quantity q_i allocated to i is defined so that

$$\begin{aligned} q_i &\in [x_i^b(T^b), x_i^b(T^b - 1)], \text{ and } q_i \geq C_i(T^b - 1), \text{ when } D_i(T^s) = 0, \\ q_i &\in [x_i^s(T^s - 1), x_i^s(T^s)], \text{ and } q_i \leq D_i(T^s - 1), \text{ when } C_i(T^b) = 0, \\ \sum_{i \leq N} q_i &= 0, \end{aligned} \tag{3.3}$$

when both T^b and T^s are finite, and otherwise $q_i = 0$.

When either submitted demands or supplies "jump" discontinuously and at no time clear the market, the market has to be rationed. Intuitively, each player is rationed down from what he demands or supplies just before the jump, in a way that market clears. Some caution is required with how rationing is defined: following Okamoto (2018), we require that the demands and supplies are executed according to an exogenous order of players, until the market clears. Importantly, we also require that the player whose price scheme is discontinuous when market clears is considered last in the rationing order.

The clock auction also determines a payment rule for the agents. Namely, the agents are charged the sum of the price clock at which they clinch each unit of the asset and the additional taxation, which is determined by the price clock and the cumulative clinches up to that point in the auction. Let $c_i(t)$ and $d_i(t)$ denote the marginal clinch/unloading at period t, which are defined by

$$\begin{aligned} c_i(t) =& C_i(t) - C_i(t-1), \\ d_i(t) =& D_i(t) - D_i(t-1). \end{aligned}$$

Agent i clinches $c_i(t)$ units at time t and unloads $d_i(t)$ units at time t. If an agent ends up being a seller, her payment is similarly determined.

$$t_i = \begin{cases} \sum_{0 \le t \le T^b} c_i(t)(\lambda^b(t) + \tau_i(\lambda^b(t), C_i(t))) & q_i > 0 \\ 0 & q_i = 0 \\ \sum_{0 \le t \le T^s} d_i(t)(\lambda^s(t) + \tau_i(\lambda^s(t), D_i(t))) & q_i < 0 \end{cases}$$

3.4 Nonparametric Setting

In this section we investigate strategic properties of the double-clock auction constructed in the previous section. Namely, we show that it is an *ex-post perfect equilibrium* for bidders to play *sincerely*, as if they were a 'price-taker' against the price path determined in the clock auction.

As a first step, we define a notion of *sincere bidding*.

Definition 3.4.1. *For an arbitrary history of play until round r, $r = 1, \cdots, MK$, player i bids sincerely at t if*

$$x_i^*(t) = \begin{cases} \min Q_i(p^*), & \text{if } \min Q_i(p^*) \geq 0, \\ \max Q_i(p^*), & \text{if } \max Q_i(p^*) \leq 0, \\ 0, & \text{if } 0 \in Q_i(p^*), \end{cases} \tag{3.4}$$

for $* \in \{b, s\}$, as long as both reports belong to the permissible interval $[x_i^s(t-1), x_i^b(t-1)]$, and otherwise $x_i^*(t)$ is the nearer boundary of the interval.

In other words, player i bids sincerely if at any time he reports truthfully his demands, given the current two clock states, and so the current two menus of prices. Given that the marginal utilities are decreasing, and the per-unit prices $\lambda^*(t) + \tau_i(\lambda^*(t), \cdot)$ are weakly increasing in quantity, at any time, the problem is concave and each player simply reports a quantity x that is locally optimal—at which his marginal utility crosses the per-unit price. In case there are several optimal quantities, the player chooses conservatively the quantity closest to zero.

Such strategy of sincere bidding, if followed from the start, is consistent: submitted demands $x_i^b(t)$ increase over time, $x_i^s(t)$ decrease over time, and $x_i^s(t) \leq x_i^b(t)$, for $t \leq MK$. However, after an arbitrary, "non-sincere" bidding history, sincere bidding means bids that are consistent with the history and closest to the optimal conservative bids.

It follows also from the previous argument that—again, when players bid sincerely—the allocation in an auction is determined by the system of prices, and in particular by the range of price clocks at which the total demand $\sum x_i$ is zero. Specifically, it does not depend on the details of the clock adjustment process. The same is true for the payments in an auction, under sincere bidding.

This is because the submitted demands and supplies, and so the clinched and unloaded quantities depend only on the clock state λ and not period t, and so the per-unit payments in (3.4) depend only on the system of prices.

Given sincere bidding, a system of prices provides a convenient instrument in the nonparametric setting of double-clock auctions. By determining both allocation as well as payments, it allows the designer to strike a balance between such objectives as efficiency, small budget deficit, or expected profit maximization. In the next section, we will see that, indeed, double-clock auctions with appropriate price schemes may achieve each of those objectives.

