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HAMILTONIAN FORMULATION OF OPEN WZW STRINGS

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Abstract

Using a Hamiltonian approach, we construct the classical and quantum theory of open WZW strings on a strip. (These are the strings which end on WZW branes.) The development involves non-abelian generalized Dirichlet images in an essential way. At the classical level, we find a new non-commutative geometry in which the equal-time coordinate brackets are non-zero at the world-sheet boundary, and the result is an intrinsically non-abelian effect which vanishes in the abelian limit. Using the classical theory as a guide to the quantum theory, we also find the operator algebra and the analogue of the Knizhnik-Zamolodchikov equations for the conformal field theory of open WZW strings.

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1 Introduction

Affine Lie algebra [1, 2, 3] has played a central role in string theory over the decades, first in open string theory, then closed string theory and now back again to open strings. We refer in particular to the theory[a] of D-branes [6, 7], open descendants [8, 9] and WZW branes [10-24]. The present paper is concerned with open WZW strings, which end on WZW branes.

In particular, we shall construct the classical and quantum theory of open WZW strings on a strip, following the principles:

- At the quantum level, open string WZW dynamics on the Lie algebra g must follow from the Hamiltonian

$$H_g = L_g(0) = L_g^{ab} \sum_{m \in \mathbb{Z}} : J_a(m) J_b(-m) : \tag{1.1}$$

  where the current modes $\{J_a(m)\}$ generate the affine Lie algebra of g and $L_g(0)$ is the zero mode of the affine-Sugawara construction [2, 25-28, 3] on g. Thus, open WZW strings are controlled by a single non-abelian chiral current.

- The dynamics of open WZW strings on the strip $0 \le \xi \le \pi$ must be locally WZW in the bulk

$$0 < \xi < \pi \tag{1.2}$$

  that is, the same as the ordinary WZW model [29, 30]. The classical group elements $g(T, \xi, t)$ of the open WZW string must satisfy generalized Dirichlet boundary conditions [14]

$$g^{-1}(T,\xi,t)\partial_+ g(T,\xi,t) = g(T,\xi,t)\partial_- g^{-1}(T,\xi,t) \quad \text{at } \xi = 0, \pi \tag{1.3}$$

  at the boundary of the strip.

As we will see, both the classical and quantum theory involve generalized non-abelian Dirichlet image charges in an essential way.

At the classical level (see Secs. 2-6), we find that the single non-abelian chiral current algebra determines both the phase space and the coordinate space formulation of open WZW theory. In the phase space formulation, we obtain the complete bracket algebra of the theory and, in particular, we find a new equal-time non-commutative geometry in which the coordinate brackets

$$\{x^i(\xi,t), x^j(\eta,t)\} = \begin{cases} \neq 0 & \text{if } \xi = \eta = 0 \text{ or } \pi \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

are non-zero at the boundary. This effect (which is not related in any simple sense to known [31-35] non-commutative effects) is intrinsically non-abelian and vanishes in the abelian limit.

[a]Early papers on Dirichlet boundary conditions in string theory include Refs. [4, 5].

Using the classical theory as a guide, the quantum theory of open strings is constructed in Sec. 7. Here we find the operator algebra of the theory, the differential equations for the vertex operators and the analogue of the Knizhnik-Zamolodchikov equations [26, 27] for the conformal field theory of open WZW strings. We mention in particular that the open string vertex operators $g(T)$ can be factorized into chiral vertex operators $g_\pm(T)$

$$g(T) = g_-(T)g_+(T), \quad \partial_- g_+(T) = \partial_+ g_-(T) = 0 \tag{1.5}$$

but, in distinction to ordinary WZW theory, $g_-(T)$ and $g_+(T)$ do not form independent subspaces and the open WZW string correlators do not factorize into left and right mover correlators.

2 Open WZW Strings

2.1 Quantum Formulation

The Hamiltonian H of any open string theory is the zero mode $L(0)$ of a single set of Virasoro generators $L(m)$, so open WZW string theory on the Lie algebra g is described by the zero mode $L_g(0)$ of the affine-Sugawara construction [2, 25-28, 3] on g

$$H_g = L_g(0) = \frac{1}{2\pi}\int_0^{2\pi} d\xi \; L_g^{ab} : J_a(\xi,t)J_b(\xi,t) := L_g^{ab}\sum_{m\in\mathbb{Z}} : J_a(m)J_b(-m) : \tag{2.1a}$$

$$[J_a(\xi,t), J_b(\eta,t)] = 2\pi i\big(f_{ab}{}^c J_c(\xi,t)\delta(\xi-\eta) + G_{ab}\partial_\xi\delta(\xi-\eta)\big) \tag{2.1b}$$

$$\delta(\xi\mp\eta) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{im(\xi\mp\eta)}, \quad 0\le\xi,\eta\le 2\pi, \quad a,b,c = 1,\ldots \dim g \; . \tag{2.1c}$$

Here $f_{ab}{}^c$ and G_{ab} are the structure constants and metric of g

$$g = \oplus_I g^I, \quad f_{ab}{}^c = \oplus_I f_{ab}^{I\;c}, \quad G_{ab} = \oplus_I k_I \eta_{ab}^I \tag{2.2}$$

where I is the semisimplicity index and k_I is the level of the simple component g^I. The coefficient L_g^{ab}

$$L_g^{ab} = \oplus_I \frac{\eta_I^{ab}}{2k_I + Q_I} \xrightarrow{\{k_I\}\gg 1} L_{g,\infty}^{ab} \equiv \frac{G^{ab}}{2}, \quad G^{ab} = \oplus_I k_I^{-1}\eta_I^{ab} \tag{2.3}$$

is the inverse inertia tensor of the affine-Sugawara construction on g.

The Hamiltonian (2.1a) guarantees the chirality of the single non-abelian chiral current $J(\xi,t)$

$$\partial_t A(\xi,t) = i[H_g, A(\xi,t)] \tag{2.4a}$$

$$\partial_- J_a(\xi,t) = 0, \quad \partial_- \equiv \partial_t - \partial_\xi \tag{2.4b}$$

$$J_a(\xi,t) = \sum_{m\in\mathbb{Z}} J_a(m)e^{-im(t+\xi)} \tag{2.4c}$$

$$[J_a(m), J_b(n)] = if_{ab}{}^c J_c(m+n) + mG_{ab}\delta_{m+n,0} \tag{2.4d}$$

and Eq. (2.4d) is the affine Lie algebra [1, 2, 3] of g.

The affine-Sugawara Hamiltonian above is defined on the cylinder $0 \le \xi \le 2\pi$, but it is conventional to consider open string theory on the strip $0 \le \xi \le \pi$. Since the single chiral current is periodic when $\xi \to \xi + 2\pi$, the affine-Sugawara Hamiltonian may be rewritten in the open string picture

$$H_g = \frac{1}{2\pi} \int_0^\pi d\xi\ L_g^{ab} : \big(J_a(\xi,t)J_b(\xi,t) + J_a(-\xi,t)J_b(-\xi,t)\big) : \tag{2.5}$$

but where are the WZW branes? In what follows we answer this question first in the classical formulation of the theory, returning to the quantum theory in Sec. 7.

2.2 Classical Currents and Stress Tensors

The classical version of the affine-Sugawara system on the cylinder is

$$H_g = \int_0^{2\pi} d\xi\ T_g(\xi,t) \tag{2.6a}$$

$$T_g(\xi,t) = \frac{1}{2\pi} L_{g,\infty}^{ab} J_a(\xi,t)J_b(\xi,t) = \frac{1}{4\pi} G^{ab} J_a(\xi,t)J_b(\xi,t) \tag{2.6b}$$

$$J_a(\xi + 2\pi n, t) = J_a(\xi,t), \quad T_g(\xi + 2\pi n, t) = T_g(\xi,t) \tag{2.6c}$$

$$\{J_a(\xi,t), J_b(\eta,t)\} = 2\pi i\big(f_{ab}{}^c J_c(\xi,t)\delta(\xi-\eta) + G_{ab}\partial_\xi\delta(\xi-\eta)\big) \tag{2.6d}$$

$$\{T_g(\xi,t), T_g(\eta,t)\} = i\big(T_g(\xi,t) + T_g(\eta,t)\big)\partial_\xi\delta(\xi-\eta) \tag{2.6e}$$

$$\{T_g(\xi,t), J_a(\eta,t)\} = iJ_a(\xi,t)\partial_\xi\delta(\xi-\eta) \tag{2.6f}$$

$$0 \le \xi,\eta \le 2\pi \tag{2.6g}$$

where $J_a(\xi,t)$ and $T_g(\xi,t)$ are the classical non-abelian chiral current and its classical stress tensor respectively. In (2.6) $\{\cdot,\cdot\}$ are Poisson brackets multiplied by a convenient extra factor of i.

To go to the open string picture, we decompose the single non-abelian cylinder current and its single stress tensor into components on the strip as follows:

$$J_a(\xi,t),\ 0 \le \xi \le 2\pi \quad \to \quad J_a(\pm\xi,t),\ 0 \le \xi \le \pi \tag{2.7a}$$

$$T_g(\xi,t),\ 0 \le \xi \le 2\pi \quad \to \quad T_g(\pm\xi,t),\ 0 \le \xi \le \pi \tag{2.7b}$$

$$T_g(\pm\xi,t) = \frac{1}{4\pi} G^{ab} J_a(\pm\xi,t)J_b(\pm\xi,t) \tag{2.7c}$$

$$H_g = \int_0^\pi d\xi\ \big(T_g(\xi,t) + T_g(-\xi,t)\big) \tag{2.7d}$$

$$= \frac{1}{4\pi} \int_0^\pi d\xi\ G^{ab}\big(J_a(\xi,t)J_b(\xi,t) + J_a(-\xi,t)J_b(-\xi,t)\big)\ . \tag{2.7e}$$

The bracket algebra of the components on the strip $0 \leq \xi, \eta \leq \pi$ then follows from (2.6c-e)

$$\{J_a(\xi,t), J_b(\eta,t)\} = 2\pi i\big(f_{ab}{}^c J_c(\xi,t)\delta(\xi-\eta) + G_{ab}\partial_\xi\delta(\xi-\eta)\big) \tag{2.8a}$$

$$\{J_a(\xi,t), J_b(-\eta,t)\} = 2\pi i\big(f_{ab}{}^c J_c(\xi,t)\delta(\xi+\eta) + G_{ab}\partial_\xi\delta(\xi+\eta)\big) \tag{2.8b}$$

$$\{J_a(-\xi,t), J_b(-\eta,t)\} = 2\pi i\big(f_{ab}{}^c J_c(-\xi,t)\delta(\xi-\eta) - G_{ab}\partial_\xi\delta(\xi-\eta)\big) \tag{2.8c}$$

$$\{T_g(\xi,t), T_g(\eta,t)\} = i\big(T_g(\xi,t) + T_g(\eta,t)\big)\partial_\xi\delta(\xi-\eta) \tag{2.9a}$$

$$\{T_g(\xi,t), T_g(-\eta,t)\} = i\big(T_g(\xi,t) + T_g(-\eta,t)\big)\partial_\xi\delta(\xi+\eta) \tag{2.9b}$$

$$\{T_g(-\xi,t), T_g(-\eta,t)\} = -i\big(T_g(-\xi,t) + T_g(-\eta,t)\big)\partial_\xi\delta(\xi-\eta) \tag{2.9c}$$

$$\{T_g(\xi,t), J_a(\eta,t)\} = iJ_a(\xi,t)\partial_\xi\delta(\xi-\eta) \tag{2.10a}$$

$$\{T_g(\xi,t), J_a(-\eta,t)\} = iJ_a(\xi,t)\partial_\xi\delta(\xi+\eta) \tag{2.10b}$$

$$\{T_g(-\xi,t), J_a(\eta,t)\} = -iJ_a(-\xi,t)\partial_\xi\delta(\xi+\eta) \tag{2.10c}$$

$$\{T_g(-\xi,t), J_a(-\eta,t)\} = -iJ_a(-\xi,t)\partial_\xi\delta(\xi-\eta)\ . \tag{2.10d}$$

In (2.8-2.10), the delta function $\delta(\xi-\eta)$ has support only at $\xi=\eta$, while the delta function $\delta(\xi+\eta)$ has support only at the strip boundary $\xi=\eta=0$ or π. We will refer to terms proportional to $\delta(\xi-\eta)$ and $\delta(\xi+\eta)$ as bulk and boundary terms respectively. In what follows, we interpret the form of the strip current algebra (2.8), remarking only that a similar interpretation applies to (2.9) and (2.10).

In discussing (2.8), it is instructive to bear in mind the equal-time current algebra of affine $(g \times g)$

$$\{J_a(\xi,t), J_b(\eta,t)\} = 2\pi i\big(f_{ab}{}^c J_c(\xi,t)\delta(\xi-\eta) + G_{ab}\partial_\xi\delta(\xi-\eta)\big) \tag{2.11a}$$

$$\{J_a(\xi,t), \bar{J}_b(\eta,t)\} = 0 \tag{2.11b}$$

$$\{\bar{J}_a(\xi,t), \bar{J}_b(\eta,t)\} = 2\pi i\big(f_{ab}{}^c \bar{J}_c(\xi,t)\delta(\xi-\eta) - G_{ab}\partial_\xi\delta(\xi-\eta)\big) \tag{2.11c}$$

$$0 \leq \xi,\eta \leq 2\pi \tag{2.11d}$$

which holds in the ordinary WZW model. One sees that the brackets (2.8a) and (2.8c) are locally isomorphic under $J(-\xi,t) \to \bar{J}(\xi,t)$ to the brackets (2.11a) and (2.11c), but (2.8b) tells us that this isomorphism fails at the boundary of the strip, where

$$\{J_a(\xi,t), J_b(-\eta,t)\} \neq 0 \quad \text{at } \xi=\eta=0 \,\text{or}\, \pi \tag{2.12}$$

due to the boundary term $\delta(\xi+\eta)$. As we will see, this difference controls many important aspects of open WZW theory.

