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Motivic Steenrod operations in characteristic  $p$

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by

Eric J. Primožic

2019

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ABSTRACT OF THE DISSERTATION

Motivic Steenrod operations in characteristic  $p$

by

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Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2019

Professor Burt Totaro, Chair

Using the recent work of Frankland and Spitzweck, we define motivic Steenrod operations  $P^n$  on the mod  $p$  motivic cohomology of smooth varieties defined over a base field of characteristic  $p$ . We show that the operations  $P^n$  satisfy expected properties such as the Adem relations and Cartan formula when restricted to mod  $p$  Chow groups. We also show that  $P^n$  is the  $p$ th power on  $CH^n(-)/p$  and prove an instability result. Using these new operations, we remove previous restrictions on the characteristic of the base field for Rost's degree formula. We also prove Hoffmann's conjecture (generalized to include quadratic forms over a base field of characteristic 2) on the possible values of the first Witt index of an anisotropic quadratic form for the special case of nonsingular anisotropic quadratic forms over a base field of characteristic 2.

The dissertation of Eric J. Primožic is approved.

Paul Balmer

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2019

To my parents

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## VITA

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# Chapter 1

## Introduction

Voevodsky constructed motivic reduced power operations  $P_F^n$  for  $n \geq 0$  where the base field  $F$  is a perfect field with  $\text{char}(F)$  not equal to the characteristic of the coefficient field [30]. These operations were used in the proof of the Bloch-Kato conjecture. Motivic reduced power operations can also be used to prove results about quadratic forms and degree formulas in algebraic geometry as in [5] and [21].

For a prime  $p$ , Voevodsky's construction of Steenrod operations for the coefficient field  $\mathbb{F}_p$  uses the calculation of the motivic cohomology of  $BS_p$ . However, when defined over a base field  $k$  of characteristic  $p$ ,  $B\mathbb{Z}/p$  is contractible [22, Proposition 3.3]. Hence, over the base field  $k$ ,  $H^{*,*}(BS_p, \mathbb{F}_p) \cong H^{*,*}(k, \mathbb{F}_p)$  and so one cannot carry out Voevodsky's construction. It has also been an open problem to just define Steenrod operations on the mod  $p$  Chow groups of smooth schemes over a field of characteristic  $p$ . Huaijun made progress on this problem by constructing the first  $p - 1$  homological Steenrod operations on Chow groups mod  $p$  and  $p$ -primary torsion [11], defining the first Steenrod square on mod 2 Chow groups over any base field [12], and constructing weak forms of the second and third Steenrod squares over a field of characteristic 2 [14]. Note that in papers where Steenrod squares (or weak forms of Steenrod squares) on mod 2 Chow groups are used, the  $n$ th Steenrod square on mod 2 Chow groups corresponds to the  $2n$ th Steenrod square on mod 2 motivic cohomology since the Bockstein homomorphism is 0 on

mod 2 Chow groups.

For  $p$  a prime, we use the results of Frankland and Spitzweck in [8] to define Steenrod operations  $P_k^n : H^{i,j}(-, \mathbb{F}_p) \rightarrow H^{i+2n(p-1), j+n(p-1)}(-, \mathbb{F}_p)$  for  $n \geq 0$  on the mod  $p$  motivic cohomology of smooth schemes over a field  $k$  of characteristic  $p$ . Note that some authors use the notation  $H^i(-, \mathbb{Z}(j))$  in place of  $H^{i,j}(-, \mathbb{Z})$  to denote motivic cohomology. For  $n \geq 1$ , we show that  $P_k^n$  is the  $p$ th power on  $H^{2n,n}(-, \mathbb{F}_p) = CH^n(-)/p$ , and we also prove an instability result for the Steenrod operations. Restricted to mod  $p$  Chow groups, we prove that the  $P_k^i$  satisfy expected properties such as Adem relations and the Cartan formula. We also show that the operations  $P_k^n$  agree with the operations  $P_K^n$ , constructed by Voevodsky for  $\text{char}(K) = 0$ , on the mod  $p$  Chow rings of flag varieties in characteristic 0.

In Chapter 9, we extend Rost's degree formula [21, Theorem 6.4] to a base field of arbitrary characteristic. The degree formula we obtain at odd primes seems to be new. In Chapter 11, we use the new operations to study quadratic forms defined over a field of characteristic 2. Previous results or proofs avoided the case of quadratic forms in characteristic 2 since Steenrod squares were not available. We prove Hoffmann's conjecture (a generalization including characteristic 2 quadratic forms) on the possible values of the first Witt index of anisotropic quadratic forms for the case of nonsingular anisotropic quadratic forms over a field of characteristic 2. In characteristic  $\neq 2$ , Hoffmann's conjecture was proved by Karpenko in [17]. Previously, Houton used a weak form of the first homological Steenrod square to prove a result on the parity of the first Witt index for nonsingular anisotropic quadratic forms over a field of characteristic 2 [13, Theorem 6.2]. Houton's result is a corollary of the case of Hoffmann's conjecture proved in this paper. Using the Steenrod squares defined in this paper, it should be possible to extend other results on quadratic forms to the case where the base field has characteristic 2.

## Chapter 2

# Prior results on the dual Steenrod algebra and setup

Let  $k$  be a field of characteristic  $p > 0$ . For a base scheme  $S$ , we let  $\mathrm{Sm}_S$  denote the category of quasi-projective separated smooth schemes of finite type over  $S$ , let  $H(S)$  denote the unstable motivic homotopy category of spaces over  $S$  defined by Morel-Voevodsky [22], let  $H_\bullet(S)$  denote the pointed unstable motivic homotopy category of spaces over  $S$ , and we let  $SH(S)$  denote the stable motivic homotopy category of spectra over  $S$  [28]. Let

$$\Sigma_+^\infty : \mathrm{Sm}_S \rightarrow SH(S),$$

$$\Sigma_+^\infty : H(S) \rightarrow H_\bullet(S) \rightarrow SH(S),$$

denote the infinite  $\mathbb{P}^1$ -suspension functors. We recall some results from [27] and [8] that hold in the categories  $H(k)$  and  $SH(k)$ . Let  $B\mu_p \in H(k)$  denote the geometric motivic classifying space of the group scheme  $\mu_p$  over  $k$  of the  $p$ th roots of unity. Let  $H\mathbb{F}_p^k \in SH(k)$  denote the motivic Eilenberg-MacLane spectrum representing mod  $p$  motivic cohomology. Let  $v \in H^{2,1}(B\mu_p, \mathbb{F}_p)$  denote the pullback of the first Chern class  $c_1 \in H^{2,1}(B\mathbb{G}_m, \mathbb{F}_p)$ . From the computation of the motivic cohomology of  $B\mu_p$  in [30, Theorem 6.10], there exists a unique  $u \in H^{1,1}(B\mu_p, \mathbb{F}_p)$  such that  $\beta(u) = v$  where  $\beta$  denotes the Bockstein homomorphism on mod  $p$  motivic cohomology.

The class  $\rho = -1$  in  $H^{1,1}(k, \mathbb{F}_p) = k^*/k^{*p}$  is 0 and the class of  $\tau = -1$  in  $H^{0,1}(k, \mathbb{F}_p) = \mu_p(k) = 0$  is also 0. We need the following computation which can be deduced from [30, Theorem 6.10] by setting  $\rho = 0$  and  $\tau = 0$ .

**Theorem 2.0.1.** *There is an isomorphism*

$$H^{*,*}(B\mu_p, \mathbb{F}_p) \cong H^{*,*}(k, \mathbb{F}_p)[[v, u]]/(u^2).$$

Let  $\mathcal{A}_{*,*}^k := \pi_{*,*}(H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k)$ . As described in [27, Chapter 10.2], there is a coaction map

$$H^{*,*}(B\mu_p, \mathbb{F}_p) \rightarrow \mathcal{A}_{-*,-*}^k \widehat{\otimes}_{\pi_{-*,-*} H\mathbb{F}_p^k} H^{*,*}(B\mu_p, \mathbb{F}_p). \quad (2.1)$$

We use the left  $H\mathbb{F}_p^k$ -module structure on  $H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k$ . For  $i \geq 0$  and  $j \geq 1$ , classes  $\tau_i \in \mathcal{A}_{2^{p^i-1}, p^{i-1}}^k$  and  $\xi_j \in \mathcal{A}_{2^{p^j-2}, p^{j-1}}^k$  are defined by the coaction map:

$$u \mapsto u + \sum_{i \geq 0} \tau_i \otimes v^{p^i},$$

$$v \mapsto v + \sum_{j \geq 1} \xi_j \otimes v^{p^j}.$$

**Proposition 2.0.2.**  $\tau_i^2 = 0$  for all  $i \geq 0$ .

*Proof.* We use the argument of [30, Theorem 12.6]. First, we assume that  $\text{char}(k) = 2$ . Under the coaction map 2.1,

$$u^2 = 0 \mapsto u^2 + \sum_{i \geq 0} \tau_i^2 \otimes v^{2^{i+1}} = 0.$$

For  $i \geq 0$ , the coefficient of  $v^{2^{i+1}}$  equals  $0 = \tau_i^2$ .

Now we assume that  $p = \text{char}(k)$  is odd. Let  $i \geq 0$ . As  $\mathcal{A}_{*,*}^k$  is graded-commutative under the first grading, we have  $\tau_i^2 = (-1)^{(2^{p^i-1})(2^{p^i-1})} \tau_i^2 = -\tau_i^2$  which implies that  $\tau_i^2 = 0$ .  $\square$

In this paper, we shall consider finite sequences  $\alpha = (\epsilon_0, r_1, \epsilon_1, r_2, \dots)$  of integers such that  $\epsilon_i \in \{0, 1\}$  and  $r_j \geq 0$  for all  $i \geq 0$  and  $j \geq 1$ . From now on, it will be assumed that any

sequence  $\alpha$  in this paper satisfies these conditions. To a sequence  $\alpha$ , we associate a monomial  $\omega(\alpha) = \tau_0^{\epsilon_0} \xi_1^{\tau_1} \tau_1^{\epsilon_1} \cdots \in \mathcal{A}_{*,*}^k$  of bidegree  $(p_\alpha, q_\alpha)$ . The sequences  $\alpha$  induce a morphism

$$\Psi_k : \bigoplus_{\alpha} \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k \rightarrow H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k$$

of left  $H\mathbb{F}_p^k$ -modules. Frankland and Spitzweck proved the following theorem [8, Theorem 1.1] which allows us to define Steenrod operations on mod  $p$  motivic cohomology over the base  $k$ .

**Theorem 2.0.3.** *The morphism*

$$\Psi_k : \bigoplus_{\alpha} \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k \rightarrow H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k$$

*is a split monomorphism.*

It is conjectured that  $\Psi_k$  is an isomorphism. Frankland and Spitzweck proved this theorem by comparing  $\Psi_k$  to the corresponding isomorphism

$$\Psi_K : \bigoplus_{\alpha} \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^K \rightarrow H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \quad (2.2)$$

of left  $H\mathbb{F}_p^K$ -modules for  $\text{char}(K) = 0$ . From now on, we will identify  $\bigoplus_{\alpha} \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^K$  with  $H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K$  as left  $H\mathbb{F}_p^K$ -modules through  $\Psi_K$  whenever  $K$  is a field of characteristic 0. Let  $D$  be a complete unramified discrete valuation ring with closed point  $i : \text{Spec}(k) \rightarrow \text{Spec}(D)$  and generic point  $j : \text{Spec}(K) \rightarrow \text{Spec}(D)$  where  $K = \text{Frac}(D)$ . For example, when  $k = \mathbb{F}_p$ , we take  $D = \mathbb{Z}_p$  and  $K = \mathbb{Q}_p$ .

For a morphism  $f : S_1 \rightarrow S_2$  of base schemes, we let  $f_* := Rf_* : SH(S_1) \rightarrow SH(S_2)$  and  $f^* := Lf^* : SH(S_2) \rightarrow SH(S_1)$  denote the right derived pushforward and left derived pullback functors respectively. Pullback  $f^*$  is strongly monoidal while  $f_*$  is lax monoidal. Furthermore,  $f_*$  commutes with all suspensions  $\Sigma^{i,j}$  [8, Lemma 7.5].

For  $S$  a separated Noetherian scheme of finite dimension, we let  $\widehat{HZ}^S \in SH(S)$  denote the motivic  $E_\infty$  ring spectrum constructed by Spitzweck in [27] and let  $\widehat{H}\mathbb{F}_p^S := \widehat{HZ}^S/p$ . Let  $D(\widehat{HZ}^S)$  denote the homotopy category of left  $\widehat{HZ}^S$ -modules. See [2, Section 7.2] for a discussion

on the homotopy category of left  $R$ -modules  $D(R)$  for a ring object  $R \in SH(S)$ . There is a forgetful functor  $U_S : D(\widehat{HZ}^S) \rightarrow SH(S)$ . A morphism  $f : X \rightarrow Y$  in  $D(\widehat{HZ}^S)$  is equivalent to giving a commutative diagram

$$\begin{array}{ccc} \widehat{HZ}^S \wedge X & \longrightarrow & X \\ \downarrow id. \wedge f & & \downarrow f \\ \widehat{HZ}^S \wedge Y & \longrightarrow & Y \end{array}$$

in  $SH(S)$ .

The spectrum  $\widehat{HZ}^S$  enjoys a number of desirable properties. The spectrum  $\widehat{HZ}^S$  is Cartesian. This means that for a morphism  $f : S_1 \rightarrow S_2$  of base schemes, the induced morphism  $f^* \widehat{HZ}^{S_2} \rightarrow \widehat{HZ}^{S_1}$  is an isomorphism in  $SH(S_1)$  of  $E_\infty$  ring spectra [27, Chapter 9]. Hence, the square

$$\begin{array}{ccc} D(\widehat{HZ}^{S_2}) & \xrightarrow{f^*} & D(\widehat{HZ}^{S_1}) \\ \downarrow U_{S_2} & & \downarrow U_{S_1} \\ SH(S_2) & \xrightarrow{f^*} & SH(S_1) \end{array}$$

commutes.

