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Publication Date

2026-06-04

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA,
IRVINE

Self-Selection, Decoys, and Complexity in Models of Heterogeneous Expected Utility

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Economics

by

Kuang-Yu (Kevin) Peng

Dissertation Committee:
Associate Professor Igor Kopylov, Chair
Professor Stergios Skaperdas
Professor John Duffy

DEDICATION

In Memoriam:

To my dear grandma *Nai-nai* and auntie *Er-gu-gu*.

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ACKNOWLEDGMENTS

My greatest gratitude goes toward you, Dr Igor Kopylov, for your guidance and support throughout my time at UCI. You have taught me invaluable lessons of diligence, resilience, collaboration, and responsibility. Your advice on how to tread the waters of the research world will greatly facilitate my academic journey in the future. I am also happy I got to work with you as your TA for multiple courses throughout the years, and I am grateful for all the tips I received regarding how to manage a class.

Dr Stergios Skaperdas, thank you for always being there for me, be it a quick catch-up chat, or a deep discussion regarding current events. Your wise insight in all aspects of life has helped me greatly and I had the pleasure of TAing for you as well. I am especially grateful for the Heinz Chair research funding that was pivotal to my experimental research.

Thank you, Dr John Duffy, for all the advice you have given me on my experiment. Your guidance through the necessary processes and provision of important text streamlined my project. I still remember the first time receiving your email of my acceptance to the program, and you have been a constant source of support ever since.

I would also like to extend my gratitude to professors who are not on my committee, but have provided me with precious feedback and advice. Thank you, Dr Miguel Zerecero and Dr David Agrawal for your interest in my research and always being there for a chat. Thank you, Dr Jan Brueckner for your generosity in sharing your wisdom in teaching public economics. Thank you, Dr Amjad Toukan for being a colleague as well as a friend.

It has been a pleasure working with each member of ESSL. Thank you, Dr Michael McBride for your enthusiasm in every student's project. Thank you Laila Delgado for your immeasurable support during each session we held for my experiment. Thank you Nidhi Pandey for showing me the ropes to using the new software for the lab.

I would also like to thank the department for the support throughout the years, specifically the wonderful staff, Katie Swindell, Laura Robledo-Vega, and Denise Nguyen. Your assistance has been of utmost importance to my experience here. Furthermore, I am grateful for the summer research and conference travel funds I have received.

I made so many lasting friends in my six years at UCI, and each and every one of you made my time here so much easier. For the wonderful friends I made in the department, it has been an honor to strive alongside you in this doctoral program. Thank you, my cohort-mates, Joanna Williams, Jyotsana Kala, and Levi Edwards for being such good buddies. We explored all over Orange County and I will bring these memories with me wherever I go. I would also like to thank Michael Fitzpatrick, Sydney Gordon, Gamy De La Torre, and Noah Kouchekinia, for having shared an office before, and always being happy to chat when I pop in for a visit after we parted offices. Kory Fenstemacher, Russ Leonard, Stefan Ebmer, thank you for not just keeping the Friday Beers tradition alive, but also all the adventures we've had. Thank you, Tengyuan Liang, Tom Wang, Hubert Fok, and Franklin Yang for being my fellow Mandarin-speaking friends and introducing me to the good food all over OC.

I am also grateful for all the friends I have outside of Economics. Firstly, I would like to thank all the roommates, Andy Dang, Subhamoy Karmakar, Elliot Yu, Jordan Finn, Sagar Patil, and Freya I have had the pleasure to share a roof with. Countless roommate dinners and outings adorned the fabric of life. I would also like to thank Bryant and Chris Jackson-Green, for welcoming me to your home each time I visit and the wonderful meals you cooked throughout the years. I will cherish every trip we have gone on together, and am looking forward to more adventures in the future. Finally, thank you David Urzua and Joseph Labagnao Ligunas for introducing me to so many board games and friends.

Last but definitely not least, I would like to thank my parents and family for their unwavering support throughout my endeavors eight time zones away from home. I cannot emphasize the importance of your belief in me and willingness to let me explore the world. I can't wait to share my experience here when I finally go home into your welcoming arms.

VITA

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ABSTRACT OF THE DISSERTATION

Self-Selection, Decoys, and Complexity in Models of Heterogeneous Expected Utility

By

Kuang-Yu (Kevin) Peng

Doctor of Philosophy in Economics

University of California, Irvine, 2026

Associate Professor Igor Kopylov, Chair

In Chapter 1, Dr Igor Kopylov and I extend *random expected utility* (REU) from Gul and Pesendorfer (2006) in two directions. First, we rewrite it in terms of *types* that combine the expected utility maximization with endogenous *tie-breaking*. Thus a salient expected utility function, such as the risk neutral one, can be maximized with a positive probability. We characterize *discrete random expected utility* (DREU) where random choices are represented *uniquely* by a probability distribution μ over finitely many types. Second, we extend DREU to *partial random expected utility* (PREU) where μ can be affected by the feasible menu, but still determined uniquely by random choice data. PREU captures various kinds of *context dependence* with both behavioral and normative motivations. We use PREU to identify *self-selection*, where a type can become more likely, but only if its feasible options are improved. Such self-selection implies the standard monotonicity for random choices. It can even agree with the random utility model of Block and Marschak (1960), which becomes misspecified in this case. Finally, we discuss applications of DREU and PREU to *random risk aversion*.

In Chapter 2, I extend the discussion on random risk attitudes in PREU from the previous chapter via an experiment to test whether the distribution of risk attitudes changes depending on the construction of the menu. Specifically, when an unattractive certain option is added to a menu, does the average decision maker exhibit higher levels of risk aversion? I

find that there is a systematic decrease in the choice of the riskier lottery when the certain option is added. What’s more, when the certain option is a decoy, there is a significant increase in the choice of the safer lottery. This violates core axioms that characterize random utility in standard literature, evidencing the need for menu-dependence to model heterogeneity. I also discuss the persistence of decoy effects, and observe that when the certain decoy is removed, the enhanced level of risk aversion persists. Finally, I identify that cognitive levels do not have an effect on susceptibility to decoy effects, supporting the belief that it is a universal phenomenon.

In Chapter 3, I discuss the difficulty to identify in stochastic choice models. Instead, I use a *multi-utility* (MU) model by Aizerman and Malishevski (1981) due to its simpler choice primitives. I consider three cases where individual types are respectively modeled by (i) an arbitrary utility, (ii) expected utility, and (iii) quasilinear utility. I discover that while the expected utility case is still computationally difficult to identify, there is promise with the other two models. I dedicate the majority of this chapter to the algorithmic identification of heterogeneity in a quasilinear framework. My algorithm allows for incompleteness as well as noise. I propose candidate indices to measure type dispersion and noisiness. What’s more, I extend the discussion to *constant threshold representations* (CTR) (Jamison and Lau, 1973), a special case of MU where agents are unable to distinguish small differences in utility. I implement a novel algorithmic approach that attempts to fit CTR into general datasets. While I do not provide any theoretical complexity guarantees for the CTR case, my initial findings suggest that my algorithm works reasonably fast in practical settings.

Chapter 1

Discrete Random Expected Utility and Self-Selection

This chapter is coauthored with Dr Igor Kopylov.

1.1 Introduction

Observed choices in consumption menus can be naturally *stochastic* when they are produced by distinct rational (i.e., utility maximizing) agents. It is common for data to be presented as an aggregation of choices from diverse agents. Such data is used in a wide range of empirical work with examples of preferred modes of transportation (McFadden, 2001) and choice of fishing sites (Train, 1998). This heterogeneity may also result from more behavioral motivations where a single individual can exhibit distinct random responses in identical choice problems (see Agranov and Ortoleva (2017)).

The classic random utility model (RUM) by Block and Marschak (1960) (henceforth, BM) aims to represent stochastic choice data by a stable probability distribution μ across *types* that include all possible utility functions. In the general case where types are unrestricted,

Falmagne (1978) characterizes RUM via non-negativity of Block and Marschak’s polynomials. Gul and Pesendorfer (2006) (henceforth, GP) refine this model to *random expected utility* (REU) where all types should maximize expected utility functions rather than arbitrary ones. Other refinements of RUM have restricted types by single-crossing conditions (Apesteguia, Ballester, and Lu, 2017), quasi-linearity (McFadden, 1973; Williams, 1977; Daly and Zachery, 1979; Yang and Kopylov, 2023), and other assumptions. All of these refinements preserve the assumption that μ is invariant of the consumption menu.

Recent literature extends REU in many meaningful ways. Frick et al. (2019) introduce dynamics to the decision making process, with discussion on the limits to history dependence. They also disentangle taste shocks and gradual learning, two separate motivations to change in choice that would be equivalent under GP’s REU. Dynamics are also introduced in Ahn and Sarver (2013), where agents first choose among menus before making a stochastic choice in that menu under REU. This allows for a unique identification which cannot be guaranteed under GP. Lu (2021) introduces ambiguity aversion to REU by adapting and relaxing its axioms. This allows for modeling fluctuating beliefs instead of tastes.

There exists debate around the monotonicity axiom for REU (and RUM in general), with criticisms on its rigidity. Apesteguia and Ballester (2018) show that structural issues occur when applying RUM to risk. Specifically, standard usage of logit/probit errors on CRRA/CARA will break monotonicity, and REU will cause identification errors. Keffert and Schweizer (2024) reconsiders these critiques and propose a weaker, more behaviorally-plausible version of monotonicity based on generalized risk premia. This allows meaningful inference about heterogeneous risk preferences under REU in some cases.

Lin (2020) studies situations where random risk preferences do not conform to expected utility theory. It was shown that the uniqueness of the underlying distribution, a key feature of REU, fails under some non-expected utility frameworks such as the betweenness class, though it can be restored under stronger restrictions such as weighted utility theory. Cerreia-

Vioglio et al. (2019) establish that stochastic choice can emerge as a deliberate reaction when vNM independence is relaxed. Under their framework, non-expected utility maximizers, such as agents with rank dependence and cautious expected utility, purposefully randomize. Usage of these alternative models also inherently violates monotonicity, further bringing scrutiny on the invariance framework of standard random utility.

REU and other variations of RUM have two broad problematic issues that we address in this chapter. First, as individual types are modeled by expected utility maximization, then *tie-breaking* must have a zero probability for any distinct consumption alternatives x and y :

$$\mu(\{u \in \mathcal{U} : u(x) = u(y)\}) = 0. \quad (1.1)$$

Otherwise, the likelihood of choice x in the menu $\{x, y\}$ is not well-defined in the absence of any endogenous tie-breaking rule. For example, the probability of risk neutrality must be zero in REU because distinct gambles can have the same expected value. It appears problematic that REU does not allow to identify a positive probability for a salient risk attitude, such as risk neutrality, or more broadly, for any standard expected utility function over monetary gambles.¹

Experimental literature found non-negligible incidences of risk neutral behavior. Albert and Duffy (2012) elicited risk attitudes using Holt and Laury (2002) and found that 11.5% (*resp.* 10.7%) of young (*resp.* old) adults exhibited risk neutrality. Oechssler and Sofianos (2019) estimated a baseline level of risk neutrality for standard laboratory settings, and found that it takes up anywhere between 10% and 35%. On the empirical side, Bombardini and Trebbi (2012) and Metrick (1995) analyzed game show data from *Affari Tuoi* (Italian version of *Deal or No Deal*) and *Jeopardy!* respectively, and found a prevalence of risk neutral behavior, specifically when stakes are small. Similar results are found in deductible choice regarding small stakes (Cohen and Einav, 2007). These findings reinforce the salience of risk neutrality

¹This issue persists whenever there are at least three distinct deterministic payoffs.

and casting doubt on using REU to properly model heterogeneous risk attitudes.

Tie-breaking becomes essential in the discretization of the type space. Piermont (2022) relaxes GP’s model to represent only those choices that are strict maximizers for corresponding types. Ties are broken when the utility values of multiple options are the same, and the tie-breaking rule can be menu-dependent and arbitrary. Piermont provides an example where agents break ties in favor of relatively interior points. Other methods of breaking ties include lexicographic measures (Yang and Kopylov, 2023) where an agent will break ties in favor of the magnitude of a specific component of a bundle/lottery. Dekel and Lipman (2012) study REU under temptation, and break ties in favor of the current utility function.

Second, there are various empirical and normative motivations for *context dependence* where type probabilities $\mu(\theta)$ should be affected by the feasible menu. For example, combining a menu A with an extra option y can increase the probability of choosing some $x \in A$ due to reason-based heuristics (e.g., Shafir, Simonson, and Tversky (1993)) and other context effects. Such non-monotonic patterns are inconsistent with any RUM.

In a more normative way, context dependence can arise because of *self-selection* when the participation of agents in the observed choices can be strategic. To illustrate, imagine that passengers can buy tickets for two daily commute trains, a and e , that run in the afternoon and the evening respectively. Suppose that the afternoon train is more popular so that

$$\rho(a, \{a, e\}) > \rho(e, \{a, e\}) \tag{1.2}$$

in terms of the corresponding random choice rule ρ . Imagine that due to construction projects, the afternoon train starts to get cancelled randomly with probability $\frac{1}{2}$ every day. Then the choice of the afternoon train ticket becomes a lottery $a^* = \frac{1}{2}a + \frac{1}{2}e$. Then it is plausible that

$$\rho(a^*, \{a^*, e\}) < \rho(e, \{a^*, e\}) \tag{1.3}$$

because those passengers who chose a in $\{a, e\}$ can decide to take some unobserved transportation option (e.g. a taxi) or change their travel destination entirely when their favorite afternoon train becomes unreliable. However, the reversal from (1.2) to (1.3) violates GP's Linearity in REU.

We discuss self-selection as a rational motivation for menu-dependence. Borjas (1987) models this behavior by studying how differences in earning distributions can factor into an endogenous decision to migrate. As origin-country conditions changed over time, entry wages declined by 10% to 15%. In econometric research, this type of self-selection introduces a classic endogeneity issue. Selection can arise in two forms as reviewed by Tucker (2010) from an accounting and finance viewpoint. First it can be due to variables that are observed to the researcher but cannot be controlled. It can also result from unobservables that affect the outcome. Propensity score matching (Rosenbaum and Rubin, 1983) can mitigate the first issue by rebalancing the sample to mimic a randomized controlled trial. The latter issue, if framed as an omitted variable problem (Heckman, 1979), can be resolved under a two-stage setting. However, this method is famously sensitive to parametrization.

To accommodate tie-breaking and context dependence, we extend REU in several steps. First, we introduce the notion of a *type*—a preference relation that satisfies completeness, transitivity, antisymmetry, and Independence. Such relations combine expected utility maximization and endogenous tie-breaking rules in a way that is formalized by Theorem 1.1. For example, a type θ^* can be risk neutral over monetary gambles, and break all ties lexicographically via probabilities of larger payoffs. Then for all distinct lotteries x and y ,

$$x\theta^*y \iff v(x) > v(y), \text{ or both } v(x) = v(y) \text{ and } x(q^*) > y(q^*) \quad (1.4)$$

where v is the expected value function, and q^* is the largest monetary payoff where the probabilities $x(q^*)$ and $y(q^*)$ are not the same. In finite-dimensional frameworks, such bi-

nary relations appear in Fishburn (1982), Myerson (1986), and Blume, Brandenburger and Dekel (1991).

Our Theorem 1.2 asserts that any REU representation can be rewritten via a *random type function* (RTF) μ —a finitely additive probability distribution over types rather than over expected utility functions in GP. This reformulation allows to drop the regularity condition (1.1) because each type is equipped with endogenous tie-breaking.

Conversely, if a random choice rule ρ is induced by some RTF μ , then it also has a REU representation, but it can become much more complicated than the original RTF. For example, let $\mu(\theta^*) = 1$ assign probability one to the risk neutral type θ^* in (1.4). Then the corresponding REU representation must assign zero probability to risk neutrality and hence, to all risk neutral types. This apparent paradox arises because GP’s identification for REU distorts the tie-breaking rule in a type like θ^* into a finitely additive distribution over expected utility functions that are not risk neutral. Occam’s razor favors the use of RTFs in such applications.

Next, our Theorem 1.3 characterizes *discrete random expected utility* (DREU), where the RTF μ has a finite support and is called a *discrete type function* (DTF) as well. This finiteness assumption cannot be rejected empirically, but it greatly improves the identification of μ . Theorem 1.3 asserts that if ρ is induced by a discrete type function μ , then such μ is unique. Moreover, its identification is novel and much faster than GP’s construction.

Theorem 1.4 is the most general of our results. It extends DREU to *partial random expected utility* (PREU), where the DTF μ can vary with the feasible menu A . Such variations, written as μ_A , can be still identified uniquely given that

- (i) all μ_A share the same finite support, and
- (ii) μ_A have a suitable continuity property with respect to A .

By weakening the invariance assumption, it is required to discuss what it means for elements to be chosen with positive probability. In empirical settings, however, observed choice frequencies may be affected by noise, making it inappropriate to treat any strictly positive empirical frequency as evidence of positive choice probability. Since even an alternative that is never chosen in the data may have a small but nonzero underlying probability, an approximate criterion is required.

A useful benchmark is provided by the Rule of Three, which states that if an event is not observed in N independent trials, then with 95% confidence its true rate of occurrence is at most $\frac{3}{N}$ (Jovanovic and Levy, 1997). Accordingly, in a population of size N , this bound yields a conservative threshold: alternatives with true choice probabilities exceeding $\frac{3}{N}$ would be expected to be chosen by at least one agent with high probability. We therefore treat $\frac{3}{N}$ as a reasonable cutoff for considering an element as effectively chosen.

Moreover, we observe that PREU can be always rewritten in a form that resembles the Tversky–Koehler *support theory* for subjective beliefs (Tversky and Koehler, 1994). Then for all menus A and types θ in a finite support Θ ,

$$\mu_A(\theta) = \frac{f_\theta(A)}{\sum_{\psi \in \Theta} f_\psi(A)} \quad (1.5)$$

where each *fitness function* f_θ is positive and suitably continuous. These functions can be interpreted in terms of *participation rates* when each type $\theta \in \Theta$ is equally likely ex ante and participates in the random choice data in the menu A with probability $f_\theta(A) \in (0, 1]$.

Theorem 1.5 refines PREU for *self-selection* that can exhibit reversals (1.2)–(1.3), but preserves the standard monotonicity principle,

$$\rho(x, A) \geq \rho(x, A \cup y),$$

so that x should not be chosen more often if the original menu A is enlarged by an extra element y . In terms of the DTFs μ_A , monotonicity is shown to be equivalent to the inequalities

$$\mu_A(\theta) \geq \mu_{A \cup y}(\theta)$$

that must hold whenever y is not the best choice for the type θ in the menu $A \cup y$.

For example, it should be reasonable for gamblers to show up in higher numbers to bookmakers that offer them more favorable odds on some sporting event. What's more, the expansion of bet types with the rise of online gambling can incentivize new customers to enter the market (Hing et al., 2022). Self-selection can also be observed in the dining industry. Garnett et al. (2019) found that doubling the options of vegetarian meals increased sales of such meals by around 15%.

Moreover, Theorem 1.5 asserts that if the fitness functions f_θ in (1.5) are non-decreasing transformations of expected utility function u_θ that are maximized by θ , then PREU implies that all Block-Marschak polynomials should be non-negative.

In other words, a natural class of representations with self-selection cannot be distinguished from RUM when data is given in subsets of some finite menu. Thus the general RUM can be *misspecified* and distort the true set of types Θ by phantom utility functions that serve to counterbalance the effects of self-selection in subsets ones. the random utility model does not hold for the true set of types Θ , but instead distorts Θ by phantom types.

Finally, we discuss applications of DREU and PREU to random risk aversion (Corollary 1.6) where the risk aversion across all possible types is revealed through a suitable constrain on the random choice rule ρ . For example, agents make more risk neutral choices when lotteries are presented in a binary format (i.e., only two states of the world have positive probabilities) instead of lotteries that are more complex or compound lotteries (Harrison et al., 2013).

This chapter is structured as follows. In Section 1.2, we introduce the preliminaries and explain the tie-breaking mechanism. Section 1.3 introduces *discrete random expected utility* and compares it to GP's REU. Section 1.4 relaxes the invariant assumption to allow for menu-dependence, and discusses extensions to the model respectively in frameworks of self-selection and risk attitudes.

1.2 Types and Tie-Breaking

Let $X = \{x, \dots\}$ be a consumption domain that is assumed to be a *convex subset* of some normed linear space L . A function $u : X \rightarrow \mathbb{R}$ is called *mixture linear* if for all $x, y \in X$ and $\gamma \in [0, 1]$,

$$u(\gamma x + (1 - \gamma)y) = \gamma u(x) + (1 - \gamma)u(y).$$

Let $\mathcal{U}$ be the set of all non-constant mixture linear functions.

For example, take any set Q of deterministic prizes (e.g. monetary ones). Let Δ^Q be the set of all *lotteries*—functions $x : Q \rightarrow [0, 1]$ for which the set $\{q \in Q : x(q) > 0\}$ is finite, and $\sum_{q \in Q} x(q) = 1$. Then Δ^Q is a convex subset of the normed linear space L of all bounded functions $x : Q \rightarrow \mathbb{R}$, and the mixture linearity implies the *expected utility* form:

$$u(x) = \sum_{q \in Q} x(q)u(q) \text{ for all } x \in \Delta^Q.$$

This framework with a finite Q is adopted in GP's REU. In our framework, X need not be finite-dimensional, closed, or bounded. Elements in X need not be lotteries, and mixture linearity need not be interpreted as expected utility. For example, Cobb-Douglas utility functions are mixture linear over consumption bundles in the logarithmic scale.

Let $\mathcal{M} = \{A, \dots\}$ be the set of all *menus*—non-empty finite subsets of X . Singleton menus

$\{x\}$ are written as x . For any $\gamma \in [0, 1]$ and menus $A, B \in \mathcal{M}$, let

$$\gamma A + (1 - \gamma)B = \{\gamma x + (1 - \gamma)y : x \in A, y \in B\}.$$

For each menu $A \in \mathcal{M}$ and function $u : X \rightarrow \mathbb{R}$, write

$$u(A) = \max_{x \in A} u(x).$$

For each menu $A \in \mathcal{M}$, its convex hull $\text{conv}(A)$ is the set of all convex combinations of its elements. If A is non-singleton, then its convex hull is not finite and hence, is not a menu.

A binary relation θ on X is called a *total order* if it is complete, transitive, and antisymmetric: for all $x, y \in X$, $x\theta y\theta x$ implies $x = y$.

A total order θ is called a *type* if θ satisfies *Independence*: for all $x, y, z \in X$ and $\gamma \in (0, 1]$,

$$x\theta y \iff [\gamma x + (1 - \gamma)z]\theta[\gamma y + (1 - \gamma)z].$$

Let $\mathcal{T}$ be the set of all types. Take any $\theta \in \mathcal{T}$. As θ is a total order, then for any menu $A \in \mathcal{M}$, there is a unique maximal element $b(\theta, A) \in A$ such that

$$b(\theta, A)\theta y \text{ for all } y \in A.$$

This element is also called *best* for θ in A .

Say that $\theta \in \Theta$ *maximizes* a function $u : X \rightarrow \mathbb{R}$ if for every menu $A \in \mathcal{M}$,

$$u(A) = u(b(\theta, A)), \tag{1.6}$$

so that θ and u share the same maximal element $b(\theta, A)$ in A . Such element is unique for θ ,

but need not be unique for u .

A type $\theta \in \mathcal{T}$ is called *stable* if there exist $x^*, y^* \in X$ such that for any $z \in X$, there exists $\gamma \in (0, 1]$ such that

$$[\gamma z + (1 - \gamma)x^*]\theta y^* \text{ and } x^*\theta[\gamma z + (1 - \gamma)y^*]. \quad (1.7)$$

For example, stability is trivial if $x^*, y^* \in X$ are such that $x^*\theta z\theta y^*$ for all $z \in X$. In general, stability imposes² a form of continuity for the comparison $x^*\theta y^*$: for any $z \in X$, the rankings (1.7) must hold for all sufficiently small $\gamma > 0$.

Theorem 1.1. *A type $\theta \in \mathcal{T}$ is stable if and only if θ maximizes some $u \in \mathcal{U}$. If θ maximizes $v \in \mathcal{U}$ as well, then $v = \alpha u + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$.*

Moreover, if L is finite-dimensional, then each type $\theta \in \mathcal{T}$ is stable.

All proofs are in the appendix.

Theorem 1.1 asserts that any stable type θ maximizes some non-constant mixture linear function $u \in \mathcal{U}$, which is unique up to a positive affine transformation. This claim is true for any type $\theta \in \mathcal{T}$ when L is finite dimensional.

By (1.6), the function $u \in \mathcal{U}$ determines best elements $b(\theta, A)$ in all menus A where u has a unique maximizer. In other menus, the type θ uses some endogenous *tie-breaking* rule to select a single element $b(\theta, A)$ among all maximizers of u in A .

A rich family of types and tie-breaking rules can be generated as follows. Fix any linear basis λ in L . Then $\lambda \subseteq L$ should be a linearly independent set such that L is the linear

²As (1.7) holds for $z = x^*$, then $x^*\theta y^*$. For all $\varepsilon \in [0, \gamma]$,

$$\varepsilon z + (1 - \varepsilon)x^* = \frac{\varepsilon}{\gamma}[\gamma z + (1 - \gamma)x^*] + \frac{1 - \varepsilon}{\gamma}x^*.$$

By Independence, $[\gamma z + (1 - \gamma)x^*]\theta y^*$ implies $[\varepsilon z + (1 - \varepsilon)x^*]\theta y^*$. Similarly, $x^*\theta[\gamma z + (1 - \gamma)y^*]$ implies $x^*\theta[\varepsilon z + (1 - \varepsilon)y^*]$.

span of λ . This definition is standard for finite dimensional L . If L is infinite dimensional, then λ is called a *Hamel basis*, and its existence follows from Zorn's lemma (see Aliprantis and Border (1999, Theorem 1.6)). Endow λ with any total order. Let $f_\lambda(0) = 0$, and for any non-zero $z \in X$, let $f_\lambda(z)$ be the first non-zero coordinate of z in the basis λ . For all $x, y \in X$, let

$$x\theta_\lambda y \iff f_\lambda(x - y) \geq 0. \quad (1.8)$$

Then θ_λ is a type³ that reflects a *lexicographic* comparison of coordinates in the basis λ . Moreover, the type θ_λ can break ties in the maximization of any $u \in \mathcal{U}$. For all $x, y \in X$, let

$$x\theta y \iff u(x) > u(y), \text{ or both } u(x) = u(y) \text{ and } x\theta_\lambda y. \quad (1.9)$$

Then θ is a type that maximizes u and breaks all ties by the lexicographic comparison of coordinates in λ . In finite-dimensional frameworks, lexicographic representations like (1.8) and (1.9) appear in Fishburn (1982), Myerson (1986), and Blume et al. (1991).

1.3 Random Expected Utility and Discreteness

Let Ω be the set of all pairs (x, A) such that $A \in \mathcal{M}$ is a menu, and $x \in A$ is feasible in A . Such pairs are called *trials*. A trial $(x, A) \in \Omega$ is called *extreme* if for any $y, z \in A$ and $\gamma \in (0, 1)$, $x = \gamma y + (1 - \gamma)z$ implies $x = y = z$.

A function $\rho : \Omega \rightarrow [0, 1]$ is called a *random choice rule* (RCR) if

$$\sum_{x \in A} \rho(x, A) = 1 \quad (1.10)$$

for all menus $A \in \mathcal{M}$. Then every value $\rho(x, A)$ can be interpreted as the probability of x

³Completeness, totality, and Independence are obvious. Show transitivity. Take any $x, y, z \in X$ such that $x\theta_\lambda y\theta_\lambda z$. As $f_\lambda(x - y) \geq 0$ and $f_\lambda(y - z) \geq 0$, then $f_\lambda(x - z) \geq 0$ because by definition of f_λ , $f_\lambda(x - z)$ equals $f_\lambda(x - y)$, or $f_\lambda(y - z)$, or the sum $f_\lambda(x - y) + f_\lambda(y - z)$. By (1.8), $x\theta_\lambda z$, and θ_λ is transitive.

being chosen when the menu A is feasible. For any RCR ρ , menu $A \in \mathcal{M}$, and set $V \subset L$, write

$$\rho(V, A) = \sum_{x \in V \cap A} \rho(x, A).$$

Types generate a basic class of RCRs. For any type $\theta \in \mathcal{T}$, define a binary function $\rho_\theta : \Omega \rightarrow \{0, 1\}$ over all trials $(x, A) \in \Omega$,

$$\rho_\theta(x, A) = \begin{cases} 1 & \text{if } x = b_\theta A \\ 0 & \text{if } x \neq b_\theta A. \end{cases}$$

Such ρ_θ is a random choice rule that describes deterministic choices of the type θ in all menus A . Moreover, if $\rho = \rho_\theta$ for some type $\theta \in \mathcal{T}$, then it satisfies the following list of conditions for all trials $(x, A) \in \Omega$, menus $B \in \mathcal{M}$, elements $y \in X$, and coefficients $\gamma \in (0, 1]$.

- (i) *Monotonicity*: $\rho(x, A) \geq \rho(x, A \cup B)$.
- (ii) *Linearity*: $\rho(x, A) = \rho(\gamma x + (1 - \gamma)y, \gamma A + (1 - \gamma)y)$.
- (iii) *Extremeness*: If $\rho(x, A) > 0$, then (x, A) is extreme.
- (iv) *Mixture Continuity* (MC): for any open set $V \subset L$,

$$\liminf_{\alpha \rightarrow \gamma} \rho(V, \alpha A + (1 - \alpha)B) \geq \rho(V, \gamma A + (1 - \gamma)B).$$

These conditions appear in GP's random expected utility model and hence, called *GP's axioms*. They can be checked explicitly for $\rho = \rho_\theta$ or derived from Theorem 1.2 below. MC applies the standard definition of weak convergence to the discrete probability measures that are imposed in menus $\gamma A + (1 - \gamma)B$ by the RCR ρ . Note that Monotonicity does not use mixtures of outcomes or menus and hence, holds in more general frameworks like BM's

random utility model.

1.3.1 Random Expected Utility via Types

A set function $\pi : 2^{\mathcal{U}} \rightarrow [0, 1]$ is called a *random utility function* (RUF) if $\pi(\mathcal{U}) = 1$, and π is *additive*: for all sets $\Theta, \Psi \subset \mathcal{U}$,

$$\Theta \cap \Psi = \emptyset \implies \pi(\Theta \cup \Psi) = \pi(\Theta) + \pi(\Psi).$$

Say that a RCR ρ is *induced* by a RUF π if for all $(x, A) \in \Omega$,

$$\rho(x, A) = \pi(\mathcal{U}(x, A)), \tag{1.11}$$

where $\mathcal{U}(x, A)$ is the set of all functions $u \in \mathcal{U}$ that are maximized by x in A ,

$$\mathcal{U}(x, A) = \{u \in \mathcal{U} : u(x) \geq u(y) \text{ for all } y \in A\}.$$

Gul and Pesendorfer (Gul and Pesendorfer, 2006, Theorem 2) axiomatize representation (1.11) when X is a finite-dimensional lottery simplex. Thus we refer to this model as *random expected utility* (REU), albeit our framework and interpretations are more general than this term originally suggests.

Rewrite REU in terms of types. Say that $\mu : 2^{\mathcal{T}} \rightarrow [0, 1]$ is a *random type function* (RTF) if $\mu(\mathcal{T}) = 1$, and μ is additive.

Say that a RCR ρ is *induced* by a RTF μ if for all $(x, A) \in \Omega$,

$$\rho(x, A) = \mu(\mathcal{T}(x, A)), \tag{1.12}$$

where $\mathcal{T}(x, A)$ is the set of all types θ for which x is best in A ,

$$\mathcal{T}(x, A) = \{\theta \in \mathcal{T} : x = b_\theta A\}.$$

It turns out that RUFs and RTFs induce the same class of RCRs.

Theorem 1.2. *For any random choice rule ρ , the following claims are equivalent:*

- (i) ρ is induced by some RUF π ,
- (ii) ρ satisfies GP's axioms,
- (iii) ρ is induced by some RTF μ .

Besides establishing the equivalence of representations (1.11) and (1.12), Theorem 1.2 extends GP's characterization to a more general framework where X need not be finite dimensional, or closed, or bounded. For example, X can be the set $\Delta^{\mathbb{R}}$ of all monetary gambles.

There are two main practical reasons to write REU in terms of RTFs rather than RUFs. First, it can become more transparent and parsimonious in this way. Second, RTFs allow for extensions that accommodate context dependence and self-selection, but would not be possible (or at least much more cumbersome) if RUFs are employed.

For example, take any type $\theta \in \mathcal{T}$. Then the RCR ρ_θ is induced by the degenerate RTF δ_θ such that for all $\Psi \subset \mathcal{T}$,

$$\delta_\theta(\Psi) = 1 \iff \theta \in \Psi.$$

By Theorem 1.2, ρ_θ is induced by some RUF π_θ as well, but π_θ is usually much more cumbersome than δ_θ . To see this, let $u \in \mathcal{U}$ be maximized by the type θ , which has probability one in δ_θ . However, the maximization of u has probability zero in π_θ in the generic case when X is non-collinear. This apparent paradox arises because GP's identification replaces the

endogenous tie-breaking of the type θ by a finitely additive distribution over mixture linear functions other than u . Occam's razor favors the use of δ_θ in this case.

To clarify the issue of tie-breaking, suppose that an RCR ρ is induced by some RUF π . Then π must satisfy the *regularity* condition

$$\pi_{xy} = \pi(\{u \in \mathcal{U} : u(x) = u(y)\}) = 0$$

because $\pi_{xy} > 0$ violates the identity (1.10) for the binary menu $\{x, y\}$, where

$$\begin{aligned} \rho(x, \{x, y\}) + \rho(y, \{x, y\}) &= [\pi(\{u \in \mathcal{U} : u(x) > u(y)\}) + \pi_{xy}] + \\ &[\pi(\{u \in \mathcal{U} : u(y) > u(x)\}) + \pi_{xy}] = 1 + \pi_{xy} > 1. \end{aligned}$$

By regularity, ties should happen with probability zero in any menu $A \in \mathcal{M}$. Therefore, if X is non-collinear, then for any $u \in \mathcal{U}$,

$$\pi(\{\alpha u + \beta : \alpha > 0, \beta \in \mathbb{R}\}) = 0,$$

that is π should assign probability zero to the maximization of u . To show this claim, take elements x, y, z such that $x - y$ and $y - z$ are linearly independent, and $u(x) \geq u(y) \geq u(z)$. Then either $u(x) = u(z)$, or $u(y) = u(\gamma x + (1 - \gamma)z)$ where $\gamma = \frac{u(y) - u(z)}{u(x) - u(z)}$. As $x - y$ and $y - z$ are linearly independent, then $y \neq \gamma x + (1 - \gamma)z$. By regularity, $\pi(\{\alpha u + \beta : \alpha > 0, \beta \in \mathbb{R}\}) = 0$.

By contrast, the specification of REU in terms of RTFs does not put any constraints on μ , except for the finite additivity. Indeed, if ρ is induced by any RTF μ , then ρ must be a RCR because for each menu A , the finite union

$$\mathcal{T} = \bigcup_{x \in A} \mathcal{T}(x, A)$$

is disjoint (i.e. a partition) and hence, $1 = \mu(\mathcal{T}) = \sum_{x \in A} \rho(x, A)$.

For example, risk neutrality is a salient model in empirical and theoretical research, but it cannot be assigned any positive probability by RUF π in representation (1.11). This constraint is removed when ρ is induced by RTFs. Let $\mathcal{V} \subset \mathcal{T}$ be the set of all types that maximize the expected value function

$$v(x) = \sum_{q \in R} qx(q)$$

over the space $X = \Delta^{\mathbb{R}}$ of all monetary lotteries. If ρ is induced by an RTF μ , then $\mu(\mathcal{V}) \in [0, 1]$ is the likelihood of risk neutrality in this interpretation. Given a mild discreteness assumption, the RTF μ and hence, the likelihood $\mu(\mathcal{V})$ are determined uniquely by the RCR ρ .