Although (2.8b) is non-zero at the boundary, the strip current system (2.8) is locally WZW in the bulk, where the boundary terms do not contribute.

We may also interpret the boundary terms proportional to $\delta(\xi+\eta)$ in (2.8-2.10) as due to the interaction of a non-abelian charge at ξ (or η) with a generalized non-abelian Dirichlet image charge at $-\eta$ (or $-\xi$).

2.3 Classical Dynamics

In classical open WZW string theory, the time dependence of the fields is determined by

$$\partial_t A(\xi,t) = i\{H_g, A(\xi,t)\} \tag{2.13}$$

where H_g is given in (2.7e). For such computations, it is useful to record the following identities

$$\int_0^\pi d\eta\; \delta(\eta-\xi)f(\eta) = \begin{cases} \frac{1}{2}f(0) & \text{if } \xi=0 \\ f(\xi) & \text{if } 0<\xi<\pi \\ \frac{1}{2}f(\pi) & \text{if } \xi=\pi, \end{cases} \tag{2.14a}$$

$$\int_0^\pi d\eta\; \delta(\eta+\xi)f(\eta) = \begin{cases} \frac{1}{2}f(0) & \text{if } \xi=0 \\ 0 & \text{if } 0<\xi<\pi \\ \frac{1}{2}f(\pi) & \text{if } \xi=\pi \end{cases} \tag{2.14b}$$

which will allow us to evaluate brackets of integrated quantities. Then we find from (2.13) and (2.14) that the strip currents and stress tensors are chiral

$$\partial_- J_a(\xi,t) = \partial_+ J_a(-\xi,t) = 0, \quad \partial_\pm \equiv \partial_t \pm \partial_\xi \tag{2.15a}$$

$$J_a(\xi,t) = \sum_{m\in\mathbb{Z}} J_a(m) e^{-im(t+\xi)}, \quad J_a(-\xi,t) = \sum_{m\in\mathbb{Z}} J_a(m) e^{-im(t-\xi)} \tag{2.15b}$$

$$\{J_a(m), J_b(n)\} = i f_{ab}{}^c J_c(m+n) + m G_{ab}\delta_{m+n,0} \tag{2.15c}$$

$$\partial_- T(\xi,t) = \partial_+ T(-\xi,t) = 0 \tag{2.15d}$$

as expected from the quantum theory.

We may also consider the natural candidate for a momentum operator, which we call the bulk momentum operator P:

$$P_g(t) \equiv \int_0^\pi d\xi\; \big(T_g(\xi,t) - T_g(-\xi,t)\big) \tag{2.16a}$$

$$= \frac{1}{4\pi}\int_0^\pi d\xi\; G^{ab}\big(J_a(\xi,t)J_b(\xi,t) - J_a(-\xi,t)J_b(-\xi,t)\big) \tag{2.16b}$$

$$\partial_t P_g(t) = \frac{G^{ab}}{2\pi}\big(J_a(\pi,t)J_b(\pi,t) - J_a(0,t)J_b(0,t)\big)\,. \tag{2.16c}$$

According to its name, iP generates ∂_ξ only in the bulk. In the case of the currents, one finds for example

$$i\{P_g(t), J_a(\pm\xi,t)\} = \partial_\xi \left\{ \begin{array}{ll} J_a(\pm\xi,t) & \text{if } 0<\xi<\pi \\ 0 & \text{if } \xi = 0,\pi \end{array} \right. \tag{2.17}$$

where we have used the equal-time current algebra (2.8) and the relations (2.14). In fact (as seen in this example) the correct form of $\partial_\xi A$ for any A can be obtained from the form of $i\{P,A\}$ in the bulk

$$\partial_\xi A = i\{P_g(t), A\}, \quad 0<\xi<\pi \tag{2.18}$$

by smoothly extending this form to the boundary. This observation will be useful in the quantum theory.

3 Phase Space Realization of the Currents

In what follows, we postulate the following phase space realization[b] of the equal-time current algebra (2.8)

$$J_a(\xi,t) \equiv 2\pi e(x(\xi,t))_a{}^i p_i(B,\xi,t) + \frac{1}{2}\partial_\xi x^i(\xi,t) e(x(\xi,t))_i{}^b G_{ba} \tag{3.1a}$$

$$J_a(-\xi,t) \equiv 2\pi \bar{e}(x(\xi,t))_a{}^i p_i(B,\xi,t) - \frac{1}{2}\partial_\xi x^i(\xi,t) \bar{e}(x(\xi,t))_i{}^b G_{ba} \tag{3.1b}$$

$$0 \le \xi \le \pi \,. \tag{3.1c}$$

The quantities which appear in (3.1) are defined as follows

$$e_i(T) \equiv e_i{}^a T_a = -ig^{-1}(T)\partial_i g(T), \quad \bar{e}_i(T) \equiv \bar{e}_i{}^a T_a = -ig(T)\partial_i g^{-1}(T) \tag{3.2a}$$

$$e_i{}^a e_a{}^j = \bar{e}_i{}^a \bar{e}_a{}^j = \delta_i^j \tag{3.2b}$$

$$p_i(B,\xi,t) = p_i(\xi,t) + \frac{1}{4\pi} B_{ij}(x(\xi,t))\partial_\xi x^j(\xi,t) \tag{3.2c}$$

$$\partial_i B_{jk}(x) + \partial_j B_{ki}(x) + \partial_k B_{ij}(x) = -i\,\mathrm{Tr}\big(\, M(k,T) e_i(x,T)[e_j(x,T), e_k(x,T)]\,\big) \tag{3.2d}$$

$$[T_a,T_b] = if_{ab}{}^c T_c, \quad \mathrm{Tr}\big(M(k,T)T_aT_b\big) = G_{ab}, \quad T_a = \oplus_I T_a^I, \quad M(k,T) \equiv \oplus_I \frac{k_I}{y_I(T^I)} \tag{3.2e}$$

where T_a is any matrix irrep of the Lie algebra g, $g(T)$ are the group elements in irrep T and $e_i{}^a$ and $\bar{e}_i{}^a$ are the left and right invariant vielbeins on the group manifold, respectively. The antisymmetric tensor B_{ij} is the B field of the open WZW string and the data matrix

[b] Other realizations of the equal time current algebra can be obtained by replacing $J_a(-\xi,t)$ on the left of (3.1b) by $J_a^\omega(-\xi,t) = \omega_a{}^b J_b(-\xi,t)$, where ω is an element of the automorphism group of g. This leads to the theory of twisted open WZW strings, which will not be discussed in this paper.

M stores information about the Dynkin indices $y_I(T)$ of the simple factors T^I. In (3.2), the matrices T and $g(T)$ are square, and matrix multiplication is defined as

$$e_i{}^a(T_a)_\alpha{}^\beta = -ig^{-1}(T)_\alpha{}^\gamma \partial_i g(T)_\gamma{}^\beta, \quad \alpha, \beta, \gamma = 1, \dots, \dim T \tag{3.3}$$

with a summation convention for repeated indices.

In Eq. (3.1), our phase space realization of the currents $J(\xi, t)$ and $J(-\xi, t)$, $0 \le \xi \le \pi$ is the usual [36] WZW phase space realization of the left and the right mover currents $J(\xi, t)$ and $\bar{J}(\xi, t)$, $0 \le \xi \le 2\pi$. As we shall see, this realization will allow us to require that the open WZW string has the same bulk dynamics ($0 < \xi < \pi$) as the WZW model.

Indeed, Eq. (3.1) gives the phase space form of the Hamiltonian

$$H_g = \int_0^\pi d\xi\ \mathcal{H}_g \tag{3.4a}$$

$$\mathcal{H}_g = \frac{G^{ab}}{4\pi}(J_a(\xi)J_b(\xi) + J_a(-\xi)J_b(-\xi)) = 2\pi G^{ij} p_i(B) p_j(B) + \frac{1}{8\pi} \partial_\xi x^i \partial_\xi x^j G_{ij} \tag{3.4b}$$

$$G_{ij} = e_i{}^a e_j{}^b G_{ab} = \bar{e}_i{}^a \bar{e}_j{}^b G_{ab} \tag{3.4c}$$

whose density $\mathcal{H}_g$ is the same as that of the WZW model. The bulk momentum operator

$$P_g(t) = \int_0^\pi d\xi\ \partial_\xi x^i(\xi, t) p_i(\xi, t) \tag{3.5}$$

also has the usual WZW momentum density.

Moreover, the equal-time current algebra (2.8) and its phase space realization (3.1) imply a system of constraints on the phase space variables, which will allow us to obtain the phase space bracket algebra itself. As a first example of these constraints, we note two equivalent forms of the boundary conditions

$$J_a(0, t) = J_a(-0, t), \quad J_a(\pi, t) = J_a(-\pi, t) \quad \text{or} \tag{3.6a}$$

$$4\pi(\bar{e}(\xi, t)_a{}^i - e(\xi, t)_a{}^i) p_i(B, \xi, t) = \partial_\xi x^i(\xi, t)(\bar{e}(\xi, t)_i{}^b + e(\xi, t)_i{}^b) G_{ba} \quad \text{at } \xi = 0, \pi \tag{3.6b}$$

which follow from the 2π periodicity of the cylinder current. We will see below that these boundary conditions are equivalent to the generalized Dirichlet boundary conditions of Ref. [14].

4 Phase Space Algebra: First Results

4.1 The Inverse Relations and $\partial_\xi g$

In this section, we will use the equal-time current algebra (2.8) and its phase space realization (3.1) to begin our analysis of the phase space brackets of open WZW theory.

In preparation for this analysis, we will need the inverse relations

$$\partial_\xi x^i(\xi,t) = J_a(\xi,t)G^{ab}e(\xi,t)_b{}^i - J_a(-\xi,t)G^{ab}\bar{e}(\xi,t)_b{}^i \tag{4.1a}$$

$$p_i(B,\xi,t) = \frac{1}{4\pi}\big(e(\xi,t)_i{}^a J_a(\xi,t) + \bar{e}(\xi,t)_i{}^a J_a(-\xi,t)\big) \tag{4.1b}$$

which follow from (3.1).

As a first application of the inverse relations, we note the following form of $\partial_\xi g$

$$\partial_\xi g(T,\xi,t) = \partial_\xi x^i \partial_i g(T,\xi,t) = i\big(g(T,\xi,t)J(T,\xi,t) + J(T,-\xi,t)g(T,\xi,t)\big) \tag{4.2a}$$

$$J(T,\pm\xi,t) \equiv J_a(\pm\xi,t)G^{ab}T_b \tag{4.2b}$$

which is obtained by chain rule from (4.1) and (3.2). The result (4.2) is the usual form (with $\bar{J}(\xi) \to J(-\xi)$) of $\partial_\xi g$ in the WZW model.

4.2 Bracket of J with x

Using the equal-time current algebra (2.8) and the inverse relations (4.1), we may derive a differential equation for the bracket of $J(\xi,t)$ with x,

$$\begin{aligned}
\partial_\eta\{J_a(\xi,t),x^i(\eta,t)\} =& \{J_a(\xi,t),\partial_\eta x^i(\eta,t)\} \\
=& \{J_a(\xi), J_b(\eta)G^{bc}e(\eta)_c{}^i - J_b(-\eta)G^{bc}\bar{e}(\eta)_c{}^i\} \\
=& 2\pi i\big(f_{ab}{}^c J_c(\xi)\delta(\xi-\eta) + G_{ab}\partial_\xi\delta(\xi-\eta)\big)G^{bd}e(\eta)_d{}^i \\
&- 2\pi i\big(f_{ab}{}^c J_c(\xi)\delta(\xi+\eta) + G_{ab}\partial_\xi\delta(\xi+\eta)\big)G^{bd}\bar{e}(\eta)_d{}^i \\
&+ \{J_a(\xi),x^j(\eta)\}\big(\partial_j e(\eta)_b{}^i G^{bc}J_c(\eta) - \partial_j\bar{e}(\eta)_b{}^i G^{bc}J_c(-\eta)\big) .
\end{aligned} \tag{4.3}$$

The same equation with $\xi \to -\xi$ is obtained for the bracket of $J(-\xi,t)$ with $x(\eta,t)$. We have checked that the ordinary WZW bracket

$$\{J_a(\xi,t),x^i(\eta,t)\}_{\mathrm{WZW}} = -2\pi i e(\eta,t)_a{}^i\delta(\xi-\eta) \tag{4.4}$$

is not a solution to (4.3). A particular solution to these equations is

$$\{J_a(\xi,t),x^i(\eta,t)\} = -2\pi i\big(e(\eta,t)_a{}^i\delta(\xi-\eta) + \bar{e}(\eta,t)_a{}^i\delta(\xi+\eta)\big) \tag{4.5a}$$

$$\{J_a(-\xi,t),x^i(\eta,t)\} = -2\pi i\big(\bar{e}(\eta,t)_a{}^i\delta(\xi-\eta) + e(\eta,t)_a{}^i\delta(\xi+\eta)\big) . \tag{4.5b}$$

Among these terms only the bulk terms (proportional to $\delta(\xi-\eta)$) are present in the ordinary WZW model, while the terms proportional to $\delta(\xi+\eta)$ are boundary terms which represent non-abelian generalized Dirichlet images.

In finding these solutions from (4.3), we used the relation

$$J_a(\xi,t)\delta(\xi+\eta) = J_a(-\eta,t)\delta(\xi+\eta), \quad 0 \le \xi,\eta \le \pi \tag{4.6}$$

which holds because the cylinder current is 2π periodic. Verification of these solutions requires considerable algebra: In particular, one needs the explicit form (3.1) of $J(\pm\xi,t)$ and the Cartan-Maurer identities, as well as the identities

$$\bar{e}_i{}^a = -e_i{}^b\Omega_b{}^a, \quad \partial_i\Omega_a{}^b = f_{ac}{}^d e_i{}^c\Omega_d{}^b, \quad \Omega_a{}^c\Omega_b{}^d G_{cd} = G_{ab} \tag{4.7}$$

where Ω is the adjoint action.

