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1-, 2-, and 3-Dimensional Effective Conductivity of Aquifers[1]

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Starting with a stochastic differential equation with random coefficients describing steady-state flow, the effective hydraulic conductivity of 1-, 2-, and 3-dimensional aquifers is derived. The natural logarithm of hydraulic conductivity (lnK) is assumed to be heterogeneous, with a spatial trend, and isotropic. The effective conductivity relates the mean specific discharge in an aquifer to the mean hydraulic gradient, thus its importance in predicting Darcian discharge when field data represent mean or average values of conductivity or hydraulic head. Effective conductivity results are presented in exact form in terms of elementary functions after the introduction of special sets of coordinate transformations in two and three dimensions. It was determined that in one, two, and three dimensions, for the type of aquifer heterogeneity considered, the effective hydraulic conductivity depends on: (i) the angle between the gradient of the trend of lnK and the mean hydraulic gradient (which is zero in the one-dimensional situation); (2) (inversely) on the product of the magnitude of the trend gradient of lnK, b, and the correlation scale of lnK, λ; and (3) (proportionally) on the variance of lnK, σ_f^2. The product bλ plays a central role in the stability of the results for effective hydraulic conductivity.

KEY WORDS: hydraulic conductivity, aquifer flow, spatial covariance, stochastic groundwater analysis, spectral analysis.

INTRODUCTION

The Theoretical Background

Consider the stochastic differential equation describing steady-state groundwater flow in an aquifer with random hydraulic conductivity $K(\mathbf{x})$ ($\mathbf{x}$ represents a three-dimensional coordinate vector and tensorial index notation is used with $i = 1$,

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2, or 3 depending on aquifer dimensionality):

$$\frac{\partial}{\partial x_i}\left[K(\mathbf{x}) \frac{\partial \phi(\mathbf{x})}{\partial x_i} \right] = 0 \tag{1}$$

where $\phi(\mathbf{x})$ is the hydraulic head. Following Loaiciga and others (1993), $K(\mathbf{x})$ is a stochastic parameter whose distribution is expressed conveniently in terms of the natural logarithm of hydraulic conductivity, $Y = \ln K$. Y is composed of a deterministic and spatially variable trend T plus a zero-mean random noise f with specified spatial statistical structure. Specifically, $Y(\mathbf{x}) = T(\mathbf{x}) + f(\mathbf{x})$. The hydraulic head $\phi(\mathbf{x})$, is modeled as the sum of a deterministic mean $H(\mathbf{x})$ and a zero-mean random noise h, that is, $\phi(\mathbf{x}) = H(\mathbf{x}) + h(\mathbf{x})$. Loaiciga and others (1993) showed that based on these (mean plus perturbation) decompositions of hydraulic head and $\ln K$, as well as on Equation (1), one arrives at a couple of partial differential equations of which the first [Eq. (2)] governs the distribution of mean hydraulic head (H) and the second [Eq. (3)] governs the perturbations of hydraulic head, [h ($b_i = \partial T / \partial x_i$ and E denotes expectation in Eqs. (2) and (3))]

$$\frac{\partial^2 H}{\partial x_i \partial x_i} + b_i \frac{\partial H}{\partial x_i} + E\left\{ \frac{\partial f}{\partial x_i} \frac{\partial h}{\partial x_i} \right\} = 0 \tag{2}$$

and

$$\frac{\partial^2 h}{\partial x_i \partial x_i} + b_i \frac{\partial h}{\partial x_i} + \frac{\partial f}{\partial x_i} \frac{\partial H}{\partial x_i} + \left\{ \frac{\partial f}{\partial x_i} \frac{\partial h}{\partial x_i} - E\left(\frac{\partial f}{\partial x_i} \frac{\partial h}{\partial x_i} \right) \right\} = 0 \tag{3}$$

Equations (2) and (3) are a system of coupled partial differential equations. Methods for the solution of these equations have been examined by Christakos, Miller, and Oliver (1993). One approach (the "small perturbation" approach, see for example, Gelhar, 1993, for a thorough discussion of this approach) decouples the two equations by neglecting the products of perturbations in the right-hand side of Equation (3). Spectral methods (Gelhar and Axness, 1983; Gelhar, 1993) then are useful to transform the simplified form of Equation (3) to the frequency domain from which it can be integrated readily to yield, for example, the variance of hydraulic head (Bakr and others, 1978).

Averaging of the Equation of Motion and Spectral Analysis

Loaiciga and others (1993, 1994) adapted the small-perturbations approach to determine the relationship between the complex-valued processes $dZ_h(\mathbf{k})$ and $dZ_f(\mathbf{k})$ defined by the spectral representations $h = \int_{-\infty}^{\infty} e^{j\mathbf{k}\cdot\mathbf{x}} dZ_h(\mathbf{k})$ and $f = \int_{-\infty}^{\infty} e^{j\mathbf{k}\cdot\mathbf{x}} dZ_f(\mathbf{k})$ where $j^2 = -1$, $\mathbf{k}$ is the wave-number vector (of components

k_i), and "·" denotes inner vectorial product. Specifically, Loaiciga and others (1993, 1994) have shown that the relationship between the complex-valued processes $dZ_h(\mathbf{k})$ and $dZ_f(\mathbf{k})$ is given by the following expression (where $\mathbf{J}$ is the mean hydraulic gradient vector whose components are $J_i = \partial H/\partial x_i$; $\mathbf{b}$ is the $\ln K$ trend gradient vector with components $b_i = \partial T/\partial x_i$; and $k^2 = \mathbf{k} \cdot \mathbf{k}$):

$$dZ_h(\mathbf{k}) = \frac{[jk^2 - \mathbf{k} \cdot \mathbf{b}][\mathbf{k} \cdot \mathbf{J}]}{[(k^2)^2 + (\mathbf{k} \cdot \mathbf{b})^2]} dZ_f(\mathbf{k}) \tag{4}$$

The relevance of Equation (4) to the effective hydraulic conductivity becomes clear if one considers Darcy's law for the specific discharge q_i:

$$q_i = -K \frac{\partial \phi}{\partial x_i} \tag{5}$$

Replacing the hydraulic head ϕ by $H + h$ and the hydraulic conductivity K by e^{T+f} in Equation (5), expanding the exponential e^f by a Taylor series up to first-order terms, and taking expectations of the resulting Equation (5), one obtains the following equation relating the mean specific discharge $\bar{q}_i$ to the mean hydraulic gradient J_i and the covariance of the $\ln K$ perturbation f and the partial derivative of the head perturbation h:

$$\bar{q}_i \approx -e^T \left[J_i + E\left(f \frac{\partial h}{\partial x_i} \right) \right] \tag{6}$$

