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### Authors

Ho, T
Peralta, AM
Russo, B





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# TERNARY WEAKLY AMENABLE C*-ALGEBRAS AND JB*-TRIPLES

*by* TONY HO†

(*13416 Sheridan Ave, Urbandale, IA 50323, USA*)

ANTONIO M. PERALTA‡

(*Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain*)

*and* BERNARD RUSSO§

(*Department of Mathematics, University of California, Irvine, CA 92697-3875, USA*)

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**Abstract**

A well-known result of Haagerup from 1983 states that every C*-algebra $A$ is weakly amenable, that is, every (associative) derivation from $A$ into its dual is inner. A Banach algebra $B$ is said to be ternary weakly amenable if every continuous Jordan triple derivation from $B$ into its dual is inner. We show that commutative C*-algebras are ternary weakly amenable, but that $B(H)$ and $K(H)$ are not, unless $H$ is finite dimensional. More generally, we inaugurate the study of weak amenability for Jordan–Banach triples, focussing on commutative JB*-triples and some Cartan factors.

## 1. Introduction

Two fundamental questions concerning derivations from a Banach algebra $A$ into a Banach $A$-bimodule $M$ are:

(i) Is an everywhere defined derivation automatically continuous?
(ii) Are all continuous derivations inner? If not, can every continuous derivation be approximated by inner derivations?

One can ask the same questions in the setting of Jordan Banach algebras (and Jordan modules), and more generally for Jordan–Banach triple systems (and Jordan–Banach triple modules). Significant special cases occur in each context when $M = A$ or when $M = A^*$.

In order to obtain a better perspective on the objectives of this paper, we shall give here a comprehensive review of the major existing results on these two problems in the contexts in which we will be interested, namely, C*-algebras, JB*-algebras and JB*-triples. Although we will be dealing with

†E-mail: zh_ho01@yahoo.com

‡E-mail: aperalta@ugr.es

§Corresponding author. E-mail: brusso@uci.edu

both the real and complex cases in this paper, in the interest of space this review will be confined to the complex case.

A derivation on a Banach algebra $A$ into a Banach $A$-bimodule $M$ is a linear mapping $D : A \to M$ such that $D(ab) = a \cdot D(b) + D(a) \cdot b$. An inner derivation, in this context, is a derivation of the form: $\mathrm{ad}_x(a) = x \cdot a - a \cdot x$ $(x \in M, a \in A)$.

In the context of C*-algebras, automatic continuity results were initiated by Kaplansky before 1950 (see [27]) and culminated in the following series of results: Every derivation from a C*-algebra into itself is continuous [47]; every derivation from a C*-algebra $A$ into a Banach $A$-bimodule is continuous [45].

The major results for C*-algebras regarding inner derivations read as follows: every derivation from a C*-algebra on a Hilbert space $H$ into itself is of the form $x \mapsto ax - xa$ for some $a$ in the weak closure of the C*-algebra in $L(H)$ [26, 48]; every amenable C*-algebra is nuclear [7]; every nuclear C*-algebra is amenable [15]; every C*-algebra is weakly amenable [15, 16]. For finite-dimensional C*-algebras, the last result follows from the work of Hochschild in 1942 [21].

As a bridge to the Jordan algebra setting, we make a slight digression. Sinclair proved in 1970 (cf. [51]) that a continuous Jordan derivation from a semisimple Banach algebra to itself is a derivation, although this result fails for derivations of semisimple Banach algebras into a Banach bimodule. (A *Jordan derivation* from a Banach algebra $A$ into a Banach $A$-module is a linear map $D$ satisfying $D(a^2) = aD(a) + D(a)a$, $(a \in A)$, or equivalently, $D(ab + ba) = aD(b) + D(b)a + D(a)b + bD(a)$, $(a, b \in A)$.) Nevertheless, Johnson proved in 1996 (cf. [25]) that every continuous Jordan derivation from a C*-algebra $A$ to a Banach $A$-bimodule is a derivation. A new proof of this fact was presented by Haagerup and Laustsen [16].

The following subsequent result partially removed the assumption of continuity from this theorem of Johnson: Every Jordan derivation from a von Neumann algebra, or from a commutative C*-algebra, into a Banach bimodule is continuous [1]. More recently, the assumption was completely removed: Every Jordan derivation from an arbitrary C*-algebra into a Banach bimodule is continuous [43, Corollary 22]. Earlier, Cusack [8], completing a study of Sinclair, showed that every Jordan derivation on a semisimple Banach algebra is continuous, and Villena [56] extended this result to semisimple Jordan–Banach algebras.

We now move to the context of Jordan–Banach algebras. A derivation from a Jordan–Banach algebra $A$ into a Jordan Banach module $M$ is a linear mapping $D : A \to M$ such that $D(a \circ b) = a \circ Db + Da \circ b$, where $\circ$ denotes both the product in the Jordan algebra and the module action. (Jordan–Banach algebras and Jordan–Banach modules will be defined below.) An inner derivation in this context is a derivation of the form: $\sum_{i=1}^m (L(x_i)L(a_i) - L(a_i)L(x_i))$ $(x_i \in M, a_i \in A)$. Here, $L(x)$ is the operator $a \mapsto a \circ x$ from $A$ to $M$ and $L(a)$ is either the operator $b \mapsto b \circ a$ from $A$ to $A$ or $x \mapsto a \circ x$ from $M$ to $M$.

In the context of JB*-algebras, the major automatic continuity results consist of the following. Every (Jordan) derivation of a reversible JC*-algebra extends to a derivation (associative) of its enveloping C*-algebra ([53]—this recovers Sinclair's result in the case of C*-algebras); every Jordan derivation from a JB*-algebra $A$ into $A$ or into $A^*$ is continuous and every Jordan derivation from a commutative or a compact C*-algebra into a Jordan–Banach bimodule is continuous [17]. This latter result was also extended to arbitrary C*-algebras in [43, Corollary 21]: Every Jordan derivation from an arbitrary C*-algebra $A$ into a Jordan–Banach $A$-bimodule is continuous.

The major results for JB*-algebras regarding inner derivations are the following: Every Jordan derivation from a finite-dimensional JB*-algebra into a Jordan–Banach module is inner (follows from

Jacobson [23, 24]); every Jordan derivation of a purely exceptional or a reversible JBW-algebra is inner [53]; every Jordan derivation of $\bigoplus L^\infty(S_j, U_j)$ ($U_j$ spin factors) is inner if and only if $\sup_j \dim U_j < \infty$ [53]. By a structure theorem for JBW-algebras, these theorems of Upmeier completely determine whether a given JBW-algebra has only inner derivations.

Finally, we move to a discussion of Jordan–Banach triples, which is the proper setting for this paper. A (triple or ternary) derivation on a Jordan–Banach triple $A$ into a Jordan–Banach triple module $M$ is a *conjugate* linear mapping $D : A \to M$ such that $D\{a, b, c\} = \{Da, b, c\} + \{a, Db, c\} + \{a, b, Dc\}$. An inner derivation in this context is a derivation of the form: $\sum_i^m (L(x_i, a_i) - L(a_i, x_i))$ ($x_i \in M, a_i \in A$), where $L(x, a)$ and $L(a, x)$ denote, respectively, the maps $b \mapsto \{x, a, b\}$ and $b \mapsto \{a, x, b\}$ arising from the module action. (Jordan–Banach triple and Jordan–Banach triple module will be defined below, after which the reason for the conjugate linearity in the complex case of derivations into a module, as opposed to linearity, will be more transparent.)

In the context of JB*-triples, automatic continuity results were initiated by Barton and Friedman in 1990 (cf. [3]) who showed that every triple derivation of a JB*-triple is continuous. Peralta and Russo in 2010 (see [43, Theorem 13]) gave necessary and sufficient conditions under which a derivation of a JB*-triple into a Jordan–Banach triple module is continuous. As shown in [43], these conditions are automatically satisfied in the case where the JB*-triple is actually a C*-algebra with the triple product $(xy^*z + zy^*x)/2$, leading to a new proof (cf. [43, Corollary 23]) of the theorem of Ringrose quoted above as well as the results of Alaminos–Brešar–Villena and Hejazian–Niknam, also quoted above.

The known results for JB*-triples regarding inner derivations are surveyed in the following statements: Every derivation from a finite-dimensional JB*-triple into itself is inner (follows from Meyberg [38]); every derivation from a finite-dimensional JB*-triple into a Jordan–Banach triple module is inner (follows from Kühn–Rosendahl [31]); every derivation of a Cartan factor of type $I_{n,n}$ ($n$ finite or infinite), type II (with underlying Hilbert space of even or infinite dimension) or type III is inner [20]. Infinite-dimensional Cartan factors of type $I_{m,n}$, $m \neq n$ and type IV have derivations into themselves which are not inner (cf. [20]).

It is worth noting that, besides the consequences for C*-algebras of the main result of Peralta and Russo [43] noted above, another consequence is the automatic continuity of derivations of a JB*-triple into its dual [43, Corollary 15], leading us to the study of weak amenability for JB*-triples, which is the main focus of this paper.

We conclude this review introduction by describing the contents of this paper. Section 2 sets down the definitions and basic properties of Jordan triples, Jordan triple modules, derivations and (ternary) weak amenability that we shall use. Sections 3 and 4 are concerned with C*-algebras, considered as JB*-triples with the triple product $(xy^*z + zy^*x)/2$. It is proved that commutative C*-algebras are ternary weakly amenable, and that the compact operators, as well as all bounded operators on a Hilbert space $H$ are ternary weakly amenable if and only if $H$ is finite dimensional.

Sections 5 and 6 are concerned with more general JB*-triples. It is proved that certain Cartan factors (Hilbert spaces and spin factors) are ternary weakly amenable if and only if they are finite dimensional, that infinite-dimensional finite rank Cartan factors of type 1 are not ternary weakly amenable, and that commutative JB*-triples are almost weakly amenable in the sense that the inner derivations into the dual are norm dense in the set of all derivations into the dual. In comparison, the existing forerunners on the approximation of derivations on C*-algebras by inner derivations (immediate consequence of the Sakai–Kadison results [26, 48]), JB*-algebras [53] and JB*-triples [3] involved the topology of pointwise convergence and not the norm topology.

## 2. Derivations on Jordan triples and Jordan triple modules

### 2.1. *Jordan triples*

A complex (respectively, real) *Jordan triple* is a complex (respectively, real) vector space $E$ equipped with a non-trivial triple product

$$E \times E \times E \to E,$$
$$(xyz) \mapsto \{x, y, z\},$$

which is bilinear and symmetric in the outer variables and conjugate linear (respectively, linear) in the middle one satisfying the so-called 'Jordan Identity'

$$L(a, b)L(x, y) - L(x, y)L(a, b) = L(L(a, b)x, y) - L(x, L(b, a)y),$$

for all $a, b, x, y$ in $E$, where $L(x, y)z := \{x, y, z\}$. When $E$ is a normed space and the triple product of $E$ is continuous, we say that $E$ is a *normed Jordan triple*. If a normed Jordan triple $E$ is complete with respect to the norm (i.e. if $E$ is a Banach space), then it is called a *Jordan–Banach triple*. Every normed Jordan triple can be completed in the usual way to become a Jordan–Banach triple. Unless otherwise specified, the term 'normed Jordan triple' (respectively, 'Jordan–Banach triple') will always mean a real or complex normed Jordan triple (respectively, a real or complex Jordan–Banach triple).

A subspace $F$ of a Jordan triple $E$ is said to be a *subtriple* if $\{F, F, F\} \subseteq F$. We recall that a subspace $J$ of $E$ is said to be a *triple ideal* if $\{E, E, J\} + \{E, J, E\} \subseteq J$. When $\{J, E, J\} \subseteq J$, we say that $J$ is an *inner ideal* of $E$.

A real (respectively, complex) *Jordan algebra* is a (non-necessarily associative) algebra over the real (respectively, complex) field whose product is abelian and satisfies $(a \circ b) \circ a^2 = a \circ (b \circ a^2)$. A normed Jordan algebra is a Jordan algebra $A$ equipped with a norm, $\|\cdot\|$, satisfying $\|a \circ b\| \leq \|a\| \ \|b\|$, $a, b \in A$. A *Jordan–Banach algebra* is a normed Jordan algebra whose norm is complete.

A Jordan algebra is called *special* if it is isomorphic to a subspace of an associative algebra which is closed under $ab + ba$. Every Jordan algebra is a Jordan triple with respect to

$$\{a, b, c\} := (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b.$$

If a Jordan triple arises from a special Jordan algebra, then the triple product reduces to $\{a, b, c\} = \frac{1}{2}(abc + cba)$. Thus, every real or complex associative Banach algebra (respectively, Jordan–Banach algebra) is a real Jordan–Banach triple with respect to the product $\{a, b, c\} = \frac{1}{2}(abc + cba)$ (respectively, $\{a, b, c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$).

A real or complex Jordan–Banach triple $E$ is said to be *commutative* or *abelian* if the identity

$$\{\{x, y, z\}, a, b\} = \{x, y, \{z, a, b\}\} = \{x, \{y, z, a\}, b\}$$

holds for all $x, y, z, a, b \in E$, equivalently, $L(a, b)L(c, d) = L(c, d)L(a, b)$ for every $a, b, c, d \in E$.

A JB$^*$-algebra is a complex Jordan–Banach algebra $A$ equipped with an algebra involution $^*$ satisfying $\|\{a, a^*, a\}\| = \|a\|^3$, $a \in A$. (Recall that $\{a, a^*, a\} = 2(a \circ a^*) \circ a - a^2 \circ a^*$.)

A (*complex*) *JB*-triple* is a complex Jordan Banach triple $E$ satisfying the following axioms:

(JB*1) for each $a$ in $E$, the map $L(a,a)$ is an hermitian operator on $E$ with non-negative spectrum;
(JB*2) $\|\{a,a,a\}\| = \|a\|^3$ for all $a$ in $A$.