Moreover, RTFs are also more convenient than RUFs in extensions of REU that accommodate context dependence and self-selection. For example, the probability $\mu_A(\mathcal{V})$ of risk neutral types can depend on the composition of the menu A and still be identified uniquely by the RCR ρ . Such extensions are not feasible (or at least, much more cumbersome) when ρ is modeled via RUFs rather than RTFs.

1.3.2 Discrete Random Expected Utility

A random type function μ is called *discrete* (or DTF for short) if the set

$$\Theta_\mu = \{\theta \in \mathcal{T} : \mu(\theta) > 0\}$$

is finite, and for all sets $\Psi \subset \mathcal{T}$,

$$\mu(\Psi) = \mu(\Psi \cap \Theta_\mu).$$

The set Θ_μ is called the *support* of the DTF μ . By definition, any DTF is additive and even countably so.

Let $\mathcal{D}$ be the set of all DTFs. Let ρ be induced by some DTF $\mu \in \mathcal{D}$. Then

$$\rho = \sum_{\theta \in \Theta_\mu} \mu(\theta) \rho_\theta \quad (1.13)$$

is a mixture of finitely many functions ρ_θ such that Every binary function $\rho_\theta : \Omega \rightarrow \{0, 1\}$ captures deterministic choices of the type θ in all menus A .

We refer to representation (1.13) as *discrete random expected utility* (DREU). To characterize DREU, we replace GP's axioms with two other conditions.

Axiom 1.1 (Additivity). *For all $(x, A) \in \Omega$, $B \in \mathcal{M}$, and $\gamma \in (0, 1]$,*

$$\rho(x, A) = \rho(\gamma x + (1 - \gamma)B, \gamma A + (1 - \gamma)B).$$

Take any trial $(x, A) \in \Omega$, menu $B \in \mathcal{M}$, and coefficient $\gamma \in (0, 1)$. The set $\mathcal{T}(x, A)$ can be partitioned into the disjoint union

$$\mathcal{T}(x, A) = \bigcup_{y \in B} \mathcal{T}(\gamma x + (1 - \gamma)y, \gamma A + (1 - \gamma)B). \quad (1.14)$$

This union holds because each type $\theta \in \mathcal{T}$ satisfies Independence and hence, x is best for θ in A if and only if there is $y \in B$ such that $\gamma x + (1 - \gamma)y$ is best for θ in $\gamma A + (1 - \gamma)B$. Such y must be best for θ in B . The union (1.14) is disjoint because for any type $\theta \in \mathcal{T}$, its best element in $\gamma A + (1 - \gamma)B$ is unique.

Figure 1.1 provides a graphical illustration of Additivity. The **green** menu is a mixture of the **orange** and **blue** menus with weights $\frac{2}{3}$ and $\frac{1}{3}$. By Additivity, the likelihood of each of the

four symbols (square, circle, asterisk, plus) is preserved across the corresponding menus. For example, let the orange items (plus and asterisk) be menu “A” and blue items (circle and square) be menu “B”. Let $\gamma = \frac{2}{3}$ and x be the orange plus. Additivity will state that the probability of the orange plus chosen in the orange menu is equal to the sum of probabilities of the green circle-plus and square-plus in the green menu.

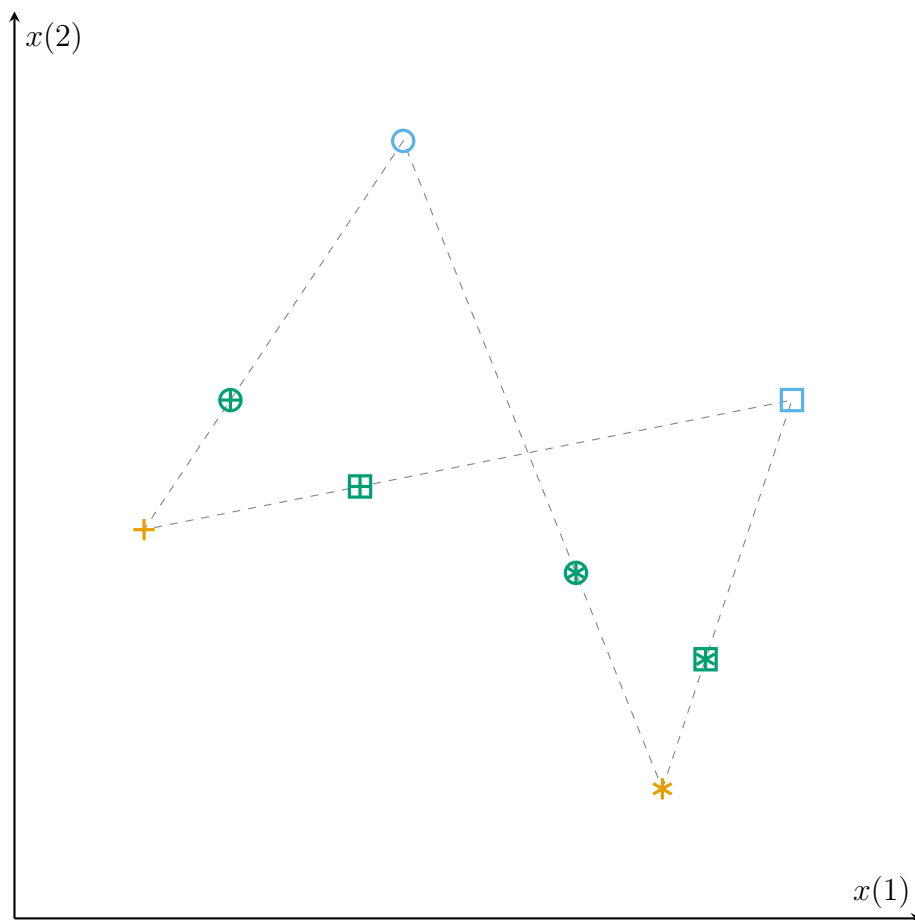


Figure 1.1: Additivity axiom for DREU.

Additivity can be also derived from its *superadditivity* part:

$$\rho(x, A) \geq \rho(\gamma x + (1 - \gamma)B, \gamma A + (1 - \gamma)B).$$

Indeed if these inequalities hold and some of them are strict, then by (1.10),

$$1 = \sum_{x \in A} \rho(x, A) > \sum_{x \in A} \sum_{y \in B} \rho(\gamma x + (1 - \gamma)y, \gamma A + (1 - \gamma)B) \geq 1.$$

By contradiction, superadditivity implies Additivity.

The discreteness of the RTF μ has a counterpart for the induced RCR ρ .

Axiom 1.2 (Discreteness). *There is $n \in \mathbb{N}$ such that for all $A \in \mathcal{M}$,*

$$|\{x \in A : \rho(x, A) > 0\}| \leq n.$$

Indeed, take any trial $(x, A) \in \Omega$. If ρ is induced by some DTF μ , then

$$\rho(x, A) > 0 \iff x \text{ is best in } A \text{ for some } \theta \in \Theta_\mu, \quad (1.15)$$

and Discreteness holds with $n = |\Theta_\mu|$.

Theorem 1.3. *A random choice rule ρ satisfies Additivity and Discreteness if and only if ρ is induced by some DTF $\mu \in \mathcal{D}$. Such μ is unique.*

The construction of the DTF μ is fast and explicit. Let ρ satisfy Additivity and Discreteness. By Discreteness, there is a menu $C = \{c_1, \dots, c_k\}$ of maximal size such that $\rho(c_i, C) > 0$ for all i . For any $x, y \in X$ and $i = 1, \dots, k$, let

$$x\theta_i y \iff \rho\left(\frac{x+c_i}{2}, \frac{x+C}{2}\right) > 0, \quad (1.16)$$

and $\mu(\theta_i) = \rho(c_i, C)$. Then μ is a DTF with the support $\Theta_\mu = \{\theta_1, \dots, \theta_k\} \subset \mathcal{T}$, and ρ is induced by μ . Moreover, such μ is identified by (1.16) uniquely. All details are in the appendix.

Neither the identification (1.16) nor the uniqueness claim extend from DTFs to RTFs. Note that Theorem 1.2 invokes GP’s proofs to find a regular RUF π that induces ρ . Then π can be translated into an RTF μ , which is never discrete even if ρ satisfies Discreteness. Therefore, such μ is distinct from the DTF that is derived via (1.16). Our identification (1.16) can be motivated by Occam’s razor because it is much faster and more transparent than GP’s construction.

The axioms in Theorem 1.3 are distinct from GP’s list as well. Discreteness need not hold in general REU, but Additivity must. Indeed, it is still implied by the partition (1.14) when ρ is induced by any additive function μ . We use Additivity because it is more parsimonious and convenient for extensions of DREU below. Note that Linearity is a special case of Additivity where $B = y$ is a singleton. Moreover, Additivity implies Monotonicity. Indeed, for all $(x, A) \in \Omega$ and $B \in \mathcal{M}$,

$$\rho(x, A) = \rho(x, A, 0.5, A \cup B) \geq \rho(x, A \cup B, 0.5, A) = \rho(x, A \cup B).$$

The other axioms in GP’s Theorem 2—Extremeness and Mixture Continuity—can be derived from the combination of Additivity and Discreteness.⁴ Conversely, the combination of Linearity, Monotonicity, and Extremeness can replace Additivity in Theorem 1.3. However, this modification requires a considerable extra effort in the proofs without any upside for the applications below. Hence we omit it.

While improving the identification of μ , Discreteness does not change the observable content of REU in finite datasets. Indeed, for any RTF μ and any finite collection of sets $\Theta_1, \dots, \Theta_k \subset \mathcal{T}$, there is a DTF $\mu^* \in \mathcal{D}$ such that

$$\mu(\Theta_i) = \mu^*(\Theta_i) \text{ for all } i = 1, \dots, k.$$

⁴See the proof of Theorem 1.2.

To show this claim, partition $\mathcal{T}$ into finitely many disjoint non-empty sets,

$$\mathcal{T} = \Psi_1 \cup \dots \cup \Psi_n$$

such that every Θ_i is a union of some sets from this partition. For each j , fix a type $\theta_j \in \Psi_j$.

Then

$$\mu = \sum_{j=1}^n \mu(\Psi_j) \delta_{\theta_j}.$$

coincides with μ on all Ψ_j and hence, on all Θ_i as well. Therefore, if $\rho(x, A) = \mu(\mathcal{T}(x, A))$ for all trials (x, A) in some finite dataset $\Omega^* \subset \Omega$, then there is a discrete $\mu^* \in \mathcal{D}$ such that $\rho(x, A) = \mu^*(\mathcal{T}(x, A))$ for all $(x, A) \in \Omega^*$.

1.4 Partial Random Expected Utility

Monotonicity—and a fortiori, Additivity—can be problematic when random choices exhibit *context effects*. For example, combining a menu A with extra options can increase the probability of choosing some $x \in A$ due to reason-based heuristics (e.g., Shafir, Simonson, and Tversky (1993)). Even if Monotonicity holds, *self-selection* can violate Linearity and hence, Additivity as well (see the example in the introduction). To accommodate such patterns, we relax DREU to a more general model that is still identified uniquely from random choice rules.

Define the Hausdorff metric over menus: for all $A, B \in \mathcal{M}$,

$$h(A, B) = \max \left\{ \max_{x \in A} \min_{y \in B} \|x - y\|, \max_{y \in B} \min_{x \in A} \|x - y\| \right\}.$$

A sequence of menus $B_1^\infty = (B_1, B_2, \dots)$ is called *simple* if it is convergent in the Hausdorff metric, and $B_i \subseteq \text{conv}(B_1)$ for all $i \in \mathbb{N}$. It is required here that all menus B_n must belong to some finite-dimensional simplex, which equals the convex hull of B_1 without any loss in

generality. This constraint makes any mixture linear function $u \in \mathcal{U}$ bounded on all menus B_n .

Call a function $f : \mathcal{M} \rightarrow \mathbb{R}$ *simply continuous* if for any simple sequence B_1^∞ ,

$$\lim_{n \rightarrow \infty} f(B_n) = f(\lim_{n \rightarrow \infty} B_n).$$

For example, simple continuity is guaranteed if for all $A \in \mathcal{M}$,

$$f(A) = g(u_1(A), \dots, u_k(A)), \quad (1.17)$$

where the functions $u_1, \dots, u_k \in \mathcal{U}$ are mixture linear, and the aggregator $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous. Indeed, suppose first that there is $u \in \mathcal{U}$ such that $f(A) = u(A)$ for all $A \in \mathcal{M}$. Then f is simply continuous because the restriction of u to any finite dimensional simplex is continuous.⁵ More generally, if f has the form (1.17), then simple continuity follows via the standard composition argument.

Take any map $\mu : \mathcal{M} \rightarrow \mathcal{D}$. For any menu $A \in \mathcal{M}$, write the DTF that μ assigns to A as μ_A rather than $\mu(A)$. For any type $\theta \in \mathcal{T}$, define a function $\mu_\theta : \mathcal{M} \rightarrow [0, 1]$ such that

$$\mu_\theta(A) = \mu_A(\theta) \text{ for all } A \in \mathcal{M}.$$

For any trial $(x, A) \in \Omega$, let

$$\rho_\mu(x, A) = \mu_A(\mathcal{T}(x, A)). \quad (1.18)$$

Such ρ_μ is a RCR because for each menu A , $\mathcal{T} = \bigcup_{x \in A} \mathcal{T}(x, A)$ is a partition, and μ_A is

⁵All norms induces the same topology in the finite dimensional case. However, if X is infinite dimensional, then mixture linearity does not imply continuity any more. For example, if v is the expected value function on $X = \Delta^\mathbb{R}$ with the uniform norm, then v is not continuous on X . Thus simple continuity does not guarantee continuity for f .

additive. To identify each μ_A uniquely in terms of ρ_μ , the map μ can be restricted as follows.

Say that $\mu : \mathcal{M} \rightarrow \mathcal{D}$ is a *partial type function* (PTF) if

- (i) the DTFs μ_A have the same support $\mathcal{T}_{\mu_A} = \mathcal{T}_\mu$ for all menus $A \in \mathcal{M}$,
- (ii) the functions μ_θ are simply continuous for all types $\theta \in \mathcal{T}_\mu$.

This terminology alludes to the fact that the likelihoods $\mu_A(\theta)$ are only partially determined by the type θ .

Say that ρ is *induced* by a PTF μ if $\rho = \rho_\mu$. This representation is called *partial random expected utility* (PREU). Obviously, DREU is a special case of PREU where μ is constant.

As all RTFs μ_A share the same support $\mathcal{T}_\mu$, then PREU still implies (1.15) and hence, Discreteness. On the other hand, PREU violates Additivity because it applies distinct DTFs μ_A and $\mu_{\gamma A + (1-\gamma)B}$ to the opposite sides of the partition (1.14). Similarly, Monotonicity can fail because the DTFs μ_A and $\mu_{A \cup B}$ need not be the same.

To characterize PREU, relax Additivity into two weaker properties

Axiom 1.3 (Boundary Additivity (BA)). *For all $(x, A) \in \Omega$, $\gamma \in (0, 1]$, and $B \in \mathcal{M}$,*

$$\rho(x, A) = 0 \iff \rho(\gamma x + (1 - \gamma)B, \gamma A + (1 - \gamma)B) = 0.$$

BA requires Additivity, but only in the boundary case when either side in the equation $\rho(x, A) = \rho(x, A, \gamma, B)$ is zero. BA follows the partition (1.14) and the assumption that the RTFs μ_A and $\mu_{\gamma A + (1-\gamma)B}$ share the same support.

Axiom 1.4 (Local Additivity (LA)). *For all $(x, A) \in \Omega$ and simple sequences B_1^∞ ,*

$$\rho(x, A) > 0 \implies \lim_{n \rightarrow \infty} \rho\left(\frac{1}{n}x + \frac{n-1}{n}B, \frac{1}{n}A + \frac{n-1}{n}B\right) > 0.$$

Note that if B_1^∞ is a simple sequence, then the sequence $\frac{1}{n}A + \frac{n-1}{n}B_n$ is also simple. Thus LA follows from the partition (1.14) and the assumption that the function μ_θ are simply continuous.

Theorem 1.4. *A random choice rule ρ satisfies BA, LA, and Discreteness if and only if ρ is induced by some PTF μ . Such μ is unique.*

This is our main characterization result for RCRs. Its corollaries include Theorem 1.3 for DREU and other applications below. The existence and uniqueness claims in Theorem 1.4 assert that the transformation

$$\mu \leftrightarrow \rho_\mu$$

is a bijection between PTFs and RCRs that satisfy BA, LA, and Discreteness.

The support Θ_μ of the PTF μ can be still constructed via (1.16). The unique assignment of probabilities $\mu_A(\theta)$ for all menus $A \in \mathcal{M}$ and types $\theta \in \Theta_\mu$ can be obtained asymptotically via

$$\mu_A(\theta) = \lim_{n \rightarrow \infty} \rho \left(b_\theta \left(\frac{1}{n}C + \frac{n-1}{n}A \right), \frac{1}{n}C + \frac{n-1}{n}A \right). \quad (1.19)$$

All details are in the appendix.

1.4.1 Self-Selection and Participation Rates

Let ρ_μ be induced by some PTF μ . *Self-selection* refers to patterns that ρ_μ can exhibit when Monotonicity is still preserved.

Monotonicity is plausible for patterns where any type θ who selects some $x \neq y$ in a menu $A \cup y$ a type θ can be become more likely likelihood

$$\mu_\theta(A \cup y) > \mu_\theta(A)$$

can happen only if y is best for θ in $A \cup y$. Such patterns have a normatively plausible motivation: some agents of type θ can be attracted to the menu $A \cup y$ because y dominates their unobserved default options, while $b_\theta A$ does not. One can also consider behavioral arguments where the type θ should become more popular in $A \cup y$ due to increased attention or wishful thinking when y is the best choice for θ in A .

To distinguish self-selection from broader behavioral context effects, assume that if $\mu_\theta(A)$ should never increase in $A \cup y$ unless y is best for θ in $A \cup y$. Say that μ is *monotonic* if for all $A \in \mathcal{M}$, $y \in X$, and $\theta \in \mathcal{T}$,

$$\mu_\theta(A \cup y) > \mu_\theta(A) \implies y = b(\theta, A \cup y). \quad (1.20)$$

To refine this notion, rewrite the PTF $\mu : \mathcal{M} \rightarrow \mathcal{D}$ similarly to the Tversky–Koehler *support theory* for subjective beliefs (Tversky and Koehler, 1994). Take a finite set of types $\mathcal{T}_\mu \subseteq \mathcal{T}$, and assume that for all menus $A \in \mathcal{M}$ and types $\theta \in \mathcal{T}_\mu$,

$$\mu_A(\theta) = \mu_\theta(A) = \frac{f_\theta(A)}{\sum_{\psi \in \mathcal{T}_\mu} f_\psi(A)} \quad (1.21)$$

where each *fitness function* $f_\theta : \mathcal{M} \rightarrow \mathbb{R}_{++}$ is positive and simply continuous.⁶ Similarly to Tversky and Koehler, each value $f_\theta(A)$ can be interpreted as a degree of subjective fitness/salience/evidence for the type θ in the menu A . In a different vein, functions f_θ can be interpreted in terms of *participation rates* when each type $\theta \in \mathcal{T}_\mu$ is equally likely ex ante and participates in the random choice data in the menu $A \in \mathcal{M}$ with probability $f_\theta(A) \in (0, 1]$.

⁶Obviously, if μ is a PTF, then it satisfies (1.21) with $f_\theta = \mu_\theta$ for all $\theta \in \mathcal{T}_\mu$. Conversely, if μ satisfies (1.21), then each μ_A is a DTF with the support $\mathcal{T}_\mu$, and each μ_θ is simply continuous as a continuous aggregation of simply continuous functions f_ψ .

Then a plausible form of self-selection can be modeled by (1.21) where for all $A \in \mathcal{M}$ and $\theta \in \mathcal{T}$,

$$f_\theta(A) = c_\theta(u_\theta(A)), \quad (1.22)$$

$u_\theta \in \mathcal{U}$ is maximized by θ , and $c_\theta : \mathbb{R} \rightarrow \mathbb{R}_+$ is continuous and non-decreasing. In other words, for each type θ , its participation rate in a menu A is a monotonic aggregation of its maximal utility $u_\theta(A)$ in this menu.

Say that μ is *selective* if it can be represented by (1.21)–(1.22). If μ is selective, then it is monotonic because

$$\mu_\theta(A \cup y) > \mu_\theta(A) \implies u_\theta(A \cup y) > u_\theta(A) \implies y = b(\theta, A \cup x).$$

As c_θ is allowed to be constant, then any DTF $\mu \in \mathcal{D}$ is a constant selective PTF. Thus the distinct classes of PTFs in our paper are nested:

$$\text{DTF} \implies \text{selective PTF} \implies \text{monotonic PTF} \implies \text{PTF}.$$

Monotonic and selective PTFs have other interesting features.

Theorem 1.5. *A random choice rule ρ satisfies BA, LA, Monotonicity, and Discreteness if and only if ρ is induced by some monotonic PTF μ .*

If μ is selective, then for all trials $(x, A) \in \Omega$ and menus $B \in \mathcal{M}$ such that $A \cap B = \emptyset$,

$$\sum_{C \subseteq B} (-1)^{|C|} \rho(x, A \cup C) \geq 0. \quad (1.23)$$

First, this result asserts that monotonicity for PTFs μ is equivalent to Monotonicity for the induced RCRs ρ_μ . Second, if μ is selective, then the induced ρ_μ satisfies the stronger

condition (1.23) known as *Block-Marschak monotonicity* (Block and Marschak, 1960). This observation explains in part why self-selection is hard to identify in the classic RUM: an entire class of selective PTFs produces stochastic choice data that is consistent with the condition (1.23) which is equivalent to the classic RUM representation. In this case, the self-selection can make μ *misspecified* unless the types are constrained by the Independence axiom. Then PREU can be identified uniquely even if Additivity and DREU does not hold.

Consider a trivial case for selective PTF. Let $f_\theta(A) = u_\theta(A)$, and hence:

$$\mu_\theta(A) = \frac{u_\theta(A)}{\sum_{\psi \in \mathcal{T}_\mu} u_\psi(A)}.$$

This can be applied as an extension of the luce model of stochastic choice (Luce, 1959) and joins many other extensions in recent literature. Gul, Natenzon, and Pesendorfer (2014) relates $\rho(x, A)$ to the proportion of the values of attributes x has relative to those of all attributes observed in a menu. Agents hence view each option as a bundle of attributes. Echenique and Saito (2019) captures just-noticeable differences by allowing dominated items still be chosen with positive probability when compared to closer alternatives. Kovach and Tserenjigmid (2022) touches upon limited attention by weighing items higher if they are in the focal set of the menu. This model also discusses menu dependence like in our PREU.

Note that if μ is monotonic, but not selective, then BM monotonicity is not guaranteed. To illustrate, let $X = \Delta^Q$ consist of lotteries on a finite set $Q = \{a, b, c, d\}$ of deterministic outcomes. Consider three types represented by expected utility functions u_1, u_2, u_3 that have a range in the strictly negative part of the real line. For all $A \in \mathcal{M}$, let

$$\begin{aligned}\mu_1(A) &= \frac{u_2(A) + u_3(A)}{2(u_1(A) + u_2(A) + u_3(A))} \\ \mu_2(A) &= \frac{u_1(A) + u_3(A)}{2(u_1(A) + u_2(A) + u_3(A))} \\ \mu_3(A) &= \frac{u_1(A) + u_2(A)}{2(u_1(A) + u_2(A) + u_3(A))}\end{aligned}$$

Then μ is a monotonic PTF, but ρ_μ does not satisfy BM monotonicity. For example, consider the types and choice data in Table 1.1. Portions of the PTF regarding menus not presented are irrelevant to this example. Note that the number in the parenthesis for the last four columns is $u_\theta(A)$ for the corresponding menu A .

θ	$u_\theta(a)$	$u_\theta(b)$	$u_\theta(c)$	$u_\theta(d)$	$\mu_\theta(\{a, b\})$	$\mu_\theta(\{a, b, c\})$	$\mu_\theta(\{a, b, d\})$	$\mu_\theta(Q)$
1	-2	-3	-4	-5	$\frac{1}{3} (-2)$	0.3 (-2)	0.3 (-2)	$\frac{2}{8} (-2)$
2	-3	-2	-1	-4	$\frac{1}{3} (-2)$	0.4 (-1)	0.3 (-2)	$\frac{3}{8} (-1)$
3	-3	-2	-4	-1	$\frac{1}{3} (-2)$	0.3 (-2)	0.4 (-1)	$\frac{3}{8} (-1)$

Table 1.1: Monotonic PTF that violates BM.

We now compute the Block–Marschak polynomial (1.23) at $x = a$, $A = \{a, b\}$, and $B = \{c, d\}$

$$\begin{aligned}
q(a, \{ab\}) &= \rho(a, \{a, b\}) - \rho(a, \{a, b, c\}) - \rho(a, \{a, b, d\}) + \rho(a, \{a, b, c, d\}) \\
&= \frac{1}{3} - \frac{3}{10} - \frac{3}{10} + \frac{2}{8} = -\frac{1}{60} < 0.
\end{aligned}$$

The negative result shows that this monotonic PTF does not have a RUM representation.

This can be generalized into the following form. Take k types each with a utility function $u_\theta : X \rightarrow \mathbb{R}_{--}$. Let

$$\mu_\theta(A) = \frac{\sum_{\psi \neq \theta} u_\psi(A)}{(k-1) \sum_{\psi} u_\psi(A)}.$$

This μ is selective as expanding the menu without adding a better option for type θ will decrease the numerator in magnitude faster than the denominator.

1.4.2 Random Risk Aversion

PREU and any of its special cases can further refined by imposing additional structure on the types $\theta \in \Theta$. Olschewski et al. (2022) discovered that the random utility model does not precisely differentiate between risk-averse and risk-seeking agents. This motivates us to next extend the discussion of menu dependence to risk attitudes.

Let $X = \Delta^Q$ be the set of all lotteries on some interval $Q \subseteq \mathbb{R}$ of monetary prizes.⁷ This interval is convex, but need not be closed or bounded. Any $q \in Q$ is associated with the degenerate lottery that delivers q with certainty. Accordingly, $Q \subseteq X$ denotes the subclass of such degenerate lotteries.

By mixture linearity, any utility function $u \in \mathcal{U}$ has the expected utility form:

$$u(x) = \sum_{q \in Q} x(q)u(q) \text{ for all } x \in X.$$

The restriction of u to Q is known as the *Bernoulli utility index*, and its concavity can be interpreted in terms of *risk aversion*. Formally, u is strictly increasing and concave on Q if and only if for all $x \in X$ and $q \in Q$,

$$q > E(x) \implies q > u(x),$$

where $E(x) = \sum_{q \in Q} x(q)q$ is the mathematical expectation (i.e. the mean) of the lottery x .

Risk aversion can be imposed on types in our model as well.

A type $\theta \in \mathcal{T}$ is called *risk averse* if it maximizes some $u \in \mathcal{U}$ with a strictly increasing and concave restriction to Q . Let $\mathcal{T}_{ra}$ be the set of all risk averse types. Note that the concavity is not required to be strict and hence, the risk neutral types θ that maximize $u(x) = E(x)$ are included in $\mathcal{T}_{ra}$.

Axiom 1.5 (Archimedean Risk Aversion (ARA)). *For any $q \in Q$ and $x, y \in \Delta^Q$, there is $\varepsilon > 0$ such that*

$$q > E(x) \implies \rho(x, \{x, \varepsilon y + (1 - \varepsilon)q\}) = 0.$$

⁷This example can be extended to the set X of all Borel distributions on Q with bounded rather than finite support. For brevity, we adopt the simplest framework here that still allows to identify a random risk aversion model.

When $y = q$, then ARA implies

$$q > E(x) \implies \rho(x, \{x, q\}) = 0.$$

This part assumes that any type $\theta \in \mathcal{T}_\mu$ should prefer the certain payoff q to any gamble x that delivers less than q on average. The Archimedean extension is necessary to guarantee that the strict risk aversion of the type θ should be reflected by the expected utility function u that is maximized by θ rather than the tie-breaking rule that is embedded in θ .

Corollary 1.6. *For any PTF μ , ρ_μ satisfies ARA if and only if $\mathcal{T}_\mu \subseteq \mathcal{T}_{ra}$.*

This result is derived from Theorem 1.1. Corollary 1.6 can be combined with Theorems 1.3–1.5 to characterize PREU and its refinements where all types are required to be risk averse.

In the next chapter, I explore the effects of a certain option (degenerate lottery) on risk behavior in an experimental setting. Results shows systemic violations of Monotonicity. Hence, GP’s standard REU would fail to model this phenomenon, and menu-dependence is needed. This phenomenon can be explained by treating the certain option as a decoy, where its presence will make some agents more risk averse.

Several papers also allow for the degree of risk neutrality to vary with the menu. Experimental work by Chen et al. (2024) also studies menu-dependent risk attitudes and estimates how risk-taking behavior changes with the available set of lotteries. Similarly, evidence from insurance choice by Ericson et al. (2021) identifies heterogeneous risk preferences from different insurance menus, allowing the effective share of risk-neutral versus risk-averse behavior to differ across choice environments.

This can be extended to more types. Discrete Arrow-Pratt indices by Baillon and L’Haridon (2021) can capture the finite nature of the type space. A discrete lognormal distribution

may be appropriate for $\mu_A(\cdot)$ as CRRA, in the format of $u(x) = \frac{x^r}{1-r}$, indices are positive and skewed right, with most agents exhibiting some level of risk aversion. The mode of μ_A will move left when a certain option is present. Desmettre and Steffensen (2023) also used random risk aversion in CARA/CRRA to solve an investor's portfolio allocation problem.

Chapter 2

Certainty as a Decoy: Testing Menu Dependence in Risk Attitudes

2.1 Introduction

Decoy effects have been widely discussed in literature as a primary example of how context-dependence can override rational choice. The effects of introducing an unattractive option to the choice set to change how decision makers choose can be observed in beer consumption (Huber et al., 1982), subscriptions to the Economist (Ariely, 2008), and household appliances (Simonson, 1989). In fact, decoys can be more than just a marketing trick, as they appear to be fundamentally rooted in our cognitive architecture, affecting decision-making at the neural level (Soltani et al., 2012). However, while these classic examples focus on static consumer preferences, there remains a critical gap in understanding how decoys, particularly certain ones, influence choices under risk and whether these shifts in preference persist over time.

Decoy effects, and many other behavioral patterns, challenge how heterogeneous (or *stochas-*

tic) choice data is modeled. As discussed in Chapter 1, the random utility model (RUM) by Block and Marschak (1960) describes stochastic choice data by a stable probability distribution across different types of decision makers. Refinements of RUM to specific utility functions to RUM include expected utility Gul and Pesendorfer (2006), single-crossing conditions (Apesteguia, Ballester, and Lu, 2017), and quasi-linearity (McFadden, 1973; Williams, 1977; Daly and Zachery, 1979; Yang and Kopylov, 2023). These refinements inherit the invariance in distribution, and may cause trouble when attempting to fit data with decoy effects.

In Chapter 1, Dr Kopylov and I adjusted the random expected utility model to allow for a distribution of types that can depend on the menu. This allows the capture of context-dependence, and hence, is a better candidate for data where decoy effects are prevalent. What’s more, due to the nature of expected utility, it also allows for discussion of choices under risk and uncertainty. This model further relaxes a core assumption in RUM, monotonicity, which I will later illustrate its violation in the experiment.

Literature has noted that the structural presentation of lotteries can shift risk attitudes. For instance, individuals tend to exhibit greater risk neutrality in simple binary formats but shift toward risk aversion when faced with more complex or compound lottery structures (Harrison et al., 2013). Specific framing of a decision can also significantly alter observed levels of risk aversion (Crosetto et al., 2023). While a significant portion of the literature on the instability of risk preferences focuses on framing effects, less attention has been paid to how the structural composition of the menu itself, specifically the inclusion of a certain decoy, can drive these shifts.

Recent literature (Chen et al., 2024) has challenged the robustness of menu-dependent risk attitudes by finding that simply varying the range of potential lottery prizes does not reliably shift preferences. My study focuses on a specific, psychologically potent menu addition: the certain decoy. By introducing an option with zero variance but unattractive payoff, I test

whether certainty acts as a unique catalyst for menu-dependency where prize-range variations often fail.

My experimental results provide a comprehensive map of how certain decoys influence risky choice. First, I find robust evidence for the certain option acting as a decoy. The introduction of a certain option systematically shifts relative preferences toward the safer lottery, constituting a significant violation of monotonicity.

Second, this bias exhibits striking persistence: subjects previously exposed to the certain decoy remain significantly more risk-averse in subsequent scenarios when the decoy is removed than those in the control group. This suggests a *stickiness* in preference architecture. This extends the work of Chen et al. (2024), whose focus on static environments did not account for the enduring impact choice architecture may have on risk attitudes even after the menu has been simplified. This suggests that the impact of choice architecture on risk attitudes may be more enduring than previously thought.

Finally, the results indicate that this susceptibility is not a function of cognitive ability. Regression analysis reveals that higher pattern-recognition (Raven, 2000) scores do not significantly mitigate the decoy effect. This supports the view that the pull of certainty is a universal heuristic that operates independently of individual cognitive capacity. This result is supported by findings from Soltani et al. (2012); Dumbalska et al. (2020). However, baseline risk preference does seem to have a potential effect on susceptibility, but is more likely due to more risk-averse agents being closer to a “switching threshold” than the more risk-loving ones.

Similar results of using safe options to change risk behavior can be seen in Lim and Silvio (2021). While they establish that “Avoidable Risk” (the presence of a safe option) increases risk aversion in a static menu, my work extends this by demonstrating that this effect is sticky across time and universal across different cognitive profiles. Furthermore, while their

study examines the impact of adding any safer alternative to a set, my design specifically utilizes certainty as a decoy, allowing for a direct test of how these targeted menu additions distort the perceived value of existing risky options. They did find that the increase in risk aversion when a safe option is added is a consistent pattern across their subject pool, aligning with my finding that these effects are not restricted to certain cognitive profiles.

With uncertainty being an integral part of everyday scenarios, I discuss applications to my model. For example, insurance choice can be affected by certain decoys in the form of an undesirable no-deductible plans. This will allow insurance companies to design policies to sway consumers to purchase options that increases producer profits. What's more, tax evasion can be mitigated through the introduction of an withholding option.

This paper is structured as follows. In Section 2, I introduce a theoretical framework that integrates menu-dependence into stochastic choice, as well as the main predictions. This inherits the model from Chapter 1. In Section 3, I describe the experimental design. In Section 4, I provide statistical results, beginning with descriptive choice patterns followed by econometric analyses of the decoy effect on certain options and individual cognitive differences. I discuss applications and alternatives in Section 5 and conclude in Section 6.

2.2 Theoretical Framework

Assume a set of non-negative monetary payoffs $Q \subseteq \mathbb{R}_+$. A *lottery* L is a probability distribution $p_L(q)$ over Q . A *menu* A is a set of such lotteries representing the options available for a decision maker to choose from. Denote the set of all observed menus by $\Sigma \subseteq \mathcal{P}(\Delta(Q))$. A lottery is considered *degenerate* (or a *certain option*) if a single payoff in Q has a probability 1.

Consider a population of expected utility maximizers with *heterogeneous* levels of constant

relative risk aversion (CRRA). Individual i 's Bernoulli utility function is hence in the form

$$u_i(q) = q^{1-r_i} \text{ where } r_i < 1,$$

and is risk averse (*resp.* neutral, seeking) if $r_i > 0$ (*resp.* $= 0$, < 0).

