Title
Toric surfaces over arbitrary fields

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Publication Date
2017

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA

Los Angeles

Toric surfaces over arbitrary fields

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Mathematics

by

Fei Xie

2017

ABSTRACT OF THE DISSERTATION

Toric surfaces over arbitrary fields

by

Fei Xie

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2017

Professor Christian Haesemeyer, Chair

We study toric varieties over arbitrary fields with an emphasis on toric surfaces in the Merkurjev-Panin category of "K-motives". We explore the decomposition of certain toric varieties as K-motives into products of central simple algebras (CSA), the geometric and topological information encoded in these CSAs, and the relationship between the decomposition of the K-motive and the semiorthogonal decomposition of the derived category. We obtain the information mentioned above for toric surfaces by classifying all minimal smooth projective toric surfaces.

The dissertation of Fei Xie is approved.

Per J Krause

Paul Balmer

Alexander Sergee Merkurjev

Christian Haesemeyer, Committee Chair

University of California, Los Angeles

2017

To my family

who has been supportive for my career

TABLE OF CONTENTS

LIST OF TABLES

ACKNOWLEDGMENTS

Graduate school has been a challenging and wonderful adventure for me. I would like to thank my advisor, Christian Haesemeyer, without whom the dissertation would not have come into existence. I would like to thank him for suggesting this problem to me and for all the math conversations we had and for many career advises he gave me over the past few years. I am grateful for his patience and encouragements during my struggled time.

I would like to thank Alexander Merkurjev for the beautiful paper "K-theory of algebraic tori and toric varieties" he wrote with Panin, which this dissertation is based on and for the useful discussions he had with me regarding the project. I benefitted greatly from the classes and seminars given by UCLA algebra group faculty, which includes the two mentioned before, Burt Totaro, Paul Balmer and Raphael Rouquier. Many tools that have been applied to the dissertation, including algebraic geometry, algebraic groups, K-theory and triangulated categories, come from these classes and seminars. In addition, I am grateful to Ben Antieau for bringing to my attention the work by Asher Auel and Marcello Bernardara on semiorthogonal decompositions of del Pezzo surfaces over arbitrary fields, which helped me improving the previous version of the work and guided me into the direction of derived categories. Finally, I want to thank Honglu Fan for teaching me toric varieties.

Women in Math group at UCLA has also been an indispensable part of my graduate life. I would not have survived graduate school without the constant support from the group. I want to give my special thanks to Laure Flapan and Jackie Lang for their help, advises and encouragements over the years.

I was partially supported by the NSF research grant DMS-#1160206.

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CHAPTER 1

Introduction

Throughout, we fix the base field k, which is arbitrary. Let X be a scheme over k and let K/k be a field extension. We say a scheme Y over k is a *K/k-form* of X if base change from k to K, the schemes $X_K := X \otimes_k K$ and Y_K are isomorphic as schemes over K [Ser97, Chapter III §1]. For k^s the separable closure of k, a k^s/k-form is simply called a *form* or *twisted form*. The scheme X_{k^s} has a natural $\Gamma = \mathrm{Gal}(k^s/k)$-action.

We will focus on the study of toric varieties over k. Let X be a normal geometrically irreducible variety over k and let T be an algebraic torus acting on X over k. The variety X is a toric T-variety if there is an open orbit U such that U is a principal homogeneous space or torsor over T. A toric T-variety is called split if the torus T is split. The case of split toric varieties have been extensively studied, for example in [Dan78][Ful93][CLS11]. Since any toric variety X has a torus action over k and is a twisted form of a split toric variety, the study of X is equivalent to the study of the split toric variety X_{k^s} with a Γ-action on the fan structure as well as the open orbit U under the torus action, see §2.2. The main result is the classification of minimal smooth projective toric surfaces:

Theorem 1 (Theorem 3.1.10)**.** *The surface X is a minimal smooth projective toric surface if and only if X is (i) a $\mathbb{P}^1$-bundle over a smooth conic curve; (ii) the Severi-Brauer surface; (iii) an involution surface; (iv) the del Pezzo surface of degree 6 with Picard rank 1.*

This paper is motivated by ideas in [MP97], which studies toric varieties over an arbitrary field in the motivic category $\mathcal{C}$ defined in *loc. cit.*, and in particular by the following question:

Question 1. *If X is a smooth projective toric variety over k, is $K_0(X_{k^s})$ always a permutation Γ-module?*

Definition 1.0.1. A Γ-module M is a *permutation* Γ*-module* if there exists a Γ-invariant $\mathbb{Z}$-basis of M. We call such a basis a *permutation* Γ*-basis* or Γ*-basis*.

The reason that we care about the Γ-action on $K_0(X_{k^s})$ is that it in some way determines X, cf §4.1. For example, if X has a rational point and $K_0(X_{k^s})$ is a permutation Γ-module, then X is isomorphic to the étale algebra corresponding to any Γ-basis of $K_0(X_{k^s})$ in the motivic category $\mathcal{C}$ [MP97, Proposition 4.5]. In general, if $K_0(X_{k^s})$ has a permutation Γ-basis of line bundles over X_{k^s}, then X has a decomposition into a finite product of finite Azumaya algebras in the motivic category $\mathcal{C}$ completely described by this Γ-basis as follows:

Theorem 2 (Theorem 4.1.5)**.** *Let X be a smooth projective toric T-variety over k that splits over l and $G = \mathrm{Gal}(l/k)$. Assume $K_0(X_l)$ has a permutation G-basis P of line bundles on X_l. Let $\{P_i \,|\, 1 \leqslant i \leqslant t\}$ be G-orbits of P, and let $\pi : X_l \to X$ be the projection. For any $S_i \in P_i$, set $B_i = \mathrm{End}_{\mathcal{O}_Y}(\pi_*(S_i))$ and $B = \prod_{i=1}^t B_i$, then the map $u = \bigoplus_{i=1}^t \pi_*(S_i) : X \to B$ gives an isomorphism in the motivic category $\mathcal{C}$.*