The covariance of perturbations in Equation (6) can be obtained by a Fourier transform of its corresponding spectrum Φ_{fh_i}:

$$E\left(f \frac{\partial h}{\partial x_i} \right) = \int_{\mathbf{R}} \Phi_{fh_i}(\mathbf{k}) d^3\mathbf{k} \tag{7}$$

in which $\mathbf{R}$ is the complete 3-dimensional space and

$$\Phi_{fh_i}(\mathbf{k}) = E[dZ_f(\mathbf{k}) dZ^*_{h_i}(\mathbf{k})] \tag{8}$$

in which

$$dZ^*_{h_i}(\mathbf{k}) = -jk_i dZ^*_h(\mathbf{k}) \tag{9}$$

In Equation (9), $dZ^*_h(\mathbf{k})$ is the complex conjugate of $dZ_h(\mathbf{k})$, and the latter was given in Equation (4) in function of the complex-valued process $dZ_f(\mathbf{k})$. From Equation (4), the cross-spectrum of Equation (8) becomes (using the fact the spectrum of $\ln K$ is given by $\Phi_{ff}(\mathbf{k}) = E[dZ_f(\mathbf{k}) dZ^*_f(\mathbf{k})]$):

$$\Phi_{fh_i}(\mathbf{k}) = \frac{jk_i[jk^2 + \mathbf{k} \cdot \mathbf{b}][\mathbf{k} \cdot \mathbf{J}]}{[(k^2)^2 + (\mathbf{k} \cdot \mathbf{b})^2]} \Phi_{ff}(\mathbf{k}) \tag{10}$$

Equations (6), (7), and (10) will be used to relate the mean specific discharge to the corresponding mean hydraulic gradient along the ith coordinate direction. The constant of proportionality relating mean specific discharge and mean hydraulic gradient is the effective hydraulic conductivity.

On the Relevance of Effective Conductivity in Aquifer Analysis

The effective hydraulic conductivity is an important parameter because it represents an average or ensemble mean of aquifer properties (Sposito, Jury, and Gupta, 1986). The effective hydraulic conductivity in the absence of trends has been examined by Gelhar and Axness (1983), Kitanidis (1990), and reviewed extensively by Gelhar (1993). The situation of effective hydraulic conductivity in the presence of structural trends has been treated previously by Loaiciga and others (1993, 1994), and by Indelman and Rubin (1995). Theoretically, the effective hydraulic conductivity relates an ensemble specific discharge to an ensemble hydraulic gradient. Practically, it relates measurable, field-scale, variables (specific discharge, hydraulic gradient) to field-scale aquifer parameters (for a discussion of flow-domain scales, field measurements, and stochastics, see Cushman, 1984; Dagan, 1986.) If the effective hydraulic conductivity is identifiable from measurable parameters and characteristics of the groundwater flow regime, then it can be useful in the calibration of groundwater flow models and in implementing stochastic models of flow and transport that rely on "effective" (that is, ensemble means) parameters. The remainder of this paper is devoted to develop exact expressions for the effective hydraulic conductivity for 1-, 2-, and 3-dimensional aquifers in terms of a number of key aquifer parameters and groundwater flow characteristics. In addition to the heterogeneous model for ln$K$ already presented, it is assumed that the the covariance of ln$K$ is exponential, that is, $\sigma_{ff}(\tau) = \sigma_f^2 \exp(-|\tau|/\lambda)$, where τ is the separation vector, and λ is the correlation scale of ln$K$.

THE MEAN SPECIFIC DISCHARGE IN 1-, 2-, AND 3-DIMENSIONAL DOMAINS

A General Expression for the Mean Specific Discharge

Starting with Equations (6), (7), and (10) it is possible to derive the mean specific discharge in terms of the spectrum of ln$K$. For the assumed exponential covariance model $\sigma_{ff}(\tau)$, the spectrum of ln$K$ can be derived by the Fourier transform relating spectra to covariances: $\Phi_{ff}(\mathbf{k}) = (2\pi)^{-n} \int_{\mathbf{R}_n} \exp(-j\mathbf{k} \cdot \tau)\sigma_{ff}(\tau)d^n\tau$, where n is the flow dimensionality ($n = 1, 2,$ or 3) and $\mathbf{R}_n$ is the complete n-dimensional space. The following spectrum for ln$K$ results:

$$\Phi_{ff}(\mathbf{k}) = \frac{\sigma_f^2\lambda^{q-1}}{\kappa_{q-1}(1 + k^2\lambda^2)^{q/2}} \qquad (11)$$

where $q = 2, 3$, and 4 for 1-, 2-, and 3-dimensional domains, respectively, and $\kappa_{q-1} = \pi, 2\pi$, and π^2 for 1-, 2-, and 3-dimensional domains, respectively. Based on Equations (6) and (11), a generic expression for the mean specific discharge is given by the following equation:

$$\bar{q}_i \approx -e^T \left[J_i + \frac{j\sigma_f^2 \lambda^{q-1}}{\kappa_{q-1}} \int_{\mathbf{R}_{q-1}} \frac{k_i[jk^2 + \mathbf{k} \cdot \mathbf{b}][\mathbf{k} \cdot \mathbf{J}]}{[(k^2)^2 + (\mathbf{k} \cdot \mathbf{b})^2](1 + k^2\lambda^2)^{q/2}} d^{q-1}\mathbf{k} \right] \tag{12}$$

where q takes the values 2, 3, and 4 for 1-, 2-, and 3-dimensional flow domains, respectively, as before, κ_{q-1} takes the values π, 2π, and π^2 for $q = 2, 3$, and 4, respectively, and $\mathbf{R}_{q-1}$ denotes the complete $q - 1$-dimensional space, for $q = 2, 3, 4$.

The One-Dimensional Situation

Integrating Equation (12) for a 1-dimensional space [i.e., $q = 2$ in Eq. (12)] leads to the following mean specific discharge:

$$\bar{q} \approx -e^T \left[1 - \frac{\sigma_f^2}{b\lambda + 1} \right] J \tag{13}$$

in which J is the mean hydraulic gradient, and $b = dT/dx$ is the gradient of the trend in ln$K$. It is clear from Equation (13) that the 1-dimensional hydraulic conductivity is given by:

$$K_e = e^T \left[1 - \frac{\sigma_f^2}{b\lambda + 1} \right] \tag{14}$$

It is seen from Equation (14) that the condition for nonnegative effective hydraulic conductivity is $\sigma_f^2 < b\lambda + 1$, and that a singular point exists at $b\lambda = -1$. (Gutjahr and others, 1978, presented an alternate expression for 1-dimensional effective hydraulic conductivity leading to different stability conditions.) Notice then how in the presence of a trend in ln$K$, the restriction on the variance of ln$K$ perturbation involves the critical product of parameters $b\lambda$. It also is possible to show by the spectral method that the variance of hydraulic head becomes arbitrarily large when $b\lambda \rightarrow -1$. Because the correlation scale is a positive parameter, this condition can arise only when the trend gradient in ln$K$ is negative and equal to $1/\lambda$. A plausible explanation for the instability of the stochastic results at $b\lambda = -1$ is that as the gradient of the trend of ln$K$ approaches the critical value $-1/\lambda$, the hydraulic conductivity declines over spatial scales equivalent to the correlation scale to a value small enough to prevent flow for finite hydraulic gradients. (See Loaiciga and others, 1993, for a further discussion of this matter.) The parameter $b\lambda$ plays an important role in 2- and 3-dimensional analysis also, as will be shown. The expression in Equation (14) also shows that, because the trend T is in general spatially variable, so is the effective hydraulic conductivity. If the trend of ln$K$ can be identified adequately from data, it is possible to construct a spatially dependent effective hydraulic conductivity field directly from Equation (14). In 2- and 3-dimensional domains this potentially is useful in calibrating numerical models of groundwater flow and mass transport. Most of the hydraulic head and aquifer properties data collected in the field represent averages or "effective" values for extended spatial domains. Numerical simulation models also are coarse and discrete spatial approximations to continuous processes. It seems reasonable, therefore, that in seeking calibrating parameters for such numerical simulation models, to focus on the theoretical effective conductivity relating the mean or average groundwater flow discharge to the mean hydraulic gradients. Average effective hydraulic conductivities over finite-difference cells and finite elements can be calculated (e.g., by integration) from the continuous-space function K_e.