For  $S = \text{Spec}(F)$  with  $F$  a field,  $\widehat{HZ}^S$  is isomorphic as an  $E_\infty$  ring spectrum to the usual Eilenberg-MacLane spectrum  $H\mathbb{Z}^S$  constructed by Voevodsky [27, Theorem 6.7]. For the discrete valuation ring  $D$ ,  $\widehat{HZ}^D$  represents Bloch-Levine motivic cohomology as defined in [20].

We briefly describe the definition of Bloch-Levine motivic cohomology in [20] for a discrete valuation ring  $D$ . Let  $X \rightarrow \text{Spec}(D)$  be a morphism of finite type with  $X$  irreducible. If the image of the generic point  $\eta_X$  of  $X$  is  $\text{Spec}(k)$ , then we define  $\dim(X) := \dim(X_{\text{Spec}(k)})$ . Otherwise, we define  $\dim(X) := \dim(X_{\text{Spec}(K)}) + 1$ . For  $n \geq 0$ , let  $\Delta^n := \text{Spec}(D[t_0, \dots, t_n]/\sum_i t_i - 1)$  denote the algebraic  $n$ -simplex over  $D$ . Let  $z_q(X, r)$  denote the free abelian group generated by all irreducible closed subschemes  $C \subset \Delta^r \times_{\text{Spec}(D)} X$  of dimension  $r + q$  such that  $C$  meets each face of  $\Delta^r \times_{\text{Spec}(D)} X$  properly. We then set  $z^q(X, r) = z_{\dim(X) - q}(X, r)$  so that we get a pullback homomorphism  $z^q(X, r) \rightarrow z^q(X, r - 1)$  for each face of  $\Delta^r$ . Then the Zariski hypercohomology of the complex  $z^q(X, *)$  with alternating face maps is Bloch-Levine motivic cohomology (with the appropriate shift).

**Theorem 2.0.4.**

The morphism  $H\mathbb{F}_p^k \cong i^*(\widehat{H}\mathbb{F}_p^D) \rightarrow i^*j_*H\mathbb{F}_p^K \cong i^*j_*j^*\widehat{H}\mathbb{F}_p^D$  in  $D(H\mathbb{F}_p^k)$  induced by adjunction induces a splitting  $i^*j_*H\mathbb{F}_p^K \cong H\mathbb{F}_p^k \oplus \Sigma^{-1,-1}H\mathbb{F}_p^k$  in  $D(H\mathbb{F}_p^k)$ . We let  $\pi : i^*j_*H\mathbb{F}_p^K \rightarrow H\mathbb{F}_p^k$  and  $\pi_0 : i^*j_*H\mathbb{F}_p^K \rightarrow \Sigma^{-1,-1}H\mathbb{F}_p^k$  denote the projections induced by this splitting. There is also a splitting  $i^*j_*H\mathbb{Z}^K \cong H\mathbb{Z}^k \oplus \Sigma^{-1,-1}H\mathbb{Z}^k$  in  $D(H\mathbb{Z}^k)$  [8, Lemma 4.10].

Let  $\eta : id. \rightarrow j_*j^*$  denote the unit map. From now on, we shall denote all adjunction morphisms  $i^*E \rightarrow i^*j_*j^*E$  for  $E \in SH(D)$  by  $i^*\eta$ . We will also denote all  $\Sigma^{s,t}\pi, \Sigma^{s,t}\pi_0$  by  $\pi$  and  $\pi_0$  respectively to make the text easier to read. The morphisms  $\Psi_k$  and  $\Psi_K$  lift to a morphism

$$\Psi_D : \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} \widehat{H}\mathbb{F}_p^D \rightarrow \widehat{H}\mathbb{F}_p^D \wedge \widehat{H}\mathbb{F}_p^D$$

in  $D(\widehat{H}\mathbb{F}_p^D)$  [8, Lemma 3.10]. Applying  $i^*\eta$  to  $\Psi_D$ , we get a commuting square

$$\begin{array}{ccc} \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k & \xrightarrow{\Psi_k} & H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \\ \downarrow i^*\eta & & \downarrow i^*\eta \\ \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} i^*j_*H\mathbb{F}_p^K & \xrightarrow{i^*j_*\Psi_K} & i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \end{array} \quad (2.3)$$

in  $D(H\mathbb{F}_p^k)$ . Let  $r : H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \rightarrow \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k$  be the retraction of  $\Psi_k$  defined by the following composite [8, Theorem 5.1]

$$\begin{array}{ccc} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k & \xrightarrow{i^*\eta} & i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \xrightarrow{i^*j_*\Psi_K^{-1}} \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} i^*j_*H\mathbb{F}_p^K \\ & & \downarrow \oplus \pi \\ & & \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k \end{array}$$

For  $S = k, K$ , or  $D$  (use  $\widehat{H}\mathbb{F}_p^D$ ), we let  $\mu_1^S : H\mathbb{F}_p^S \wedge H\mathbb{F}_p^S \rightarrow H\mathbb{F}_p^S$  denote the multiplication morphism. There is also a multiplication morphism

$$\mu_2^S : (H\mathbb{F}_p^S \wedge H\mathbb{F}_p^S) \wedge (H\mathbb{F}_p^S \wedge H\mathbb{F}_p^S) \rightarrow H\mathbb{F}_p^S \wedge H\mathbb{F}_p^S$$

defined in the standard way by interchanging the two middle  $H\mathbb{F}_p^S$  terms and then applying  $\mu_1^S \wedge \mu_1^S$ .

For a sequence  $\alpha_0$ , we define  $i^*\eta_{\alpha_0} : H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \rightarrow \Sigma^{p\alpha_0, q\alpha_0} H\mathbb{F}_p^k$  in  $D(H\mathbb{F}_p^k)$  to be the composite

$$\begin{array}{ccc}
H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k & \xrightarrow{i^*\eta} & i^*j_*(H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) \xrightarrow{i^*j_*\Psi_K^{-1}} \bigoplus_{\alpha} \Sigma^{p\alpha, q\alpha} i^*j_*H\mathbb{F}_p^k & \xrightarrow{proj.} & \Sigma^{p\alpha_0, q\alpha_0} i^*j_*H\mathbb{F}_p^k \\
& & & & \downarrow \pi \\
& & & & \Sigma^{p\alpha_0, q\alpha_0} H\mathbb{F}_p^k.
\end{array} \tag{2.4}$$

The morphism  $i^*\eta_{\alpha_0}$  is a retract of the morphism  $H\mathbb{F}_p^k \wedge \omega(\alpha_0) : \Sigma^{p\alpha_0, q\alpha_0} H\mathbb{F}_p^k \rightarrow H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k$ .

From the work of Voevodsky [29] and Friedlander-Suslin [7, Corollary 12.2], Bloch's higher Chow groups are isomorphic to motivic cohomology as defined by Voevodsky. The isomorphism between motivic cohomology and Bloch's higher Chow groups is compatible with product structures [27, Theorem 6.7]. See also [19].

**Theorem 2.0.5.** *Let  $F$  be a field and let  $X \in \text{Sm}_F$ . Then*

$$H^{n,i}(X, \mathbb{Z}) \cong CH^i(X, 2i - n)$$

for all  $n$  and  $i \geq 0$ .

Let  $n, i \geq 0$  such that  $n > 2i$ . From the above theorem, we get that  $H^{n,i}(X, A) = 0$  for any coefficient ring  $A$  and  $X \in \text{Sm}_F$ .

## Chapter 3

# Definition of operations

In this chapter, we use the results of Frankland and Spitzweck in [8] to define new Steenrod operations  $P_k^n$  for  $n \geq 0$ . Let

$$i_L, i_R : H\mathbb{F}_p^S \rightarrow H\mathbb{F}_p^S \wedge H\mathbb{F}_p^S$$

denote the left and right unit maps respectively for  $S = D$  (use  $\widehat{H}\mathbb{F}_p^D$ ),  $k$ , or  $K$ . Motivated by the corresponding duality in characteristic 0, we want to define operations  $P_k^n \in H\mathbb{F}_p^{k*,*} H\mathbb{F}_p^k$  for  $n \geq 0$  by taking operations dual to the  $\xi_1^n$ .

**Definition 3.0.1.** *Let  $\alpha$  be a sequence. Define  $P_k^\alpha \in H\mathbb{F}_p^{k*,*} H\mathbb{F}_p^k$  by  $P_k^\alpha := i^* \eta_\alpha \circ i_R$ . For  $n \geq 0$ , we let  $P_k^n = P_k^{(0,n,0,\dots)}$ . Let  $\beta_k = P_k^{(1,0,\dots)}$ .*

There are corresponding operations  $P_K^\alpha$  in characteristic 0 defined from 2.2 by

$$H\mathbb{F}_p^K \xrightarrow{i_R} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \xrightarrow{proj.} \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^K.$$

**Definition 3.0.2.** *To define a homomorphism  $\Phi : H\mathbb{F}_p^{K*,*} H\mathbb{F}_p^K \rightarrow H\mathbb{F}_p^{k*,*} H\mathbb{F}_p^k$  of graded additive groups, let  $f : H\mathbb{F}_p^K \rightarrow \Sigma^{k,l} H\mathbb{F}_p^K$  be given. Define  $\Phi(f) : H\mathbb{F}_p^k \rightarrow \Sigma^{k,l} H\mathbb{F}_p^k$  by  $\Phi(f) = \pi \circ i^* j_*(f) \circ i^* \eta$ .*

$$H\mathbb{F}_p^k \xrightarrow{i^* \eta} i^* j_* H\mathbb{F}_p^K \xrightarrow{i^* j_*(f)} \Sigma^{k,l} i^* j_* H\mathbb{F}_p^K \xrightarrow{\pi} \Sigma^{k,l} H\mathbb{F}_p^k \quad (3.1)$$

From the definition of  $\Phi$ , it is clear that  $\Phi(id.) = id.$ . The following lemma will be important for proving that the operations  $P_k^n$  restricted to mod  $p$  Chow groups satisfy the Adem relations and Cartan formula.

**Lemma 3.0.1.** *Let  $X \in \text{Sm}_k$  and let  $f : \Sigma_+^\infty X \rightarrow \Sigma^{2m,m} H\mathbb{F}_p^k$  be given.*

1. *Let  $\alpha_0$  be a sequence. Consider the morphism*

$$g_{\alpha_0} : H\mathbb{F}_p^k \rightarrow \Sigma^{p_{\alpha_0}-1, q_{\alpha_0}-1} H\mathbb{F}_p^k$$

*given by the following composite.*

$$H\mathbb{F}_p^k \xrightarrow{i^*\eta} i^*j_* H\mathbb{F}_p^K \xrightarrow{i^*j^*(P_K^{\alpha_0})} i^*j_* \Sigma^{p_{\alpha_0}, q_{\alpha_0}} H\mathbb{F}_p^K \xrightarrow{\pi_0} \Sigma^{p_{\alpha_0}-1, q_{\alpha_0}-1} H\mathbb{F}_p^k.$$

*Then  $\Sigma^{2m,m} g_{\alpha_0} \circ f = 0$ .*

2. *The composite*

$$\begin{array}{ccc} \Sigma_+^\infty X & \xrightarrow{f} \Sigma^{2m,m} H\mathbb{F}_p^k & \xrightarrow{i_R} \Sigma^{2m,m} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k & \xrightarrow{i^*\eta} i^*j_*(\Sigma^{2m,m} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \\ & & & \downarrow i^*j_*\Psi_K^{-1} \\ & & & \bigoplus_{\alpha} \Sigma^{p_{\alpha}+2m, q_{\alpha}+m} i^*j_* H\mathbb{F}_p^K \\ & & & \downarrow \bigoplus \pi_0 \\ & & & \bigoplus_{\alpha} \Sigma^{2m+p_{\alpha}-1, m+q_{\alpha}-1} H\mathbb{F}_p^k \end{array}$$

*is equal to 0.*

*Proof.* Note that for any sequence  $\alpha$  of bidegree  $(p_{\alpha}, q_{\alpha})$ ,  $p_{\alpha} \geq 2q_{\alpha}$  which implies that  $p_{\alpha} - 1 > 2(q_{\alpha} - 1)$ . For (1) and (2), Theorem 2.0.5 implies that

$$\text{Hom}_{SH(k)}(\Sigma_+^\infty X, \Sigma^{2m+p_{\alpha}-1, m+q_{\alpha}-1} H\mathbb{F}_p^k) = H^{2m+p_{\alpha}-1, m+q_{\alpha}-1}(X, \mathbb{F}_p) = 0$$

for any sequence  $\alpha$ .

□

**Theorem 3.0.2.** 1. *We have  $\Phi(H^{*,*}(K, \mathbb{F}_p)) \subset H^{*,*}(k, \mathbb{F}_p)$ .*

2. Let  $\alpha$  be a sequence. Then  $\Phi(P_K^\alpha) = P_k^\alpha$ . In particular, for the Bockstein  $\beta_K$  and reduced power operations  $P_K^n$  constructed by Voevodsky in characteristic 0,  $\Phi(P_K^n) = P_k^n$  for  $n \geq 0$  and  $\Phi(\beta_K) = \beta_k$ . Also,  $P_k^0$  is the identity since  $P_K^0$  is the identity.
3. Let  $X \in \text{Sm}_k$  and let  $f : \Sigma_+^\infty X \rightarrow \Sigma^{2m,m} H\mathbb{F}_p^k$  be given. Let  $\alpha$  be a sequence and let  $h : H\mathbb{F}_p^K \rightarrow \Sigma^{i,j} H\mathbb{F}_p^K$  be given. Then

$$\Phi(h \circ P_K^\alpha)(f) = \Phi(h)(P_k^\alpha(f)).$$

*Proof.* We first prove (1). Let  $a \in H^{*,*}(K, \mathbb{F}_p)$ . The element  $a$  corresponds to a morphism  $f_a : H\mathbb{F}_p^K \rightarrow \Sigma^{m,n} H\mathbb{F}_p^K$  in  $D(H\mathbb{F}_p^K)$ . The functors  $i^*, j_*$  restrict to functors  $i^* : D(\widehat{H}\mathbb{F}_p^D) \rightarrow D(H\mathbb{F}_p^k)$  and  $j_* : D(H\mathbb{F}_p^K) \rightarrow D(\widehat{H}\mathbb{F}_p^D)$ . Hence,  $i^*j_*(f_a)$  is a morphism in  $D(H\mathbb{F}_p^k)$ . From the definition of  $\Phi$ , it follows that  $\Phi(f_a)$  is a morphism in  $D(H\mathbb{F}_p^k)$ . Thus,  $\Phi(a) := \Phi(f_a) \in H^{*,*}(k, \mathbb{F}_p)$ .