In a similar vein, we shall assume a demanding notion of incentive compatibility for the players. Evaluating strategic incentives of the players in a Bayesian framework would require a specification of prior beliefs over the types (utilities) of other players, as well as details of the games information structure, and so what exactly players learn in the course of the game; formally, information structure would be reflected in how histories H_i^t evolve over time. Instead, we require that each player's strategy remains optimal *ex post*, even if the utilities of the opponents were known. This guarantees optimality given any beliefs about the opponents' types, and so for any distribution over the types and information structures (see Ausubel (2004)). Moreover, reflecting the dynamic nature of the game, we require the strategies to be optimal after any history of play. Formally,

Definition 3.4.2. *A profile $(X_1, ..., X_N)$ of mappings from utility functions into strategies, $X_i(U_i) = (x_i^b(t), x_i^s(t))_{t\geq 0}$, for every i and U_i, is an* ex post perfect equilibrium *if for every realized utilities $(U_1, ..., U_N)$, at every time T, and after every history the profile of continuation strategies $(X_1(U_1)_{t\geq \tau}, ..., X_N(U_N)_{t\geq T})$ is a Nash equilibrium of the game in which the realized utilities are common knowledge.*

Besides incentive compatibility, we also require participation in the auction to be a better alternative than autarky.

Definition 3.4.3. *A mapping X_i from utility functions into strategies is* dominance individually rational *for player i if for every utility function U_i and any strategies of other players, participating in the auction with strategy $X_i(U_i)$ results in utilities at least $U_i(0)$.*

Next, we argue that the sincere strategy constitutes an equilibrium in our double clock auction with individualized price. In addition, it ensures that each agent is not worse-off from joining the exchange.

Proposition 3.4.1. *i) Sincere bidding by every player is an ex post perfect equilibrium. ii) Sincere bidding is dominance individually rational for every player.*

Proof of Proposition 3.4.1. During the double clock auction, any payoff-relevant event happens only through clinching and unloading. The definition of clinches and unloads

(3.1) ensures that agent i's clinching and unloading happen independent of her own demand reports x_i. In addition, the rationing rule, which uses an exogenous order of players to assign any excess demand or supply kills the room for strategic behavior from each agent. Thus, any deviation from the sincere strategy affects agent i's payoff if i) changes the final allocation q_i^* or ii) affects strategies of other agents.

We argue that the sincere strategy constitutes an ex-post perfect equilibrium of the clock auction game. Suppose that agents other than i employs the sincere strategy, so that agent i's deviation from the sincere strategy does not prompt other agents to play otherwise. Thus, the deviation has payoff impacts only if it changes the final allocation.

Consider any history—price states and previous reports—of the double clock auction. It is a best response for agent i to play sincerely since it maximizes her payoff given the price and tax path, which is determined by other agents being sincere. Agent i's decreasing marginal utility and increasing marginal cost of acquiring an additional unit of the good ensures that agent i would stick to the sincere strategy at any history.

$\square$

There are two key intuitions behind the proof of the incentive compatibility. First, bidders other than bidder i bid sincerely, they determine a menu of quantities and nonlinear transfers that bidder i effectively choose from in the course of the dynamic auction. As a buyer, a unit x becomes available to bidder i at the buyer-state at which the total demand from others drops below $-x$. The price for this unit is determined by bidder i's menu of prices at that state. The feature of other bidders determining a menu for bidder i to choose from is shared with the clinching auction in Ausubel

(2004). A new feature is that the price for a unit at the moment of clinching is not common to all players, but determined by a private menu.

However, the second feature of the auction is that, as we argue below, a unit is clinched only when it is demanded at that moment, and unloaded only when supplied. Incentive compatibility of sincere bidding then follows because the menu of prices at any given moment single-crosses from above (or, is flatter than) the menu that determines the allocation, and the two cross at the clinched (unloaded) unit. This single-crossing holds as long as the menus are weakly increasing at any state—rather than just constant, as in Ausubel (2004)—and increasing over time. In a sense, if a player ends up clinching a unit, he gets an additional discount relative to the schedule he faced; clinching more units would require paying an additional penalty relative to the schedule faced.

It follows from the rules of the auction that if a unit is clinched or unloaded strictly before a clock stops, it must have been demanded (or supplied) at that time. When strategies are discontinuous and overshoot a market clearing quantity, a player may also clinch or unload an "atom" of units at the end of the auction, in the process of rationing. We show in the proof that if price schedule is discontinuous at this moment (which presumably resulted in a discontinuous sincere bidding), a player also only clinches units that he demands, and unloads what he supplies. (When price schedule is continuous, a player clinches at a supremum of prices at which he demanded the unit.)

3.5 Direct Mechanisms

In the previous section, we have argued that our dynamic auction possesses a sincere ex-post perfect equilibrium, like in Ausubel (2004). However, it remains to explore whether our double clock auction could be considered optimal for the auctioneer in some sense. We proceed with a parametric approach by modeling a one-dimensional type, with convex support and a quasi-linear utility with single crossing, which is a standard approach in mechanism design.

Similarly to the previous section, let us assume that the agents have the quasi-linear utility that is indexed by a single-dimensional type $\theta_i \in \Theta_i$, where the type space Θ_i is a compact interval on $\mathbb{R}$. By the revelation principle (see Myerson (1979)), we focus on a *direct mechanism*, which is a collection of allocation and transfer functions $q_i(\theta), t_i(\theta)$ or, equivalently, allocation and surplus functions $q(\theta), s(\theta)$, defined for every agent i and every realization of $\theta = (\theta_i, \theta_{-i})$ in the Cartesian product of supports of individual types. We will refer to the domains of θ_i, θ_{-i} and θ as simply *the support*, and we will refer to the distribution functions of θ_i and θ_{-i} as F_i and $\mathcal{F}_i$ correspondingly.