The general solution to the differential equations for these brackets is obtained by adding to the particular solution (4.5a-b) the terms

$$\delta\{J_a(\xi,t), x^i(\eta,t)\} = f(\xi,t)B(\eta,t)_a{}^i \tag{4.8a}$$

$$\delta\{J_a(-\xi,t), x^i(\eta,t)\} = h(\xi,t)B(\eta,t)_a{}^i \tag{4.8b}$$

$$f(0,t) = h(0,t), \quad f(\pi,t) = h(\pi,t) \tag{4.8c}$$

$$\partial_\eta B(\eta,t)_a{}^i = B(\eta,t)_a{}^j\big(\partial_j e(\eta,t)_b{}^i G^{bc}J_c(\eta,t) - \partial_j \bar{e}(\eta,t)_b{}^i G^{bc}J_c(-\eta,t)\big) \tag{4.8d}$$

where $f(\xi,t)$ and $h(\xi,t)$ are arbitrary except for the boundary conditions in (4.8c).

Following our program, we set these terms to zero in order to maintain the ordinary WZW term $-2\pi i e_a{}^i\delta(\xi-\eta)$ in the bulk.

Note that the solutions (4.5) are consistent with the generalized Dirichlet boundary conditions (3.6a) because

$$\delta(-\eta) = \delta(\eta), \quad \delta(\pi-\eta) = \delta(\pi+\eta). \tag{4.9}$$

Moreover, the boundary terms proportional to $\delta(\xi+\eta)$ are necessary for this consistency.

By chain rule from the brackets of $J(\pm\xi,t)$ with x, we also find the brackets of the currents with the group elements

$$\{J_a(\xi,t), g(T,\eta,t)\} = 2\pi\big(\delta(\xi-\eta)g(T,\eta,t)T_a - \delta(\xi+\eta)T_a g(T,\eta,t)\big) \tag{4.10a}$$

$$\{J_a(-\xi,t), g(T,\eta,t)\} = 2\pi\big(-\delta(\xi-\eta)T_a g(T,\eta,t) + \delta(\xi+\eta)g(T,\eta,t)T_a\big)\,. \tag{4.10b}$$

As seen for $\{J(\pm\xi), x\}$, these brackets are also consistent with the generalized Dirichlet boundary conditions (3.6a). Here the first relation shows a right rotation in the bulk and a left rotation at the boundary, and vice-versa for the second relation. The brackets (4.10) will be central in the quantization of the open WZW string in Sec. 7.

As an application of Eqs. (4.10) and (2.14), we may compute the action of the bulk momentum operator (2.16) on the group elements

$$i\{P_g(t), g(T,\xi,t)\} = \begin{cases} i\big(g(T,\xi,t)J(T,\xi,t) + J(T,-\xi,t)g(T,\xi,t)\big) & \text{if } 0<\xi<\pi \\ 0 & \text{if } \xi = 0,\pi. \end{cases} \tag{4.11}$$

Comparing this result with $\partial_\xi g$ in Eq. (4.2), we see that the correct form of $\partial_\xi g$ can be obtained by smoothly extending to the boundary the form of $i\{P,g\}$ obtained in the bulk. The same conclusion was obtained for the currents in (2.17).

5 Coordinate Space

5.1 Equations of Motion

We postpone further study of the phase space bracket algebra, because we already know enough to make the transition to coordinate space.

Consider the computation

$$\begin{aligned}
\partial_t x^i(\xi,t) =& i\{H_g, x^i(\xi,t)\} \\
=& G^{ab}\int_0^\pi d\eta\,\big[\big(e(\xi)_a{}^i J_b(\eta) + \bar{e}(\xi)_a{}^i J_b(-\eta)\big)\delta(\eta-\xi) \\
&\qquad + \big(\bar{e}(\xi)_a{}^i J_b(\eta) + e(\xi)_a{}^i J_b(-\eta)\big)\delta(\eta+\xi)\big] \\
=& G^{ab}\big(e(\xi)_a{}^i J_b(\xi) + \bar{e}(\xi)_a{}^i J_b(-\xi)\big) \\
=& 4\pi G^{ij}(\xi,t)p_j(B,\xi,t)
\end{aligned} \tag{5.1a}$$

$$p_i(B,\xi,t) = \frac{1}{4\pi}G_{ij}(\xi,t)\partial_t x^j(\xi,t) \tag{5.1b}$$

where we have used the results (4.5), (2.14) and (3.6a). This is the usual local relation between $\partial_t x$ and p in the WZW model. Then the relations (5.1), (4.1) and the chain rule give the time-derivative of the group elements in terms of the currents. We record this result along with the space-derivative of g derived in (4.2):

$$\partial_t g(T,\xi,t) = i\big(g(T,\xi,t)J(T,\xi,t) - J(T,-\xi,t)g(T,\xi,t)\big) \tag{5.2a}$$

$$\partial_\xi g(T,\xi,t) = i\big(g(T,\xi,t)J(T,\xi,t) + J(T,-\xi,t)g(T,\xi,t)\big) \tag{5.2b}$$

$$\partial_+ g(T,\xi,t) = 2ig(T,\xi,t)J(T,\xi,t),\quad \partial_- g(T,\xi,t) = -2iJ(T,-\xi,t)g(T,\xi,t)\ . \tag{5.2c}$$

The same result for $\partial_t g$ can be obtained from (2.13) and the brackets $\{J(\pm\xi), g\}$. From the phase space realization of the currents and the relation between p and $\partial_t x$, we obtain the coordinate space form of J:

$$J_a(\xi,t) = \frac{1}{2}\partial_+ x^i(\xi,t)e(\xi,t)_i{}^b G_{ba},\quad J_a(-\xi,t) = \frac{1}{2}\partial_- x^i(\xi,t)\bar{e}(\xi,t)_i{}^b G_{ba} \tag{5.3a}$$

$$J(T,\xi,t) = -\frac{i}{2}g^{-1}(T,\xi,t)\partial_+ g(T,\xi,t),\quad J(T,-\xi,t) = -\frac{i}{2}g(T,\xi,t)\partial_- g^{-1}(T,\xi,t) \tag{5.3b}$$

$$\partial_-(g^{-1}(T,\xi,t)\partial_+ g(T,\xi,t)) = \partial_+(g(T,\xi,t)\partial_- g^{-1}(T,\xi,t)) = 0\ . \tag{5.3c}$$

The results in (5.3b) are equivalent to Eq. (5.2c). The relations in (5.3c) (which are the usual WZW equations of motion) follow from (5.3b) and the chirality (2.15a) of the currents.

5.2 Generalized Dirichlet Boundary Conditions

Using (5.1b), we see that the boundary conditions (3.6) can now be written in the following three equivalent forms:

$$J(0,t) = J(-0,t), \quad J(\pi,t) = J(-\pi,t) \quad \text{or} \tag{5.4a}$$

$$\partial_t x^i(\xi,t)(\bar{e}(\xi,t)_i{}^a - e(\xi,t)_i{}^a) = \partial_\xi x^i(\xi,t)(\bar{e}(\xi,t)_i{}^a + e(\xi,t)_i{}^a) \quad \text{or} \tag{5.4b}$$

$$g^{-1}(T,\xi,t)\partial_+ g(T,\xi,t) = g(T,\xi,t)\partial_- g^{-1}(T,\xi,t) \quad \text{at } \xi = 0,\pi \tag{5.4c}$$

where (5.4c) is the generalized Dirichlet boundary condition of Ref. [14]. The result (5.4) shows that our open WZW strings end on WZW branes.

5.3 The Bulk Lagrange Density

We define a bulk Lagrange density $\mathcal{L}_g$ for the open WZW string by the usual Legendre transformation

$$\mathcal{L}_g \equiv \partial_t x^i p_i - \mathcal{H}_g = \frac{1}{8\pi}(G_{ij} + B_{ij})\partial_+ x^i \partial_- x^j, \quad 0 < \xi < \pi \tag{5.5a}$$

$$\frac{1}{8\pi}G_{ij}\partial_+ x^i \partial_- x^j = -\frac{1}{8\pi}\mathrm{Tr}\big(M(k,T)g^{-1}(T)\partial_+ g(T) g^{-1}(T)\partial_- g(T)\big) . \tag{5.5b}$$

The bulk density $\mathcal{L}_g$ in (5.5a) is the sigma model form of the usual WZW Lagrange density, and M in (5.5b) is the data matrix defined in (3.2). The local equations of motion of this density agree in the bulk with our Hamiltonian equations of motion (5.3c) so, again, our Hamiltonian formulation is locally WZW in the bulk.

6 New Equal-Time Non-Commutative Geometry

In this section, we continue our phase space construction to find the equal-time coordinate brackets

$$\Delta^{ij}(\xi,\eta,t) \equiv \{x^i(\xi,t), x^j(\eta,t)\} . \tag{6.1}$$

In particular, we may use the inverse relations (4.1) and the equal-time brackets (4.5) of $J(\pm\xi)$ with x to find the differential equation

$$\begin{aligned}\partial_\eta \Delta^{ij}(\xi,\eta,t) =& \{x^i(\xi,t), \partial_\eta x^j(\eta,t)\} \\ =& \{x^i(\xi), J_a(\eta)G^{ab}e(\eta)_b{}^i - J_a(-\eta)G^{ab}\bar{e}(\eta)_b{}^i\} \\ =& 2\pi i \Psi^{ij}(\xi,\eta)\delta(\xi+\eta) + \Delta^{ik}(\xi,\eta)\Lambda(\eta)_k{}^j\end{aligned} \tag{6.2a}$$

$$\Psi^{ij}(\xi,\eta,t) \equiv \bar{e}(\xi,t)_a{}^i G^{ab} e_b{}^j(\eta,t) - e(\xi,t)_a{}^i G^{ab}\bar{e}_b{}^j(\eta,t) = -\Psi^{ji}(\eta,\xi,t) \tag{6.2b}$$

$$\Lambda(\xi,t)_i{}^j \equiv J_a(\xi,t)G^{ab}\partial_i e_b{}^j(\xi,t) - J_a(-\xi,t)G^{ab}\partial_i \bar{e}_b{}^j(\xi,t) \tag{6.2c}$$

and a similar equation for $\partial_\xi \Delta$.

In a matrix notation, these two equations read

$$\partial_\eta \Delta(\xi,\eta,t) = 2\pi i \Psi(\xi,\eta,t)\delta(\xi+\eta) + \Delta(\xi,\eta,t)\Lambda(\eta,t) \tag{6.3a}$$

$$\partial_\xi \Delta(\xi,\eta,t) = 2\pi i \Psi(\xi,\eta,t)\delta(\xi+\eta) + \Lambda^{\mathrm{T}}(\xi,t)\Delta(\xi,\eta,t) \tag{6.3b}$$

$$\Psi^{\mathrm{T}}(\eta,\xi) = -\Psi(\xi,\eta) \tag{6.3c}$$

where T is matrix transpose. The integrability condition for this system is

$$\partial_\xi \partial_\eta \Delta(\xi,\eta,t) = \partial_\eta \partial_\xi \Delta(\xi,\eta,t) \quad \text{iff} \tag{6.4a}$$

$$\big(\partial_\eta \Psi(\xi,\eta,t) - \Psi(\xi,\eta,t)\Lambda(\eta,t)\big)\delta(\xi+\eta) = \big(\partial_\xi \Psi(\xi,\eta,t) - \Lambda^{\mathrm{T}}(\xi,t)\Psi(\xi,\eta,t)\big)\delta(\xi+\eta) \ . \tag{6.4b}$$

Using the definitions of Ψ and Λ in Eq. (6.2), we find after some algebra that the integrability condition is satisfied. The inhomogeneous terms in (6.3) are boundary terms associated to the interaction between a non-abelian charge at ξ (or η) and a generalized non-abelian Dirichlet image charge at $-\eta$ (or $-\xi$). Note that, because of the inhomogeneous terms, the WZW bracket $\Delta_{\mathrm{WZW}}(\xi,\eta,t) = 0$ is not a solution of the equations (6.3).

By standard methods, one finds the solution for the coordinate brackets

$$\begin{aligned} \Delta(\xi,\eta,t) =& U^{\mathrm{T}}(\xi,t)\Delta(0,0,t)U(\eta,t) \\ & + \pi i \int_0^\eta d\eta' \, \big(\Psi(\xi,\eta',t)\delta(\xi+\eta') + U^{\mathrm{T}}(\xi,t)\Psi(0,\eta',t)\delta(\eta')\big)U^{-1}(\eta',t)U(\eta,t) \\ & + \pi i U^{\mathrm{T}}(\xi,t) \int_0^\xi d\xi' \, U^{-1\,\mathrm{T}}(\xi',t)\big(\Psi(\xi',\eta,t)\delta(\xi'+\eta) + \Psi(\xi',0,t)\delta(\xi')U(\eta,t)\big) \end{aligned} \tag{6.5}$$

where $\Delta(0,0,t)$ is so far undetermined and we have introduced the ordered product U of Λ

$$\partial_\xi U(\xi,t) = U(\xi,t)\Lambda(\xi,t), \quad \partial_\xi U^{\mathrm{T}}(\xi,t) = \Lambda^{\mathrm{T}}(\xi,t)U^{\mathrm{T}}(\xi,t) \tag{6.6a}$$

$$U(0,t) = U^{\mathrm{T}}(0,t) = 1 \ . \tag{6.6b}$$

To check, for example, that (6.5) solves (6.3), one needs the identity

$$\begin{aligned} \partial_\xi \int_0^\eta d\eta' \, \Psi(\xi,\eta',t)U^{-1}(\eta',t)\delta(\xi+\eta') =& \Lambda^{\mathrm{T}}(\xi,t) \int_0^\eta d\eta' \, \Psi(\xi,\eta',t)U^{-1}(\eta',t)\delta(\xi+\eta') \\ & + \Psi(\xi,\eta,t)U^{-1}(\eta,t)\delta(\xi+\eta) - \Psi(\xi,0,t)\delta(\xi) \end{aligned} \tag{6.7}$$

which itself follows from the integrability condition (6.4). If we assume that the coordinate brackets are matrix antisymmetric at $\xi = \eta = 0$

$$\Delta^{\mathrm{T}}(0,0,t) = -\Delta(0,0,t) \tag{6.8}$$

then (6.5) shows the correct antisymmetry of the coordinate brackets for all ξ, η

$$\Delta^{\mathrm{T}}(\eta, \xi, t) = -\Delta(\xi, \eta, t) \tag{6.9}$$

because Ψ has the same antisymmetry. We may also evaluate (6.5) explicitly, with the result

$$\Delta(\xi, \eta, t) = \begin{cases} \Delta(0,0,t), & \text{if } \xi = \eta = 0 \\ U^{\mathrm{T}}(\pi, t)\big(\Delta(0,0,t) + i\pi\Psi(0,0,t)\big)U(\pi, t) + i\pi\Psi(\pi,\pi,t) & \text{if } \xi = \eta = \pi \\ U^{\mathrm{T}}(\xi, t)\big(\Delta(0,0,t) + i\pi\Psi(0,0,t)\big)U(\eta, t) & \text{otherwise.} \end{cases} \tag{6.10}$$

This expression is suitable for the computation of ξ-derivatives in the bulk, but one must return to (6.5) to compute ξ-derivatives at the boundary.