Using the classification of minimal toric surfaces, we obtain that any smooth projective toric surface satisfies the conditions of the above theorem:

Theorem 3 (Theorem 3.2.2)**.** *Let X be a smooth projective toric T-surface over k that splits over l and $G = \mathrm{Gal}(l/k)$. Then $K_0(X_l)$ has a permutation G-basis of line bundles on X_l.*

The original motivation for finding the decomposition of a smooth projective variety over k into a product of finite Azumaya algebras in $\mathcal{C}$ is to compute higher algebraic K-theory of the variety. Quillen [Qui73] computed higher algebraic K-theory for Severi-Brauer varieties, cf Example 2.2.5, and Swan [Swa85] for quadric hypersurfaces. Panin [Pan94] generalized their results by finding the decomposition in $\mathcal{C}$ for twisted flag varieties.

As a matter of fact, these Azumaya algebras also encode arithmetic/geometric information about the variety, and in nice cases, classify all its twisted forms. Blunk investigated del Pezzo surfaces of degree 6 over k in [Blu10] in this direction, cf Example 2.2.6. He showed that a del Pezzo surface of degree 6 is determined by a pair of Azumaya algebras and the surface has a rational point if and only if the corresponding pair of Azumaya algebras are both split. We will investigate the same information for all smooth projective toric surfaces over k, cf §4.2. More generally, if the Picard group $\mathrm{Pic}(X_{k^s})$ of a smooth projective toric variety X is a permutation Γ-module, the open orbit U is determined by a set of Azumaya algebras, each corresponding to a Γ-orbit of $\mathrm{Pic}(X_{k^s})$, see Corollary 4.2.3. This implies that the toric variety X has a rational point if and only if every Azumaya algebra in the set is split. For example, we obtain that a $\mathbb{P}^1$-bundle over a smooth conic curve is isomorphic to $k \times Q \times k \times Q$ in $\mathcal{C}$ and the surface is determined by the quaternion algebra Q corresponding to the conic curve.

Moreover, since Tabuada [Tab14, Theorem 6.10] showed that the motivic category $\mathcal{C}$ is a part of the category of noncommutative motives Hmo_0, it implies that certain semiorthogonal decompositions of the derived category of a smooth projective variety will give a decomposition of the variety in $\mathcal{C}$ as follows:

Theorem 4 (Theorem 5.1.4)**.** *Let X be a smooth projective variety over k. Assume $D^b(X)$ has a full exceptional collection of objects $\{V_1, \dots, V_n\}$ where V_i is A_i-exceptional and A_i is a finite simple k-algebra, then $X \cong \prod_{i=1}^n A_i$ in the motivic category $\mathcal{C}$.*

We will also briefly discuss the possibility of lifting the decomposition in the motivic category of a smooth projective toric variety to the derived category, cf §5.1. A partial result is the following:

Theorem 5 (Lemma 5.1.6, Theorem 5.1.7)**.** *Using the same notation as Theorem 2 and assume $\dim X \geqslant 3$, then any G-orbit P_i forms an exceptional block. If there is an ordering for G-orbits $\{P_i\}_{i=1}^t$ of P such that $\{P_1, \dots, P_t\}$ forms a full exceptional collection of $D^b(X_l)$, then for any $S_i \in P_i$, $\{\pi_* S_1, \dots, \pi_* S_t\}$ is a full exceptional collection of $D^b(X)$.*

By the classification of minimal toric surfaces and the known results of semiorthogonal decompositions of geometrically rational surfaces, there is always such an ordering for smooth projective toric surfaces. However, for higher dimensional toric varieties, it is still unclear.

The organization of the dissertation is as follows:

Sections 2.1 and 2.2 introduce the background on the motivic category $\mathcal{C}$ and toric varieties over k, including some basic facts and examples needed for the paper. For more details about $\mathcal{C}$, see [MP97, §1] or [Mer05, §3]. Section 3.1 classifies minimal smooth projective toric surfaces over k via toric geometry. Section 3.2 verifies that $K_0(X_{k^s})$ has a permutation Γ-basis of line bundles for toric surfaces. In section 4.1, we consider smooth projective toric varieties X of all dimensions where $K_0(X_{k^s})$ has a permutation Γ-basis of line bundles. We reinterpret the construction for the separable algebra corresponding to a toric variety in [MP97], and deduce that it provides exactly a decomposition of a smooth projective toric variety with the aforementioned property in terms of such a basis. In section 4.2, we apply the construction in §4.1 to toric surfaces. Moreover, we relate the constructed algebras to the open orbit U via Galois cohomology. For details of Galois cohomology, see [Ser97][KMR98][GS06]. In section 5.1, we discuss the relationship between the semiorthogonal decomposition of the derived category and the decomposition in the motivic category of toric varieties via noncommutative motives and descent theory for derived categories.

Most of the time, instead of working with X_{k^s} and Γ-action, we work with X_l and $G = \mathrm{Gal}(l/k)$-action where l is the splitting field of the torus T.

We will use the following notations:

Fix the base field k and a separable closure k^s of k. Let $\Gamma = \mathrm{Gal}(k^s/k)$. T is an algebraic torus over k. l is the splitting field of T and $G = \mathrm{Gal}(l/k)$ unless otherwise stated. For any object Z (algebraic groups, varieties, algebras, maps) over k and any extension K/k, write $Z \otimes_k K$ as Z_K. For a split toric variety Y, we denote Σ as the fan structure and Aut_Σ as the group of fan automorphisms. We will freely use the same notation for the ray in the fan, the minimal generator of the ray in the lattice and the Weil divisor corresponding to the ray when the context is clear. For an algebra A, denote A^{op} as its opposite algebra. Denote S_n as the permutation group of a set of n elements.