Due to the nature of expected utility maximization, this individual will choose lottery L such that

$$L \in \arg \max_{M \in A} \sum_{q \in Q} p_M(q) q^{1-r_i} \quad (2.1)$$

when faced with some menu A . I will refer to this property as L *maximizes* A . A tie-breaking rule such as in Chapter 1 can be imposed to ensure singleton choices. However, since the design of the experiment forces subjects to only make one choice, I shall assume that preferences over the lotteries are linear.

For notational simplicity, I shall assume that each lottery only has a finite number of payoffs that have positive probability. The population is characterized by the distribution of CRRA levels r_i on the interval $(-\infty, 1)$. Again, for notational simplicity, I shall assume a finite set of possible CRRA levels $\mathcal{R} \subseteq (-\infty, 1)$.

2.2.1 Menu-Dependence

Consider instead that the level of risk aversion for an individual can change depending on the menu presented. In other words, treat r_i as a function that maps Σ to $\mathcal{R}$. This individual will no longer be considered CRRA across menus, but can still be modeled using the same functional form:

$$u_i(q, A) = q^{1-r_i(A)} \text{ where } r_i(A) < 1 \text{ for all } A \in \Sigma,$$

and make choices using Equation 2.1.

This menu-dependence can violate some classical properties in “rational”¹ choice. For example, let non-degenerate lotteries be $R = (0.5, \$10; 0.5, \$0)$ and $S = (0.5, \$7; 0.5, \$2)$, where R is the relatively *riskier* lottery and S is the relatively *safer* lottery. Let a certain option be $c = \$2$ and menus be $A = \{R, S\}$ and $B = \{R, S, c\}$. Assume an individual i has relative risk aversion levels $r_i(A) = 0.1$ and $r_i(B) = 0.2$. In menu A , she will choose lottery R but in menu B , she will choose lottery S . This is a violation of Sen’s α , and in turn, also violates the weak axiom of revealed preference (Sen, 1971). Therefore, this behavior is not consistent with (expected) utility maximization. However, such revealed preference can describe the scenario where an individual becomes more risk averse when a certain option (c) is introduced.

The population distribution is now also menu-dependent. For instance, one menu may consist of a higher proportion of the population that is risk averse than the other. I shall now properly denote this distribution by $\mu_A(r)$ over the support $\mathcal{R}$, which is the mass of the population that exhibits a relative risk aversion level of r when faced with menu A . Following axioms of probability,

$$\mu_A(r) \geq 0 \text{ for all } A \in \Sigma \text{ for all } r \in \mathcal{R} \text{ and } \sum_{r \in \mathcal{R}} \mu_A(r) = 1 \text{ for all } A \in \Sigma.$$

2.2.2 Stochastic Choice and Random Utility

Choice data sometimes does not reveal individual choices but instead comes in aggregation, i.e., it shows what proportion of the population chooses some lottery L in some menu A . Such data is in the form of *stochastic choice rules* (SCR) and denoted by $\rho(L, A)$. SCRs can also be used to describe randomness in a single individual’s choice, and was referred to as a *random choice rule* (RCR) in the previous chapter. As such, all three axioms of probability follow.

¹I loosely use rational as an umbrella term for choices consistent with utility maximization.

As mentioned in Chapter 1, the *random utility model* (RUM) is the standard model to describe the behavior of such a population. Specifically an SCR belongs to a *random expected utility* model (REU) if there is a some distribution $\mu(r)$ on $\mathcal{R}$ such that

$$\rho(L, A) = \sum_{r \in \mathcal{R}: L \text{ maximizes } A} \mu(r). \quad (2.2)$$

To allow for menu-dependence, each menu has its own distribution $\mu_A(r)$ on $\mathcal{R}$ similar to the previous subsection. Such a distribution follows a *partial random expected utility* (PREU) model defined in the previous chapter where

$$\rho(L, A) = \sum_{r \in \mathcal{R}: L \text{ maximizes } A} \mu_A(r). \quad (2.3)$$

Evidently, REU is a special case of PREU where $\mu_A = \mu_B$ for all $A, B \in \Sigma$.

Monotonicity (also known as stochastic α or regularity) states that when new options are added to the menu, previously available options will not be chosen with a higher probability:

$$\rho(L, A) \geq \rho(L, B) \text{ for all } L \in A \subseteq B. \quad (2.4)$$

In terms of rationality, the weak decrease in probability is due to new options potentially being more attractive to decision makers than previously available ones.

Monotonicity is a requirement for the standard RUM and its refinements. However, it can be violated under PREU. Take lotteries and menus from the example in the previous subsection. If there is a general increase in risk aversion, some individuals that chose R in A will switch to S in B . Individuals that chose S in A will continue to choose S in B since they already made the “safer” choice. S has first-order stochastic dominance (FOSD) over c , and hence

c will not be chosen. In summary, it may be observed that

$$\rho(S, A) < \rho(S, B),$$

which is a clear violation of monotonicity.

2.2.3 Predictions

Let $\{R, S\}$ be a *pair* of lotteries. To ensure that all CRRA functions will “agree” on the order of riskiness of lotteries, I shall restrict the pairs such that they satisfy the *single-crossing property*. This states that for any pair, there exists some cutoff $\bar{r} \in (-\infty, 1)$ such that the “riskier” lottery R (*resp.* “safer” lottery S) maximizes $\{R, S\}$ for individuals with $r_i(\{R, S\}) < \bar{r}$ (*resp.* $> \bar{r}$).

The over-arching prediction of this paper is that individuals systematically have a higher level of risk aversion when a certain option is introduced. As such, no one will should switch from S to R when a certain option c becomes available.

Hypothesis 2.1 (Increased Risk Aversion). *The proportion of the population that chooses the riskier lottery will decrease with the introduction of a certain option.*

$$\rho(R, \{R, S\}) > \rho(R, \{R, S, c\}) \text{ for all pairs of } \{R, S\} \text{ for all } c.$$

On the other hand, individuals can switch from R to S if their increase of risk aversion crosses over $\bar{r}$. When either R or S first-order stochastically dominates c , then no individual will switch to c . I shall call such a certain option a *certain decoy*.

Hypothesis 2.2 (Violation of Monotonocity). *The proportion of the population that chooses*

the safer lottery will increase with the introduction of a certain decoy.

$$\rho(S, \{R, S\}) < \rho(S, \{R, S, c\}) \text{ for all pairs of } \{R, S\} \text{ for all dominated } c.$$

This hypothesis describes a violation of monotonicity.

When c is not a certain decoy, then some individuals may switch to c from either R or S , as there is no FOSD. However, since there is still no individuals switching from S to R , there can still be an increase in the proportion of the population that chooses S when we only focus on the two non-degenerate lotteries.

Hypothesis 2.3 (Conditional Safe Switch). *The conditional proportion of the population that chooses the safer lottery will increase with the introduction of a certain option.*

$$\rho(S, \{R, S\}) < \frac{\rho(S, \{R, S, c\})}{\rho(S, \{R, S, c\}) + \rho(R, \{R, S, c\})} \text{ for some pairs of } \{R, S\} \text{ for some } c.$$

Hypothesis 2.3 represents a weaker, more general version of Hypothesis 2.2, allowing for cases where the certain option c is attractive enough to “steal” mass from the S as long as it steals more from R .

Expanding to the discourse of dynamics, a persistence, or *stickiness*, of the decoy effect may exist. As mentioned in the introduction, there is evidence that decoy effects last beyond its removal. The stickiness may arise because the “reason” generated by the decoy is remembered (Simonson, 1989), or because the decoy induces an update in the valuation of the items (Sen, 1993).

Hypothesis 2.4 (Stickiness). *When a certain decoy is removed, some individuals’ risk aversion level will remain elevated.*

Finally, individuals may have differences in their susceptibility to the decoy effects depending on their levels of cognitive abilities (Mao and Oppewal, 2012).

Hypothesis 2.5 (Cognition). *Individuals with higher cognitive abilities are less likely to change their risk attitudes with the presence of a certain decoy.*

2.3 Experimental Design

I detail how the experiment is designed and implemented with a comprehensive set of screenshots for the program available in Appendix B.1. The program is coded in oTree (Chen et al., 2016). This experiment meets all conditions of a *typical* experiment as defined in the IRB Protocol narrative.

The experiment is organized into a hierarchical structure: a *scenario* is a single choice task, where the subject is faced with a menu with two or three lotteries; a *set* consists of two scenarios sharing the same non-degenerate lottery pair similar to the $\{R, S\}$ structure in the previous section; and a *group* is a collection of four sets characterized by a similar risk-aversion *threshold* $\bar{r}$.

2.3.1 Main Tasks

Subjects face 50 choice scenarios, with each scenario being a *main task*, where they choose an option among two to three lotteries. Each lottery is presented in the form of a pie chart, with each slice detailing the payoff and probability. I use color-blind friendly colors [orange](#), [sky blue](#), and [bluish green](#), respectively with the hex codes #E69F00, #56B4E9, and #009E73 for the slices. This follows the Wong Palette (Wong, 2011), which adds contrast to the colors. The terminology “lottery” is never used in the program to avoid usage of unnecessary jargon. Payoffs are represented in points, with a conversion rate of \$0.01 per point. An on-screen calculator is available for subjects.

The 50 scenarios can be split into 25 *sets* of two scenarios each. The two scenarios in a set share the same pair of non-degenerate lotteries $\{R, S\}$ where R is the relatively “riskier” lottery and S is the relatively “safer” one. One scenario in each set only contains $\{R, S\}$, which I shall call B-menus (for binary, Figure 2.1). The other scenario contains an extra certain option, which I shall call T-menus (for ternary, Figure 2.2). The order of the options on the screen is randomized.

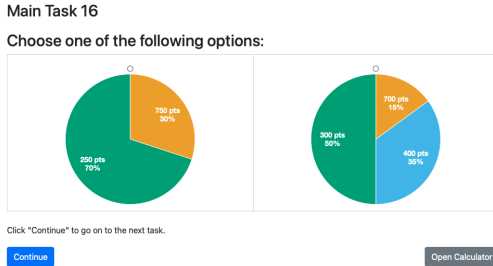


Figure 2.1: Binary “B”-menu

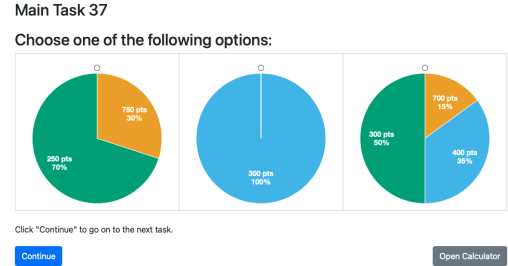


Figure 2.2: Ternary “T”-menu

One of the sets includes an “ S ” option (0.1, 600 pts; 0.6; 300 pts; 0.3, 0 pts) that is first-order stochastically dominated by the “ R ” option (0.2, 600 pts; 0.7; 300 pts; 0.1, 0 pts). This set of two scenarios is designed to be used as an exclusion criterion for attention check, which I shall go in depth in Section 2.3.5. Since R should never be chosen regardless of the RRA level given $r_i < 1$, this set is not used in the main analysis.

Table 2.1 shows the lotteries used in many experiments to elicit risk attitudes (Holt and Laury, 2002). I call each pair here a *group*. Below the lotteries, I list the intervals of r_i (levels of RRA) which would motivate an individual to switch from choosing option S to R at that *threshold*. For example, an individual with $r_i = 0$ would choose S in groups 1 to 4, then switch and stick to R starting from group 5. Such an individual is in group 5. The group terminology will be repeated below to categorize sets from the tasks with a similar concept of thresholds.

The remaining 24 sets can be partitioned into 6 *groups* of 4 sets (8 scenarios) each. Let

Group	Payoffs	1	2	3	4	5	6	7	8	9	10
Option S	\$2.00 w.p.	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
	\$1.60 w.p.	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
Option R	\$3.85 w.p.	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
	\$0.10 w.p.	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
r_i 's if	lower bd.	$-\infty$	-1.71	-0.95	-0.49	-0.14	0.15	0.41	0.68	0.97	≈ 1
$S \rightarrow R$	upper bd.	-1.71	-0.95	-0.49	-0.14	0.15	0.41	0.68	0.97	≈ 1	1

Table 2.1: Risk Elicitation by Holt and Laury (2002)

a scenario be in group $G \in \{3, 4, \dots, 8\}$, then all individuals that fall in groups G and above should choose the S option, and the remaining should choose R . In other words, the threshold $\bar{r}$ that induces switching would fall in group G in Table 2.1.

In each group, the four sets have unique combinations of the number of payoffs for the $\{R, S\}$ pair as shown in Table 2.2. Out of the four sets, three of them will include a certain decoy in the T-menu, while the T-menu other set includes a certain option that is almost dominated. A detailed list of all scenarios is available in Appendix B.2.

Option	# of payoffs			
R	2	2	3	3
S	2	3	2	3

Table 2.2: Combinations of the lotteries

2.3.2 Treatments

I designed three treatments for this experiment. The purpose of the treatments is mainly to test Hypothesis 2.4, whether there exists some stickiness to the decoy effect. Subjects are randomly assigned to a treatment, so the data collection is balanced across the three.

T1: (Binary-First) Subjects make decisions in all 25 B-menus first, and then make decisions in all 25 T-menus.

T2: (Ternary-First) Subjects make decisions in all 25 T-menus first, and then make deci-

sions in all 25 B-menus.

T3: (Alternating) Subjects alternate between the two types of menus, with odd tasks being B-menus and even tasks being T-menus.

For Treatments 1 and 2, I define the first 25 menus and the last 25 menus each as *blocks*.

The order of the menus are fully randomized for each subject other than the restriction on the treatment. This includes the order of each block. For example, for *T1* and *T2*, tasks t and $t + 25$ are likely not in the same set, and for *T3*, tasks t and $t + 1$ are likely not in the same set. This prevents agents from making choices based on memory.

2.3.3 Implementation and Other Tasks

I recruited 276 participants from the Experimental Social Science Laboratory subject pool at the University of California, Irvine. The sessions are conducted over Zoom and subjects are required to use a laptop or desktop with a stable Internet connection. The experiment is conducted in the following phases as shown in Table 2.3. Subjects are allowed to finish the experiment at their own pace without needing to wait for other subjects to finish a task/phase. Sessions are set to 75 minutes with a verbal overview of the experiment for the first 5 minutes. The mean time spent on the experiment was 24.78 minutes (SD = 10.21 minutes) with a median of 21.68 minutes.

Phase	Component	Description
1A	Instructions	Overview of rules and procedures
1B	Comprehension Check	Verification of subject understanding
2	Main Tasks	50 choice scenarios (B-menus and T-menus)
3A	Questionnaire	Self-reported demographics
3B	Cognitive Pattern Recognition	8 problems from Matzen et al. (2010)
4	Risk Elicitation	10 tasks (Holt and Laury, 2002)

Table 2.3: Experimental Procedure Phases

The experiment starts with screens dedicated to instructions, including the choice tasks, usage of an on-screen calculator, as well as the payment structure. After viewing the instructions, the subjects are asked five multiple choice questions to verify their comprehension of the experiment. Subjects need to answer all five questions correctly to continue on to the main tasks. The number of tries is another exclusion criteria and will be discussed in Section 2.3.5.

The main tasks are described in detail in the previous subsections and will not be repeated here. After the 50 tasks, subjects complete a questionnaire with demographic prompts including major (at the school level), year at UCI, gender, ethnicity, race, number of ECON courses taken, and GPA.

Subjects then complete 8 cognitive pattern recognition problems from Matzen et al. (2010) (Figure 2.3). Each problem has a 3×3 grid of “items” where the bottom-right one is missing. Subjects need to select among 8 options what the missing “item” should be. Subjects are given 60 seconds to complete each problem and the problems appear in increasing difficulty. Each problem has five versions, and the versions chosen for each problem is randomized. Data from this phase is used for Hypothesis 2.5 as a measure of cognition. I opted to not use the common cognitive reflection test questions, such as the \$1.10 bat and ball problem, as many participants in the subject pool have likely already seen these questions before in other experiments (Stieger and Reips, 2016).

The final phase of the experiment includes the 10 tasks shown in Table 2.1 for the purpose of eliciting a baseline level of risk aversion. This is conducted after the questionnaire and pattern recognition phases so subjects will be desensitized to decoy and other effects in the main tasks. The tasks are designed using the same interface as the main tasks to avoid needing a second set of instructions for choice between lotteries.

At the conclusion of the experiment, one random main task is selected and realized. Points

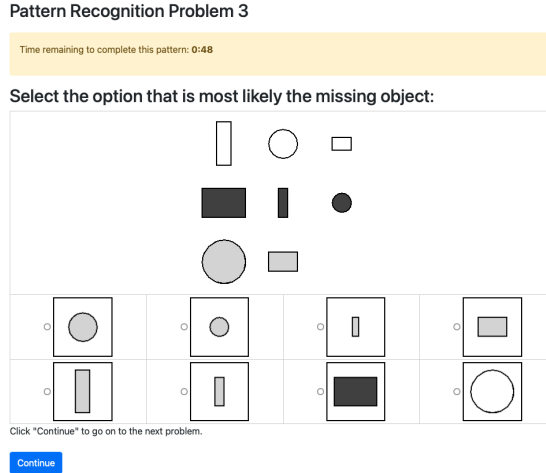


Figure 2.3: Pattern recognition problem

are converted to US dollars at a \$0.01 per point rate. Subjects also receive an additional \$0.10 for each correct response to the pattern recognition problems. A \$10 participation fee is guaranteed on top of the two sources of payments. The mean payment was \$14.46 (SD = \$2.19) while the median payment for the subjects was \$14.10. The distribution of payments was moderately skewed (skew = 1). This likely reflects the underlying distribution of risk preferences in the sample, with few subjects being risk loving; high earners are subjects who choose options with higher possible outcomes in riskier lotteries.

2.3.4 Preregistration

This experiment was preregistered on AsPredicted with documentation available in Appendix B.3. A few key differences are as follows:

- A fifth hypothesis (Hypothesis 2.5) is specified.
- Instead of 26 sets, there are now 25 sets. 1 set of attention check was removed.
- Time taken on a task and whether the on-screen calculator was used are added as covariates to the analysis.

- Non-parametric methods are predominantly used.
- The number of menus that include a first-order stochastically dominated option, instead of 10, was expanded to include: 2 menus are attention checks, with the lotteries having the same possible outcomes but one menu have worse probabilities. 36 menus have a certain option that is at most the lowest possible outcome of one of the other lotteries. The exclusion criterion is slightly adjusted and is explained in the next subsection.

2.3.5 Exclusion Criteria

I preregistered the following two exclusion criteria. Data from a subject whose ...

- number of attempts at the comprehension quiz exceeds two standard deviations above the mean, or
- number of times choosing a dominated option exceeds two standard deviations above the mean will not be included in the main analysis.

During the preregistration, about 11.1% of the data was expected to be excluded. The mean quiz attempts is 4.44 tries with a standard deviation of 7.48. According to my preregistration, subjects that attempted the quiz 20 times or more should be excluded from the main analysis. This excludes 12 subjects.

In terms of the number of times of dominated options, I decided to split it into two categories. First, if the subject failed either of the attention checks, their data is excluded (11 subjects). Second, if the subject chose a certain decoy from the other tasks over the cutoff determined in the preregistration, their data is excluded (9 subjects). This cutoff, which is 4 or more tasks, is calculated with a mean of 0.56 tasks and a standard deviation of 1.63 tasks. Curiously, these three exclusions are partitioned, as no subject is excluded due to more than one reason.

The total exclusions is 32 subjects, which takes up around 11.6% of the data, matching the expectation in preregistration.

2.4 Results

2.4.1 Descriptive Statistics

Demographics

In this section, I report the summary for each question from the self-reported demographics. All subjects that participated in the experiment are considered for this part.

Figure 2.4 shows the distribution of majors categorized by different schools at UCI within the subjects. About a third of the subjects are from Social Sciences and a quarter are from Engineering. Figure 2.5 shows the distribution of years at UCI within the subjects. Each year takes up about a quarter of the sample. Figure 2.6 shows the distribution of race for the subjects. Figure 2.7 illustrates the distribution of gender identity in the sample, where female subjects outnumber male subjects by 2:1. 22.83% of the participants are of Hispanic origin.

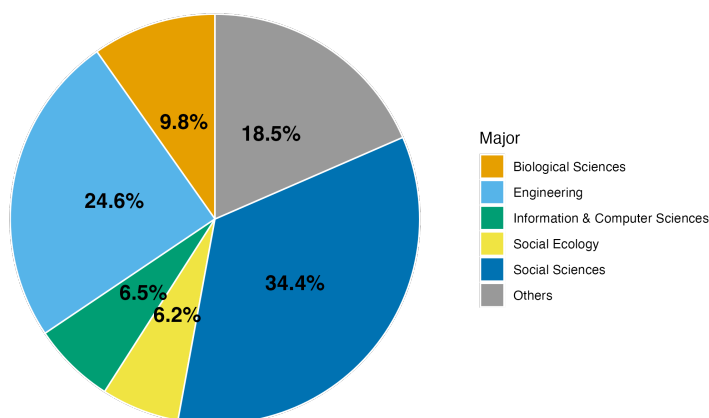


Figure 2.4: Majors

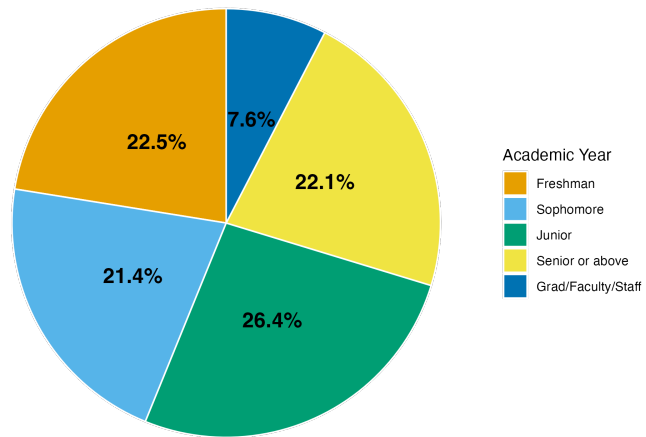


Figure 2.5: Years

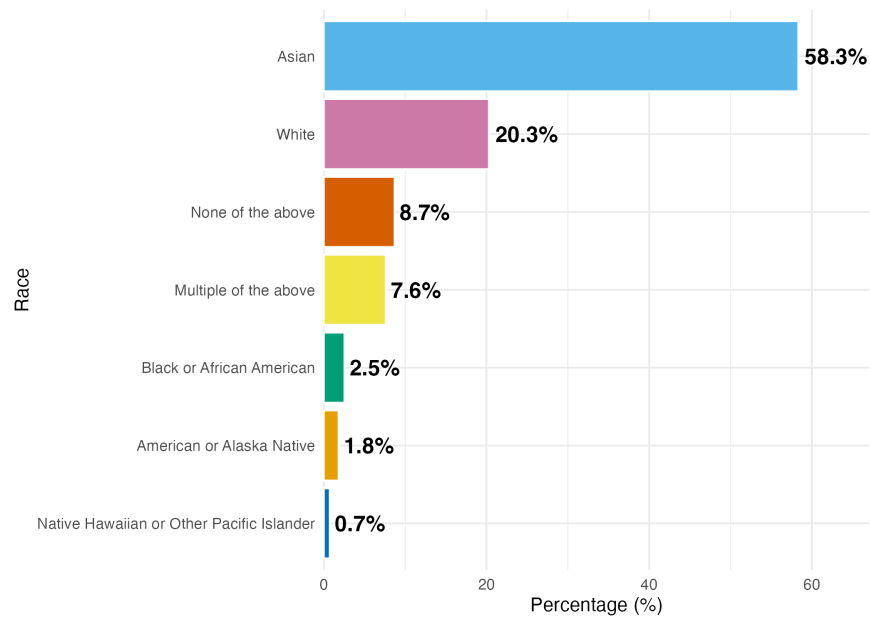


Figure 2.6: Race

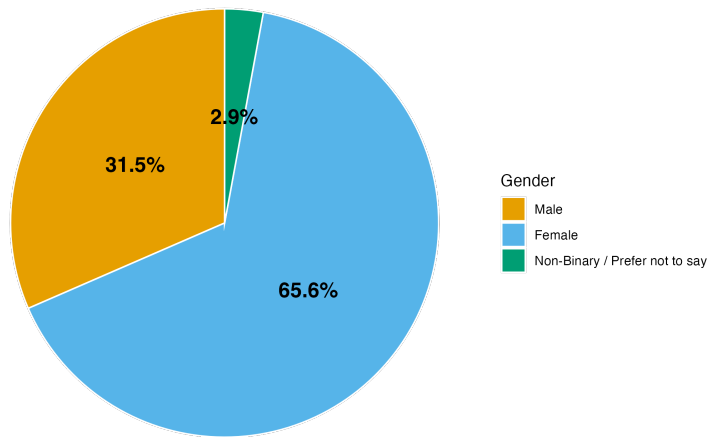


Figure 2.7: Gender

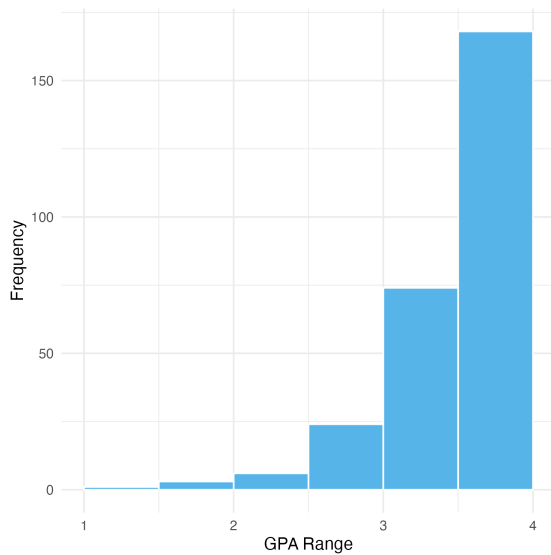


Figure 2.8: GPA

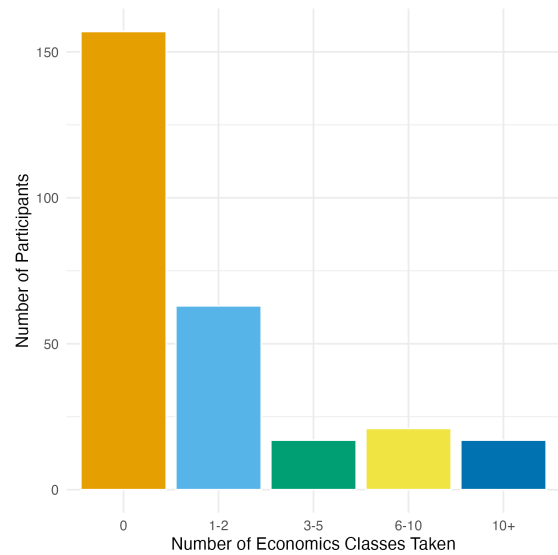


Figure 2.9: Economics courses

The average GPA for the subjects is 3.54 ($SD = 0.43$) with a median of 3.6. The average number of classes offered by the economics department taken is 2.19 courses ($SD = 4.41$ courses). 57% of the subjects have not taken any classes offered by the economics department.² Among students that have taken courses in the economics department, the average

²One observation (80) was excluded from the summary statistics, likely representing credit hours instead of individual courses. Note that the questionnaire states “Type in an estimate of the number of courses in the economics department you have taken thus far:” and hence does not capture economic-oriented courses outside of the department, such as those offered in the School of Business.

is 5.09 (SD = 5.53) and a median of 2 courses. Histograms for GPA and coursework done are provided in Figures 2.8 and 2.9.

Treatments and Risky Choice

In this portion, as well as the main analyses, I focus on the 242 subjects that survived the exclusion criteria as well as the non-attention check menus. After exclusions, there are respectively 82, 82, and 80 subjects for Treatments 1, 2, and 3.

Table 2.4 shows for each treatment and for each option, how many times and what proportion (in parentheses) it is chosen respectively in the binary “B”-menus and ternary “T”-menus. The **Cond.** row shows the proportions when only considering R and S choices.

(#)	riskier (R) option		safer (S) option		certain option (c)
	B-menu	T-menu	B-menu	T-menu	
$T1$ (82)	1057 (51.6%)	849 (41.4%)	993 (48.4%)	1063 (51.9%)	138 (6.7%)
$T2$ (82)	950 (46.3%)	835 (40.7%)	1100 (53.7%)	1089 (53.1%)	126 (6.1%)
$T3$ (80)	877 (43.9%)	826 (41.3%)	1123 (56.2%)	1072 (53.6%)	102 (5.1%)
Pool	2884 (47.3%)	2510 (41.1%)	3216 (52.7%)	3224 (52.9%)	366 (6.0%)
Cond.	47.3%	43.8%	52.9%	56.2%	—

Table 2.4: Choice Counts and Proportions by Treatment

Some preliminary observations can be made from this table. Regarding Hypothesis 2.1 (Increased Risk Aversion), the proportion of risky choices dropped from 47.3% to 41.1%. Regarding Hypotheses 2.3 (Conditional Safe Switch) the condition proportion of safe choices rose from 52.9% to 56.2%. Regarding Hypothesis 2.4 (Stickiness), the proportion of Treatment 2 subjects, ones that see the certain options first before they are removed, that chose the safe option in the B-menus (53.7%) is higher than that of Treatment 1 subjects (48.4%).

Pattern Recognition

Figure 2.10 shows how many subjects answered a certain pattern recognition problem correctly. Recall that each problem has 5 versions and is randomized. The downwards trend

shows the increase in difficulty as latter problems are harder under the categorization by Matzen et al. (2010).³ Figure 2.11 shows the distribution of the number of correct responses, which has a mean of 4.79 problems ($SD = 1.65$) and a median of 5 problems. The distribution appears relatively balanced (skewness of -0.17 and excess kurtosis of -0.55), with most subjects scoring between 3 and 6, suggesting that the task successfully captured a wide range of cognitive variance without suffering from extreme floor or ceiling effects.

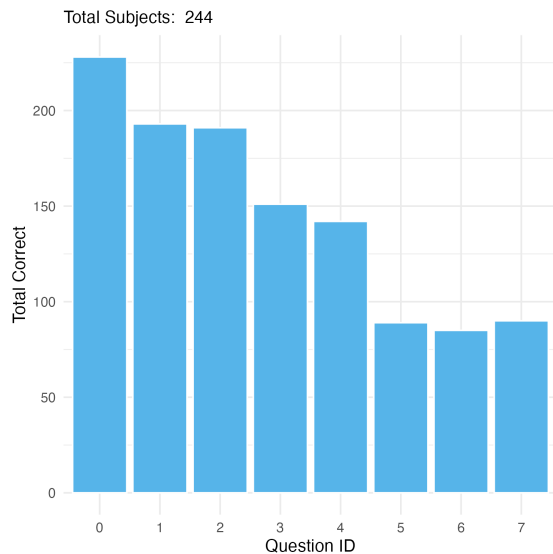


Figure 2.10: Correctness for each problem

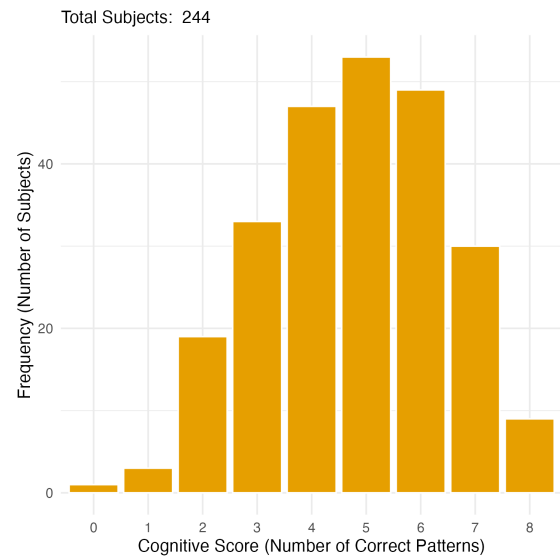


Figure 2.11: Number of correct responses

2.4.2 Hypothesis 2.1: Increased Risk Aversion

Global Analysis

To test the over-arching prediction of whether individuals systematically have a higher level of risk aversion with the introduction of a certain option, I employed a mixed-effects logistic regression of the choice of the riskier lottery R on a few covariates. To account for the inherent correlation between choices made by the same individual, standard errors are clustered at the subject level via random intercepts.

³In the original validation study, these questions were categorized by difficulty with the following correct rate: Question 0 (100%), Questions 1 and 2 (75%), Questions 3 and 4 (50%), Questions 5 and 6 (25%), and Question 7 (0%)

$$\text{logit}(P_{ij}) = \beta_0 + \beta_1 \text{three}R_{ij} + \beta_2 \text{three}S_{ij} + \beta_3 \text{certain}_{ij} + \beta_4 \text{group}_{ij} + \beta_5 \text{time}_{ij} + \beta_6 \text{calc}_{ij} + b_i \quad (2.5)$$

where $P_{ij} = \Pr(Y_{ij} = 1)$.

- Y_{ij} : whether subject i chooses the riskier lottery R in task j .
- $\text{three}R_{ij}$: whether the riskier lottery R in the task has three possible payoffs.
- $\text{three}S_{ij}$: whether the safer lottery S in the task has three possible payoffs.
- certain_{ij} : whether a certain lottery is present in the task.
- group_{ij} : which group this task falls in.
- time_{ij} : the time (in seconds) subject i spends on this task.
- calc_{ij} : whether subject i used the on-screen calculator for this task.
- $b_i \sim \mathcal{N}(0, \sigma_b^2)$: random intercept for subject i . This accounts for unobserved individual-level heterogeneity in baseline risk preferences.

From Table 2.5, the main observation is a statistically significant 31% decrease (odds-ratio of 0.69) in risky choice when a certain option is introduced. It is also note-worthy that having three possible payoffs also statistically decrease risky choice, with a 32% decrease (odds-ratio of 0.68) when R has three payoffs and 26% decrease (odds-ratio of 0.74) with S . This suggests that having more complex lotteries, more-so in the riskier one, may lead individuals to default to the safer option. This is in line with the works of Sonsino et al. (2002); Huck and Weizsäcker (1999).

When the group that the task belongs in goes up, more subjects should choose the risky

Variable	Estimate	Std. Error	z-value	Pr(> z)
Fixed Effects				
(Intercept)	−3.9506	0.1171	−33.74	< 0.001 ***
<i>R</i> with Three Payoffs	−0.3818	0.0458	−8.33	< 0.001 ***
<i>S</i> with Three Payoffs	−0.2989	0.0460	−6.50	< 0.001 ***
Certain Decoy Present	−0.3671	0.0458	−8.02	< 0.001 ***
Group of the Task	0.7055	0.0156	45.21	< 0.001 ***
Time (seconds)	0.0060	0.0018	3.38	0.0007 ***
Used Calculator?	0.4075	0.1620	2.52	0.0119 *
Random Effects				
	Variance	Std. Dev.		
Subject (Intercept)	0.9574	0.9785		
Model Fit				
	Observations	Clusters	Log-like	AIC
	11,712	244	−6082.4	12180.8

Significance codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05

Table 2.5: Mixed-Effects Logistic Regression of Risk Choice (Hypothesis 1)

option as the cutoff threshold $\bar{r}$ for switching goes up. This resulted in almost double (odds-ratio of 2.02) the odds of the risky option being chosen when the group increases by 1. Calculator usage also increases the chances of risky choice by about 50% (odds-ratio of 1.50), as using a calculator may reveal that the risk option has a higher expected value, prompting the subject to make the risky choice. Finally, time spent on a task has a negligible effect on the risky choice of the subjects.