Every C*-algebra (respectively, every JB*-algebra) is a JB*-triple with respect to the product $\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)$ (respectively, $\{a,b,c\} := (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$).

A summary of the basic facts about JB*-triples, an important and well-understood class of Jordan–Banach triples, some of which are recalled here, can be found in [46] and some of the references therein, such as [13, 14, 28, 54, 55].

We recall that a *real JB*-triple* is a norm-closed real subtriple of a complex JB*-triple (cf. [22]). The class of real JB*-triples includes all complex JB*-triples, all real and complex C*- and JB*-algebras and all JB-algebras.

A complex (respectively, real) *JBW*-triple* is a complex (respectively, real) JB*-triple which is also a dual Banach space (with a unique isometric predual [4, 37]). It is known that the triple product of a real or complex JBW*-triple is separately weak* continuous (cf. [4, 37]). The second dual of a JB*-triple $E$ is a JBW*-triple with a product extending the product of $E$ [9, 22].

JB-algebras are precisely the self-adjoint parts of JB*-algebras [33], and a JBW-algebra is a JB-algebra that is a dual space.

When $E$ is a (complex) JB*-triple or a real JB*-triple, a subtriple $I$ of $E$ is a triple ideal if and only if $\{E,E,I\} \subseteq I$ or $\{E,I,E\} \subseteq I$ or $\{E,I,I\} \subseteq I$ (cf. [5, Proposition 1.3]).

## 2.2. *Jordan triple modules*

Let $A$ be an associative algebra. Let us recall that an *$A$-bimodule* is a vector space $X$, equipped with two bilinear products $(a,x) \mapsto ax$ and $(a,x) \mapsto xa$ from $A \times X$ to $X$ satisfying the axioms

$$a(bx) = (ab)x, \quad a(xb) = (ax)b \quad \text{and} \quad (xa)b = x(ab),$$

for every $a, b \in A$ and $x \in X$.

Let $A$ be a Jordan algebra. A *Jordan $A$-module* is a vector space $X$, equipped with two bilinear products $(a,x) \mapsto a \circ x$ and $(x,a) \mapsto x \circ a$ from $A \times X$ to $X$, satisfying

$$a \circ x = x \circ a, \quad a^2 \circ (x \circ a) = (a^2 \circ x) \circ a$$

and

$$2((x \circ a) \circ b) \circ a + x \circ (a^2 \circ b) = 2(x \circ a) \circ (a \circ b) + (x \circ b) \circ a^2,$$

for every $a, b \in A$ and $x \in X$. The space $A \oplus X$ is a Jordan algebra with respect to the product

$$(a,x) \circ (b,y) := (a \circ b, a \circ y + b \circ x).$$

The Jordan algebra $(A \oplus X, \circ)$ is called the Jordan split null extension of $A$ and $X$ (cf. [24, Section II.5, p. 82]). When $A$ is a Jordan–Banach algebra, $X$ is a Banach space and the mapping $A \times X \to X$, $(a,x) \mapsto a \circ x$ is continuous, then $X$ is said to be a Jordan–Banach module. The Jordan split null extension is never a JB-algebra since $(0,x)^2 = 0$.

Let $E$ be a complex (respectively, real) Jordan triple. A *Jordan triple $E$-module* (also called *triple $E$-module*) is a vector space $X$ equipped with three mappings

$$\{\cdot,\cdot,\cdot\}_1 : X \times E \times E \to X, \quad \{\cdot,\cdot,\cdot\}_2 : E \times X \times E \to X$$

and

$$\{\cdot,\cdot,\cdot\}_3 : E \times E \times X \to X$$

satisfying the following axioms:

(JTM1) $\{x, a, b\}_1$ is linear in $a$ and $x$ and conjugate linear in $b$ (respectively, trilinear), $\{abx\}_3$ is linear in $b$ and $x$ and conjugate linear in $a$ (respectively, trilinear) and $\{a, x, b\}_2$ is conjugate linear in $a, b, x$ (respectively, trilinear);

(JTM2) $\{x, b, a\}_1 = \{a, b, x\}_3$, and $\{a, x, b\}_2 = \{b, x, a\}_2$ for every $a, b \in E$ and $x \in X$;

(JTM3) denoting by $\{\cdot,\cdot,\cdot\}$ any of the products $\{\cdot,\cdot,\cdot\}_1$, $\{\cdot,\cdot,\cdot\}_2$ or $\{\cdot,\cdot,\cdot\}_3$, the identity $\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\}$, holds whenever one of the elements $a, b, c, d, e$ is in $X$ and the rest are in $E$.

It is obvious that every real or complex Jordan triple $E$ is a *real* triple $E$-module. It is problematical whether every complex Jordan triple $E$ is a complex triple $E$-module for a suitable triple product. This is partly why we have defined (in Sections 1 and 2.3) a derivation of a complex JB$^*$-triple into a Jordan–Banach triple module to be conjugate linear.

When $E$ is a Jordan–Banach triple and $X$ is a triple $E$-module which is also a Banach space and, for each $a, b$ in $E$, the mappings $x \mapsto \{a, b, x\}_3$ and $x \mapsto \{a, x, b\}_2$ are continuous, we shall say that $X$ is a triple $E$-module with *continuous module operations*. When the products $\{\cdot,\cdot,\cdot\}_1$, $\{\cdot,\cdot,\cdot\}_2$ and $\{\cdot,\cdot,\cdot\}_3$ are (jointly) continuous, we shall say that $X$ is a *Banach* (*Jordan*) *triple $E$-module*.

Hereafter, the triple products $\{\cdot,\cdot,\cdot\}_j$, $j = 1, 2, 3$, will be simply denoted by $\{\cdot,\cdot,\cdot\}$ whenever the meaning is clear from the context.

Every (associative) Banach $A$-bimodule (respectively, Jordan Banach $A$-module) $X$ over an associative Banach algebra $A$ (respectively, Jordan–Banach algebra $A$) is a real Banach triple $A$-module (respectively, $A$-module) with respect to the '*elementary*' product

$$\{a, b, c\} := \tfrac{1}{2}(abc + cba)$$

(respectively, $\{a, b, c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$), where one element of $a, b, c$ is in $X$ and the other two are in $A$.

It is easy but laborious to check that the dual space $E^*$ of a complex (respectively, real) Jordan–Banach triple $E$ is a complex (respectively, real) triple $E$-module with respect to the products

$$\{a, b, \varphi\}(x) = \{\varphi, b, a\}(x) := \varphi\{b, a, x\} \tag{1}$$

and

$$\{a, \varphi, b\}(x) := \overline{\varphi\{a, x, b\}}, \quad \forall\, x \in X, a, b \in E. \tag{2}$$

Given a triple $E$-module $X$ over a Jordan triple $E$, the space $E \oplus X$ can be equipped with a structure of a real Jordan triple with respect to the product $\{a_1 + x_1, a_2 + x_2, a_3 + x_3\} = \{a_1, a_2, a_3\} + \{x_1, a_2, a_3\} + \{a_1, x_2, a_3\} + \{a_1, a_2, x_3\}$. Consistent with the terminology in [24, Section II.5], $E \oplus X$ will be called the *triple split null extension* of $E$ and $X$. It is never a JB$^*$-triple.

A subspace $S$ of a triple $E$-module $X$ is said to be a *Jordan triple submodule* or a *triple submodule* if and only if $\{E, E, S\} + \{E, S, E\} \subseteq S$. Every triple ideal $J$ of $E$ is a Jordan triple $E$-submodule of $E$.

### 2.3. *Derivations*

Let $X$ be a Banach $A$-bimodule over an (associative) Banach algebra $A$. A linear mapping $D : A \to X$ is said to be a (*binary or associative*) *derivation* if $D(ab) = D(a)b + aD(b)$ for every $a, b$ in $A$. The symbol $\mathcal{D}_b(A, X)$ will denote the set of all continuous binary derivations from $A$ to $X$.

When $X$ is a Jordan–Banach module over a Jordan–Banach algebra $A$, a linear mapping $D : A \to X$ is said to be a *Jordan derivation* if $D(a \circ b) = D(a) \circ b + a \circ D(b)$ for every $a, b$ in $A$. We denote the set of continuous Jordan derivations from $A$ to $X$ by $\mathcal{D}_J(A, X)$. Although Jordan derivations also are binary derivations, we use the word 'binary' only for associative derivations.

In the setting of Jordan–Banach triples, a *triple* or *ternary derivation* from a (real or complex) Jordan–Banach triple $E$ into a Banach triple $E$-module $X$ is a conjugate linear mapping $\delta : E \to X$ satisfying

$$\delta\{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}, \tag{3}$$

for every $a, b, c$ in $E$. The set of all continuous ternary derivations from $E$ to $X$ will be denoted by $\mathcal{D}_t(E, X)$. According to [3, 20], a *ternary derivation* on $E$ is a linear mapping $\delta : E \to E$ satisfying the identity (3).

It should be remarked here that, unlike derivations from $E$ to itself, derivations from $E$ to $E^*$, when the latter is regarded as a Jordan triple $E$-module, are defined to be conjugate linear maps (in the complex case). The words Jordan or ternary may seem redundant in the expressions 'Jordan derivation on a Jordan algebra' and 'ternary derivation on a Jordan triple'; nevertheless, we shall make use of them for clarity.

If $E$ is a real or complex Jordan–Banach triple, we can easily conclude, from the Jordan identity, that $\delta(a, b) := L(a, b) - L(b, a)$ is a ternary derivation on $E$ for every $a, b \in E$. A triple or ternary derivation $\delta$ on $E$ is said to be *inner* if it can be written as a finite sum of derivations of the form $\delta(a, b)$ $(a, b \in E)$. Following [3, 20], we shall say that $E$ has the *inner derivation property* if every ternary derivation on $E$ is inner. The just quoted papers study the inner derivation property in the setting of real and complex JB*-triples.

The following technical result will be needed later.

PROPOSITION 2.1 *Let $E$ be a real or complex JB*-triple and let $\delta : E \to E^*$ be a ternary derivation. Then $\delta^{**} : E^{**} \to E^{***}$ is a weak*-continuous ternary derivation satisfying $\delta^{**}(E^{**}) \subseteq E^*$.*

*Proof*. Let $\delta$ be a ternary derivation from a real (or complex) JB*-triple to its dual, which is automatically bounded by Peralta and Russo [43, Corollary 15]. It is known that every bounded linear operator from a real JB*-triple to the dual of another real JB*-triple factors through a real Hilbert space (cf. [42, Lemma 5]). Thus, $\delta$ factors though a real Hilbert space and hence it is weakly compact. By Hille and Phillips [19, Lemma 2.13.1], we have $\delta^{**}(E^{**}) \subset E^*$.

We will prove now that $\delta^{**}$ is a ternary derivation. Clearly, the mapping $\delta^{**} : E^{**} \to E^{***}$ is $\sigma(E^{**}, E^*)$-to-$\sigma(E^{***}, E^{**})$-continuous. Let $a, b$ and $c$ be elements in $E^{**}$. By Goldstine's Theorem, there exist (bounded) nets $(a_\lambda)$, $(b_\mu)$ and $(c_\beta)$ in $E$ such that $(a_\lambda) \to a$, $(b_\mu) \to b$ and $(c_\beta) \to c$ in the weak*-topology of $E^{**}$.

It should be noted here that, for every net $(\phi_\lambda)$ in $E^{***}$, converging to some $\phi \in E^{***}$ in the $\sigma(E^{***}, E^{**})$-topology, the nets $(\{\phi_\lambda, a, b\})$ and $(\{a, \phi_\lambda, b\})$ converge in the $\sigma(E^{***}, E^{**})$-topology to $(\{\phi, a, b\})$ and $(\{a, \phi, b\})$, respectively. Having this fact in mind, it follows from the separate weak$^*$-continuity of the triple product in $E^{**}$ together with the weak$^*$-continuity of $\delta^{**}$ that

$$\begin{aligned}\delta^{**}\{a, b_\mu, c_\beta\} &= w^*\text{-}\lim_\lambda \delta\{a_\lambda, b_\mu, c_\beta\}\\ &= w^*\text{-}\lim_\lambda \{\delta(a_\lambda), b_\mu, c_\beta\} + \{a_\lambda, \delta(b_\mu), c_\beta\} + \{a_\lambda, b_\mu, \delta(c_\beta)\},\end{aligned}$$

$$w^*\text{-}\lim_\lambda \{\delta(a_\lambda), b_\mu, c_\beta\} = \{\delta^{**}(a), b_\mu, c_\delta\},$$

$$w^*\text{-}\lim_\lambda \{a_\lambda, \delta(b_\mu), c_\beta\} = \{a, \delta(b_\mu), c_\beta\}$$

and

$$w^*\text{-}\lim_\lambda \{a_\lambda, b_\mu, \delta(c_\beta)\} = \{a, b_\mu, \delta(c_\beta)\},$$

for every $\mu$ and $\beta$. Therefore,

$$\delta^{**}\{a, b_\mu, c_\beta\} = \{\delta^{**}(a), b_\mu, c_\beta\} + \{a, \delta(b_\mu), c_\beta\} + \{a, b_\mu, \delta(c_\beta)\}, \tag{4}$$

for every $\mu$ and $\beta$. By a similar argument, taking weak$^*$-limits in (4) first in $\mu$ and later in $\beta$, we get

$$\delta^{**}\{a, b, c\} = \{\delta^{**}(a), b, c\} + \{a, \delta^{**}(b), c\} + \{a, b, \delta^{**}(c)\},$$

which concludes the proof. □

### 2.4. *Weakly amenable Jordan–Banach triples*

Let $X$ be a Banach $A$-bimodule over an associative Banach algebra $A$. Given $x_0$ in $X$, the mapping $D_{x_0} : A \to X$, $D_{x_0}(a) = x_0 a - a x_0$ is a bounded (associative or binary) derivation. Derivations of this form are called *inner*. The set of all inner derivations from $A$ to $X$ will be denoted by $\mathcal{I}nn_b(A, X)$.