Given a direct mechanism, agent i's payoff after a type profile θ was reported is denoted by $s_i(\theta)$, which is defined as

$$s_i(\theta) = u_i(\theta_i, q_i(\theta)) - t_i(\theta),$$

where $q_i \in \mathbb{R}$ is the quantity allocated, $t_i \in \mathbb{R}$ are monetary transfers. Throughout the paper, we will refer to $s_i \in \mathbb{R}$ as the *surplus*, $\tilde{s} = s - u(\theta_i, 0)$ as the *net surplus* and $\tilde{u} = u - u(\theta_i, 0)$ as the *net utility*.

Assumption 3.5.1. *i) θ_i are one-dimensional, independently distributed on a compact interval Θ_i, ii) $\tilde{u}_i$ are strictly single crossing and strictly concave, for all types in the support.*

Similar to the previous section, we impose two types of ex-post constraints on our mechanism: incentive compatibility (IC) and individual rationality (IR). Ex-post means that these constraints are satisfied pointwise, as in Andreyanov and Sadzik (2021), rather than on average, as in Lu and Robert (2001).

Definition 3.5.1. *A robust mechanism satisfies the following ex-post constraints:*

IC: $s_i(\theta) \geqslant u_i(\theta_i, q(\theta_i', \theta_{-i})) - t(\theta_i', \theta_{-i})$

IR: $\inf_{\theta_i} \tilde{s}_i(\theta_i, \theta_{-i}) \geqslant 0$

for all i and all types in the support.

The structure imposed by the robustness requirement is summarized in Lemma 3.5.1. By the standard envelope argument, the IC constraints tie the slope of net surplus for agent i to the slopes of his supporting utilities, and the IR constraint normalizes the intercept. The shape of the net surplus function is therefore described by a simple integral formula which we refer to as *envelope conditions*:

$$\tilde{s}_i(\theta_i, \theta_{-i}) = \int_{\theta_i^*(\theta_{-i})}^{\theta_i} \frac{\partial}{\partial \theta} \tilde{u}_i(x, q_i(x, \theta_{-i}))dx, \tag{3.5}$$

Note that in the envelope conditions, the bound of integration $\theta_i^*(\theta_{-i}) \in \Theta_i$ has to be a *net surplus minimizer*. It is endogenous and depends on the realization of types of other agents, but it can be associated with types θ_i such that $q_i(\theta) = 0$, in other words, types that are *excluded from trade*. This will be helpful for the characterization of certain mechanisms, most notably, profit maximizing ones.

For notational simplicity, we denote (mixed) partial derivatives as follows: $\tilde{s}'_{\theta_i}(\theta) = \frac{\partial \tilde{s}_i}{\partial \theta_i}(\theta)$, $mu_i(\theta_i, q_i) = \frac{\partial \tilde{u}_i}{\partial q_i}(\theta_i, q_i)$, $mv_i(\theta_i, q_i) = \frac{\partial v_i}{\partial q_i}(\theta_i, q_i)$, $mv_{i,x}(\theta_i, q_i) = \frac{\partial^2 v_i}{\partial q_i \partial x}(\theta_i, q_i)$ for $x = q_i$ or $x = \theta_i$.

Lemma 3.5.1. *Under Assumption 3.5.1, in a robust mechanism, for all i and all types in the support: i) allocation $q_i(\theta)$ is weakly increasing in θ_i, ii) the envelope conditions (3.5) hold, iii) net surplus $\tilde{s}_i(\theta)$ is weakly convex in θ_i, and iv) the set of types excluded from trade is a subset of net surplus minimizers.*

Proof of Lemma 3.5.1. The monotonicity of allocation i) is a standard necessary condition under single crossing, and the envelope conditions ii) can be traced back to Theorem 2 in Milgrom and Segal (2002). To prove iii) observe that, by the envelope conditions, net surplus is a.e. differentiable, thus at points of differentiability we can write:

$$\tilde{s}'_{\theta_i}(\theta) = (s(\theta) - u(\theta, 0))'_\theta = u'_\theta(\theta, x)|_{x=q(\theta)} - u'_\theta(\theta, x)|_{x=0} = \int_0^{q(\theta)} u''_{12}(\theta, x)dx$$

which, more generally, can be interpreted as a condition on the subgradient $\tilde{s}'_{\theta_i}$. This, together with the monotonicity of allocation, implies convexity of net surplus.

Finally, to prove iv) observe that if the set of types excluded from trade is empty, the claim holds trivially. If it is not empty, then the subgradient of $\tilde{s}$ at the type excluded from trade necessarily contains zero, by the envelope formula. Since $\tilde{s}$ is convex, this point also has to be one of its minimizers.

□

We are interested in robust mechanisms that maximize the following objective:

$$\max_{q(\theta)} \int \int \left[\sum_i v_i(\theta_i, q_i) \right] dF(\theta_i) d\mathcal{F}(\theta_{-i}), \quad \text{s.t.} \quad \sum q_i(\theta) = 0 \tag{3.6}$$

for a collection of functions $v_i(\theta_i, q)$, which can be interpreted as individual contributions of each agent to a certain social utility. We will broadly refer to these mechanisms as *optimal robust*.