When the regression is only done on students that took at least one course offered in the economics department (105 subjects), there are a few observations. These students are less likely to be impacted by the complexity of the lotteries (7% and 6% increase in odds-ratio on *R* and *S*). Calculator usage has a higher impact with these students (37% more likely calculator usage leads to risky choice). These observations reflect the training on computing uncertainty in economics courses. The effect of a certain option is negligible (3% less likely to choose *R*), suggesting that the behavioral response to certain options is robust across different academic

cohorts. The detailed results for this specification are provided in Table C.1 in the technical appendix for this chapter.

Set-Specific Analysis

To properly test Hypothesis 2.1, I construct the proportions of choice R in the binary menu of a set $\rho(R, \{R, S\})$ and that in the ternary menu $\rho(R, \{R, S, c\})$. Out of the 24 sets, 18 of them see a decrease. Figure 2.12 highlights the sets that did not see a decrease.⁴ It is evident that most did not have a change with the exception of one of the sets. The overall trend (dashed line) is downward sloping, implying that the 6 sets did not have a significant effect on the overall effect.

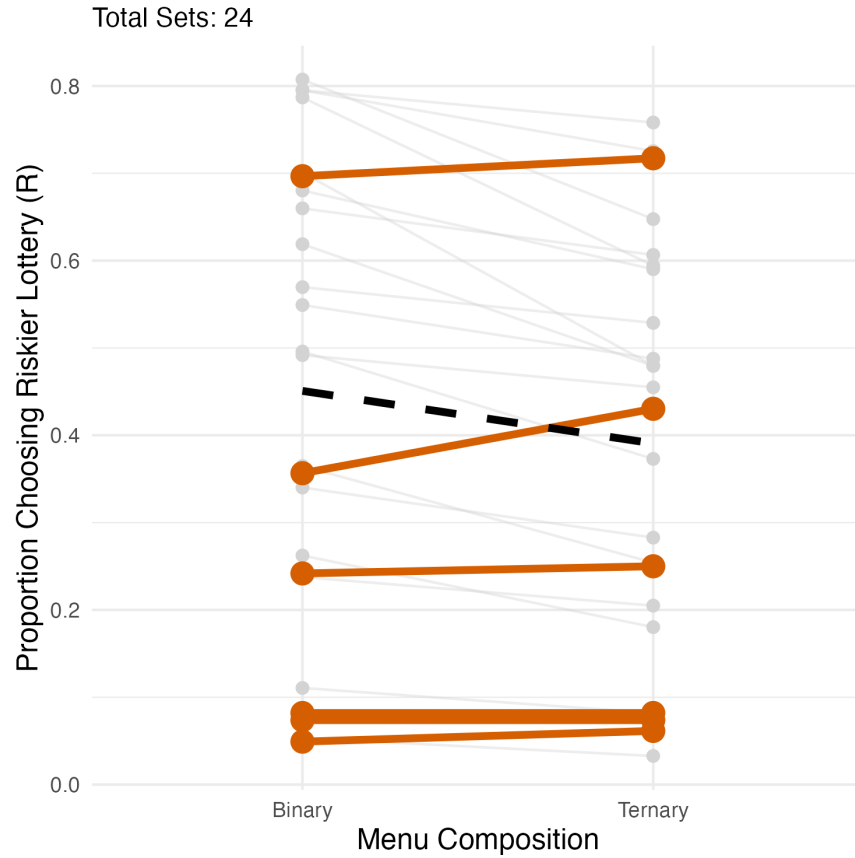


Figure 2.12: Decrease in risky choice across sets

⁴Of the six sets, two of them have zero change and four saw an increase

I conduct a Wilcoxon signed-rank test to compare $\rho(R, \{R, S\})$ and $\rho(R, \{R, S, c\})$ of the 24 sets to formally test the consistency of this shift across each set. The test confirms a highly significant directional shift ($V = 270^5$ and $p = 0.0006$). This confirms that the accumulation of the downward shifts far outweigh the minor upwards shifts.

Individual Analysis

A natural extension to this topic is whether the changes are purely random noise. Therefore, I bring the analysis to an individual level. The following table shows the joint proportion of shifts in the entire dataset in the format similar to a transition matrix.

Binary Menu (B)	Ternary Menu (T)			Total
	R (Risk)	S (Safe)	c (Certain)	
R (Risk)	29.2%	14.8%	1.1%	45.0%
S (Safe)	9.9%	40.3%	4.8%	55.0%
Total	39.1%	55.1%	5.9%	100%

Table 2.6: Joint Proportions of Choice Transitions

First note that 69.5% of the observations remained consistent. These choices are unaffected by the certain option and suggest that most subjects are not randomly selecting. Specifically, 14.8% of the total observations shifted from R to S upon the introduction of the certain decoy, whereas only 9.9% moved in the opposite direction ($S \rightarrow R$). To test for systematic menu-dependence, I apply McNemar’s test of symmetry to the choice cross-tabulation. This test evaluates whether the propensity for choice reversal is symmetric between the two conditions. The results ($\chi^2 = 56.69, p = 5.11 \times 10^{-14}$) indicate a significant directional asymmetry: the combination of $\{B : R, T : S\}$ occurs significantly more frequently than the reverse $\{B : S, T : R\}$. This confirms that the introduction of a certain option is associated with a non-random divergence toward safer choices, regardless of the order in which the menus were presented.

⁵With $n = 24$, the maximum possible V for this scenario is 300.

Next, I construct the conditional proportions based on the choice in the binary menu.

Binary Menu (B)	Ternary Menu (T) given B			Total
	R (Risk)	S (Safe)	c (Certain)	
R (Risk)	64.8%	32.8%	2.5%	100%
S (Safe)	18.0%	73.4%	8.7%	100%

Table 2.7: Conditional Choice Transitions (Row Percentages)

This magnifies the effects. An agent that would choose the risky option in the binary menu will switch with a 32.8% probability, while one that would choose the safer option in the binary menu will switch with a 18% probability.

2.4.3 Hypothesis 2.2: Violation of Monotonicity

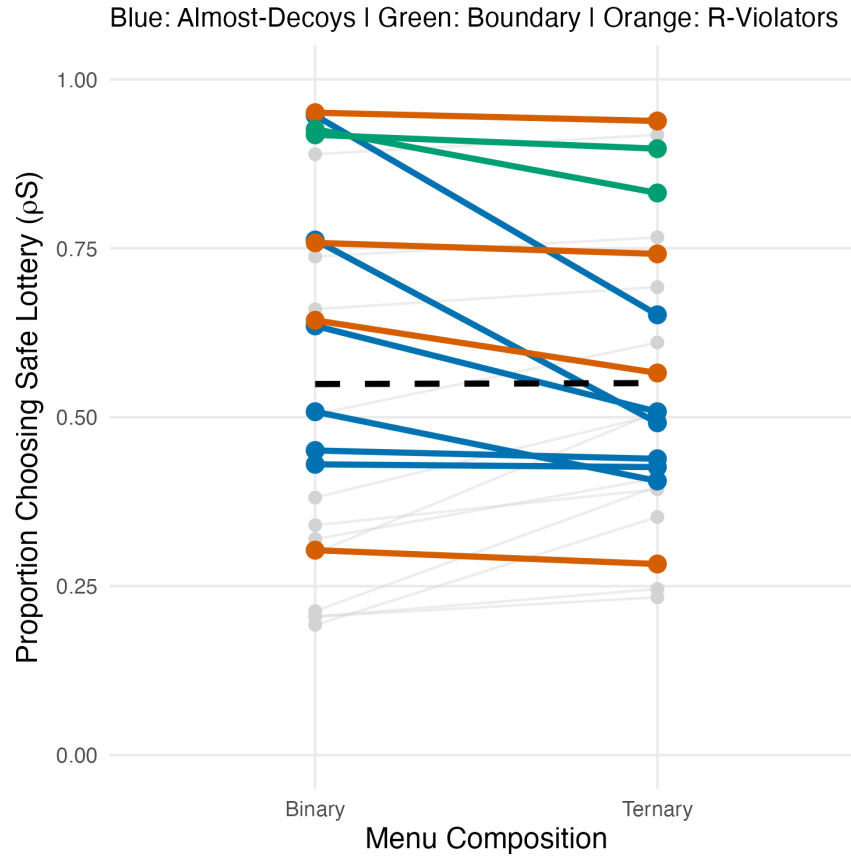


Figure 2.13: Increase in safe choice across sets

To see violations of monotonicity, I move to the analysis on changes in the proportions of choosing S : $\rho(S, \{R, S\})$ versus $\rho(S, \{R, S, c\})$. Before considering whether c is a decoy, 12 out of 24 sets saw an increase.

Figure 2.13 focuses on the ten sets that did not violate monotonicity for S . Six of them (blue) have a certain option that is an “almost” decoy, which means that it can still be attractive to particularly risk averse individuals. The four orange sets are the ones that saw an upwards shift in the R choice, which still saw a violation of monotonicity via $\rho(R, \{R, S\}) < \rho(R, \{R, S, c\})$. The two green sets correspond to the two sets that did not see any change in R choice. The orange and green sets have flatter slopes compared to the blue ones, which implies that they are boundary cases.

Out of the 24 sets, monotonicity is violated two-thirds of the time (12 from S and another 4 from R , 16 sets total). This supports my claim that the random utility model, which requires monotonicity as a core axiom, is insufficient to describe scenarios like these.

Certain Decoys

To show Hypothesis 2.2, I first restrict the dataset to only the sets where c acts as a decoy (dominated either by S or R) and conduct a Wilcoxon signed-rank test comparing $\rho(S, \{R, S\})$ and $\rho(S, \{R, S, c\})$ of the 18 sets. The test confirms a significant directional shift ($V = 137$, $p = 0.013$). This confirms that violations of monotonicity is prevalent.

2.4.4 Hypothesis 2.3: Conditional Safe Switch

When a similar Wilcoxon signed-rank test is conducted on all 24 sets, the results are not statistically significant ($p = 0.30$). It proves that the “almost” decoys are effective at drawing share away from S that the monotonicity violations in the other 12 sets are eclipsed. This also explains the flat dashed line in Figure 2.13 as it suggest the certain option does not have an overall effect on the choice of S .

To include the sets with “almost” decoys, I first compute the proportion of the population that chooses the safer lottery conditional on not choosing the certain option as seen in the right hand side of Hypothesis 2.3. This effectively ignores the effects of the “almost” decoy. I then compare it with $\rho(S, \{R, S\})$.

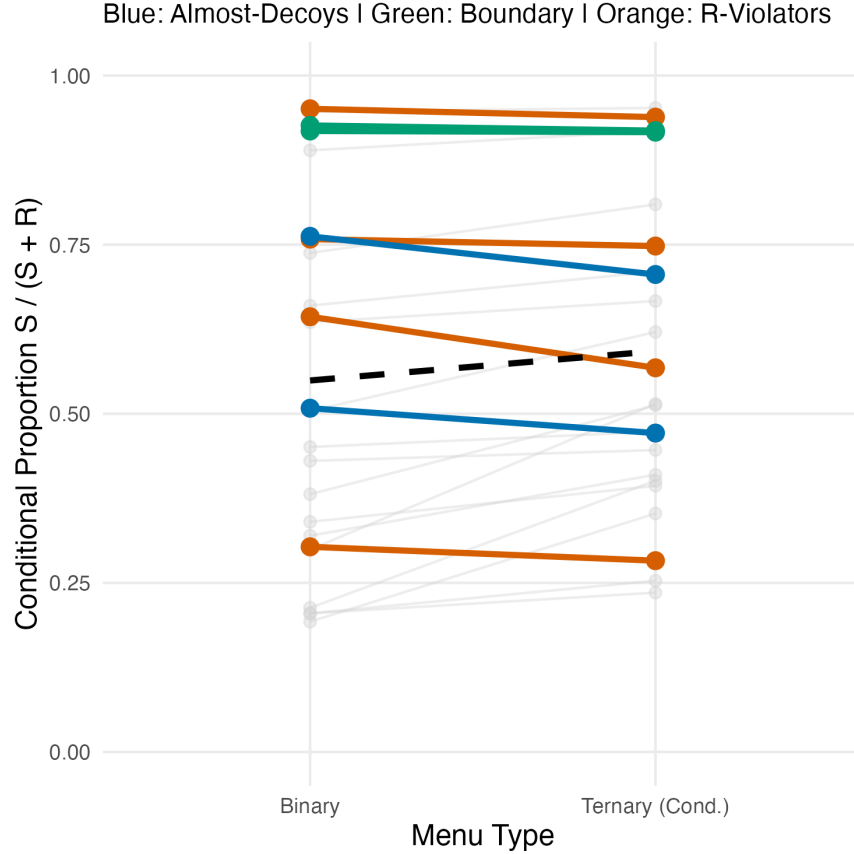


Figure 2.14: Considering conditional proportions

As shown in Figure 2.14, there are still eight sets that does not violate conditional monotonicity. Two of them (blue) are the sets with “almost” decoys. Four (orange) are the ones that saw an upwards shift in the R choice and the green two ones are a boundary case, also being the sets closest to violating conditional monotonicity. It can also be observed that none of these decreases are large in magnitude, which contributes to the dashed line pointing upwards instead of flat like in Figure 2.13.

I conduct one final Wilcoxon signed-rank test comparing $\rho(S, \{R, S\})$ and

$$\frac{\rho(S, \{R, S, c\})}{\rho(S, \{R, S, c\}) + \rho(R, \{R, S, c\})}$$

of all 24 sets. The test confirms a highly significant directional shift ($V = 234$ and $p = 0.0075$). This confirms that the switches from R to S outweighs the shifts from S to c from the “almost” decoys.

2.4.5 Hypothesis 2.4: Stickiness

If the certain decoys are sticky, then subjects may maintain a higher level of risk aversion relative to before being exposed to the decoy, even after it is removed. To test whether the decoy effect is sticky, I turn to the three different treatments.

Treatment 1 versus Treatment 2

Treatment 1 subjects start with only binary menus and move on to ternary ones, while Treatment 2 subjects start with ternary menus and move on to binary menus. If stickiness does exist, then Treatment 2 subjects should pick the riskier option with a lower probability in the binary menus than the Treatment 1 subjects. To test this, I conduct a Wilcoxon rank-sum by grouping observations at a subject level. With a $W = 3985$ and $p = 0.02$, there is evidence to suggest that there is a significantly lower proportion of risky choices in the binary rounds of Treatment 2 compared to Treatment 1.

Descriptive statistics confirm the direction of this effect. Subjects in Treatment 1 (Binary-First) exhibited a baseline risky-choice proportion of 51.6%, indicating a near-equal preference between the safe and risky lotteries in the absence of decoys. However, subjects in Treatment 2 (Ternary-First) opted for the risky lottery in only 46.3% of binary trials. This decrease suggests that the initial exposure to the certain decoy “shifted” the subjects’ internal risk-safety calibration, leading to a persistent preference for the safe lottery even after

the certain option was removed.

Treatment 3

Treatment 3 subjects face alternating menus. This allows for discussion of whether the decoy effect can build up over time in increasing risk aversion. I run a mixed-effects (at the subject and version level) logistic regression of odds ratio of choosing R on the binary menu rounds.

Variable	Estimate	Std. Error	z -value	$\Pr(> z)$
Fixed Effects				
(Intercept)	−0.4403	0.3711	−1.19	0.2354
round	−0.0071	0.0041	−1.74	0.0811 .
Random Effects				
	Variance	Std. Dev.		
Subject Intercept	1.0790	1.0390		
Set Intercept	2.6370	1.6240		
Model Fit				
	Observations	Clusters (S/V)	Log-likelihood	AIC
	1,920	80 / 24	−974.5	1957.0
Significance codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1				

Table 2.8: Mixed-Effects Regression with Subject and Set Controls on T3

Table 2.8 reveals a marginally significant trend toward increased risk aversion as the experiment progresses ($p = 0.081$). This suggests that while the certain decoy induces an immediate shift in the preference baseline, there is a secondary, albeit negligible (odds-ratio of 99.3%) cumulative effect of exposure. Given the magnitude of this effect relative to the primary treatment variables, the findings suggest that baseline risk preferences remain largely stable across the duration of the experiment.

Recovery

While Treatment 3 showed no compounding effect on repeated exposure, I explore whether Treatment 2 (Ternary-First, or Block) will retain the level of stickiness by running the same model but using Treatment 2 data.

Variable	Estimate	Std. Error	z -value	$\Pr(> z)$
Fixed Effects				
(Intercept)	−1.5470	0.5073	−3.05	0.0023 **
Round	0.0279	0.0083	3.37	< 0.001 ***
Random Effects				
	Variance	Std. Dev.		
Subject Intercept	1.709	1.307		
Version Intercept	3.155	1.776		
Model Fit				
	Observations	Clusters (S/V)	Log-like	AIC
	1,968	82 / 24	−973.0	1953.9

Significance codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05

Table 2.9: Mixed-Effects Regression on Treatment 2: Evidence of Risk Recovery

The round estimate is now significant. This shows that for Treatment 2 subjects, there is a recovery of 2.8% (odds-ratio of 0.0279) in the chances of choosing R . Treatment 1 (Binary-First, or Control) should show no effect in its binary rounds, as it has not been exposed to the decoy. The results of this robustness check is included in the technical appendix (Table C.3). The scatterplot in 2.15 shows the trends of the three treatments.

2.4.6 Robustness Check to Hypothesis 2.1

To provide a definitive test of the increase in risk aversion from Hypothesis 2.1, I estimate a global mixed-effects model on the full dataset (including the excluded subjects). This model (Table C.2 in the technical appendix) accounts for the presence of the all covariates in Estimation (2.5), the passage of time ($round_{ij}$), and treatment-level effects ($treat$), while specifically controlling for the inherent variance of each stimulus ($version_{ij}$) and individual subject heterogeneity.

Even under this highly conservative specification, the presence of the certain option remains a powerful and highly significant predictor of reduced risk-taking ($\beta = -0.3958, odds = 0.67, p < 0.001$). The robustness of this coefficient across all models confirms that the

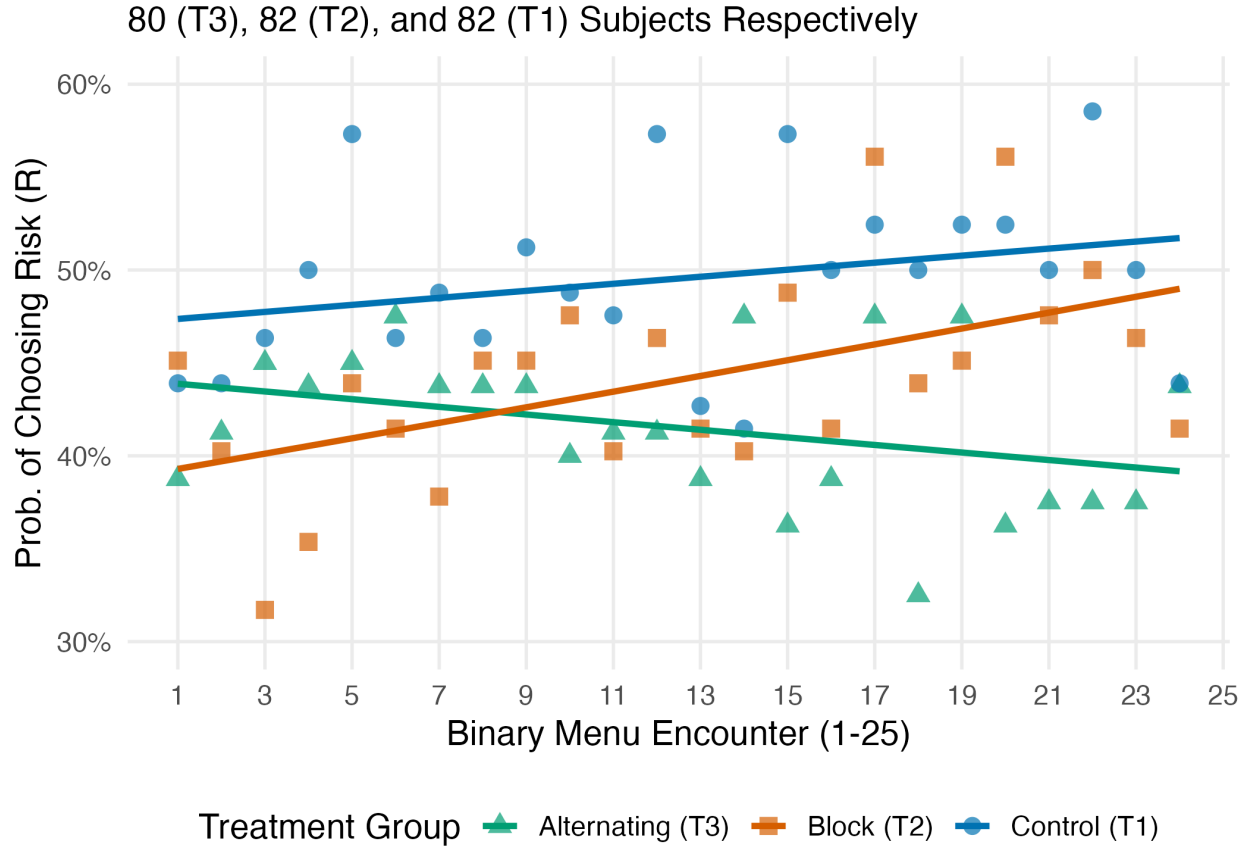


Figure 2.15: Trends for treatments in B-Menus

effect is a systemic behavioral shift rather than an artifact of specific lottery parameters or subject-level outliers.

Note that the three-payoff indicators are no longer significant. This is expected, as the variance previously attributed to complexity is now captured by the set-level random effects, which account for the inherent characteristics of each of the 24 unique versions.

2.4.7 Hypothesis 2.5: Cognition

To investigate whether higher cognitive ability serves as a buffer against the decoy effect, I regressed the individual “switch rate” (the frequency with which a subject abandoned a

risky choice for a safe one upon the introduction of a certain decoy)

$$switchRate = \frac{\text{number of times switching from } R \text{ in B to } S \text{ in } T}{\text{number of times picked } R \text{ in B minus}}. \quad (2.6)$$

on their cognitive performance score (number of correct responses from the pattern recognition problems).

Variable	Estimate	Std. Error	<i>t</i> -value	Pr(> <i>t</i>)
Fixed Effects				
(Intercept)	0.3982	0.0500	7.96	< 0.001 ***
Cognitive Score	−0.0082	0.0099	−0.83	0.4083
Model Fit				
Observations	<i>R</i> ²	Adj. <i>R</i> ²	F-stat	
244	0.0028	−0.0013	0.686	

Significance codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05

Table 2.10: Linear Regression of Individual Switch Rates on Cognitive Performance

As shown in Table 2.10, the coefficient for cognitive score is negligible and statistically non-significant ($\beta = -0.0082, p = 0.41$). The R^2 of nearly zero (0.0028) indicates that cognitive ability explains almost none of the variation in decoy susceptibility. These results support the view that the certainty effect is a robust behavioral phenomenon that affects subjects uniformly, rather than a manifestation of bounded rationality that diminishes with higher cognitive skill.

2.5 Discussion

2.5.1 Heterogeneous Susceptibility

In Section 2.4.7, a null effect is observed on the relationship between cognitive abilities and how susceptible an agent is to the decoy effect. First, observe in Figure 2.16 the distribution of switch rates. A clear level of heterogeneity can be observed (interquartile range of 0.32).

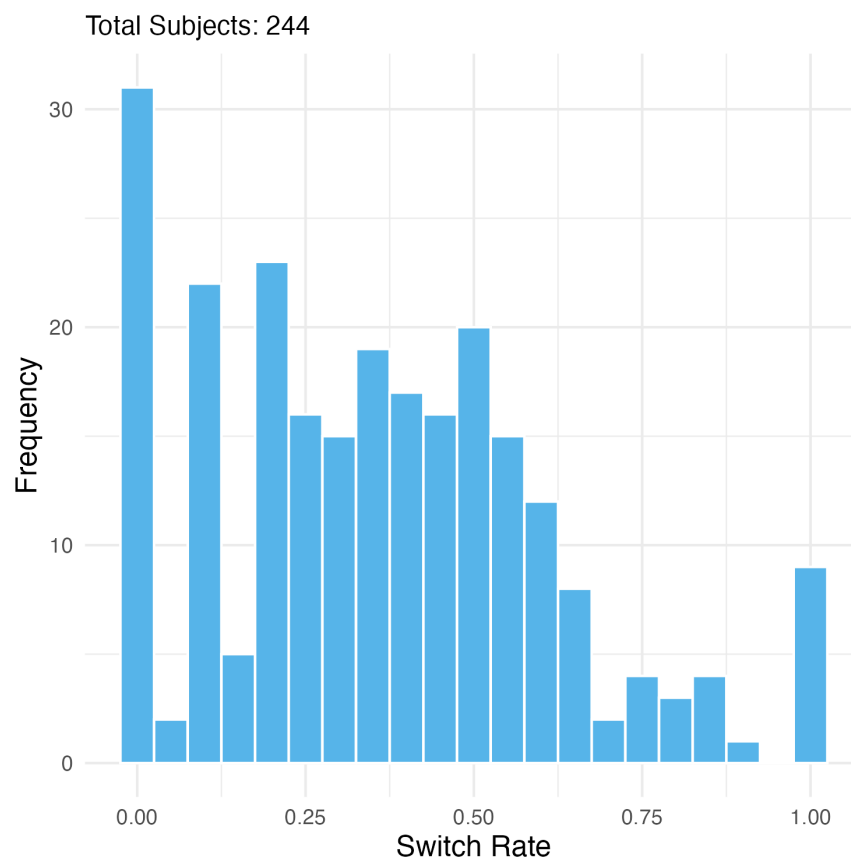


Figure 2.16: Distribution of switch rates

Treatments and Exposure

I now explore other possible explanations behind this heterogeneity. Table 2.11 shows great statistical significance in the treatments. The results here suggest that Treatment 3 individuals are least likely to fall victim to the decoy effect, and can be attributed to the lower rate of choosing R in the binary menus to begin with (see Table 2.4).

Baseline Risk and Thresholds

The number of times the subject chose R in the binary menus is a proxy for the baseline risk aversion level, with a higher number associated with less risk-averse (more risk-loving). Its negative effect on “susceptibility” can be explained as such: when an agent is risk-loving, the increase in risk aversion from the certain decoy may not be enough to induce them to

Variable	Estimate	Std. Error	<i>t</i> -value	Pr(> <i>t</i>)
Fixed Effects				
(Intercept)	0.8432	0.0582	14.49	< 0.001 ***
Economics Background	−0.0068	0.0034	−1.99	0.0473 *
Time Spent (minutes)	−0.0047	0.0015	−3.21	0.0015 **
Treatment 2 (Ternary-First)	−0.0847	0.0350	−2.42	0.0163 *
Treatment 3 (Alternating)	−0.1457	0.0350	−4.17	< 0.001 ***
Binary Risk Baseline	−0.0323	0.0046	−7.06	< 0.001 ***
Model Fit				
	Observations	R^2	Adj. R^2	F-stat
	244	0.2731	0.2578	17.88 ***

Binary Risk Baseline *is the denominator in (2.6)*.

Significance codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05

Table 2.11: Linear Regression of Switch Rates on Other Covariates

switch from R to S . In other words, they do not surpass the switching threshold ($\bar{r}$). While these result show that risk-loving agents are less likely to be susceptible to the decoy effect in my experiment, it does not directly show that they are less likely to have an increase in risk aversion. To explore this further, more granular calibration is needed and such experiments may be more cumbersome to perform.

Time Spent and System II Thinking

Time spent on the experiment is also shown to have a negative and statistically significant effect on the switch rate. When an agent spends more time on the experiment, it is more likely for them to demonstrate System II Thinking (Kahneman, 2011) where an agent exert effort in reasoning. Therefore, these agents would be more likely to notice the decoy effect. However, the effects are below moderate at about 0.5% decrease per extra minute spent.

Economics Background and Other Demographics

How many courses offered by the economics department can be a measure of a subject’s familiarity with lotteries and uncertainty. Therefore, it is reasonable to see a negative and

somewhat significant effect, as these subjects are more likely to notice the decoy effect.

A full regression with variables insignificant here is included in the technical appendix (Table C.4). GPA and quiz attempts do not have an effect, supporting the null results in the previous section where susceptibility to the decoy effect is universal and not affected by an individual's abilities. No other demographic predictor offered a significant effect other than a few dummy variables with very low counts (such as select majors).

2.5.2 Insurance Choice

Insurance companies can design policies with various terms and conditions to optimize producer profits (Cohen and Einav, 2007). Firms tend to prefer high-premium-low-deductible options (Sydnor, 2010), which aligns better with risk averse agents. To capture more profit from relatively risk-seeking individuals, firms can leverage choice patterns like the decoy effect to introduce specific contract features, such as zero-deductible “certain” options but with unreasonably high premia, to steer consumers towards more profitable products.

For example, assume an individual has wealth \$10,000 and faces a one-time risk with probability 1% at a loss of all \$10,000. Let two policies be designed as such:

P_1 : Premium of \$200 with deductible of \$100.

P_2 : Premium of \$150 with deductible of \$1,000.

Under P_1 , the company's expected profit is

$$200 - 0.01 \times (10000 - 100) = 101$$

while for P_2 it is

$$150 - 0.01 \times (10000 - 1000) = 60.$$

Obviously, the insurance company would prefer the individual to purchase the first policy. When faced with these two policies (a menu of $\{P_1, P_2\}$), the individual has a CRRA level of r . She will choose the first policy if and only if

$$\begin{aligned} 0.99 \times (10000 - 200)^{1-r} + 0.01 \times (10000 - 200 - 100)^{1-r} \\ \geq 0.99 \times (10000 - 150)^{1-r} + 0.01 \times (10000 - 150 - 1000)^{1-r} \\ \implies r \geq 0.52. \end{aligned}$$

This means that if her level of risk aversion is low, she will choose P_2 instead.

Now assume the insurance firm adds a third choice: P_c : Premium of \$1,000 with no deductible. This is considered a no-risk option as regardless of risk level, the individual is left with wealth of $\$10,000 - \$1,000 = \$9,000$. However, it is actuarially unfavorable, and its utility is lower across a wide range of risk aversion level. Hence, it can be used as a certain decoy to increase the individual's risk aversion level when faced with the menu $\{P_1, P_2, P_c\}$, prompting her to switch from choosing P_2 to the more producer-profitable P_1 .

2.5.3 Tax Evasion

Using withholding as a mechanism to decrease tax evasion has been celebrated in literature Kleven et al. (2011); Slemrod (2007); Lederman (2010). I show here that “certainty-as-a-decoy” can be a possible explanation to its success.

First, assume an individual that has income \$100,000 and owes \$10,000 in taxes. Let the rate of auditing be 0.2 and fine for evasion be \$20,000 on top of repayment. Some individuals are tempted to evade tax⁶ if

$$0.8 \times (100,000)^{1-r} + 0.2(100,000 - 10,000 - 20,000)^{1-r} \geq 90,000^{1-r} \implies r \geq 0.385.$$

⁶For simplicity, assume an all-or-nothing tax evasion.

Now assume that the government introduces a tax withholding option of 14%, and filers will receive a refund for over-withheld amounts. This should be considered a less optimal strategy than no withholding due to delayed income and discounting.⁷ Therefore, withholding can be considered the certain decoy to persuade tax evaders to be more risk averse, and hence choose to properly file.

2.5.4 Alternative Theories

While the decoy effect is an appropriate candidate for the behavioral shift from R to S , I explore a few other potential candidates, and argue why the decoy effect, along with the increase in risk aversion, is the most reasonable among these explanations.

Extremeness Aversion

Simonson (1989) explores the scenario where an agent avoids picking extreme options. In my experiment, S is considered the “middle” option in $\{R, S, c\}$. However, this would not explain why there is heterogeneity in susceptibility driven by baseline risk levels. What’s more, Chen et al. (2024) found that agents do not shift their preferences when an extremely risky option is presented supports my argument that extremeness aversion may be less at play here.

Probability Weighting

Probability weighting (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992) dictates that agents would distort probabilities close to extremes, and may overvalue certain options. However, while a minor proportion of subjects to choose certain decoys (extreme cases were excluded from the dataset), probability weighting would imply that behavioral shifts would go to the certain options instead. This is not widely observed as the majority of the shifts

⁷Tax payers are still observed to opt to withhold as a method of self-control (Thaler, 1994) or due to stickiness (Jones, 2012).

(see Table 2.6) go from R to S instead. Therefore, probability weighting cannot explain the sudden attractiveness of S when c is included.

Cognitive Load and Fatigue

When a menu is more complex, agents prefer safer options (Benjamin et al., 2013). It is true that in Table 2.5, significance is seen when menus include three-payoff lotteries. However, this only implies that subjects systematically choose S more often in these cases, but does not explain the switch from R to S . One may also argue that when there are more items in the menu (T menus), cognitive load is increased and agents fall back on a safer option. However, time spent, which can be associated with more cognitive load, saw a lower likelihood of switching to the safe option, suggesting that a heavier cognitive load did not increase risk aversion as much.

Regret Theory

Agents can “anticipate” regret on future loss (Loomes and Sugden, 1982; Bell, 1982), and hence make choices such that it could be minimize. For my experiment, this would imply that when presented with the certain option, it enhances the future regret, making the potential low outcomes from the risky lottery feel more painful. This model can be complementary to the decoy effect, and be used as an explanation of the switch from R to S . If the choice framework for agents does not follow expected utility, then regret theory can be a strong candidate for the micro-foundation of choice.

2.6 Conclusion

In this chapter, I explored the effects of the introduction of a no-risk-low-reward option. Using my experimental design, I found evidence of an increase of risk aversion and the presence of a decoy effect. What’s more, I showed that monotonicity, a core requirement to

the standard random utility model, is violated in two-thirds of the scenarios.

I also found evidence that the decoy effect is “sticky”, in that agents do not return to their baseline risk aversion level when the certain option is removed. Null results suggest that varying levels of susceptibility to the decoy effect does not arise from cognitive or academic performance. However, small effects are found from time spent and economic backgrounds.

I finish the paper by proposing applications and alternative explanations. Insurance companies can introduce an actuarially unfavorable no-deductible option to induce higher levels of risk aversion in consumers. Governments can propose withholding rate to dissuade tax evasion. I explain why the decoy effect, instead of extremeness aversion and other alternatives, would be the dominant reason behind this behavior.

Chapter 3

Algorithmic Identification of Heterogeneity in Incomplete Choice

3.1 Introduction

Various utility representations can be *verified* in a choice dataset ($\mathcal{D}$) via revealed preference theory. In other words, models in this theory provide conditions (or *axioms*) that are necessary and sufficient. However, finding practical (i.e., *computationally efficient*) algorithms that can *identify* such representations usually requires more than the theoretical verification task, especially when data is *incomplete*. For example, the standard utility maximization model with no extra assumptions on the functional form can be verified via the generalized axiom of revealed preference (GARP) (Suzumura, 1976), and is solvable in $O(|X|^2)$ quadratic time (Varian, 1982) where X is the *consumption set*. It can be identified via Tarjan’s strongly connected components algorithm (Tarjan, 1972) and is solvable in $O(|X| + |\Sigma|)$ linear time where $|\Sigma|$ is the number of observations.

More broadly, identification algorithms depend on structural assumptions about the utility

function, and can become more sophisticated when datasets are generated by *heterogeneous* types (Liang, 2019). Even in finite frameworks ($|X| < \infty$), where brute-force methods must eventually provide a solution, its identification algorithms can be prohibitively complex for application. Examples include non-polynomial algorithms such as the simplex method in linear programming (Klee and Minty, 1972). On the other hand, intelligent guess-and-verify approaches can work reasonably fast in practical settings, although they do not always have theoretically guaranteed solutions.