Recall that a Banach algebra $A$ is said to be *amenable* if every bounded derivation of $A$ into a dual $A$-module is inner, and *weakly amenable* if every (bounded) derivation from $A$ to $A^*$ is inner. In [**15**], Haagerup making use of preliminary work of Bunce and Paschke [**6**] and the Pisier–Haagerup Grothendieck's inequality for general C$^*$-algebras, proved that every C$^*$-algebra is weakly amenable. In [**16**], Haagerup and Laustsen gave a simplified proof of this result.

When $x_0$ is an element in a Jordan–Banach $A$-module $X$, over a Jordan–Banach algebra $A$, for each $b \in A$, the mapping $\delta_{x_0,b} : A \to X$,

$$\delta_{x_0,b}(a) := (x_0 \circ a) \circ b - (b \circ a) \circ x_0 \quad (a \in A)$$

is a bounded derivation. Finite sums of derivations of this form are called *inner*. The symbol $\mathcal{I}nn_J(A, X)$ will stand for the set of all inner Jordan derivations from $A$ to $X$.

The Jordan–Banach algebra $A$ is said to be *weakly amenable*, or *Jordan weakly amenable*, if every (bounded) derivation from $A$ to $A^*$ is inner. It is natural to ask whether every JB$^*$-algebra is weakly amenable. The answer is no, as Lemma 4.1 or 4.3 shows.

In the more general setting of Jordan–Banach triples, the corresponding definitions read as follows: Let $E$ be a complex (respectively, real) Jordan triple and let $X$ be a triple $E$-module. For each $b \in E$ and each $x_0 \in X$, we conclude, via the main identity for Jordan triple modules (JTM3), that the mapping $\delta = \delta(b, x_0) : E \to X$, defined by

$$\delta(a) = \delta(b, x_0)(a) := \{b, x_0, a\} - \{x_0, b, a\} \quad (a \in E) \tag{5}$$

is a ternary derivation from $E$ into $X$. Finite sums of derivations of the form $\delta(b, x_0)$ are called *(ternary) inner derivations*. Henceforth, we shall write $\mathcal{I}nn_t(E, X)$ for the set of all inner ternary derivations from $E$ to $X$.

A Jordan–Banach triple $E$ is said to be *weakly amenable* or *ternary weakly amenable* if every continuous triple derivation from $E$ into its dual space is necessarily inner.

In the next step, we explore the connections between ternary weak amenability in a real JB*-triple and its complexification. Let $E$ be a real JB*-triple. By Isidro *et al.* [**22**, Proposition 2.2], there exists a unique complex JB*-triple structure on the complexification $\hat{E} = E \oplus \mathrm{i}E$, and a unique conjugation (i.e. conjugate linear isometry of period 2) $\tau$ on $\hat{E}$ such that $E = \hat{E}^\tau := \{x \in \hat{E} : \tau(x) = x\}$, that is, $E$ is a real form of a complex JB*-triple. Let us consider

$$\tau^\sharp : \hat{E}^* \to \hat{E}^*,$$

defined by

$$\tau^\sharp(\phi)(z) = \overline{\phi(\tau(z))}.$$

The mapping $\tau^\sharp$ is a conjugation on $\hat{E}^*$. Furthermore, the map

$$(\hat{E}^*)^{\tau^\sharp} \longrightarrow (\hat{E}^\tau)^* \ (= E^*),$$
$$\phi \mapsto \phi|_E$$

is an isometric bijection, where $(\hat{E}^*)^{\tau^\sharp} := \{\phi \in \hat{E}^* : \tau^\sharp(\phi) = \phi\}$, and thus $\hat{E}^* = E^* \oplus \mathrm{i}E^*$ (cf. [**22**, p. 316]).

Our next result is a module version of Martinez-Moreno *et al.* [**20**, Proposition 1]. We shall only include a sketch of the proof.

PROPOSITION 2.2 *A real JB*-triple is ternary weakly amenable if and only if its complexification has the same property.*

*Proof*. Let $E$ be a real JB*-triple, whose complexification is denoted by $\hat{E} = E \oplus \mathrm{i}E$, and let $\tau$ denote the conjugation on $\hat{E}$ satisfying $E = \hat{E}^\tau$.

According to [**43**, Remark 13], given a triple derivation $\delta : E \to E^*$, the mapping $\hat{\delta} : \hat{E} \to \hat{E}^*$, $\hat{\delta}(x + \mathrm{i}y) := \delta(x) - \mathrm{i}\delta(y)$ is a (conjugate linear) triple derivation from $\hat{E}$ into $\hat{E}^*$. It can be easily checked that the identity

$$\delta(a + \mathrm{i}b, \phi_1 + \mathrm{i}\phi_2) = \delta(a, \phi_1) - \delta(b, \phi_2) - \mathrm{i}\delta(a, \phi_2) - \mathrm{i}\delta(b, \phi_1) \tag{6}$$

holds for every $a, b \in E \subseteq \hat{E}$ and $\phi_1, \phi_2 \in E^* \subseteq \hat{E}^*$.

Having in mind the facts proved in the above paragraph, we can see that $E$ is ternary weakly amenable whenever $\hat{E}$ satisfies the same property.

For the reciprocal implication, we note that if $\hat{\delta} : \hat{E} \to \hat{E}^*$ is a ternary derivation from $\hat{E} = E \otimes \mathrm{i}E$ to $\hat{E}^* = E^* \otimes \mathrm{i}E^*$, it can be easily checked that the identity

$$(P \circ \hat{\delta})\{a, b, c\} = \{(P \circ \hat{\delta})a, b, c\} + \{a, (P \circ \hat{\delta})b, c\} + \{a, b, (P \circ \hat{\delta})c\}$$

holds for every $a, b, c \in E = \hat{E}^\tau$, where $P$ denotes $(\mathrm{Id}_{E^*} + \tau^\sharp)/2$ or $(\mathrm{Id}_{E^*} - \tau^\sharp)/2\mathrm{i}$. Therefore, the mapping $P \circ \hat{\delta}|_E : E \to E^*$ is a ternary derivation. Since every ternary inner derivation $\delta$ from $E$ to $E^*$ defines a ternary inner derivation from $\hat{E}$ to $\hat{E}^*$, we can guarantee that $\hat{E}$ is ternary weakly amenable when $E$ has this property. □

Note that if $A$ is a Banach $^*$-algebra, $A$ is ternary weakly amenable if each continuous ternary derivation from $A$ (considered as a Jordan–Banach triple system) into $A^*$ is inner. Thus, a Banach $^*$-algebra can be weakly amenable and/or ternary weakly amenable, and the two concepts do not necessarily coincide (cf. Proposition 4.2).

We emphasize again that, unlike derivations from $A$ to itself, derivations from $A$ to $A^*$ are defined to be conjugate linear maps (in the complex case).

## 3. Commutative C*-algebras are ternary weakly amenable

In this section, we prove that every commutative (real or complex) C*-algebra is ternary weakly amenable. Our next results establish some technical connections between associative and ternary derivations from a Banach $^*$-algebra $A$ to a Jordan $A$-module (respectively, associative $A$-bimodule).

Following standard notation, given a Banach algebra $A$, $a \in A$ and $\varphi \in A^*$, $a\varphi$, $\varphi a$ will denote the elements in $A^*$ given by

$$a\varphi(y) = \varphi(ya) \quad \text{and} \quad \varphi a(y) = \varphi(ay) \ (y \in A).$$

LEMMA 3.1 *Let $A$ be an associative unital (Banach) *-algebra (which we consider as a Jordan–Banach algebra), $X$ be a unital Jordan $A$-module and $\delta : A_{sa} \to X$ be a (real) linear mapping. The following assertions are equivalent*:

(a) *$\delta$ is a ternary derivation and $\delta(1) = 0$;*
(b) *$\delta$ is a Jordan derivation.*

*Further, a conjugate linear mapping $\delta : A \to X$ is a ternary derivation with $\delta(1) = 0$ if, and only if, the linear mapping $D : A \to X$, $D(a) := \delta(a^*)$ is a Jordan derivation.*

*Proof.* (a) ⇒ (b) Since $X$ is a unital real Jordan $A_{sa}$-module and $\delta(1) = 0$, the identity

$$\begin{aligned}\delta(a \circ b) &= \delta\{a, 1, b\} = \{\delta(a), 1, b\} + \{a, \delta(1), b\} + \{a, 1, \delta(b)\}\\ &= \{\delta(a), 1, b\} + \{a, 1, \delta(b)\} = \delta(a) \circ b + a \circ \delta(b)\end{aligned}$$

gives the desired statement.

For every Jordan derivation $\delta : A_{sa} \to X$, we have $\delta(1) = \delta(1 \circ 1) = 2(1 \circ \delta(1)) = 2\delta(1)$, and hence $\delta(1) = 0$. The implication (b) ⇒ (a) follows straightforwardly.

To prove the last statement, we observe that a conjugate linear mapping $\delta : A \to X$ is a ternary derivation with $\delta(1) = 0$ if, and only if, $\delta|_{A_{sa}} : A_{sa} \to X$ is a ternary derivation with $\delta(1) = 0$, which, by (a)$\Leftrightarrow$ (b), is equivalent to saying that $\delta|_{A_{sa}}$ is a Jordan derivation. It is easy to check that $\delta|_{A_{sa}} = D|_{A_{sa}}$ is a Jordan derivation if and only if $D$ is a Jordan derivation from $A$ to $X$. □

Henceforth, given a unital associative *-algebra $A$ and a Jordan $A$-module $X$, we shall write $\mathcal{D}_t^o(A, X)$ for the set of all (continuous) ternary derivations from $A$ to $X$ vanishing at the unit element. We have seen in Lemma 3.1 that, when $A$ and $X$ are unital, we have

$$\mathcal{D}_J(A, X) \circ * = \mathcal{D}_t^o(A, X) := \{\delta \in \mathcal{D}_t(A, X) : \delta(1) = 0\}. \tag{7}$$

Given a Banach *-algebra $A$, we consider the involution $^*$ on $A^*$ defined by $\varphi^*(a) := \overline{\varphi(a^*)}$ ($a \in A$, $\varphi \in A^*$). An element $\delta \in \mathcal{D}_J(A, A^*)$ is called a **-derivation* if $\delta(a^*) = \delta(a)^*$ for every $a \in A$. The symbols $\mathcal{D}_J^*(A, A^*)$ and $\mathcal{I}nn_J^*(A, A^*)$ (respectively, $\mathcal{D}_b^*(A, A^*)$ and $\mathcal{I}nn_b^*(A, A^*)$) will, respectively, denote the sets of all Jordan and Jordan-inner (respectively, associative and inner) *-derivations from $A$ to $A^*$.

LEMMA 3.2 *Let $X$ be an $A$-bimodule over a Banach $^*$-algebra $A$. Then the following statements hold*:

(i) $\mathcal{I}nn_J(A, X) \subset \mathcal{I}nn_b(A, X)$. *In particular*, $\mathcal{I}nn_J^*(A, A^*) \subset \mathcal{I}nn_b^*(A, A^*)$;
(ii) *let $D$ be an element in $\mathcal{I}nn_b(A, A^*)$, that is, $D = D_\varphi$ for some $\varphi$ in $A^*$. Then $D$ is a *-derivation whenever $\varphi^* = -\varphi$. Further, if the linear span of all commutators of the form $[a, b]$ with $a, b$ in $A$ is norm-dense in $A$, then $D$ is a *-derivation if, and only if, $\varphi^* = -\varphi$.*

*Proof*. (i) Let us consider a Jordan derivation of the form $\delta_{x_0,b}$, where $x_0 \in X$ and $b \in A$. For each $a$ in $A$, we can easily check that

$$\delta_{x_0,b}(a) = (x_0 \circ a) \circ b - (b \circ a) \circ x_0 = \tfrac{1}{4}([b, x_0]a - a[b, x_0]) = D_{(1/4)[b,x_0]}(a),$$

where the Lie bracket $[\cdot, \cdot]$ is defined by $[b, x_0] = (bx_0 - x_0b)$ for every $b \in A$, $x_0 \in X$. Since every inner Jordan derivation $D$ from $A$ to $X$ must be a finite sum of the form $D = \sum_{j=1}^n \delta_{x_j,b_j}$, with $x_j \in X$ and $b_j \in A$, it follows that $D = \sum_{j=1}^n D_{(1/4)[b_j,x_j]} = D_{(1/4)\sum_{j=1}^n [bj,x_j]}$ is an inner (associative) binary derivation.

(ii) Let $D = D_\varphi$, where $\varphi \in A^*$ and $\varphi^* = -\varphi$. Let us fix two arbitrary elements $a, b$ in $A$. The identities

$$D_\varphi(a^*)(b) = (\varphi a^* - a^*\varphi)(b) = \varphi(a^*b - ba^*)$$

and

$$D_\varphi(a)^*(b) = (\varphi a - a\varphi)^*(b) = (a^*\varphi^* - \varphi^*a^*)(b) = \varphi^*(ba^* - a^*b)$$

give $D_\varphi(a^*) = D_\varphi(a)^*$, proving that $D$ is a $^*$-derivation.

Conversely, suppose now that the linear span of all commutators of the form $[a, b]$ with $a, b$ in $A$ is norm-dense in $A$ and $D = D_\varphi$ is a $^*$-derivation. The identity $D_\varphi(a^*) = D_\varphi(a)^*$ ($a \in A$) implies that $\varphi[a^*, b] = -\varphi^*[a^*, b]$ for every $a, b \in A$, therefore $\varphi = -\varphi^*$ as required. □

REMARK 3.3 There exist many examples of Banach algebras $A$ in which the linear span of all commutators of the form $[a, b]$ with $a, b$ in $A$ is norm-dense in $A$. This property is never satisfied by a commutative Banach algebra. However, the list of examples of C*-algebras satisfying this property includes all properly infinite C*-algebras, all properly infinite von Neumann algebras and the C*-algebra of all compact operators on an infinite-dimensional complex Hilbert space [40] (see also the survey [57]). Fack [10] proved that if the unit of a (unital) C*-algebra $A$ is properly infinite (i.e. there exist two orthogonal projections $p, q$ in $A$ Murray–von Neumann equivalent to 1), then any hermitian element is a sum of at most five self-adjoint commutators. Many other results have been established to show that all elements in a C*-algebra that have trace zero with respect to all tracial states can be written as a sum of finitely many commutators (cf. [34–36, 44], among others).