We now prove (2). Let  $\alpha$  be a sequence. Applying the natural transformation  $i^* \rightarrow i^*j_*j^*$  to the right unit map  $i_R : \widehat{H}\mathbb{F}_p^D \rightarrow \widehat{H}\mathbb{F}_p^D \wedge \widehat{H}\mathbb{F}_p^D$ , we obtain the following commuting square in  $SH(k)$ .

$$\begin{array}{ccc} H\mathbb{F}_p^k & \xrightarrow{i_R} & H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \\ \downarrow i^*\eta & & \downarrow i^*\eta \\ i^*j_*H\mathbb{F}_p^K & \xrightarrow{i^*j_*(i_R)} & i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \end{array}$$

From the definition of  $i^*\eta_\alpha$  2.4, the following diagram commutes.

$$\begin{array}{ccccc} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k & \xrightarrow{i^*\eta_\alpha} & \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k & \xrightarrow{id.} & \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k \\ \downarrow i^*\eta & & & & \downarrow id. \\ i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) & \xrightarrow{proj.} & i^*j_*\Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^K & \xrightarrow{\pi} & \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^K \end{array}$$

Putting these 2 diagrams together, we get the following commuting diagram.

$$\begin{array}{ccccccc} H\mathbb{F}_p^k & \xrightarrow{i_R} & H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k & \xrightarrow{i^*\eta_\alpha} & \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k & \xrightarrow{id.} & \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k \\ \downarrow i^*\eta & & \downarrow i^*\eta & & & & \downarrow id. \\ i^*j_*H\mathbb{F}_p^K & \xrightarrow{i^*j_*(i_R)} & i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) & \xrightarrow{proj.} & i^*j_*\Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^K & \xrightarrow{\pi} & \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^K \end{array} \quad (3.2)$$

The top row of this diagram gives  $P_k^\alpha$  while the composite starting at  $H\mathbb{F}_p^k$  in the top left and continuing along the bottom row gives  $\Phi(P_K^\alpha)$ . Hence,  $\Phi(P_K^\alpha) = P_k^\alpha$ .

Now, we prove (3). Consider the following diagram.

$$\begin{array}{ccccc}
\Sigma_+^\infty X & & & & \\
\downarrow f & & & & \\
\Sigma^{2m,m} H\mathbb{F}_p^k & \xrightarrow{P_k^\alpha} & \Sigma^{2m+p_\alpha, m+q_\alpha} H\mathbb{F}_p^k & \xrightarrow{\Phi(h)} & \Sigma^{i+2m+p_\alpha, j+m+q_\alpha} H\mathbb{F}_p^k \\
\downarrow i^*\eta & & \downarrow i^*\eta & & \downarrow i^*\eta \\
i^*j_*\Sigma^{2m,m} H\mathbb{F}_p^K & \xrightarrow{i^*j_*P_K^\alpha} & i^*j_*\Sigma^{2m+p_\alpha, m+q_\alpha} H\mathbb{F}_p^K & \xrightarrow{i^*j_*h} & i^*j_*\Sigma^{i+2m+p_\alpha, j+m+q_\alpha} H\mathbb{F}_p^K \\
& & & & \downarrow \pi \\
& & & & \Sigma^{i+2m+p_\alpha, j+m+q_\alpha} H\mathbb{F}_p^k.
\end{array} \tag{3.3}$$

As  $\Phi(P_K^\alpha) = P_k^\alpha$ , Lemma 3.0.1 implies that the composite

$$i^*\eta \circ P_k^\alpha \circ f : \Sigma_+^\infty X \rightarrow i^*j_*\Sigma^{2m+p_\alpha, m+q_\alpha} H\mathbb{F}_p^K$$

in diagram 3.3 is equal to

$$i^*j_*P_K^\alpha \circ i^*\eta \circ f.$$

Equivalently,

$$\pi_0 \circ i^*j_*P_K^\alpha \circ i^*\eta \circ f = 0 : \Sigma_+^\infty X \rightarrow \Sigma^{2m+p_\alpha-1, m+q_\alpha-1} H\mathbb{F}_p^k.$$

Thus, from diagram 3.3,

$$\Phi(h)(P_k^\alpha(f)) = \pi \circ i^*\eta \circ \Phi(h) \circ P_k^\alpha \circ f = \pi \circ i^*j_*(h) \circ i^*j_*(P_K^\alpha) \circ i^*\eta \circ f = \Phi(h \circ P_K^\alpha)(f)$$

as desired. □

We next prove that the operations  $P_k^n$  commute with base change of the field  $k$  on mod  $p$  Chow groups. For a morphism of fields  $f : \text{Spec}(F_1) \rightarrow \text{Spec}(F_2)$ , the pullback functor  $f^* : SH(F_2) \rightarrow SH(F_1)$  induces a homomorphism  $H\mathbb{F}_p^{F_2, *, *}, H\mathbb{F}_p^{F_2} \rightarrow H\mathbb{F}_p^{F_1, *, *}, H\mathbb{F}_p^{F_1}$ . For  $\text{char}(F_2) \neq p$ ,  $f^*(P_{F_2}^n) = P_{F_1}^n$  since the dual Steenrod algebra has the expected form in this case [16, Theorem 1.1]. However, for our situation where the base field is of characteristic  $p$ , we do not yet know the full structure of the dual Steenrod algebra.

Let  $f_1 : \text{Spec}(k) \rightarrow \text{Spec}(\mathbb{F}_p)$  be the structure map. In the following commuting diagram,

$f_2, f_3, i_0,$  and  $j_0$  are maps compatible with  $f_1$ .

$$\begin{array}{ccc} \text{Spec}(k) & \xrightarrow{f_1} & \text{Spec}(\mathbb{F}_p) \\ \downarrow i & & \downarrow i_0 \\ \text{Spec}(D) & \xrightarrow{f_2} & \text{Spec}(\mathbb{Z}_p) \\ \uparrow j & & \uparrow j_0 \\ \text{Spec}(K) & \xrightarrow{f_3} & \text{Spec}(\mathbb{Q}_p) \end{array}$$

**Proposition 3.0.3.** *Let  $X \in \text{Sm}_k$  and let  $g : \Sigma_+^\infty X \rightarrow \Sigma^{2m,m} H\mathbb{F}_p^k$  be given. Then  $P_k^n(g) = f_1^*(P_{\mathbb{F}_p}^n)(g)$  for all  $n \geq 0$ .*

*Proof.* Let  $\eta_0 : 1 \rightarrow j_{0*}j_0^*$  denote the unit map. Let  $f_2^* \widehat{H}\mathbb{F}_p^{\mathbb{Z}_p} \rightarrow f_2^* j_{0*} H\mathbb{F}_p^{\mathbb{Q}_p}$  be the map  $f_2^* \eta_0$  induced by the isomorphism  $j_0^* \widehat{H}\mathbb{F}_p^{\mathbb{Z}_p} \rightarrow H\mathbb{F}_p^{\mathbb{Q}_p}$ . The exchange transformation  $f_2^* j_{0*} \rightarrow j_* f_3^*$  induces a morphism  $f_2^* j_{0*} H\mathbb{F}_p^{\mathbb{Q}_p} \rightarrow j_* f_3^* H\mathbb{F}_p^{\mathbb{Q}_p}$ . Let  $f_2^* \widehat{H}\mathbb{F}_p^{\mathbb{Z}_p} \rightarrow j_* f_3^* H\mathbb{F}_p^{\mathbb{Q}_p}$  be the map  $\eta f_2^*$  induced by the isomorphism

$$j^* f_2^* \widehat{H}\mathbb{F}_p^{\mathbb{Z}_p} = f_3^* j_0^* \widehat{H}\mathbb{F}_p^{\mathbb{Z}_p} \rightarrow f_3^* H\mathbb{F}_p^{\mathbb{Q}_p}.$$

Putting these maps together, we get the following square which commutes by adjunction.

$$\begin{array}{ccc} f_2^* \widehat{H}\mathbb{F}_p^{\mathbb{Z}_p} & \xrightarrow{f_2^* \eta_0} & f_2^* j_{0*} H\mathbb{F}_p^{\mathbb{Q}_p} \\ \downarrow id. & & \downarrow \\ f_2^* \widehat{H}\mathbb{F}_p^{\mathbb{Z}_p} & \xrightarrow{\eta f_2^*} & j_* f_3^* H\mathbb{F}_p^{\mathbb{Q}_p} \end{array} \quad (3.4)$$

Applying the exchange transformation  $f_2^* j_{0*} \rightarrow j_* f_3^*$  to  $P_{\mathbb{Q}_p}^n$ , we get the following commuting square.

$$\begin{array}{ccc} f_2^* j_{0*} H\mathbb{F}_p^{\mathbb{Q}_p} & \xrightarrow{f_2^* j_{0*} P_{\mathbb{Q}_p}^n} & f_2^* j_{0*} \Sigma^{2n(p-1), n(p-1)} H\mathbb{F}_p^{\mathbb{Q}_p} \\ \downarrow & & \downarrow \\ j_* H\mathbb{F}_p^K & \xrightarrow{j_* P_K^n} & j_* \Sigma^{2n(p-1), n(p-1)} H\mathbb{F}_p^K \end{array}$$

Applying  $i^*$  to these two squares and combining with  $g : \Sigma_+^\infty X \rightarrow \Sigma^{2m,m} H\mathbb{F}_p^k$ , we obtain the following commuting diagram.

$$\begin{array}{ccccccc}
\Sigma_+^\infty X & \xrightarrow{g} & \Sigma^{2m,m} H\mathbb{F}_p^k & \xrightarrow{f_1^* i_0^* \eta_0} & f_1^* i_0^* j_0^* \Sigma^{2m,m} H\mathbb{F}_p^{\mathbb{Q}_p} & \xrightarrow{f_1^* i_0^* j_0^* P_p^n} & f_1^* i_0^* j_0^* \Sigma^{2(m+n(p-1)), m+n(p-1)} H\mathbb{F}_p^{\mathbb{Q}_p} \\
\downarrow id. & & \downarrow id. & & \downarrow & & \downarrow \\
\Sigma_+^\infty X & \xrightarrow{g} & \Sigma^{2m,m} H\mathbb{F}_p^k & \xrightarrow{i^* \eta} & i^* j_* \Sigma^{2m,m} H\mathbb{F}_p^K & \xrightarrow{i^* j_* P_K^n} & i^* j_* \Sigma^{2(m+n(p-1)), m+n(p-1)} H\mathbb{F}_p^K
\end{array} \tag{3.5}$$

Let  $\pi' : i_0^* j_0^* H\mathbb{F}_p^{\mathbb{Q}_p} \rightarrow H\mathbb{F}_p^{\mathbb{F}_p}$  and  $\pi_0 : i_0^* j_0^* H\mathbb{F}_p^{\mathbb{Q}_p} \rightarrow \Sigma^{-1,-1} H\mathbb{F}_p^{\mathbb{F}_p}$  be projection morphisms induced by the isomorphism  $i_0^* j_0^* H\mathbb{F}_p^{\mathbb{Q}_p} \cong H\mathbb{F}_p^{\mathbb{F}_p} \oplus \Sigma^{-1,-1} H\mathbb{F}_p^{\mathbb{F}_p}$  of Theorem 2.0.4. From Theorem 2.0.5, the two composites  $\Sigma_+^\infty X \rightarrow \Sigma^{2(m+n(p-1))-1, m+n(p-1)-1} H\mathbb{F}_p^k$  given by the following diagram are equal to 0.

$$\begin{array}{ccccccc}
& & & & & & \Sigma^{2(m+n(p-1))-1, m+n(p-1)-1} H\mathbb{F}_p^k \\
& & & & & & \uparrow f_1^* \pi_0 \\
\Sigma_+^\infty X & \xrightarrow{g} & \Sigma^{2m,m} H\mathbb{F}_p^k & \xrightarrow{f_1^* i_0^* \eta_0} & f_1^* i_0^* j_0^* \Sigma^{2m,m} H\mathbb{F}_p^{\mathbb{Q}_p} & \xrightarrow{f_1^* i_0^* j_0^* P_p^n} & f_1^* i_0^* j_0^* \Sigma^{2(m+n(p-1)), m+n(p-1)} H\mathbb{F}_p^{\mathbb{Q}_p} \\
\downarrow id. & & \downarrow id. & & \downarrow & & \downarrow \\
\Sigma_+^\infty X & \xrightarrow{g} & \Sigma^{2m,m} H\mathbb{F}_p^k & \xrightarrow{i^* \eta} & i^* j_* \Sigma^{2m,m} H\mathbb{F}_p^K & \xrightarrow{i^* j_* P_K^n} & i^* j_* \Sigma^{2(m+n(p-1)), m+n(p-1)} H\mathbb{F}_p^K \\
& & & & & & \downarrow \pi_0 \\
& & & & & & \Sigma^{2(m+n(p-1))-1, m+n(p-1)-1} H\mathbb{F}_p^k
\end{array} \tag{3.6}$$

Consider the following diagram.

$$\begin{array}{ccccccc}
& & & & & & \Sigma^{2(m+n(p-1)), m+n(p-1)} H\mathbb{F}_p^k \\
& & & & & & \uparrow f_1^* \pi' \\
\Sigma_+^\infty X & \xrightarrow{g} & \Sigma^{2m,m} H\mathbb{F}_p^k & \xrightarrow{f_1^* i_0^* \eta_0} & f_1^* i_0^* j_0^* \Sigma^{2m,m} H\mathbb{F}_p^{\mathbb{Q}_p} & \xrightarrow{f_1^* i_0^* j_0^* P_p^n} & f_1^* i_0^* j_0^* \Sigma^{2(m+n(p-1)), m+n(p-1)} H\mathbb{F}_p^{\mathbb{Q}_p} \\
\downarrow id. & & \downarrow id. & & \downarrow & & \downarrow \\
\Sigma_+^\infty X & \xrightarrow{g} & \Sigma^{2m,m} H\mathbb{F}_p^k & \xrightarrow{i^* \eta} & i^* j_* \Sigma^{2m,m} H\mathbb{F}_p^K & \xrightarrow{i^* j_* P_K^n} & i^* j_* \Sigma^{2(m+n(p-1)), m+n(p-1)} H\mathbb{F}_p^K \\
& & & & & & \downarrow \pi \\
& & & & & & \Sigma^{2(m+n(p-1)), m+n(p-1)} H\mathbb{F}_p^k
\end{array} \tag{3.7}$$

From Theorem 3.0.2, the composite  $\Sigma_+^\infty X \rightarrow \Sigma^{2(m+n(p-1)), m+n(p-1)} H\mathbb{F}_p^k$  given by the upper half of diagram 3.7 is equal to  $f_1^*(P_{\mathbb{F}_p}^n)(g)$  and the composite  $\Sigma_+^\infty X \rightarrow \Sigma^{2(m+n(p-1)), m+n(p-1)} H\mathbb{F}_p^k$

given by the lower half of diagram 3.7 is equal to  $P_k^n(g)$ . As diagram 3.5 commutes and the 2 composite morphisms from diagram 3.6 are 0, we then obtain that  $f_1^*(P_{\mathbb{F}_p}^n)(g) = P_k^n(g)$ .  $\square$

We can now prove that the Steenrod operations  $P_k^n$  commute with base change on mod  $p$  Chow groups. Let  $f : \text{Spec}(k_1) \rightarrow \text{Spec}(k_2)$  be given where  $k_1, k_2$  are fields of characteristic  $p$ . Let  $h : \text{Spec}(k_2) \rightarrow \text{Spec}(\mathbb{F}_p)$  be the structure map.