CHAPTER 2

Background

2.1 Motivic Category $\mathcal{C}$

Definition 2.1.1. The *motivic category* $\mathcal{C} = \mathcal{C}_k$ over a field k has:

- Objects: The pair (X, A) where X is a smooth projective variety over k, A is a finite separable k-algebra
- Morphisms: $\mathrm{Hom}_{\mathcal{C}}((X, A), (Y, B)) = K_0(X \times Y, A^{\mathrm{op}} \otimes_k B)$

The Grothendieck group K_0 of a pair is defined below. A k-algebra A is *finite separable* if $\dim_k(A)$ is finite and for any field extension K of k, A_K is semisimple. Equivalently we have:

Definition 2.1.2. The algebra A is a *finite separable k-algebra* if it is a finite product of central simple algebras A_i where centers l_i are finite separable field extensions of k.

Let $u : (X, A) \to (Y, B)$ and $v : (Y, B) \to (Z, C)$ be morphisms in $\mathcal{C}$. Since $u \in K_0(X \times Y, A^{\mathrm{op}} \otimes_k B) \cong K_0(Y \times X, B \otimes_k A^{\mathrm{op}})$, the map u can also be viewed as $u^{\mathrm{op}} : (Y, B^{\mathrm{op}}) \to (X, A^{\mathrm{op}})$. The *composition* $v \circ u : (X, A) \to (Z, C)$ is given by

$$\pi_*(q^* v \otimes_B p^* u)$$

where $p : X \times Y \times Z \to X \times Y$, $q : X \times Y \times Z \to Y \times Z$, $\pi : X \times Y \times Z \to X \times Z$ are projections.

We write X for (X, k) and A for $(\mathrm{Spec}\, k, A)$. Since the morphisms are defined in K_0, the category is also called *K-correspondences*.

2.1.1 Algebraic K-theory of a pair

The algebraic K-theory of a pair (X, A) is defined in the following way and it generalizes the Quillen K-theory of varieties:

Let $\mathcal{P}(X, A)$ be the exact category of left $\mathcal{O}_X \otimes_k A$-modules which are locally free $\mathcal{O}_X$-modules of finite rank and morphisms of $\mathcal{O}_X \otimes_k A$-modules. The group $K_n(X, A)$ of the pair (X, A) is defined as $K_n^Q(\mathcal{P}(X, A))$, the Quillen K-theory of $\mathcal{P}$. Let $\mathcal{M}(X, A)$ be the exact category of left $\mathcal{O}_X \otimes_k A$-modules which are coherent $\mathcal{O}_X$-modules and morphisms of $\mathcal{O}_X \otimes_k A$-modules. The group $K'_n(X, A)$ of the pair (X, A) is defined as $K_n^Q(\mathcal{M}(X, A))$. The embedding $\mathcal{P} \subset \mathcal{M}$ induces a map $K_n(X, A) \to K'_n(X, A)$ and it is an isomorphism if X is regular (resolution theorem). Note that $K_n(X, k)$ is the usual $K_n(X)$ and $K_n(\operatorname{Spec} k, A) = K_n(\operatorname{Rep}(A))$ is the K-theory of representations of A.

In fact, K_n defines a functor $K_n : \mathcal{C} \to Ab$ which sends (X, A) to $K_n(X, A)$. For $u : (X, A) \to (Y, B)$, $x \in K_n(X, A)$, we can define

$$K_n(u)(x) = q_*(u \otimes_A p^* x)$$

where $p : X \times Y \to X, q : X \times Y \to Y$ are projections.

Similarly we can define, for any variety V over k, a functor $K_n^V : \mathcal{C} \to Ab$ where on objects $K_n^V(X, A) = K'_n(V \times X, A)$.

Example 2.1.3. [MP97, Example 1.6(1)] $\mathrm{M}_n(k) \simeq k$ in $\mathcal{C}$.

Example 2.1.4. [MP97, Example 1.6(3)], see also [Tab14, Theorem 9.1]. Let A and B be two central simple k-algebras. Then $A \cong B$ in $\mathcal{C}$ if and only if $[A] = [B] \in \operatorname{Br}(k)$.

Proof. Previous example indicates that Brauer equivalences give isomorphisms in $\mathcal{C}$, so $[A] = [B] \in \operatorname{Br}(k)$ implies $A \cong B$ in $\mathcal{C}$.

For the opposite direction, since each central simple k-algebra is Brauer equivalent to a unique division k-algebra, we can assume A, B are division algebras. Let $M : A \to B$ and $N : B \to A$ be inverse maps in $\mathcal{C}$. Since $K_0(A^{\mathrm{op}} \otimes_k B) \cong \mathbb{Z}R$ and $K_0(B^{\mathrm{op}} \otimes_k A) \cong \mathbb{Z}R^{\mathrm{op}}$ for R the unique simple A-B-bimodule, we have $M = nR$ and $N = mR^{\mathrm{op}}$ for some $m, n \in \mathbb{Z}$.

$N \circ M = N \otimes_B M \cong mn\, R^{\mathrm{op}} \otimes_B R \cong A$, $M \circ N = M \otimes_A N \cong mn\, R \otimes_A R^{\mathrm{op}} \cong B$. Since A, B are simple modules, we have $mn = 1$ and we can assume $M = R, N = R^{\mathrm{op}}$. As a right A-module and a left B-module respectively, we have $M_A \cong A^r$ and ${}_B M \cong B^s$. Similarly, ${}_A N \cong A^p$ and $N_B \cong B^q$. The left A-module isomorphism $N \otimes_B M \cong N \otimes_B B^s \cong N^s \cong A^{ps} \cong A$ implies that $p = s = 1$. Similarly $r = q = 1$. In particular, this implies $\dim_k A = \dim_k B$.