An Integral Approach to Effective Conductivity in Two and Three Dimensions

Loaiciga and others (1994) developed an integration method to evaluate integrals of the type in Equation (12) for three-dimensional groundwater flow domains [$q = 4$ in Eq. (12)]. Results for 2-dimensional domains are derived in this article and one example of 2-D effective conductivity calculations is presented herein. Recently, Indelman and Rubin (1995) derived effective hydraulic conductivities when the trend is linear for domains of arbitrary dimensionality using linearization techniques.

The integrals in Equation (12) contain numerous parameters (**b**, **J**, λ) and a variety of rather discordant, although individually simple, functional expressions. In this respect, they resemble quantum mechanics problems (Kallen, 1950), a prolific source of physically important but exceptionally puzzling integration problems. The integrals of Equation (12) are taken for large or infinite regions (in wave-vector number) and either are slowly convergent or contain divergent portions that approximately cancel and are renormalizable by subtraction and domain truncation. All this makes normal numerical integration costly or impractical, or both, so that preliminary reductions to known (though seldom encountered) functions by exact methods is highly desirable, if not unavoidable, in this type of stochastic groundwater problem. Define the generic integral:

$$I_{l,n,p,q}(\mathbf{b}, \mathbf{J}, \lambda, \mathbf{S}) = \int_{\mathbf{S}} \frac{k_p^l F(k^2, \mathbf{k} \cdot \mathbf{b})(\mathbf{k} \cdot \mathbf{J})^n}{[(k^2)^2 + (\mathbf{k} \cdot \mathbf{b})^2](1 + \lambda^2 k^2)^{q/2}} d^{q-1}\mathbf{k} \qquad (15)$$

where F is a suitably defined function; **S** is a spherical region in **k**-space with center at the origin and of radius ρ; l, n, p, q are index integers suitably selected for any given function F. The functions F in the integral of Equation (15) are typically simple. For example, if $F(k^2, \mathbf{k} \cdot \mathbf{b}) = \mathbf{k} \cdot \mathbf{b} + jk^2$, $l = 1$, $n = 1$, $p = i$ ($i = 1, 2, 3$), and $q = 3$, or 4, then $I_{l,n,p,q}$ so defined would represent the integral in Equation (12) for 2- and 3-dimensional domains, respectively. The integration method presented herein does not depend strongly on the nature of F, given that F depends only on k^2 and $\mathbf{k} \cdot \mathbf{b}$. The integration method to be developed does apply to more general integrals than those represented by Equation (15), such as those that might arise in anisotropic porous media.

The basic strategy to integrate Equation (15) in two and three dimensions follows the approach of Loaiciga and others (1994), where an integration method was proposed in three dimensions for finite and infinite domains. The approach consists of determining geometric transformations of the vectors **b, k,** and **J** to reduce the triple integrals of interest to single integrals with elementary functions as integrands. Radial, conical, and spherical trigonometry will play a central role in carrying out these geometric transformations. It also is convenient to carry out the integrations in a finite, symmetric, domain (the spherical region **S** of radius ρ) defined by Equation (15) and then take the limit in the finite-domain results to obtain answers for the infinite-domain situation. This procedure allows mutual cancellation of individually divergent functions that arise through the integration procedure.

If the function $F(k^2, \mathbf{b} \cdot \mathbf{k})$ in the integral of Equation (15) is an even function of $\mathbf{b} \cdot \mathbf{k}$, assign F the parity $\nu = 0$, and if it is an odd function of $\mathbf{b} \cdot \mathbf{k}$, assign it the parity $\nu = 1$. Consider the effect of the transformation $\mathbf{k} \to -\mathbf{k}$ on the integral of Equation (15). If the integration region **S** is invariant under this transformation, it is seen from Equation (15) that if $l + n + \nu$ is odd, then $I_{l,n,p,q} = 0$, and if $l + n + \nu$ is even, then $I_{l,n,p,q}$ is likely to be nonzero. The dependence of Equation (15) on l and p can be clarified by differentiation of a two-index family of integrals, $G_{n,q}$, l times with respect to J_p, the pth component of the mean vector gradient **J**. Let

$$G_{n,q} = \int_{\mathbf{S}} \frac{F(k^2, \mathbf{k} \cdot \mathbf{b})(\mathbf{k} \cdot \mathbf{J})^n}{[(k^2)^2 + (\mathbf{k} \cdot \mathbf{b})^2](1 + \lambda^2 \mathbf{k}^2)^{q/2}} \, d^{q-1}\mathbf{k} \tag{16}$$

Clearly, $G_{n,q} = I_{0,n,p,q}$ (which is independent of p). Differentiation of $G_{n,q}$ in Equation (16) l times with respect to J_p establishes that

$$\frac{\partial^l G_{n,q}}{\partial J_p^l} = \frac{n!}{(n - l)!} I_{l,n-l,p,q} \tag{17}$$

so, that by letting $n' = n - l$, then:

$$I_{l,n',p,q} = \frac{(n')!}{(n' + l)!} \frac{\partial^l G_{n'+l,q}}{\partial J_p^l} \tag{18}$$

Thus, if $G_{n,q}$ is known as a function of **J** for some specific F, formal differentiation as indicated yields $I_{l,n-l,p,q}$.

Further progress is difficult unless the integral $I_{l,n,p,q}$ is particularized. Let us focus on the example

$$I_{1,1,p,q} = \int_S \frac{k_p(\mathbf{k} \cdot \mathbf{b} + jk^2)(\mathbf{k} \cdot \mathbf{J})}{[(k^2)^2 + (\mathbf{k} \cdot \mathbf{b})^2](1 + \lambda^2 k^2)^{q/2}} d^{q-1}\mathbf{k} \tag{19}$$

which enters the integral in Equation (12). Obviously, $I_{1,1,p,q}$ is a sum of two integrals of the $I_{l,n,p,q}$ type for different selections of F. The first integral has $l = 1$, $n = 1$, $q = 3$ or 4 (depending on whether the domain is 2- or 3-dimensional, respectively), $F_1 = \mathbf{k} \cdot \mathbf{b}$, $\nu = 1$ (i.e., F_1 is an odd function of $\mathbf{k} \cdot \mathbf{b}$, thus the parity 1 assigned to it), and, therefore, $l + n + \nu = 3$; so, if the region of integration **S** is a sphere with center at the origin, then the first integral is zero. The second integral has $l = 1$, $n = 1$, $q = 3$ or 4, $F_2 = jk^2$, with parity $\nu = 0$ (e.g., the function is F_2 is even), and, therefore, $l + n + \nu = 2$; so, the second integral probably does not vanish. With the selection of F as in Equation (19), the following relation exists between the two-index integral $G_{2,q}$ and $I_{1,1,p,q}$, according to the result of Equation (18):

$$\begin{aligned} I_{1,1,p,q} &= \frac{1}{2}\frac{\partial G_{2,p}}{\partial J_p} \\ &= \frac{1}{2}\frac{\partial}{\partial J_p}\int_S \frac{jk^2(\mathbf{k} \cdot \mathbf{J})^2}{[(k^2)^2 + (\mathbf{k} \cdot \mathbf{b})^2](1 + \lambda^2 k^2)^{q/2}} d^{q-1}\mathbf{k} \end{aligned} \tag{20}$$