While restrictive, this formulation covers a number of important families of mechanisms. In particular, two such families have been studied before. The first family, studied in Gresik and Satterthwaite (1989), Lu and Robert (2001), can be informally defined via $v_i = u_i - \alpha s_i$, and can be thought of as a convex combination of efficient and profit maximizing mechanisms. The second family, studied in Andreyanov and Sadzik (2021), is $v_i = u_i - \sigma q_i^2$ and can be thought of as a nearly efficient mechanism, capable of balancing the budget ex-post through partial demand reduction. By coincidence, if the utility is quadratic: $u_i(\theta_i, q) = \theta_i q - \frac{\mu}{2} q^2$, the second family also contains (for $\sigma = \frac{\mu}{n-2}$) the uniform-price double auction, studied, among others, in Kyle (1989) and Rostek and Weretka (2012).

Assuming concavity of the v_i functions, we can attempt to solve the problem using the method of Lagrange multipliers, ignoring the monotonicity constraints. This gives rise to the following necessary first order conditions:

$$mv_i(\theta_i, q_i) = p, \quad \sum_{i=1}^{n} q_i(\theta) = 0. \tag{3.7}$$

The basic properties of the solution are derived in Lemma 3.5.2. To derive the slopes of allocation and price in types, linearize (3.7) around the equilibrium point (p, q), where p is a market clearing price and $q \neq 0$:

$$mv'_{i,\theta_i} + mv'_{i,q_i} q'_{i,\theta_i} = p'_{\theta_i}, \quad mv'_{j,q_j} q'_{j,\theta_i} = p'_{\theta_i}, \quad j \neq i.$$

We can then solve for the slopes using market clearing:

$$p'_{\theta_i} = \frac{mv'_{i,\theta_i}}{mv'_{i,q_i}} (\sum \frac{1}{mv'_{k,q_k}})^{-1}, \quad q'_{j,\theta_i} = \frac{mv'_{i,\theta_i}}{mv'_{j,q_j}} \left(\frac{1/mv'_{j,q_j}}{\sum 1/mv'_{k,q_k}} - \mathbb{I}(j = i) \right). \tag{3.8}$$

Clearly, under single crossing of the v_i functions, the allocation of any given agent is weakly increasing in own type and weakly decreasing in types of others, but the market clearing price is weakly increasing in all types.

Assumption 3.5.2. *i) v_i is continuous in both arguments, ii) strictly concave in q and iii) strictly single crossing iv) continuously differentiable for all, for all $q \neq 0$ and all types in the support.*

Lemma 3.5.2. *Under Assumptions 3.5.1 and 3.5.2, in an optimal robust mechanism, i) for any realization of types θ in the support, there exists (p, q) such that (3.7) holds, and the allocation function q_i is ii) weakly decreasing in θ_{-i} and iii) upper hemi-continuous in both θ_i and θ_{-i}, iv) for any $x \neq 0$, there exists is at most one solution $z = q_i^{-1}(x, \theta_{-i})$ to the equation $q_i(z, \theta_{-i}) = x$.*

Proof of Lemma 3.5.2. To prove i), observe that by mere concavity of the v_i function in the neighborhood of $q = 0$, for any profile of types θ, there exist finite prices $\underline{p}$ and $\overline{p}$ such that $\underline{p} < mv_i(\theta_i, 0) < \overline{p}$, for all i. Consequently, for $\overline{p}$ total demand is non-positive, while for $\underline{p}$ total demand is non-negative. Since the demands are also upper hemi-continuous in p, by Intermediate Value Theorem, there exist a price in the $[\underline{p}, \overline{p}]$ segment, such that the market is cleared.

To prove ii) observe that by single crossing of v_i, from the perspective of player i, an increase in θ_i can be interpreted as shift of demand, while an increase in θ_{-i} as a shift of supply. Moreover, by convexity of v_i, all demand functions are non-increasing and upper hemi-continuous. Consequently, the allocation q_i in the intersection of supply and demand is weakly increasing in θ_i and weakly decreasing in θ_{-i}.

To prove iii), observe that (3.6) is a continuous and convex optimization problem, since the monotonicity constraint is not binding. Thus, by Maximum Theorem, the solution is upper hemi-continuous in all parameters. □

Before we proceed to the first main result of this section, there is one technical, yet important assumption that we need to make, for the mechanism to be tractable. This assumption is related to the interiority of the net surplus minimizer, often referred to as the "worst-off type", see Lu and Robert (2001), and requires either a full symmetry of the model, or a full range of certain marginal values, or full support and stronger versions of concavity and single crossing of v functions.

Assumption 3.5.3. *The type distribution F_i has a density function f_i that is strictly positive on any $\theta_i \in \Theta_i$ and $mv_{i,\theta_i}, mv_{i,q_i}$ are bounded away from zero.*

Lemma 3.5.3. *Under Assumptions 3.5.1 to 3.5.3, for any fixed θ_{-i} in the support: i) the allocation $q_i(z, \theta_{-i})$ attains zero for some z in the support, and ii) the transfers can be written as:*

$$t_i(q) = \int_0^q mu_i(q_i^{-1}(x, \theta_{-i}), x)dx. \tag{3.9}$$

Proof of Lemma 3.5.3. Pick a trader i, and fix a profile of types θ_{-i}. Next, consider the economy without trader i, that is, solve a system of first order conditions

$mv_j(\theta_j, \tilde{q}_j) = \tilde{p}$ for all $j \neq i$, and $\sum_{j\neq i} \tilde{q}_j = 0$. This solution exists by Lemma 3.5.2 for some $\tilde{p}$. Then there has to be a type z in the support, such that $\tilde{p} = mv_i(z, 0)$, which means that trader i is excluded from trade in the original economy with the profile of types (z, θ_{-i}).