Finally consider the k-algebra homomorphism $f : B \to \mathrm{End}_A(M_A) \cong A$ by sending b to l_b left multiplication by b. This is obviously injective, and it is surjective because A, B have the same dimension, so $A \cong B$ as k-algebras. $\square$

2.2 Toric Varieties

Let T be an algebraic torus over k.

Definition 2.2.1. A *toric T-variety* X over k is a normal geometrically irreducible variety with an action of the torus T and an open orbit U which is a principal homogeneous space over T.

By definition, the torus $T_{k^s} \cong \mathbb{G}^n_{m,k^s}$ splits where $n = \dim X$. The torus T corresponds to a cocycle class $[\rho] \in H^1(\Gamma, \mathrm{Aut}_{\mathrm{gp},k^s}(\mathbb{G}^n_{m,k^s})) = H^1(\Gamma, \mathrm{GL}(n, \mathbb{Z}))$. Moreover, the torus T splits over a finite Galois extension l of k ($T_l \cong \mathbb{G}^n_{m,l}$), which is called the *splitting field* of T.

Explicitly, tori T_{k^s} and $\mathbb{G}^n_{m,k^s}$ have natural Galois actions with Γ acting on the factor k^s. This Galois action gives group automorphisms of T_{k^s} over k, but not over k^s because Γ also acts on the scalar k^s. Let $\phi : T_{k^s} \to \mathbb{G}^n_{m,k^s}$ be an isomorphism, then we obtain $\rho : \Gamma \to \mathrm{GL}(n, \mathbb{Z})$ by sending g to $\phi g \phi^{-1} g^{-1}$, and we have $\ker(\rho) = \mathrm{Gal}(k^s/l)$ where l is the splitting field. Conversely, the torus T can be constructed from ρ as follows, cf [VE83, §1]. Let $\rho' : G = \mathrm{Gal}(l/k) \to \mathrm{GL}(n, \mathbb{Z})$ be induced by ρ and let $\mu : G \to \mathrm{Aut}_k(\mathbb{G}^n_{m,l})$ act on $\mathbb{G}^n_{m,k} \otimes_k l$ via $\mu(g) = \rho'(g) \otimes g, g \in G$, then $T \cong \mathbb{G}^n_{m,l}/\mu(G)$.

Definition 2.2.2. A toric T-variety X over k is called a *toric T-model* if the open orbit U has a rational point.

In this case, the open orbit $U \cong T$ and there is an T-equivariant embedding $T \hookrightarrow X$. If X is smooth, by [VA85, §4 Proposition 4], the set $U(k)$ is nonempty if and only if $X(k)$ is.

Definition 2.2.3. A toric T-variety is *split* if T splits, and is *non-split* otherwise.

Let X_{k^s} (or X_l) be the split toric variety with the fan structure Σ. Since the Γ-action on T_{k^s} is compatible with the one on X_{k^s}, the image of ρ is contained in Aut_Σ, namely

$$\rho(\Gamma) = \mathrm{Gal}(l/k) \subseteq \mathrm{Aut}_\Sigma \subset \mathrm{GL}(n, \mathbb{Z}).$$

Let X_Σ be the split toric variety over k with fan structure Σ. If X is a toric T-model, then similarly as the torus T, the variety X can be recovered from ρ and Σ as $(X_\Sigma \otimes_k l)/\mu(G)$. In general, for each toric T-variety X, there is a unique (up to T-isomorphism) toric T-model X^* such that $X_{k^s} \cong (X^*)_{k^s}$. We call X^* the associated toric T-model of X. In detail, the toric T-model $X^* = (X \times U)/T$ where T acts on $X \times U$ diagonally, and the toric T-variety $X = (X^* \times U)/T$ where T acts on $X^* \times U$ via $t \cdot (x, y) = (tx, yt^{-1})$, cf [VA85, §4].

In summary, an algebraic torus T is uniquely determined by a 1-cocycle (class) $\rho : \Gamma \to \mathrm{GL}(n, \mathbb{Z})$. A toric T-model X is uniquely determined by ρ and fan Σ with the restriction $\rho(\Gamma) \subseteq \mathrm{Aut}_\Sigma$. A toric T-variety is uniquely determined by its associated T-model X^* and a principal homogeneous space $U \in H^1(k, T)$.

Lemma 2.2.4. *Let $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ be a toric morphism of split smooth projective toric varieties over k^s, and let $\bar{\phi} : N_1 \to N_2$ be the induced $\mathbb{Z}$-linear map of lattices that is compatible with fans Σ_1, Σ_2 for each i. Let $\rho_i : \Gamma \to \mathrm{Aut}(N_i)$ be a Galois action on N_i that is compatible with fan Σ_i ($\rho_i(\Gamma) \subseteq \mathrm{Aut}_{\Sigma_i}$) such that $\bar{\phi}$ is Γ-equivariant. Let T_i be the torus corresponding to ρ_i. Then, for any $U_1 \in H^1(k, T_1)$, there exists $U_2 \in H^1(k, T_2)$ such that ϕ descends to a map $X_1 \to X_2$ where X_i is the toric variety corresponding to (ρ_i, Σ_i, U_i) for $i = 1, 2$.*

Proof. Restrict ϕ to tori $\phi|_{T_{N_1}} : T_{N_1} \to T_{N_2}$. Since $\bar{\phi}$ is Γ-equivariant, maps ϕ and $\phi|_{T_{N_1}}$ descend to $\varphi : X_1^* \to X_2^*$ where X_i^* is the toric T_i-model corresponding to Σ_i and $\psi : T_1 \to T_2$. ψ induces $H^1(k, T_1) \to H^1(k, T_2)$ and let U_2 be the image of U_1 under this map. Set $X_i = (X_i^* \times U_i)/T_i$, then ϕ descends to a map $X_1 \to X_2$. □

Example 2.2.5. Severi-Brauer variety X ($X_{k^s} \cong \mathbb{P}^n$) . Let A be a central simple k-algebra of degree $n+1$, then $X = \mathrm{SB}(A)$ is a toric variety with torus $T = \mathrm{R}_{E/k}(\mathbb{G}_{m,E})/\mathbb{G}_{m,k}$ where E is a maximal étale k-subalgebra of A. X has a rational point if and only if $A = M_{n+1}(k)$ if and only if $X \cong \mathbb{P}^n$.