Note that $I_{1,1,p,q}$ is the integral in Equation (12). Therefore, if $G_{2,q}$ is known, for $q = 3$ or 4, its derivative with respect to J_p gives the integral of Equation (12). In the next sections the multiple integral of Equation (20) is reduced to single integrals that are then evaluated exactly in terms of elliptic functions in the 2-dimensional situation ($q = 3$) and in terms of logarithmic functions for 3-dimensional aquifers ($q = 4$).

TWO-DIMENSIONAL EFFECTIVE HYDRAULIC CONDUCTIVITY

Geometric Transformations and Analysis of Stability

Consider Equation (20) with $q = 3$ that defines the integral

$$G_{2,3} = \int_S \frac{jk^2(\mathbf{k} \cdot \mathbf{J})^2}{[(k^2)^2 + (\mathbf{k} \cdot \mathbf{b})^2](1 + k^2\lambda^2)^{3/2}} d^2\mathbf{k} \tag{21}$$

in which the region of integration is $S = [\mathbf{k}\colon 0 \leq |\mathbf{k}| \leq \rho]$. Define the radial coordinates $r = \sqrt{k^2}$ (where $k^2 = \mathbf{k} \cdot \mathbf{k}$); ψ is the angle between the vectors **b** and **k**; and let θ be the angle between **b** and **J**. Integration of Equation (21) in the radial coordinates ($0 \leq r \leq \rho$; $0 \leq \psi \leq 2\pi$) leads to the following expression for $G_{2,3}$ (where $J^2 = \mathbf{J} \cdot \mathbf{J}$ and $b^2 = \mathbf{b} \cdot \mathbf{b}$):

$$G_{2,3} = jJ^2(L_1 \cos^2\theta + L_2 \sin^2\theta) \tag{22}$$

where:

$$L_1 = \int_0^{2\pi} \int_0^{\rho} \frac{r^3 \cos^2 \psi \, dr \, d\psi}{(r^2 + b^2 \cos^2 \psi)(1 + \lambda^2 r^2)^{3/2}} \tag{23}$$

and

$$L_2 = \int_0^{2\pi} \int_0^{\rho} \frac{r^3 \sin^2 \psi \, dr \, d\psi}{(r^2 + b^2 \cos^2 \psi)(1 + \lambda^2 r^2)^{3/2}} \tag{24}$$

The integrals of Equations (23) and (24) are integrated with respect to ψ first and then with respect to ρ using standard integration methods. Integration results for an infinite domain are obtained from the finite-domain situation by letting $\rho \to \infty$, after cancellation of individually divergent terms that cancel out pairwise in passing to the limit ($\rho \to \infty$). Reduction of L_1 and L_2 in Equations (23) and (24), respectively, to single integrals in an infinite domain leads to (where $t = \lambda r$):

$$L_1 = \frac{2\pi}{b^2}\left[\frac{1}{\lambda^4}[(1 + (\lambda\rho)^2)^{1/4} - (1 + (\lambda\rho)^2)^{-1/4}]^2\right] - \frac{2\pi}{b^2}\left[\frac{1}{\lambda^4}\int_0^{\lambda\rho} \frac{t^4 \, dt}{(t^2 + (b\lambda)^2)^{1/2}(1 + t^2)^{3/2}}\right] \tag{25}$$

$$L_2 = 2\pi\left[\frac{1}{\lambda^2}\int_0^{\lambda\rho} \frac{t^2 \, dt}{(t^2 + (b\lambda)^2)^{1/2}(1 + t^2)^{3/2}}\right] - L_1 \tag{26}$$

The integrals in Equations (25) and (26) are of the elliptic type (introduced by Abel and Jacobi about 1826 in the context of the theory of rigid bodies and pendulums; see Greenhill, 1959) for $(b\lambda)^2 \neq 1$.

Clearly, the first term within brackets in the right-hand side of Equation (25) tends to $O(\lambda^{-3}\rho)$ for large ρ. This term then is divergent for $\rho \to \infty$. It also shows up in the right-hand side of Equation (26) because L_1 occurs there. Consequently, the second expression within brackets in the right-hand side of Equation (25) also must contain a term of order $O(\lambda^{-3}\rho)$ to ensure cancellation of individually divergent terms for $\rho \to \infty$. Mathematical analysis of the expressions of Equations (25) and (26) was used to establish the following: (i) there is mutual cancellation of terms of order $O(\lambda^{-3}\rho)$ as $\rho \to \infty$ in Equations (25) and (26); (ii) the elliptic integrals that occur in Equations (25) and (26) are continuous at $b\lambda = \pm 1$; and (iii) a discontinuity in the results occurs only at b

$= 0$, a situation which is not considered here because the trend in ln$K$ is by definition nonzero. These findings (i)–(iii) are of significance, because they prove the stability of results for an infinite-domain in two and three dimensions. Note, in particular, that the discontinuity caused by the product of parameters $b\lambda$ in 1-dimensional domains [see Eq. (14)] disappears in passing from one dimension to two and three dimensions. Also, in 2- and 3-dimensional domains the magnitude of the vector gradient of the trend of ln$K$, b, is positive and so is the correlation scale λ; therefore, we are interested in the corner point $b\lambda = 1$ in the analysis to follow since the point $b\lambda = -1$ is not physically realizable.

In spite of the continuity of L_1 and L_2 at $b\lambda = 1$, their integration for the situations $b\lambda < 1$, $b\lambda > 1$, and $b\lambda = 1$ is considered separately, because they lead to different analytical results (which coincide, as required by continuity, at the point $b\lambda = 1$ as shown later).