Let $X$ be a unital Banach $A$-bimodule over a unital Banach algebra $A$. Regarding $X$ as a real Banach triple $A$-module with respect to the induced triple product $\{a, x, c\} = \frac{1}{2}(axc + cxa)$, $\{x, a, c\} = \frac{1}{2}(xac + cax)$ $(a, c \in A, x \in X)$, we can easily see that every ternary derivation $\delta : A \to X$ annihilates at 1, that is,

$$\mathcal{D}_t(A, X) = \mathcal{D}_t^o(A, X).$$

Indeed, since

$$\delta(1) = \delta(\{1, 1, 1\}) = \{\delta(1), 1, 1\} + \{1, \delta(1), 1\} + \{1, 1, \delta(1)\} = 3\delta(1),$$

we have $\delta(1) = 0$. When we consider Banach $A$-bimodules equipped with ternary products that differ from the previous one, the identity $\mathcal{D}_t(A, X) = \mathcal{D}_t^o(A, X)$ does not hold in general. Our next lemmas study the case $X = A^*$, where $A$ is a unital Banach $^*$-algebra.

LEMMA 3.4 *Let $A$ be a unital Banach $^*$-algebra equipped with the ternary product given by $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$. Every ternary derivation $\delta$ in $\mathcal{D}_t(A, A^*)$ satisfies the identity $\delta(1)^* = -\delta(1)$, that is, $\overline{\delta(1)(a^*)} = -\delta(1)(a)$, for every $a$ in $A$.*

*Proof*. Let $\delta : A \to A^*$ be a ternary derivation. Since the identity

$$\begin{aligned}\delta(1)(a) &= \delta(\{1, 1, 1\})(a) = \{\delta(1), 1, 1\}(a) + \{1, \delta(1), 1\}(a) + \{1, 1, \delta(1)\}(a)\\ &= 2\delta(1)\{1, 1, a\} + \overline{\delta(1)\{1, a, 1\}} = 2\delta(1)(a) + \delta(1)^*(a)\end{aligned}$$

holds for every $a \in A$, we do have $\delta(1)^* = -\delta(1)$. □

LEMMA 3.5 *Let $A$ be a unital Banach $^*$-algebra equipped with the ternary product given by $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$. Then*

$$\mathcal{D}_t(A, A^*) = \mathcal{D}_t^o(A, A^*) + \mathcal{I}nn_t(A, A^*).$$

*More precisely, if $\delta \in \mathcal{D}_t(A, A^*)$, then $\delta = \delta_0 + \delta_1$, where $\delta_0 \in \mathcal{D}_t^o(A, A^*)$ and $\delta_1$, defined by $\delta_1(a) := \delta(1) \circ a^* = \frac{1}{2}(\delta(1)a^* + a^*\delta(1))$, is the inner derivation $-\frac{1}{2}\delta(1, \delta(1))$.*

*Proof*. Let $\delta : A \to A^*$ be a ternary derivation. The mapping $\delta_1 : A \to A^*$ $\delta_1(a) := \delta(1) \circ a^*$ is a conjugate linear mapping with $\delta_1(1) = \delta(1)$. We will show that $\delta_1 = -\frac{1}{2}\delta(1, \delta(1))$. Then, the mapping $\delta_0 = \delta - \delta_1$ is a triple derivation with $\delta_0(1) = 0$ and $\delta = \delta_0 + \delta_1$, proving the lemma.

Lemma 3.4 implies that $\delta(1)^* = -\delta(1)$.

Now we consider the inner triple derivation $-\frac{1}{2}\delta(1,\delta(1))$. For each $a$ and $b$ in $A$, we have

$$\begin{aligned} -\tfrac{1}{2}\delta(1,\delta(1))(a)(b) &= -\tfrac{1}{2}(\{1,\delta(1),a\} - \{\delta(1),1,a\})(b) \\ &= -\tfrac{1}{2}(\overline{\delta(1)(\{1,b,a\})} - \delta(1)(\{1,a,b\})) \\ &= -\tfrac{1}{2}(\delta(1)^*(\{1,a,b\}) - \delta(1)(\{1,a,b\})) \\ (\text{since } \delta(1)^* = -\delta(1)) &= -\tfrac{1}{2}(-\delta(1)(\{1,a,b\}) - \delta(1)(\{1,a,b\})) \\ &= \delta(1)(\{1,a,b\}) = \delta(1)(a^* \circ b) = \delta_1(a)(b). \end{aligned}$$

Thus, $\delta_1 = -\frac{1}{2}\delta(1,\delta(1))$, as promised. □

LEMMA 3.6 *Let $A$ be a unital Banach $^*$-algebra equipped with the ternary product given by $\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)$ and the Jordan product $a \circ b = (ab+ba)/2$, let $D : A \to A^*$ be a linear mapping and let $\delta : A \to A^*$ denote the conjugate linear mapping defined by $\delta(a) := D(a^*)$. Then $D$ lies in $\mathcal{D}_J(A,A^*)$ if, and only if, $\delta\{a,1,b\} = \{\delta(a),1,b\} + \{a,1,\delta(b)\}$ for all $a,b \in A$. Moreover,*

$$\mathcal{D}_t^o(A,A^*) = \{\delta : A \to A^* : \exists D \in \mathcal{D}_J^*(A,A^*) \text{ s.t. } \delta(a) = D(a^*), (a \in A)\}.$$

*Proof*. The first statement follows immediately from the definitions, that is, $\{\delta a, 1, b\} = D(a^*) \circ b^*$, $\{a,1,\delta b\} = D(b^*) \circ a^*$ and $\delta\{a,1,b\} = D(a^* \circ b^*)$.

Suppose next that $\delta \in \mathcal{D}_t^o(A,A^*)$. From the first statement, $D$ lies in $\mathcal{D}_J(A,A^*)$. Actually $D$ is $^*$-derivation; if $a \in A$, then $\delta(a^*) = \delta\{1,a,1\} = \{1,\delta(a),1\}$, and so, for all $y \in A$, we have

$$\langle \delta(a^*), y\rangle = \langle \{1,\delta(a),1\}, y\rangle = \overline{\langle \delta(a), \{1,y,1\}\rangle} = \langle (\delta(a))^*, y\rangle,$$

and hence $D(a^*) = \delta(a) = (\delta(a^*))^* = (Da)^*$.

Suppose now that $D \in \mathcal{D}_J^*(A,A^*)$. It follows from the definitions and the fact that $D \in \mathcal{D}_J(A,A^*)$ that the following three equations hold:

$$\begin{aligned} \delta\{a,b,a\} &= 2(D(a^*) \circ b) \circ a^* + 2(a^* \circ D(b)) \circ a^* + 2(a^* \circ b) \circ D(a^*) \\ &\quad - 2(D(a^*) \circ a^*) \circ b - (a^* \circ a^*) \circ D(b), \\ \{\delta(a),b,a\} &= D(a^*) \circ (b \circ a^*) + (D(a^*) \circ b) \circ a^* - (D(a^*) \circ a^*) \circ b \end{aligned}$$

and

$$\{a,\delta(b),a\} = 2((D(b^*))^* \circ a^*) \circ a^* - D(b) \circ (a^* \circ a^*).$$

From these three equations, we have

$$\delta\{a,b,a\} - 2\{\delta(a),b,a\} - \{a,\delta(b),a\} = 2(a^* \circ D(b)) \circ a^* - 2((D(b^*))^* \circ a^*) \circ a^*.$$

Since $D$ is self-adjoint, the right-hand side of the last equation vanishes, and the result follows. □

PROPOSITION 3.7 *Let $A$ be a unital Banach $^*$-algebra equipped with the ternary product given by $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$ and the Jordan product $a \circ b = (ab + ba)/2$. Then*

$$\mathcal{D}_t(A, A^*) \subset \mathcal{D}_J^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*).$$

*If $A$ is Jordan weakly amenable, then*

$$\mathcal{D}_t(A, A^*) = \mathcal{I}nn_b^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*).$$

*Proof.* Let $\delta : A \to A^*$ be a ternary derivation. By Lemma 3.5, $\delta = \delta_0 + \delta_1$, where $\delta_0 \in \mathcal{D}_t^o(A, A^*)$, $\delta_1(a) = -\frac{1}{2}\delta(1, \delta(1))(a) = \delta(1) \circ a^*$. Lemmas 3.1 and 3.6 assure that $D = \delta_0 \circ *$ is a Jordan $^*$-derivation. This proves the first statement.

The assumed Jordan weak amenability of $A$, together with Lemma 3.2 implies that $D = \delta_0 \circ *$ lies in $\mathcal{I}nn_b^*(A, A^*)$, which gives $\delta = D \circ * + \delta_1 \in \mathcal{I}nn_b^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*)$. Since a simple calculation shows that $\mathcal{I}nn_b^*(A, A^*) \subset \mathcal{D}_t(A, A^*)$, the reverse inclusion holds, proving the second statement. $\square$

When a Banach $^*$-algebra $A$ is commutative, we have $\mathcal{I}nn_b(A, A^*) = \{0\}$. In the setting of unital and commutative Banach $^*$-algebras, Proposition 3.7 implies the following.

COROLLARY 3.8 *Let $A$ be a unital and commutative Banach $^*$-algebra. Then $A$ is ternary weakly amenable whenever it is Jordan weakly amenable.*

Every C$^*$-algebra $A$ is binary weakly amenable (cf. [15]), and by Peralta and Russo [43, Theorem 19 or Corollary 21], every Jordan derivation $D : A \to A^*$ is continuous, and hence an associative derivation by Johnson's Theorem [25]. This gives us the next corollary.

COROLLARY 3.9 *Every unital and commutative (real or complex) C$^*$-algebra is ternary weakly amenable.*

The following corollary of Proposition 3.7 will be used in the next section. The proof consists in observing that Lemmas 3.1, 3.5 and 3.6 are valid in this context and using [6, Theorem 3.2].

COROLLARY 3.10 *Let $M$ be a semifinite von Neumann algebra and consider the submodule $M_* \subset M^*$. Then*

$$\mathcal{D}_t(M, M_*) = \mathcal{I}nn_b^*(M, M_*) \circ * + \mathcal{I}nn_t(M, M_*).$$

Our next proposition shows that Corollary 3.9 remains valid in the setting of non-necessarily-unital abelian C$^*$-algebras.

PROPOSITION 3.11 *Every commutative (real or complex) C$^*$-algebra is ternary weakly amenable.*

*Proof.* Let $A$ be a commutative C$^*$-algebra and $\delta : A \to A^*$ be a ternary derivation. By Proposition 2.1, $\delta^{**} : A^{**} \to A^{***}$ is a weak$^*$-continuous ternary derivation with $\delta^{**}(A^{**}) \subseteq A^*$. Since $A^{**}$ is a unital and commutative (real or complex) C$^*$-algebra, $\mathcal{D}_t(A^{**}, A^{***}) = \mathcal{I}nn_t(A^{**}, A^{***})$ and $\delta^{**}$ may be written in the form $\delta^{**} = -\frac{1}{2}\delta(1, \delta^{**}(1))$ (cf. Corollary 3.9 and Lemma 3.5).

Since every C*-algebra admits a bounded approximate unit (cf. [41, Theorem 1.4.2]), by Cohen's Factorization Theorem (cf. [18, Theorem VIII.32.22]), there exist $b \in A$ and $\varphi \in A^*$ such that $\frac{1}{2}\delta^{**}(1) = \varphi b$. Finally, for each $a$ in $A$ we have

$$\begin{aligned}\delta(a) &= -\tfrac{1}{2}\delta(1, \delta^{**}(1))(a) = \delta(1, \varphi b)(a) = \{1, \varphi b, a\} - \{\varphi b, 1, a\} \\ &= \text{(by the commutativity of } A\text{)} = \{b, \varphi, a\} - \{\varphi, b, a\} = \delta(b, \varphi)(a),\end{aligned}$$

which gives $\delta = \delta(b, \varphi)$. □

## 4. C*-algebras are not ternary weakly amenable

In this section, we present some examples of C*-algebras that are not ternary weakly amenable.

LEMMA 4.1 *The C*-algebra $A = K(H)$ of all compact operators on an infinite-dimensional Hilbert space $H$ is not Jordan weakly amenable.*

*Proof*. We shall identify $A^*$ with the trace-class operators on $H$.

Supposing that $A$ were Jordan weakly amenable, let $\psi \in A^*$ be arbitrary. Then $D_\psi$ would be an inner Jordan derivation, so there would exist $\varphi_j \in A^*$ and $b_j \in A$ such that $D_\psi(x) = \sum_{j=1}^n [\varphi_j \circ (b_j \circ x) - b_j \circ (\varphi_j \circ x)]$ for all $x \in A$.