The preferred choice of $\Delta(0,0,t)$ is

$$\Delta(0,0,t) = -i\pi\Psi(0,0,t) \tag{6.11}$$

because in this case, as in the WZW model, the coordinate brackets vanish in the bulk:

$$\begin{aligned} \Delta(\xi, \eta, t) = &- i\pi U^{\mathrm{T}}(\xi, t)\Psi(0,0,t)U(\eta, t) \\ &+ \pi i \int_0^\eta d\eta' \, \big(\Psi(\xi, \eta', t)\delta(\xi + \eta') + U^{\mathrm{T}}(\xi, t)\Psi(0, \eta', t)\delta(\eta')\big)U^{-1}(\eta', t)U(\eta, t) \\ &+ \pi i U^{\mathrm{T}}(\xi, t) \int_0^\xi d\xi' \, U^{-1\,\mathrm{T}}(\xi', t)\big(\Psi(\xi', \eta, t)\delta(\xi' + \eta) + \Psi(\xi', 0, t)\delta(\xi')U(\eta, t)\big) \end{aligned} \tag{6.12a}$$

$$\{x^i(\xi, t), x^j(\eta, t)\} = \Delta^{ij}(\xi, \eta, t) = \begin{cases} -i\pi\Psi^{ij}(0,0,t) & \text{if } \xi = \eta = 0 \\ +i\pi\Psi^{ij}(\pi,\pi,t) & \text{if } \xi = \eta = \pi \\ 0 & \text{otherwise} \end{cases} \tag{6.12b}$$

$$\Psi^{ij}(\xi, \eta, t) = \bar{e}(\xi, t)_a{}^i G^{ab} e_b{}^j(\eta, t) - e(\xi, t)_a{}^i G^{ab} \bar{e}_b{}^j(\eta, t) \, . \tag{6.12c}$$

Eq. (6.12), which shows the new non-commutative geometry of open WZW strings, is a central result of this paper.

We emphasize that this new non-commutative geometry is an intrinsically non-abelian effect, because the antisymmetric tensor Ψ vanishes in the abelian limit:

$$f_{ab}{}^c = 0, \quad e_i{}^a = \delta_i{}^a, \quad \bar{e}_i{}^a = -\delta_i{}^a \tag{6.13a}$$

$$\Psi^{ij} = 0, \quad \{x^i(\xi, t), x^j(\eta, t)\} = 0 \, . \tag{6.13b}$$

In this abelian limit, our coordinates become Dirichlet and no non-commutativity is expected for Dirichlet coordinates. Our new non-abelian non-commutativity is therefore unrelated in any simple sense to the standard [31-35] non-commutativity found for Neumann coordinates in the presence of a constant magnetic field. At $B = 0$, other aspects of the abelian limit are noted in App. A.

Using (6.12), (3.2a) and the chain rule, we have also computed the brackets of the group elements among themselves

$$\{g(\xi,t)_\alpha{}^\beta, g(\eta,t)_\gamma{}^\delta\} = \begin{cases} i\pi(g(0,t)T_a)_\alpha{}^\beta G^{ac}(\Omega^{-1}(0,t)-\Omega(0,t))_c{}^b(g(0,t)T_b)_\gamma{}^\delta & \text{if } \xi=\eta=0 \\ -i\pi(g(\pi,t)T_a)_\alpha{}^\beta G^{ac}(\Omega^{-1}(\pi,t)-\Omega(\pi,t))_c{}^b(g(\pi,t)T_b)_\gamma{}^\delta & \text{if } \xi=\eta=\pi \\ 0 & \text{otherwise} \end{cases} \tag{6.14a}$$

$$\alpha,\beta,\gamma,\delta = 1,\ldots \dim T \tag{6.14b}$$

where Ω is the adjoint action in (4.7).

All other phase space brackets can now be straightforwardly obtained from the known brackets $\{J(\pm\xi),x\}$ and $\{x,x\}$, without solving any further differential equations. In particular, App. B gives the explicit form of the brackets:

$$\{x^i(\xi,t),p_j(B,\eta,t)\},\quad \{x^i(\xi,t),p_j(\eta,t)\},\quad \{J_a(\pm\xi,t),p_j(B,\eta,t)\} \tag{6.15a}$$

$$\{p_i(B,\xi,t),p_j(B,\eta,t)\},\quad \{p_i(\xi,t),p_j(\eta,t)\}\ . \tag{6.15b}$$

The last set of brackets $\{p,p\}$ is again non-zero only at the boundary, and this effect again vanishes in the abelian limit.

7 The Conformal Field Theory of Open WZW Strings

7.1 The Quantum Vertex Operators $g(T)$

The Hamiltonian of the quantum theory of open WZW strings was given in Sec. 1

$$H_g = L_g(0) = \frac{1}{2\pi}\int_0^\pi d\xi\ L_g^{ab} : J_a(\xi,t)J_b(\xi,t) + J_a(-\xi,t)J_b(-\xi,t) : \tag{7.1a}$$

$$= L_g^{ab}\sum_{m\in\mathbb{Z}} : J_a(m)J_b(-m) : \tag{7.1b}$$

$$= L_g^{ab}\big(J_a(0)J_b(0) + 2\sum_{m=1}^{\infty} J_a(-m)J_b(m)\big) \tag{7.1c}$$

$$: J_a(m)J_b(n) :\equiv \theta(m\ge 0)J_b(n)J_a(m) + \theta(m<0)J_a(m)J_b(n) \tag{7.1d}$$

$$[J_a(m),J_b(n)] = if_{ab}{}^c J_c(m+n) + mG_{ab}\delta_{m+n,0} \tag{7.1e}$$

$$\partial_t A(\xi,t) = i[A(\xi,t),t] \tag{7.1f}$$

where the current modes $J_a(m)$ generate the affine Lie algebra [1, 2, 3] of g and $L_g(0)$ is the zero mode of the affine-Sugawara construction [2, 25, 26, 27, 28, 3] on g.

Using the classical theory as a guide, and in particular (4.10), we may now augment the quantum system (7.1) with the equal-time commutators

$$[J_a(\xi,t),g(T,\eta,t)] = 2\pi\big(g(T,\eta,t)T_a\delta(\xi-\eta) - T_a g(T,\eta,t)\delta(\xi+\eta)\big) \tag{7.2a}$$

$$[J_a(-\xi,t),g(T,\eta,t)] = 2\pi\big(-T_a g(T,\eta,t)\delta(\xi-\eta) + g(T,\eta,t)T_a\delta(\xi+\eta)\big) \tag{7.2b}$$

of the currents with the open string quantum vertex operators $g(T)$. We emphasize that, as in the classical theory, these commutators are consistent with the generalized Dirichlet boundary conditions (5.4a). Moreover, using the mode expansions (2.15b) of the currents, we find that the combined system (7.2) is equivalent to the algebra

$$[J_a(m), g(T,\xi,t)] = g(T,\xi,t)T_a e^{im(t+\xi)} - T_a g(T,\xi,t)e^{im(t-\xi)} \tag{7.3}$$

of the current modes with the vertex operators. We have checked that the commutator (7.3) satisfies the J, J, g Jacobi identity, as it must. The Hamiltonian (7.1) and the commutators (7.2) or (7.3) will allow us to find the analogues of the Knizhnik-Zamolodchikov equations for the conformal field theory of open WZW strings.

7.2 Time Dependence

Towards this goal, we first follow standard methods (see e.g. Halpern and Obers [37]) to obtain the time differential equation for the open string vertex operators. The result is

$$\partial_t g(T,\xi,t) = i[H_g, g(T,\xi,t)] \tag{7.4a}$$

$$\begin{aligned}\partial_t g(T,\xi,t) =& 2iL_g^{ab} : g(T,\xi,t)J_a(\xi,t)T_b - J_a(-\xi,t)T_b g(T,\xi,t) : \\ &- 2iL_g^{ab}T_a g(T,\xi,t)T_b + 2i\Delta_g(T)g(T,\xi,t) \ .\end{aligned} \tag{7.4b}$$

Here the normal ordering is defined as

$$: g(T,\xi,t)J_a(\pm\xi,t) : \equiv J_a^{(-)}(\pm\xi,t)g(T,\xi,t) + g(T,\xi,t)J_a^{(+)}(\pm\xi,t) \tag{7.5a}$$

$$J_a^{(+)}(\pm\xi,t) \equiv \sum_{m\geq 0} J_a(m)e^{-im(t\pm\xi)} \qquad J_a^{(-)}(\pm\xi,t) \equiv \sum_{m<0} J_a(m)e^{-im(t\pm\xi)} \tag{7.5b}$$

$$J_a^{(+)}(\pm\xi,t) + J_a^{(-)}(\pm\xi,t) = J_a(\pm\xi,t) \tag{7.5c}$$

and $\Delta_g(T)$ is the conformal weight of irrep T under the affine Sugawara-construction on g

$$L_g^{ab}T_aT_b = \Delta_g(T)\mathbb{1} \ . \tag{7.6}$$

In Eq. (7.4b), the normal-ordered terms have the same form as the classical result Eq. (5.2a), while the extra terms are quantum effects from the normal ordering.

In order to study correlators of the open string vertex operators, we introduce the usual affine ground state

$$J_a(m \geq 0)|0\rangle = \langle 0|J_a(m \leq 0) = 0 \tag{7.7a}$$

$$J_a^{(+)}(\pm\xi,t)|0\rangle = \langle 0|J_a^{(-)}(\pm\xi,t) = 0 \ . \tag{7.7b}$$

This gives immediately the g-global Ward identities

$$A(T,\xi,t)_{\alpha}{}^{\beta} \equiv A(T,\xi,t)_{\alpha_1\ldots\alpha_n}{}^{\beta_1\ldots\beta_n} \equiv \langle g(T^1,\xi_1,t_1)_{\alpha_1}{}^{\beta_1}\ldots g(T^n,\xi_n,t_n)_{\alpha_n}{}^{\beta_n}\rangle \tag{7.8a}$$

$$\alpha_i,\beta_i = 1,\ldots \dim T^i \tag{7.8b}$$

$$A(T,\xi,t) \equiv \langle g(T^1,\xi_1,t_1)\ldots g(T^n,\xi_n,t_n)\rangle, \quad q_a \equiv \sum_{i=1}^{n} T_a^i \tag{7.8c}$$

$$\langle [J_a(0), g(T^1,\xi_1,t_1)\ldots g(T^n,\xi_n,t_n)]\rangle = 0 \quad \Rightarrow \quad [A(T,\xi,t),q_a] = 0 \tag{7.8d}$$

for the open string WZW correlators $A(T,\xi,t)$.

Continuing to follow Ref. [37], we next obtain the KZ-like equations

$$\begin{aligned}\partial_{t_i} A =& 2i\Delta_g(T^i)A - 2iL_g^{ab}T_a^iAT_b^i \\ &+ 2iL_g^{ab}\sum_{j\neq i}\Big(\frac{AT_a^jT_b^i}{1-e^{i(\phi_j-\phi_i)}} - \frac{T_a^jAT_b^i}{1-e^{i(\bar\phi_j-\phi_i)}} \\ &\qquad - \frac{T_a^iAT_b^j}{1-e^{i(\phi_j-\bar\phi_i)}} + \frac{T_a^iT_b^jA}{1-e^{i(\bar\phi_j-\bar\phi_i)}}\Big)\end{aligned} \tag{7.9a}$$

$$\phi_i \equiv t_i + \xi_i \quad \bar\phi_i \equiv t_i - \xi_i, \quad i,j = 1,\ldots,n \tag{7.9b}$$

for the time dependence of the correlators $A \equiv A(T,\xi,t)$ in open WZW theory. Tensor products are assumed in (7.9), as illustrated in the example:

$$(AT^iT^j)_{\alpha_i\alpha_j}{}^{\beta_i\beta_j} \equiv (AT^i\otimes T^j)_{\alpha_i\alpha_j}{}^{\beta_i\beta_j} = A_{\alpha_i\alpha_j}{}^{\gamma_i\gamma_j}(T^i)_{\gamma_i}{}^{\beta_i}(T^j)_{\gamma_j}{}^{\beta_j} \tag{7.10a}$$

$$[T_a^i, T_b^j] = i\delta^{ij}f_{ab}{}^cT_c^i\ . \tag{7.10b}$$

To obtain these differential equations, we used the vertex operator equation (7.4b), the commutators

$$[J_a^{(+)}(\pm\xi_i,t_i), g(T^j,\xi_j,t_j)] = \frac{g(T^j,\xi_j,t_j)T_a^j}{1-e^{i(t_j+\xi_j-(t_i\pm\xi_i))}} - \frac{T_a^jg(T^j,\xi_j,t_j)}{1-e^{i(t_j-\xi_j-(t_i\pm\xi_i))}} \tag{7.11a}$$

$$[J_a^{(-)}(\pm\xi_i,t_i), g(T^j,\xi_j,t_j)] = \frac{T_a^jg(T^j,\xi_j,t_j)}{1-e^{i(t_j-\xi_j-(t_i\pm\xi_i))}} - \frac{g(T^j,\xi_j,t_j)T_a^j}{1-e^{i(t_j+\xi_j-(t_i\pm\xi_i))}} \tag{7.11b}$$

and the ground state condition (7.7).