Two-Dimensional Effective Hydraulic Conductivity When $b\lambda > 1$

The elliptic integrals of Equations (25) and (26) are presented for the situation of an infinite domain ($\rho \to \infty$.) In the finite-domain situation, these integrals have been widely tabulated (see Gradshteyn and Ryzhik, 1993). In the infinite domain situation there are a number of algebraic relationships and series expansions of elliptical functions that simplify calculations while maintaining excellent accuracy (e.g., twelfth-decimal precision). Let:

$$K[u] = \frac{\pi}{2}\,[1 + 2h(u) + 2h^4(u) + 2h^9(u) + \cdots]^2 \tag{27}$$

be a series expansion of the complete elliptic integral of the first type (Greenhill, 1959), $K[u]$, related to the complete elliptic integral of the second type (Greenhill, 1959), $E[u]$, by the following differential equation:

$$E[u] = (1 - u^2)\left(u\,\frac{dK[u]}{du} + K[u]\right) \tag{28}$$

where in Equation (27) $h(u)$ is expressed by the series:

$$h(u) = \frac{1}{2}\,l(u) + \left(\frac{1}{2}\right)^5 l^5(u) + 15\left(\frac{1}{2}\right)^9 l^9(u) + 150\left(\frac{1}{2}\right)^{13} l^{13}(u) + \cdots \tag{29}$$

The function $l(u)$ in Equation (29) is given by:

$$l(u) = \frac{1 - (1 - u^2)^{1/4}}{1 + (1 - u^2)^{1/4}} \tag{30}$$

The argument u in Equations (27)–(30) depends on the parameter $b\lambda$ as shown next. In Equation (28), the derivative term can be obtained by differentiation of the series in Equation (27). With these definitions, the elliptic integrals of Equations (25) and (26) take the following values when $\rho \to \infty$ and $b\lambda > 1$:

$$L_1 = \frac{2\pi}{b\lambda^3}\left[\left(1 - \frac{1}{(b\lambda)^2 - 1}\right)E[u] + \frac{1}{(b\lambda)^2 - 1}K[u]\right] \tag{31}$$

$$L_2 = \frac{2\pi}{b\lambda^3}\left[\frac{K[u]}{b\lambda + 1} - \left(1 + \frac{1}{b\lambda + 1}\right)E[u]\right] \tag{32}$$

with the argument u in Equations (31) and (32) being equal to (for $b\lambda > 1$):

$$u = \left[1 - \frac{1}{(b\lambda)^2}\right]^{1/2} \tag{33}$$

Substitution of Equations (31) and (32) into Equation (22) results in the integral $G_{2,3}$; taking the derivative of the resulting expression with respect to J_p as indicated by Equation (20) to yield the integral $I_{1,1,p,3}$, and substituting this integral into Equation (12) leads to the 2-dimensional expression for effective conductivity for the situation $b\lambda > 1$:

$$K_e(x, y) = e^{T(x,y)}\left[1 - \frac{\sigma_f^2\lambda^2}{2\pi}(L_1\cos^2\theta + L_2\sin^2\theta)\right] \tag{34}$$

Results for 2-dimensional effective conductivity when $b\lambda < 1$ and $b\lambda = 1$ are presented next. These are followed by the 3-dimensional example and a discussion of the significance of these theoretical findings.

Two-Dimensional Effective Hydraulic Conductivity When $b\lambda < 1$

The expression for effective hydraulic conductivity in this instance is given by:

$$K_e(x, y) = e^{T(x,y)}\left[1 - \frac{\sigma_f^2\lambda^2}{2\pi}(L_1\cos^2\theta + L_2\sin^2\theta)\right] \tag{35}$$

in which:

$$L_1 = \frac{2\pi}{b^2\lambda^4}\left[\left(1 + \frac{1}{1 - (b\lambda)^2}\right)E[u] - \frac{(b\lambda)^2}{1 - (b\lambda)^2}K[u]\right] \tag{36}$$

and

$$L_2 = \frac{2\pi}{b^2\lambda^4}\left[(b\lambda)^2K[u] - 2E[u]\right] \tag{37}$$

where the elliptic functions $K[u]$ and $E[u]$ are given by Equations (27) and (28), respectively, with the only difference being that the argument $u = [1 - (b\lambda)^2]^{1/2}$ for $b\lambda < 1$.

Two-Dimensional Effective Hydraulic Conductivity for $b\lambda = 1$

If $b\lambda = 1$, the 2-dimensional effective hydraulic conductivity simplifies to (in this situation $L_1 = -L_2 = \pi^2/\lambda^2$):

$$K_e(x, y) = e^{T(x,y)} \left[1 - \frac{\sigma_f^2 \pi}{2} (2 \cos^2 \theta - 1) \right] \tag{38}$$

It can be verified easily that the effective conductivity of Equation (38) represents the limit as $b\lambda \to 1$ of the effective conductivity expressions (34) and (35) for $b\lambda < 1$ and $b\lambda > 1$, respectively, as required by continuity. For the situation $b\lambda = 1$, Equation (38) indicates that for values of the angle $0 \leq \theta < \pi/4$ the variance σ_f^2 is restricted. For example, the most restrictive situation occurs if $\theta = 0$, that is, if the trend **b** and mean hydraulic gradient **J** vectors are parallel, then $\sigma_f^2 \leq 2/\pi$. Otherwise, σ_f^2 is unrestricted. Restrictions on σ_f^2 when $b\lambda = 1$ and $\theta \to 0$ simply indicate that, under the mathematical premises of our theory, it is not possible to maintain flow under such special conditions when the variance of ln$K$ exceeds values consistent with the small perturbation approach.

THREE-DIMENSIONAL EFFECTIVE HYDRAULIC CONDUCTIVITY

It should be clear from the developments leading to Equation (20) that the 3-dimensional situation requires setting $q = 4$ in that equation, and upon integration, the effective conductivity can be obtained directly from Equation (12). This was in essence the approach followed in the 1- and 2-dimensional situations. The 3-dimensional domain, however, requires a more involved set of geometric transformation to reduce the triple integral in Equation (20) to single integrals that then are evaluated exactly. The procedure consists of introducing a set of coordinate transformations that permit the simplification of the triple integral in Equation (20) to single integrals for a finite integration domain **S**, a sphere of radius ρ. Individually divergent terms that arise in the finite-domain situation cancel out mutually when one takes the limit $\rho \to \infty$ in passing to the infinite domain.

Biplanar Radial, Biconical Radial, and Biconical Mixed Coordinates

Let us introduce the biplanar radial coordinates $r = \sqrt{k^2}$, $u = \mathbf{k} \cdot \mathbf{b}$, $v = \mathbf{k} \cdot \mathbf{J}$, suggested by the structure of the integrand in Equation (20) for the three-dimensional situation ($q = 4$.) Integration of $G_{2,4}$ in the r, u, v coordinates requires the Jacobian, $J_{r,u,v}$, of the transformation from **k** space to r, u, v space. It is assumed that the vectors **b** and **J** are neither parallel nor antiparallel for calculations to be nondegenerate in biplanar radial coordinates. After careful analysis of the geometry of the r, u, v space, it follows that the absolute value of the Jacobian of the transformation is (with $\mathbf{c} = \mathbf{b} \times \mathbf{J}$ the vector product of **b** and **J**; and $w^2 = \|w\|^2 = \mathbf{w} \cdot \mathbf{w}$ for any vector **w**, $\mathbf{w} = \mathbf{b}, \mathbf{c}, \mathbf{J}$):

$$|J_{r,u,v}| = \frac{r}{\sqrt{r^2c^2 + 2uv\mathbf{b} \cdot \mathbf{J} - v^2b^2 - u^2J^2}} \tag{39}$$

The integral $G_{2,q}$ of Equation (20) with $q = 4$ in r, u, v coordinates becomes:

$$G_{2,4} = \int_\Omega j \frac{r^2v^2}{(r^4 + u^2)(1 + \lambda^2r^2)^2} \frac{r\, dr\, du\, dv}{\sqrt{r^2c^2 + 2uv\mathbf{b} \cdot \mathbf{J} - b^2v^2 - J^2u^2}} \tag{40}$$

where Ω is the set of triples (r, u, v) such that $r^2c^2 > b^2v^2 + J^2u^2 - 2\mathbf{b} \cdot \mathbf{J}uv$ and $r \leq \rho$, where ρ has been defined as the radius of the region of integration **S**.