**Corollary 3.0.4.** *Let  $X \in \text{Sm}_{k_2}$ . Let  $n \geq 0$ . The following square commutes.*

$$\begin{array}{ccc} CH^*(X)/p & \xrightarrow{P_{k_2}^n} & CH^*(X)/p \\ \downarrow f^* & & \downarrow f^* \\ CH^*(X_{k_1})/p & \xrightarrow{P_{k_1}^n} & CH^*(X_{k_1})/p \end{array}$$

*Proof.* From Proposition 3.0.3,  $h^*P_{\mathbb{F}_p}^n$  agrees with  $P_{k_2}^n$  on  $CH^*(X)/p$  and  $f^*h^*P_{\mathbb{F}_p}^n$  agrees with  $P_{k_1}^n$  on  $CH^*(X_{k_1})/p$ . Let  $g : \Sigma_+^\infty X \rightarrow \Sigma^{2m,m} H\mathbb{F}_p^{k_2}$  be given. Then

$$f^*(P_{k_2}^n(g)) = f^*(h^*P_{\mathbb{F}_p}^n(g)) = f^*h^*(P_{\mathbb{F}_p}^n)(f^*g) = P_{k_1}^n(f^*g)$$

as required.  $\square$

**Proposition 3.0.5.** *The morphism  $\beta_k = P_k^{(1,0,\dots)}$  defined above is equal to the Bockstein homomorphism  $\beta$  on mod  $p$  motivic cohomology.*

*Proof.* We let  $\beta$  denote the Bockstein homomorphism on mod  $p$  motivic cohomology over any base scheme. The Bockstein homomorphism  $\beta$  in characteristic 0 is known to be dual to  $\tau_0$ . Hence,  $\beta = P_K^{(1,0,\dots)} = \beta_K$ . Applying the natural transformation  $i^* \rightarrow i^*j_*j^*$  to the diagram

$$\begin{array}{ccccccc} & & & & \beta & & \\ & & & & \curvearrowright & & \\ \widehat{H}\mathbb{Z}^D & \xrightarrow{\cdot p} & \widehat{H}\mathbb{Z}^D & \longrightarrow & (\widehat{H}\mathbb{Z}^D)/p & \longrightarrow & \Sigma^{1,0}\widehat{H}\mathbb{Z}^D \xrightarrow{proj.} \Sigma^{1,0}\widehat{H}\mathbb{Z}^D/p \end{array}$$

in  $SH(D)$ , we get the following commuting diagram in  $SH(k)$ .

$$\begin{array}{ccccccc}
& & & & \beta & & \\
& & & & \curvearrowright & & \\
H\mathbb{Z}^k & \xrightarrow{p} & H\mathbb{Z}^k & \xrightarrow{proj.} & H\mathbb{F}_p^k & \longrightarrow & \Sigma^{1,0}H\mathbb{Z}^k \xrightarrow{proj.} \Sigma^{1,0}H\mathbb{F}_p^k \\
\downarrow i^*\eta & & \downarrow i^*\eta & & \downarrow i^*\eta & & \downarrow i^*\eta \\
i^*j_*H\mathbb{Z}^K & \xrightarrow{p} & i^*j_*H\mathbb{Z}^K & \xrightarrow{proj.} & i^*j_*H\mathbb{F}_p^K & \longrightarrow & \Sigma^{1,0}i^*j_*H\mathbb{Z}^K \xrightarrow{proj.} \Sigma^{1,0}i^*j_*H\mathbb{F}_p^K \\
& & & & \downarrow i^*\eta & & \downarrow i^*\eta \\
& & & & & & \Sigma^{1,0}H\mathbb{F}_p^k \\
& & & & \curvearrowleft i^*j_*\beta_K & & \downarrow \pi
\end{array} \quad (3.8)$$

From Theorem 3.0.2,  $\Phi(\beta_K) = \beta_k$ . The composite in diagram 3.8 that starts at  $H\mathbb{F}_p^k$  in the top row and goes immediately down to  $\Sigma^{1,0}H\mathbb{F}_p^k$  is equal to  $\Phi(\beta_K)$ . As the diagram commutes and  $\pi \circ i^*\eta = id.$ , it follows that  $\Phi(\beta_K) = \beta = \beta_k$ .  $\square$

## Chapter 4

# Adem relations

In this chapter, we use the map  $\Phi : H\mathbb{F}_p^{K^*,*} H\mathbb{F}_p^K \rightarrow H\mathbb{F}_p^{k^*,*} H\mathbb{F}_p^k$  3.1 and Theorem 3.0.2 to show that the operations  $P_k^n$  for  $n \geq 0$  satisfy the expected Adem relations when restricted to mod  $p$  Chow groups. The proof uses the corresponding Adem relations in characteristic 0 which can be found in [24, Théorème 4.5.1] for  $p = 2$  and [24, Théorème 4.5.2 ] for odd  $p$ . First, we state the Adem relations for  $p = 2$  over the base  $K$  of characteristic 0. Let  $\tau \in H^{0,1}(K, \mathbb{F}_2)$  denote the class of  $-1 \in \mu_2(K)$  and let  $\rho \in H^{1,1}(K, \mathbb{F}_2)$  denote the class of  $-1 \in K^*/K^{*2}$ . Set  $\text{Sq}_k^{2n} := P_k^n$  and  $\text{Sq}_k^{2n+1} = \beta_k \text{Sq}_k^{2n}$  for  $n \geq 0$ .

**Theorem 4.0.1.** *Let  $a, b \in \mathbb{N}$  with  $a < 2b$ .*

1.

$$\text{Sq}_K^a \text{Sq}_K^b = \sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \text{Sq}_K^{a+b-j} \text{Sq}_K^j + \sum_{\substack{j=1 \\ j \text{ odd}}}^{\lfloor \frac{a}{2} \rfloor} \rho \binom{b-1-j}{a-2j} \text{Sq}_K^{a+b-j-1} \text{Sq}_K^j$$

*if  $a$  is even and  $b$  is odd.*

2.

$$\text{Sq}_K^a \text{Sq}_K^b = \sum_{\substack{j=0 \\ j \text{ odd}}}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \text{Sq}_K^{a+b-j} \text{Sq}_K^j$$

*if  $a$  and  $b$  are odd.*

3.

$$\mathrm{Sq}_K^a \mathrm{Sq}_K^b = \sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \tau^{j \bmod 2} \binom{b-1-j}{a-2j} \mathrm{Sq}_K^{a+b-j} \mathrm{Sq}_K^j$$

if  $a$  and  $b$  are even.

4.

$$\mathrm{Sq}_K^a \mathrm{Sq}_K^b = \sum_{\substack{j=0 \\ j \text{ even}}}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \mathrm{Sq}_K^{a+b-j} \mathrm{Sq}_K^j + \sum_{\substack{j=1 \\ j \text{ odd}}}^{\lfloor \frac{a}{2} \rfloor} \rho \binom{b-1-j}{a-1-2j} \mathrm{Sq}_K^{a+b-j-1} \mathrm{Sq}_K^j$$

if  $a$  is odd and  $b$  is even.

Next, we state the characteristic 0 Adem relations for  $p$  odd.

**Theorem 4.0.2.** 1. Let  $a, b \in \mathbb{N}$  with  $a < pb$ . Then

$$P_K^a P_K^b = \sum_{j=0}^{\lfloor \frac{a}{p} \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P_K^{a+b-j} P_K^j.$$

2. Let  $a, b \in \mathbb{N}$  with  $a \leq pb$ . Then

$$P_K^a \beta_K P_K^b = \sum_{j=0}^{\lfloor \frac{a}{p} \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} \beta_K P_K^{a+b-j} P_K^j +$$

$$\sum_{j=0}^{\lfloor \frac{a-1}{p} \rfloor} (-1)^{a+j+1} \binom{(p-1)(b-j)-1}{a-pj-1} P_K^{a+b-j} \beta_K P_K^j.$$

We can now prove the Adem relations for the operations  $P_k^n$  restricted to mod  $p$  Chow groups.

**Theorem 4.0.3.** Let  $X \in \mathrm{Sm}_k$  and let  $x \in H^{2m,m}(X, \mathbb{F}_p) = CH^m(X)/p$  for some  $m \geq 0$ . Let  $a, b \in \mathbb{N}$  such that  $a < pb$ . Then

$$P_k^a(P_k^b(x)) = \sum_{j=0}^{\lfloor \frac{a}{p} \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P_k^{a+b-j}(P_k^j(x)).$$

*Proof.* From Theorem 3.0.2,  $P_k^a(P_k^b(x)) = \Phi(P_K^a P_K^b)(x)$ . We then use the Adem relations in characteristic 0 to rewrite  $P_K^a P_K^b \in H\mathbb{F}_p^{K,*} H\mathbb{F}_p^K$ . Note that the Bockstein  $\beta_k$  is the 0

homomorphism on mod  $p$  Chow groups. If  $p = 2$ ,  $\Phi(\mathrm{Sq}_K^n)(x) = \mathrm{Sq}_K^n(x) = 0$  whenever  $n$  is odd.

Thus, applying Theorem 3.0.2, we get

$$\begin{aligned}
 P_k^a(P_k^b(x)) &= \Phi(P_K^a P_K^b)(x) = \Phi\left(\sum_{j=0}^{\lfloor \frac{a}{p} \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P_K^{a+b-j} P_K^j\right)(x) \\
 &= \sum_{j=0}^{\lfloor \frac{a}{p} \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P_k^{a+b-j}(P_k^j(x)).
 \end{aligned}$$

□

## Chapter 5

# Coaction map for smooth $X$

In this chapter, for  $X \in \text{Sm}_k$ , we describe a coaction map

$$\lambda_X : H^{*,*}(X, \mathbb{F}_p) \rightarrow \pi_{-*, -*} \left( \bigoplus_{\alpha} \Sigma^{p\alpha, q\alpha} H\mathbb{F}_p^k \right) \otimes_{\pi_{-*, -*} H\mathbb{F}_p^k} H^{*,*}(X, \mathbb{F}_p)$$

such that the actions of the cohomology operations  $P_k^n$  defined in Chapter 3 on  $H^{*,*}(X, \mathbb{F}_p)$  are determined by  $\lambda_X$ . We show that  $\lambda_X$  is a ring homomorphism when restricted to mod  $p$  Chow groups. This will allow us to prove the Cartan formula in the next chapter.

There is a multiplication morphism

$$m : \left( \bigoplus_{\alpha} \Sigma^{p\alpha, q\alpha} H\mathbb{F}_p^k \right) \wedge \left( \bigoplus_{\alpha} \Sigma^{p\alpha, q\alpha} H\mathbb{F}_p^k \right) \rightarrow \bigoplus_{\alpha} \Sigma^{p\alpha, q\alpha} H\mathbb{F}_p^k \quad (5.1)$$

defined as  $m = r \circ \mu_2^k \circ (\Psi_k \wedge \Psi_k)$ . The morphism  $m$  defines multiplication on

$$\left( \bigoplus_{\alpha} \Sigma^{p\alpha, q\alpha} H\mathbb{F}_p^k \right)^{*,*} (\Sigma_+^{\infty} X)$$

and

$$\pi_{*,*} \left( \bigoplus_{\alpha} \Sigma^{p\alpha, q\alpha} H\mathbb{F}_p^k \right).$$

For sequences  $\alpha_1, \alpha_2$ , Proposition 2.0.2 allows us to calculate the product

$$r_*(\omega(\alpha_1))r_*(\omega(\alpha_2)) \in \pi_{*,*} \left( \bigoplus_{\alpha} \Sigma^{p\alpha, q\alpha} H\mathbb{F}_p^k \right)$$

in terms of another sequence  $\alpha_1 + \alpha_2$  by using the relations  $\tau_i^2 = 0$  for  $i \geq 0$ .

**Proposition 5.0.1.** *The natural ring homomorphism*

$$\pi_{-*,-*} \left( \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k \right) \otimes_{\pi_{-*,-*} H\mathbb{F}_p^k} H^{*,*}(X, \mathbb{F}_p) \rightarrow \left( \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k \right)^{*,*} (\Sigma_+^{\infty} X)$$

is an isomorphism.