The system (7.9) resembles the usual [27] KZ equations, but it is in fact quite different:

- In (7.9), the variables ϕ_i and $\bar\phi_i$ are the locations of the i-th non-abelian charge and the i-th generalized non-abelian Dirichlet image charge respectively.
- The ordinary KZ equations for the time dependence of the left and right mover WZW

correlators $A_+^{\rm WZW}$ and $A_-^{\rm WZW}$ on the cylinder are

$$\partial_{t_i} A_+^{\rm WZW} = i\Delta_g(T^i) A_+^{\rm WZW} + A_+^{\rm WZW}\, 2iL_g^{ab} \sum_{j\neq i} \frac{T_a^j T_b^i}{1-e^{i(\phi_j-\phi_i)}} \tag{7.12a}$$

$$\partial_{t_i} A_-^{\rm WZW} = i\Delta_g(T^i) A_-^{\rm WZW} + 2iL_g^{ab} \sum_{j\neq i} \frac{T_a^i T_b^j}{1-e^{i(\bar\phi_j-\bar\phi_i)}} A_-^{\rm WZW} \,. \tag{7.12b}$$

The coefficients of $A_\pm^{\rm WZW}$ in (7.12) also appear in (7.9): However in (7.9) the AT^jT^i terms represent the interactions among the non-abelian charges, while the T^iT^jA terms represent the interactions among the non-abelian image charges.

- In (7.9), the terms proportional to T^jAT^i and T^iAT^j are interactions between the charges and the image charges. Such terms, with a simultaneous left and right action, are unfamiliar in standard KZ theory.
- The extra term $-2iL_g^{ab}T_a^iAT_b^i$ in (7.9) will be interpreted below.

It will be convenient to write the system (7.9) in the more conventional form

$$\partial_{t_i} A = 2i\Delta_g(T^i)A + A\omega_i + \bar\omega_i A - \omega_i^a A T_a^i - T_a^i A \bar\omega_i^a \tag{7.13a}$$

$$\omega_i \equiv 2iL_g^{ab} \sum_{j\neq i} T_a^i T_b^j f(\phi_j-\phi_i), \quad \bar\omega_i \equiv 2iL_g^{ab} \sum_{j\neq i} T_a^i T_b^j f(\bar\phi_j-\bar\phi_i) \tag{7.13b}$$

$$\omega_i^a \equiv 2iL_g^{ab} \sum_j T_b^j f(\bar\phi_j-\phi_i), \quad \bar\omega_i^a \equiv 2iL_g^{ab} \sum_j T_b^j f(\phi_j-\bar\phi_i) \tag{7.13c}$$

$$f(x) \equiv \frac{1}{1-e^{ix}}, \quad f(x)+f(-x)=1 \tag{7.13d}$$

where we have used the identity (7.13d) to re-express the $-2iL_g^{ab}T_a^iAT_b^i$ term in (7.9) as the $j=i$ terms of the completed sums in (7.13c). In this way, we interpret the $-2iL_g^{ab}T_a^iAT_b^i$ term in (7.9) as equivalent to two types of interaction between a given charge at ϕ_i and its own image at $\bar\phi_i$. In what follows, the ω's of Eq. (7.13) are referred to as connections.

We have checked explicitly that these KZ-like differential equations satisfy the appropriate integrability condition

$$[\partial_{t_i}, \partial_{t_j}]A = 0, \quad \forall\, i,j \tag{7.14}$$

and that the differential equations are also consistent with the g-global Ward identities (7.8). We will postpone the details of this discussion, however, while we develop a simpler and more comprehensive description of this system.

7.3 Constituent Vertex Operators

In this subsection, we introduce constituent vertex operators $g_\pm(T)$ which provide an enlightening alternative derivation of the vertex operator differential equation (7.4).

The constituent vertex operators are defined as follows[c]:

$$g(T,\xi,t) \equiv g_-(T,\xi,t)g_+(T,\xi,t) \tag{7.15a}$$

$$[J_a(m), g_+(T,\xi,t)] = g_+(T,\xi,t)T_a e^{im(t+\xi)} \tag{7.15b}$$

$$[J_a(m), g_-(T,\xi,t)] = -T_a g_-(T,\xi,t)e^{im(t-\xi)} \ . \tag{7.15c}$$

Taken together, the simple commutation relations in (7.15) reproduce the algebra (7.3) of the current modes with the full vertex operators $g(T)$.

By direct computation with (7.15), we find the simpler time differential equations

$$\partial_t g_\pm(T,\xi,t) = i[H_g, g_\pm(T,\xi,t)] \tag{7.16a}$$

$$\partial_t g_+(T,\xi,t) = 2iL_g^{ab} : J_a(\xi,t)g_+(T,\xi,t)T_b : +i\Delta_g(T)g_+(T,\xi,t) \tag{7.16b}$$

$$\partial_t g_-(T,\xi,t) = -2iL_g^{ab} : J_a(-\xi,t)T_b g_-(T,\xi,t) : +i\Delta_g(T)g_-(T,\xi,t) \tag{7.16c}$$

for the constituent vertex operators. The normal ordering here is the same as in (7.5) with $g \to g_\pm$. Then we find that the time derivative (7.4b) of the full $g(T)$ is a consequence of the constituent equations (7.16) as follows:

$$\partial_t g(T,\xi,t) = \partial_t g_-(T,\xi,t)g_+(T,\xi,t) + g_-(T,\xi,t)\partial_t g_+(T,\xi,t) \tag{7.17a}$$

$$\begin{aligned} &= -2iL_g^{ab}\big(:J_a(-\xi,t)T_b g_-(T,\xi,t): g_+(T,\xi,t) - g_-(T,\xi,t) :J_a(\xi,t)g_+(T,\xi,t)T_b: \big) \\ &\quad + 2i\Delta_g(T)g(T,\xi,t) \end{aligned} \tag{7.17b}$$

$$\begin{aligned} &= 2iL_g^{ab} : g(T,\xi,t)J_a(\xi,t)T_b - J_a(-\xi,t)T_b g(T,\xi,t) : \\ &\quad - 2iL_g^{ab}T_a g(T,\xi,t)T_b + 2i\Delta_g(T)g(T,\xi,t) \ . \end{aligned} \tag{7.17c}$$

In this computation, the expression in (7.17b) was not completely normal ordered, so we used the equal-time commutators

$$[J_a^{(+)}(-\xi,t), g_+(T,\xi,t)] = \frac{g_+(T,\xi,t)T_a}{1-e^{2i\xi}} \tag{7.18a}$$

$$[J_a^{(-)}(\xi,t), g_-(T,\xi,t)] = \frac{T_a g_-(T,\xi,t)}{1-e^{-2i\xi}} \tag{7.18b}$$

$$\frac{1}{1-e^{-2i\xi}} + \frac{1}{1-e^{2i\xi}} = 1 \tag{7.18c}$$

to obtain the final completely normal ordered form in (7.17c).

We emphasize that the $-2iL_g^{ab}T_a^i A T_b^i$ term in Eq. (7.4b) or (7.17c) is a result of this final normal ordering.

[c]Constituent vertex operators can also be introduced in the same way for the classical theory, and all the same properties obtained below for the quantum constituents can be obtained as well at the classical level (see also Eq. (7.29)). In particular, we will not need to know the explicit multiplication law for g_- times g_+, which presumably involves quantum groups.

We also consider the action of the Virasoro generators $L_g(m)$ of the affine-Sugawara construction

$$L_g(m) \equiv L_g^{ab} \sum_{p\in\mathbb{Z}} : J_a(p) J_b(m-p) : \tag{7.19a}$$

$$[L_g(m), L_g(n)] = (m-n) L_g(m+n) + \frac{c_g}{12} m(m^2-1)\delta_{m+n,0} \tag{7.19b}$$

on the constituent vertex operators. By direct computation with (7.15b-c) and (7.16b-c) we find that

$$[L_g(m), g_\pm(T,\xi,t)] = e^{im(t\pm\xi)}\big(-i\partial_t + m\Delta_g(T)\big) g_\pm(T,\xi,t) \ . \tag{7.20}$$

We have checked that these commutators satisfy the $L, L, g_\pm$ Jacobi identities.

7.4 The Constituents are Chiral

We begin this discussion with the quantum version of the bulk momentum operator

$$P_g(t) \equiv \frac{1}{2\pi}\int_0^\pi d\xi \; L_g^{ab} : J_a(\xi,t) J_b(\xi,t) - J_a(-\xi,t) J_b(-\xi,t) : \tag{7.21a}$$

$$= \frac{i}{\pi} L_g^{ab} \sum_{m+n\neq 0} e^{-i(m+n)t} \left(\frac{(-1)^{m+n}-1}{m+n}\right) : J_a(m) J_b(n) : \tag{7.21b}$$

$$= -\frac{2i}{\pi}\sum_{m\in\mathbb{Z}} \frac{e^{-i(2m+1)t}}{2m+1} L_g(2m+1) \tag{7.21c}$$

$$\partial_t P_g(t) = \frac{L_g^{ab}}{\pi} : J_a(\pi,t) J_b(\pi,t) - J_a(0,t) J_b(0,t) : \tag{7.21d}$$

$$L_g(m)^\dagger = L_g(-m) \quad \Rightarrow \quad P_g(t)^\dagger = P_g(t) \tag{7.21e}$$

where $L_g(m)$ are the Virasoro generators in (7.19a). By direct computation with (7.20), we find that

$$i[P_g(t), g_+(T,\xi,t)] = 4\Big(\int_0^\xi d\eta \; e^{i\eta}\delta(2\eta)\Big)\partial_t g_+(T,\xi,t) + 4e^{i\xi}\delta(2\xi)\Delta_g(T) g_+(T,\xi,t) \tag{7.22a}$$

$$i[P_g(t), g_-(T,\xi,t)] = -4\Big(\int_0^\xi d\eta \; e^{-i\eta}\delta(2\eta)\Big)\partial_t g_-(T,\xi,t) + 4e^{-i\xi}\delta(2\xi)\Delta_g(T) g_-(T,\xi,t) \tag{7.22b}$$

$$4\int_0^\xi d\eta \; e^{i\eta}\delta(2\eta) = 4\int_0^\xi d\eta \; e^{-i\eta}\delta(2\eta) = \begin{cases} 1 & \text{if } 0<\xi<\pi \\ 0 & \text{if } \xi = 0,\pi. \end{cases} \tag{7.22c}$$

The summation identities

$$\sum_{n\in\mathbb{Z}} \frac{e^{\pm i(2n+1)\xi}}{2n+1} = \pm 2\pi i \int_0^\xi d\eta \; e^{\pm i\eta}\delta(2\eta) \tag{7.23}$$

were used to obtain these results.

The last terms in (7.22a-b) are quantum effects which contribute only at the boundary, so that

$$\partial_\xi g_\pm(T,\xi,t) = i[P_g(t), g_\pm(T,\xi,t)] = \pm\partial_t g_\pm(T,\xi,t) \quad 0<\xi<\pi \tag{7.24}$$

is obtained in the bulk. Following our classical intuition, we extend this result smoothly to include the boundary

$$\partial_- g_+(T,\xi,t) = \partial_+ g_-(T,\xi,t) = 0, \quad \partial_\pm = \partial_t \pm \partial_\xi, \quad 0\le\xi\le\pi \tag{7.25}$$

which tells us that the constituent vertex operators $g_+(T)$ and $g_-(T)$ are respectively chiral and antichiral.

It is simple to check that this extension to the boundary is consistent: First, the chirality of $g_\pm(T)$ in the form

$$\frac{1}{2}\partial_\pm g_\pm(T,\xi,t) = \partial_t g_\pm(T,\xi,t) \tag{7.26}$$

allows us to rewrite the equations for $\partial_t g_\pm$ in (7.16b-c) as light-cone differential equations for the constituent vertex operators

$$\frac{1}{2}\partial_+ g_+(T,\xi,t) = 2iL_g^{ab} : J_a(\xi,t) g_+(T,\xi,t) T_b : + i\Delta_g(T) g_+(T,\xi,t) \tag{7.27a}$$

$$\frac{1}{2}\partial_- g_-(T,\xi,t) = -\,2iL_g^{ab} : J_a(-\xi,t) T_b g_-(T,\xi,t) : + i\Delta_g(T) g_-(T,\xi,t)\,. \tag{7.27b}$$

Then one easily checks that these equations are consistent with the chirality conditions (7.25)

$$\partial_+\partial_- g_\pm(T,\xi,t) = \partial_-\partial_+ g_\pm(T,\xi,t) = 0 \tag{7.28}$$

because $g_+(T)$, $J(+\xi)$ are chiral and $g_-(T)$, $J(-\xi)$ are antichiral.

As another check on the consistency of the chiralities (7.25), we remark that the relations in (7.15), (7.25) and (7.27) are nothing but the quantum versions of the classical relations

$$g(T,\xi,t) \equiv g_-(T,\xi,t) g_+(T,\xi,t) \tag{7.29a}$$

$$\{J_a(\pm\xi,t), g_+(T,\eta,t)\} = 2\pi\delta(\xi\mp\eta) g_+(T,\xi,t) T_a \tag{7.29b}$$

$$\{J_a(\pm\xi,t), g_-(T,\xi,t)\} = -\,2\pi\delta(\xi\pm\eta) T_a g_-(T,\xi,t) \tag{7.29c}$$

$$\partial_- g_+(T,\xi,t) = \partial_+ g_-(T,\xi,t) = 0, \quad 0\le\xi\le\pi \tag{7.29d}$$

$$\partial_+ g_+(T,\xi,t) = 2i g_+(T,\xi,t) J(T,\xi,t) \tag{7.29e}$$

$$\partial_- g_-(T,\xi,t) = 2iJ(T,-\xi,t) g_-(T,\xi,t) \tag{7.29f}$$

which should be taken to supplement our classical discussion above. As in Ref. [30], the relations (7.29a) and (7.29d-f) solve the classical relations for $\partial_\pm g(T)$ in (5.2c).