For fixed r, $b^2v^2 - 2\mathbf{b} \cdot \mathbf{J}uv + J^2u^2 < r^2c^2$ is the interior of an elliptical region in (u, v) space. As r differs from 0 to ρ, the regions are geometrically similar, with the same principal axes. Thus the integration region Ω in Equation (40) is a solid elliptical cone. This geometry suggests replacement of the biplanar radial coordinate system with a biconical radial system, wherein (u, v) is replaced by angular variables A, B, leading to a 3-dimensional system (r, A, B). Specifically, $u = \mathbf{k} \cdot \mathbf{b} = br \cos B$, $v = \mathbf{k} \cdot \mathbf{J} = Jr \cos A$, supplemented by a third angular quantity θ defined by $\mathbf{b} \cdot \mathbf{J} = bJ \cos \theta$, which remains fixed in later calculations.

Clearly, A, B, θ are the angles between the edges **b**, **k**, **J** of a trihedron, where **k** is variable and **b**, **J** are fixed. (The angles A, B, θ can be selected to range in the interval $[0, \pi]$.) Because $r = \sqrt{k^2}$, it is evident that the coordinates r, A, B constitute a type of spherical coordinate system. Further simplifications arise in passing from (r, u, v) space to (r, A, B) in Equation (40). The absolute value of the Jacobian of the transformation $(r, u, v) \to (r, A, B)$ is given by $|J_{r,A,B}| = bJr^2|\sin A|\,|\sin B|$. Substitution of the biconical radial coordinates and their absolute Jacobian in Equation (40) transforms the integral to:

$$G_{2,4} = J^2 \int_\Omega j \frac{r^4 \cos A\,|\sin A|\,|\sin B|}{\Delta(A, B, \theta)(r^2 + b^2\cos^2 B)(1 + \lambda^2r^2)^2}\, dr\, dA\, dB \tag{41}$$

where $\Delta(A, B, \theta) = \sqrt{\sin^2 \theta + 2 \cos \theta \cos A \cos B - \cos^2 A - \cos^2 B}$.

The formula for $\Delta(A, B, \theta)$ can be simplified by the use of the coangle α to A, which goes back to Ptolemy. The coangles (α, β, γ) to (A, B, θ) are the angles between the three planes determined by the pairs of vectors (**k**, **J**), (**k**, **b**), and (**b**, **J**). Ptolemy's formula (second century AD) $\cos A = \cos B \cos \theta$

$+ \cos\alpha \sin B \sin\theta$ and its analogs are useful in determining (α, β, γ) from (A, B, θ). One key result is that $\Delta(A, B, \theta) = |\sin\theta|\,|\sin B|\,|\sin\alpha|$, showing that $\Delta(A, B, \theta)$ is separable in (r, α, B) coordinates.

The last in the series of geometric transformations aimed at simplifying the original triple integrals in Cartesian space, is to derive the Jacobian $J_{r,\alpha,B}$ of the transformation from biconical radial coordinates (r, A, B) to the mixed biconical coordinates (r, α, B). The absolute value of this Jacobian can be shown to be $|J_{r,\alpha,B}| = |\sin\alpha|\,|\sin B|\,|\sin\theta|/|\sin A|$, which upon substitution in Equation (41) along with the results for $\Delta(A, B, \theta)$ and Ptolemy's formula for $\cos A$, yields:

$$G_{2,4} = J^2 \int_\Omega j \frac{r^4 |\sin B| \cos^2 A}{(r^2 + b^2\cos^2 B)(1 + \lambda^2 r^2)^2}\, dr\, d\alpha\, dB \tag{42}$$

where $\Omega = (0 \le \alpha \le 2\pi;\ 0 \le B \le \pi;\ 0 \le r \le \rho)$ is the integration region; and $\cos^2 A = \cos^2 B \cos^2\theta + \tfrac{1}{2}\cos\alpha \sin 2B \sin 2\theta + \cos^2\alpha \sin^2 B \sin^2\theta$.

Equation (42) is integrated with respect to α from 0 to 2π and with respect to B from 0 to π to yield a single integral in a finite domain $0 \le r \le \rho$:

$$\begin{aligned} G_{2,4} = {} & 2j\pi J^2 \cos^2\theta \int_0^\rho \frac{2r^4}{(1 + \lambda^2 r^2)^2 b^2}\left[1 - \frac{r}{b}\tan^{-1}\left(\frac{b}{r}\right)\right] dr \\ & + j\pi J^2 \sin^2\theta \int_0^\rho \frac{2r^4}{b(1 + \lambda^2 r^2)^2} \\ & \cdot \left[\frac{1}{r}\tan^{-1}\left(\frac{b}{r}\right) - \frac{1}{b}\left(1 - \frac{r}{b}\tan^{-1}\left(\frac{b}{r}\right)\right)\right] dr \end{aligned} \tag{43}$$

Loaiciga and others (1994) developed a method for the integration of Equation (43) in the finite-domain and infinite-domain situations ($\rho \to \infty$). Their analysis showed that: (i) the integral is continuous at $b\lambda = \pm 1$ and convergent for $\rho \to \infty$, and (ii) the only singularity of the integral occurs at $b\lambda = 0$, a situation which is ruled out because the trend in $\ln K$ is nonzero by definition. Similarly to findings related to 2-dimensional domains, the singularities related to critical values of $b\lambda$ in one dimension disappear in three dimensions.

Results for K_e in 3-Dimensional Domains

Integration of Equation (43) and substitution of the resulting expression in Equation (20) after taking the limit $\rho \to \infty$, indicates, according to Equation (12), that the effective hydraulic conductivity in three dimensions is (where $m = b\lambda$):

$$K_e = e^T \left\{1 - \frac{\sigma_f^2 \lambda^3 b^3}{\pi}\left[(4\cos^2\theta - 2\sin^2\theta)F_1 - 2\sin^2\theta F_2\right]\right\} \tag{44}$$

in which

$$F_1 = \frac{\pi}{m^6}\left[\ln(1 + |m|) + \frac{|m|}{4}\frac{(m^2 - 2|m| - 4)}{1 + |m|}\right] \tag{45}$$

and

$$F_2 = \frac{\pi}{2m^4}\ln(1 + |m|) - \frac{\pi}{4|m|^3(1 + |m|)} \tag{46}$$

Evidently, the effective hydraulic conductivity involves elementary polynomial and logarithmic functions and key parameters of stochastic groundwater flow analysis.