For $x, y \in A$, a direct calculation yields

$$\psi(xy - yx) = -\frac{1}{4}\left(\sum_{j=1}^n b_j\varphi_j - \varphi_j b_j\right)(xy - yx).$$

It is known [40, Theorem 1] (see also the excellent survey [57]) that every compact operator on a separable infinite-dimensional Hilbert space is a finite sum of commutators of compact operators. Let $z$ be any element in $A = K(H)$. By standard spectral theory, we can find a separable infinite-dimensional Hilbert subspace $H_0 \subseteq H$ such that $z \in K(H_0)$, that is, $z = pz = zp$, where $p$ is the orthogonal projection of $H$ onto $H_0$. By the just quoted theorem of Pearcy and Topping, $z$ can be written as a finite sum of commutators $[x, y] = xy - yx$ of elements $x, y$ in $K(H_0) = pK(H)p \subseteq K(H)$. Thus, it follows that the trace-class operator $\psi = -\frac{1}{4}(\sum_{j=1}^n b_j\varphi_j - \varphi_j b_j)$ is a finite sum of commutators of compact and trace-class operators, and hence has trace zero. This is a contradiction, since $\psi$ was arbitrary. □

PROPOSITION 4.2 *The C*-algebra $A = K(H)$ of all compact operators on an infinite-dimensional Hilbert space $H$ is not ternary weakly amenable.*

*Proof*. Let $\psi$ be an arbitrary element in $A^*$. The binary inner derivation $D_\psi : x \mapsto \psi x - x\psi$ may be viewed as a map from either $A$ or $A^{**}$ into $A^*$. Considered as a map on $A^{**}$, it belongs to $\mathcal{I}nn_b(A^{**}, A^*)$, and so, by Corollary 3.10, $D_\psi \circ * : a \mapsto D_\psi(a^*)$, belongs to $\mathcal{D}_t(A^{**}, A^*)$.

Assuming that $A$ is ternary weakly amenable, the restriction of $D_\psi \circ *$ to $A$ belongs to $\mathcal{I}nn_t(A, A^*)$. Thus, there exist $\varphi_j \in A^*$ and $b_j \in A$ such that $D_\psi \circ * = \sum_{j=1}^n (L(\varphi_j, b_j) - L(b_j, \varphi_j))$ on $A$.

For $x, a \in A$, direct calculations yield

$$\psi(a^*x - xa^*) = \frac{1}{2}\sum_{j=1}^{n}(\varphi_j b_j - b_j^*\varphi_j^*)(a^*x) + \frac{1}{2}\sum_{j=1}^{n}(b_j\varphi_j - \varphi_j^*b_j^*)(xa^*).$$

We can and do set $x = 1$ to get

$$0 = \frac{1}{2}\sum_{j=1}^{n}(\varphi_j b_j - b_j^*\varphi_j^*)(a^*) + \frac{1}{2}\sum_{j=1}^{n}(b_j\varphi_j - \varphi_j^*b_j^*)(a^*), \tag{8}$$

and therefore

$$\psi(a^*x - xa^*) = \frac{1}{2}\sum_{j=1}^{n}(\varphi_j b_j - b_j^*\varphi_j^*)(a^*x - xa^*), \tag{9}$$

for every $a, x \in A$.

Using [40, Theorem 1] as in the proof of Lemma 4.1, and taking note of (9) and (8), we have

$$\psi = \frac{1}{2}\sum_{j=1}^{n}(\varphi_j b_j - b_j^*\varphi_j^*) = \frac{1}{2}\sum_{j=1}^{n}(\varphi_j^*b_j^* - b_j\varphi_j).$$

Hence,

$$\begin{aligned}2\psi &= \sum_{j=1}^{n}(\varphi_j b_j - b_j\varphi_j + b_j\varphi_j - \varphi_j^*b_j^* + \varphi_j^*b_j^* - b_j^*\varphi_j^*)\\ &= \sum_{j=1}^{n}[\varphi_j, b_j] - 2\psi + \sum_{j=1}^{n}[\varphi_j^*, b_j^*].\end{aligned}$$

Finally, the argument given at the end of the proof of Lemma 4.1 shows that $\psi$ has trace 0, which is a contradiction, since $\psi$ was arbitrary. □

Next we study the ternary weak amenability of the C$^*$-algebra $L(H)$ of all bounded linear operators on a complex Hilbert space $H$. We shall recall first some standard theory of von Neumann algebras.

Given a von Neumann algebra $M$, with predual $M_*$ and dual $M^*$, there exists a (unique) central projection $z_0$ in $M^{**}$ satisfying $M_* = M^*z_0$. Moreover, denoting $M_*^{\perp} = M^*(1 - z_0)$ we have $M^* = M_* \oplus^{\ell_1} M_*^{\perp}$ (cf. [52, Theorem III.2.14]). Here $M_*$ (respectively, $M_*^{\perp}$) is called the *normal* (respectively, the *singular*) part of $M^*$. Every functional $\phi$ in $M^*$ is uniquely decomposed into the sum

$$\phi = \phi_n + \phi_s, \quad \phi_n \in M_*,\ \phi_s \in M_*^{\perp}.$$

The functionals $\phi_n$ and $\phi_s$ are called, respectively, the normal part and the singular part of $\phi$. Since $z_0$ is a central projection in $M^{**}$, we can easily see that

$$\begin{gathered}(\phi a)_n = \phi_n a, \quad (\phi a)_s = \phi_s a, \quad (a\phi)_n = a\phi_n, \quad (a\phi)_s = a\phi_s,\\ \{\phi, a, b\}_n = \{\phi_n, a, b\}, \quad \{\phi, a, b\}_s = \{\phi_s, a, b\},\\ \{a, \phi, b\}_n = \{a, \phi_n, b\} \quad \text{and} \quad \{a, \phi, b\}_s = \{a, \phi_s, b\},\end{gathered}$$

for every $a, b \in M$ and $\phi \in M^*$.

LEMMA 4.3 *The $C^*$-algebra $M = L(H)$ of all bounded linear operators on an infinite-dimensional Hilbert space $H$ is not Jordan weakly amenable.*

*Proof*. Let $B = K(H)$ denote the ideal of all compact operators on $H$. We note that $B^{**} = M = L(H)$ and hence $M_* = B^*$ coincides with the trace-class operators on $H$. Let $\psi$ be an element in $B^*$ whose trace is not zero. The argument given in the proof of Lemma 4.1 guarantees that the derivation $D_\psi : B \to B^*$, $a \mapsto \psi a - a\psi$ does not belong to $\mathcal{I}nn_J(B, B^*)$.

By Proposition 2.1 and its proof, $D_\psi^{**} : B^{**} = M \to B^* = M_* \subseteq M^*$ is a Jordan derivation whose image is contained in $M_*$. It can be easily checked that $D_\psi^{**}(x) = \psi x - x\psi$ for every $x \in B^{**} = M$. We claim that $D_\psi^{**}$ is not an inner Jordan derivation. Otherwise, there exist $\varphi_j \in M^*$ and $b_j \in M$ such that $D_\psi^{**}(x) = \sum_{j=1}^n [\varphi_j \circ (b_j \circ x) - b_j \circ (\varphi_j \circ x)]$ for all $x \in M$. For each $j$, let us write $\varphi_j = \phi_j + \psi_j$, where $\phi_j \in M_*$ and $\psi_j \in M_*^\perp$ are the normal and singular part of $\varphi$, respectively. Since $D_\psi^{**}$ remains $M_*$-valued and, for each $x$ in $M$, $\sum_{j=1}^n [\phi_j \circ (b_j \circ x) - b_j \circ (\phi_j \circ x)] \in M_*$ and $\sum_{j=1}^n [\psi_j \circ (b_j \circ x) - b_j \circ (\psi_j \circ x)] \in M_*^\perp$, it follows that $D_\psi^{**}(x) = \sum_{j=1}^n [\phi_j \circ (b_j \circ x) - b_j \circ (\phi_j \circ x)]$ for all $x \in M$, where, in this case, $\phi_j \in M_*$ and $b_j \in M$.

Now, we can mimic the argument in the proof of Lemma 4.1 to show that $\psi = -\frac{1}{4}(\sum_{j=1}^n b_j\phi_j - \phi_j b_j)$ is a finite sum of commutators of bounded and trace-class operators, and hence has trace zero, which is impossible. $\square$

Similar ideas to those applied in the previous lemma give us the following result.

PROPOSITION 4.4 *The $C^*$-algebra $M = L(H)$ of all bounded linear operators on an infinite-dimensional Hilbert space $H$ is not ternary weakly amenable.*

*Proof*. Let $B = K(H)$ denote the ideal of all compact operators on $H$ (note that $B^{**} = L(H) = M$). Let $\psi$ be an element in $B^*$ whose trace is not zero. From Proposition 4.2 and its proof, we know that the mapping $D_\psi \circ * : B \to B^*$, $a \mapsto \psi a^* - a^*\psi$ is a ternary derivation (see Corollary 3.10) which does not belong to $\mathcal{I}nn_t(B, B^*)$.

Applying Proposition 2.1 and its proof, the bitranspose $D_\psi^{**} : B^{**} = M \to B^* = M_* \subseteq M^*$ is an associative derivation whose image is contained in $M_*$. Moreover, $D_\psi^{**}(x) = \psi x - x\psi$ for every $x \in B^{**} = M$. We will prove that $D_\psi^{**} \circ *$ is ternary derivation from $M$ to $M^*$ (cf. Corollary 3.10) which is not inner. Suppose, on the contrary, that there exist $\varphi_j \in M^*$ and $b_j \in M$ such that $D_\psi^{**} \circ * = \sum_{j=1}^n (L(\varphi_j, b_j) - L(b_j, \varphi_j))$ on $M$.

For each $j$, we write $\varphi_j = \phi_j + \psi_j$, where $\phi_j \in M_*$ and $\psi_j \in M_*^\perp$ are the normal and singular part of $\varphi$, respectively. Since $D_\psi^{**}(M) \subseteq M_*$, and for each $x \in M$, $\sum_{j=1}^n \{\phi_j, b_j, x\} - \{b_j, \phi_j, x\} \in M_*$ and $\sum_{j=1}^n \{\psi_j, b_j, x\} - \{b_j, \psi_j, x\} \in M_*^\perp$, we have $D_\psi^{**} \circ * = \sum_{j=1}^n (L(\phi_j, b_j) - L(b_j, \phi_j))$ on $M$, where $\phi_j \in M_*$ and $b_j \in M$.

Following the lines in the last part of the proof of Proposition 4.2, we derive

$$4\psi = \sum_{j=1}^n [\phi_j, b_j] + \sum_{j=1}^n [\phi_j^*, b_j^*],$$

which is impossible because $\psi$ has non-zero trace. $\square$

The techniques in this subsection can be used to show that the Cartan factor $M_n(\mathbb{C})$ of all operators on a finite-dimensional Hilbert space is ternary weakly amenable. This is of course a special case of

Proposition 5.1, but we are able to give a direct proof here. Note also that $M_n(\mathbb{C})$ is Jordan weakly amenable by Jacobson [23] or [24].

Although Lemma 4.1 states that $\mathcal{I}nn_J(A, A^*) \neq \mathcal{D}_J(A, A^*)$ when $A = K(H)$ with $H$ infinite dimensional, nevertheless for $A = M_n(\mathbb{C})$, we shall show directly in the next lemma that the equality does hold for the subsets of $^*$-derivations. This is included in the next lemma (as well as following from [23] or [24]).

LEMMA 4.5 *Let $A$ denote the JB$^*$-triple $M_n(\mathbb{C})$. Then*

$$\mathcal{I}nn_b^*(A, A^*) = \mathcal{I}nn_J^*(A, A^*) = \mathcal{D}_J^*(A, A^*).$$

*Proof*. Let $D \in \mathcal{I}nn_b^*(A, A^*)$ so that $D(x) = \psi x - x\psi$ for some $\psi \in A^*$. Recall [40, Theorem 1] that every compact operator is a finite sum of commutators of compact operators. Therefore, by Lemma 3.2(ii), $\psi^* = -\psi$. Also, since every matrix of trace 0 is a commutator [2, 49], we have $\psi = [\varphi, b] + (\mathrm{Tr}(\psi)/n)I$. Expanding $\varphi = \varphi_1 + \mathrm{i}\varphi_2$ and $b = b_1 + \mathrm{i}b_2$ into hermitian and skew-symmetric parts and using $\psi^* = -\psi$ leads to

$$\psi = [\varphi_1, b_1] - [\varphi_2, b_2] + \frac{\mathrm{Tr}(\psi)}{n} I.$$

For $x, y \in A$, direct calculation yields

$$D(x) = \varphi_1 \circ (b_1 \circ x) - b_1 \circ (\varphi_1 \circ x) - \varphi_2 \circ (b_1 \circ x) + b_2 \circ (\varphi_2 \circ x),$$

so that $D \in \mathcal{I}nn_J^*(A, A^*)$.

From the theorems of Haagerup (alternatively [21, Theorem 2.2]) and Johnson, and what was just proved, we have

$$\mathcal{D}_J^*(A, A^*) = \mathcal{D}_b^*(A, A^*) = \mathcal{I}nn_b^*(A, A^*) \subseteq \mathcal{I}nn_J^*(A, A^*) \subseteq \mathcal{D}_J^*(A, A^*).$$

□

PROPOSITION 4.6 *The JB$^*$-triple $A = M_n(\mathbb{C})$ is ternary weakly amenable and Jordan weakly amenable.*

*Proof*. We have noted above that $M_n(\mathbb{C})$ is Jordan weakly amenable.

By Proposition 3.7,

$$\mathcal{D}_t(A, A^*) = \mathcal{I}nn_b^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*),$$

so it suffices to prove that $\mathcal{I}nn_b^*(A, A^*) \circ * \subset \mathcal{I}nn_t(A, A^*)$.

As in the proof of Lemma 4.5, if $D \in \mathcal{I}nn_b^*(A, A^*)$ so that $Dx = \psi x - x\psi$ for some $\psi \in A^*$, then $\psi = [\varphi_1, b_1] - [\varphi_2, b_2] + (\mathrm{Tr}\,(\psi)/n)I$, where $b_1, b_2$ are self-adjoint elements of $A$ and $\varphi_1$, and $\varphi_2$ are self-adjoint elements of $A^*$. It is easy to see that, for each $x \in A$, we have

$$D(x^*) = \{\varphi_1, 2b_1, x\} - \{2b_1, \varphi_1, x\} - \{\varphi_2, 2b_2, x\} + \{2b_2, \varphi_2, x\},$$

so that $D \circ * \in \mathcal{I}nn_t(A, A^*)$. □

## 5. The case of Cartan factors

Contrary to what happens for (binary) weak amenability in the setting of C*-algebras, not every JB*-triple is ternary weakly amenable. In this section, we shall study weak amenability for some examples of JB*-triples.