The constituent vertex operator equations (7.27) are cylindrical analogues of the vertex operator equations for the left and right mover vertex operators in ordinary WZW theory (see e.g. Ref. [37]), and indeed these equations guarantee that g_+ correlators and g_- correlators

$$g_\pm(i) \equiv g_\pm(T^i, \xi_i, t_i), \quad A_\pm \equiv \langle g_\pm(1) \dots g_\pm(n) \rangle \tag{7.30}$$

obey cylindrical analogues of the ordinary KZ equations

$$\frac{1}{2}\partial_{i+} A_+ = i\Delta_g(T^i) A_+ + A_+ \omega_i \tag{7.31a}$$

$$\frac{1}{2}\partial_{i-} A_- = i\Delta_g(T^i) A_- + \bar{\omega}_i A_- \tag{7.31b}$$

$$\partial_{i-} A_+ = \partial_{i+} A_- = 0 \tag{7.31c}$$

$$\partial_{i+} \equiv \partial_{t_i} + \partial_{\xi_i} \quad \partial_{i-} \equiv \partial_{t_i} - \partial_{\xi_i} \ . \tag{7.31d}$$

The connections $\omega_i, \bar{\omega}_i$ are defined in Eq. (7.13b). The commutator identities

$$[J_a^{(+)}(\pm\xi_i, t_i), g_+(T^j, \xi_j, t_j)] = \frac{g_+(T^j, \xi_j, t_j) T_a^j}{1 - e^{i(t_j + \xi_j - (t_i \pm \xi_i))}} \tag{7.32a}$$

$$[J_a^{(+)}(\pm\xi_i, t_i), g_-(T^j, \xi_j, t_j)] = -\frac{T_a^j g_-(T^j, \xi_j, t_j)}{1 - e^{i(t_j - \xi_j - (t_i \pm \xi_i))}} \tag{7.32b}$$

$$[J_a^{(-)}(\pm\xi_i, t_i), g_+(T^j, \xi_j, t_j)] = -\frac{g_+(T^j, \xi_j, t_j) T_a^j}{1 - e^{i(t_j + \xi_j - (t_i \pm \xi_i))}} \tag{7.32c}$$

$$[J_a^{(-)}(\pm\xi_i, t_i), g_-(T^j, \xi_j, t_j)] = \frac{T_a^j g_-(T^j, \xi_j, t_j)}{1 - e^{i(t_j - \xi_j - (t_i \pm \xi_i))}} \tag{7.32d}$$

are needed in the derivation of (7.31). Because the correlators $A_\pm$ are chiral, the equations in (7.31) are identical to the cylindrical KZ equations given in (7.12).

But in open WZW theory, the full correlators A of the full vertex operators g cannot be factorized into the constituent correlators $A_\pm$:

$$g(i) \equiv g(T^i, \xi_i, t_i) = g_-(i) g_+(i) \tag{7.33a}$$

$$A = \langle g(1) \dots g(n) \rangle \neq \langle g_-(1) \dots g_-(n) \rangle \langle g_+(1) \dots g_+(n) \rangle = A_- A_+ \ . \tag{7.33b}$$

This follows because, in open string theory, the current modes $J(m)$ have non-trivial action on both g_+ and g_- so that g_+ and g_- do not form independent subspaces [37] as in the WZW model. Looking back, this phenomenon can also be understood as the fact that the strip current $J(\xi, t)$ does not commute with the strip current $J(-\xi, t)$ at the boundary.

Finally, we may use Eq. (7.26) to recast (7.20) in the form

$$[L_g(m), g_\pm(T, \xi, t)] = e^{i(t \pm \xi)}\big(-\frac{i}{2}\partial_\pm + m\Delta_g(T)\big) g_\pm(T, \xi, t) \tag{7.34}$$

and this form also satisfies the $L, L, , g_\pm$ Jacobi identities. Here $\{L_g(m)\}$ acts on $g_+(T)$ as a left mover Virasoro acts on a left mover Virasoro primary, but $\{L_g(m)\}$ also acts on $g_-(T)$ as a right mover Virasoro $\{\bar{L}_g(m)\}$ acts on a right mover Virasoro primary.

7.5 The Full Open String Vertex Operator Equations

In this subsection, we use the results above for the chiral vertex operators $g_\pm(T)$ to find the differential equations for the full vertex operators $g(T)$.

Our first step is to use (7.25) and (7.27) to obtain the light-cone differential equations for $g(T)$

$$\frac{1}{2}\partial_+ g(T,\xi,t) = 2iL_g^{ab} : J_a(\xi)g(T,\xi,t)T_b : -2iL_g^{ab}\frac{T_a g(T,\xi,t)T_b}{1-e^{-2i\xi}} + i\Delta_g(T)g(T,\xi,t) \tag{7.35a}$$

$$\frac{1}{2}\partial_- g(T,\xi,t) = -2iL_g^{ab} : J_a(-\xi)T_b g(T,\xi,t) : -2iL_g^{ab}\frac{T_a g(T,\xi,t)T_b}{1-e^{2i\xi}} + i\Delta_g(T)g(T,\xi,t)\ . \tag{7.35b}$$

In the final step of this computation, the commutators (7.18) are needed again to obtain the fully normal ordered form. This result is the quantum version of the classical result in Eq. (5.2c): The normal ordered terms have the same form as the classical result, and the remaining terms are quantum effects from the normal ordering. The consistency of the system (7.35)

$$[\partial_+, \partial_-]g(T,\xi,t) = 0 \tag{7.36}$$

follows by construction from $g = g_- g_+$.

Taking linear combinations of Eqs. (7.35a-b), we also find the ∂_t and ∂_ξ equations for the vertex operators $g(T)$:

$$\begin{aligned}\partial_t g(T,\xi,t) =& 2iL_g^{ab} : J_a(\xi)g(T,\xi,t)T_b - J_a(-\xi)T_b g(T,\xi,t) : \\ & - 2iL_g^{ab}T_a g(T,\xi,t)T_b + 2i\Delta_g(T)g(T,\xi,t)\end{aligned} \tag{7.37a}$$

$$\begin{aligned}\partial_\xi g(T,\xi,t) =& 2iL_g^{ab} : J_a(\xi)g(T,\xi,t)T_b + J_a(-\xi)T_b g(T,\xi,t) : \\ & - 2L_g^{ab}T_a g(T,\xi,t)T_b \cot\xi\end{aligned} \tag{7.37b}$$

$$\frac{1}{1-e^{2i\xi}} + \frac{1}{1-e^{-2i\xi}} = 1, \quad \frac{1}{1-e^{2i\xi}} - \frac{1}{1-e^{-2i\xi}} = i\cot\xi \tag{7.37c}$$

where we have used the relations in (7.37c). The equation in (7.37a) agrees of course with the earlier result in (7.4).

As a simple application, we use the vertex operator equations (7.37), the ground state conditions (7.7) and the g-global Ward identity to compute the one-point correlators of the open WZW string

$$[T_a, \langle g(T,\xi,t)\rangle] = 0 \tag{7.38a}$$

$$\partial_t\langle g(T,\xi,t)\rangle = 0, \quad \partial_\xi\langle g(T,\xi,t)\rangle = -2\Delta_g(T)\cot(\xi)\langle g(T,\xi,t)\rangle\ . \tag{7.38b}$$

The solution of (7.38) is

$$\langle g(T,\xi,t)\rangle = \mathbb{1}\, C(\sin\xi)^{-2\Delta_g(T)} \tag{7.39}$$

where C is an undetermined number. As another application, the vertex operator equations (7.35) or (7.37) can be used to study the operator product $g(1)g(2)$ and its operator product expansion.

7.6 The Full Open String KZ Equations

Using (7.7), (7.11) and the vertex operator equations (7.35), we now obtain the full open string KZ equations

$$A = \langle g(T^1, \xi_1, t_1) \dots g(T^n, \xi_n, t_n) \rangle \tag{7.39a}$$

$$\partial_{\phi_i} A = i\Delta_g(T^i)A + A\omega_i - \omega_i^a A T_a^i \tag{7.39b}$$

$$\partial_{\bar{\phi}_i} A = i\Delta_g(T^i)A + \bar{\omega}_i A - T_a^i A \bar{\omega}_i^a \tag{7.39c}$$

$$\phi_i = t_i + \xi_i, \quad \bar{\phi}_i = t_i - \xi_i, \quad i = 1, \dots, n \tag{7.39d}$$

$$\omega_i \equiv 2iL_g^{ab} \sum_{j \neq i} T_a^i T_b^j f(\phi_j - \phi_i), \quad \bar{\omega}_i \equiv 2iL_g^{ab} \sum_{j \neq i} T_a^i T_b^j f(\bar{\phi}_j - \bar{\phi}_i) \tag{7.39e}$$

$$\omega_i^a \equiv 2iL_g^{ab} \sum_{j} T_b^j f(\bar{\phi}_j - \phi_i), \quad \bar{\omega}_i^a \equiv 2iL_g^{ab} \sum_{j} T_b^j f(\phi_j - \bar{\phi}_i) \tag{7.39f}$$

$$f(x) \equiv \frac{1}{1 - e^{ix}} \tag{7.39g}$$

$$[\sum_{i=1}^{n} T_a^i, A] = 0 \tag{7.39h}$$

for the correlators A of open WZW theory. This system is a central result of this paper.

The n-point correlators in this system satisfy $2n$ partial differential equations in the $2n$ independent variables $\{\phi_i, \bar{\phi}_i\}$, so the complexity of the n-point correlators in open WZW string theory is comparable to the complexity of the $2n$-point correlators of the ordinary KZ equations. The solution for the open string one-point correlators is given in (7.39).

Because the form of this system is unfamiliar, we have checked its integrability conditions carefully. Aside from considerable algebra using the form of the connections ω, the only identities needed are

$$f(x-y)f(z-x) + f(z-y)f(y-x) - f(z-y)f(z-x) = 0, \quad \forall\, x, y, z \tag{7.40a}$$

$$f(x) + f(-x) = 1\ . \tag{7.40b}$$

In detail, we find that the integrability conditions are satisfied in the following way:

$$[\partial_{\phi_i}, \partial_{\phi_j}]A = 0 \quad \text{because :} \tag{7.41a}$$

$$\partial_{\phi_i}\omega_j - \partial_{\phi_j}\omega_i = 0, \quad \partial_{\phi_i}\omega_j^a D T_a^j - \partial_{\phi_j}\omega_i^a D T_a^i = 0 \tag{7.41b}$$

$$[\omega_i, \omega_j] = 0, \quad [\omega_i^a, \omega_j^b] D T_b^j T_a^i + \omega_j^a D[\omega_i, T_a^j] - \omega_i^a D[\omega_j, T_a^i] = 0, \quad \forall\, i, j \tag{7.41c}$$

$$[\partial_{\bar\phi_i},\partial_{\bar\phi_j}]A=0 \quad \text{because :} \tag{7.42a}$$

$$\partial_{\bar\phi_i}\bar\omega_j-\partial_{\bar\phi_j}\bar\omega_i=0,\quad T^j_aD\partial_{\bar\phi_i}\bar\omega^a_j-T^i_aD\partial_{\bar\phi_j}\bar\omega^a_i=0 \tag{7.42b}$$

$$[\bar\omega_i,\bar\omega_j]=0,\quad T^i_aT^j_bD[\bar\omega^a_i,\bar\omega^b_j]+[\bar\omega_i,T^j_a]D\bar\omega^a_j-[\bar\omega_j,T^i_a]D\bar\omega^a_i=0,\quad \forall\, i,j \tag{7.42c}$$

$$[\partial_{\phi_i},\partial_{\bar\phi_j}]A=0 \quad \text{because :} \tag{7.43a}$$

$$\partial_{\phi_i}\bar\omega_j=\partial_{\bar\phi_j}\omega_i=0,\quad T^j_aD\partial_{\phi_i}\bar\omega^a_j-\partial_{\bar\phi_j}\omega^a_iDT^i_a=0 \tag{7.43b}$$

$$[\omega^a_i,\bar\omega_j]DT^i_a-T^j_aD[\omega_i,\bar\omega^a_j]+T^j_a\omega^b_iDT^i_b\bar\omega^a_j-\omega^b_iT^j_aD\bar\omega^a_jT^i_b=0,\quad \forall\, i,j\ . \tag{7.43c}$$

Here, D is an arbitrary square matrix in the space of correlators.

The compatibility between the open string KZ equations (7.39) and the g-global Ward identity (7.8)

$$q_a=\sum_i T^i_a,\quad [q_a,\omega_i]=[q_a,\bar\omega_i]=0 \tag{7.44a}$$

$$[q_a,T^i_b]D\omega^b_i+T^i_bD[q_a,\omega^b_i]=\bar\omega^b_iD[q_a,T^i_b]+[q_a,\bar\omega^b_i]DT^i_b=0,\quad \forall\, i \tag{7.44b}$$

can also be checked in the same way.

We also give the full open string KZ equations in the alternate form

$$\partial_{t_i}A=2i\Delta_g(T^i)A+A\omega_i+\bar\omega_iA-\omega^a_iAT^i_a-T^i_aA\bar\omega^a_i \tag{7.45a}$$

$$\partial_{\xi_i}A=A\omega_i-\bar\omega_iA-\omega^a_iAT^i_a+T^i_aA\bar\omega^a_i,\quad i=1,\dots,n \tag{7.45b}$$

where the derivatives are now with respect to the basic world-sheet variables $\{t_i,\xi_i\}$. In this system, Eq. (7.45a) is the same as (7.13).

Still another form of our open string KZ system is given in App. C, where we use dual matrix representations to present the equations for the open n-point correlators as a single "chiral" KZ system in $2n$ variables.