Quillen [Qui73, §8 Theorem 4.1] showed that $K_m(\mathrm{SB}(A)) \cong K_m(k) \times \prod_{i=1}^n K_m(A^{\otimes i})$ for $m \geqslant 0$, and Panin [Pan94] further showed that $\mathrm{SB}(A) \cong k \times \prod_{i=1}^n A^{\otimes i}$ in $\mathcal{C}$.

Example 2.2.6. Let X be a del Pezzo surface of degree 6 over k (K_X is anti-ample with $K_X^2 = 6$, $X_{k^s} \cong \mathrm{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)$ where p_1, p_2, p_3 are not collinear). It is a toric T-variety where T is the connected component of the identity of $\mathrm{Aut}_k(X)$.

Blunk [Blu10] showed that $X \cong k \times P \times Q$ in $\mathcal{C}$ where P is an Azumaya K-algebra of rank 9 ($\dim_k(P)/\dim_k(K) = 9$) and Q is an Azumaya L-algebra of rank 4 where K, L are étale k-algebras of degree 2 and 3 respectively.

Example 2.2.7. Involution surface X ($X_{k^s} \cong \mathbb{P}^1 \times \mathbb{P}^1$). The surface X corresponds to a central simple algebra (A, σ) of degree 4 with a quadratic pair σ, the even Clifford algebra $C_0(A, \sigma)$ is a quaternion algebra over K, the discriminant extension of X. Write $B = C_0(A, \sigma)$, then X is the Weil restriction $\mathrm{R}_{K/k}\mathrm{SB}(B)$, cf [AB15, Example 3.3]. Denote the torus of $\mathrm{SB}(B)$ in Example 2.2.5 as T, then X is a toric variety with torus $\mathrm{R}_{K/k}T$.

Panin [Pan94] showed that $X \cong k \times B \times A$ in $\mathcal{C}$.

2.2.1 K_0 of split toric varieties

Let Y be a split smooth proper toric T-variety with fan Σ.

For $\sigma \in \Sigma$, denote $\mathcal{O}_\sigma$ the closure of the T-orbit corresponding to σ and J_σ the sheaf of ideals defining $\mathcal{O}_\sigma$. Write $\sigma(1)$ as the set of rays span σ. For $\sigma, \tau \in \Sigma$, if $\sigma(1) \cap \tau(1) = \emptyset$ and $\sigma(1) \cup \tau(1)$ span a cone in Σ, denote the cone by $\langle \sigma, \tau \rangle$, otherwise $\langle \sigma, \tau \rangle = 0$.

From [AA92], we have

Theorem 2.2.8 (Klyachko, Demazure). *As an abelian group, $K_0(Y)$ is generated by $\mathcal{O}_\sigma =$*

$1 - J_\sigma$ *with relations*

$$\mathcal{O}_\sigma \cdot \mathcal{O}_\tau = \begin{cases} \mathcal{O}_{\langle \sigma, \tau \rangle}, & \langle \sigma, \tau \rangle \neq 0 \\ 0, & \textit{otherwise} \end{cases} \tag{2.2.1}$$

$$\prod_{e \in \Sigma(1)} J_e^{f(e)} = 1, f \in \mathrm{Hom}(N, \mathbb{Z}) = M = T^* \tag{2.2.2}$$

where T^* *is the group of characters of* T*.*

Theorem 2.2.9 (Klyachko)**.** *The abelian group* $K_0(Y)$ *is free with rank equal to the number of the maximal cones. In addition, sheaves* $\mathcal{O}_y$ *and* $\mathcal{O}_{y'}$ *coincide in* $K_0(Y)$ *for any rational closed points* $y, y' \in Y$*.*

CHAPTER 3

Main Results for Toric Surfaces

3.1 Minimal Toric Surfaces

Let X be a smooth projective toric surface over k. We say X is *minimal* if any birational morphism $f : X \to X'$ from X to another smooth projective surface X' defined over k is an isomorphism. In this section, we will classify minimal smooth projective toric surfaces.

Definition 3.1.1. Define $\mathcal{T}_{G,K} = \{Y \,|\, G$ can be embedded into $\mathrm{Aut}_\Sigma(Y)\}/$ isomorphisms over K, where Y is a split smooth projective toric surface with fan Σ over a field K, the group $\mathrm{Aut}_\Sigma(Y)$ is the group of fan automorphisms of Y and G is a finite subgroup of $\mathrm{GL}(2,\mathbb{Z})$.

We will simply write $\mathcal{T}_G$ if the base field K is not important. Note that $\mathcal{T}_G$ only depends on the conjugacy classes of G in $\mathrm{GL}(2,\mathbb{Z})$ and for a surface $Y \in \mathcal{T}_G$, there is an induced G-action on Y. For any two varieties Y_1, Y_2 with G-actions, we say a morphism $h : Y_1 \to Y_2$ is a *G-morphism* if h is G-equivariant.

Definition 3.1.2. The split smooth projective toric surface Y is *G-minimal over K* if $Y \in \mathcal{T}_{G,K}$ and any birational G-morphism $f : Y \to Y'$, $Y' \in \mathcal{T}_{G,K}$ defined over K is a G-isomorphism.