DISCUSSION OF THEORETICAL RESULTS

All the expressions for effective hydraulic conductivity, be it in one [see Eq. (14)], two [see Eqs. (34), (35), and (38)], or three [see Eq. (44)] dimensions, have the generic form:

$$K_e(x, y) = e^{T(x, y)}[1 - \sigma_f^2\, \Xi\, (b\lambda, \theta)] \tag{47}$$

where θ is the angle between the gradient of the trend in ln$K$, **b**, and the mean hydraulic gradient, **J**; $T(x, y)$ is the trend in ln$K$; and Ξ is a function of aquifer parameters ($b = |\mathbf{b}|$, the correlation scale λ, and θ), and depends on the flow-domain dimensionality. For example, for 1-dimensional domains, $\Xi = 1/(b\lambda + 1)$. Equation (47) shows that the effective conductivity in the presence of trends of ln$K$ is equal to the geometric mean e^T times a factor introduced by the stochastic analysis ($= 1 - \sigma_f^2\, \Xi$) that depends on aquifer parameters. When Ξ is positive, nonnegativity of K_e requires that

$$\sigma_f^2 < \frac{1}{\Xi(b\lambda, \theta)}, \text{ (if } \Xi > 0) \tag{48}$$

In that situation, that is when $\Xi(b\lambda, \theta) > 0$, Equation (48) represents a generalization of the condition $\sigma_f^2 < 1$ widely used in stochastic groundwater analyses relying on the small-perturbations assumption.

Because the trend of ln$K$ generally is space-dependent, the effective conductivity generally also is, heterogeneous. According to Equation (47), K_e is isotropic, although it depends on the angle between the vector gradient of the trend T and the mean hydraulic gradient, both of which in general, are spatially variable. For each location **x**, K_e(**x**) represents a spatial average centered at that point. If areal or volumetric averages of hydraulic conductivity are needed, say, for the purpose of calibrating numerical models that rely on spatial discretizations, the effective conductivities given here can be averaged for a domain Ω by carrying out the integral $(1/\Omega)\int_\Omega K_e\, d\Omega$ (see Desbarats, 1992).

Numerical implementation of Equation (47) requires the trend $T(x, y)$, which must be identified from $\ln K(x, y)$ data. Trend identification consists essentially of fitting suitable functions to spatially indexed data. Having the trend $T(x, y)$, its vector gradient **b** follows directly by differentiation with respect to spatial coordinates. The variance of $\ln K$, σ_f^2, can be obtained from the second moment of the residuals $f = \ln K - T$, or jointly with the correlation scale λ by estimation methods such as maximum likelihood (Hoeksema and Kitanidis, 1985; Loaiciga and Marino, 1987). The correlation scale can be estimated by variogram analysis if a geostatistical estimation method is considered adequate for that purpose (Task Committee on Geostatistics in Geohydrology, 1989a, 1989b). In regards to the angle θ, its value depends on the trend of $\ln K$ as well as on the orientation of the mean hydraulic gradient vector at any point, **J**. Contours maps of hydraulic head and piezometric head data are key to calculating the angle θ through the flow domain.

AN EXAMPLE OF EFFECTIVE HYDRAULIC CONDUCTIVITY CALCULATION IN TWO DIMENSIONS

Sudicky (1986) presented hydraulic conductivity data acquired during the Borden aquifer tracer experiment; these data were analyzed further by Woodbury and Sudicky (1991). Figure 1A shows a plot of the hydraulic conductivity data along cross-section A-A' of the Borden experiment. This plot was composed from 720 conductivity values collected along a vertical cross section 20-m long and 1.80-m deep; along the horizontal coordinate K was determined at 1.0 m intervals; the vertical sampling spatial frequency was 1 sample every 0.05 m.

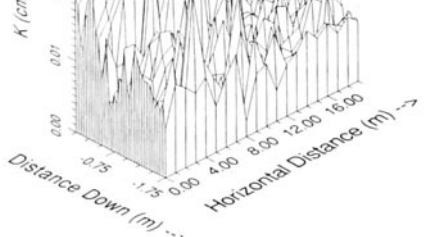

Figure 1A. Plot of hydraulic conductivity data (cm/s) for section A-A' of Borden aquifer (Sudicky, 1986).

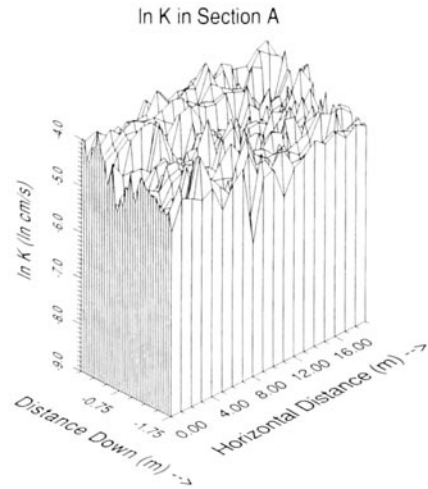

Figure 1B. Plot of ln*K* data for cross-section A-A' of Borden aquifer (Sudicky, 1986).

Figure 1B shows the plot of ln*K* for cross-section A-A'. Although masked by the irregular distribution of *K* and ln*K* in Figures 1A and 1B, a decreasing trend in both *K* and ln*K* can be observed with highest gradient approximately oriented from north (near the surface, where *K* and ln*K* appear to be highest) to south (along the deepest side of the cross section) in the plots of Figures 1A and 1B. In order to "filter out" the effective conductivity from the ln*K* field, a third-order polynomial trend was fitted to the ln*K* data, $T(x, y) = -4.12 + 0.152x - 1.06y - 0.0170x^2 + 0.515y^2 - 0.0946xy + 0.000408x^3 + 0.00524x^2y$, where $\mathbf{x} = (x, y, z)$, with all spatial dimensions in meters; modeling the ln*K* data as isotropic led to a correlation scale estimate of approximately 1 m; σ_f^2 was estimated to be about 0.30. Based on the approximate orientation of groundwater velocity at the Borden site reported in the series of papers that described the Borden experiment (see Freyberg, 1986), the angle θ between the ln*K* trend gradient **b** and the mean hydraulic gradient **J** could be calculated at every location. Figure 2 shows that the effective hydraulic conductivity (in cm/s) for section A-A' estimated by the equations developed for 2-dimensional domains [Eqs. (34), (35), and (38)]. In Figure 2 the decline in effective hydraulic conductivity with depth from the surface is apparent. Pronounced variations in K_e occur along the longitudinal direction at shallow depths. The longitudinal variability in K_2 decreases considerably with depth. The effective hydraulic conductivity is a deterministic and continuous function, hence its rather smooth shape in Figure 2, in contrast to the irregular distribution of *K* and ln*K* in Figures 1A and 1B.