In this chapter, I provide identification algorithms for several models of heterogeneous types in choice functions. The common approach to model heterogeneity is making use of the random utility model (RUM) (Block and Marschak, 1960; Falmagne, 1978) as discussed in the previous chapters, where the type of an agent is a random variable. It is well known, however, that the identification of a RUM is complex (Kitamura and Stoye, 2018; Boxall and Adamowicz, 2002; Liang, 2019) with unique identification even more challenging (Turansick, 2022). In Chapter 1, we showed that when the invariance assumption is relaxed, uniqueness can be achieved. However, identification is made difficult with each menu having its own distribution.

Instead, I focus on a simpler framework with deterministic choice functions where heterogeneity is modeled using the *multi-utility* (MU) model from Aizerman and Malishevski (1981). I discuss the verification of the MU model and identification with various constructions where data is incomplete. These include the general (i.e., *non-parametric*) case, expected utility (EU), and quasilinear utility.

I argue that a natural refinement of this model for the EU framework cannot be identified in polynomial time. This negative example illustrates that even innocuous assumptions like EU can make identification intractable.

On a more positive note, the general case and *quasilinear* frameworks have more efficient

algorithms. In fact, a multi-utility model with quasilinear utilities, or *multiple quasilinear utility* (MQU) model can be verified in polynomial time. My algorithm also identifies the maximal set of types that can be fit into the dataset. Moreover, it is possible to *purify* noisy data. I can construct a *maximal* part of $\mathcal{D}$ that is consistent with MQU. My purification objective is similar to the results by Apesteguia and Ballester (2021). However, my algorithm is both conceptually and technically distinct because I fit an MU model for choice functions instead of an RUM for stochastic choice rules.

Finally, I study the identification of *constant threshold representations* (CTR) (Jamison and Lau, 1973). These models instead accommodate *individual imperception*, but can still be interpreted as a special case of the MU model. Here, my findings are mixed: while my algorithm works reasonably well in simulated data, there is no theoretical guarantee due to the multiplicity of maximizers perceived by the individual.

3.1.1 Related Literature

Existing literature mainly focuses on axiomatic approaches to heterogeneity. It includes frameworks with or without (Eden, 2020) applied contexts, such as labor (Veneziani and Yoshihara, 2015). These papers start on the assumption that a choice dataset already adheres to some heterogeneous utility model, and present important attributes and properties of the data. I will drop this assumption to instead add a verification step.

Much contemporary empirical work uses experiments in behavioral contexts such as altruism (Visser and Roelofs, 2011) and ambiguity (Stahl, 2014). These papers include discussion in dispersion of types for different demographics or environments. Unlike these papers, I strive to develop an algorithm that can find levels of dispersion from real-world data instead of data from labs and experiments.

Predominant views on heterogeneity either does not use a specific functional form (the non-

parametric case) (Hoderlein and Stoye, 2014; Matzkin, 2007; Cosaert and Demuynck, 2018) or follow a random utility framework (Block and Marschak, 1960; McFadden, 1974; Boxall and Adamowicz, 2002). As I mentioned previously, identification has been found to be difficult in heterogeneous frameworks (Boxall and Adamowicz, 2002; Liang, 2019) and more so when the choice dataset is incomplete (Crawford and De Rock, 2014). Multi-utility models have not been as thoroughly discussed and there is hope to use such models to combat the complexities in identification.

The concept of imperceptible differences is not new. Weber (1834) first proposed *just-noticeable differences* (JND) in the 19th Century. Much of such research in different fields take an experimental approach to measure JND (Wandell, 1982; Ragain Jr and Johnston, 2001). In economics, Luce (1956) created the *semiorder* using an axiomatic approach to account for JND. Identifying semiorder preferences in a binary relation setup is simple and quick (Pirlot, 1990).

Threshold representation, first explored by Jamison and Lau (1973), is an extension to the semiorder model to accommodate menus with more than two options. The basic form of threshold representation dictates that a decision maker will accept alternatives that provide a utility of at most one util below that of the true-best option. This is referred to as a *constant threshold representation* (CTR). Modifications can also be made to the threshold representation model (Frick, 2016).

Semiorders and threshold representations have contemporary applications in other fields of economics. Balart (2021) introduced semiorder preferences in a product-differentiation model (Hotelling, 1929) where consumers are unable to perfectly differentiate product features. This allows firms to design and price products based on this information in order to maximize profits.

3.2 Non-Parametric Multi-Utility

Definitions and terminology in this section follow closely to Chambers and Echenique (2016a).

3.2.1 Primitives and Notation

Let X be a finite *consumption set*. Each element $x \in X$ is called an *option* (or *alternative/item*). A non-empty subset of the consumption space $A \subseteq X$ is called a *menu*. A menu (or a *budget set*, particularly when there are prices involved) lists options that are *feasible* (or *available*) to the decision maker. $\mathcal{M} \equiv \mathcal{P}(X) \setminus \emptyset$ is the set of all menus, which is equivalent to the power set of X minus the empty set.

Let $\Sigma \subseteq \mathcal{M}$ denote the set of all *observable menus*. It includes all menus where a decision maker is observed to make a choice in the dataset $\mathcal{D}$. If $\Sigma = \mathcal{M}$, then the choice dataset is complete. Define $\phi : \Sigma \rightarrow \mathcal{M}$ to be a choice function (or *choice rule*) that selects a non-empty subset of each menu: $\emptyset \subsetneq \phi(A) \subseteq A$. It can be interpreted as options found *acceptable* by the decision maker.¹ This allows us to summarize the choice dataset: $\mathcal{D} = \{(A, \phi(A)) | A \in \Sigma\}$ as a list of each observable menu and its corresponding acceptable options.

The criterion for acceptability can vary depending on the context. In the framework of *utility maximization*, an option is acceptable in a menu if and only if a utility function $u : X \rightarrow \mathbb{R}$ can be found such that for all observable menus $A \in \Sigma$:

$$\phi(A) = \arg \max_{x \in A} u(x). \quad (3.1)$$

If opting out is possible, an *outside option* $0 \in X$ is included to allow for individual rationality. An option not accepted by a choice function in a menu is *rejected*. I denote the set of rejected options for a menu A by $\rho(A) \equiv A \setminus \phi(A)$.

¹Colloquially, options *chosen* by the decision maker. I avoid using this terminology as a way to clarify that a decision maker may find multiple items acceptable but can only choose one.

Revealed preference theory, first conceptualized by Samuelson (1938) and later refined by Sen (1971), allows preference relations in the settings of a choice function. A pair of revealed preference relations ($\succeq^*$, $\succ^*$) is defined as follows. If there exists an observable menu A where x is *feasible* and also *acceptable*, then x is *revealed* to be *weakly preferred* to all alternatives in A :

$$\exists A[\{x, y\} \subseteq A \in \Sigma \wedge x \in \phi(A)] \iff x \succeq^* y \quad (3.2)$$

If there exists an observable menu B where x is *feasible* and *acceptable*, and y is *feasible* but *rejected*, then x is *revealed* to be *strictly preferred* to y :

$$\exists B[\{x, y\} \subseteq B \in \Sigma \wedge x \in \phi(B) \wedge y \in \rho(B)] \iff x \succ^* y \quad (3.3)$$

3.2.2 Verification

The standard utility maximization does not accommodate heterogeneity, which can be observed across subjects as well as in a single agent (Kopylov, 2022). Choice functions that include heterogeneous choices of a *family* of k types $\Theta = \{u_i : X \rightarrow \mathbb{R}\}_{i=1}^k$ can be instead modeled by the MU model proposed by Aizerman and Malishevski (1981). Then, every observed $\phi(A)$ consists of *all options that are maximizers* for at least one type u_i , i.e.:

$$\phi(A) = \bigcup_{i=1}^k \arg \max_{x \in A} u_i(x). \quad (3.4)$$

Each *type* $i \in \{1, \dots, k\}$ can be interpreted as an individual maximizing their own utility represented by their respective utility function $u_i \in \Theta$. For each observable menu $A \in \Sigma$, the choice function will “ask” each type of decision maker what options in A are acceptable to them, and output the union of their responses.

Unlike RUM and stochastic choice, an MU model does not assign weights or probabilities

to different types as it only takes into account the union across maximizers. Hence, this simplicity can help combat some complexities in identifying heterogeneity.

Aizerman and Malishevski (1981) found that a choice function is represented by a multi-utility model if and only if for any two observable menus $\{A, B\} \subseteq \Sigma$, the *path-independence* (originally *independence of path*) axiom by Plott (1973) is satisfied:

$$\phi(A \cup B) = \phi(\phi(A) \cup B). \quad (3.5)$$

The equality above describes path-independence as the invariance between finding acceptable options from (i) the union of two menus, and (ii) one menu first and then among those accepted earlier and options of the other menu.

Path-independence is also equivalent to the conjunction of two simpler axioms: (a) *Sen's α* (Sen, 1971),² which states that *an acceptable option in a menu should still be acceptable in any subset menu whenever feasible*:

$$\forall x \in B \subseteq A [x \in \phi(A) \implies x \in \phi(B)] \quad (3.6)$$

and (b) *Reduction* (Chernoff, 1954),³ which states that *if a menu contains all acceptable options from some superset menu, then the choice function will return the same set of acceptable options*:

$$\phi(A) \subseteq B \subseteq A \implies \phi(A) = \phi(B). \quad (3.7)$$

Both properties can be checked for arbitrary orders in polynomial time (see pseudocodes in D.2.1). Therefore, to verify that a choice dataset $\mathcal{D}$ has an MU representation takes polynomial time when there is no parametric form for the utility functions. Note that since

²While commonly attributed to Sen, it was first formally introduced by Chernoff (1954) as Postulate 4.

³Postulate 5* by Chernoff. Also called *independence of rejecting the outcast variants* by Aizerman and Malishevski (1981).

completeness is not necessary⁴ for either Sen’s α or Reduction, $\mathcal{D}$ can be incomplete for verification.

Theorem 3.1. *It takes a dataset $O(|\Sigma|^4)$ quartic time to verify if it satisfies the axioms of a non-parametric multi-utility model.*

MU models have an interesting property:

Axiom 3.1 (Conditional Dominance, CD).

$$A \neq \bigcup_{B \subsetneq A} \rho(B). \quad (3.8)$$

In other words, for any observable menu, there must exist at least one option that is *never rejected in any of its strict subset menus* when available. I shall define such an option as a *conditionally-dominant option*. The existence of such an option in any menu of a multi-utility model is shown in D.1.1.

3.2.3 Identification

Unlike verification, finding a multi-utility identification without a parametric form generally requires close-to brute force methods, which is in an inefficient $O(|X|!)$ factorial time. The naïve algorithm is to first list all possible types. Even if the scope is limited to linear orders, the algorithm then check each of the $|X|!$ types to see if its preferences can be accommodated in the dataset.

Aizerman and Malishevski (1981) developed an algorithm that finds the *maximal rationalization* for a complete choice dataset. This algorithm finds the maximal set of types Θ such that adding in any other order would break the MU model. For example, the maximal ra-

⁴To be specific, an incomplete dataset can at most *not violate* either of these axioms. For brevity I shall assume non-violation as a form of adherence.

tionalization for a choice function $\phi(\{a, b, c\}) = \{a, c\}$ and $\phi(A) = A$ for all binary menus A is $\{a \succ b \succ c; a \succ c \succ b; c \succ a \succ b; c \succ b \succ a\}$.

A second problem to this method is that it fails Occam's razor since it can overestimate the number of types required to represent a choice data set. The example in the previous paragraph can be rationalized using just $a \succ b \succ c$ and $c \succ b \succ a$. With certain constraints on capacity, identification can be found in polynomial time for arbitrary orders using *minimal rationalization* (Kopylov, 2022), whereby taking away any order in Θ would break the MU model.

3.3 Expected Utility: a Negative Result

When utility frameworks are required, identification (and sometimes even verification) can potentially become even more cumbersome. For example, assume a *subjective expected utility* framework (Savage, 1972) for heterogeneous agents with *beliefs* over possible states of the world Ω . Here, I shall consider each possible belief as a type.

Assume first $|\Omega| = 3$. The set of all possible types, or the *type space*, can be represented by a two-dimensional (triangular) *simplex* Δ . Each type is a dot on the simplex with the belief for each state be the perpendicular distance from the dot to the axis across from the node of the state. This can be visualized in Figure 3.1 with a type that has a belief of $T = (0.5, 0.4, 0.1)$ for the three states.

Definition 3.1 (Rejection Subset). *A subset of the type space that is mutually exclusive with the family of types Θ that describes the dataset in a multi-utility model.*

Each observation may create some rejection subsets. In order to verify that a choice dataset built using beliefs adheres to multi-utility, *the union of all rejection subsets cannot cover the entire type space*. However, determining if multiple convex polygons (the rejection subsets)

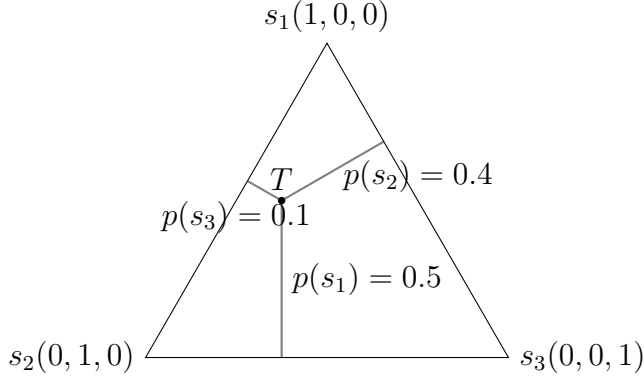


Figure 3.1: Type space of beliefs

cover another convex polygon (the type space/simplex) is NP-Hard (Culberson and Reckhow, 1994),⁵ and hence *verifying if a choice dataset is a multi-utility model under subjective expected utility is not efficient*.

3.4 Quasilinear Utility: a Positive Result

Quasilinearity is commonplace in many fields of economics including theory work (Yang and Kopylov, 2023), demand estimation (Nocke and Schutz, 2017), mechanism design (Makowski and Ostroy, 1987), and more. A *quasilinear* framework views each option as an ordered pair of *item* and *price*: (x, p) using a two-dimensional consumption set: $Z = X \times \mathbb{R}$. The options in a given menu consist of each feasible item x along with its menu-dependent price $p(x, A)$. There is no double-pricing or price discrimination for any item in any menu. This allows the prices to be exogenous to the decision maker. Not all items in X need to be feasible, and ones that are not can be excluded from the menu. Alternatively, I shall set the prices of non-feasible items to some very large number for the purpose of this paper.

A decision maker's choice function can be represented by a quasilinear framework (Chambers

⁵In fact, it's even worse than expected (Abrahamsen, 2022).

and Echenique, 2016b) if there exists some $u : X \rightarrow \mathbb{R}$ such that for each menu $A \in \Sigma$,

$$\phi(A) = \arg \max_{x \in X} u(x) - p(x, A). \quad (3.9)$$

$u(x)$ is the *reservation utility function*, which can be interpreted as the utility obtained from consuming x for free. Unlike ordinal utility models with arbitrary orders, *utils* are no longer meaningless. Here it is equivalent to owning one unit of money, and for each unit paid, it decreases the utility by one. Therefore, this decision maker will find an option acceptable if its item provides the *highest net utility*: $u(x) - p(x, A)$.

Offsetting the utility function by a constant preserves quasilinear representation. I define $v(x) \equiv u(x) - u(0)$, where item 0 is a *normalizer*. A decision maker is now viewing the utility of each item as relative to the normalizer. This does not change the output of the choice function due to the preservation of representation. I shall call $v(x)$ the *normalized utility*.

Item 0 can be any of the items in X . However, if the agent is allowed to opt out, item 0 has nice contextual properties if it is considered to be the *outside option* or not making a choice and $u(0)$ is the *status-quo utility* with $p(0) = 0$.

What's more, normalization can also be done to the price for analogous reasons: $c(x, A) \equiv p(x, A) - p(0, A)$. This is the *normalized price* of x in menu A . However, it is required that item 0 have a finite price, or equivalently, be feasible, in each observable menu. When it is impossible to find such an item, one should revert to the model without normalization.⁶

Using the normalization processes above, the decision maker finds an option acceptable in a menu if its item has the *highest normalized net utility*:

$$\phi(A) = \arg \max_{x \in X} v(x) - c(x, A) \equiv u(x) - u(0) - [p(x, A) - p(0, A)]. \quad (3.10)$$

⁶One could also just normalize the utility function but not the prices. To make the algorithms more reader-friendly, I shall limit the menus to have all items always feasible.

To account for a tie-breaking rule, I shall arbitrarily assume that the decision maker favors items with a *lower item index*. This makes the choice function for each agent singleton, and can be expressed as:

$$\phi(A) = \arg \min \left[\arg \max_{x \in X} v(x) - c(x, A) \right] \Big| X = \{0, 1, 2, \dots, |X| - 1\}. \quad (3.11)$$

To assist with identification, take note of the following property of quasilinear preferences. Assume item z at price $p(z, A)$ is rejected in menu A . It is intuitive that item z will also be rejected in any menu where item z is more expensive and all other items are cheaper. A brief proof can be seen in D.1.2. This can be summarized as below along with the consideration of the tie-breaker rule:

Axiom 3.2 (Rejection of Unilateral Inflation , RUI).

$$\begin{aligned} p(z, A) \leq p(z, B) \text{ and } \forall x < z [p(x, A) \geq p(x, B)] \\ \text{and } \forall y > z [p(y, A) > p(y, B)] \\ \implies [(z, p(z, A)) \in \rho(A) \implies (z, p(z, B)) \in \rho(B)] \quad . \end{aligned} \quad (3.12)$$

3.4.1 Visualization

I shall start with a visualization using a three-item environment: $X = \{0, 1, 2\}$. Using item 0 as the normalizer, the type space only requires two dimensions, a horizontal axis representing *the normalized utility of item 1*: $v(1)$ and a vertical axis *the normalized utility of item 2*: $v(2)$. Such a type space can be visualized using the Euclidean $\mathbb{R}^2$ space as shown in Figure 3.2. An agent with type (or reserved utility function) $\{u(0), u(1), u(2)\}$ is located at the coordinates $\langle v(1) \equiv u(1) - u(0), v(2) \equiv u(2) - u(0) \rangle$.

Each menu A consists of prices for items 0, 1, and 2. Normalized prices can be constructed

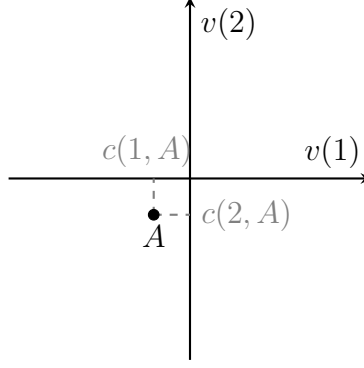


Figure 3.2: Type space and menu price

again using item 0 as the normalizer. The prices for each menu can be represented on the type space using the normalized prices: $\langle c(1, A) \equiv p(1, A) - p(0, A), c(2, A) \equiv p(2, A) - p(0, A) \rangle$. I shall call this vector the *menu price* of A and can be located in Figure 3.2. The normalized net utility for agent i from choosing item x is $v_i(x) - c(x, A)$. To also accommodate the tie-breaker rule, this decision maker will:

$$\text{accept } x \text{ in } A \text{ such that } x \in \arg \min \left[\arg \max_{z \in \{0,1,2\}} v_i(z) - c(z, A) \right]. \quad (3.13)$$

For brevity, I shall denote $c(1, A) \equiv p$ and $c(2, A) \equiv q$. If an agent accepts item 0, then it must be that she gets the highest net normalized utility from consuming item 0:

$$v_i(0) - c(0) = 0 - 0 \geq v_i(1) - p \text{ and } 0 \geq v_i(2) - q. \quad (3.14)$$

The equation above is equivalent to $v_i(1) \leq p$ and $v_i(2) \leq q$. Using the partition in Figure 3.3, this agent's type $\langle v_i(1), v_i(2) \rangle$ is in the blue hull. Similarly, if an agent accepts item 1 (*resp.* 2), her type is in the red (*resp.* green) hull.

I shall now tie this back to multi-utility. If an item is rejected in a menu, it creates a rejection subset similar to that in the previous section. For example, if $\phi(A) = \{1, 2\}$, then *no type can maximize their net utility by consuming item 0, and hence no type is in the blue hull in*

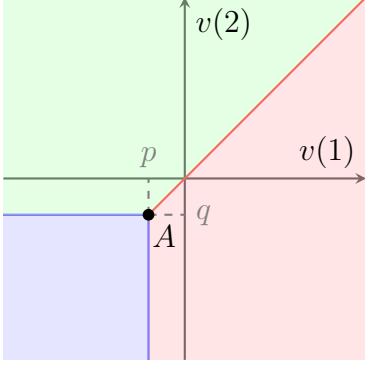


Figure 3.3: Partition of type space

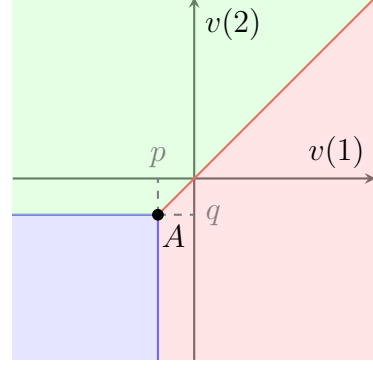


Figure 3.4: Rejection subset

Figure 3.4. Hints of Axiom 3.2 can be seen in the construction of rejection subsets. Relative to menu prices A , all points in the blue hull has a lower normalized price for items 1 and 2 while the normalized price for item 0 stays the same (at 0). Since all agents rejected item 0 at menu prices A , they would also reject it in any point of the blue hull.

3.4.2 The Multiple Quasilinear Utility Model

In the general case with $N + 1$ items: $X = \{0, \dots, N\}$, the type space is in the Euclidean hyperspace $\mathbb{R}^N$. I shall denote the rejection subset created from menu prices A by the rejection of item x by $\mathcal{R}(x, A)$. It can be expressed as a conjunction of inequalities:

$$\mathcal{R}(x, A) = \left\{ \bigwedge_{y < x} [v(x) - c(x, A) > v(y) - c(y, A)] \right\} \wedge \left\{ \bigwedge_{z > x} [v(x) - c(x, A) \geq v(z) - c(z, A)] \right\}. \quad (3.15)$$

I shall call the complement of the union of all rejection subsets created in a dataset $\mathcal{D}$ the potential cloud:

Definition 3.2 (Potential Cloud).

$$\mathcal{C}(\mathcal{D}) = \left(\bigcup_{(x,A)|x \in \rho(A), A \in \Sigma} \mathcal{R}(x, A) \right)^c. \quad (3.16)$$

Since all types cannot be in rejection subsets, they must all be in the potential cloud. The potential cloud also shows the minimum part of the type space that includes all types using the information from the dataset. Consider this following example in a three-item environment $X = \{0, 1, 2\}$ with four menus $\Sigma = \{A, B, C, D\}$ along with the menu prices, choice function, and rejection function:

$\cdot$	$c(1, \cdot)$	$c(2, \cdot)$	$\phi(\cdot)$	$\rho(\cdot)$
A	-2	-2	$\{1, 2\}$	$\{0\}$
B	0	-4	$\{2\}$	$\{0, 1\}$
C	-6	3	$\{1\}$	$\{0, 2\}$
D	2	5	$\{0\}$	$\{1, 2\}$

Figure 3.5: Example choice dataset

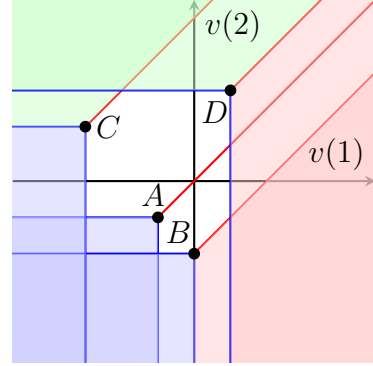


Figure 3.6: Potential cloud

Menu A created one rejection subset $\mathcal{R}(0, A)$ since item 0 was rejected at menu price $\langle -2, -2 \rangle$. All other menus created two rejection subsets. Each rejection subset is represented by a hashed area in Figure 3.6. The area uncovered by the hashes is the potential cloud given this dataset. All types in Θ must be in this cloud.

Analogous to the previous section, for a rejection subsets cannot cover the entire type space:

$$\bigcup_{(x,A)|x \in \rho(A), A \in \Sigma} \mathcal{R}(x, A) \subsetneq \mathbb{R}^N. \quad (3.17)$$

This is equivalent to saying the *potential cloud cannot be non-empty*.

Axiom 3.3 (Non-Empty Potential Cloud, NPC).

$$\mathcal{C}(\mathcal{D}) \neq \emptyset. \quad (3.18)$$

3.4.3 The Potential Cloud

While checking *NPC* is identical to the covering-the-triangle problem in the previous section, there is promise to avoid the complexity due to all rejection subsets being predictable shapes, i.e. edges intersecting at 90 or 135 degrees. This subsection illustrate this method. By plugging (3.15) into (3.16), and ignoring the tiebreakers, the potential cloud can be re-expressed as:

$$\mathcal{C}(\mathcal{D}) = \bigwedge_{(x,A) | x \in \rho(A), A \in \Sigma} \left\{ \bigvee_{z \neq x} [v(x) - c(x, A) \leq v(z) - c(z, A)] \right\} \quad (3.19)$$

via De Morgan's Law of Complements. This is a *conjunctive-normal form* (Cormen and et al., 2022). When $N > 2$, it is NP-Complete (Cook, 1971; Levin, 1973). This means that it is very inefficient to find a point in the potential cloud from scratch. However,

Theorem 3.2. *To verify that $\mathcal{C}(\mathcal{D}) \neq \emptyset$, it is sufficient to show that one point in $\mathbb{R}^N$ satisfies the linear inequalities in (3.19).*

Fortunately, due to the NP-Completeness, a *certificate* of a solution can be verified in polynomial time (Cormen and et al., 2022). The algorithm to construct candidates and checking if each is indeed in $\mathcal{C}(\mathcal{D})$ is as follows:

1. *Construct rejection subsets to build (3.19):* this takes at most $|\Sigma| \times N \times N$ operations, which is better than cubic time in $|\Sigma|$.
2. *Find the smallest difference between distinct prices of each item across menus.* Denote

this as d . This takes at most $N \times |\Sigma| \log |\Sigma|$ operations, which is also better than cubic time in $|\Sigma|$. Note that ties in prices won't affect the algorithm as one of the rejection subsets will be eclipsed by the other.

3. *Find all menu prices that rejected item 0*: the reason for using this is due to all sides of $\mathcal{R}(0, A)$ being orthogonal.
4. *Construct type-candidates*: for each item that a menu accepted, add $\frac{d}{2}$ to its price. Since d is the smallest price difference and items where prices are added are accepted, this addition will not move into a different rejection subset if Axiom 3.3 is satisfied.
5. *Check if any type-candidate satisfies (3.19)*: Theorem 3.2 says that only one is needed for verification. Checking all type-candidate takes at most $|\Sigma| \times (N + |\Sigma| \times N \times N)$ operations, which is better than quartic time. In fact, since only one is needed, it is closer to cubic time.

This algorithm shows that verification that a potential cloud is non-empty can be done in approximately cubic time, which demonstrates efficiency. The reason it does not run into the same complexity problem in the expected utility case is that each rejection subset are not arbitrary shapes, and hence, properties of the regularity can be exploited. I shall use the example from the previous subsection to illustrate the verification algorithm.

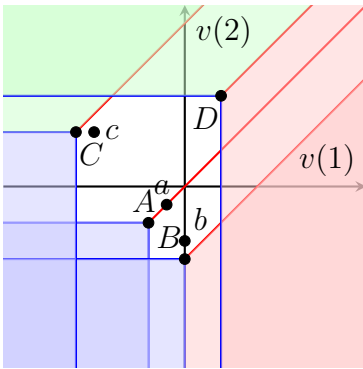


Figure 3.7: Verification

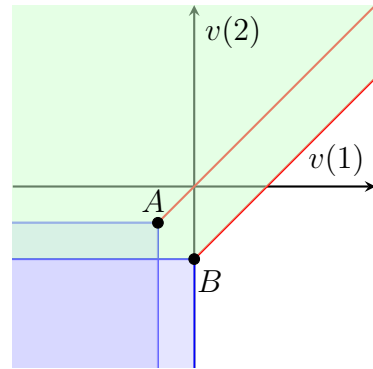


Figure 3.8: Axiom RUI failed

1. The hash zones represent the rejection subsets.

2. The smallest price difference for the normalized prices is $2 = d$.
3. There are three item 0 rejection subsets: $\mathcal{R}(0, A)$, $\mathcal{R}(0, B)$, and $\mathcal{R}(0, C)$.
4. For menu A , both 1 and 2 are accepted, so we add $1 = \frac{d}{2}$ to both values of the menu price to create a type-candidate: $\langle -1, -1 \rangle$. For menu B (*resp.* C), only 2 (*resp.* 1) is accepted so its type-candidate is $\langle 0, -3 \rangle$ (*resp.* $\langle -5, 3 \rangle$). They shown as lower case letters in Figure 3.7.
5. Upon observation,⁷ it is evident that all type-candidates are in the un-hashed area. This means that $\mathcal{C}(\mathcal{D}) \neq \emptyset$.

Once verification is completed, and if the dataset truly belongs to MQU, then the description of the potential cloud in (3.19) is the smallest possible set that includes all the types in Θ for sure from the information given by the dataset. It is considered the *maximal representation* of the dataset since any expansion of the potential cloud would overlap with a rejection subset. Assuming that $|\mathcal{C}(\mathcal{D})| \equiv \delta$ is somewhat proportional to $|\Theta|$, it is also a candidate measure of heterogeneity.

Due to the construction of the potential cloud being a CNF, finding the volume is a #P-Hard problem, which are at least as hard as NP-Complete (Dyer and Frieze, 1988). Instead, approximation methods can be used by considering the convex hull of menu prices that border $\mathcal{C}$. Such methods include efficient algorithms like Qhull (Barber et al., 1996).

The algorithm so far only accounts for Axiom 3.3 (*NPC*). Axiom 3.2 (*RUI*) needs to also be considered. Consider the menus A, B in Figure 3.8 and associated choice function: $\phi(A) = \{1, 2\}$, $\phi(B) = \{1\}$.

This dataset does produce a non-empty potential cloud. However, it cannot be an MQU since *the rejection subset $\mathcal{R}(2, B)$ fully covers the types that would choose 2 in menu A* . This

⁷A computer will insert the type-candidates into (3.19).

is a violation of *RUI*.

Define $\mathcal{A}(x, A)$ as an *acceptance subset* if $x \in \phi(A)$. Mathematically, it uses the same equations as $\mathcal{R}(x, A)$ seen in (3.15). It is instead interpreted as there must be a type in this subset since it is the maximizing choice for some type.

Conjecture 3.3. *A dataset $\mathcal{D}$ follows *MCU* if and only if all acceptance subsets intersect with the potential cloud:*

$$\forall x \in \phi(A) \forall A [\mathcal{A}(x, A) \cap \mathcal{C}(\mathcal{D}) \neq \emptyset]. \quad (3.20)$$

I append the following steps to the algorithm above to check for *NPC* as well:

6. *For each acceptance subset $\mathcal{A}(x, A)$, construct testers:* extend $\frac{d}{2}$ on each menu price at A other than x into the acceptance subset.
7. *Check if each tester is in the potential cloud:* since there are at most $|\Sigma| \times N \times N$ inequalities, and there are at most $|\Sigma| \times N$ acceptance subsets, this step takes quintic time $O(|\Sigma|^5)$ or better.

3.4.4 Purification

A choice dataset could contain *noise* where some menus included the maximizers of types not included in Θ . To allow for such a relaxation, I consider the following extensions to the standard model.

Utility maximization is considered a *strong rationalization*. It requires completeness, transitivity, (Cantor, 1895) and when X is uncountable, continuity (Debreu, 1954). Relaxing this condition can assist in explaining preferences with error, limited cognition, or in this case, noisy data. One of such relaxations is subrationalizability introduced by Fishburn (1976). Preferences are *subrationalizable* if the choice function *at least accepts the “best” options*,

and may admit other options as well. In other words, there is a utility function $u : X \rightarrow \mathbb{R}$ where for each observable menu $A \in \Sigma$,

$$\arg \max_{x \in A} u(x) \subseteq \phi(A). \quad (3.21)$$

To introduce multi-utility to this relaxed versions of rationality, I shall first define two categories of agents:

Definition 3.3 (Residents and Visitors). (a) residents u_i ($i = 1, \dots, k$) where for each menu, their maximizers are always observed;

(b) visitors $\hat{u}_j$ ($j = 1, \dots, v$) where their maximizers are observed in only some of the menus.

In other words, for each menu, the choice function for the dataset will admit acceptable options in the following set-inclusive fashion:

$$\bigcup_{i=1}^k \arg \max_{x \in A} u_i(x) \subseteq \phi(A) \subseteq \left[\bigcup_{i=1}^k \arg \max_{x \in A} u_i(x) \right] \cup \left[\bigcup_{j=1}^v \arg \max_{x \in A} \hat{u}_j(x) \right] \quad (3.22)$$

The choice function at least admits the maximizers of all residents, and sometimes include choices made by some visitors.

I shall define a *subrationalizable multi-utility model* when $\Theta = \{u_i\}_{i=1}^k$. This just considers the left part of (3.22). Options that were chosen by visitors are considered the *noise* of the dataset. Note that the discrepancy in the example from Figure 3.8 is caused by a *visitor* (as in Definition 3.3(b)) who accepted item 2 in menu A but was not present in menu B . Item 2 should not have been accepted by the choice function in menu A since it is from a visitor. In this case, this accepted item needs to be *purified* from the choice dataset. The purified data is considered the *noise*, which I shall denote by $\mathcal{N}(\mathcal{D}) = \{(2, A)\}$. Since a subrationalizable MQU only consider residents as true types, the potential cloud for an MQU after purification

will be the same as the one seen in Figure 3.8.

I propose a candidate measure for noise: $\varepsilon \equiv \frac{|\mathcal{N}(\mathcal{D})|}{|\mathcal{D}|}$ by using the proportion of acceptance subsets that are purified from the dataset.

3.5 Constant Thresholds: a Practical Result

ϕ has a *threshold representation* if there exists some $u : X \rightarrow \mathbb{R}$ such that for all $A \in \Sigma$,

$$\phi(A) = \{x \in A \mid u(x) \geq \max_{z \in A} u(z) - \delta(A)\} \quad (3.23)$$

where $\delta : \Sigma \rightarrow \mathbb{R}_+$ is menu dependent. ϕ is (strongly) *rationalizable* when $\delta(A) = 0$ for all $A \in \Sigma$, as it collapses into the case in for utility rationalization.

As defined by Jamison and Lau (1973), ϕ has a *constant threshold representation* (CTR) when, without loss of generality, $\delta(A) = 1$ for all $A \in \mathcal{M}$. It can be shown that *weak revealed preference* ($\succeq^*$) as defined in Section 3.2.1 can explain why an alternative is acceptable in a menu. Hence,

$$x \succeq^* y \implies u(x) \geq u(y) - 1. \quad (3.24)$$

In a CTR framework, an option is rejected by a choice function in a menu if it is more than 1 util below the true-best alternative. Since $\arg \max_{z \in A} u(z)$ has to be non empty for any menu A , there must be at least one alternative in A that is more than 1 util higher than, if exists, a rejected alternative. I shall define *domination*, denoted by $\gg$, as follows. x *dominates* y if the following holds:

$$x \gg y \iff u(x) > u(y) + 1. \quad (3.25)$$

There must be an acceptable alternative (the true-best) that dominates all discarded ones

in a menu. Domination ($\gg$) explains why an alternative is rejected from a menu.