We can redefine minimal toric surfaces as follows:

Definition 3.1.3. Let X be a smooth projective toric T-surface over k and let $\rho : \Gamma \to \mathrm{GL}(2,\mathbb{Z})$ be the map corresponding to the torus T. We say X is a *minimal toric surface* if X_{k^s} is $G = \rho(\Gamma)$-minimal over k^s.

This definition is equivalent to the one given at the beginning of the section. This is because for a smooth projective toric T-surface X, the exceptional locus of a birational morphism $f : X \to X'$ is T-invariant. Thus, the surface X' also has a toric structure with the same torus T. On the other hand, the existence of a birational morphism $f : X \to X'$ where X' is smooth projective is equivalent to the existence of a G-birational morphism $h : X_{k^s} \to Y$ where Y is a split smooth projective toric surface such that the G-action on Y is torus invariant, i.e the group G can be embedded into $\mathrm{Aut}_\Sigma(Y)$.

In general, there is a finite chain of blow-ups of toric T-surfaces

$$X = X_0 \to X_1 \to \cdots \to X_n = X'$$

where $(X')_{k^s}$ is G-minimal.

Now, to classify all minimal smooth projective toric surfaces over k is the same as classifying, for each finite subgroup G of $\mathrm{GL}(2, \mathbb{Z})$, G-minimal surfaces over k^s. It is well known that, when G is trivial, the minimal (toric) surfaces are $\mathbb{P}^2$ and Hirzebruch surfaces $F_a = \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$ for $a \geqslant 0, a \neq 1$.

There are 13 non-conjugate classes of finite subgroups of $\mathrm{GL}(2, \mathbb{Z})$ and they can only be either cyclic or dihedral groups. See Table 3.1 below.

Table 3.1: Non-conjugate classes of finite subgroups of $\mathrm{GL}(2, \mathbb{Z})$ and their generators

Cyclic	Dihedral	Generators
$C_1 = \langle I \rangle$	$D_2 = \langle C \rangle$ $D_2' = \langle C' \rangle$	$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$
$C_2 = \langle -I \rangle$	$D_4 = \langle -I, C \rangle$ $D_4' = \langle -I, C' \rangle$	$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
$C_3 = \langle A^2 \rangle$	$D_6 = \langle A^2, C \rangle$ $D_6' = \langle A^2, -C \rangle$	$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$C_4 = \langle B \rangle$ $C_6 = \langle A \rangle$	$D_8 = \langle B, C \rangle$ $D_{12} = \langle A, C \rangle$	$C' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

We will make use of the following simple fact from toric geometry [OM78, Proposition 6.7]:

Proposition 3.1.4. *Let Y be a split smooth projective toric surface with the fan structure Σ. Counterclockwise label (the minimal generators of) the rays of Σ as $y_1, ..., y_n$ and denote D_i the divisor corresponding to y_i, then $y_{i-1} + y_{i+1} + a_i y_i = 0$ where $a_i = D_i^2$ ($y_{n+1} = y_1$).*

Definition 3.1.5. Let Y be a split smooth projective toric surface. We can assign a sequence $a = (a_1, ..., a_n)$ to Y where a_i comes from Proposition 3.1.4. We refer this sequence as the *self-intersection sequence* of Y.

There is an induced action of $\mathrm{Aut}_\Sigma(Y)$ on $a = (a_1, ..., a_n)$ that fixes this sequence: Let $\alpha \in \mathrm{Aut}_\Sigma(Y)$ and define $\alpha(i)$ so that $\alpha(y_i) = y_{\alpha(i)}$, then $\alpha(a) = (a_{\alpha(1)}, ..., a_{\alpha(n)})$. Applying α to the relation $y_{i-1} + y_{i+1} + a_i y_i = 0$, we get $a_i = a_{\alpha(i)}$ and thus $\alpha(a) = a$.

More specifically, consider the case where $\mathrm{Aut}_\Sigma(Y) \cap \mathrm{SL}(2, \mathbb{Z}) = C_t$ is nontrivial and let us look at the action of C_t on a. As indicated in Table 3.1, the cyclic group C_t is generated by powers of A or B where B is the rotation by $\pi/4$ and A is conjugate in $\mathrm{GL}(2, \mathbb{R})$ to the rotation by $\pi/3$. In particular, the action of C_t on the fan Σ is free which implies $t \,|\, n$. Let $n = tm$ and let σ be the generator of C_t that rotates counterclockwise, then $\sigma(a) = (a_{m+1}, ..., a_m) = a$.

Lemma 3.1.6. *Let $\mathrm{Aut}_\Sigma(Y) \cap \mathrm{SL}(2, \mathbb{Z}) = C_t$ be nontrivial (i.e, $t = 2, 3, 4, 6$). If the number of rays of the fan $> \max\{4, t\}$, then Y is not C_t-minimal, that is, there exists a split smooth projective toric surface Y' such that $Y \to Y'$ is a blow-up along torus invariant points and the group of fan automorphisms of Y' contains C_t.*

Therefore, C_t-minimal surfaces have the number of rays $\leqslant \max\{4, t\}$.

Proof. Denote counterclockwise $y_1, ..., y_n$ as rays of Σ and let $a = (a_1, ..., a_n)$ be its self-intersection sequence. If $n > 4$, Y is not $\mathbb{P}^2$ or F_a, then there exists i such that $a_i = -1$. Let σ be a generator of C_t and as discussed above, σ rotates the rays. If $n > t$, then the ray $\sigma(y_i)$ is not adjacent to y_i and thus $\sigma(y_i)$ stays a (-1)-curve after Y blowing down y_i. Hence Y' can be obtained by successively blowing down $y_i, \sigma(y_i), \dots, \sigma^{t-1}(y_i)$. □

Lemma 3.1.7. *D_2 fixes rays generated by $\pm(1, 1)$ or maximal cones generated by $(1, 0)$ and $(0, 1)$ or by $(-1, 0)$ and $(0, -1)$; D_2' fixes rays generated by $\pm(1, 0)$.*

Now we are ready to classify G-minimal toric surfaces for G a finite subgroup of $\mathrm{GL}(2,\mathbb{Z})$.