To prove ii) recall the envelope formula:

$$t(\theta) = \tilde{u}(\theta_i, q(\theta)) - \int_{\theta^*}^{\theta_i} \frac{\partial}{\partial \theta} \tilde{u}(x, q(x, \theta_{-i}))dx =$$
$$= \int_{\theta^*}^{\theta_i} \frac{d}{dx} \tilde{u}(x, q(x, \theta_{-i}))dx - \int_{\theta^*}^{\theta_i} \frac{\partial}{\partial \theta} \tilde{u}(x, q(x, \theta_{-i}))dx =$$
$$= \int_{\theta^*}^{\theta_i} \frac{\partial}{\partial q} \tilde{u}(x, q(x, \theta_{-i}))dq(x) = \int_{\theta^*}^{\theta_i} \frac{\partial}{\partial q} u(x, q(x, \theta_{-i}))dq(x)$$

Finally, since for any θ_{-i} there exist a type excluded from trade by i), and since it has to be one of net surplus minimizers by Lemma 3.5.1, we get the formula below:

$$t(\theta) = \int_0^q mu(q^{-1}(x, \theta_{-1}), x)dx.$$

□

We are now ready to derive the taxation scheme $\tau(p, q)$ that would implement the optimal robust mechanism via our auction design. Define the boundary condition for the taxes as $\tau_i(p, 0) = 0$ and the law of motion via any of the fixed points $\hat{\theta}_i(q, p)$ such that the system of equations below holds:

$$m\tau_i(p, q_i) = mu_i(\hat{\theta}_i, q) - p = mu_i(\hat{\theta}_i, q_i) - mv_i(\hat{\theta}_i, q_i). \quad (3.10)$$

Intuitively, this means that when the first order condition (3.7) holds, marginal costs of acquiring an additional unit will be matched to marginal benefits, for each trader, so that they will be incentivized to convey their demands truthfully throughout the auction. Put differently, the cumulative tax function is exactly equal to:

$$\tau(p, q) = \int_0^q [mu_i(\hat{\theta}(x, p), x) - p]dx. \tag{3.11}$$

However, what the player would pay under sincere bidding is the following quantity:

$$t_i(q) = \int_0^q (m\tau(p_{-i}(x), x) + p_{-i}(x))\, dx = \int_0^q mu_i(\hat{\theta}(x, p_{-i}(x)), x)dx, \tag{3.12}$$

where $p_{-i}(x)$ is the residual supply curve, formed by the demands of all other agents.

Finally, note that for any $z \in q^{-1}(x, \theta_{-i})$, the first order condition $p \in mv_i(z, x)$ holds, from which it follows that z is one of the possible fixed points $\hat{\theta}(x, p)$, and the converse is also true. In other words, the range of $\hat{\theta}(x, p_{-i}(x))$ spanned by all possible fixed points, coincides with the range of $q_i^{-1}(x, \theta_{-i})$ from the Taxation Principle, spanned by all possible selections.

This means that the transfers under sincere bidding in the auction coincide with that of the mechanism at points where it is differentiable and, consequently at all points, since they are a.e. continuous in q. It then immediately follows that the sincere equilibrium of our clock auction where the individual price is given by $p + \tau_i(p, q)$ implements this mechanism, as long as the activity rules are not binding.

The construction of the fixed point $\hat{\theta}_i(p, q)$ is formally treated in Proposition 3.5.1, but the slopes can be heuristically derived, by linearizing (3.10):

$$\hat{\theta}'_p = \frac{1}{mv_{i,\theta}} \geqslant 0, \quad \hat{\theta}'_q = -\frac{mv_{i,q}}{mv'_{i,\theta}} \geqslant 0, \quad m\tau'_{i,p} = mu'_{i,\theta}\hat{\theta}'_p - 1, \quad m\tau'_{i,q} = mu'_{i,\theta}\hat{\theta}'_q + mu_{i,q}.$$

Interestingly the slope of demand is proportional to the curvature of the v function:

$$\theta^*(p,q) = \theta \quad \Rightarrow \quad \frac{dq}{dp} = \frac{1}{mv_{i,q}}$$

Proposition 3.5.1. *Under Assumptions 3.5.1 to 3.5.3, there exist a continuous and a.e. differentiable in q taxation function $\tau(p,q)$, such that the optimal mechanism is implemented by the clock implementation.*

Proof of Proposition 3.5.1. Define the boundary condition for the taxes as $\tau_i(p,0) = 0$ and the law of motion via fixed point:

$$m\tau_i(p,q_i) := mu_i(\hat{\theta}_i,q) - p = mu_i(\hat{\theta}_i,q_i) - mv_i(\hat{\theta}_i,q_i). \tag{3.13}$$

From eq. (3.13), one can see that there exists a unique $\hat{\theta}_i$ given p and q_i since the net utility function $\tilde{u}$ is strictly single crossing. Thus, eq. (3.13) defines an ordinary differential equation for the tax function along with the boundary condition above.