As was mentioned in Section 1, every finite-dimensional JB*-triple has the inner derivation property (cf. [38, Chapter 11] or [32, Chapter 8]), and indeed is 'super amenable', meaning that every derivation into a Jordan–Banach triple module is inner (see [31, III.Korollar 1.6]). In particular, we have the following proposition.

PROPOSITION 5.1 *Every finite-dimensional JB*-triple is ternary weakly amenable.*

### 5.1. *Hilbert spaces and finite rank type I Cartan factors*

Let $X$ be a real Hilbert space considered as a real Cartan factor of type I, with respect to the product

$$\{x, y, z\} := \tfrac{1}{2}((x|y)z + (z|y)x) \quad (x, y, z \in X), \tag{10}$$

where $(\cdot|\cdot)$ denotes the inner product of $X$. Henceforth, $J = J_X : X \to X^*$ will denote the Riesz mapping. We begin with a useful observation.

PROPOSITION 5.2 *Let $\delta : X \to X^*$ be linear mapping. Then denoting $T = J^{-1}\delta : X \to X$, the following are equivalent:*

(a) *$\delta$ is a ternary derivation*;
(b) *$T$ is a bounded linear operator with $T^* = -T$.*

*Proof*. (a) $\Rightarrow$ (b) By Peralta and Russo [43, Corollary 15], we may assume that $\delta$ (and hence $T$) is continuous. Let us suppose that $\delta$ is a ternary derivation. For each $x$, $y$ and $z$ in $X$, we have

$$\delta\{x, y, z\} = \{\delta(x), y, z\} + \{x, \delta(y), z\} + \{x, y, \delta(z)\}. \tag{11}$$

Applying the definition (10) to (11) results in

$$0 = \tfrac{1}{2}\delta(x)(y)J(z) + \tfrac{1}{2}\delta(y)(z)J(x) + \tfrac{1}{2}\delta(y)(x)J(z) + \tfrac{1}{2}\delta(z)(y)J(x),$$

for every $x, y, z \in X$. Taking $x = z$, we see that

$$0 = (\delta(x)(y) + \delta(y)(x))J(x),$$

for every $x, y \in X$, which gives $\delta(x)(y) + \delta(y)(x) = 0$ for any $x, y \in X$, or equivalently, $(y|T(x)) = -(x|T(y)) = -(T(y)|x)$ for any $x, y \in X$, which proves (b).

(b) $\Rightarrow$ (a) By a direct calculation using (10) and the definition of $T$, we have $\delta\{x, y, z\} = \{\delta(x), y, z\} + \{x, \delta(y), z\} + \{x, y, \delta(z)\}$. □

In the terminology employed above, let $x, y$ be two elements in $X$. It is not hard to see that the inner derivation $\delta(J(x), y) = L(J(x), y) - L(y, J(x)) : X \to X^*$ is the mapping given by

$\delta(J(x), y)(a) = \frac{1}{2}(a|y)J(x) - \frac{1}{2}(a|x)J(y)$. Therefore, every inner derivation from $X$ to $X^*$ is a finite rank operator. This argument shows the following corollary.

COROLLARY 5.3 *Let $\delta : X \to X^*$ be a linear mapping. Then denoting $T = J^{-1}\delta : X \to X$, the following are equivalent*:

(a) *$\delta$ is a ternary inner derivation*;
(b) *$T$ is a finite rank operator with $T^* = -T$.*

We can prove now that every infinite-dimensional real Hilbert space is not ternary weakly amenable.

PROPOSITION 5.4 *A real or complex Hilbert space $X$ regarded as a type* I *Cartan factor is ternary weakly amenable if, and only if, it is finite dimensional.*

*Proof*. The if implication follows from Propositions 5.1 and 2.2. To see the other implication, suppose that $X$ is infinite dimensional. Then we can find a bounded linear operator $T : X \to X$ having infinite-dimensional range and satisfying $T^* = -T$. Proposition 5.2 and Corollary 5.3 imply that $\delta = J_X T$ is a ternary derivation which is not inner. □

The above results also give new ideas to deal with the ternary weak amenability in other Cartan factors of type I.

Suppose that $H_1$ and $H_2$ are Hilbert spaces. The symbol $K(H_1, H_2)$ will denote the set of all compact operators from $H_1$ to $H_2$. It is known that every $a$ in $K(H_1, H_2)$ can be written (uniquely) as a (possibly finite) sum of the form

$$a = \sum_{n=1}^{\infty} \sigma_n(a) k_n \otimes h_n,$$

where $(\sigma_n(a)) \subset \mathbb{R}_0^+$ is the sequence of *singular values* of $a$, $(h_n)$ and $(k_n)$ are orthonormal systems in $H_1$ and $H_2$, respectively, and given $\xi \in H_2$, $\eta, h \in H_1$, we define $\xi \otimes \eta(h) = (h|\eta)\xi$ (cf. [50, Section 1.2]). We denote by $S^1(H_1, H_2)$ the set of all compact operators $\phi$ from $H_1$ to $H_2$, whose sequence of singular values $(\sigma_i(\phi))_{i\in\mathbb{N}} \in \mathbb{R}_0^+$ lies in $\ell_1$. For each $\xi \in H_2$ and $\eta \in H_1$, we can define an element $\omega_{\xi,\eta} \in K(H_1, H_2)^*$, given by $\omega_{\xi,\eta}(x) = (x(\eta)|\xi)$ $(\forall x \in K(H_1, H_2))$. When we equip $S^1(H_1, H_2)$ with the norm $\|\phi\|_1 = \sum_i \sigma_i(\phi)$, $(S^1(H_1, H_2), \|.\|_1)$ is a Banach space and $S^1(H_1, H_2)$ can be identified with $K(H_1, H_2)^*$, via the assignment

$$\xi \otimes \eta \mapsto \omega_{\xi,\eta}$$

(cf. [50]). We omit the straightforward proof of the following lemma.

LEMMA 5.5 *Let $X$ and $Y$ be two real Hilbert spaces. Suppose that $Y_1$ and $Y_2$ are two closed subspaces of $Y$ such that $Y = Y_1 \oplus^{\perp} Y_2$. Then the polar $K(X, Y_1)^\circ$, of $K(X, Y_1)$ in $K(X, Y)^* = S^1(X, Y)$ coincides with $S^1(X, Y_2)$.*

The following easily verified formulas will facilitate the proof of the next theorem. For real Hilbert spaces $X$ and $Y$, and vectors $\xi, a, c \in Y$ and $\eta, b, d \in X$,

$$2\{\omega_{\xi,\eta}, a \otimes b, c \otimes d\} = (b|d)(\xi|a)\omega_{c,\eta} + (a|c)(\eta|b)\omega_{\xi,d}$$

and

$$2\{a \otimes b, \omega_{\xi,\eta}, c \otimes d\} = (\eta|d)(a|\xi)\omega_{c,b} + (\eta|b)(\xi|c)\omega_{a,d}.$$

THEOREM 5.6 *Let $X$ and $Y$ be real Hilbert spaces with* $\dim(Y) < \infty = \dim(X)$. *Then the real Cartan factor $L(X, Y)$ is not ternary weakly amenable.*

*Proof*. Since $\dim(Y) < \infty$, $L(X, Y) = K(X, Y) = S^1(X, Y)$ as linear spaces and $L(X, Y) = K(X, Y)$ as Banach spaces. We can also pick $T \in L(X)$ with infinite-dimensional range and $T^* = -T$. Since the elements $k \otimes h$ (respectively, $\omega_{k,h}$) with $h \in X$, $k \in Y$ generate the whole $L(X, Y)$ (respectively, $S^1(X, Y)$), the assignment $k \otimes h \mapsto \delta(k \otimes h) := \omega_{k,T(h)}$ defines a linear operator $\delta : L(X, Y) \to L^1(X, Y) = K(X, Y)^*$. We claim that $\delta$ is a ternary derivation. Indeed, it is enough to prove that

$$\begin{aligned}\delta\{k_1 \otimes h_1, k_2 \otimes h_2, k_3 \otimes h_3\} &= \{\delta(k_1 \otimes h_1), k_2 \otimes h_2, k_3 \otimes h_3\}\\ &\quad + \{k_1 \otimes h_1, \delta(k_2 \otimes h_2), k_3 \otimes h_3\}\\ &\quad + \{k_1 \otimes h_1, k_2 \otimes h_2, \delta(k_3 \otimes h_3)\},\end{aligned}$$

for every $k_1, k_2, k_3 \in Y$, $h_1, h_2, h_3 \in X$, which follows by direct calculation.

We will finally prove that $\delta$ is not inner. Suppose, on the contrary, that $\delta = \sum_{j=1}^{p} \delta(\phi_j, a_j)$ for suitable $\phi_1, \ldots, \phi_p \in S^1(X, Y)$, $a_1, \ldots, a_p \in K(X, Y)$. Let us fix a norm-1 element $k_0 \in Y$ and an arbitrary $h \in X$, so that we have $\delta(k_0 \otimes h) = \omega_{k_0,T(h)}$. On the other hand, each $\phi_j$ can be written in the form

$$\phi_j = \sum_{n=1}^{m_j} \alpha_n^j \omega_{k_n^j, h_n^j},$$

where $m_j \leq \dim(Y)$, $\alpha_n^j > 0$, and $(k_n^j)_n$ and $(h_n^j)_n$ are orthonormal systems in $Y$ and $X$, respectively. Now, we can check that

$$\begin{aligned}\omega_{k_0,T(h)} &= \delta(k_0 \otimes h) = \sum_{j=1}^{p} \delta(\phi_j, a_j)(k_0 \otimes h) = \sum_{j=1}^{p} \delta\left(\sum_{n=1}^{m_j} \alpha_n^j \omega_{k_n^j,h_n^j}, a_j\right)(k_0 \otimes h)\\ &= \sum_{j=1}^{p} \sum_{n=1}^{m_j} \alpha_n^j \left(\frac{1}{2}(a_j(h_n^j)|k_0)\, \omega_{k_n^j,h} + \frac{1}{2}(a_j(h)|k_n^j)\, \omega_{k_0,h_n^j}\right)\\ &\quad + \sum_{j=1}^{p} \sum_{n=1}^{m_j} \alpha_n^j \left(-\frac{1}{2}(h_n^j|h)\, \omega_{k_0,a_j^*(k_n^j)} - \frac{1}{2}(k_0|k_n^j)\, \omega_{a_j(h_n^j),h}\right).\end{aligned}$$

Write $Y = \mathbb{R}k_0 \oplus^{\perp} Y_2$, where $Y_2 = \{k_0\}^{\perp}$. Since, for every $\xi \in X$, $\omega_{k_0,\xi}$ lies in $K(X, Y_2)^{\circ} = S^1(X, \mathbb{R}k_0)$ (cf. Lemma 5.5), it follows from the above identities that the functional

$$\psi := \sum_{j=1}^{p} \sum_{n=1}^{m_j} \left( \frac{1}{2}(a_j(h_n^j)|k_0)\, \omega_{k_n^j,h} - \frac{1}{2}\alpha_n^j (k_0|k_n^j)\omega_{a_j(h_n^j),h} \right)$$

belongs to $K(X, Y_2)^{\circ} = S^1(X, \mathbb{R}k_0)$.

Therefore, there exists a scalar $\lambda$ such that $\psi = \omega_{\lambda k_0,h} = \lambda\omega_{k_0,h}$. Thus,

$$\omega_{k_0,T(h)} = \lambda\omega_{k_0,h} + \sum_{j=1}^{p} \sum_{n=1}^{m_j} \alpha_n^j \left( \frac{1}{2}(a_j(h)|k_n^j)\omega_{k_0,h_n^j} + \frac{1}{2}(h_n^j|h)\omega_{k_0,a_j^*(k_n^j)} \right).$$

In particular,

$$T(h) = \lambda h + \sum_{j=1}^{p} \sum_{n=1}^{m_j} \alpha_n^j \left( \frac{1}{2}(a_j(h)|k_n^j)h_n^j + \frac{1}{2}(h_n^j|h)a_j^*(k_n^j) \right).$$

Since $h$ was arbitrary, $T$ is a multiple of the identity plus a finite rank operator, that is, $T = \lambda \mathrm{Id}_X + F$, where $F : X \to X$ is a finite rank operator. Finally, applying that $T^* = -T$, we get $\lambda = 0$, and hence $T = F$ is a finite rank operator, which is impossible. □

Let $H$ and $K$ be two complex Hilbert spaces. Every rectangular complex Cartan factor of type I of the form $L(H, K)$ with $\dim(H) = \infty > \dim(K)$ admits a real form that coincides with $L(X, Y)$, where $X$ and $Y$ are real Hilbert spaces with $\dim(X) = \infty > \dim(Y)$ (cf. [29]). The following corollary follows straightforwardly from Proposition 2.2 and Theorem 5.6.

COROLLARY 5.7 *Let $H$ and $K$ be two complex Hilbert spaces with* $\dim(H) = \infty > \dim(K)$. *Then the rectangular complex Cartan factor of type* I, $L(H, K)$ *and all its real forms are not ternary weakly amenable.*

### 5.2. *Spin factors*

A (complex) JB*-triple $A$, which can be equipped with an inner product $(\cdot|\cdot)$ and a conjugation $\sharp$, satisfying the following conditions is called a (complex) spin factor:

(a) the norm on $A$ is given by $\|x\|^2 = (x|x) + \sqrt{(x|x)^2 - |(x|x^{\sharp})|^2}$;
(b) the triple product satisfies

$$\{a, b, c\} = \tfrac{1}{2}[(a|b)c + (c|b)a - (a|c^{\sharp})b^{\sharp}].$$

Throughout this section, $A$ will be a (complex) spin factor and the duality of $A$ with $A^*$ will be denoted by $\langle \cdot, \cdot \rangle$, while $J : A \to A^*$ will stand for the Riesz map.