Figures 3A and 3B show the plots of *K* and ln*K*, respectively, for cross-

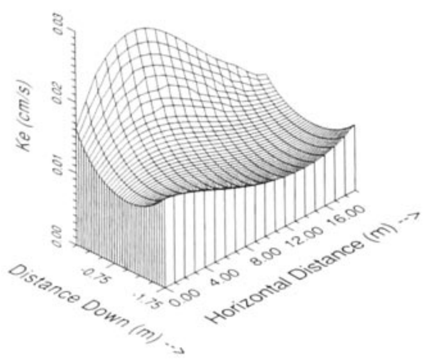

Figure 2. Effective hydraulic conductivity for section A-A' of Borden aquifer (in cm/s).

section B-B' of the Borden aquifer experiment as reported by Sudicky (1986). The sampling intervals along the longitudinal and vertical sides were 1m and 0.05 m, respectively, with the longitudinal dimension having a total length of 13 m and the vertical one being 1.80 m. The trend of ln$K$ for section B-B' was estimated to be a fourth-order polynomial, $T(x, y) = -3.93 + 0.0215x -$

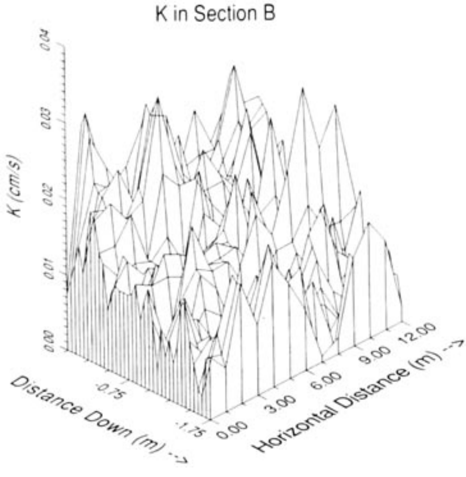

Figure 3A. Plot of hydraulic conductivity data (cm/s) for section B-B' of Borden aquifer (Sudicky, 1986).

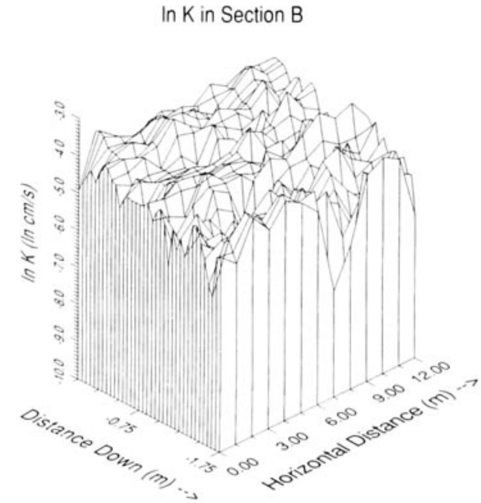

Figure 3B. Plot of ln$K$ data for section B-B' of Borden aquifer (Sudicky, 1986).

$1.49y^2 - 0.364xy + 0.663xy^2 + 0.432y^4 - 0.218xy^3$; the correlation scale and variance of ln$K$ were estimated to be, respectively, 1.0 m and 0.28, similar to the cross-section A-A' data, which is not surprising given that both cross sections were located within the same type of geologic material and in close proximity. Figure 4 shows the effective hydraulic conductivity (in cm/s) for cross-section B-B'. There is an overall declining trend in K_e with depth, and its variability along the longitudinal direction is more pronounced than that calculated for cross-section A-A' in Figure 2.

The results for K_e in Figures 2 and 4 demonstrate the methods for deriving the effective hydraulic conductivity in 2-dimensional domains. Our results are approximate, because, for example, Sudicky (1986), identified different horizontal and vertical anisotropies for sections A-A' and B-B'. In fact, the analysis of the hydraulic conductivity data from the Borden site as being 2-dimensional is somewhat arbitrary because the two cross sections evidently, are part of a three-dimensional aquifer. Nevertheless, the experimental design leading to the collection of K data along planar sections lends itself well for the 2-dimensional analysis proposed in this work.

Calculations of 3-D nonstationary effective conductivities based on Equation (44) have been presented in Loaiciga and others (1994). The reader is referred to that work for further details.

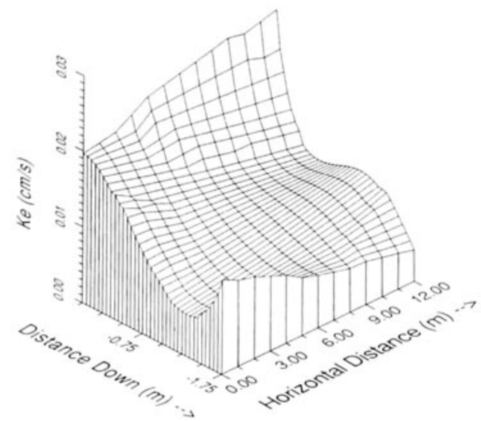

Figure 4. Effective hydraulic conductivity (cm/s) for section B-B' of Borden aquifer.

SUMMARY AND CONCLUSIONS

This paper has developed a method to calculate the effective hydraulic conductivity of in 1-, 2-, and 3-dimensional groundwater flow domains. Starting with the assumption of an isotropic and trend-heterogeneous ln$K$, equations for effective hydraulic conductivity were developed and an example was given for a 2-D log-conductivity field based on the Borden aquifer conductivity data. Examples of 3-D effective conductivity calculations are available from Loaiciga and others (1994).

It was determined that the effective hydraulic conductivity is spatially variable and isotropic. Assuming an exponential covariance for ln$K$, it was determined that the effective hydraulic conductivity is continuous in two- and three-dimensional aquifers for all values of the parameter product $b\lambda$. Singularities that arise at $b\lambda = 0$ are of no practical or theoretical interest because by definition our analysis is concerned with log-conductivities that exhibit structural trends. In 2-D flow domains, restrictions on the variance of ln$K$ occur if the angle between the log-conductivity trend gradient and the mean hydraulic gradient vectors approaches zero if the corner condition $b\lambda = 1$ holds. These restrictions, however, are consistent with the small perturbation approach. In 3-D domains, parallelism or antiparallelism between the ln$K$ trend gradient vector and the mean hydraulic gradient vector are not permissible. In one dimension, a singularity in K_e exists at $b\lambda = -1$.

The results of effective hydraulic conductivity in two and three dimensions show that this "ensemble" parameter depends on the flow regime and on aquifer parameters. The flow regime determines K_e, somewhat indirectly, through the angle formed by the ln$K$ trend gradient and the mean hydraulic gradient. The magnitude of the ln$K$ trend gradient, b, the variance of ln$K$, σ_f^2, and the correlation scale of ln$K$, λ, are aquifer parameters that directly effect K_e. In the one dimension, the effective hydraulic conductivity depends on σ_f^2, λ, and b exclusively.

A general condition for the feasibility of K_e according to our results in 1-, 2-, and 3-dimensional domains is that the variance of ln$K$ satisfies the inequality $\sigma_f^2 < 1/\Xi(b\lambda, \theta)$ when $\Xi > 0$, where the function $\Xi(b\lambda, \theta)$ depends on flow-domain dimensionality, and it was given for one, two, and three dimensions. It was determined that the effective hydraulic conductivity is proportional to the geometric mean e^T, where T is the trend in ln$K$; the proportionality factor is $1 - \sigma_f^2 \Xi$.

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