Since the sincere equilibrium of the auction game is ex-post IC, by the envelope conditions, the transfers t_i are uniquely defined for any allocation function, up to a constant. However, $\tau(p,0) = 0$ guarantees that, in this equilibrium, the net surplus $\tilde{s}$ attains minimum at $q = 0$. In other words, the aforementioned constant is set to satisfy the IR constraint. Thus, if we prove that the allocation function in the auction coincides with that of the v-optimal mechanism, the equivalence of transfers would automatically follow.

To see the equivalence of allocations, note first that by ex post IC, for any potential stop-off price p, the marginal utility at the final allocation is equal to the stop-off price plus the marginal tax, which is captured by the first order condition

$m\tau_i(p, q_i) = mu_i(q) - p$. Second, by definition of taxes, this implies $p = mv_i(q_i)$ for all agents, which is exactly the first order condition for the v-optimality of the allocation.

It only remains to show that the sincere demands satisfy the activity rules. Note first, that while the v_i functions are formally non-differentiable in the neighborhood of $q_i = 0$, we can still use standard theorems to characterize their behavior separately for $q_i > 0$ and $q_i < 0$, and then extend it to the full support. By applying the Inverse Function Theorem to the first order condition $mv_i(\theta_i, q) = p$, we get that $\frac{dp}{dq} = v''_{qq} < \varepsilon \leqslant 0$, thus the sincere demand is continuous and strictly monotone. □

3.6 Expected Profit Maximizing Mechanisms

Proposition 3.5.1 shows that the mechanism designer may implement objectives other than full efficiency. In this section, we show that expected profit-maximization could be an instance of such objectives that our clock auction implements. There are a couple of additional assumptions that we have to make. One is related to the integrability[2] of net surplus, which is necessary for a proper integration by parts, and requires either a compact support or a finite expected net surplus in the economy.

Assumption 3.6.1. *Either the support is compact, or*

$$\sum \tilde{u}_i(\theta_i, q) \leqslant C(\theta_i) \tag{3.14}$$

for any $q : \sum q_i = 0$ *and some function* $C(x)$*, such that* $\int C(x)dF_i(x) < \infty$*.*

[2]Riemann if dF is continuous

The other is related to the sign of higher order derivative of the net utility function that ensures the local incentive compatibility implies global ones. This is a standard assumption.

Assumption 3.6.2. *The higher order derivatives of the net utility function $\tilde{u}$ satisfies the following properties.*

$$\frac{\partial^3 \tilde{u}_i}{\partial \theta_i^2 \partial q_i}(\theta_i, q_i) \begin{cases} > 0 & q_i > 0 \\ < 0 & q_i < 0 \end{cases} \tag{3.15}$$

$$\frac{\partial^3 \tilde{u}_i}{\partial \theta_i \partial q_i^2}(\theta_i, q_i) \begin{cases} > 0 & q_i > 0 \\ < 0 & q_i < 0 \end{cases} \tag{3.16}$$

With these two additional assumptions on the net utility function, we are ready to state the main point of this section that our double clock auction implements an expected profit maximizing mechanism.

Proposition 3.6.1. *Under Assumptions 3.5.1, 3.5.3, 3.6.1 and 3.6.2 the robust profit maximizing mechanism is optimal with:*

$$v_i(\theta_i, q) = u(\theta_i, q) - \frac{\mathbb{I}(q_i > 0) - F(\theta_i)}{f(\theta_i)} \frac{\partial}{\partial \theta}[u(\theta_i, q) - u(\theta_i, 0)].$$

Proof of Proposition 3.6.1. We start by reminding ourselves that, by definition, a transfer can be rewritten as $t_i = u_i - s_i = \tilde{u}_i - \tilde{s}_i$, therefore in a profit-maximizing mechanism, the allocation maximizes the following objective:

$$\int_{\Theta_{-i}} \int_{\Theta_i} \tilde{u}_i(\theta_i, q_i) - \tilde{s}_i(\theta_i, \theta_{-i}) dF_i(\theta_i) d\mathcal{F}_{-i}(\theta_{-i})$$

subject to the ex-post IC, IR and the MC constraints. Note first that the net surplus function is integrable with respect to dF_i, which means that we can separate it from the net utility. Using the ex-post envelope conditions eq. (3.5), we shall now replace the inner integral over net surplus with something that does not involve transfers, thus effectively eliminating the IC constraints from the analysis. The procedure is fairly similar to the application of integration by parts in mechanism design problems, but adapted to the two-sided setting with the ex-post, rather than Bayesian constraints. Assume, for simplicity, the full support case.

First, split the integral at one of the net surplus minimizers $\theta^*(\theta_{-i})$, which exists by Lemma 3.5.3, and plug in the envelope condition:

$$\int_{\Theta_i} \tilde{s}(\theta_i, \theta_{-i}) dF(\theta_i) = \lim_{N\to\infty} \int_{-N}^{\theta^*} \tilde{s}(\theta_i, \theta_{-i}) dF(\theta_i) + \lim_{N\to\infty} \int_{\theta^*}^{N} \tilde{s}(\theta_i, \theta_{-i}) d(F(\theta_i) - 1)$$

$$\int_{\Theta_i} \tilde{s}(\theta_i, \theta_{-i}) dF(\theta_i) = \lim_{N\to\infty} \int_{-N}^{\theta^*} \int_{\theta^*}^{\theta_i} \tilde{u}_1'(x, q(x, \theta_{-i})) dx dF(\theta_i) + \lim_{N\to\infty} \int_{\theta^*}^{N} \int_{\theta^*}^{\theta_i} \tilde{u}_1'(x, q(x, \theta_{-i})) dx d(F(\theta_i) -$$