The following lemma shows that ternary derivations from $A$ to $A^*$ are in bijective correspondence with the (linear) ternary derivations on $A$.

LEMMA 5.8 *Let $D : A \to A$ be a linear mapping. Then denoting $\delta = J \circ D$, the identities*

(i) $\langle\{\delta(a), b, a\}, c\rangle = (c|\{D(a), b, a\})$,
(ii) $\langle\{a, \delta(b), a\}, c\rangle = (c|\{a, D(b), a\})$,

*hold for every $a, b, c$ in $A$. Consequently, $D$ is a (linear) ternary derivation on $A$ if, and only if, $\delta$ lies in $\mathcal{D}_t(A, A^*)$.*

*Proof*. To prove the first two statements, note that, for $a, b, c \in A$,

$$\begin{aligned}2\langle\{\delta(a), b, a\}, c\rangle &= 2\langle\delta(a), \{b, a, c\}\rangle = 2(\{b, a, c\}|D(a))\\ &= ((b|a)c + (c|a)b - (b|c^\sharp)a^\sharp|D(a))\\ &= ((b|a)c|D(a)) + ((c|a)b|D(a)) - ((b|c^\sharp)a^\sharp|D(a))\\ &= (c|(a|b)D(a)) + (c|(D(a)|b)a) - (c|(D(a)|a^\sharp)b^\sharp)\\ &= (c|[(a|b)D(a) + (D(a)|b)a - (D(a)|a^\sharp)b^\sharp])\\ &= 2(c|\{D(a), b, a\})\end{aligned}$$

and

$$\begin{aligned}\langle\{a, \delta(b), a\}, c\rangle &= \overline{\langle\delta(b), \{a, c, a\}\rangle} = (D(b)|\{a, c, a\})\\ &= (D(b)|[(a|c)a - \tfrac{1}{2}(a|a^\sharp)c^\sharp])\\ &= (c|[(a|D(b))a - \tfrac{1}{2}(a|a^\sharp)D(b)^\sharp]) = (c|\{a, D(b), a\}).\end{aligned}$$

Finally, if $D$ is a (linear) ternary derivation on $A$ and $x \in A$, by (i) and (ii), we have

$$\begin{aligned}\langle\delta\{a, b, a\}, x\rangle &= (x|D\{a, b, a\})\\ &= (x|2\{D(a), b, a\}) + (x|\{a, D(b), a\})\\ &= \langle 2\{\delta(a), b, a\}, x\rangle + \langle\{a, \delta(b), a\}, x\rangle,\end{aligned}$$

so that $\delta \in \mathcal{D}_t(A, A^*)$. Similarly, if $\delta \in \mathcal{D}_t(A, A^*)$, then $D$ is a (linear) ternary derivation on $A$. □

We deal now with inner ternary derivations.

LEMMA 5.9 *For each element $a$ in $A$, let $\varphi = J(a) \in A^*$. Then, for all $b, x, y \in A$, we have:*

(i) $\langle\{b, \varphi, x\}, y\rangle = (y|\{b, a, x\})$,
(ii) $\langle\{\varphi, b, x\}, y\rangle = (y|\{a, b, x\})$.

*It follows that a linear mapping $D : A \to A$ is an inner ternary derivation if, and only if, $\delta = J \circ D \in \mathcal{I}nn_t(A, A^*)$.*

*Proof*. The first two statements follow by straightforward calculations. If $D$ is an inner (linear) derivation on $A$ of the form $D = \delta(b, a)$ with $a, b \in A$, then, for $x, y \in A$,

$$\begin{aligned}\langle \delta(x), y\rangle &= (y|D(x))\\ &= (y|\{b, a, x\} - \{a, b, x\})\\ &= \langle\{b, J(a), x\}, y\rangle - \langle\{J(a), b, x\}, y\rangle\\ &= \langle\delta(b, J(a))(x), y\rangle,\end{aligned}$$

so that $\delta \in \mathcal{I}nn_t(A, A^*)$. Similarly, if $\delta \in \mathcal{I}nn_t(A, A^*)$, then $D$ is an inner (linear) derivation on $A$. □

PROPOSITION 5.10 *A spin factor is ternary weakly amenable if, and only if, it is finite dimensional.*

*Proof*. Combining Lemmas 5.8 and 5.9, $A$ is ternary weakly amenable if, and only if, $A$ has the inner derivation property. Thus, applying [**20**, Theorem 3] and the fact that every finite-dimensional JB$^*$-triple has the inner derivation property (cf. Proposition 5.1), we get the desired equivalence. □

The real forms of (complex) spin factors are called real spin factors. The next corollary is a direct consequence of Propositions 2.2 and 5.10.

COROLLARY 5.11 *A real spin factor is ternary weakly amenable if, and only if, it is finite dimensional.*

The following questions have been intractable up to this moment.

PROBLEM 5.12 *Are Cartan factors of type* II *and* III *ternary weakly amenable?*

PROBLEM 5.13 *Does there exist an infinite-rank rectangular Cartan factor of type* I *which is ternary weakly amenable?*

## 6. Commutative JB*-triples are almost ternary weakly amenable

In this section, we prove that every commutative real or complex JB$^*$-triple is almost ternary weakly amenable. More concretely, we prove that every ternary derivation from a commutative real or complex JB$^*$-triple into its dual can be approximated in norm by an inner derivation.

We shall make use of the Gelfand representation theory for commutative JB$^*$-triples (cf. [**12**, **28**, Section 1]). Let us define $\mathbb{T} := \{\alpha \in \mathbb{C} : |\alpha| = 1\}$. Given a commutative (complex) JB$^*$-triple $E$, there exists a principal $\mathbb{T}$-bundle $\Lambda = \Lambda(E)$, i.e. a locally compact Hausdorff space $\Lambda$ together with a continuous mapping $\mathbb{T} \times \Lambda \to \Lambda, (t, \lambda) \mapsto t\lambda$ such that $s(t\lambda) = (st)\lambda$, $1\lambda = \lambda$ and $t\lambda = \lambda \Rightarrow t = 1$, satisfying that $E$ is JB$^*$-triple isomorphic to

$$\mathcal{C}_0^{\mathbb{T}}(\Lambda) := \{f \in \mathcal{C}_0(\Lambda) : f(t\lambda) = tf(\lambda), \forall t \in \mathbb{T}, \lambda \in \Lambda\}.$$

We note that $\mathcal{C}_0^{\mathbb{T}}(\Lambda)$ is a JB$^*$-subtriple of the commutative C$^*$-algebra $\mathcal{C}_0(\Lambda)$. Every commutative JB$^*$-triple is a $C_\Sigma$-space and hence a complex Lindenstrauss space in the terminology of Olsen [**39**].

An element $e$ in a Jordan triple $E$ is called *tripotent* if $\{e,e,e\}=e$. Each tripotent $e$ in $E$ induces a decomposition of $E$ (called *Peirce decomposition*) in the form

$$E = E_0(e) \oplus E_1(e) \oplus E_2(e),$$

where $E_k(e) = \{x \in E : L(e,e)x = (k/2)x\}$ for $k = 0, 1, 2$.

The Peirce space $E_2(e)$ is a unital JB*-algebra with unit $e$, product $x \circ_e y := \{x, e, y\}$ and involution $x^{\sharp_e} := \{e, x, e\}$.

A tripotent $e$ in $E$ is said to be *unitary* if $L(e,e)$ coincides with the identity map on $E$, equivalently, $E_2(e) = E$. When $E_0(e) = \{0\}$, the tripotent $e$ is called *complete*.

The proof given in Lemma 3.4 remains valid in the following setting.

LEMMA 6.1 *Let $E$ be a JB*-triple containing a unitary tripotent $u$. Every ternary derivation $\delta$ in $\mathcal{D}_t(E,E^*)$ satisfies the identity $\delta(u)^{\sharp_u} = -\delta(u)$, that is, $\overline{\delta(u)(a^{\sharp_u})} = -\delta(u)(a)$ for every $a$ in $E$.*

The following lemma summarizes some basic properties of commutative JB*-triples; an implicit proof can be found by combining Theorems 2 and 4 in [11].

LEMMA 6.2 *Let $u$ be a norm-1 element in a commutative JB*-triple $E \cong \mathcal{C}_0^{\mathbb{T}}(\Lambda(E))$. The following statements are equivalent*:

(a) *$u$ is a complete tripotent*;
(b) *$u$ is a unitary element*;
(c) *$u$ is an extreme point of the unit ball of $E$.*

*If $u$ satisfies one of the above conditions, then $E$ is a commutative C*-algebra with unit $u$, product and involution given by $a \circ_u b := \{a, u, b\}$ and $a^{\sharp_u} := \{u, a, u\}$ $(a, b \in E)$, respectively.*

COROLLARY 6.3 *Every commutative JB*-triple $E$ containing a complete tripotent $u$ is ternary weakly amenable. Further, every ternary derivation $\delta : E \to E^*$ can be written in the form $\delta = -\frac{1}{2}\delta(u, \delta(u)) = \{\delta(u), u, \cdot\}$.*

*Proof*. Lemma 6.2 shows that $E$ is a commutative C*-algebra with product and involution given by $a \circ_u b := \{a, u, b\}$ and $a^{\sharp_u} := \{u, a, u\}$ $(a, b \in E)$, respectively. By the proof of Lemma 3.5 (see also Corollary 3.9), every ternary derivation $\delta : E \to E^*$ may be written in the form $\delta = -\frac{1}{2}\delta(u, \delta(u))$. Given $a, b$ in $E$, since $\{uba\}^{\sharp_u} = \{uab\}$, we have

$$\begin{aligned}\delta(a)(b) &= -\tfrac{1}{2}\delta(u,\delta(u))(a)(b) = -\tfrac{1}{2}\{u,\delta(u),a\}(b) + \tfrac{1}{2}\{\delta(u),u,a\}(b)\\ &= \tfrac{1}{2}(\delta(u)\{u,a,b\} - \overline{\delta(u)\{u,b,a\}}) = \tfrac{1}{2}(\delta(u)\{u,a,b\} - \delta(u)^{\sharp_u}\{u,a,b\})\\ &= \text{(by Lemma 6.1) } \delta(u)\{u,a,b\} = \{\delta(u),u,a\}(b),\end{aligned}$$

which proves the last identity. □

COROLLARY 6.4 *Every commutative JBW*-triple $E$ is (isometrically JB*-triple isomorphic to) a commutative von Neumann algebra, and thus it is ternary weakly amenable. Moreover, every ternary derivation $\delta : E \to E^*$ can be written in the form $\delta = -\frac{1}{2}\delta(u, \delta(u)) = \{\delta(u), u, \cdot\}$, where $u$ is any complete tripotent in $E$.*

*Proof*. Let $E$ be a commutative JBW$^*$-triple. The first assertion follows, indirectly, from [12, Remark 2.7] (see also [11] or Lemma 6.2). Since $E$ is a dual Banach space, it follows from the Krein–Milman Theorem that the closed unit ball of $E$ contains an extreme point. Lemma 6.2 assures that every extreme point of the unit ball of $E$ is a complete tripotent in $E$. Therefore, $E$ is a commutative JB$^*$-triple containing a complete tripotent and the desired statement follows from Corollary 6.3. $\square$

COROLLARY 6.5 *Let $E$ be a commutative JB$^*$-triple. Then every derivation $\delta$ in $\mathcal{D}_t(E, E^*)$ may be written in the form $\delta = -\frac{1}{2}\delta(u, \delta^{**}(u)) = \{\delta^{**}(u), u, \cdot\}$, where $u$ is any complete (unitary) tripotent in $E^{**}$ and $\delta^{**}(u) \in E^*$.*

*Proof*. Let $\delta : E \to E^*$ be a ternary derivation. Since the triple product of $E^{**}$ is separately weak$^*$-continuous, it follows from Goldstine's Theorem that $E^{**}$ is a commutative JBW$^*$-triple. Proposition 2.1 guarantees that $\delta^{**} : E^{**} \to E^{***}$ is a weak$^*$-continuous ternary derivation with $\delta^{**}(E^{**}) \subseteq E^*$. Corollary 6.4 gives the desired statement. $\square$

From now on, let $E$ be a commutative JB$^*$-triple that is identified with $\mathcal{C}_0^{\mathbb{T}}(\Lambda)$. For later use, we highlight the following properties: Let $\mathcal{C}_0^1(\Lambda)$ denote the C$^*$-subalgebra of $\mathcal{C}_0(\Lambda)$ of all $\mathbb{T}$-invariant functions, that is,

$$\mathcal{C}_0^1(\Lambda) := \{f \in \mathcal{C}_0(\Lambda) : f(t\lambda) = f(\lambda),\ \forall\, t \in \mathbb{T}, \lambda \in \Lambda\}.$$

It is clear that, for every $a, b$ in $E$ and $c$ in $\mathcal{C}_0^1(\Lambda)$, the products $ab^*$ and $ac$ lie in $\mathcal{C}_0^1(\Lambda)$ and in $E$, respectively.

The mapping

$$\begin{aligned} E \times E &\to \mathcal{C}_0^1(\Lambda), \\ (a, b) &\mapsto ab^* \end{aligned}$$

is sesquilinear and positive ($aa^* \geq 0$ and $aa^* = 0 \iff a = 0$). The products

$$\begin{aligned} E \times \mathcal{C}_0^1(\Lambda) &\to E, \\ (a, c) &\mapsto ac \end{aligned}$$

and

$$\begin{aligned} E^* \times \mathcal{C}_0^1(\Lambda) &\to E^*, \\ (\phi, c) &\mapsto (\phi c)(a) = \phi(ac) \end{aligned}$$

equip $E$ and $E^*$ with a structure of Banach $\mathcal{C}_0^1(\Lambda)$-bimodules. We also have two mappings $E^* \times E \to \mathcal{C}_0^1(\Lambda)^*$ and $\mathcal{C}_0^1(\Lambda)^* \times E \to E^*$ defined by $\phi a(c) := \phi(ac)$ and $\psi a(b) = \psi(a^* b)$ ($\phi \in E^*$, $\psi \in \mathcal{C}_0^1(\Lambda)^*$, $c \in \mathcal{C}_0^1(\Lambda)$, $a, b \in E$), respectively.