Note first that domination is defined to be independent of menus. It is a reflection of strict preference in the semiorder case (Luce, 1956) as it only considers two alternatives. Equally important is that *strict revealed preference* ($\succ^*$) does not imply domination.⁸ The underlying reason is when a choice function admits more than one acceptable alternative, those close to the threshold but above it may not be able to dominate those that are slightly under it. For example, let ϕ be a CTR with a unit threshold and $u(x) = \frac{2x}{3}$. Then $\phi(\{0, 1, 2\}) = \{1, 2\}$ since both of them provide utilities no more than 1 below the maximum utility of the menu. $2 \gg 0$ since $u(2) > u(0) + 1$. However, while $1 \succ^* 0$, 1 does not dominate 0 as $u(1) \leq u(0) + 1$. The multiplicity of acceptable alternatives in a menu is a main motivator to this section of the research, and will be discussed in detail in the next subsection.

3.5.1 Complexity Issues

Pirlot (1990) showed that verifying if a dataset with binary preference follows a semiorder model is efficient regardless of completeness. Identifying a semiorder can be seen as a sorting process. If all binary menus are complete, then a CTR can be derived from a semiorder (Jamison and Lau (1973)). This implies that identification of a CTR is efficient if $\{A : |A| = 2\} \subseteq \Sigma$. This leaves the question of identification of CTR with incomplete binary data.

Identification of CTR in incomplete data has yet to be fully explored. In order to fit a utility function, it is crucial to form all $(\succeq^*, \gg)$ such that these binary relations agree with each menu. $\succeq^*$ can be derived from the observations themselves since it is equivalent to weak revealed preference. $\gg$ is not as straight-forward. Consider the example $\phi(\{0, 1, 2\}) = \{1, 2\}$. Since the choice function assigns both 1 and 2 to be acceptable in the menu, at least one of them dominates 0, and hence either $u(2) > u(0) + 1$ or $u(1) > u(0) + 1$. However, it is not clear which one does. It is also possible for both of them to dominate 0.

⁸However, $x \gg y \implies x \succ^* y$ as long as there is an observable menu where x and y are both observable.

For each menu that has at least one rejected option, all acceptable options are candidates for the possible true-best. Without loss of generality, assume that all menus rejected something. To find the true-best option in each menu, $\prod_{A \in \Sigma} |\phi(A)|$ possible outcomes need to be considered.⁹ If each menu admits even just 3 acceptable alternatives, brute force performs with time complexity $O(3^{|\Sigma|})$, which is an exponential issue.

Provided that ϕ has a CTR, brute force does guarantee identification since at least one of the possibilities prescribes a logical true-best option in each menu. However, it is not time efficient given its exponential complexity. In order to reduce time complexity, an intelligent guess can reduce the number of iterations needed for identification by ruling out utility functions that fail measures which CTRs would imply. The following steps illustrate the general format of the overall algorithm.

- (1) Propose a utility ranking using an intelligent guess.
- (2) Use utility ranking to determine the true-best alternative in each menu.
- (3) Construct binary relations $(\succeq^*, \gg)$.
- (4) Attempt to fit a CTR utility function using the binary relations.
- (5) Check if the utility function agrees with the binary relations. If disagrees, return to Step 1.

Steps 3 through 5 verify if this certificate is correct in polynomial time, with details in D.2.2. Therefore, if the formulation of the *certificate* (Steps 1 and 2) is also polynomial, then each guess will be polynomial.

⁹A menu that doesn't discard anything would not have any dominance relations, and would contribute just 1 to their portion in the product. A general formula for possibilities is $\prod_{A \in \Sigma} \{1 + (|\phi(A)| - 1) \cdot \mathbf{1}[\rho(A) \neq \emptyset]\}$.

3.5.2 Subrationalizability

Recall *subrationalizability* by Fishburn (1976). Choice data with CTR must be subrationalizable, as the true-best element is always acceptable. Therefore, for the utility ranking u^R , an option must always be acceptable in menus where no other option is ranked higher. The following algorithm proposes a subrationalizable u^R .

- (1) Set all menus as *active*.
- (2) Construct three sets:
 - $\mathcal{V}$: the set of alternatives that are *available* in at least one active menu, and
 - $\mathcal{J}$: the set of alternatives that are *rejected* in at least one active menu, and
 - $\mathcal{K}$: the set of alternatives that are *never rejected* in any active menu.
- (3) Determine if choice data is subrationalizable.
 - If $|\mathcal{J}| = |\mathcal{V}| > 0$, then at this iteration, all alternatives are discarded in some menu. This violates subrationalizability, as in any subset of menus, there must be at least a utility maximizer that is always acceptable when available. End algorithm here.
 - If $|\mathcal{V}| = 0$, then all menus are inactive and $\mathcal{C}$ is subrationalizable. Skip to Step 6.
 - Otherwise, it's undeterminable. Continue to Step 4.
- (4) Choose one alternative $k \in \mathcal{K}$ and set $u^R(k)$ as the current iteration number.
- (5) Set all menus with k as *inactive*, and return to Step 2 for the next iteration.
- (6) For all unassigned alternatives, randomize rankings between last iteration plus 1 and N .

Step 1 takes $|\Sigma|$ operations. For each iteration, Step 2 takes $2|\Sigma|N$ operations, Step 3 takes 2 operations, Step 4 takes 2 operations, and Step 5 takes at most $2|\Sigma|$ operations. Step 6 takes at most $N - 1$ operations. In total, the worst case scenario is cubic time in $|\Sigma|$.

In this paper, I used R to simulated choice data. For each environment (set of parameters) and intelligent guess measure, I shall generate 1000 choice data sets. Each choice data set will run the designated algorithm and propose guesses until it fits a correct CTR or it uses up 100 guesses. The algorithms provided here cannot have a false positive. However, an incorrect guess is a false negative in the simulation as data is generated through a CTR process. The 100-guess cutoff is a mock-up of assuming that there is no convergence when given a choice data that does not have a CTR.

The data generation process used here takes exponential time relative to N as it requires going through all possible menus. For the simulations below, N is designed to be at most 15 in order to avoid computational burden.

First, I control for threshold size of $k = 2$ and μ that returns 30 menus in expectation while varying the size of consumption space N . Each row of Table 3.1 shows the results of each N . Each column is the convergence rate range. In each cell, it shows the number of simulations out of 1000 for the environment of N prescribed in the row that converges within the range noted in the column.

Upon observation of each row, the subrationalizability measure has similar performance across different consumption spaces, *ceteris paribus*, as long as the number of observable menus is similar across the different environments of N .

Next, I control for consumption space $N = 12$ and the same μ while varying the size of the threshold k . Each row of 3.2 now instead shows the results for each k . As seen here, when k gets large, performance drops drastically as convergence speed takes longer. When threshold gets to 6, which in the case of $N = 12$ is when the “decision maker” have trouble telling half

N	[1, 9]	[10, 24]	[25, 39]	[40, 54]	[55, 69]	[70, 84]	[85, 99]	≥ 100
6	489	289	119	43	23	16	10	11
7	446	286	121	43	37	13	14	40
8	477	269	105	59	25	14	14	37
9	502	266	104	43	27	14	5	39
10	492	271	101	46	27	19	10	34
11	480	243	112	56	32	17	15	45
12	478	283	112	43	21	12	8	43
13	504	248	97	50	38	17	12	34
14	474	267	96	59	30	18	12	43
15	486	270	88	44	34	24	13	40

Table 3.1: Subrationalizability Measure with Varying Consumption Space Sizes

of the alternatives apart, the algorithm does not converge within 100 guesses more than half of the time.

k	[1, 9]	[10, 24]	[25, 39]	[40, 54]	[55, 69]	[70, 84]	[85, 99]	≥ 100
1	843	127	24	3	1	0	1	1
2	478	283	112	43	21	12	8	43
3	284	230	114	84	61	36	25	166
4	180	154	122	86	49	48	36	325
5	129	127	74	74	67	48	23	458
6	95	123	79	54	40	39	38	532

Table 3.2: Subrationalizability Measure with Varying Threshold Sizes

3.5.3 Binary Cycles

To counter this problem of non-convergence, I introduce a second measure by eliminating *binary cycles*. If x is accepted in a menu A , then no alternative in A can dominate x . This is reminiscent of WARP, and can be formulated as:

Axiom 3.4 (Weak Axiom of Revealed CTR, WACTR).

$$x \succeq^* y \implies y \not\succ x. \quad (3.26)$$

To formulate a utility ranking that does not violate WACTR, I first go through each menu and determine that all acceptable alternatives cannot be dominated by any alternative in that menu. The worst case is $|\Sigma|N^2$ operations, which is cubic time in $|\Sigma|$.

The true-best alternative of each menu must (a) be acceptable in that menu and (b) does not cause an undesired domination relation from another menu. If there are multiple possible ones, randomly pick one. The worst case is also proportional to $|\Sigma|N^2$ operations, which is cubic time in $|\Sigma|$.

Overall, this algorithm takes cubic time, and replaces Steps 1 and 2 in Section 3.5.1. Pseudocodes for this algorithm can be found in D.2.3.

k	[1, 9]	[10, 24]	[25, 39]	[40, 54]	[55, 69]	[70, 84]	[85, 99]	≥ 100
1	992	4	3	0	0	1	0	0
2	948	38	5	5	2	0	0	2
3	939	48	8	1	0	1	0	3
4	829	119	20	13	3	2	2	12
5	774	151	36	20	12	4	7	26
6	652	174	67	24	18	13	9	43

Table 3.3: Binary-Cycle Elimination Measure with Varying Threshold Sizes

When keeping the same environments as from Table 3.2, the results in Table 3.3 shows a drastic improvement in performance, as an almost first-order stochastic dominance can be observed for each row when compared to the subrationalizability measure.

3.6 Conclusion

In this chapter, I have identified what is missing from current literature in the discussion of heterogeneity. At first, I start by deviating from the common framework of stochastic choice in favor of a multi-utility model for its robustness. I showed that verification with arbitrary orders is relatively simple but identification can be problematic.

Once I introduced parametric forms, I immediately noticed that not all of them are created equal. I ran into problems trying to verify expected utility as the framework for a multi-utility model since there is no efficient method of covering a big triangle with smaller ones. However, not all is lost. A quasilinear framework, which is used in many aspects of economics, has nice properties that allows for a simpler approach in identification.

I then created an efficient algorithm that can both verify and identify the maximal part of the type space that includes all types in Θ . However, this algorithm does not take into account choices made by “visitors” (ones who make decision only some of the time). Therefore, purification is needed to delete these extra choices. On the other hand, recovery is also possible until an identification is achieved.

Identification, purification, and recovery allows me to construct three different measures. The first is an index ($|\mathcal{C}|$) of heterogeneity or dispersion and the other two are indices of noise ($|\mathcal{N}|, |\mathcal{M}|$). These measures have analytical and empirical potential. Once created with real-life datasets, they can be regressed against other variables for possible inference.

The algorithm used for CTR provide a reasonably fast method to fit CTR into choice data that is incomplete. While this is not a theoretical guarantee, just by eliminating binary cycles can performance be polynomial in expectation.

3.6.1 Future Work

Scanner datasets include how much of an item a retailer in the United States sold at what price for each week across categories. I plan to focus on the category of beauty and health products, as it is common for agents to just purchase one of a certain item (for example, shampoo or lipstick). This provides a discrete choice setup and X is each brand of a certain item. My algorithm in the previous section allows me to build $|\mathcal{C}|, |\mathcal{N}|, |\mathcal{M}|$ across different counties.

Using $|\mathcal{C}|$, I can regress it on demographic data from the US Census. This provides a preview of how dispersed preferences can be given different demographics. I can also compare this to pre-existing measures of heterogeneity from labs and other research.

Using either $|\mathcal{N}|$ or $|\mathcal{M}|$, I can regress them on how “rural” a county is. I am still in the search for an appropriate measure for rurality, with average distance from a big city as a good candidate.

For CTR, while exclusion of binary cycles greatly improves performance, it is still not perfect. A decrease, however slow, in convergence speed can still be observed as threshold size increase. Consider $X = \{a, b, c, d, e\}$ and

$$\begin{aligned}\mathcal{D} &= \Sigma \times \phi(\Sigma) \\ &= \{(\{a, b, c\}, \{a, b\}), (\{c, e\}, \{c\}), (\{d, e\}, \{d, e\}), (\{b, d\}, \{b, d\})\}.\end{aligned}$$

Since no two alternatives share more than 1 menu, binary cycles vacuously cannot be created. This means that any subrationalizable u^R would be allowed under the algorithms in this paper. R_{bacde} is an example of a subrationalizable u^R . The first menu implies $b \gg c$, the second $c \gg e$, the third $e \succeq^* d$, and the fourth $d \succeq^* b$. This creates a relation cycle of size 4: $b \gg c \gg e \succeq^* d \succeq^* b$. Such a cycle of size 4 violates Luce’s semiorder axioms Luce (1956), and hence R_{bacde} cannot be used to fit a CTR. However, the choice data does have a CTR: $u = \langle 5, 3, 2, 1, -1 \rangle$ with threshold 2.

It can be seen that eliminating cycles of size 4 or higher would refine convergence speed. However, it is yet to be discussed whether this may change the performance of each guess as ruling out higher-order cycles would be more algorithmically complicated.

So far, each trial uses a random guess permitted by the measure. This means that there is a chance that it could propose a utility ranking already rejected before. Another possible

refinement is to keep track of previous bad guesses and form new and better ones based off of existing priors. While this should also increase convergence speed, it comes at the cost of increasing memory complexity and the complexity of Step 1 in the main algorithm of Section 3.5.1.

Bibliography

- M. Abrahamsen. Covering polygons is even harder. *In 2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 375–386, 2022. doi: 10.48550/arXiv.2106.02335.
- M. Agranov and P. Ortoleva. Stochastic choice and preferences for randomization. *Journal of Political Economy*, 125:40–68, 2017.
- D. S. Ahn and T. Sarver. Preference for flexibility and random choice. *Econometrica*, 81(1): 341–361, 2013. ISSN 00129682, 14680262.
- M. Aizerman and A. Malishevski. General theory of best variants choice: Some aspects. *IEEE Trans. on Autom. Control.*, 26(5):1030–1040, 1981. doi: 10.1109/TAC.1981.1102777.
- S. Albert and J. Duffy. Differences in risk aversion between young and older adults. *Neurosci Neuroecon*, 1:3–10, 2012.
- C. Aliprantis and K. Border. *Infinite Dimensional Analysis*. Springer, 1999.
- J. Apesteguia and M. A. Ballester. Monotone stochastic choice models: The case of risk and time preferences. *Journal of Political Economy*, 126(1):74–106, 2018. doi: 10.1086/695504.
- J. Apesteguia and M. A. Ballester. Separating predicted randomness from residual behavior. *J. of the Eur. Econ. Assoc.*, 19(2):1041–1076, 2021. doi: 10.1093/jeea/jvaa016.
- J. Apesteguia, M. Ballester, and J. Lu. Single-crossing random utility models. *Econometrica*, 85:661–674, 2017.
- D. Ariely. *Predictably Irrational: The Hidden Forces That Shape Our Decisions*. Harper-Collins, New York, 2008.
- A. Baillon and O. L’Haridon. Discrete arrow-pratt indexes for risk and uncertainty. *Economic Theory*, 72(4):1375–1393, 2021.
- P. Balart. Semiorder preferences and price-oriented buyers in a hotelling model. *J. of Econ. Behav. & Organ.*, 188:394–407, 2021. doi: 10.1016/j.jebo.2021.05.015.
- C. B. Barber, D. P. Dobkin, and H. Huhdanpaa. The quickhull algorithm for convex hulls. *ACM Trans. on Math. Softw.*, 22(4):469–483, 1996. doi: 10.1145/235815.235821.
- D. E. Bell. Regret in decision making under uncertainty. *Operations Research*, 30(5):961–981, 1982.

- D. J. Benjamin, S. A. Brown, and J. M. Shapiro. Who is 'behavioral'? cognitive ability and anomalous preferences. *Journal of the European Economic Association*, 11(6):1231–1255, 2013.
- H. Block and J. Marschak. Random orderings and stochastic theories of response. In Olkin, Ghurye, Hoeffding, Madow, and Mann, editors, *Contributions to Probability and Statistics*, pages 97–132. Stanford University Press, Palo Alto, 1960.
- L. Blume, A. Branderburger, and E. Dekel. Lexicographic probabilities and choice under uncertainty. *Econometrica*, pages 61–79, 1991.
- M. Bombardini and F. Trebbi. Risk aversion and expected utility theory: An experiment with large and small stakes. *Journal of the European Economic Association*, 10(6):1348–1399, 12 2012. ISSN 1542-4766. doi: 10.1111/j.1542-4774.2012.01086.x. URL <https://doi.org/10.1111/j.1542-4774.2012.01086.x>.
- G. J. Borjas. Self-selection and the earnings of immigrants. *The American Economic Review*, 77(4):531–553, 1987. ISSN 00028282.
- P. C. Boxall and W. L. Adamowicz. Understanding heterogeneous preferences in random utility models: The use of latent class analysis. *Environ. and Resour. Econ.*, 23:421–446, 2002. doi: 10.1023/A:1021351721619.
- G. Cantor. Beiträge zur begründung der transfiniten mengenlehre [contributions to the foundation of transfinite set theory]. *Mathematische Annalen*, 46(4):481–512, 1895. doi: 10.1007/BF02124929.
- S. Cerreia-Vioglio, D. Dillenberger, P. Ortoleva, and G. Riella. Deliberately stochastic. *American Economic Review*, 109(7):2425–2445, 2019. doi: 10.1257/aer.20171120.
- C. P. Chambers and F. Echenique. *Revealed preference theory*, volume 56, chapter 2, pages 15–33. Cambridge University Press, 2016a. doi: 10.1017/CBO9781316104293.
- C. P. Chambers and F. Echenique. *Revealed preference theory*, volume 56, chapter 4, pages 57–70. Cambridge University Press, 2016b. doi: 10.1017/CBO9781316104293.
- D. Chen, M. Schonger, and C. Wickens. otree - an open-source platform for laboratory, online and field experiments. *Journal of Behavioral and Experimental Finance*, 9:88–97, 2016.
- Z. Chen, R. Golman, and J. Somerville. Menu-dependent risk attitudes: Theory and evidence. *Journal of Risk and Uncertainty*, 68(1):77–105, 2024.
- H. Chernoff. Rational selection of decision functions. *Econometrica: J. of the Econom. Soc.*, 22:422–443, 1954. doi: 10.2307/1907435.
- A. Cohen and L. Einav. Estimating risk preferences from deductible choice. *American Economic Review*, 97(3):745–788, 2007. doi: 10.1257/aer.97.3.745.
- S. A. Cook. The complexity of theorem-proving procedures. In *Proceedings of the third annual ACM symposium on Theory of computing*, pages 151–158, May 1971. doi: 10.1145/800157.805047.

- T. H. Cormen and et al. *Introduction to Algorithms*, pages 1082–1085. The MIT Press, 4 edition, 2022. ISBN 9780262046305.
- S. Cosaert and T. Demuynck. Nonparametric welfare and demand analysis with unobserved individual heterogeneity. *Rev. of Econ. and Stat.*, 100(2):349–361, 2018. doi: 10.1162/REST_a_00672.
- I. Crawford and B. De Rock. Empirical revealed preference. *Annu. Rev. Econ.*, 6(1):503–524, 2014. doi: 10.1146/annurev-economics-080213-041238.
- P. Crosetto, M. Hermanns, and J. Kokot. Framing effects in risk-taking: The role of risk aversion and loss aversion. *Journal of Behavioral and Experimental Economics*, 104:102008, 2023.
- J. C. Culberson and R. A. Reckhow. Covering polygons is hard. *J. Algorithms*, 17(1):2–44, 1994. doi: 10.1006/jagm.1994.1025.
- A. Daly and S. Zachery. Improved multiple choice models. In D. Hensher and Q. Dalvi, editors, *Identifying and Measuring the Determinants of Mode Choice*. London: Teakfield, 1979.
- G. Debreu. Representation of a preference ordering by a numerical function. In R. Thrall, C. H. Coombs, and R. L. Davis, editors, *Decision Processes*, chapter 6, pages 159–165. Wiley, New York, 1954. doi: 10.1017/CCOL052123736X.007.
- E. Dekel and B. Lipman. Costly self-control and random self-indulgence. *Econometrica*, 80: 1271–1302, 2012.
- S. Desmettre and M. Steffensen. Equilibrium investment with random risk aversion. *Mathematical Finance*, 33(3):946–975, 2023.
- T. Dumbalska, V. Li, K. Tsetsos, and C. Summerfield. A map of decoy influence in human multialternative choice. *Proceedings of the National Academy of Sciences*, 117(40):25169–25178, 2020.
- M. E. Dyer and A. M. Frieze. On the complexity of computing the volume of a polyhedron. *SIAM J. on Comput.*, 17(5):967–974, 1988. doi: 10.1137/0217060.
- F. Echenique and K. Saito. General luce model. *Economic Theory*, 68(4):811–826, 2019.
- M. Eden. Welfare analysis with heterogeneous risk preferences. *J. of Polit. Econ.*, 128(12): 4574–4613, 2020. doi: 10.1086/710561.
- K. M. Ericson, P. Kircher, J. Spinnewijn, and A. Starc. Inferring risk perceptions and preferences using choice from insurance menus: Theory and evidence. *The Economic Journal*, 131(634):713–744, 2021. doi: 10.1093/ej/ueaa069.
- J. Falmagne. A representation theorem for finite random scale systems. *Journal of Mathematical Psychology*, 18:52–72, 1978.
- P. C. Fishburn. Representable choice functions. *Econometrica: J. of the Econom. Soc.*, 44 (5):1033–1043, 1976. doi: 10.2307/1911543.

- P. C. Fishburn. *The Foundations of Expected Utility*. D. Reidel Publishing Company, Dordrecht, 1982.
- M. Frick. Monotone threshold representations. *Theor. Econ.*, 11(3):757–772, 2016. doi: 10.3982/TE1547.
- M. Frick, R. Iijima, and T. Strzalecki. Dynamic random utility. *Econometrica*, 87(6):1941–1981, 2019. doi: 10.3982/ECTA14855.
- E. E. Garnett, A. Balmford, C. Sandbrook, M. A. Pilling, and T. M. Marteau. Impact of increasing vegetarian availability on meal selection and sales in cafeterias. *Proceedings of the National Academy of Sciences*, 116(42):20923–20929, 2019.
- F. Gul and W. Pesendorfer. Random expected utility. *Econometrica*, 74:121–146, 2006.
- F. Gul, P. Natenzon, and W. Pesendorfer. Random choice as behavioral optimization. *Econometrica*, 82(5):1873–1912, 2014.
- G. W. Harrison, J. Martínez-Correa, and J. T. Swarthout. Inducing risk neutral preferences with binary lotteries: A reconsideration. *Journal of Economic Behavior & Organization*, 94:145–159, 2013.
- J. J. Heckman. Sample selection bias as a specification error. *Econometrica*, 47(1):153–161, 1979. ISSN 00129682, 14680262.
- N. Hing, M. Smith, M. Rockloff, H. Thorne, A. M. T. Russell, N. A. Dowling, and B. Breen. How structural changes in online gambling are shaping the contemporary experiences and behaviours of online gamblers: an interview study. *BMC Public Health*, 22(1620), 2022.
- S. Hoderlein and J. Stoye. Revealed preferences in a heterogeneous population. *Rev. of Econ. and Stat.*, 96(2):197–213, 2014. doi: 10.1162/REST_a_00397.
- C. A. Holt and S. K. Laury. Risk aversion and incentive effects. *American Economic Review*, 92(5):1644–1655, December 2002. doi: 10.1257/000282802762024700.
- H. Hotelling. Stability in competition. *The Econ. J.*, 39(153):41–57, 1929. doi: 10.2307/2224214.
- J. Huber, J. W. Payne, and C. Puto. Adding asymmetrically dominated alternatives: Violations of regularity and the similarity hypothesis. *Journal of Consumer Research*, 9(1): 90–98, 1982.
- S. Huck and G. Weizsäcker. Risk, complexity, and deviations from expected-value maximization: Results of a lottery choice experiment. *Journal of Economic Psychology*, 20(6): 699–715, 1999.
- D. T. Jamison and L. J. Lau. Semiorders and the theory of choice. *Econometrica: J. of the Econom. Soc.*, 41(5):901–912, 1973. doi: 10.2307/1913813.
- D. Jones. Inertia and overwithholding: Explaining the tax refund puzzle. *American Economic Journal: Economic Policy*, 4(1):158–197, 2012.

- B. D. Jovanovic and P. S. Levy. A look at the rule of three. *The American Statistician*, 51(2):137–139, 1997.
- D. Kahneman. *Thinking, Fast and Slow*. Farrar, Straus and Giroux, New York, 2011.
- D. Kahneman and A. Tversky. Prospect theory: An analysis of decision under risk. *Econometrica*, 47(2):263–291, 1979.
- H. Keffer and N. Schweizer. Stochastic monotonicity and random utility models: The good and the ugly. *arXiv preprint arXiv:2409.00704*, 2024. doi: 10.2139/ssrn.4943373.
- Y. Kitamura and J. Stoye. Nonparametric analysis of random utility models: testing. *Econometrica: J. of the Econom. Soc.*, 86(6):1883–1909, 2018. doi: 10.48550/arXiv.1606.04819.
- V. Klee and G. J. Minty. How good is the simplex algorithm? *Inequalities.*, 3(3):159–175, 1972.
- H. J. Kleven, M. B. Knudsen, C. T. Kreiner, S. Pedersen, and E. Saez. Unwilling or unable to cheat? evidence from a tax audit experiment in denmark. *Econometrica*, 79(3):651–692, 2011.
- I. Kopylov. Minimal rationalizations. *Econ. Theory*, 73(4):859–879, 2022. doi: 10.1007/s00199-021-01345-w.
- M. Kovach and G. Tserenjigmid. The focal luce model. *American Economic Journal: Microeconomics*, 14(3):378–413, 2022.
- L. Lederman. Statutory speed bumps: The role of third-party information reporting in voluntary compliance. *Stanford Law Review*, 63(4):695–747, 2010.
- L. A. Levin. Universal sequential search problems. *Probl. peredachi inf.*, 9(3):115–116, 1973.
- A. Liang. Inference of preference heterogeneity from choice data. *J. of Econ. Theory.*, 179:275–311, 2019. doi: 10.1016/j.jet.2018.09.010.
- X. Lim and R. Silvio. Avoiding risk in the lab: An experiment on avoidable risk. Working paper, 2021. URL <https://silvioravaioli.github.io/website-documents/Research/Avoidable%20Risk/LimRavaioli2021AvoidableRisk.pdf>.
- Y.-H. Lin. Random non-expected utility: Non-uniqueness. *arXiv*, 2020. doi: 10.48550/arxiv.2009.04173.
- G. Loomes and R. Sugden. Regret theory: An alternative theory of rational choice under uncertainty. *The Economic Journal*, 92(368):805–824, 1982.
- J. Lu. Random ambiguity. *Theoretical Economics*, 16(2):539–570, 2021.
- D. Luce. *Individual Choice Behavior: A Theoretical Analysis*. Wiley, New York, 1959.
- R. D. Luce. Semiorders and a theory of utility discrimination. *Econometrica: J. of the Econom. Soc.*, 24(2):178–191, 1956. doi: 10.2307/1905751.
- L. Makowski and J. M. Ostroy. Vickrey-clarke-groves mechanisms and perfect competition. *J. of Econ. Theory*, 42(2):244–261, 1987. doi: 10.1016/0022-0531(87)90087-1.

- W. Mao and H. Oppewal. The Attraction Effect is more pronounced for consumers who rely on intuitive reasoning. *Marketing Letters*, 23(1):339–351, 2012. doi: 10.1007/s11002-011-9157-y.
- L. E. Matzen, Z. O. Benz, K. R. Dixon, J. Posey, J. K. Kroger, and A. E. Speed. Recreating raven’s: Software for systematically generating large numbers of raven-like matrix problems with normed properties. *Behavior Research Methods*, 42(2):525–541, 2010. doi: 10.3758/BRM.42.2.525.
- R. L. Matzkin. Heterogeneous choice. *Econom. Soc. Monogr.*, 43, 2007. doi: 10.1017/CBO9780511607547.005.
- D. McFadden. Conditional logit analysis of qualitative choice behavior. In P. Zarembka, editor, *Frontiers in Economics*. Academic Press, New York, 1973.
- D. McFadden. The measurement of urban travel demand. *J. of Public Econ.*, 3(4):303–328, 1974. doi: 10.1016/0047-2727(74)90003-6.
- D. McFadden. Economic choices. *American Economic Review*, 91(3):351–378, 2001.
- A. Metrick. A natural experiment in “jeopardy!”. *The American Economic Review*, 85(1): 240–253, 1995. ISSN 00028282. URL <http://www.jstor.org/stable/2118006>.
- R. Myerson. Multistage games with communication. *Econometrica*, pages 323–358, 1986.
- V. Nocke and N. Schutz. Quasi-linear integrability. *J. of Econ. Theory*, 169:603–628, 2017. doi: 10.1016/j.jet.2017.03.006.
- J. Oechssler and A. Sofianos. The binary lottery procedure: An ineffective equalizer. Discussion Paper Series 663, Heidelberg University, Department of Economics, 2019. URL <https://doi.org/10.1111/jeea.12086>.
- S. Olschewski, P. Sirotkin, and J. Rieskamp. Empirical underidentification in estimating random utility models: The role of choice sets and standardizations. *British Journal of Mathematical and Statistical Psychology*, 75(2):252–292, 2022.
- E. Piermont. Disentangling strict and weak choice in random expected utility models. *Journal of Economic Theory*, 202:201–230, 2022.
- M. Pirlot. Minimal representation of a semiorder. *Theory and Decis.*, 28:109–141, 1990. doi: 10.1007/BF00160932.
- C. R. Plott. Path independence, rationality, and social choice. *Econometrica: J. of the Econom. Soc.*, 41(6):1075–1091, 1973. doi: 10.2307/1914037.
- J. C. Ragain Jr and W. M. Johnston. Minimum color differences for discriminating mismatch between composite and tooth color. *J. of Esthet. and Restor. Dent.*, 13(1):41–48, 2001. doi: 10.1111/j.1708-8240.2001.tb00250.x.
- J. Raven. The raven’s progressive matrices: Change and stability over culture and time. *Cognitive Psychology*, 41(1):1–48, 2000.

- P. R. Rosenbaum and D. B. Rubin. The central role of the propensity score in observational studies for causal effects. *Biometrika*, 70(1):41–55, 04 1983. ISSN 0006-3444. doi: 10.1093/biomet/70.1.41. URL <https://doi.org/10.1093/biomet/70.1.41>.
- P. A. Samuelson. A note on the pure theory of consumer’s behaviour. *Economica*, 5(17):61–71, 1938. doi: 10.2307/2548836.
- L. J. Savage. *The foundations of statistics*. Courier Corporation, 1972.
- A. Sen. Internal consistency of choice. *Econometrica: Journal of the Econometric Society*, 61(3):495–521, 1993.
- A. K. Sen. Choice functions and revealed preference. *The Rev. of Econ. Stud.*, 38(3):307–317, 1971. doi: 10.2307/2296384.
- E. Shafir, I. Simonson, and A. Tversky. Reason-based choice. *Cognition*, 49:11–36, 1993.
- I. Simonson. Choice based on reasons: The case of attraction and compromise effects. *Journal of Consumer Research*, 16(2):158–174, 1989.
- J. Slemrod. Cheating ourselves: The economics of tax evasion. *Journal of Economic Perspectives*, 21(1):25–48, 2007.
- A. Soltani, B. De Martino, and C. Camerer. A range-normalization model of context-dependent decision making. *PLoS Computational Biology*, 8(7):e1002607, 2012.
- D. Sonsino, U. Benzion, and G. Mader. On complexity and uncertainty. *Journal of Economic Behavior & Organization*, 48(4):333–354, 2002.
- D. O. Stahl. Heterogeneity of ambiguity preferences. *Rev. of Econ. and Stat.*, 96(4):609–617, 2014. doi: 10.1162/REST_a.00392.
- S. Stieger and U.-D. Reips. A limitation of the Cognitive Reflection Test: familiarity. *PeerJ*, 4:e2395, 2016. doi: 10.7717/peerj.2395.
- K. Suzumura. Rational choice and revealed preference. *The Rev. of Econ. Stud.*, 43(1):149–158, 1976. doi: 10.2307/2296608.
- J. Sydnor. (over)insuring modest risks. *American Economic Journal: Applied Economics*, 2(4):177–199, 2010.
- R. Tarjan. Depth-first search and linear graph algorithms. *SIAM J. on Comput.*, 1(2):146–160, 1972. doi: 10.1137/0201010.
- R. H. Thaler. Psychology and savings policies. *The American Economic Review*, 84(2):186–192, 1994.
- K. E. Train. Recreation demand models with taste differences over people. *Land economics*, pages 230–239, 1998.
- J. Tucker. Selection bias and econometric remedies in accounting and finance research. *Journal of Accounting Literature*, 29:31–57, 2010.

- C. Turansick. Identification in the random utility model. *J. of Econ. Theory.*, 203, 2022. doi: 10.1016/j.jet.2022.105489.
- A. Tversky and D. Kahneman. Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, 5(4):297–323, 1992.
- A. Tversky and D. Koehler. Support theory: A nonextensional representation of subjective probability. *Psychological Review*, 101:547–567, 1994.
- H. R. Varian. The nonparametric approach to demand analysis. *Econometrica: J. of the Econom. Soc.*, 50(4):945–973, 1982. doi: 10.2307/1912771.
- R. Veneziani and N. Yoshihara. Exploitation in economies with heterogeneous preferences, skills and assets: An axiomatic approach. *J. of Theor. Polit.*, 27(1):8–33, 2015. doi: 10.1177/0951629814538911.
- M. S. Visser and M. R. Roelofs. Heterogeneous preferences for altruism: Gender and personality, social status, giving and taking. *Exp. Econ.*, 14:490–506, 2011. doi: 10.1007/s10683-011-9278-4.
- B. A. Wandell. Measurement of small color differences. *Psychol. Rev.*, 89(3):281–302, 1982. doi: 10.1037/0033-295X.89.3.281.
- E. H. Weber. *De Pulsu, resorptione, auditu et tactu: Annotationes anatomicae et physiologicae [On Pulse, Resorption, Hearing, and Touch: Anatomical and Physiological Annotations]*. 1834.
- H. Williams. On the formulation of travel demand models and economic measures of user benefit. *Environment and Planning A*, 9:285–344, 1977.
- B. Wong. Points of view: Color blindness. *Nature Methods*, 8(6):441–441, 2011. doi: 10.1038/nmeth.1618. URL <https://doi.org/10.1038/nmeth.1618>.
- E. Yang and I. Kopylov. Random quasi-linear utility. *J. of Econ. Theory.*, 209, 2023. doi: 10.1016/j.jet.2023.105650.

Appendix A

Proofs for Chapter 1

Show Theorem 1.1. In this proof, we revert to the more standard notation for binary relations. Take any type $\succeq \in \mathcal{T}$. Write its strict part as $\succ$.

Let $\succeq$ maximize $u \in \mathcal{U}$. Then for all $x, y \in X$,

$$u(x) > u(y) \implies x \succ y. \quad (\text{A.1})$$

As u is non-constant, then $u(x^*) > u(y^*)$ for some $x^*, y^* \in X$. For any $z \in X$, take $\gamma \in (0, 1]$ such that

$$\gamma u(z) + (1 - \gamma)u(x^*) > u(y^*) \text{ and } u(x^*) > \gamma u(z) + (1 - \gamma)u(y^*).$$

By (A.1), $\gamma z + (1 - \gamma)x^* \succ y^*$ and $\gamma z + (1 - \gamma)y^* \succ x^*$. Thus θ is stable.