Second, observe that we have two terms of the form $\int adb$, where a is absolutely continuous by the Envelope Theorem, and db is a density function, thus Lebesgue integrable. This means, that integration by parts can be applied, under the sign of the limit.

$$-\int_{\Theta_i} \tilde{s}(\theta_i,\theta_{-i})dF(\theta_i) = \lim_{N\to\infty}\int_{-N}^{\theta^*} \tilde{u}_1'(\theta_i, q(\theta_i,\theta_{-i}))F(\theta_i)d\theta_i$$

$$+\lim_{N\to\infty}\int_{\theta^*}^{N} \tilde{u}_1'(\theta_i, q(\theta_i,\theta_{-i}))(F(\theta_i)-1)d\theta_i + +\lim_{N\to\infty} A_N(\theta_{-i}) - \lim_{N\to\infty} B_N(\theta_{-i})$$

where $A_N(\theta_{-i}) = \tilde{s}(-N,\theta_{-i})F(-N)$ and $B_N(\theta_{-i}) = \tilde{s}(N,\theta_{-i})(1-F(N))$.

Third, because $\tilde{s}_i$ convex, we can majorize both A_N and B_N like this:

$$\tilde{s}(N,\theta_{-i})(1-F(N)) \leqslant \int_N^\infty \tilde{s}dF \leqslant \sum\int_N^\infty \tilde{s}_i dF \leqslant \sum\int_N^\infty \tilde{u}_i(q)dF \leqslant \int_N^\infty C(\theta_i)dF(\theta_i)$$

with the latter converging to zero, by absolute continuity of a finite Lebesque integral. What we get is:

$$v_i(\theta_i,\theta_{-i}) = u(\theta_i,q) - u(\theta_i,0) - \frac{\mathbb{I}(\theta_i > \theta^*(\theta_{-i})) - F(\theta_i)}{f(\theta_i)}\frac{\partial}{\partial\theta}[u(\theta_i,q) - u(\theta_i,0)].$$

Note that this is not the final formula yet. Observe that the first appearance (but not the second) of the $u(\theta_i,0)$ term integrates to a constant in the original problem, thus can be dropped. The final ingredient is the claim that we made in Lemma 1 v). Because the set of types excluded from trade, is a subset of surplus minimizers, we can replace the indicator with $\mathbb{I}(q>0)$, without loss of generality. It only remains to verify that this function satisfies Assumption 3.6.1, so that the premise of Proposition 3.5.1 is satisfied.

Note first, that, after the substitution of the indicator, the v_i function is clearly continuous in q. To see this, observe that while $\mathbb{I}(q_i>0)/f(\theta_i)$ creates a downwards jump at $q=0$, the $\frac{\partial}{\partial\theta_i}\tilde{u}$ term is equal to zero at that point. Consequently, the v_i

function experiences a concave kink, rather than a downwards jump.

The resulting function is single crossing and concave due to Assumption 3.6.2.

□

3.7 Conclusion

This paper constructed a clock implementation of direct mechanisms of exchange economies whose objective is not limited to the exact efficiency. At the core of our auction design, there are two concurrent price clocks that run on both—buying and selling—side of the market, which gradually reveals the market price over the course of the implementation and endogenously sorts the traders into their after-trade positions. In addition, we employ individualized price clocks so that the auctioneer may pursue other objectives, such as profit maximization from marketmaking.

Bibliography

Mohammad Akbarpour and Shengwu Li. Credible auctions: A trilemma. *Econometrica*, 88(2):425–467, 2020.

Edward J Anderson and Peter Nash. *Linear programming in infinite-dimensional spaces: theory and applications.* John Wiley & Sons, 1987.

Pasha Andreyanov and Tomasz Sadzik. Robust mechanism design of exchange. *The Review of Economic Studies*, 2021.

Mark Armstrong and David EM Sappington. Recent developments in the theory of regulation. *Handbook of industrial organization*, 3:1557–1700, 2007.

Mark Armstrong and Jidong Zhou. Consumer information and the limits to competition. 2020.

Lawrence M Ausubel. An efficient ascending-bid auction for multiple objects. *American Economic Review*, 94(5):1452–1475, 2004.

Lawrence M Ausubel. An efficient dynamic auction for heterogeneous commodities. *American Economic Review*, 96(3):602–629, 2006.

Felix Bierbrauer and Justus Winkelmann. All or nothing: State capacity and optimal public goods provision. *Journal of Economic Theory*, 185:104955, 2020.

Felix J Bierbrauer and Martin F Hellwig. Robustly coalition-proof incentive mechanisms for public good provision are voting mechanisms and vice versa. *The Review of Economic Studies*, 83(4):1440–1464, 2016.

Tilman Börgers and Daniel Krahmer. *An introduction to the theory of mechanism design*. Oxford University Press, USA, 2015.

Benjamin Brooks and Songzi Du. Maxmin auction design with known expected values. 2021a.

Benjamin Brooks and Songzi Du. Optimal auction design with common values: An informationally robust approach. *Econometrica*, 89(3):1313–1360, 2021b.

Benjamin A Brooks and Songzi Du. Optimal auction design with common values: An informationally-robust approach. *Available at SSRN 3137227*, 2019.