Proposition 3.1.8. *Let Y be a split smooth projective toric surface and let G be a finite subgroup of* $\mathrm{GL}(2,\mathbb{Z})$*. If Y is G-minimal, then Y must be one of the following varieties:*

1. *$G = D_2$: $Y = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, F_{2a+1}, a \geqslant 1$;*
2. *$G = D_2'$: $Y = F_{2a}, a \geqslant 0$;*
3. *$G = C_2$: $Y = \mathbb{P}^1 \times \mathbb{P}^1$;*
4. *$G = D_4$ or D_4': $Y = \mathbb{P}^1 \times \mathbb{P}^1$;*
5. *$G = C_3$: $Y = \mathbb{P}^2$;*
6. *$G = D_6$: $Y = \mathbb{P}^2$;*
7. *$G = D_6'$: $Y = S$;*
8. *$G = C_4$: $Y = \mathbb{P}^1 \times \mathbb{P}^1$;*
9. *$G = D_8$: $Y = \mathbb{P}^1 \times \mathbb{P}^1$;*
10. *$G = C_6$: $Y = S$;*
11. *$G = D_{12}$: $Y = S$*

where $F_a = Proj(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$ is the Hirzebruch surface and S is the blow-up $\mathrm{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)$ *of $\mathbb{P}^2$ along three torus invariant points.*

Proof. Let Σ be the fan structure of Y.

$G = D_2$ If D_2 fixes at least one maximal cone, then $Y = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1 = F_0$. Otherwise Σ has rays $\pm(1,1)$, and the rays counterclockwise before and after $(1,1)$ must be $(a+1,a)$ and $(a,a+1)$ respectively. Y is isomorphic to F_{2a+1} in this case. For F_1, it has a G-invariant (-1)-curve so it is the blow-up of a rational point on $\mathbb{P}^2$, not minimal. So we have $a \geqslant 1$.

$G = D_2'$ Σ has rays $\pm(1,0)$, and the rays counterclockwise before and after $(1,0)$ are $(a,-1)$ and $(a,1)$ respectively. Y is isomorphic to $F_{2a}, a \geqslant 0$.

$G = C_2$ Σ should have rays $x, y, -x, -y$ and x, y form a basis of the lattice, thus $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$.

$G = D_4, D_4'$ $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$ which follows from the case $G = C_2$.

For cases $G \supseteq C_t, t > 2$, let n be the number of rays of Σ. Recall that $t \mid n$ and by Lemma 3.1.6, $n \leqslant \max\{4, t\}$.

$G = C_3$ $3 \mid n, n \leqslant 4$, so $n = 3$ and $Y \cong \mathbb{P}^2$.

$G = C_4$ $4 \mid n, n \leqslant 4$, so $n = 4$ and $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$.

$G = C_6$ $6 \mid n, n \leqslant 6$, so $n = 6$. Following from the case $G = C_3$, Y is the blow up of $\mathbb{P}^2$ along three torus invariant closed points.

$G = D_6$ Following from the case $G = C_3$, $Y \cong \mathbb{P}^2$.

$G = D_6'$ Following from the case $G = C_3$, Y is either $\mathbb{P}^2$ or the blow-ups of $\mathbb{P}^2$ along three torus invariant points. As $\mathrm{Aut}_\Sigma(\mathbb{P}^2)$ is isomorphic to the conjugacy class of D_6, Y can not be $\mathbb{P}^2$.

$G = D_8$ Following from the case $G = C_4$, $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$.

$G = D_{12}$ Following from the case $G = C_6$, Y is the blow up of $\mathbb{P}^2$ along three torus invariant closed points.

$\square$

Lemma 3.1.9. *Let X be a toric surface that is a form of $F_a, a \geqslant 1$, then X is a $\mathbb{P}^1$-bundle over a smooth conic curve. If X has a rational point, then $X \cong F_a$.*

Proof. Let X correspond to (ρ_1, Σ_1, U_1) and let Σ_1 be the fan of F_a with rays $(1,0)$, $(0,1)$, $(-1,a)$, $(0,-1)$. Let $\bar{\phi} : \mathbb{Z}^2 \to \mathbb{Z}$ be the projection to the first factor, which corresponds to $\phi : F_a \to \mathbb{P}^1$. Let $\rho_2 = \det \circ \rho_1 : \Gamma \to \mathrm{GL}(1, \mathbb{Z})$, then $\bar{\phi}$ is Galois equivariant with respect to ρ_1 and ρ_2. By Lemma 2.2.4, ϕ descends to $X \to C$. As a form of $\mathbb{P}^1$, C is a smooth plane conic curve ([GS06, Corollary 5.4.8] for characteristic not 2 and [EKM08, §45A] for any characteristic).

Either $X \cong F_a$ or ρ_1 permutes rays $(1,0),(-1,a)$. Let D' be the Cartier divisor corresponding to the ray $(0,-1)$, then it is Galois invariant in both cases. Thus, D' descends to a Cartier divisor D on X, and $X \cong \mathrm{Proj}(\mathcal{O}_C \oplus \mathcal{O}_C(D))$ is a $\mathbb{P}^1$-bundle over C. If X has a rational point, so does C. Therefore, $C \cong \mathbb{P}^1, X \cong F_a$. □

By Proposition 3.1.8, a minimal smooth projective toric surface X is a form of (i) $F_a, a \geqslant 2$; (ii) $\mathbb{P}^2$; (iii) $\mathbb{P}^1 \times \mathbb{P}^1$; (iv) $\mathrm{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)$ where p_1, p_2, p_3 are not collinear. Furthermore, we have