We shall regard $E = \mathcal{C}_0^{\mathbb{T}}(\Lambda)$ and $\mathcal{C}_0^1(\Lambda)$ as norm-closed JB$^*$-subtriples of the C$^*$-algebra $A = \mathcal{C}_0(\Lambda)$. We shall identify the weak$^*$ closure, in $A^{**}$, of a closed subspace $Y$ of $A$ with $Y^{**}$. It follows from the separate weak$^*$-continuity of the triple product in $A^{**}$, that, for every $a, b$ in $E^{**}$ and $c$ in $\mathcal{C}_0^1(\Lambda)^{**}$, the products $ab^*$ and $ac$ lie in $\mathcal{C}_0^1(\Lambda)^{**}$ and in $E^{**}$, respectively. Clearly, the mappings defined in the previous paragraph extend to $E^{**} \times E^{**}$ and $E^{**} \times \mathcal{C}_0^1(\Lambda)^{**}$ and $E^* \times E^{**}$, respectively.

One of the main consequences of the Gelfand theory for JB*-triples provides a structure theorem for the JB*-subtriples generated by a single element. Concretely speaking, for each element $a$ in a JB*-triple $F$, the JB*-subtriple of $F$ generated by the element $a$, $F_a$, is JB*-triple isomorphic (and hence isometric) to $C_0(L)$ for some locally compact Hausdorff space $L$ contained in $(0, \|a\|]$, such that $L \cup \{0\}$ is compact, where $C_0(L)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0. It is also known that there exists a triple isomorphism $\Psi$ from $F_a$ onto $C_0(L)$, such that $\Psi(a)(t) = t$ $(t \in L)$ (cf. [28, Corollary 1.15]). Consequently, for each element $a$ in $F$ there exists a (unique) element $b \in F_a$ satisfying $\{b, b, b\} = a$. This element $b$ is usually called the *cube root* of $a$.

The following proposition shows that, in the parlance of ternary rings of operators (TROs [30]), $\mathcal{C}_0^1(\Lambda)$ identifies with the left and right linking $C^*$-algebras of the TRO $\mathcal{C}_0^{\mathbb{T}}(\Lambda)$. The proof is included here for reasons of completeness.

PROPOSITION 6.6 *Let $E = \mathcal{C}_0^{\mathbb{T}}(\Lambda)$ be a commutative JB*-triple. The norm-closed linear span of the set*

$$\bar{E} \cdot E := \{a^* b : a, b \in E\}$$

*coincides with $\mathcal{C}_0^1(\Lambda)$.*

*Proof*. Let $B$ denote the norm-closed linear span of the set $\bar{E} \cdot E$. Given $a, b, c$ and $d$ in $E$, the product $bc^*d$ lies in $E$ and hence $(a^*b)(c^*d) = a^*(bc^*d)$ belongs to $\bar{E} \cdot E$. Thus, $\bar{E} \cdot E$ is multiplicatively closed and clearly self-adjoint (i.e. $(\bar{E} \cdot E)^* = \bar{E} \cdot E$). We deduce that $B$ is a norm-closed *-subalgebra of $\mathcal{C}_0^1(\Lambda)$.

We observe that $\mathcal{C}_0^1(\Lambda)$ is triple isometrically isomorphic to $\mathcal{C}_0(\Lambda/\mathbb{T})$ (via the canonical identification $c \mapsto \hat{c}$, where $\hat{c}(\lambda + \mathbb{T}) := c(\lambda)$ for every $c \in \mathcal{C}_0^1(\Lambda)$, $\lambda \in \Lambda$). We shall identify $\mathcal{C}_0^1(\Lambda)$ and $\mathcal{C}_0(\Lambda/\mathbb{T})$. We claim that, under this identification, $B$ is a norm-closed *-subalgebra of $\mathcal{C}_0(\Lambda/\mathbb{T})$, separates the points of $\Lambda/\mathbb{T}$ and vanishes nowhere.

To this end, we claim first that $E$ separates the points of $\Lambda$ and vanishes nowhere; that is, given $\lambda \in \Lambda$, there exists $a \in E$ with $a(\lambda) = 1$. By Urysohn's lemma, there exists $f \in \mathcal{C}_0(\Lambda)$ satisfying $f(t\lambda) = 1$ for every $t \in \mathbb{T}$. Let $\mathrm{d}\mu$ denote the unit Haar measure on $\mathbb{T}$; the assignment $g \mapsto \pi(g)(\lambda) := \int_{\mathbb{T}} t^{-1} g(t\lambda)\, \mathrm{d}\mu(t)$ defines a contractive projection on $\mathcal{C}_0(\Lambda)$ whose image coincides with $E$. It is clear that $a = \pi(f) \in E$ and $a(\lambda) = \pi(f)(\lambda) = 1$. Take $\lambda_1 \neq \lambda_2$ in $\Lambda$. We may assume that $\lambda_1 + \mathbb{T} \neq \lambda_2 + \mathbb{T}$, that is, the orbits of $\lambda_1$ and $\lambda_2$ are two compact disjoint subsets of $\Lambda$. Applying Urysohn's lemma, we find an element $f \in \mathcal{C}_0(\Lambda)$ satisfying $f(t\lambda_1) = 1$ and $f(t\lambda_2) = 0$ for every $t \in \mathbb{T}$. The element $a = \pi(f)$ satisfies $a(\lambda_1) = 1$ and $a(\lambda_2) = 0$.

Let us now take $\lambda_1 + \mathbb{T} \neq \lambda_2 + \mathbb{T}$ in $\Lambda/\mathbb{T}$. Suppose that $a^*b(\lambda_1) = a^*b(\lambda_2)$ for every $a, b \in E$. In particular, $a^*a(\lambda_1) = a^*a(\lambda_2)$, and hence $aa^*a(\lambda_1) = aa^*a(\lambda_2)$ for every $a \in E$. Since every element $b$ in $E$ admits a cube root $a \in E_b$ satisfying $\{a, a, a\} = b$, we deduce that $b(\lambda_1) = b(\lambda_2)$ for every $b \in E$, which is impossible because $\lambda_1 \neq \lambda_2$. By the same argument, $B$ vanishes nowhere, so the Stone–Weierstrass theorem assures that $B = \mathcal{C}_0^1(\Lambda)$. □

THEOREM 6.7 *Every commutative (real or complex) JB*-triple E is almost ternary weakly amenable, that is, $\mathcal{I}nn_t(E, E^*)$ is a norm-dense subset of $\mathcal{D}_t(E, E^*)$.*

*Proof*. By Proposition 2.2, we may assume that $E$ is a commutative complex JB$^*$-triple. We write $E = \mathcal{C}_0^{\mathbb{T}}(\Lambda(E))$ and $A = \mathcal{C}_0(\Lambda(E))$. Let $\delta : E \to E^*$ be a ternary derivation. By Corollary 6.5, $\delta^{**} = -\frac{1}{2}\delta(u, \delta^{**}(u))$, where $u$ is a unitary in $E^{**} \subseteq A^{**}$ and $\psi = \delta^{**}(u) \in E^*$. In this case,

$$\delta(a)(b) = -\tfrac{1}{2}(\overline{\psi(ub^*a)} - \psi(ua^*b)),$$

for every $a, b \in E$, where the products are taken in $\mathcal{C}_0(\Lambda(E))^{**}$.

The mapping $c \mapsto \psi(uc)$ defines a functional in the dual of $\mathcal{C}_0^1(\Lambda(E))$. Since the latter is a C$^*$-algebra, by Cohen's Factorization Theorem (cf. [18, Theorem VIII.32.22]), there exist $\varphi \in \mathcal{C}_0^1(\Lambda(E))^*$ and $d \in \mathcal{C}_0^1(\Lambda(E))$ such that $\psi(uc) = \varphi(dc)$ for every $c \in \mathcal{C}_0^1(\Lambda(E))$. Therefore, for each $a, b \in E$ we have

$$\delta(a)(b) = -\tfrac{1}{2}(\overline{\psi(ub^*a)} - \psi(ua^*b)) = -\tfrac{1}{2}(\overline{\varphi(db^*a)} - \varphi(da^*b)).$$

Given $\varepsilon > 0$, by Proposition 6.6, there exist $x_1, y_1, \ldots, x_n, y_n \in E$ satisfying $\|d - \sum_{j=1}^n y_j^* x_j\| < \varepsilon$. Let $\phi_j = \varphi y_j^* \in E^*$ $(j = 1, \ldots, n)$. The sum $-\frac{1}{2}\sum_{j=1}^n \delta(x_j, \phi_j)$ defines an inner ternary derivation from $E$ to $E^*$. Given $a, b \in E$, we have

$$\begin{aligned}
&\left|\delta(a)(b) + \frac{1}{2}\sum_{j=1}^n \delta(x_j, \phi_j)(a)(b)\right| \\
&= \left|-\frac{1}{2}(\overline{\varphi(db^*a)} - \varphi(da^*b)) + \frac{1}{2}\sum_{j=1}^n (\overline{\phi_j(x_j b^*a)} - \phi_j(x_j a^*b))\right| \\
&= \left|-\frac{1}{2}(\overline{\varphi(db^*a)} - \varphi(da^*b)) + \frac{1}{2}\sum_{j=1}^n (\overline{\varphi(y_j^* x_j b^*a)} - \varphi(y_j^* x_j a^*b))\right| \\
&= \left|\frac{1}{2}\overline{\varphi\left(\left(\sum_{j=1}^n y_j^* x_j - d\right) b^*a\right)} + \frac{1}{2}\varphi\left(\left(d - \sum_{j=1}^n y_j^* x_j\right) a^*b\right)\right| \\
&\leq \frac{1}{2}\|\varphi\|\,\|a\|\,\|b\|\left\|d - \sum_{j=1}^n y_j^* x_j\right\| < \varepsilon\|\varphi\|\,\|a\|\,\|b\|/2.
\end{aligned}$$

Thus, $\|\delta - (-\frac{1}{2}\sum_{j=1}^n \delta(x_j, \phi_j))\| < \varepsilon\|\varphi\|/2$. □

The proof given in the above theorem shows that, under additional hypothesis on the set $E \cdot \bar{E} := \{ab^* : a, b \in E\}$, a commutative JB$^*$-triple $E$ is ternary weakly amenable.

COROLLARY 6.8 *Let $E = \mathcal{C}_0^{\mathbb{T}}(\Lambda(E))$ be a commutative JB$^*$-triple. Suppose that the linear span of the set $E \cdot \bar{E} := \{ab^* : a, b \in E\}$ coincides with $\mathcal{C}_0^1(\Lambda(E))$. Then $E$ is ternary weakly amenable.*

The question clearly is whether the additional hypothesis in Corollary 6.8 is automatically satisfied for every commutative JB$^*$-triple $E$. We do not know the answer; the best result we could obtain in this line is Proposition 6.6.

Related to this topic, we can say that, given a commutative JB*-triple $E = \mathcal{C}_0^{\mathbb{T}}(\Lambda(E))$, the mapping $E \times E \to \mathcal{C}_0^1(\Lambda(E))$, $(a, b) \mapsto ab^*$ need not be, in general, surjective. Indeed, let $L$ be a locally compact Hausdorff space. We shall say that $L$ is a locally compact *principal* $\mathbb{T}$-*bundle* if there exists a continuous mapping $\mathbb{T} \times L \to L, (t, \lambda) \mapsto t\lambda$ satisfying $s(t\lambda) = (st)\lambda$ and $1\lambda = \lambda$ for every $s, t \in \mathbb{T}$, $\lambda \in L$. We write

$$\mathcal{C}_0^{\mathbb{T}}(L) := \{f \in \mathcal{C}_0(L) : f(t\lambda) = tf(\lambda)\ (t \in \mathbb{T}, \lambda \in L)\}.$$

It is known that $\mathcal{C}_0^{\mathbb{T}}(L)$ is isometrically isomorphic to $C_0(L')$ for some locally compact $L'$ if, and only if, $L$ is a trivial $\mathbb{T}$-bundle, i.e. $L/\mathbb{T} \times \mathbb{T} \cong L$ (cf. [28, Corollary 1.13]). The set $S := \{z \in \mathbb{C}^{n+1} : \|z\|_2 = 1\}$ is compact and a non-trivial principal $\mathbb{T}$-bundle. Let $E = \mathcal{C}^{\mathbb{T}}(S) \subset \mathcal{C}(S)$ and $\mathcal{C}^1(S) := \{f \in \mathcal{C}(S) : f(t\ z) = f(z)\ (t \in \mathbb{T}, z \in S)\}$. We can obviously identify $S$ with a closed subset of $\Lambda(E)$ which satisfies $\mathbb{T}S = S$.

If the mapping $E \times E \to \mathcal{C}_0^1(\Lambda(E))$, $(a, b) \mapsto ab^*$ were surjective, then, applying Urysohn's lemma, there would exist functions $a, b \in E$ satisfying $ab^* = v$, where $v \in \mathcal{C}_0^1(\Lambda(E)) \cong \mathcal{C}_0(\Lambda(E)/\mathbb{T})$ is a function satisfying $v(z) = 1$ for every $z \in S$. In this case, the function $z \mapsto u(z) = a(z)/|a(z)|$ $(z \in S)$ would be a unitary element in $E$, and hence, by Lemma 6.2, $E$ would be an abelian C*-algebra, which is impossible because $S$ is a non-trivial principal $\mathbb{T}$-bundle.

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