Benjamin A Brooks and Songzi Du. A strong minimax theorem for informationally-robust auction design. 2020.

Vinicius Carrasco, Vitor Farinha Luz, Nenad Kos, Matthias Messner, Paulo Monteiro, and Humberto Moreira. Optimal selling mechanisms under moment conditions. *Journal of Economic Theory*, 177:245–279, 2018.

Ethan Che. Distributionally robust optimal auction design under mean constraints. *arXiv preprint arXiv:1911.07103*, 2019.

Ottorino Chillemi and Benedetto Gui. Team human capital and worker mobility. *Journal of Labor Economics*, 15(4):567–585, 1997.

Daniele Condorelli, Balazs Szentes, et al. Surplus bounds in cournot monopoly and competition. 2020.

Jacques Crémer and Richard P McLean. Full extraction of the surplus in bayesian and dominant strategy auctions. *Econometrica: Journal of the Econometric Society*, pages 1247–1257, 1988.

Enrica Detragiache and Paolo G Garella. Debt restructuring with multiple creditors and the role of exchange offers. *Journal of Financial Intermediation*, 5(3):305–336, 1996.

Piotr Dworczak and Giorgio Martini. The simple economics of optimal persuasion. *Journal of Political Economy*, 127(5):1993–2048, 2019.

Loren K Fryxell. Public good provision with no extortion. 2019.

Matthew Gentzkow and Emir Kamenica. A rothschild-stiglitz approach to bayesian persuasion. *American Economic Review*, 106(5):597–601, 2016.

Thomas A Gresik and Mark A Satterthwaite. The rate at which a simple market converges to efficiency as the number of traders increases: An asymptotic result for optimal trading mechanisms. *Journal of Economic theory*, 48(1):304–332, 1989.

Werner Güth and Martin Hellwig. The private supply of a public good. *Journal of Economics*, 46(1):121–159, 1986.

Emir Kamenica and Matthew Gentzkow. Bayesian persuasion. *American Economic Review*, 101(6):2590–2615, 2011.

Narayana R Kocherlakota and Yangwei Song. Public goods with ambiguity in large economies. *Journal of Economic Theory*, 182:218–246, 2019.

Anton Kolotilin. Optimal information disclosure: A linear programming approach. *Theoretical Economics*, 13(2):607–635, 2018.

Albert S Kyle. Informed speculation with imperfect competition. *The Review of Economic Studies*, 56(3):317–355, 1989.

Yunan Li and Xingtan Zhang. Collective decision through an intermediary. *Available at SSRN 3761473*, 2021.

Simon Loertscher and Leslie M Marx. Asymptotically optimal prior-free clock auctions. *Journal of Economic Theory*, 187:105030, 2020a.

Simon Loertscher and Leslie M Marx. A dominant-strategy asset market mechanism. *Games and Economic Behavior*, 120:1–15, 2020b.

Simon Loertscher and Claudio Mezzetti. A dominant strategy double clock auction with estimation-based tâtonnement. *Theoretical Economics*, 16(3):943–978, 2021.

Hu Lu and Jacques Robert. Optimal trading mechanisms with ex ante unidentified traders. *Journal of Economic Theory*, 97(1):50–80, 2001.

George J Mailath and Andrew Postlewaite. Asymmetric information bargaining problems with many agents. *The Review of Economic Studies*, 57(3):351–367, 1990.

Paul Milgrom and Ilya Segal. Envelope theorems for arbitrary choice sets. *Econometrica*, 70(2):583–601, 2002.

Roger B Myerson. Incentive compatibility and the bargaining problem. *Econometrica: journal of the Econometric Society*, pages 61–73, 1979.

Zvika Neeman. Property rights and efficiency of voluntary bargaining under asymmetric information. *The Review of Economic Studies*, 66(3):679–691, 1999.

Noriaki Okamoto. An efficient ascending-bid auction for multiple objects: Comment. *American Economic Review*, 108(2):555–60, 2018.

Martin Pesendorfer. Pollution claim settlements under correlated information. *journal of economic theory*, 79(1):72–105, 1998.

Rafael Rob. Pollution claim settlements under private information. *Journal of Economic Theory*, 47(2):307–333, 1989.

Anne-Katrin Roesler and Balázs Szentes. Buyer-optimal learning and monopoly pricing. *American Economic Review*, 107(7):2072–80, 2017.

Marzena Rostek and Marek Weretka. Price inference in small markets. *Econometrica*, 80(2):687–711, 2012.

Michael Rothschild and Joseph E Stiglitz. Increasing risk ii: Its economic consequences. *Journal of Economic theory*, 3(1):66–84, 1971.

Patrick W Schmitz. Simple contracts, renegotiation under asymmetric information, and the hold-up problem. *European Economic Review*, 46(1):169–188, 2002.

Moshe Shaked and J George Shanthikumar. *Stochastic orders*. Springer Science & Business Media, 2007.

Alex Suzdaltsev. An optimal distributionally robust auction. *arXiv preprint arXiv:2006.05192*, 2020.

Jean Tirole. *The theory of industrial organization*. MIT press, 1988.

Jin Xi and Haitian Xie. Strength in numbers: Robust mechanisms for public goods with many agents. *arXiv preprint arXiv:2101.02423*, 2021.

Wanchang Zhang. Robust bilateral trade mechanisms with known expectations. *arXiv preprint arXiv:2105.05427*, 2021.