Suppose instead that θ is stable and show that θ maximizes some $u \in \mathcal{U}$. Take any menu $A = \{a_1, a_2, \dots, a_n\}$ such that

$$a_1 \succeq a_2 \succeq \dots \succeq a_n.$$

Let $X_A = \text{conv}(A)$. As X is convex, then $X_A \subset X$. Show that for all $z \in X_A$,

$$a_1 \succeq z \succeq a_n. \quad (\text{A.2})$$

The proof is by induction with respect to $n \geq 2$. If $n = 2$, then by Independence, (A.2) holds for all $z \in X_A$ because $z = \gamma a_1 + (1 - \gamma)a_2$ for some $\gamma \in [0, 1]$. Let $n > 2$. Take any $z \in X_A$ and write it as $z = \gamma y + (1 - \gamma)a_n$ for some $\gamma \in [0, 1]$ and $y \in X_{A \setminus a_n}$, where $X_{A \setminus a_n} = \text{conv}(A \setminus a_n)$. By the inductive assumption, $a_1 \succeq y \succeq a_{n-1} \succeq a_n$. By Independence,

$$a_1 \succeq \gamma y + (1 - \gamma)a_1 \succeq \gamma y + (1 - \gamma)a_n \succeq \gamma a_n + (1 - \gamma)a_n = a_n$$

and hence, (A.2) holds.

Let $\mathcal{U}_A$ be the set of all non-constant mixture linear functions $u : X_A \rightarrow \mathbb{R}$.

Lemma A.1. *If $a_1 \succ a_n$, then there is $u \in \mathcal{U}_A$ such that for all $x, y \in X_A$,*

$$x \succeq y \implies u(x) \geq u(y). \quad (\text{A.3})$$

Moreover, such u is unique up to a positive linear transformation.

Proof. Suppose that $a_1 \succ a_n$. For each $\alpha \in [0, 1]$, let $z_\alpha = \alpha a_1 + (1 - \alpha)a_n$. Show that for all $\alpha, \beta \in [0, 1]$,

$$\alpha \geq \beta \iff z_\alpha \succeq z_\beta. \quad (\text{A.4})$$

As $a_1 \succ a_n$, then by Independence,

$$\alpha > \beta \implies z_\alpha = \alpha a_1 + (1 - \alpha)a_n \succ a_n \implies$$

$$z_\alpha = \frac{\beta}{\alpha} z_\alpha + \frac{\alpha - \beta}{\alpha} z_\alpha \succ \frac{\beta}{\alpha} z_\alpha + \frac{\alpha - \beta}{\alpha} a_n = z_\beta.$$

If $\alpha \geq \beta$, then either $z_\alpha = z_\beta$ or $z_\alpha \succ z_\beta$. Thus (A.4) holds.

For all $x \in X_A$, let

$$\begin{aligned} u(x) &= \inf\{\alpha \in [0, 1] : z_\alpha \succeq x\} \\ v(x) &= \sup\{\beta \in [0, 1] : x \succeq z_\beta\}. \end{aligned} \tag{A.5}$$

By (A.2), these values are well-defined. Show that $u = v$. Indeed, $z_\alpha \succeq x \succeq z_\beta$ implies $\alpha \geq \beta$ and hence, $u(x) \geq v(x)$. Suppose that $u(x) > v(x)$. Then there is γ such that $u(x) > \gamma > v(x)$. By (A.5), $u(x) > \gamma$ implies $x \succ z_\gamma$. By (A.5), $\gamma > v(x)$ implies $z_\gamma \succ x$. By contradiction, $u(x) = v(x)$.

By (A.5), u is not constant because $u(a_1) = 1$ and $u(a_n) = 0$. Show that u is mixture linear.

By Independence, for any $x, y \in X$ and $\alpha, \beta, \gamma \in [0, 1]$ such that $z_\alpha \succeq x$ and $z_\beta \succeq y$,

$$z_{\gamma\alpha + (1-\gamma)\beta} = \gamma z_\alpha + (1-\gamma)z_\beta \succeq \gamma x + (1-\gamma)z_\beta \succeq \gamma x + (1-\gamma)y.$$

Thus $u(\gamma x + (1-\gamma)y) \leq \gamma\alpha + (1-\gamma)\beta$. By definition of u , α and β can be arbitrarily close to $u(x)$ and $u(y)$ respectively. Thus

$$u(\gamma x + (1-\gamma)y) \leq \gamma u(x) + (1-\gamma)u(y).$$

Analogously, $v(\gamma x + (1-\gamma)y) \geq \gamma v(x) + (1-\gamma)v(y)$. As $u = v$, then $u \in \mathcal{U}_A$.

For all $x, y \in X_A$, let $\gamma = \frac{1}{2}u(x) + \frac{1}{2}u(y)$. By (A.5),

$$u(x) > u(y) \implies u(x) > \gamma > u(y) \implies x \succ z_\gamma \succ y \implies x \succ y.$$

Thus representation (A.3) holds.

Suppose that (A.3) holds with another $w \in \mathcal{U}_A$. As $a_1 \succeq a_n$, then $w(a_1) \geq w(a_n)$. If $w(a_1) = w(a_n)$, then by (A.2), $w(a_1) \geq w(z) \geq w(a_n)$ for all $z \in X_A$, and w is constant on

X_A . By contradiction, $w(a_1) > w(a_n)$. For all $x \in X_A$,

$$w'(x) = \frac{w(x) - w(a_n)}{w(a_1) - w(a_n)}.$$

Then $w' \in \mathcal{U}_A$ is a positive linear transformation of w such that $w'(a_1) = 1$ and $w'(a_n) = 0$. Representation (A.3) still holds for w' . As $\alpha = w'(z_\alpha) \geq w'(x)$ for all α such that $z_\alpha \succeq x$, then $u(x) \geq w'(x)$. As $b = w'(z_\beta) \leq w'(x)$ for all β such that $x \succeq z_\beta$, then $w'(x) \geq v(x)$. Thus $w'(x) = u(x)$, and w is a positive linear transformation of u . $\square$

As $\succeq$ is stable, then there exist $x^*, y^* \in X$ that satisfy (1.7). Then $x^* \succeq y^*$ because $\gamma x^* + (1 - \gamma)x^* \succeq y^*$ for some $\gamma \in (0, 1)$. Moreover, if $x^* = y^*$, then (1.7) cannot hold for $z \neq x^*$. Thus $x^* \succ y^*$. Take any menu $A = \{a_1, a_2, \dots, a_n\}$ such that $x^*, y^* \in A$, and

$$a_1 \succeq a_2 \succeq \dots \succeq a_n.$$

As $x^* \succ y^*$, then $a_1 \succ a_n$. By Lemma A.1, there is $u \in \mathcal{U}_A$ such that (A.3) holds. By (A.3), $u(x^*) \geq u(y^*)$. Suppose that $u(x^*) = u(y^*)$. If $u(a_1) > u(x^*)$, then $\gamma a_1 + (1 - \gamma)y^* \succ x^*$ for all $\gamma \in (0, 1)$. If $u(a_n) < u(x^*)$, then $y^* \succ \gamma a_n + (1 - \gamma)x^*$ for all $\gamma \in (0, 1)$. Either case is a contradiction. Thus, $u(x^*) > u(y^*)$.

By Lemma A.1, there is a unique $u_A \in \mathcal{U}_A$ that satisfies (A.3), and

$$u_A(x^*) = 1 > 0 = u_A(y^*).$$

For each $x \in X$, define

$$u(x) = u_{\{x^*, y^*, x\}}(x).$$

Take any $x, y \in X$ and $\gamma \in [0, 1]$. Let $B = \{x^*, y^*, x, y, \gamma x + (1 - \gamma)y\}$. The uniqueness of the functions u_B and $u_{\{x^*, y^*, x\}}$ implies that $u_B = u_{\{x^*, y^*, x\}}$ on $X_{\{x^*, y^*, x\}}$. Thus $u(x) = u_B(x)$.

Similarly, $u(y) = u_B(y)$ and

$$u(\gamma x + (1 - \gamma)y) = u_B(\gamma x + (1 - \gamma)y).$$

As u_B satisfies (A.3) on X_B , then

$$x \succeq y \implies u(x) = u_B(x) \geq u_B(y) = u(y) \implies u(x) \geq u(y).$$

As u_B is mixture linear, then

$$u(\gamma x + (1 - \gamma)y) = u_B(\gamma x + (1 - \gamma)y) = \gamma u_B(x) + (1 - \gamma)u_B(y) = \gamma u(x) + (1 - \gamma)u(y)$$

and hence, u is mixture linear as well. Thus $u \in \mathcal{U}$ and (A.3) holds.

Suppose that L is finite dimensional. Then there is a menu A such that the affine hull of A equals the affine hull of X . Index $A = \{a_1, \dots, a_n\}$ so that

$$a_1 \succeq a_2 \succeq \dots \succeq a_n,$$

and let $a_0 = \sum_{i=1}^n \frac{1}{n} a_i$. As X belongs to the affine hull of A , then any $x \in X$ can be written as an affine combination

$$x = \sum_{i=1}^n \gamma_i a_i \text{ such that } \sum_{i=1}^n \gamma_i = 1.$$

Thus for any $x \in X$, there is $\varepsilon > 0$ such that

$$0 \leq \gamma \leq \varepsilon \implies \gamma x + (1 - \gamma)a_0 \in X_A. \quad (\text{A.6})$$

As θ is non-null, then there are $x, y \in X$ such that $x \succ y$. Take $\varepsilon > 0$ such that X_A contains

both $\varepsilon x + (1 - \varepsilon)a_0$ and $\varepsilon y + (1 - \varepsilon)a_0$. By Independence and (A.2),

$$x \succ y \implies \varepsilon x + (1 - \varepsilon)a_0 \succ \varepsilon y + (1 - \varepsilon)a_0 \implies a_1 \succ a_n.$$

By Lemma A.1, there is $u \in \mathcal{U}_A$ that satisfies (A.3) on X_A .

Take $x^* = \frac{a_0 + a_1}{2}$ and $y^* = \frac{a_0 + a_n}{2}$. Take any $z \in X$. By (A.6), there is $\varepsilon > 0$ such that X_A contains $\varepsilon z + (1 - \varepsilon)x^*$. As u is non-constant, then by (A.2), $u(a_1) > u(a_n)$. As u is mixture linear, then $u(x_*) > u(y^*)$ and hence,

$$u(\varepsilon z + (1 - \varepsilon)x^*) > u(y^*) \text{ and } u(x^*) > u(\varepsilon z + (1 - \varepsilon)y^*)$$

for sufficiently small ε . By (A.3), θ is stable.

A.1 Proof of Theorem 1.2

Take any RCR ρ . Suppose that ρ is induced by some RTF μ , Then ρ satisfies Additivity and MC.

Take any open set $V \subset L$. Show that

$$\liminf_{\alpha \rightarrow \gamma} \rho(V, \alpha A + (1 - \alpha)B) \leq \rho(V, \gamma A + (1 - \gamma)B)$$

Lemma A.2. *Additivity and MC imply Linearity, Monotonicity, and Extremeness.*

Next, suppose that ρ satisfies GP's axioms. Invoke GP's Theorem 2 to show that ρ is induced by some RUF π .

Finally, suppose that ρ is induced by some RUF π . Show that ρ is induced by some RTF μ . Fix some type θ^* . For each $u \in \mathcal{U}$, define a type θ_u that maximizes u and uses θ^* to break

ties. For each $\Theta \subset \mathcal{T}$, let

$$\mu(\Theta) = \pi(\{u \in \mathcal{U} : \theta_u \in \Theta\}).$$

Then ρ is induced by μ .

We have shown the implications $(i) \implies (ii) \implies (iii) \implies (i)$ for the claims of Theorem 1.2. Thus the proof is complete.

A.2 Proofs of Theorems 1.3 and 1.4

First, we show that a discrete map $\mu \in \mathcal{D}^*$ can be derived uniquely from the stochastic choice rule ρ_μ . Then this identification procedure is used to prove Theorems 1.3, 1.4, and 1.5.

To derive μ from ρ_μ , we need some preliminary steps.

Lemma A.3. *For any menu $A \in \mathcal{M}$ and converging sequences $\{0 < \gamma_n \leq 1\}$ and $\{B_n \in \mathcal{M}\}$ such that $\gamma = \lim_{n \rightarrow \infty} \gamma_n$ and $B = \lim_{n \rightarrow \infty} B_n$,*

$$\lim_{n \rightarrow \infty} \gamma_n A + (1 - \gamma_n) B_n = \gamma A + (1 - \gamma) B.$$

Proof. Take any $A \in \mathcal{M}$ and converging sequences $\{0 < \gamma_n \leq 1\}$ and $\{B_n \in \mathcal{M}\}$ such that $\gamma = \lim_{n \rightarrow \infty} \gamma_n$ and $B = \lim_{n \rightarrow \infty} B_n$. Show that for all $n \in \mathbb{N}$,

$$d_h(\gamma A + (1 - \gamma) B, \gamma_n A + (1 - \gamma_n) B_n) \leq R|\gamma - \gamma_n| + d_h(B, B_n) \quad (\text{A.7})$$

where $R = \max_{x \in A, y \in B} \|x - y\|$. Take any $z \in \gamma A + (1 - \gamma) B$. Then $z = \gamma x + (1 - \gamma) y$ for some $x \in A$ and $y \in B$. By the definition of the Hausdorff metric d_h , there is $y_n \in B_n$ such

that $\|y - y_n\| \leq d_h(B, B_n)$. Let $z_n = \gamma_n x + (1 - \gamma_n)y_n$. Then $z_n \in \gamma_n A + (1 - \gamma_n)B_n$, and

$$\begin{aligned} \|z - z_n\| &\leq \|z - (\gamma_n x + (1 - \gamma_n)y)\| + \|(\gamma_n x + (1 - \gamma_n)y) - z_n\| \leq \\ &|\gamma - \gamma_n| \|x - y\| + \gamma_n \|y - y_n\| \leq R|\gamma - \gamma_n| + d_h(B, B_n). \end{aligned}$$

Similarly, for any $z_n \in \gamma_n A + (1 - \gamma_n)B_n$, there is $z \in \gamma A + (1 - \gamma)B$ such that

$$\|z - z_n\| \leq R|\gamma - \gamma_n| + d_h(B, B_n).$$

Thus (A.11) holds. By (A.11), $\lim_{n \rightarrow \infty} \gamma_n A + (1 - \gamma_n)B_n = \gamma A + (1 - \gamma)B$. □

For any menu $C \in \mathcal{M}$, define a function $b_C : \mathcal{T} \rightarrow C$ such that

$$b_C(\theta) = b_\theta C \text{ for all } \theta \in \mathcal{T}.$$

Say that b_C is bijective on $\Theta \subset \mathcal{T}$ if for any $x \in C$, there is a unique $\theta \in \Theta$ such that $x = b_C(\theta)$.

Lemma A.4. *For any finite set $\Theta \subset \mathcal{T}$, there is $C \in \mathcal{M}$ such b_C is bijective on Θ .*

Proof. Let $\Theta = \{\theta_1, \dots, \theta_k\}$ consist of $k = |\Theta|$ distinct types. Let I be the set of all pairs $i, j \in \{1, \dots, k\}$ such that $i > j$. For each pair $(i, j) \in I$, the types $\theta_i, \theta_j \in \mathcal{D}$ are distinct and total. Thus there are distinct elements $x_{ij}, x_{ji} \in X$ such that $x_{ij}\theta_i^*x_j$ and $x_{ji}\theta_j^*x_i$. For any $n \in \{1, \dots, k\}$, let

$$c_n = \sum_{(i,j) \in I} \frac{2}{k(k-1)} x_{ij}^n,$$

where $x_{ij}^n = x_{ij}$ if $x_{ij}\theta_n^*x_{ji}$ and $x_{ji}^n = x_{ji}$ if $x_{ji}\theta_n^*x_{ij}$. For any $(m, n) \in I$, the types θ_n, θ_m

satisfy Independence and hence,

$$\begin{aligned} c_m &= \left(\sum_{(i,j) \in I} \frac{2}{k(k-1)} x_{ij}^m \right) \theta_m^* \left(\sum_{(i,j) \in I} \frac{2}{k(k-1)} x_{ij}^n \right) = c_n \\ c_n &= \left(\sum_{(i,j) \in I} \frac{2}{k(k-1)} x_{ij}^n \right) \theta_n^* \left(\sum_{(i,j) \in I} \frac{2}{k(k-1)} x_{ij}^m \right) = c_m \end{aligned} \quad (\text{A.8})$$

because $x_{ij}^n \theta_n x_{ij}^m$ and $x_{ij}^m \theta_m x_{ij}^n$ for all $(i, j) \in I$, and $x_{ij}^n \theta_n^* x_{ij}^m$ and $x_{ij}^m \theta_m^* x_{ij}^n$ when $(i, j) = (m, n)$. Let $C = \{c_1, \dots, c_k\}$. By (A.12), for all $i = 1, \dots, k$, $c_i = b_C(\theta_i)$. Thus b_C is bijective on Θ . $\square$

Take any PTF μ and any finite set $\Theta \subset \mathcal{T}$ such that $\Theta_\mu \subset \Theta$. By Lemma A.7, there is a menu $C \in \mathcal{M}$ such that b_C is bijective on Θ . For any $\gamma \in (0, 1]$ and $A \in \mathcal{M}$, let

$$A_\gamma = \gamma C + (1 - \gamma)A.$$

Show that for all $\theta \in \Theta$,

$$\mu_{A_\gamma}(\theta) = \rho_\mu(b_\theta A_\gamma, A_\gamma). \quad (\text{A.9})$$

Take any $\psi \in \Theta_\mu$ such that $\psi \neq \theta$. As b_C is bijective on Θ_μ , then $b_\theta C \neq b_\psi C$ and hence, $b_\theta C \theta^* b_\psi C$ because θ is total. By Independence,

$$b_\theta A_\gamma = [\gamma b_\theta C + (1 - \gamma)b_\theta A] \theta^* [\gamma b_\psi C + (1 - \gamma)b_\psi A] = b_\psi A_\gamma.$$

As θ is the only type in Θ for which $b_\theta A_\gamma$ is best in A_γ , then (A.13) holds by definition of ρ_μ .

As μ is continuous, then by Lemma A.6, for all $A \in \mathcal{M}$ and $\theta \in \Theta$,

$$\mu_A(\theta) = \lim_{\gamma \rightarrow 0} \mu_{A_\gamma}(\theta) = \lim_{\gamma \rightarrow 0} \rho_\mu(b_\theta A_\gamma, A_\gamma). \quad (\text{A.10})$$

This identification formula has several implications in our results.

1. First, μ is determined uniquely by ρ_μ . Indeed, if $\mu' \in \mathcal{D}^*$ is such that $\rho_\mu = \rho_{\mu'}$, then by (A.14), for all menus $A \in \mathcal{M}$ and types $\theta \in \Theta = \Theta_\mu \cup \Theta_{\mu'}$,

$$\mu_A(\theta) = \lim_{\gamma \rightarrow 0} \rho_\mu(b_\theta A_\gamma, A_\gamma) = \mu'_A(\theta).$$

Thus $\mu = \mu'$.

2. Second, if ρ_μ satisfies Additivity, then $\mu \in \mathcal{D}$ is constant. Let $\Theta = \Theta_\mu$. By Additivity, for all $\theta \in \Theta_\mu$ and $\gamma \in (0, 1]$

$$\rho_\mu(b_\theta C, C) = \rho_\mu(b_\theta C, C, \gamma, A) \geq \rho_\mu(b_\theta A_\gamma, A_\gamma)$$

because $b_\theta A_\gamma \in \gamma b_\theta C + (1 - \gamma)A$. As

$$\sum_{\theta \in \Theta} \rho_\mu(b_\theta C, C) = \sum_{\theta \in \Theta} \rho_\mu(b_\theta A_\gamma, A_\gamma) = 1$$

then $\rho_\mu(b_\theta C, C) = \rho_\mu(b_\theta A_\gamma, A_\gamma)$ for all $\theta \in \Theta$. By (A.14), $\mu_A(\theta) = \rho_\mu(b_\theta C, C)$ is invariant of A .

3. Third, if ρ_μ satisfies Monotonicity, then μ is selective. Let $\Theta = \Theta_\mu$. Take any menus $A, B \in \mathcal{M}$ and type $\theta \in \Theta$ such that $b_\theta A \cup B \in A$. Then $b(\theta, A) = b(\theta, A \cup B)$. By Independence, for all $\gamma \in (0, 1]$,

$$b(\theta, A_\gamma) = \gamma b(\theta, C) + (1 - \gamma)b(\theta, A) = \gamma b(\theta, C) + (1 - \gamma)b(\theta, A \cup B) = b(\theta, A_\gamma \cup B_\gamma).$$

By Monotonicity,

$$\rho_\mu(b(\theta, A_\gamma), A_\gamma) \geq \rho_\mu(b(\theta, A_\gamma \cup B_\gamma), A_\gamma \cup B_\gamma).$$

By (A.14) $\mu_A(\theta) \geq \mu A \cup B(\theta)$. Thus μ is selective.

We have shown already that if μ is selective, then ρ_μ satisfies Monotonicity, and if μ is constant, then ρ_μ satisfies Additivity. Thus Theorems 1.3 and 1.5 follow from Theorem 1.4, which we prove next.

Theorem 1.4 is shown first. Theorem 1.3 follows as a corollary.

Let $\rho = \rho_\mu$ be induced by some PTF μ with support Θ_μ . Discreteness follows from (1.15). Show BA and LA.

Take any trial $(x, A) \in \Omega$. For any $\gamma \in (0, 1)$, the DTFs μ_A and $\mu_{\gamma A + (1-\gamma)B}$ share the same support and hence,

$$\mu_A(\mathcal{T}(x, A)) = 0 \iff \mu_{\gamma A + (1-\gamma)B}(\mathcal{T}(x, A)) = 0.$$

Thus partition (1.14) implies BA.

Take any simple sequence B_1^∞ . Let $B = \lim_{n \rightarrow \infty} B_n$. Suppose that $\rho(x, A) > 0$. As μ_A and μ_B have the same support, then

$$\rho(x, A) > 0 \implies \mu_A(\mathcal{T}(x, A)) > 0 \implies \mu_B(\mathcal{T}(x, A)) > 0.$$

Let $C_1 = A \cup B_1$ and $C_n = \frac{1}{n}A + \frac{n-1}{n}B_n$ for all $n = 2, 3, \dots$. Then

$$\lim_{n \rightarrow \infty} C_n = \lim_{n \rightarrow \infty} B_n = B.$$

As $B_n \subset \text{conv}(B_1)$, then $C_n \subset \text{conv}(C_1)$. Thus C_1^∞ is simple. For each type $\theta \in \mathcal{T}$, the

function μ_θ is simply continuous. Thus

$$\lim_{n \rightarrow \infty} \rho(x, A, \frac{1}{n}, B_n) = \lim_{n \rightarrow \infty} \sum_{\theta \in \mathcal{T}(x, A)} \mu_\theta(C_n) = \sum_{\theta \in \mathcal{T}(x, A)} \mu_\theta(B) = \mu_B(\mathcal{T}(x, A)) > 0,$$

and LA holds.

Conversely, suppose that ρ satisfies BA, LA, and Discreteness. For all $A \in \mathcal{M}$, let

$$P(A) = \{x \in A : \rho(x, A) > 0\}$$

be the set of all elements in A that are chosen with a positive probability.

For all $(x, A) \in \Omega$, $B \in \mathcal{M}$, $y \in X$, and $\gamma \in (0, 1]$, BA implies two familiar properties for the filter P .

$$\text{P1 } P(\gamma A + (1 - \gamma)y) = \gamma P(A) + (1 - \gamma)y,$$

$$\text{P2 if } x \in P(A \cup B), \text{ then } x \in P(A).$$

P1 follows from BA by taking a singleton $B = y$. P2 is the standard Chernoff property. It follows from BA as well:

$$\rho(x, A \cup B) > 0 \implies \rho(x, A \cup B, \frac{1}{2}, A) > 0 \implies$$

$$\rho(x, A, \frac{1}{2}, A \cup B) > 0 \implies \rho(x, A) > 0,$$

where $\rho(x, A \cup B, \frac{1}{2}, A) \leq \rho(x, A, \frac{1}{2}, A \cup B)$ because $\frac{1}{2}x + \frac{1}{2}A \subset \frac{1}{2}x + \frac{1}{2}(A \cup B)$.

By Discreteness, there is a menu $B \in \mathcal{M}$ such that for all $A \in \mathcal{M}$,

$$|P(B)| \geq |P(A)|.$$

Let $C = P(B)$. By F2, $P(C) = C$. Write C as $C = \{c_1, c_2, \dots, c_K\}$, where

$$K = |C| = |P(C)| = |P(B)|$$

is the maximal number of elements that can be chosen with a positive probability in some menu.

Lemma A.5. *There is a unique set $\Theta \subset \mathcal{T}$ such that for all $A \in \mathcal{M}$,*

$$P(A) = \bigcup_{\theta \in \Theta} b_\theta A.$$

Moreover, $|\Theta| = K$.

Proof. To show this claim, assume first that X is bounded, and there is $R > 0$ such that $\|x\| < R$ for all $x \in X$. Take $\alpha \in (0, 1)$ such that

$$2\alpha R < (1 - \alpha) \min_{1 \leq i < j \leq K} \|c_i - c_j\|.$$

Then for any menu $A \in \mathcal{M}$,

$$|\alpha A + (1 - \alpha)C| = |A|K.$$

Indeed, take any $a, b \in A$ and $c_i, c_j \in C$ such that $\alpha a + (1 - \alpha)c_i = \alpha b + (1 - \alpha)c_j$. Then

$$\alpha(a - b) = (1 - \alpha)(c_i - c_j) = 0$$

because otherwise, $\alpha\|a - b\| \leq 2\alpha R < (1 - \alpha)\|c_i - c_j\|$ is a contradiction. Therefore all $|A|K$ mixtures $\alpha a + (1 - \alpha)c_i$ where $a \in A$ and $c_i \in C$ are distinct.

For any $x, y \in X$, define a binary relation θ_i such that $x\theta_i y$ if and only if

$$\alpha x + (1 - \alpha)c_i \in P(\alpha\{x, y\} + (1 - \alpha)C).$$

□

First, we show that a discrete map $\mu \in \mathcal{D}^*$ can be derived uniquely from the stochastic choice rule ρ_μ . Then this identification procedure is used to prove Theorems 1.3, 1.4, and 1.5.

To derive μ from ρ_μ , we need some preliminary steps.

Lemma A.6. *For any menu $A \in \mathcal{M}$ and converging sequences $\{0 < \gamma_n \leq 1\}$ and $\{B_n \in \mathcal{M}\}$ such that $\gamma = \lim_{n \rightarrow \infty} \gamma_n$ and $B = \lim_{n \rightarrow \infty} B_n$,*

$$\lim_{n \rightarrow \infty} \gamma_n A + (1 - \gamma_n) B_n = \gamma A + (1 - \gamma) B.$$

Proof. Take any $A \in \mathcal{M}$ and converging sequences $\{0 < \gamma_n \leq 1\}$ and $\{B_n \in \mathcal{M}\}$ such that $\gamma = \lim_{n \rightarrow \infty} \gamma_n$ and $B = \lim_{n \rightarrow \infty} B_n$. Show that for all $n \in \mathbb{N}$,

$$d_h(\gamma A + (1 - \gamma) B, \gamma_n A + (1 - \gamma_n) B_n) \leq R|\gamma - \gamma_n| + d_h(B, B_n) \quad (\text{A.11})$$

where $R = \max_{x \in A, y \in B} \|x - y\|$. Take any $z \in \gamma A + (1 - \gamma) B$. Then $z = \gamma x + (1 - \gamma)y$ for some $x \in A$ and $y \in B$. By the definition of the Hausdorff metric d_h , there is $y_n \in B_n$ such that $\|y - y_n\| \leq d_h(B, B_n)$. Let $z_n = \gamma_n x + (1 - \gamma_n)y_n$. Then $z_n \in \gamma_n A + (1 - \gamma_n) B_n$, and

$$\begin{aligned} \|z - z_n\| &\leq \|z - (\gamma_n x + (1 - \gamma_n)y)\| + \|(\gamma_n x + (1 - \gamma_n)y) - z_n\| \leq \\ &|\gamma - \gamma_n| \|x - y\| + \gamma_n \|y - y_n\| \leq R|\gamma - \gamma_n| + d_h(B, B_n). \end{aligned}$$

Similarly, for any $z_n \in \gamma_n A + (1 - \gamma_n)B_n$, there is $z \in \gamma A + (1 - \gamma)B$ such that

$$\|z - z_n\| \leq R|\gamma - \gamma_n| + d_h(B, B_n).$$

Thus (A.11) holds. By (A.11), $\lim_{n \rightarrow \infty} \gamma_n A + (1 - \gamma_n)B_n = \gamma A + (1 - \gamma)B$. $\square$

For any menu $C \in \mathcal{M}$, define a function $b_C : \mathcal{T} \rightarrow C$ such that

$$b_C(\theta) = b_\theta C \text{ for all } \theta \in \mathcal{T}.$$

Say that b_C is bijective on $\Theta \subset \mathcal{T}$ if for any $x \in C$, there is a unique $\theta \in \Theta$ such that $x = b_C(\theta)$.

Lemma A.7. *For any finite set $\Theta \subset \mathcal{T}$, there is $C \in \mathcal{M}$ such b_C is bijective on Θ .*

Proof. Let $\Theta = \{\theta_1, \dots, \theta_k\}$ consist of $k = |\Theta|$ distinct types. Let I be the set of all pairs $i, j \in \{1, \dots, k\}$ such that $i > j$. For each pair $(i, j) \in I$, the types $\theta_i, \theta_j \in \mathcal{D}$ are distinct and total. Thus there are distinct elements $x_{ij}, x_{ji} \in X$ such that $x_{ij}\theta_i^*x_j$ and $x_{ji}\theta_j^*x_i$. For any $n \in \{1, \dots, k\}$, let

$$c_n = \sum_{(i,j) \in I} \frac{2}{k(k-1)} x_{ij}^n,$$

where $x_{ij}^n = x_{ij}$ if $x_{ij}\theta_n^*x_{ji}$ and $x_{ij}^n = x_{ji}$ if $x_{ji}\theta_n^*x_{ij}$. For any $(m, n) \in I$, the types θ_n, θ_m satisfy Independence and hence,

$$\begin{aligned} c_m &= \left(\sum_{(i,j) \in I} \frac{2}{k(k-1)} x_{ij}^m \right) \theta_m^* \left(\sum_{(i,j) \in I} \frac{2}{k(k-1)} x_{ij}^n \right) = c_n \\ c_n &= \left(\sum_{(i,j) \in I} \frac{2}{k(k-1)} x_{ij}^n \right) \theta_n^* \left(\sum_{(i,j) \in I} \frac{2}{k(k-1)} x_{ij}^m \right) = c_m \end{aligned} \tag{A.12}$$

because $x_{ij}^n \theta_n x_{ij}^m$ and $x_{ij}^m \theta_m x_{ij}^n$ for all $(i, j) \in I$, and $x_{ij}^n \theta_n^* x_{ij}^m$ and $x_{ij}^m \theta_m^* x_{ij}^n$ when $(i, j) = (m, n)$.

Let $C = \{c_1, \dots, c_k\}$. By (A.12), for all $i = 1, \dots, k$, $c_i = b_C(\theta_i)$. Thus b_C is bijective on Θ . $\square$

Take any PTF μ and any finite set $\Theta \subset \mathcal{T}$ such that $\Theta_\mu \subset \Theta$. By Lemma A.7, there is a menu $C \in \mathcal{M}$ such that b_C is bijective on Θ . For any $\gamma \in (0, 1]$ and $A \in \mathcal{M}$, let

$$A_\gamma = \gamma C + (1 - \gamma)A.$$

Show that for all $\theta \in \Theta$,

$$\mu_{A_\gamma}(\theta) = \rho_\mu(b_\theta A_\gamma, A_\gamma). \quad (\text{A.13})$$

Take any $\psi \in \Theta_\mu$ such that $\psi \neq \theta$. As b_C is bijective on Θ_μ , then $b_\theta C \neq b_\psi C$ and hence, $b_\theta C \theta^* b_\psi C$ because θ is total. By Independence,

$$b_\theta A_\gamma = [\gamma b_\theta C + (1 - \gamma)b_\theta A] \theta^* [\gamma b_\psi C + (1 - \gamma)b_\psi A] = b_\psi A_\gamma.$$

As θ is the only type in Θ for which $b_\theta A_\gamma$ is best in A_γ , then (A.13) holds by definition of ρ_μ .

As μ is continuous, then by Lemma A.6, for all $A \in \mathcal{M}$ and $\theta \in \Theta$,

$$\mu_A(\theta) = \lim_{\gamma \rightarrow 0} \mu_{A_\gamma}(\theta) = \lim_{\gamma \rightarrow 0} \rho_\mu(b_\theta A_\gamma, A_\gamma). \quad (\text{A.14})$$

This identification formula has several implications in our results.

1. First, μ is determined uniquely by ρ_μ . Indeed, if $\mu' \in \mathcal{D}^*$ is such that $\rho_\mu = \rho_{\mu'}$, then by (A.14), for all menus $A \in \mathcal{M}$ and types $\theta \in \Theta = \Theta_\mu \cup \Theta_{\mu'}$,

$$\mu_A(\theta) = \lim_{\gamma \rightarrow 0} \rho_\mu(b_\theta A_\gamma, A_\gamma) = \mu'_A(\theta).$$

Thus $\mu = \mu'$.

2. Second, if ρ_μ satisfies Additivity, then $\mu \in \mathcal{D}$ is constant. Let $\Theta = \Theta_\mu$. By Additivity, for all $\theta \in \Theta_\mu$ and $\gamma \in (0, 1]$

$$\rho_\mu(b_\theta C, C) = \rho_\mu(b_\theta C, C, \gamma, A) \geq \rho_\mu(b_\theta A_\gamma, A_\gamma)$$

because $b_\theta A_\gamma \in \gamma b_\theta C + (1 - \gamma)A$. As

$$\sum_{\theta \in \Theta} \rho_\mu(b_\theta C, C) = \sum_{\theta \in \Theta} \rho_\mu(b_\theta A_\gamma, A_\gamma) = 1$$

then $\rho_\mu(b_\theta C, C) = \rho_\mu(b_\theta A_\gamma, A_\gamma)$ for all $\theta \in \Theta$. By (A.14), $\mu_A(\theta) = \rho_\mu(b_\theta C, C)$ is invariant of A .

3. Third, if ρ_μ satisfies Monotonicity, then μ is selective. Let $\Theta = \Theta_\mu$. Take any menus $A, B \in \mathcal{M}$ and type $\theta \in \Theta$ such that $b_\theta A \cup B \in A$. Then $b(\theta, A) = b(\theta, A \cup B)$. By Independence, for all $\gamma \in (0, 1]$,

$$b(\theta, A_\gamma) = \gamma b(\theta, C) + (1 - \gamma)b(\theta, A) = \gamma b(\theta, C) + (1 - \gamma)b(\theta, A \cup B) = b(\theta, A_\gamma \cup B_\gamma).$$

By Monotonicity,

$$\rho_\mu(b(\theta, A_\gamma), A_\gamma) \geq \rho_\mu(b(\theta, A_\gamma \cup B_\gamma), A_\gamma \cup B_\gamma).$$

By (A.14) $\mu_A(\theta) \geq \mu_{A \cup B}(\theta)$. Thus μ is selective.

We have shown already that if μ is selective, then ρ_μ satisfies Monotonicity, and if μ is constant, then ρ_μ satisfies Additivity. Thus Theorems 1.3 and 1.5 follow from Theorem 1.4, which we prove next.