*Proof.* The suspension spectrum  $\Sigma_+^{\infty} X \in SH(k)$  is compact. Hence,

$$\mathrm{Hom}_{SH(k)}(\Sigma^{s,t} \Sigma_+^{\infty} X, \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k) \cong \bigoplus_{\alpha} \mathrm{Hom}_{SH(k)}(\Sigma^{s,t} \Sigma_+^{\infty} X, \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k)$$

for all  $s, t \in \mathbb{Z}$ . □

**Definition 5.0.1.** *Using the isomorphism*

$$\left( \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k \right)^{*,*} (\Sigma_+^{\infty} X) \cong \pi_{-*,-*} \left( \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k \right) \otimes_{\pi_{-*,-*} H\mathbb{F}_p^k} H^{*,*}(X, \mathbb{F}_p)$$

from Proposition 5.0.1, define an additive homomorphism of graded abelian groups

$$\lambda_X : H^{*,*}(X, \mathbb{F}_p) \rightarrow \pi_{-*,-*} \left( \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k \right) \otimes_{\pi_{-*,-*} H\mathbb{F}_p^k} H^{*,*}(X, \mathbb{F}_p)$$

by the composite

$$\begin{array}{ccc} H\mathbb{F}_p^{k*,*}(\Sigma_+^{\infty} X) & \xrightarrow{i_{R*}} & (H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k)^{*,*}(\Sigma_+^{\infty} X) \\ & & \downarrow r_* \\ & & \pi_{-*,-*} \left( \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k \right) \otimes_{\pi_{-*,-*} H\mathbb{F}_p^k} H^{*,*}(X, \mathbb{F}_p). \end{array} \quad (5.2)$$

**Proposition 5.0.2.** *Restricted to mod  $p$  Chow groups,  $\lambda_X$  preserves multiplication.*

*Proof.* Let  $f : \Sigma_+^{\infty} X \rightarrow \Sigma^{2m,m} H\mathbb{F}_p^k$  and  $g : \Sigma_+^{\infty} X \rightarrow \Sigma^{2n,n} H\mathbb{F}_p^k$  be given. We need to show that  $\lambda_X(fg) = \lambda_X(f)\lambda_X(g)$ . The right unit map  $i_R$  is a morphism of commutative ring spectra. Hence,  $i_{R*}$  is a homomorphism of rings. Hence, we need to prove that  $r_*(i_{R*}(f)i_{R*}(g)) = r_*(i_{R*}(f))r_*(i_{R*}(g))$ .

Applying the natural transformation  $i^* \rightarrow i^* j_* j^*$  to  $\mu_2^D$ , we get a commuting diagram.

$$\begin{array}{ccc} (H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) \wedge (H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) & \xrightarrow{\mu_2^k} & H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \\ \downarrow i^* \eta & & \downarrow i^* \eta \\ i^* j_* ((H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K)) & \xrightarrow{i^* j_* \mu_2^K} & i^* j_* (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \\ & & \downarrow \oplus \pi \\ & & \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k \end{array} \quad (5.3)$$

We will factor the left vertical morphism in this diagram. Consider the following triangle

$$\begin{array}{ccc}
(\widehat{H}\mathbb{F}_p^D \wedge \widehat{H}\mathbb{F}_p^D) \wedge (\widehat{H}\mathbb{F}_p^D \wedge \widehat{H}\mathbb{F}_p^D) & \xrightarrow{\eta \wedge \eta} & j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \\
\downarrow \eta & \swarrow & \\
j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) & & 
\end{array} \tag{5.4}$$

where the morphism on the hypotenuse is defined by the lax monoidal property of  $j_*$ . Note that the counit morphism  $\epsilon : j^*j_* \rightarrow id.$  is an isomorphism since  $j$  is open. By adjunction, the morphism on the hypotenuse of diagram 5.4 is induced by the isomorphism

$$\epsilon \wedge \epsilon : j^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge j^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \rightarrow (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K).$$

The morphism  $\eta$  on the left leg of the triangle 5.4 is induced by the isomorphism

$$j^*\eta : j^*((\widehat{H}\mathbb{F}_p^D \wedge \widehat{H}\mathbb{F}_p^D) \wedge (\widehat{H}\mathbb{F}_p^D \wedge \widehat{H}\mathbb{F}_p^D)) \rightarrow (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K).$$

Using that pullback is strongly monoidal, we then have the following commuting triangle.

$$\begin{array}{ccc}
j^*(\widehat{H}\mathbb{F}_p^D \wedge \widehat{H}\mathbb{F}_p^D) \wedge j^*(\widehat{H}\mathbb{F}_p^D \wedge \widehat{H}\mathbb{F}_p^D) & \xrightarrow{j^*\eta \wedge j^*\eta} & j^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge j^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \\
\downarrow j^*\eta & \swarrow \epsilon \wedge \epsilon & \\
(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) & & 
\end{array}$$

Thus, by adjunction, the triangle 5.4 commutes.

Applying  $i^*$  to triangle 5.4, we then see that the commuting diagram 5.3 is a sub-diagram of the commuting diagram

$$\begin{array}{ccc}
(H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) \wedge (H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) & \xrightarrow{\mu_2^k} & H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \\
\downarrow i^*\eta \wedge i^*\eta & & \downarrow i^*\eta \\
i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) & & i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \\
\downarrow & \xrightarrow{i^*j_*\mu_2^K} & \downarrow \oplus \pi \\
i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) & & \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k.
\end{array} \tag{5.5}$$

From diagram 2.3,

$$(i^*\eta \wedge i^*\eta) \circ (\Psi_k \wedge \Psi_k) : \left( \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k \right) \wedge \left( \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k \right) \rightarrow i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K)$$

is equal to the composite  $(i^*j_*\Psi_K \wedge i^*j_*\Psi_K) \circ (i^*\eta \wedge i^*\eta)$ . Hence, diagram 5.5 implies that the multiplication morphism  $m = r \circ \mu_2^k \circ (\Psi_k \wedge \Psi_k)$  on

$$\left( \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k \right) \wedge \left( \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k \right)$$

is equal to the following composite.

$$\begin{aligned} & \left( \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k \right) \wedge \left( \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k \right) \\ & \quad \downarrow ((i^*j_*\Psi_K) \circ i^*\eta) \wedge ((i^*j_*\Psi_K) \circ i^*\eta) \\ & i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \\ & \quad \downarrow \\ & i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \xrightarrow{i^*j_*\mu_2^K} i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \\ & \quad \quad \quad \downarrow \oplus \pi \\ & \quad \quad \quad \bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} H\mathbb{F}_p^k \end{aligned} \tag{5.6}$$

From Lemma 3.0.1, the composites

$$\begin{aligned} \Sigma_+^{\infty} X & \xrightarrow{f} \Sigma^{2m, m} H\mathbb{F}_p^k \xrightarrow{i_R} \Sigma^{2m, m} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \xrightarrow{i^*\eta} \Sigma^{2m, m} i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \\ & \quad \quad \quad \downarrow i^*j_*\Psi_K^{-1} \\ & \quad \quad \quad \bigoplus_{\alpha} \Sigma^{p_{\alpha} + 2m, q_{\alpha} + m} i^*j_* H\mathbb{F}_p^K \\ & \quad \quad \quad \downarrow \oplus \pi_0 \\ & \quad \quad \quad \bigoplus_{\alpha} \Sigma^{p_{\alpha} + 2m - 1, q_{\alpha} + m - 1} H\mathbb{F}_p^k \end{aligned}$$

and

$$\begin{aligned} \Sigma_+^{\infty} X & \xrightarrow{g} \Sigma^{2n, n} H\mathbb{F}_p^k \xrightarrow{i_R} \Sigma^{2n, n} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \xrightarrow{i^*\eta} \Sigma^{2n, n} i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \\ & \quad \quad \quad \downarrow i^*j_*\Psi_K^{-1} \\ & \quad \quad \quad \bigoplus_{\alpha} \Sigma^{p_{\alpha} + 2n, q_{\alpha} + n} i^*j_* H\mathbb{F}_p^K \\ & \quad \quad \quad \downarrow \oplus \pi_0 \\ & \quad \quad \quad \bigoplus_{\alpha} \Sigma^{p_{\alpha} + 2n - 1, q_{\alpha} + n - 1} H\mathbb{F}_p^k \end{aligned}$$

are equal to 0. It follows that  $i^*\eta \circ r \circ i_R \circ f = i^*\eta \circ i_R \circ f$  and  $i^*\eta \circ r \circ i_R \circ g = i^*\eta \circ i_R \circ g$

in the following two diagrams.

$$\begin{array}{ccc}
& \Sigma_+^\infty X & \\
& \downarrow f & \\
& \Sigma^{2m,m} H\mathbb{F}_p^k & \\
& \downarrow i_R & \\
& \Sigma^{2m,m} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k & \\
& \downarrow i^*\eta & \\
\bigoplus_{\alpha} \Sigma^{p_{\alpha}+2m, q_{\alpha}+m} H\mathbb{F}_p^k & \xrightarrow{i^*\eta} & \Sigma^{2m,m} i^* j_* (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \\
& \swarrow r & \downarrow i^*\eta \\
& & \Sigma^{2m,m} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k
\end{array} \tag{5.7}$$

$$\begin{array}{ccc}
& \Sigma_+^\infty X & \\
& \downarrow g & \\
& \Sigma^{2n,n} H\mathbb{F}_p^k & \\
& \downarrow i_R & \\
& \Sigma^{2n,n} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k & \\
& \downarrow i^*\eta & \\
\bigoplus_{\alpha} \Sigma^{p_{\alpha}+2n, q_{\alpha}+n} H\mathbb{F}_p^k & \xrightarrow{i^*\eta} & \Sigma^{2n,n} i^* j_* (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \\
& \swarrow r & \downarrow i^*\eta \\
& & \Sigma^{2n,n} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k
\end{array} \tag{5.8}$$

To show that  $r_*(i_{R^*}(f)i_{R^*}(g)) = r_*(i_{R^*}(f))r_*(i_{R^*}(g))$ , we consider the following commuting diagram where  $\Delta$  is the diagonal morphism.

$$\begin{array}{ccc}
\Sigma_+^\infty X & & \\
\downarrow \Delta & & \\
\Sigma_+^\infty X \wedge \Sigma_+^\infty X & & \\
\downarrow f \wedge g & & \\
\Sigma^{2m,m} H\mathbb{F}_p^k \wedge \Sigma^{2n,n} H\mathbb{F}_p^k & & \\
\downarrow i_R \wedge i_R & & \\
(\Sigma^{2m,m} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) \wedge (\Sigma^{2n,n} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) & \xrightarrow{\mu_2^k} & \Sigma^{2m,m} H\mathbb{F}_p^k \wedge \Sigma^{2n,n} H\mathbb{F}_p^k \\
\downarrow i^* \eta \wedge i^* \eta & & \downarrow i^* \eta \\
i^* j_*(\Sigma^{2m,m} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge i^* j_*(\Sigma^{2n,n} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) & & \\
\downarrow & & \downarrow \oplus \pi \\
i^* j_*(\Sigma^{2(m+n),m+n} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) & \xrightarrow{i^* j_* \mu_2^K} & i^* j_*(\Sigma^{2(m+n),m+n} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \\
& & \downarrow \oplus \pi \\
& & \bigoplus_{\alpha} \Sigma^{p_\alpha + 2m + 2n, q_\alpha + m + n} H\mathbb{F}_p^k
\end{array} \tag{5.9}$$

The composite  $\oplus \pi \circ i^* \eta \circ \mu_2^k \circ (i_R \wedge i_R) \circ (f \wedge g) \circ \Delta$  in this diagram is equal to  $r_*(i_{R*}(f)i_{R*}(g))$ .

From diagrams 5.7 and 5.8, the composite given by

$$\begin{array}{c}
\Sigma_+^\infty X \\
\downarrow \Delta \\
\Sigma_+^\infty X \wedge \Sigma_+^\infty X \\
\downarrow f \wedge g \\
\Sigma^{2m,m} H\mathbb{F}_p^k \wedge \Sigma^{2n,n} H\mathbb{F}_p^k \\
\downarrow i_R \wedge i_R \\
(\Sigma^{2m,m} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) \wedge (\Sigma^{2n,n} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) \\
\downarrow r \wedge r \\
\left( \bigoplus_\alpha \Sigma^{p_\alpha+2m, q_\alpha+m} H\mathbb{F}_p^k \right) \wedge \left( \bigoplus_\alpha \Sigma^{p_\alpha+2n, q_\alpha+n} H\mathbb{F}_p^k \right) \\
\downarrow i^* \eta \wedge i^* \eta \\
i^* j_* (\Sigma^{2m,m} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge i^* j_* (\Sigma^{2n,n} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \\
\downarrow \\
i^* j_* (\Sigma^{2(m+n), m+n} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \xrightarrow{i^* j_* \mu_2^K} i^* j_* (\Sigma^{2(m+n), m+n} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \\
\downarrow \oplus \pi \\
\bigoplus_\alpha \Sigma^{p_\alpha+2m+2n, q_\alpha+m+n} H\mathbb{F}_p^k
\end{array} \tag{5.10}$$

is equal to the composite given by diagram 5.9. From diagram 5.6, the composite given by diagram 5.10 is equal to  $\Sigma^{2(m+n), m+n} m \circ (r \wedge r) \circ (i_R \wedge i_R) \circ (f \wedge g) \circ \Delta = r_*(i_{R^*}(f))r_*(i_{R^*}(g))$ .

Thus,  $r_*(i_{R^*}(f))i_{R^*}(g) = r_*(i_{R^*}(f))r_*(i_{R^*}(g))$  as desired.

□

## Chapter 6

# Cartan formula

In this chapter, we use the coaction map constructed in the previous chapter to prove a Cartan formula for the operations  $P_k^n$  restricted to mod  $p$  Chow groups. Let  $X \in \text{Sm}_k$ . Let  $\langle \cdot, \cdot \rangle$  denote the pairing between  $\mathcal{A}_{*,*}^k$  and  $H\mathbb{F}_p^{k*,*}H\mathbb{F}_p^k$ . Let  $n \geq 0$ . For  $x \in H^{*,*}(X, \mathbb{F}_p)$  with  $\lambda_X(x) = \sum y_i \otimes x_i$ , we have  $P_k^n(x) = \sum \langle y_i, P_k^n \rangle x_i$ .