Finally, we record the action of the Virasoro generators $L_g(m)$ on the full vertex operators $g(T)$

$$[L_g(m),g(T,\xi,t)]=\big(e^{im\bar\phi}(-i\partial_{\bar\phi}+m\Delta_g(T))+e^{im\phi}(-i\partial_\phi+m\Delta_g(T))\big)g(T,\xi,t) \tag{7.46a}$$

$$=e^{imt}\big(\cos(m\xi)(-i\partial_t+2m\Delta_g(T))+\sin(m\xi)\partial_\xi\big)g(T,\xi,t)\ . \tag{7.46b}$$

The result follows from Eq. (7.34) and the chirality (7.25) of the constituents.

7.7 General Open String CFT

In this subsection, we extend our discussion to general open string conformal field theory, using what we now know about open WZW theory as a model. In particular, we assume that the general open string CFT is governed by a single set of Virasoro generators $L(m)$

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{7.47a}$$

$$L(|m| \leq 1)|0\rangle = \langle 0|L(|m| \leq 1) = 0 \tag{7.47b}$$

with an $SL(2)$ invariant ground state $|0\rangle$. For open CFT's based on a current algebra, $|0\rangle$ is the affine ground state. The set of all open CFT's is very large, including open analogues of coset constructions [2, 25, 38, 3], affine-Virasoro constructions [39, 40, 3] and conformal sigma models.

In each open CFT, we also consider the set of open string Virasoro primary fields $\{\Phi_i\}$, which are defined to satisfy

$$[L(m), \Phi_i(\xi,t)] = e^{imt}\big(\cos(m\xi)(-i\partial_t + 2m\Delta_i) + \sin(m\xi)\partial_\xi\big)\Phi_i(\xi,t) \ . \tag{7.48}$$

Open string Virasoro quasiprimary fields $\chi_i(\xi,t)$ are defined to satisfy (7.48) for $|m| \leq 1$.

Then the $SL(2)$ Ward identities

$$A \equiv \langle \chi_1 \cdots \chi_n \rangle \tag{7.49a}$$

$$L(0): \quad \sum_{i=1}^{n} \partial_{t_i} A = 0 \tag{7.49b}$$

$$L(\pm 1): \quad \sum_{i=1}^{n} e^{\pm it_i}\big(\cos\xi_i(-i\partial_{t_i} \pm 2\Delta_i) \pm \sin\xi_i\partial_{\xi_i}\big)A = 0 \tag{7.49c}$$

follow in the usual way for open string correlators of sets of Virasoro quasiprimaries. The relation (7.49b) expresses the time translation invariance of the correlators.

The open string WZW vertex operators $g(T)$ above are examples of open string Virasoro primary fields with $\Delta_i = \Delta_g(T^i)$ and, indeed, the $SL(2)$ Ward identities (7.49) are satisfied by the correlators of the open string KZ system (7.39). In particular, we have checked that the compatibility conditions

$$L_g(0): \sum_{i=1}^{n}(2i\Delta_g(T^i)A + A\omega_i + \bar{\omega}_i A - \omega_i^a A T_a^i - T_a^i A \bar{\omega}_i^a) = 0 \tag{7.50a}$$

$$L_g(-1): \sum_{i=1}^{n}\big(e^{-i(t_i-\xi_i)}(\bar{\omega}_i A - T_a^i A \bar{\omega}_i^a) + e^{-i(t_i+\xi_i)}(A\omega_i - \omega_i^a A T_a^i)\big) = 0 \tag{7.50b}$$

$$L_g(+1): \sum_{i=1}^{n}\big(e^{i(t_i-\xi_i)}(i\Delta_g(T^i)A + \bar{\omega}_i A - T_a^i A \bar{\omega}_i^a) + e^{i(t_i+\xi_i)}(i\Delta_g(T^i)A + A\omega_i - \omega_i^a A T_a^i)\big) = 0 \tag{7.50c}$$

are satisfied by the connections ω, using also the g-global Ward identity for (7.50a) and (7.50c).

The $SL(2)$ Ward identities (7.49) can be put in a more recognizable form

$$A \equiv \left(\prod_{i=1}^{n} (z_i \bar{z}_i)^{\Delta_i} \right) F \tag{7.51a}$$

$$L(-1): \quad \sum_{i=1}^{n} (\partial_i + \bar{\partial}_i) F = 0 \tag{7.51b}$$

$$L(0): \quad \sum_{i=1}^{n} ((z_i \partial_i + \Delta_i) + (\bar{z}_i \bar{\partial}_i + \Delta_i)) F = 0 \tag{7.51c}$$

$$L(1): \quad \sum_{i=1}^{n} (z_i (z_i \partial_i + 2\Delta_i) + \bar{z}_i (\bar{z}_i \bar{\partial}_i + 2\Delta_i)) F = 0 \tag{7.51d}$$

$$z_i \equiv e^{i(t_i + \xi_i)}, \quad \bar{z}_i \equiv e^{i(t_i - \xi_i)}, \quad \partial_i \equiv \frac{\partial}{\partial z_i}, \quad \bar{\partial}_i \equiv \frac{\partial}{\partial \bar{z}_i} \tag{7.51e}$$

so the $SL(2)$ Ward identities for the open string n-point F factor have the form of the ordinary $SL(2)$ Ward identities, now for a correlator with $2n$ points $z_1, \bar{z}_1 \ldots z_n \bar{z}_n$. The solutions to these equations are easily read off from the general solution of $SL(2)$ Ward identities given in Ref. [41].

As a simple example, we find for the open string one-point correlators

$$\langle \chi_i(\xi_i, t_i) \rangle \propto \frac{(z_i \bar{z}_i)^{\Delta_i}}{(z_i - \bar{z}_i)^{2\Delta_i}} \propto \frac{1}{(\sin \xi_i)^{2\Delta_i}} \tag{7.52}$$

in agreement with our solution (7.39) of the open string KZ equations and the g-global Ward identity. In the solution (7.52), the open string one-point F factor

$$F \propto \frac{1}{(z_i - \bar{z}_i)^{2\Delta_i}} \tag{7.53}$$

has the form of the usual two-point correlator between a closed string quasiprimary field at z_i and another closed string quasiprimary field at a point called $\bar{z}_i$: In open string theory, however, the points z_i and $\bar{z}_i$ are the locations of the charge and the image charge respectively.

The $SL(2)$ forms of the open string correlators will be helpful in solving the open string KZ equations for the multi-point correlators on the strip.

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After completion of this work, a paper "Canonical Quantization of the Boundary Wess-Zumino-Witten Model", hep-th/0101170 appeared on the net by Gawedzki, Todorov and Tran-Ngoc-Bich. This paper also considers the dynamics of open WZW strings, and, although the approach is quite different, there is an explicit overlap with our result Eq. (7.2).

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Appendix A. The Abelian Case

In order to compare with the discussion in the text, we consider the classical action formulation of the simplest abelian Dirichlet string:

$$S = \int_0^\pi d\xi \ \mathcal{L} \tag{A.1a}$$

$$\mathcal{L} = \frac{1}{8\pi}\partial_+ x \, \partial_- x = \frac{1}{8\pi}((\partial_t x)^2 - (\partial_\xi x)^2) \tag{A.1b}$$

$$(\partial_t^2 - \partial_\xi^2)x(\xi,t) = 0, \quad \partial_t x(0,t) = \partial_t x(\pi,t) = 0 \ . \tag{A.1c}$$

The solution of (A.1c) is

$$\frac{1}{2}x(\xi,t) = q + J(0)\xi + \sum_{m\neq 0} J(m)\frac{\sin(m\xi)}{m}e^{-imt} \tag{A.2a}$$

$$\partial_t x(\xi,t) = -2i\sum_{m\in\mathbb{Z}} J(m)\sin(m\xi)e^{-imt} = J(\xi,t) - J(-\xi,t) \tag{A.2b}$$

$$\partial_\xi x(\xi,t) = 2\sum_{m\in\mathbb{Z}} J(m)\cos(m\xi)e^{-imt} = J(\xi,t) + J(-\xi,t) \tag{A.2c}$$

where the current modes $J(m)$ satisfy the usual abelian current algebra

$$J(\xi,t) = \sum_{m\in\mathbb{Z}} = J(m)e^{-im(t+\xi)}, \quad J(-\xi,t) = \sum_{m\in\mathbb{Z}} = J(m)e^{-im(t-\xi)} \tag{A.3a}$$

$$\{J(\xi,t), J(\eta,t)\} = 2\pi i\partial_\xi\delta(\xi-\eta) \tag{A.3b}$$

$$\{J(\xi,t), J(-\eta,t)\} = 2\pi i\partial_\xi\delta(\xi+\eta) \tag{A.3c}$$

$$\{J(-\xi,t), J(-\eta,t)\} = -2\pi i\partial_\xi\delta(\xi-\eta) \tag{A.3d}$$

$$\{J(m), J(n)\} = m\delta_{m+n,0} \ . \tag{A.3e}$$

We also assume that

$$\{q, J(m\neq 0)\} = 0 \tag{A.4}$$

but we leave the bracket $\{q, J(0)\}$ undetermined for the moment. The string coordinate $x(\xi, t)$ can also be written as

$$x(\xi,t) = 2q + \big(J(0)(t+\xi) + i\sum_{m\neq 0} J(m)\frac{e^{-im(t+\xi)}}{m}\big) - \big(J(0)(t-\xi) + i\sum_{m\neq 0} J(m)\frac{e^{-im(t-\xi)}}{m}\big) \tag{A.5}$$

and exponentials of $x(\xi, t)$ show the abelian analogue of the right and left mover product structure of the non-abelian vertex operator $g = g_- g_+$ discussed in the text.

The momentum p canonical to x follows from the Lagrange density (A.1):

$$p(\xi,t) \equiv \frac{1}{4\pi}\partial_t x(\xi,t) = \frac{1}{4\pi}(J(\xi,t) - J(-\xi,t)) \ . \tag{A.6}$$

Using the forms (A.2c) and (A.6) for $\partial_\xi x$ and p, we find the phase space realization of the currents

$$J(\xi,t) = 2\pi p(\xi) + \frac{1}{2}\partial_\xi x(\xi), \quad J(-\xi,t) = -2\pi p(\xi) + \frac{1}{2}\partial_\xi x(\xi) \tag{A.7a}$$

and from the current algebra (A.3), we compute the phase space brackets

$$\{x(\xi,t), x(\eta,t)\} = 4(\eta - \xi)\{q, J(0)\} \tag{A.8a}$$

$$\{x(\xi,t), p(\eta,t)\} = i(\delta(\xi-\eta) - \delta(\xi+\eta)) \tag{A.8b}$$

$$\{p(\xi,t), p(\eta,t)\} = 0 \tag{A.8c}$$

$$\{J(\xi,t), x(\eta,t)\} = -2\pi i(\delta(\xi-\eta) - \delta(\xi+\eta)) + 2\{J(0), q\} \tag{A.8d}$$

$$\{J(-\xi,t), x(\eta,t)\} = 2\pi i(\delta(\xi-\eta) - \delta(\xi+\eta)) + 2\{J(0), q\} \tag{A.8e}$$

in terms of the unknown quantity $\{q, J(0)\}$. We remark in particular that the second term in (A.8b) corresponds to the presence of a Dirichlet image charge.

To determine the unknown quantity $\{q, J(0)\}$, we require the consistency of the action and Hamiltonian formulations of the system. The Hamiltonian of the Dirichlet string follows by Legendre transformation

$$H = \int_0^\pi d\xi \, \mathcal{H} \tag{A.9a}$$

$$\mathcal{H} = \partial_t x P - \mathcal{L} = 2\pi p^2 + \frac{1}{8\pi}(\partial_\xi x)^2 = \frac{1}{4\pi}(J^2(\xi) + J^2(-\xi)) \tag{A.9b}$$

and then we may recompute $\partial_t x$ from the Hamiltonian equations of motion

$$\partial_t x(\xi,t) = i\{H, x(\xi,t)\} = 4\pi p(\xi,t) + 2i\{J(0), q\}J(0) \tag{A.10}$$

using the brackets in (A.8). For agreement with (A.6) we must set

$$\{J(0), q\} = 0 \tag{A.11}$$

and then all the previous relations of this appendix are in agreement with the abelian limit

$$x^i \to x, \quad p^i \to p, \quad B_{ij} \to 0 \tag{A.12a}$$

$$G_{ab} \to 1, \quad f_{ab}{}^c \to 0, \quad e_i{}^a \to 1, \quad \bar{e}_i{}^a \to -1 \tag{A.12b}$$

of our discussion in the text.

In the abelian case, there is a second possible phase space realization of the currents

$$J(\xi, t) \equiv 2\pi p(\xi) + \frac{1}{2}\partial_\xi x(\xi), \quad J(-\xi, t) \equiv 2\pi p(\xi) - \frac{1}{2}\partial_\xi x(\xi) \tag{A.13}$$

which differs from Eq. (A.7) only by the overall sign in the realization of $J(-\xi, t)$. Periodicity of the cylinder current in this case gives the Neumann boundary conditions

$$J(0, t) = J(-0, t), \quad J(\pi, t) = J(-\pi, t) \quad \text{or} \tag{A.14a}$$

$$\partial_\xi x(0, t) = \partial_\xi x(\pi, t) = 0 \ . \tag{A.14b}$$

The realization (A.13) must be taken with the same Hamiltonian (A.9) and the same current algebra (A.3). (The Neumann system is T-dual to the Dirichlet system.) The solution to the equations of motion with Neumann boundary conditions is

$$\frac{1}{2}x(\xi, t) = q + J(0)t + i\sum_{m \neq 0} J(m)\frac{\cos(m\xi)}{m}e^{-imt} \ . \tag{A.15}$$

In this case all is consistent when

$$\{q, J(0)\} = i \tag{A.16}$$

and then (A.15) and (A.3) give the phase space brackets

$$\{x(\xi, t), x(\eta, t)\} = 0 \tag{A.17a}$$

$$\{x(\xi, t), p(\eta, t)\} = i(\delta(\xi - \eta) + \delta(\xi + \eta)) \tag{A.17b}$$

$$\{p(\xi, t), p(\eta, t)\} = 0 \tag{A.17c}$$

which show a Neumann image charge in the second term of (A.17b).