A.3 Monotonicity of Selective μ

First, we shorthand $\sum_{\psi \in \mathcal{T}_\mu} f_\psi(A) \equiv W(A)$ as the weight of a menu. We also shorthand $\sum_{t \in \mathcal{T}(x,A)} f_t(A) \equiv V(x, A)$ as the value of item x in menu A where we sum up the fitness functions of types where x maximizes A . Since $\rho(x, A) = \mu_A(\mathcal{T}(x, A))$, it must be that

$$\rho(x, A) = \frac{V_x(A)}{W(A)}. \quad (\text{A.15})$$

Consider adding an item y to menu A to construct menu $B = A \cup y$. Some types in $\mathcal{T}(x, A)$ will switch from x to y if $y \theta x$. Otherwise, they will continue choosing x . Therefore, we can create a partition

$$V_x(A) = V_x(B) + V_\sigma(A) \quad (\text{A.16})$$

where σ represents the switching types ($V_\sigma(A)$ is the sum of the fitness functions for these types in A). This allows us to rewrite (A.15) as

$$\begin{aligned} \rho(x, A) &= \frac{V_x(B) + V_\sigma(A)}{W(A)} = \frac{V_x(B)W(B)}{W(A)W(B)} + \frac{V_\sigma(A)W(B)V_y(B)}{W(A)W(B)V_y(B)} \\ &= \frac{W(B)}{W(A)} \left[\rho(x, B) + \rho(y, B) \frac{V_\sigma(A)}{V_y(A)} \right]. \end{aligned} \quad (\text{A.17})$$

By rearranging the terms, we get a clean identity:

$$\rho(x, A) - \rho(x, B) = \frac{W(B) - W(A)}{W(A)} \rho(x, B) + \frac{W(B)}{W(A)} \left[\rho(y, B) \frac{V_\sigma(A)}{V_y(A)} \right]. \quad (\text{A.18})$$

Note that it must be that $W(B) > W(A)$. This follows through with $f_\theta(\cdot)$ increasing as the menu expands, given that each type will have a weakly better fitness function. So far, this shows that a selective μ is monotonic.

A.4 Proofs of Corollary 1.6

Let $X = \Delta^Q$ be the set of all lotteries on some interval $Q \subset \mathfrak{R}$ of monetary prizes. Suppose that ρ_μ satisfies ARA. Take any type $\theta \in \Theta_\mu$. Fix any payoffs $q, r \in Q$ such that $q > r$. By ARA, $q\theta^*r$ because $\rho_\mu(r, \{q, r\}) = 0$. Fix any lotteries $z_q, z_r \in X$. By ARA, there is $\varepsilon > 0$ such that

$$\varepsilon z_q + (1 - \varepsilon)q\theta^*\frac{q+r}{2}.$$

By Independence, for all $\alpha \in [0, \varepsilon]$, $\alpha z_q + (1 - \alpha)q\theta^*\frac{q+r}{2}$ is preserved. Take α sufficiently small so that

$$E(\alpha z_r + (1 - \alpha)r) = \alpha E(r) + (1 - \alpha)r < \frac{q+r}{2}.$$

Then by ARA,

$$\alpha z_q + (1 - \alpha)q\theta^*\frac{q+r}{2}\theta^*\alpha z_r + (1 - \alpha)r.$$

Thus the type θ satisfies stability for $x^* = q$ and $y^* = r$. By Theorem 1.1, θ maximizes some $u \in \mathcal{U}$. As u is non-constant and has the expected utility form, then u is non-constant on Q . By Theorem 1.1, there are two possible cases. First, if θ is stable

Moreover, if μ is monotonic, then ρ_μ satisfies Monotonicity. Indeed, for any $(x, A) \in \Omega$ and $y \in X$, the inclusion $\mathcal{T}(x, A) \supset \mathcal{T}(x, A \cup y)$ holds. Thus,

$$\begin{aligned} \rho_\mu(x, A) &= \mu_A(\mathcal{T}(x, A)) \geq \mu_A(\mathcal{T}(x, A \cup y)) = \\ &= \sum_{\theta \in \Theta_\mu: x=b(\theta, A \cup y)} \mu_\theta(A) \geq \sum_{\theta \in \Theta_\mu: x=b(\theta, A \cup B)} \mu_{A \cup B}(\theta) = \rho_\mu(x, A \cup B). \end{aligned}$$

Note that if μ is monotonic, then ρ_μ satisfies Monotonicity. Conversely, if ρ_μ satisfies Monotonicity, then μ is selective.

Appendix B

Materials for Chapter 2

B.1 Screens for Experimental Program

Welcome!

In this experiment, you will be making decisions regarding uncertain outcomes.

This experiment has four phases:

- Phase 1: Instructions. There will be a comprehension check at the end to ensure you understand the process of the experiment.
 - Phase 2: Main Tasks. There will be 50 tasks in which you will make a choice between two to three options.
 - Phase 3: Questionnaire. This will survey your demographics, identity, and pattern recognition skills.
 - Phase 4: Secondary Tasks. There will be 10 tasks in which you will make a choice between two options.
-

Click "Next" to view the instructions for this experiment.

Next

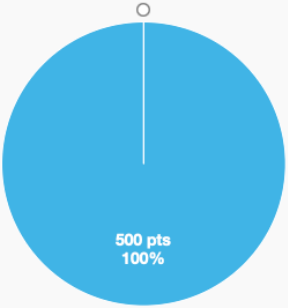
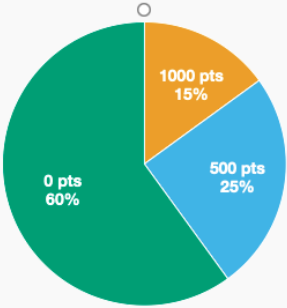
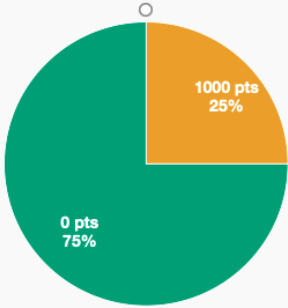
Figure B.1: Welcome Page

Instructions

In each main and secondary task, you will choose between two or three options.
An option could be a certain outcome or it could represent uncertain outcomes.
Each option will be presented in the format of a pie chart.
The following is a sample task similar to those in the experiment.

Sample Task

Choose one of the following options:



[Continue](#) ← these buttons are only for visualization purposes in this sample → [Open Calculator](#)

- The option on the left shows two uncertain outcomes. There is a 25% chance you will receive 1000 points, and a 75% chance you will receive 0 points.
- The center option shows three uncertain outcomes. There is a 15% chance you will receive 1000 points, a 25% chance you will receive 500 points, and a 60% chance you will receive 0 points.
- The option on the right is a certain outcome. You will receive 500 points for sure.

The probabilities for the outcomes of a single option must add up to 100%.
Take note that each task will have a different set of options and values.

For each task, choose the option you like the most by clicking on the corresponding pie chart. Try it!
This experiment is designed to survey your preferences when faced with uncertainty. There are no tricks or deception.
In general you would like to get more points. However, higher point values generally have a lower probability.
For the main tasks, a calculator is available for you to use on the bottom-right corner. More instructions regarding the calculator will be on the next page.
Once you are ready to go to the next task in the experiment, click "Continue".
You will not be able to return to previous tasks to change your answer.

Click "Next" to view the instructions for the calculator.

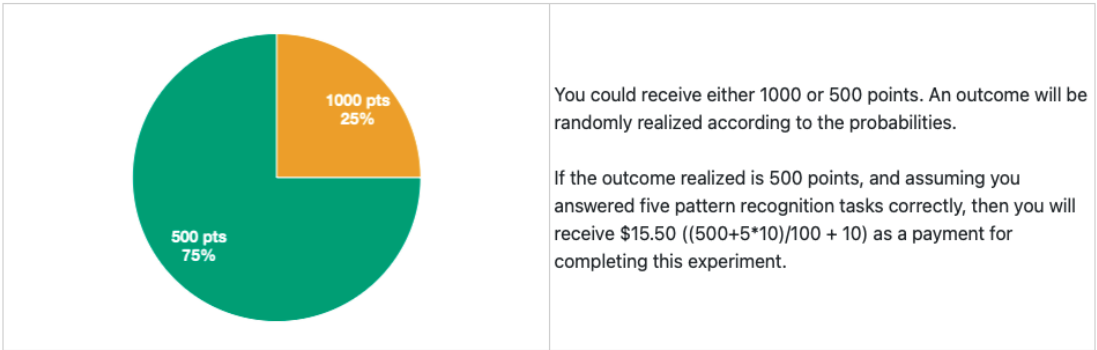
[Next](#)

Figure B.2: Tutorial Page

Payment Structure

At the end of the experiment, a random main task will be selected and implemented.
The option you chose in that task will play out and you will receive points according to the outcome that actually happens.
You will receive 10 extra points for each correct pattern recognition response in the questionnaire phase.
The conversion rate is 1 cent for each point you receive. Therefore, the more points you receive, the higher your payment will be.
You will also earn an additional \$10 for arriving at the session on time.

For example, assume the option you chose in the randomly selected main task looks like this:



Once we confirm your completion of the experiment, you will receive the payment via your preferred platform (Venmo/Zelle/PayPal) within 24 hours.

Click on "Next" to answer some comprehension questions.

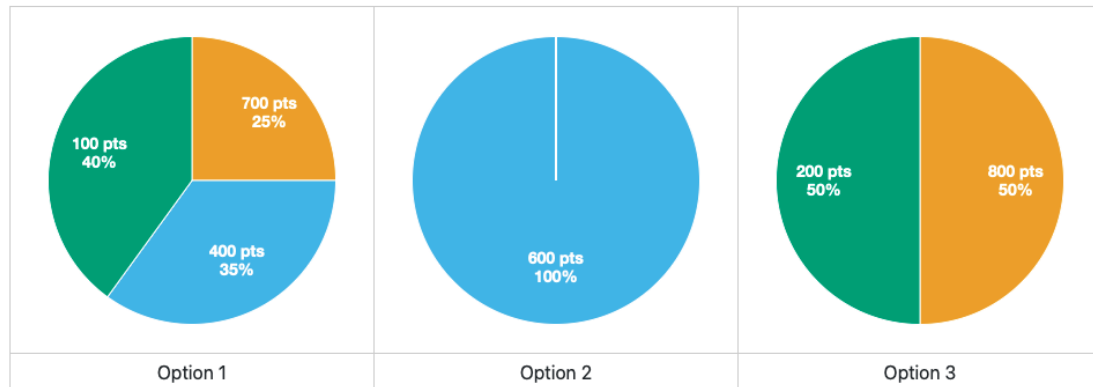
Next

Figure B.3: Payment Structure

Comprehension Check

Please answer each of the following questions.

You must get all questions correct before starting Phase 2: Main Tasks.



1. Look at the three pie charts above. Which one represents a certain outcome?

- ☐ Option 1
- ☐ Option 2
- ☐ Option 3

2. How many points, if selected, is worth a dollar?

- ☐ 1 point
- ☐ 50 points
- ☐ 100 points

3. Assume you see the pie charts above in one main task. You chose Option 1, and this task was randomly selected at the end to play out. Assume you have 4 correct pattern recognition responses. Which of the following is NOT a possible payment you will receive at the end of the experiment? (Remember that there is a show up payment.)

- ☐ \$7.40
- ☐ \$11.40
- ☐ \$14.40

4. Assume your friend earned \$18 today. Your friend did not answer any of the pattern recognition tasks correctly. If the pie charts above are the main task randomly selected at the end to play out, what must have been your friend's choice?

- ☐ Option 1
- ☐ Option 2
- ☐ Option 3

5. True or False? The experiments concludes right after the questionnaire phase.

- ☐ True
- ☐ False

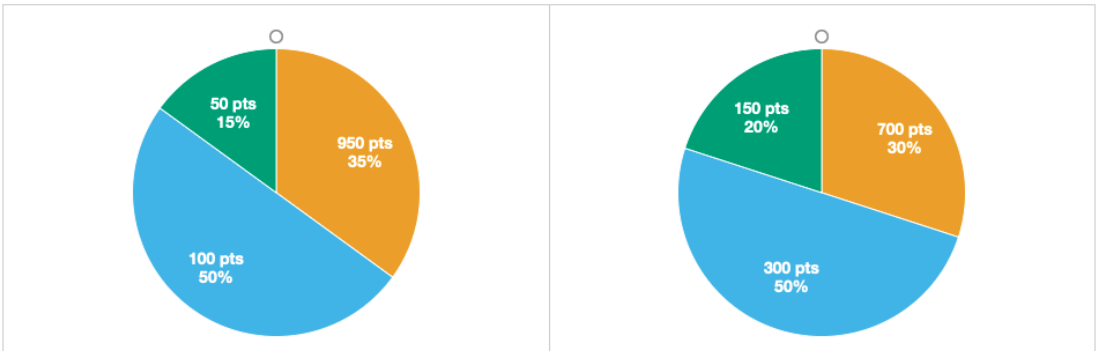
Click "Submit" when you answer all questions.

Submit

Figure B.4: Comprehension Check

Main Task 1

Choose one of the following options:



Click "Continue" to go on to the next task.

Continue

Open Calculator

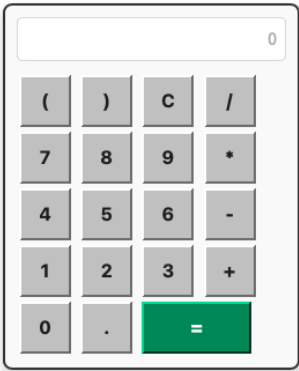


Figure B.5: Main Task with On Screen Calculator

Demographics and Identity

1. Select the gender you identify as:

- ☐ Male
- ☐ Female
- ☐ Non-Binary
- ☐ Prefer not to say

2. Are you of Hispanic, Latino, or of Spanish origin?

- ☐ Yes
- ☐ No

3. How would you describe yourself?

- ☐ American or Alaska Native
- ☐ Asian
- ☐ Black or African American
- ☐ Native Hawaiian or Other Pacific Islander
- ☐ White
- ☐ Multiple of the above
- ☐ None of the above

4. Select the school you are mainly enrolled as a student in:

5. Select the year you are currently in:

- ☐ Freshman
- ☐ Sophomore
- ☐ Junior
- ☐ Senior or above
- ☐ Graduate
- ☐ Faculty or staff

6. Type in an estimate of the number of courses in the economics department you have taken thus far:

7. Type in your approximate cumulative GPA:

Click "Next" after answering each of the questions above.

Next

Figure B.6: Demographics Questionnaire

Pattern Recognition Tutorial

You will see 8 pattern recognition problems (in increasing difficulty) similar to the one below:

Select the option that is most likely the missing object:

Continue

← this button is only for visualization purposes in this sample

The top image is a three-by-three grid of objects. The bottom-right object is missing.

Using the patterns of the other objects in the grid, choose one of the bottom eight options that would most likely be the missing object.

For example, in this problem, the missing object is likely .

This is because the bottom row are all gray objects, and each row has one triangle, one rectangle, and one square.

Note that the outer border for each option is not part of the pattern itself.

You will have 60 seconds to complete each pattern recognition problem. Once the timer is up, the response currently selected will be submitted. Empty responses will be recorded as empty.

Recall that for each pattern recognition problem you answer correctly, you will receive 10 extra points. Therefore, all else equal, the more correct responses you have, the higher your payoff.

Click "Next" to start the pattern recognition problems.

Next

Figure B.7: Pattern Recognition Tutorial

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This is the end of the experiment.

Task 21 has been randomly selected.

Based off of your choice from that task, you earned 250 points.

Currently, your responses to your pattern recognition will be manually scored, and will be revealed to you during payment.

Therefore, you will receive a payment of \$12.50 plus the pattern recognition section.

Once we confirm your completion, you will receive your payment within 24 hours. Feel free to take a screenshot of this page.

Thank you for your participation in this experiment at UC Irvine's Experimental Social Science Laboratory.

Figure B.8: Conclusion Page

B.2 List of Main Task Lotteries

Group	Set	Riskier Lottery (R)			Safer Lottery (S)			Certain (c) Payoff
		State 1	State 2	State 3	State 1	State 2	State 3	
—	0 [†]	0.20, 600 pts	0.70, 300 pts	0.10, 0 pts	0.10, 600 pts	0.60, 300 pts	0.30, 0 pts	100 pts
3	1	0.20, 900 pts	0.80, 0 pts	—	0.50, 500 pts	0.50, 100 pts	—	275 pts*
	2	0.30, 800 pts	0.70, 100 pts	—	0.20, 600 pts	0.70, 400 pts	0.10, 200 pts	200 pts
	3	0.20, 700 pts	0.40, 200 pts	0.40, 0 pts	0.60, 400 pts	0.40, 100 pts	—	100 pts
	4	0.30, 650 pts	0.60, 50 pts	0.10, 0 pts	0.70, 400 pts	0.20, 100 pts	0.10, 0 pts	100 pts
4	5	0.40, 700 pts	0.60, 200 pts	—	0.70, 500 pts	0.30, 200 pts	—	200 pts
	6	0.30, 800 pts	0.70, 200 pts	—	0.30, 600 pts	0.50, 350 pts	0.20, 250 pts	250 pts
	7	0.20, 1000 pts	0.60, 100 pts	0.20, 0 pts	0.20, 700 pts	0.80, 200 pts	—	200 pts
	8	0.10, 900 pts	0.70, 200 pts	0.20, 50 pts	0.15, 600 pts	0.75, 200 pts	0.10, 100 pts	230 pts*
5	9	0.40, 800 pts	0.60, 100 pts	—	0.35, 600 pts	0.65, 250 pts	—	250 pts
	10	0.30, 750 pts	0.70, 250 pts	—	0.15, 700 pts	0.35, 400 pts	0.50, 300 pts	300 pts
	11	0.35, 850 pts	0.50, 300 pts	0.15, 0 pts	0.45, 500 pts	0.55, 400 pts	—	400 pts
	12	0.35, 950 pts	0.50, 100 pts	0.15, 50 pts	0.30, 700 pts	0.50, 300 pts	0.20, 150 pts	345 pts*
6	13	0.30, 1000 pts	0.70, 200 pts	—	0.30, 700 pts	0.70, 300 pts	—	300 pts
	14	0.25, 900 pts	0.75, 225 pts	—	0.20, 800 pts	0.55, 300 pts	0.25, 225 pts	325 pts*
	15	0.20, 800 pts	0.60, 350 pts	0.20, 100 pts	0.25, 600 pts	0.75, 300 pts	—	300 pts
	16	0.20, 950 pts	0.45, 300 pts	0.35, 200 pts	0.20, 600 pts	0.60, 350 pts	0.20, 250 pts	200 pts
7	17	0.20, 950 pts	0.80, 350 pts	—	0.25, 600 pts	0.75, 400 pts	—	350 pts
	18	0.15, 900 pts	0.85, 250 pts	—	0.20, 500 pts	0.35, 325 pts	0.45, 250 pts	300 pts*
	19	0.10, 850 pts	0.75, 400 pts	0.15, 110 pts	0.30, 600 pts	0.70, 300 pts	—	300 pts
	20	0.15, 800 pts	0.70, 450 pts	0.15, 300 pts	0.30, 600 pts	0.45, 450 pts	0.25, 365 pts	350 pts
8	21	0.15, 950 pts	0.85, 350 pts	—	0.20, 475 pts	0.80, 400 pts	—	375 pts
	22	0.20, 900 pts	0.80, 225 pts	—	0.20, 575 pts	0.40, 275 pts	0.40, 250 pts	250 pts
	23	0.15, 875 pts	0.55, 425 pts	0.30, 100 pts	0.25, 700 pts	0.75, 250 pts	—	250 pts
	24	0.15, 1000 pts	0.70, 475 pts	0.15, 100 pts	0.20, 750 pts	0.55, 500 pts	0.25, 200 pts	350 pts*

Note: Payoffs marked with a star (*) indicate an “almost” decoy setup.

[†]Set 0 is an attention check where lottery “ S ” is First-Order Stochastically Dominated (FOSD) by lottery “ R ”.

Table B.1: Lottery Parameter Sets

B.3 Preregistration Document

'Certainty as Decoy: an Experiment to Test Menu Dependence in Risk Attitudes' (AsPredicted #251,483)

Author(s)

This pre-registration is currently anonymous to enable blind peer-review.
It has one author.

Pre-registered on

2025/10/12 18:05 (PT)

1) Have any data been collected for this study already?

No, no data have been collected for this study yet.

2) What's the main question being asked or hypothesis being tested in this study?

When a certain option (degenerate lottery) is added to a menu with uncertain prospects (non-degenerate lotteries), does the

H1: proportion of agents that choose the riskier lottery decrease, and

H2: proportion of agents that choose the less risky lottery increase, and

H3: proportion of agents that choose the less risky lottery at least increase relative to the riskier lottery?

(H2 implies H3 implies H1)

H4: Do the effects of exposure to a certain option persist beyond its removal?

3) Describe the key dependent variable(s) specifying how they will be measured.

Relative level of riskiness of preferred lottery in the menu, categorized as riskier, less-risky, and certain. It is measured by asking subjects to choose between 2-3 lotteries with varying levels of risk.

Proportions of subjects that chose the riskier/less-risky/certain option in each menu.

4) How many and which conditions will participants be assigned to?

3. Either the test subject will

T1: make choices from 26 menus with only uncertain prospects (type A) first, and then 26 menus including a certain option (type B), or

T2: make choices in the reverse order (26 type B menus, and then 26 type A menus), or

T3: make choices with the menus alternating between type A and B.

5) Specify exactly which analyses you will conduct to examine the main question/hypothesis.

Four menus are screening menus with a stochastically dominated option. They are dropped from analysis. Six more menus include a stochastically dominated certain option and are kept in the analysis.

Each type A menu has a corresponding type B menu that is its superset. I call each correspondence a pair.

H1/2/3: Each menu has independent variables: number of states for the riskier option (2-3), number of states for the less risky option (2-3), the switching level of risk aversion from riskier to less-risky (aforementioned three variables can be used as an identifier for each pair of menus), and most importantly, whether a certain option is present. I will regress individual choice on the covariates. I will also run a difference-in-proportions test for the pairs of menus, though non-parametric methods may be considered.

H4: I will test if there is a significance difference in the proportions of agents that chose the less risky option in type A menus across the three treatments. Time series tests will also be run on the responses of T3.

6) Describe exactly how outliers will be defined and handled, and your precise rule(s) for excluding observations.

10 menus include a stochastically dominated option. Subjects who chose the dominated option two standard deviations above the mean number of times a dominated option is chosen will be excluded from analysis.

Subjects who took the comprehension quiz at the beginning two standard deviations above the mean number of tries will also be excluded from analysis.

7) How many observations will be collected or what will determine sample size?

No need to justify decision, but be precise about exactly how the number will be determined.

Using a significance level of 5%, power of 80%, a moderate effect size, and assuming a 30% rate of the riskier option being chosen, a power analysis of logistic regression of individual choice on if a certain option is present suggests a sample size of 240. I will aim for a slightly larger sample of 270 in case of potential exclusions. The sample size will be equally split among 3 conditions.

8) Anything else you would like to pre-register?

(e.g., secondary analyses, variables collected for exploratory purposes, unusual analyses planned?)

A pattern recognition test (Matzen, et al., 2010) of Raven-like matrices to explore potential differences in risk attitudes between cognitive abilities.

There will be a risk elicitation activity to calibrate the baselines.

Graduate students and Large Language Models will be used to pilot the study.

Appendix C

Technical Supplements to Chapter 2

Variable	Estimate	Std. Error	<i>z</i> -value	Pr(> <i>z</i>)
Fixed Effects				
(Intercept)	−4.2300	0.1850	−22.86	< 0.001 ***
<i>R</i> with Three Payoffs	−0.2837	0.0706	−4.02	< 0.001 ***
<i>S</i> with Three Payoffs	−0.2244	0.0711	−3.16	0.0016 **
Certain Decoy Present	−0.3217	0.0707	−4.55	< 0.001 ***
Group of the Task	0.7715	0.0247	31.28	< 0.001 ***
Time (seconds)	0.0049	0.0025	2.01	0.0446 *
Used Calculator?	0.6282	0.2307	2.72	0.0065 **
Random Effects				
	Variance	Std. Dev.		
Subject (Intercept)	1.0910	1.0445		
Model Fit				
	Observations	Clusters	Log-like	AIC
	5,040	105	−2569.5	5155.0

Significance codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05

Table C.1: Mixed-Effects Logistic Regression of Risk Choice (Econ-trained Subsample)

Variable	Estimate	Std. Error	z -value	$\Pr(> z)$
Fixed Effects				
(Intercept)	−4.1778	0.5012	−8.34	< 0.001 ***
R with Three Payoffs	−0.3891	0.2687	−1.45	0.1476
S with Three Payoffs	−0.1613	0.2688	−0.60	0.5484
Certain Decoy Present	−0.3958	0.0436	−9.08	< 0.001 ***
Group of the Task	0.7276	0.0792	9.19	< 0.001 ***
Time (seconds)	0.0060	0.0017	3.52	0.0004 ***
Used Calculator?	0.3579	0.1564	2.29	0.0221 *
Round	−0.0005	0.0015	−0.30	0.7623
Random Effects				
	Variance	Std. Dev.		
Subject Intercept	0.9934	0.9967		
Set Intercept	0.4200	0.6480		
Model Fit				
	Observations	Clusters (S/V)	Log-like	AIC
	13,248	276 / 24	−6716.5	13453.0

Significance codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05

Table C.2: Global Robustness Check: Mixed-Effects Logistic Regression of Risk Choice

Variable	Estimate	Std. Error	z -value	$\Pr(> z)$
Fixed Effects				
(Intercept)	−0.2717	0.3604	−0.75	0.4510
Round	0.0109	0.0078	1.39	0.1650
Random Effects				
	Variance	Std. Dev.		
Subject Intercept	0.7593	0.8714		
Version Intercept	2.5550	1.5984		
Model Fit				
	Observations	Clusters (S/V)	Log-like	AIC
	1,968	82 / 24	−1043.2	2094.5

Significance codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05

Table C.3: Mixed-Effects Regression on Group 1 (Control): Baseline Stability

Variable	Estimate	Std. Error	<i>t</i> -value	Pr(> <i>t</i>)
Experimental Factors				
(Intercept)	0.8355	0.2967	2.816	0.005 **
Treatment 2 (Ternary-First)	−0.0837	0.0380	−2.204	0.029 *
Treatment 3 (Alternating)	−0.1499	0.0373	−4.014	< 0.001 ***
Individual Preferences				
Binary Risk Baseline	−0.0335	0.0051	−6.601	< 0.001 ***
Economics Background	−0.0064	0.0041	−1.552	0.122
Time Spent (minutes)	−0.0044	0.0016	−2.764	0.006 **
GPA	−0.0255	0.0410	−0.622	0.535
Demographics				
Gender (Female)	0.0380	0.0339	1.120	0.264
Gender (Non-Binary)	−0.1181	0.1020	−1.157	0.249
Ethnicity	−0.0039	0.0508	−0.077	0.939
Model Fit				
	Observations	R^2	Adj. R^2	F-stat
	244	0.3540	0.2416	3.150 ***

Note: Only selected coefficients shown. Full major/year/race dummies included in model.

Significance codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05

Table C.4: Appendix: Comprehensive Regression Analysis of Switch Rates

Appendix D

Materials for Chapter 3

D.1 Mathematical Proofs

D.1.1 Conditionally-Dominant Options

Let $y \in \phi(A)$. By definition of a multi-utility model, there exists some $u_i : X \rightarrow \mathbb{R}$ where $u_i(y) \geq u_i(z)$ for all $z \in A$. For subset menus B that do not include y , $y \notin \rho(B)$ trivially since $\rho(B) \subseteq B$. For those that do include y , $y \in \arg \max_{z \in B} u_i(z)$ since $u_i[B] \subseteq u_i[A]$. By definition of a multi-utility model again, $y \in \phi(B)$, which in turn says $y \notin \rho(B)$. Therefore, there cannot exist a subset menu B that rejects y , which tells us that $y \notin \bigcup_{B \subseteq A} \rho(B)$. Since $y \in A$, it must be that $y \in A \setminus \bigcup_{B \subseteq A} \rho(B)$. This shows that the union of rejected options is a strict subset of A . It also shows that all $y \in \phi(A)$ are CD.

D.1.2 Rejection of Unilateral Inflation

If item z is rejected at price $p(z, A)$, then there exists some other x such that $u(x) - p(x, A) \geq u(z) - p(z, A)$, assuming ties break in favor of x . If $p(x, A) \geq p(x, B)$ and $p(z, A) \leq p(z, B)$, then $u(x) - p(x, B) \geq u(z) - p(z, B)$. Therefore, x is still a better item than z , and hence z

is still rejected.

D.2 Pseudocodes

D.2.1 Sen's α and Reduction

ALGORITHM 1: Verification of Sen's α

```

1  $\Sigma \leftarrow$  array of observable menus;
2  $\phi \leftarrow$  array of choice function outputs;
3  $\alpha \leftarrow T$ , assume  $\alpha$  is satisfied;
4 for  $A$  in  $\Sigma$  do
5   for  $B$  in  $\Sigma$  do
6     if  $B$  is a subset of  $A$  then
7       for  $k$  in  $\phi(A)$  do
8         if  $k$  in  $B$  and  $k$  not in  $\phi(B)$  then
9            $\alpha \leftarrow F$ ;
10          end program;
11        end
12      end
13    end
14  end
15 end

```

The **if** condition at line 6 takes at most $|X|^2$ operations using a naïve approach. The **if** condition at line 8 takes at most $2|X|$ operations.

The worst case scenario for this algorithm is $|\Sigma| \times |\Sigma| \times (|X|^2 + |X| \times 2|X|)$ operations. If we assume that there are more observable menus than options, then this algorithm runs in quartic time $O(|\Sigma|^4)$. Note that checking for subsets can be done in $O(|X| \log |X|)$ time if

we use binary search, provided that the options are ordered in the dataset, which further shows that there are more efficient ways than just a pure-polynomial algorithm.

ALGORITHM 2: Verification of Reduction

```

1 Sigma  $\leftarrow$  array of observable menus;
2 phi  $\leftarrow$  array of choice function outputs;
3 reduction  $\leftarrow$  T, assume Reduction is satisfied;
4 for A in Sigma do
5   for B in Sigma do
6     if  $\phi(A)$  is a subset of B is a subset of A then
7       if  $\phi(A)$  is not the same as  $\phi(B)$  then
8         reduction  $\leftarrow$  F;
9         end program;
10      end
11    end
12  end
13 end

```

The **if** condition at line 6 takes at most $2|X|^2$ operations using a naïve approach. The **if** condition at line 7 takes at most $|X| \log |X| + |X|$ operations if we sort the options using merge or heap sort, and $|X|$ if the dataset is already sorted.

The worst case scenario for this algorithm is $|\Sigma| \times |\Sigma| \times (2|X|^2 + |X| \log |X|)$, which is in quartic time $O(|\Sigma|^4)$, and can be even better if the dataset is sorted.

Since path-independence is a conjunction of Sen's α and Reduction, verification has a worst case of $O(|\Sigma|^4 + |\Sigma|^4)$ quartic time.

D.2.2 Verifying a CTR Certificate

ALGORITHM 3: Verification of CTR

```

1  $X \leftarrow$  array of options;
2  $Sigma \leftarrow$  array of observable menus;
3  $phi \leftarrow$  array of choice function outputs;
4  $u \leftarrow$  array of options ordered by proposed certificate;
5  $BIN \leftarrow$  array to populate binary relations;
6 for  $(x, y)$  in  $X^2$  do
7   if  $x = y$  then
8      $BIN(x, y) = \succeq^*$ ;
9   end
10  for  $A$  in  $Sigma$  do
11    if  $\{x, y\} \in A$  and  $x \in \arg \max u(A)$  and  $y \notin phi(A)$  then
12       $BIN(x, y) \leftarrow \gg$ ;
13      break;
14    else if  $\{x, y\} \in A$  and  $x \in phi(A)$  then
15       $BIN(x, y) \leftarrow \succeq^*$ ;
16    end
17  end
18  $G \leftarrow$  an  $N \times N$  array of  $-N$ ;
19 for  $(x, y)$  in  $X^2$  do
20   if  $BIN(x, y) = \gg$  then
21      $G(x, y) \leftarrow 1$ ;
22   else if  $BIN(x, y) \in X^2$  then
23      $G(x, y) \leftarrow 0$ ;
24   end
25  $Q \leftarrow$  an  $N \times N$  array of 0;
26  $Q(\text{rand}(N), 1) \leftarrow -N$ ;
27 for  $x$  in  $\{2, \dots, N\}$  do
28   for  $k$  in  $X$  do
29      $Q(x, k) = \max_{y \in \{1, 2, \dots, N\}} [G(x, y) + Q(y, k - 1)]$ ;
30   end
31 end
32 if  $N \bmod 2 = 0$  then
33    $(m, n) \leftarrow (\frac{N}{2}, \frac{N}{2})$ ;
34 else
35    $(m, n) \leftarrow (\frac{N-1}{2}, \frac{N+1}{2})$ ;
36 end
37  $v[x] \leftarrow \max_{k \in \{1, 2, \dots, N\}} [(m + n)Q(x, k) - m(k - 1)]$  array of utility based on certificate;
38 for  $(x, y) \in X^2$  do
39   if  $BIN(x, y) = \gg$  and  $v(x) \leq v(y) + 1$  then
40     this certificate is incorrect. Go back to Step 1;
41   else if  $BIN(x, y) = \succeq^*$  and  $v(x) < v(y) - 1$  then
42     this certificate is incorrect. Go back to Step 1;
43   end
44 this certificate is correct. End program;

```

Step 3 (lines **6** through **17**) determines the relation for each binary pair $(x, y) \in X^2$:

- If there is at least one menu where x, y are available, and x is the true-best while y is rejected, then $x \gg y$. This explains why y was rejected and why x was accepted in these menu.
- If the above doesn't exist, but there is at least one menu where both x, y are available, and x is acceptable, then $x \succeq^* y$ but $x \not\gg y$. This will explain why x was accepted in these menus.
- If neither of the above exist, then it could be that $y \gg x$ or the data is incomplete.

For each pair of options, the worst case scenario is $6|\Sigma|$ operations. There are N^2 pairs. This step takes cubic time $O(6|\Sigma| \times N^2) \sim O(|\Sigma|^3)$.

Step 4 (lines **18** through **37**) uses the binary relations from Step 3 to first construct the optimal path between two options (N^3 operations). This information is then used to construct a utility function based on the information of the certificate (N^2 operations). This step is better than cubic time in $|\Sigma|$, assuming that $N < |\Sigma|$.

Step 5 (lines **38** through **44**) checks if the utility function agrees with the binary relations from Step 3. For each pair of alternatives, if $x \succeq^* y$ then $v(x) \geq v(y) - 1$. If $v(x) < v(y) - 1$ is observed, then the trial fails and the algorithm returns to Step 1. Similarly, if $x \gg y$ then $v(x) > v(y) + 1$. If $v(x) \leq v(y) + 1$, then the trial also fails and the algorithm returns to Step 1. This step takes $2N^2$ operations if succeeds and fewer if fails, which is better than quadratic time in $|\Sigma|$.

D.2.3 Eliminating Binary Cycles

ALGORITHM 4: Finding a Utility Ranking with No Binary Cycles

```
1  $X \leftarrow$  array of options;
2  $Sigma \leftarrow$  array of observable menus;
3  $phi \leftarrow$  array of choice function outputs;
4  $E \leftarrow$  an  $N \times N$  array of 1's;
5 for  $A$  in  $Sigma$  do
6   for  $(x, y)$  in  $X^2$  do
7     if  $x \in phi(A)$  and  $y \in A$  then
8        $E(x, y) \leftarrow 0$ ;
9     end
10  end
11 end
12  $best \leftarrow$  array to populate true-best candidate for each menu;
13 for  $A$  in  $Sigma$  do
14    $temp \leftarrow X$ ;
15   for  $x$  in  $X$  do
16     if  $x \notin phi(A)$  then
17        $remove(x, temp)$ ;
18     for  $y$  in  $X$  do
19       if  $y \in phi(A)$  and  $E(x, y) = 0$  then
20          $remove(y, temp)$ ;
21       end
22     end
23   end
24 end
25    $best(A) \leftarrow random(temp)$ ;
26 end
```