**Proposition 6.0.1.** *Let  $x, y \in CH^*(X)/p$  and  $i \geq 0$ . Then*

$$P_k^i(xy) = \sum_{j=0}^i P_k^j(x)P_k^{i-j}(y).$$

*Proof.* From the definition of  $P_k^i$ ,  $\langle \xi_1^i, P_k^i \rangle = 1$  and  $\langle \omega(\alpha), P_k^i \rangle = 0$  for all sequences  $\alpha \neq (0, i, 0, 0, \dots)$ . Using the coaction map 5.2, we write

$$\lambda_X(x) = \sum_q \omega(\alpha_q^1) \otimes x_q$$

and

$$\lambda_X(y) = \sum_r \omega(\alpha_r^2) \otimes y_r$$

for some sequences  $\alpha_q^1, \alpha_r^2$ . Then

$$\lambda_X(xy) = \sum_{q,r} ((\omega(\alpha_q^1)\omega(\alpha_r^2) \otimes x_q y_r).$$

For any 2 sequences  $\alpha_q^1, \alpha_r^2$  appearing in these sums, we have  $\omega(\alpha_q^1)\omega(\alpha_r^2) = 0$  if the relation  $\tau_m^2 = 0$  from Proposition 2.0.2 applies for some  $m \geq 0$ , or else  $\omega(\alpha_q^1)\omega(\alpha_r^2) = \pm\omega(\alpha_q^1 + \alpha_r^2)$ .

From the definition of  $\lambda_X$ ,

$$P_k^i(xy) = \sum_{q,r} \langle (\omega(\alpha_q^1)\omega(\alpha_r^2), P_k^i) x_q y_r.$$

Proposition 2.0.2 implies that if  $\omega(\alpha_1)\omega(\alpha_2) = a\xi_1^i$  for two sequences  $\alpha_1, \alpha_2$  and  $a \neq 0 \in H^{*,*}(k, \mathbb{F}_p)$ , then  $a = 1$  and  $\omega(\alpha_1) = \xi_1^j$ ,  $\omega(\alpha_2) = \xi_1^{i-j}$  for some  $0 \leq j \leq i$ . As  $P_k^i$  is dual to  $\xi_1^i$ , the only terms for which  $\langle \omega(\alpha_q^1 + \alpha_r^2), P_k^i \rangle \neq 0$  are of the form  $\omega(\alpha_{q_j}^1) = \xi_1^j$ ,  $\omega(\alpha_{r_j}^2) = \xi_1^{i-j}$  for  $0 \leq j \leq i$ . Hence,

$$P_k^i(xy) = \sum_{j=0}^i \langle \omega(\alpha_{q_j}^1 + \alpha_{r_j}^2), P_k^i \rangle x_{q_j} y_{r_j} = \sum_{j=0}^i P_k^j(x) P_k^{i-j}(y) \quad (6.1)$$

as required. □

# Chapter 7

## $p$ th power and instability

In this chapter, for  $n \in \mathbb{N}$ , we prove that  $P_k^n$  is the  $p$ th power on  $CH^n(-)/p$ . Letting  $f : \mathrm{Spec}(k) \rightarrow \mathrm{Spec}(\mathbb{F}_p)$  denote the structure map, it suffices to prove that  $f^*(P_{\mathbb{F}_p}^n)(\iota_n) = \iota_n^p$  for the canonical element  $\iota_n \in H^{2n,n}(K_{n,k}, \mathbb{F}_p)$  where  $K_{n,k} \in H(k)$  is the motivic Eilenberg-MacLane space representing  $H^{2n,n}(-, \mathbb{F}_p)$ . Our proof makes use of Morel's  $S^1$ -recognition principle.

We refer to [6, Section 3] as a reference for the  $S^1$ -recognition principle. For a base scheme  $S$ , let  $\mathrm{PSh}_{\mathrm{nis}}(\mathrm{Sm}_S)$  denote the category of Nisnevich local presheaves of spaces on  $\mathrm{Sm}_S$ . The unstable motivic homotopy category  $H(S)$  can be described as the full subcategory of  $\mathrm{PSh}_{\mathrm{nis}}(\mathrm{Sm}_S)$  of presheaves that are  $\mathbb{A}^1$ -invariant. Let  $L_{\mathrm{mot}} : \mathrm{PSh}_{\mathrm{nis}}(\mathrm{Sm}_S) \rightarrow H(S)$  denote the  $\mathbb{A}^1$ -localization functor. Let  $SH^{S^1}(S)$  denote the stable motivic homotopy category of  $S^1$ -spectra. For a morphism  $f : S_1 \rightarrow S_2$  of base schemes, we have the adjoint functors of pullback  $f^* := Lf^*$  and pushforward  $f_* := Rf_*$ :

$$f^* : H(S_2) \rightleftarrows H(S_1) : f_*$$

For  $f : S_1 \rightarrow S_2$  smooth,  $f^*$  admits a left adjoint  $f_{\#}$  such that  $f_{\#}(X) = X \in H(S_2)$  for any  $X \in \mathrm{Sm}_{S_1}$ .

For  $C = \mathrm{PSh}_{\mathrm{nis}}(\mathrm{Sm}_S)$  or  $H(S)$ , we consider the  $n$ -fold bar constructions  $B_C^n$  that are adjoint to the  $n$ th  $S^1$ -deloopings  $\Omega^n$ :

$$B_C^n : \mathrm{Mon}_{\mathcal{E}_n}(C) \rightleftarrows C : \Omega^n.$$

For  $C = \text{PSh}_{\text{nis}}(\text{Sm}_S)$  or  $H(S)$ , we let  $\text{Stab}(C) := C \otimes \text{Spt}$  denote the  $S^1$ -stabilization of  $C$ . We also consider the infinite bar construction

$$B_C^\infty : \text{CMon}(C) = \text{Mon}_{\mathcal{E}_\infty}(C) \rightleftharpoons \text{Stab}(C) : \Omega^\infty.$$

For  $C = \text{PSh}_{\text{nis}}(\text{Sm}_S)$ , we denote  $B_C^n$  by  $B_{\text{nis}}^n$  and we denote  $B_C^\infty$  by  $B_{\text{nis}}^\infty$ . Similarly, for  $C = H(S)$ , we denote  $B_C^n$  by  $B_{\text{mot}}^n$  and we denote  $B_C^\infty$  by  $B_{\text{mot}}^\infty$ . For later use, we note that  $B_{\text{nis}}^n$  and  $B_{\text{nis}}^\infty$  commute with pullbacks.

**Definition 7.0.1.** Define  $X \in \text{Mon}(H(S))$  to be strongly  $\mathbb{A}^1$ -invariant if  $B_{\text{nis}}X \simeq B_{\text{mot}}X$ . Define  $X \in \text{CMon}(H(S))$  to be strictly  $\mathbb{A}^1$ -invariant if  $B_{\text{nis}}^nX \simeq B_{\text{mot}}^nX$  for all  $n \geq 0$ .

Most of the proof of the following proposition was suggested to us by Marc Hoyois.

**Proposition 7.0.1.** Let  $k$  be a perfect field of characteristic  $p$  and let  $i : \text{Spec}(k) \rightarrow \text{Spec}(D)$  be a closed embedding where  $D$  is a DVR with generic point  $j : \text{Spec}(K) \rightarrow \text{Spec}(D)$ . Fix  $n > 0$ . We let  $K_{n,D} := \Omega_{\mathbb{F}_p}^\infty \Sigma^{2n,n} \widehat{H}\mathbb{F}_p^D$ . Then the morphism  $i^*K_{n,D} \rightarrow K_{n,k}$  induced by  $i^* \Sigma^{2n,n} \widehat{H}\mathbb{F}_p^D \cong \Sigma^{2n,n} H\mathbb{F}_p^k$  is an isomorphism in  $H(k)$ .

*Proof.* We first prove that  $K_{n,D}$  is connected. Let  $R$  be a Henselian local ring that is essentially smooth over  $D$ . From [10, Corollary 4.2], the Bloch-Levine Chow groups  $CH^m(R)$  of  $R$  vanish for  $m \geq 1$ . Thus,  $\pi_0^{\text{nis}}(K_{n,D}(\text{Spec}(R))) \simeq *$  since  $K_{n,D} \in H(D)$  represents the codimension  $n$  mod  $p$  Bloch-Levine Chow group.

Now we prove that  $i^*K_{n,D}$  is connected. As  $j : \text{Spec}(K) \rightarrow \text{Spec}(D)$  is smooth,  $j^*K_{n,D} \simeq K_{n,K}$ . Consider the homotopy pushout  $P$  in  $\text{PSh}_{\text{nis}}(\text{Sm}_D)$  of the following diagram.

$$\begin{array}{ccc} j\#K_{n,K} & \longrightarrow & K_{n,D} \\ \downarrow & & \\ \text{Spec}(K) & & \end{array}$$

The morphism  $j\#K_{n,K} \rightarrow \text{Spec}(K)$  induces a bijection on  $\pi_0^{\text{nis}}$ . Hence,  $\pi_0^{\text{nis}}(K_{n,D}) \simeq \pi_0^{\text{nis}}(P)$ . From the gluing square [22, Theorem 2.21],  $L_{\text{mot}}(P) \simeq i_*i^*(K_{n,D})$ . From [22, Corollary 3.22], it follows that  $i_*i^*(K_{n,D})$  is connected since  $K_{n,D}$  is connected. Let  $k \rightarrow S_k$  be an essentially

smooth homomorphism of rings where  $S_k$  is Henselian local. The ring  $S_k$  admits a lift  $S_D$  where  $D \rightarrow S_D$  is essentially smooth and  $S_D$  is Henselian local. Hence,  $i^*(K_{n,D})(S_k) \simeq i_*i^*(K_{n,D})(S_D)$  is connected. Thus,  $i^*K_{n,D} \in H(k)$  is connected. In particular,  $\pi_0^{\text{nis}}(i^*(K_{n,D}))$  is strongly  $\mathbb{A}^1$ -invariant. The  $S^1$ -recognition principle [6, Theorem 3.1.12] then implies that  $i^*K_{n,D}$  is strictly  $\mathbb{A}^1$ -invariant.

From [27, Theorem 8.18], we have

$$\mathrm{B}_{\text{mot}}^\infty i^*(K_{n,D}) \cong i^*(\mathrm{B}_{\text{mot}}^\infty K_{n,D}) \cong i^*(\Omega_{\mathbb{G}_m}^\infty \Sigma^{2n,n} \widehat{H}\mathbb{F}_p^D) \cong \Omega_{\mathbb{G}_m}^\infty \Sigma^{2n,n} H\mathbb{F}_p^k \cong \mathrm{B}_{\text{mot}}^\infty K_{n,k}$$

in  $SH^{S^1}(k)$ . Then [6, Corollary 3.1.15] implies that  $i^*K_{n,D} \cong K_{n,k}$  in  $H(k)$ .  $\square$

**Proposition 7.0.2.** *Let  $k$  be a field of characteristic  $p$  with structure map  $f : \mathrm{Spec}(k) \rightarrow \mathrm{Spec}(\mathbb{F}_p)$  and let  $\iota_n \in H^{2n,n}(K_{n,k}, \mathbb{F}_p)$  be the canonical element. Then  $f^*P_{\mathbb{F}_p}^n(\iota_n) = \iota_n^p$ .*

*Proof.* First, we assume that  $k$  is perfect. Let  $D$  be a DVR having  $k$  as a residue field with inclusion morphism  $i : \mathrm{Spec}(k) \rightarrow \mathrm{Spec}(D)$  and generic point  $j : \mathrm{Spec}(K) \rightarrow \mathrm{Spec}(D)$ . From Proposition 7.0.1,  $i^*K_{n,D} \cong K_{n,k}$ . Over all base schemes  $S$ , we let  $\iota_n$  denote the canonical element in  $H^{2n,n}(K_{n,S}, \mathbb{F}_p)$ . Apply  $i^* \rightarrow i^*j_*j^*$  to the natural morphism  $\iota_n : \Sigma_+^\infty K_{n,D} \rightarrow \Sigma^{2n,n} \widehat{H}\mathbb{F}_p^D$  to get the following commuting square.

$$\begin{array}{ccc} \Sigma_+^\infty K_{n,k} & \xrightarrow{\iota_n} & \Sigma^{2n,n} H\mathbb{F}_p^k \\ \downarrow i^*\eta & & \downarrow i^*\eta \\ i^*j_*\Sigma_+^\infty K_{n,K} & \xrightarrow{i^*j_*\iota_n} & \Sigma^{2n,n} i^*j_*H\mathbb{F}_p^K \end{array}$$

Apply  $i^*\eta : i^* \rightarrow i^*j_*j^*$  to the morphism  $\Sigma_+^\infty K_{n,D} \rightarrow \Sigma^{2pn,pn} \widehat{H}\mathbb{F}_p^D$  in  $SH(D)$  corresponding to  $\iota_n^p$  to get the commutative diagram

$$\begin{array}{ccc} \Sigma_+^\infty K_{n,k} & \xrightarrow{\iota_n^p} & \Sigma^{2pn,pn} H\mathbb{F}_p^k \\ \downarrow i^*\eta & & \downarrow i^*\eta \\ i^*j_*\Sigma_+^\infty K_{n,K} & \xrightarrow{i^*j_*\iota_n^p} & i^*j_*\Sigma^{2pn,pn} H\mathbb{F}_p^K \xrightarrow{\pi} \Sigma^{2pn,pn} H\mathbb{F}_p^k. \end{array} \quad (7.1)$$

From [30, Lemma 9.8],  $i^*j_*\iota_n^p = i^*j_*P_K^n(\iota_n)$ . Hence, we can rewrite the bottom row of 7.1 as

$$i^*j_*\Sigma_+^\infty K_{n,K} \xrightarrow{i^*j_*\iota_n^p} i^*j_*\Sigma^{2n,n} H\mathbb{F}_p^K \xrightarrow{i^*j_*P_K^n} i^*j_*\Sigma^{2pn,pn} H\mathbb{F}_p^K \xrightarrow{\pi} \Sigma^{2pn,pn} H\mathbb{F}_p^k.$$

From Theorem 3.0.2 and the above commuting diagrams,  $P_k^n(\iota_n) = \pi \circ (i^* j_* P_K^n) \circ (i^* j_* \iota_n) \circ i^* \eta$ .