Appendix B. The Remaining Phase Space Brackets

In this appendix, we use the results of the text and the chain rule to compute the rest of the phase space brackets of open WZW theory. In particular we need Eqs. (4.1), (4.5) and (6.3), and the results are given in terms of the coordinate brackets $\Delta = \{x, x\}$ in (6.12):

$$
\begin{aligned}
\{x^i(\xi,t), p_j(B,\eta,t)\} =&\{x^i(\xi), \frac{1}{4\pi}\big(e(\eta)_j{}^a J_a(\eta) + \bar{e}(\eta)_j{}^a J_a(-\eta)\big)\} \\
=& i\delta_j{}^i \delta(\xi-\eta) + \frac{i}{2}\big(\bar{e}(\eta)_j{}^a e(\xi)_a{}^i + e(\eta)_j{}^a \bar{e}(\xi)_a{}^i\big)\delta(\xi+\eta) \\
&+ \frac{1}{4\pi}\Delta^{ik}(\xi,\eta)\big(\partial_k e(\eta)_j{}^a J_a(\eta) + \partial_k \bar{e}(\eta)_j{}^a J_a(-\eta)\big)
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
\{x^i(\xi,t), p_j(\eta,t)\} =&\{x^i(\xi), p_j(B,\eta) - \frac{1}{4\pi}B_{jk}(\eta)\partial_\eta x^k\} \\
=& i\delta_j{}^i\delta(\xi-\eta) + \frac{i}{2}\big(\bar{e}(\eta)_j{}^a e(\xi)_a{}^i + e(\eta)_j{}^a \bar{e}(\xi)_a{}^i + \Psi^{ik}(\xi,\eta)B_{jk}(\eta)\big)\delta(\xi+\eta) \\
&+ \frac{1}{4\pi}\Delta^{ik}(\xi,\eta)\big(\partial_k e(\eta)_j{}^a J_a(\eta) + \partial_k \bar{e}(\eta)_j{}^a J_a(-\eta)\big) \\
&+ \frac{1}{4\pi}\Delta^{ik}(\xi,\eta)\big(\Lambda(\eta)_k{}^l B_{jl}(\eta) + \partial_k B_{jl}(\eta)\partial_\eta x^l\big)
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
\{J_a(\xi,t), p_i(B,\eta,t)\} =&\{J_a(\xi), \frac{1}{4\pi}\big(e(\eta)_i{}^b J_b(\eta) + \bar{e}(\eta)_i{}^b J_b(-\eta)\big)\} \\
=& \frac{i}{2}e(\eta)_i{}^b G_{ba}\partial_\xi \delta(\xi-\eta) + 2\pi i \partial_i e(\eta)_a{}^j p_j(B,\eta)\delta(\xi-\eta) \\
&+ \frac{i}{2}\bar{e}(\eta)_i{}^b G_{ba}\partial_\xi\delta(\xi+\eta) + 2\pi i\partial_i \bar{e}(\eta)_a{}^j p_j(B,\eta)\delta(\xi+\eta)
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
\{p_i(B,\xi,t), p_j(B,\eta,t)\} =&\{\frac{1}{4\pi}\big(e(\xi)_i{}^a J_a(\xi) + \bar{e}(\xi)_i{}^a J_a(-\xi)\big), p_j(B,\eta)\} \\
=& -\frac{i}{4\pi}f_{ab}{}^d G_{dc} e_i{}^a e_j{}^b e_k{}^c \partial_\xi x^k \delta(\xi-\eta) \\
&+ \frac{i}{8\pi}\big(e(\xi)_i{}^a G_{ab}\bar{e}(\eta)_j{}^b - \bar{e}(\xi)_i{}^a G_{ab} e(\eta)_j{}^b\big)\partial_\xi\delta(\xi+\eta) \\
&+ \frac{i}{2}\Big[\big(e(\xi)_i{}^a \partial_j \bar{e}(\eta)_a{}^k + \bar{e}(\xi)_i{}^a \partial_j e(\eta)_a{}^k\big)p_k(B,\eta) \\
&- \big(e(\eta)_j{}^a\partial_i\bar{e}(\xi)_a{}^k + \bar{e}(\eta)_j{}^a \partial_i e(\xi)_a{}^k\big)p_k(B,\xi)\Big]\delta(\xi+\eta) \\
&- \frac{i}{8\pi}f_{ab}{}^c\big(J_c(\xi)e(\xi)_i{}^a\bar{e}(\eta)_j{}^b + J_c(\eta)\bar{e}(\xi)_i{}^a e(\eta)_j{}^b\big)\delta(\xi+\eta) \\
&+ \frac{1}{16\pi^2}\Delta^{kl}(\xi,\eta)\tilde{\Lambda}_{ki}(\xi)\tilde{\Lambda}_{lj}(\eta) \; ;
\end{aligned}
\tag{B.4a}
$$

$$
\tilde{\Lambda}_{ij}(\xi) \equiv \partial_i e(\xi)_j{}^a J_a(\xi) + \partial_i \bar{e}(\xi)_j{}^a J_a(-\xi) \; . \tag{B.4b}
$$

The momentum bracket $\{p_i(\xi,t),p_j(\eta,t)\}$ follows from Eq. ((B.4)) and the definition ((3.2c)):

$$\begin{aligned}\{p_i(\xi,t),p_j(\eta,t)\} =& \frac{i}{8\pi}\big(e(\xi)_i{}^a G_{ab}\bar{e}(\eta)_j{}^b - \bar{e}(\xi)_i{}^a G_{ab} e(\eta)_j{}^b\big)\partial_\xi \delta(\xi+\eta)\\
&- \frac{i}{8\pi}\big[B_{ik}(\xi)\big(\bar{e}(\eta)_j{}^a e(\xi)_a{}^k + e(\eta)_j{}^a \bar{e}(\xi)_a{}^k\big)\\
&+ \big(\bar{e}(\eta)_i{}^a e(\xi)_a{}^k + e(\eta)_i{}^a \bar{e}(\xi)_a{}^k\big)B_{kj}(\eta)\big]\partial_\xi\delta(\xi+\eta)\\
&+ \frac{i}{2}\big[\big(e(\xi)_i{}^a\partial_j\bar{e}(\eta)_a{}^k + \bar{e}(\xi)_i{}^a\partial_j e(\eta)_a{}^k\big)p_k(B,\eta)\\
&- \big(\partial_i\bar{e}(\xi)_a{}^k e(\eta)_j{}^a + \partial_i e(\xi)_a{}^k\bar{e}(\eta)_j{}^a\big)p_k(B,\xi)\big]\delta(\xi+\eta)\\
&- \frac{i}{8\pi}f_{ab}{}^c\big(J_c(\xi)e(\xi)_i{}^a\bar{e}(\eta)_j{}^b + J_c(\eta)\bar{e}(\xi)_i{}^a e(\eta)_j{}^b\big)\delta(\xi+\eta)\\
&+ \frac{i}{8\pi}\big[\big(\bar{e}(\xi)_i{}^a e(\eta)_a{}^l + e(\xi)_i{}^a\bar{e}(\eta)_a{}^l\big)\partial_l B_{jk}(\eta)\partial_\eta x^k(\eta)\\
&- \partial_l B_{ik}(\xi)\partial_\xi x^k(\xi)\big(\bar{e}(\eta)_j{}^a e(\xi)_a{}^l + e(\eta)_j{}^a\bar{e}(\xi)_a{}^l\big)\big]\delta(\xi+\eta)\\
&+ \frac{i}{8\pi}\big[\big(\bar{e}(\xi)_i{}^a\partial_k e(\eta)_a{}^l + e(\xi)_i{}^a\partial_k\bar{e}(\eta)_a{}^l\big)\partial_\eta x^k(\eta)B_{jl}(\eta)\\
&- B_{il}(\xi)\big(\bar{e}(\eta)_j{}^a\partial_k e(\xi)_a{}^l + e(\eta)_j{}^a\partial_k\bar{e}(\xi)_a{}^l\big)\partial_\xi x^k(\xi)\big]\delta(\xi+\eta)\\
&+ \frac{1}{16\pi^2}\Delta^{kl}(\xi,\eta)\tilde{\Lambda}_{ki}(\xi)\tilde{\Lambda}_{lj}(\eta)\\
&- \frac{1}{4\pi}B_{ik}(\xi)\partial_l B_{jm}(\eta)\partial_\eta x^m(\eta)\big(2\pi i\Psi^{kl}(\xi,\eta)\delta(\xi+\eta) + \Delta^{kn}(\xi,\eta)\Lambda(\eta)_n{}^l\big)\\
&- \frac{1}{4\pi}\partial_k B_{im}(\xi)\partial_\xi x^m(\xi)B_{jl}(\eta)\big(2\pi i\Psi^{kl}(\xi,\eta)\delta(\xi+\eta) + \Lambda(\xi)_n{}^l\Delta^{kn}(\xi,\eta)\big)\\
&+ \frac{1}{16\pi^2}\partial_m B_{ik}(\xi)\partial_\xi x^k(\xi)\partial_n B_{jl}(\eta)\partial_\eta x^l(\eta)\Delta^{mn}(\xi,\eta)\\
&+ \frac{1}{16\pi^2}B_{ik}(\xi)B_{jl}(\eta)\partial_\xi\partial_\eta\Delta^{kl}(\xi,\eta).\end{aligned} \tag{B.5}$$

This final result for $\{p,p\}$ contains no bulk terms (proportional to $\delta(\xi-\eta)$), so it vanishes except at the boundary. Moreover, as in the case of $\{x,x\}$, this non-commutativity is essentially non-abelian and vanishes (with Δ) in the abelian limit.

Appendix C. Presentation as a Single "Chiral" KZ system

In this appendix, we present our open string KZ system for the open n-point correlators A as a single "chiral" KZ system in $2n$ variables μ:

$$\mu \equiv \{\phi_i,\bar{\phi}_i\}, \quad \partial_\mu \equiv \{\partial_{\phi_i},\partial_{\bar{\phi}_i}\} \tag{C.1a}$$

$$\phi_i = t_i+\xi_i, \quad \bar{\phi}_i = t_i-\xi_i, \quad \partial_{\phi_i} = \frac{1}{2}\partial_{\phi_i}, \quad \partial_{\bar{\phi}_i} = \frac{1}{2}\partial_{i-}\ . \tag{C.1b}$$

The variables ϕ_i and $\bar{\phi}_i$ were defined in (7.9). The result is

$$\partial_\mu \tilde{A} = \tilde{A} W_\mu \tag{C.2a}$$

$$W_{\phi_i} = i\Delta_g(T^i) + 2iL_g^{ab}\Big(\sum_{j\neq i} T_a^i T_b^j f(\phi_j - \phi_i) + \sum_j T_a^i \bar{T}_b^j f(\bar{\phi}_j - \phi_i)\Big) \tag{C.2b}$$

$$W_{\bar{\phi}_i} = i\Delta_g(T^i) + 2iL_g^{ab}\Big(\sum_{j\neq i} \bar{T}_a^j \bar{T}_b^i f(\bar{\phi}_i - \bar{\phi}_j) + \sum_j T_a^j \bar{T}_b^i f(\phi_j - \bar{\phi}_i)\Big) \tag{C.2c}$$

$$\tilde{A} Q_a = 0, \quad Q_a \equiv \sum_{i=1}^{n} (T_a^i + \bar{T}_a^i) \tag{C.2d}$$

where the function f is defined in (7.13). In (C.2) the matrices T^i are associated to the charges at ϕ_i, and the matrices $\bar{T}^i$ are associated to the image charges at $\bar{\phi}_i$. The matrix $\bar{T}^i$ is the representation dual to the irrep T^i

$$\bar{T}_\alpha{}^\beta \equiv -T_\beta{}^\alpha \; . \tag{C.3}$$

In this notation, the matrix A of the text is treated as a single large row $\tilde{A}$ and the matrices on the right of $\tilde{A}$ act to the left as a tensor product. As an example, we work out a representative term of the $\tilde{A}T^i\bar{T}^j$ type in (C.2), starting with the notation of the text,

$$\begin{aligned}(T^i A T^j)_\alpha{}^\beta =& (T^i)_\alpha{}^\gamma A_\gamma{}^\delta (T^j)_\delta{}^\beta \equiv -\tilde{A}^{\delta\gamma} (T^j)_\delta{}^\beta (\bar{T}^i)_\gamma{}^\alpha \\ \equiv& -(\tilde{A} T^j \otimes \bar{T}^i)^{\beta\alpha} \equiv -(\tilde{A} T^j \bar{T}^i)^{\beta\alpha}\end{aligned} \tag{C.4a}$$

$$\tilde{A}^{\beta\alpha} \equiv A_\alpha{}^\beta \tag{C.4b}$$

where $A_\alpha{}^\beta$ is the correlator (7.8a) of the text. In this example, we have suppressed inactive indices for simplicity. More generally, one finds that

$$\tilde{A}^{\beta_1\ldots\beta_n,\alpha_1\ldots\alpha_n} = A_{\alpha_1\ldots\alpha_n}{}^{\beta_1\ldots\beta_n} \tag{C.5}$$

where, in the chiral form $\tilde{A}$, the T's operate to the left on the β indices and the $\bar{T}$'s operate to the left on the α indices.

In this presentation one finds that the consistency conditions (7.41-7.44) take the simple form

$$\partial_\mu W_\nu - \partial_\nu W_\mu = [W_\mu, W_\nu] = 0 \tag{C.6a}$$

$$[Q_a, W_\mu] = 0 \tag{C.6b}$$

so that the connection W_μ is abelian flat.

This "chiral" description of our system is now in standard KZ form, and one may apply standard [42] formal methods in KZ theory to obtain solutions of our open string KZ system as integral representations. This appendix was worked out in a conversation with N. Reshetikin.

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