Hence, from diagram 7.1, we get  $P_k^n(\iota_n) = \pi \circ (i^* j_* \iota_n^p) \circ i^* \eta = \iota_n^p$ .

For  $k$  not perfect, we have an essentially smooth morphism  $f : \text{Spec}(k) \rightarrow \text{Spec}(\mathbb{F}_p)$  and  $f^*(K_{n, \mathbb{F}_p}) \cong K_{n, k}$  [16, Theorem 2.11]. As  $\mathbb{F}_p$  is perfect, we then have  $f^*(P_{\mathbb{F}_p}^n(\iota_n)) = f^*(P_{\mathbb{F}_p}^n)(\iota_n) = f^*(\iota_n^p) = \iota_n^p$ .  $\square$

From Proposition 3.0.3, we have the following corollary.

**Corollary 7.0.3.** *Let  $X \in \text{Sm}_k$ . Then  $P_k^n$  is the  $p$ th power on  $CH^n(X)/p$ .*

Now that we know  $f^*(P_{\mathbb{F}_p}^n)$  is the  $p$ th power on  $H^{2n, n}(-, \mathbb{F}_p)$  for all  $n \geq 1$ , we can prove an instability result. Let  $f : \text{Spec}(k) \rightarrow \text{Spec}(\mathbb{F}_p)$  be the structure morphism.

**Proposition 7.0.4.** *Let  $p, q, n \geq 0$  be integers such that  $n > p - q$  and  $n \geq q$ . Let  $X \in H(k)$  and let  $x \in H^{p, q}(X, \mathbb{F}_p)$ . Then  $f^*(P_{\mathbb{F}_p}^n)(x) = 0$ .*

*Proof.* Voevodsky's proof in [30, Lemma 9.9] works here since  $f^*(P_{\mathbb{F}_p}^n)$  is the  $p$ th power on  $H^{2n, n}(-, \mathbb{F}_p)$  by Proposition 7.0.2.  $\square$

**Corollary 7.0.5.** *Let  $X \in \text{Sm}_k$ . Then  $P_k^n$  is the 0 map on  $CH^m(X)/p$  for  $m < n$ .*

## Chapter 8

# Proper pushforward

In this chapter, we restrict our attention to mod  $p$  Chow groups on  $\mathrm{Sm}_k$ . The ring of mod  $p$  Chow groups is an oriented cohomology pretheory in the sense of [23, Section 1] with perfect integration given by proper pushforward on Chow groups. Consider the total cohomological Steenrod operation  $P_k := P_k^0 + P_k^1 + P_k^2 + \cdots : CH^*(-)/p \rightarrow CH^*(-)/p$ . From the Cartan formula 6,  $P_k$  is a ring morphism of oriented cohomology pretheories in the sense of [23, Definition 1.1.7].

Let  $\mathbb{Z}[[c_1, c_2, \dots]]$  denote the power series ring on Chern classes  $c_i$  for  $i \geq 1$  and let  $w \in \mathbb{Z}[[c_1, c_2, \dots]]$  denote the total characteristic class corresponding to the polynomial  $f(x) = 1 + x^{p-1}$ . For  $p = 2$ ,  $w$  is the total Chern class. Let  $X \in \mathrm{Sm}_k$ . For a line bundle  $L$  on  $X$ ,  $w(L) = 1 + c_1^{p-1}(L) \in CH^*(X)$ . For a vector bundle  $V$  on  $X$  that has a filtration by subbundles with quotients given by line bundles  $L_1, \dots, L_m$ ,  $w(V) = w(L_1) \cdots w(L_m)$ . Let  $w_i$  denote the  $i$ th homogeneous component of  $w$  for  $i \geq 0$ . We have  $w_i = 0$  if  $p-1$  does not divide  $i$ . Define the total homological Steenrod operation  $P^X := w(-T_X) \cdot P_k : CH^*(X)/p \rightarrow CH^*(X)/p$  where  $T_X$  is the tangent bundle on  $X$ . For  $i \geq 0$ , let  $P_i^X$  denote the  $(p-1)$ th homogeneous component of  $P^X$ . The following proposition is a consequence of the general Riemann-Roch formulas proved by Panin in [23]

**Proposition 8.0.1.** *Let  $f : X \rightarrow Y$  be a morphism of smooth projective varieties over  $k$ . Then*

$$\begin{array}{ccc} CH^*(X)/p & \xrightarrow{P^X} & CH^*(X)/p \\ \downarrow f_* & & \downarrow f_* \\ CH^*(Y)/p & \xrightarrow{P^Y} & CH^*(Y)/p \end{array}$$

*commutes.*

*Proof.* This is [23, Theorem 2.5.4]. See [23, Section 2.6] for a discussion relevant to our situation.

The main ingredients are that the operations  $P_k^n$  satisfy the Cartan formula and that  $P_k^n$  is the  $p$ th power on  $CH^n(-)/p$ . □

Restricting to the case  $p = \text{char}(k) = 2$ , we obtain a Wu formula from the work of Panin [23, Theorem 2.5.3]. Here,  $w = c$  is the total Chern class and we let  $\text{Sq}$  denote the total Steenrod square  $P_k$  on  $CH^*(-)/2$ .

**Proposition 8.0.2.** *Let  $X, Y$  be smooth projective varieties over  $k$ , and let  $i : X \hookrightarrow Y$  be a closed embedding with normal bundle  $N$ . Then*

$$i_*(c(N)) = \text{Sq}([X])$$

*in  $CH^*(Y)/2$  where  $[X] \in CH^*(Y)/2$  denotes the mod 2 cycle class of  $X$ .*

## Chapter 9

# Rost's degree formula

Now that we have Steenrod operations on mod  $p$  Chow groups of  $\mathrm{Sm}_k$ , we can prove Rost's degree formula [21, Theorem 6.4] without any restrictions on the characteristic of the base field. We closely follow the presentation of Merkurjev [21] where Steenrod operations (assuming restrictions on the characteristic of the base field) are used to prove degree formulas. In [15], Haution was able to extend the Rost degree formulas to base fields of characteristic 2.

For a variety  $X$  over  $k$ , let  $n_X$  denote the greatest common divisor of  $\deg(x)$  over all closed points  $x \hookrightarrow X$ . Let  $X \in \mathrm{Sm}_k$  be projective of dimension  $d > 0$ . Applying Proposition 8.0.1 to the structure morphism  $X \rightarrow \mathrm{Spec}(k)$  and  $[X] \in CH_d(X)/p$ , we see that  $p \mid \deg(w_d(-T_X))$ .

**Proposition 9.0.1.** *Let  $f : X \rightarrow Y$  be a morphism of projective varieties  $X, Y \in \mathrm{Sm}_k$  of dimension  $d > 0$ . Then  $n_Y \mid n_X$  and*

$$\frac{\deg(w_d(-T_X))}{p} \equiv \deg(f) \cdot \frac{\deg(w_d(-T_Y))}{p} \pmod{n_Y}.$$

*Proof.* The proof in [21, Theorem 6.4] works here. From Proposition 8.0.1,  $f_*(w_d(-T_X)) \equiv \deg(f)w_d(-T_Y) \in CH_0(Y)/p$ . We then take the degree homomorphism to finish the proof.  $\square$

# Chapter 10

## Specialization map

Fix a DVR  $D$  with residue field  $k$  and fraction field  $K$ . Let  $i : \text{Spec}(k) \rightarrow \text{Spec}(D)$  and  $j : \text{Spec}(K) \rightarrow \text{Spec}(D)$  as before. Let  $X \in \text{Sm}_D$  be projective with special fiber  $X_k$  and generic fiber  $X_K$ . As described in [9, Chapter 20.3], there are specialization maps  $\sigma_n : CH^n(X_K) \rightarrow CH^n(X_k)$  defined for all  $n \geq 0$ . The specialization maps can be defined at the level of cycles. Namely, for an irreducible closed subvariety  $Z_K \subset X_K$  of codimension  $n$ , we let  $Z_k$  denote the special fiber of the reduced closed subscheme  $\overline{Z_K} \subset X$  associated to  $Z_K \subset X$ . Then  $\sigma_n(\langle Z_K \rangle) = \langle Z_k \rangle \in CH^n(X_k)$ . We also let  $\sigma_n$  denote the specialization map induced on mod  $p$  Chow groups.

We now show that the Steenrod operations  $P_k^n$  defined on  $CH^*(X_k)$  are compatible with the operations  $P_K^n$  defined on  $CH^*(X_K)$ .

**Proposition 10.0.1.** *Let  $m \geq 0$  and let  $Z_K \subset X_K$  be a closed subvariety of codimension  $n$ .*

*Let  $\langle Z_K \rangle \in CH^n(X_K)/p$  denote the mod  $p$  cycle class of  $Z_K$ . Then*

$$P_k^m(\sigma_n(\langle Z_K \rangle)) = \sigma_{n+m(p-1)}(P_K^m(\langle Z_K \rangle)) \in CH^{n+m(p-1)}(X_k)/p.$$

*Proof.* The mod  $p$  cycle class of  $\overline{Z_K} \subset X$  induces a map

$$f_D : \Sigma_+^\infty X \rightarrow \Sigma^{2n,n} \widehat{H}\mathbb{F}_p^D$$

in  $SH(D)$ . The map  $i^* f_D$  gives the mod  $p$  cycle class of  $Z_k$  (the special fiber of  $\overline{Z_K} \subset X$ ) and

$j^* f_D$  gives the mod  $p$  cycle class of  $Z_K$ . Applying the natural transformation  $i^* \eta : i^* \rightarrow i^* j_* j^*$  to  $f_D$ , we get a commuting square.

$$\begin{array}{ccc}
\Sigma_+^\infty X_k & \xrightarrow{i^* f_D} & \Sigma^{2n, n} H\mathbb{F}_p^k \\
\downarrow i^* \eta & & \downarrow i^* \eta \\
i^* j_* \Sigma_+^\infty X_K & \xrightarrow{i^* j_* j^* f_D} & i^* j_* \Sigma^{2n, n} H\mathbb{F}_p^K
\end{array} \tag{10.1}$$

From Theorem 3.0.2,  $P_k^m = \Phi(P_K^m) = \pi \circ i^* j_* P_K^m \circ i^* \eta$ . Hence, from diagram 10.1, we get that

$$\pi \circ i^* \eta \circ P_k^m \circ i^* f_D = \pi \circ i^* j_* P_K^m \circ i^* j_* j^* f_D \circ i^* \eta$$

in the following commuting diagram.

$$\begin{array}{ccccc}
\Sigma_+^\infty X_k & \xrightarrow{i^* f_D} & \Sigma^{2n, n} H\mathbb{F}_p^k & \xrightarrow{P_k^m} & \Sigma^{2(n+m(p-1)), n+m(p-1)} H\mathbb{F}_p^k \\
\downarrow i^* \eta & & \downarrow i^* \eta & & \downarrow i^* \eta \\
i^* j_* \Sigma_+^\infty X_K & \xrightarrow{i^* j_* j^* f_D} & i^* j_* \Sigma^{2n, n} H\mathbb{F}_p^K & \xrightarrow{i^* j_* P_K^m} & i^* j_* \Sigma^{2(n+m(p-1)), n+m(p-1)} H\mathbb{F}_p^K \\
& & & & \downarrow \pi \\
& & & & \Sigma^{2(n+m(p-1)), n+m(p-1)} H\mathbb{F}_p^k
\end{array} \tag{10.2}$$

Write  $P_K^m(\langle Z_K \rangle) = \sum_{l=1}^q a_l \langle Z_K^l \rangle$  for some  $q, a_l \in \mathbb{Z}$  and closed subvarieties  $Z_K^l \subset X_K$  of codimension  $n + m(p - 1)$ . Taking the associated reduced closed subschemes in  $X$ , we get an element  $\sum_{l=1}^q a_l \langle \bar{Z}_K^l \rangle \in H^{2(n+m(p-1)), n+m(p-1)}(X, \mathbb{F}_p)$  which corresponds to a morphism  $g : \Sigma_+^\infty X \rightarrow \Sigma^{2(n+m(p-1)), n+m(p-1)} \widehat{H}\mathbb{F}_p^D$ . For  $1 \leq l \leq q$ , let  $Z_k^l$  denote the special fiber of  $\bar{Z}_K^l$ . Taking pullbacks,  $i^* g$  gives  $\sum_{l=1}^q a_l \langle Z_k^l \rangle \in H^{2(n+m(p-1)), n+m(p-1)}(X_k, \mathbb{F}_p)$  and  $j^* g = \sum_{l=1}^q a_l \langle Z_K^l \rangle = P_K^m(\langle Z_K \rangle)$ . Applying  $i^* \eta$  to  $g$ , we get a commuting diagram.

$$\begin{array}{ccc}
\Sigma_+^\infty X_k & \xrightarrow{i^* g} & \Sigma^{2(n+m(p-1)), n+m(p-1)} H\mathbb{F}_p^k \\
\downarrow i^* \eta & & \downarrow i^* \eta \\
i^* j_* \Sigma_+^\infty X_K & \xrightarrow{i^* j_* j^* g} & i^* j_* \Sigma^{2(n+m(p-1)), n+m(p-1)} H\mathbb{F}_p^K \\
& & \downarrow \pi \\
& & \Sigma^{2(n+m(p-1)), n+m(p-1)} H\mathbb{F}_p^k
\end{array} \tag{10.3}$$

From diagrams 10.2 and 10.3, we get

$$\begin{aligned} i^*g &= \sum_{l=1}^q a_l \langle Z_k^l \rangle = \pi \circ i^*j_*j^*g \circ i^*\eta = \pi \circ i^*j_*(P_K^m(\langle Z_K \rangle)) \circ i^*\eta \\ &= \pi \circ i^*j_*P_K^m \circ i^*j_*j^*f_D \circ i^*\eta = P_k^m(\langle Z_k \rangle) \end{aligned}$$

as required.  $\square$

We recall some facts about flag varieties, using [18] as a reference. Let  $G_k$  be a split reductive group over  $k$  with Borel subgroup  $B_k$  and Weyl group  $W$ . From the Bruhat decomposition, we have

$$G_k/B_k = \coprod_{w \in W} B_k w B_k / B_k.$$

For  $w \in W$ , the closure  $X_k^w$  of  $B_k w B_k / B_k$  in  $G_k/B_k$  is called a Schubert variety and  $B_k w B_k / B_k \cong \mathbb{A}_k^{l(w)}$  where  $l(w)$  is the length of  $w$  in  $W$ . Let  $P_k \supseteq B_k$  be a parabolic subgroup of  $G_k$ .

We have  $P_k = BW_P B$  for some subgroup  $W_P \leq W$ . There is a related  $W^P \subset W$ , such that for each  $w \in W^P$ ,  $B_k w B_k / B_k$  is isomorphic to  $B_k w B_k / P_k$  under the quotient morphism  $G_k/B_k \rightarrow G_k/P_k$  [18, Lemma 1.2]. We also have a cell decomposition

$$G_k/P_k = \coprod_{w \in W^P} B_k w B_k / P_k.$$

This cell decomposition is independent of the field  $k$ . It follows that the total chow group  $CH^*(G_k/P_k)$  is freely generated as an additive group by the cycle classes  $\langle Y_k^w \rangle$  of the images  $Y_k^w$  of the Schubert varieties  $X_k^w$  for  $w \in W^P$ .

Chevalley [1] and Demazure [4] showed that the chow rings

$$CH^*(G_{F_1}/P_{F_1}) \text{ and } CH^*(G_{F_2}/P_{F_2})$$

are isomorphic for any two fields  $F_1, F_2$ . The isomorphism is given by mapping the class of a Schubert subscheme  $Y_{F_1}^w$  to  $Y_{F_2}^w$  for  $w \in W^P$ . We now prove that the Steenrod operations  $P_k^n$  and  $P_K^n$  give the same action on  $H^{2*,*}(G_k/P_k, \mathbb{F}_p) \cong CH^*(G_k/P_k)/p \cong CH^*(G_K/P_K)/p \cong H^{2*,*}(G_K/P_K, \mathbb{F}_p)$ .

**Corollary 10.0.2.** *Let  $n \geq 0$  and let  $w_0 \in W^P$ . We have*

$$P_K^n(\langle Y_{w_0}^K \rangle) = \sum_{w \in W^P} a_w \langle Y_w^K \rangle$$

*in  $CH^*(G_K/P_K)/p$  for some  $a_w \in \mathbb{Z}$ . Then*

$$P_k^n(\langle Y_{w_0}^k \rangle) = \sum_{w \in W^P} a_w \langle Y_w^k \rangle.$$

*Proof.* We refer to [3] for facts about integral models of split reductive groups. Let  $w \in W$  and let  $X_D^w$  be the reduced closed subscheme of  $G_D/B_D$  associated to  $B_D w B_D/B_D$ . Note that  $X_D^w$  is flat over  $\text{Spec}(D)$ . For any field  $F$  and morphism  $\text{Spec}(F) \rightarrow \text{Spec}(D)$ , the fiber  $X_D^w \times_{\text{Spec}(D)} \text{Spec}(F)$  in  $G_F/B_F$  is isomorphic to  $X_F^w$  [26, Theorem 2]. The main point to check is that the fibers of  $X_D^w$  over  $\text{Spec}(D)$  are reduced.

Now assume that  $w \in W^P$ . Let  $Y_D^w$  denote the image of  $X_D^w$  in  $G_D/P_D$ . Then  $Y_D^w \times_{\text{Spec}(D)} \text{Spec}(F) \cong Y_F^w$  for any field  $F$  and morphism  $\text{Spec}(F) \rightarrow \text{Spec}(D)$ . Proposition 10.0.1 then applies to finish the proof.

□

# Chapter 11

## Applications to quadratic forms

In this chapter, we use the Steenrod squares  $Sq_k^{2n}$  to prove new results about nonsingular quadratic forms over a field  $k$  of characteristic 2. The results we prove have analogues in characteristic  $\neq 2$  conveniently found in [5, Sections 79-82] where the only missing ingredient for extending to characteristic 2 was the existence of Steenrod squares satisfying expected properties.

Recall that a quadratic form  $(q, V)$  over  $k$  is nonsingular if the associated radical  $V^\perp$  is of dimension at most 1 and  $q$  is nonzero on  $V^\perp \setminus 0$ . Equivalently,  $(q, V)$  is nonsingular if the associated projective quadric is smooth. Note that nonsingular quadratic forms are called nondegenerate in [5]. In characteristic 2, anisotropic quadratic forms are not necessarily nonsingular. Let  $(q, V)$  be a nonsingular anisotropic quadratic form of dimension  $D$  defined over  $k$  and let  $X$  be the associated quadric. Over some field extension  $F$  of  $k$ , the quadric  $X_F$  becomes split. A computation of  $CH^*(X_F)$  can be found in [5, Chapter XIII]. Let  $h \in CH^1(X_F)$  denote the pullback of the hyperplane class in  $\mathbb{P}(V)$  and let  $l_d \in CH_d(X_F)$  denote the class of a  $d$ -dimensional subspace in  $X_F$  where  $d = \lfloor (D-1)/2 \rfloor$ . Let  $l_i = h^i \cdot l_d$  for  $0 \leq i \leq d$ .

**Proposition 11.0.1.** *As an additive group,  $CH^*(X_F)$  is freely generated by  $h^i, l_i$  for  $0 \leq i \leq d$ .*

*For the ring structure,  $h^{d+1} = 2l_{D-d-1}$ ,  $l_d^2 = 0$  if 4 does not divide  $D$ , and  $l_d^2 = l_0$  if 4 divides  $D$ .*

From Corollary 10.0.2, the action of the Steenrod squares  $\text{Sq}_F^{2n}$  on  $CH^*(X_F)/2$  agrees with the action of Steenrod squares on the mod 2 Chow ring of a split quadric in characteristic 0. We refer to [5, Corollary 78.5] for the calculation of the action of Steenrod squares on the mod 2 Chow ring of a split quadric in characteristic 0.

**Proposition 11.0.2.** *For any  $0 \leq i \leq d$  and  $j \geq 0$ ,*

$$\text{Sq}_F^{2j}(h^i) = \binom{i}{j} h^{i+j} \text{ and } \text{Sq}_F^{2j}(l_i) = \binom{D+1-i}{j} l_{i-j}.$$

To state our results, we recall the definition of higher Witt indices. Let  $\varphi$  be a nonsingular quadratic form over a field  $F$  and let  $F(\varphi)$  denote the function field of the associated quadric. Let  $\varphi_{an}$  denote the anisotropic part of  $\varphi$  and let  $i_0(\varphi)$ , the Witt index of  $\varphi$ , denote the dimension of a maximal isotropic subspace for  $\varphi$ . Start with  $\varphi_0 := \varphi_{an}$  and  $F_0 := F$ . Inductively define  $F_i := F_{i-1}(\varphi_{i-1})$  and  $\varphi_i := (\varphi_{F_i})_{an}$  for  $i > 0$ . There exists  $m$  such that  $\dim \varphi_m \leq 1$ . For  $1 \leq j \leq m$ , we then define the  $j$ th relative higher Witt index  $i_j(\varphi)$  to be  $i_0(\varphi_{F_j}) - i_0(\varphi_{F_{j-1}})$ .

We recall Hoffmann's conjecture on the possible values of the first Witt index of an anisotropic quadratic form. Hoffmann's conjecture was originally restricted to quadratic forms over a field of characteristic  $\neq 2$  but it makes sense to consider the conjecture in characteristic 2 as well. For an integer  $n$ , let  $v_2(n)$  denote the 2-adic exponent of  $n$ .

**Conjecture 11.0.3.** *Let  $\varphi$  be an anisotropic quadratic form over a field  $F$  such that  $\dim \varphi \geq 2$ . Then  $i_1(\varphi) \leq 2^{v_2(\dim \varphi - i_1(\varphi))}$ .*

Hoffmann's original conjecture for characteristic  $\neq 2$  was proved by Karpenko in [17]. Karpenko's proof makes use of Steenrod squares on mod 2 Chow groups. With our construction of Steenrod squares on the mod 2 Chow groups over a base field of characteristic 2, we can now prove Hoffmann's conjecture for nonsingular anisotropic quadratic forms over a field of characteristic 2.

**Proposition 11.0.4.** *Let  $\varphi$  be a nonsingular anisotropic quadratic form over  $k$  such that  $\dim \varphi \geq 2$ . Then  $i_1(\varphi) \leq 2^{v_2(\dim \varphi - i_1(\varphi))}$ .*

*Proof.* The proof of [5, Proposition 79.4] works in this case. We only need to use the Cartan formula 6, the action of the Steenrod squares on the mod 2 Chow ring of a split quadric given by Proposition 11.0.2, Corollary 3.0.4 on base change, and other facts that can be found in [5] that are true for nonsingular anisotropic quadratic forms over a field of characteristic 2.  $\square$

Proposition 11.0.4 provides further evidence for the validity of Hoffmann’s conjecture in characteristic 2. Scully has proved that Hoffmann’s conjecture is valid for totally singular quadratic forms over a field of characteristic 2 [25].

The conclusions of [5, Corollary 81.19] which describes an inequality on higher Witt indices and [5, Corollary 82.2] on “holes in  $I^n$ ” should also now hold for nonsingular anisotropic quadratic forms in characteristic 2. The proofs of these results given in [5] could be imitated for nonsingular anisotropic quadratic forms in characteristic 2 using the properties we have established for the Steenrod squares  $Sq_k^{2n}$ .

# Bibliography

- [1] C. Chevalley. Sur les décompositions cellulaires des espaces  $G/B$ . Proc. Sympos. Pure Math., 56, Part 1, *Algebraic groups and their generalizations: classical methods (University Park, PA, 1991)*, 1-23, Amer. Math. Soc. (1994).
- [2] D.-C. Cisinski and F. Déglise. Triangulated categories of mixed motives (2012), Preprint, <https://arxiv.org/abs/0912.2110>.
- [3] B. Conrad. Reductive group schemes, from *Autour des schémas en groupes. Vol. I*, Panoramas et synthèses, 42-43, Soc. Math. France, Paris (2014) 94-444.
- [4] M. Demazure. Invariants symétriques entiers des groupes de Weyl et torsion. *Invent. Math.* 21 (1973): 287-302.
- [5] R. Elman, N. Karpenko, and A. Merkurjev. *The algebraic and geometric theory of quadratic forms*. Vol. 56 Amer. Math Soc. (2008).
- [6] E. Elmanto, M. Hoyois, A. A Khan, V. Sosnilo, and M. Yakerson. Motivic infinite loop spaces (2018), Preprint, <https://arxiv.org/abs/1711.05248>.
- [7] E. M. Friedlander and A. Suslin. The spectral sequence relating algebraic  $K$ -theory to motivic cohomology. *Annales scientifiques de l'École Normale Supérieure*, Série 4, Volume 35 (2002), 773-875.
- [8] M. Frankland and M. Spitzweck. Towards the dual motivic Steenrod algebra in positive characteristic (2018), Preprint, <https://arxiv.org/pdf/1711.05230.pdf>.

- [9] W. Fulton. *Intersection theory*. Springer (1998).
- [10] T. Geisser. Motivic cohomology over Dedekind rings. *Math. Z.* **248** (2004), 773-794.
- [11] O. Haution. Integrality of the Chern character in small codimension. *Adv. Math.* 231 (2012), no. 2, 855-878.
- [12] O. Haution. Duality and the topological filtration. *Math. Ann.* 357 (2013), no. 4, 1425-1454.
- [13] O. Haution. On the first Steenrod square for Chow groups. *Amer. J. Math.* 135 (2013), no. 1, 53-63.
- [14] O. Haution. Detection by regular schemes in degree two. *Algebr. Geom.* 2 (2015), no. 1, 44-61.
- [15] O. Haution. Involutions of varieties and Rost's degree formula. *J. Reine Angew. Math.* 745 (2018), 231-252.
- [16] M. Hoyois, S. Kelly, and P. A. Østvær. The motivic Steenrod algebra in positive characteristic. *J. Eur. Math. Soc.* 19 (2017), no. 12.
- [17] N. Karpenko. On the first Witt index of quadratic forms. *Invent. Math.* **153** (2003), no. 2, 455-462.
- [18] B. Köck. Chow motif and higher Chow theory of G/P. *Manuscripta mathematica* **70.4** (1991), 363-372.
- [19] S. Kondo and S. Yasuda. Product structures in motivic cohomology and higher Chow groups. *J. Pure Appl. Algebra* **215** (2011), 511-522.
- [20] M. Levine. Techniques of localization in the theory of algebraic cycles. *J. Algebraic Geom.* **10** (2001), 299-363.
- [21] A. Merkurjev. Steenrod operations and degree formulas. *J. Reine Angew. Math.* 565 (2003), 12-26.

- [22] F. Morel and V. Voevodsky.  $A^1$ -homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.* **90** (1999), 45-143.
- [23] I. Panin. Riemann-Roch theorems for oriented cohomology. *Axiomatic, enriched and motivic homotopy theory* (J.P.C. Greenlees, ed.) NATO Sci. Ser. II Math. Phys. Chem. **131** (2004), 261-334, Kluwer Acad. Publ., Dordrecht.
- [24] J. Riou. Opérations de Steenrod motiviques (2012), Preprint, <https://www.math.u-psud.fr/~riou/doc/steenrod.pdf>.
- [25] S. Scully. Hoffmann’s conjecture for totally singular forms of prime degree. *Algebra Number Theory* Volume 10, no. 5 (2016), 1091-1132.
- [26] C. S. Seshadri. Standard monomial theory and the work of Demazure. *Algebraic Varieties and Analytic Varieties*, Mathematical Society of Japan, Tokyo (1983) 355-384.
- [27] M. Spitzweck. A commutative  $\mathbb{P}^1$ -spectrum representing motivic cohomology over Dedekind domains. *Mém. Soc. Math. Fr. (N.S.)* No. 157 (2018), 1-110.
- [28] V. Voevodsky.  $\mathbf{A}^1$ -homotopy theory. *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, Extra Vol. I (1998), 579-604.
- [29] V. Voevodsky. Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic. *Int. Math. Res. Not.* 7 (2002), 351-355.
- [30] V. Voevodsky. Reduced power operations in motivic cohomology. *Publ. Math. Inst. Hautes Études Sci.* **98** (2003), 1-57.