Theorem 3.1.10. *The surface X is a minimal smooth projective toric surface if and only if X is (i) a $\mathbb{P}^1$-bundle over a smooth conic curve; (ii) the Severi-Brauer surface; (iii) an involution surface; (iv) the del Pezzo surface of degree 6 with Picard rank 1.*

Proof. It follows from Lemma 3.1.9, Example 2.2.5, 2.2.6, 2.2.7 and the fact that a minimal del Pezzo surface of degree not equal to 8 has Picard rank 1 [CKM08, Theorem 2.4]. □

3.2 K_0 of Toric Surfaces

In this section, we will show that $K_0(X_{k^s})$ is a permutation Γ-module for X smooth projective toric surface over k. First recall how K_0 behaves under blow-ups:

Theorem 3.2.1. *[GI71, VII 3.7] Let X be a noetherian scheme and let $i : Y \to X$ be a regular closed immersion of pure codimension d. Let $p : X' \to X$ be the blow up of X along Y and $Y' = p^{-1}Y$. There is a split short exact sequence*

$$0 \to K_0(Y) \xrightarrow{u} K_0(Y') \oplus K_0(X) \xrightarrow{v} K_0(X') \to 0$$

and the splitting map w for u is given by $w(y', x) = p|_{Y'*}(y'), y' \in K(Y'), x \in K(X)$.

This gives us an isomorphism $K_0(X') \cong \ker(w) \cong K_0(X) \oplus \bigoplus^{d-1} K_0(Y)$ which fits into the split short exact sequence

$$0 \to K_0(X) \xrightarrow{p^*} K_0(X') \to \bigoplus^{d-1} K_0(Y) \to 0.$$

Now let X be a smooth projective toric T-surface over k that splits over l. Let Y be a T-invariant reduced subscheme of X of dimension 0, then Y_l is a disjoint union of T_l-invariant points fixed by $G = \mathrm{Gal}(l/k)$. Set $X' = \mathrm{Bl}_Y X$. We have

$$0 \to K_0(X_l) \xrightarrow{p^*} K_0(X'_l) \to K_0(Y_l) = \bigoplus \mathbb{Z} \to 0$$

where p^* is a G-homomorphism. Each $\mathbb{Z}$ is generated by exceptional divisors E_i corresponding to the points in Y_l and G permutes E_i the same way as G permutes the points in Y_l.

If we know $K_0(X_l)$ has a permutation G-basis γ, then $K(X'_l)$ has a permutation G-basis consisting of $p^*\gamma$ (total transforms of γ) and E_i.

Theorem 3.2.2. *Let X be a smooth projective toric T-surface over k that splits over l and $G = \mathrm{Gal}(l/k)$. Then $K_0(X_l)$ has a permutation G-basis of line bundles on X_l.*

Proof. By previous discussion and the fact that $G \subseteq \mathrm{Aut}_\Sigma$, it suffices to prove that $K_0(X_l)$ has a permutation Aut_Σ-basis of line bundles for X minimal. By Theorem 3.1.10, we only need to consider the following cases for X_l:

(i) $F_a, a \geqslant 2$, $\mathrm{Aut}_\Sigma = S_2$.

(ii) $\mathbb{P}^2$, $\mathrm{Aut}_\Sigma = D_6$.

(iii) $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathrm{Aut}_\Sigma = D_8$.

(iv) del Pezzo surface of degree 6, $\mathrm{Aut}_\Sigma = D_{12}$.

We will use Equation (2.2.2) in Theorem 2.2.8 with $f = (1,0)$ and $(0,1)$ in producing relations and finding a permutation basis. We will write x_i for rays in the fan and $J_i = \mathcal{O}(-D_i)$ where D_i is the divisor corresponding to x_i.

(i): Rays $x_1 = (1, 0)$, $x_2 = (0, 1)$, $x_3 = (-1, a)$, $x_4 = (0, -1)$. S_2 fixes x_2, x_4 and permutes x_1, x_3. Relations are:

$$\begin{cases} J_3 = J_1 \\ J_4 = J_2 J_3^a \end{cases}$$

Let x be a rational point of X_l, then the sheaf $\mathcal{O}_x = (1 - J_1)(1 - J_2)$ in K_0. A permutation basis is $1, J_1, J_2, J_1 J_2$.

(ii): Rays $x_1 = (1, 0)$, $x_2 = (0, 1)$, $x_3 = (-1, -1)$. D_6 rotates x_i and reflects along lines in x_1, x_2, x_3. Relations are $J_1 = J_2 = J_3$. A permutation basis is $1, J_1, J_1^2$.

(iii): Rays $x_1 = (1, 0)$, $x_2 = (0, 1)$, $x_3 = (-1, 0)$, $x_4 = (0, -1)$. D_8 rotates x_i and reflects along lines in $x_1, x_2, (1, 1), (-1, 1)$. Relations are:

$$\begin{cases} J_3 = J_1 \\ J_4 = J_2 \end{cases}$$

A permutation basis is $1, J_1, J_2, J_1 J_2$.

(iv): Rays $x_1 = (1, 0)$, $x_2 = (0, 1)$, $x_3 = (-1, -1)$, $y_1 = (-1, 0)$, $y_2 = (0, -1)$, $y_3 = (1, 1)$. $D_{12} \cong S_2 \times S_3$ (S_2, S_3 permutation groups). $S_2 = \langle -1 \rangle$ switches between x_i and y_i. S_3 permutes the pair of rays (x_i, y_i). Let J_i' correspond to y_i. Relations are

$$\frac{J_1}{J_1'} = \frac{J_2}{J_2'} = \frac{J_3}{J_3'}$$

As proved in [Blu10, Theorem 4.2], we have a permutation basis $1, P_1, P_2, P_3, Q_1, Q_2$ where

$$\begin{cases} P_1 = J_1 J_2' \\ P_2 = J_2 J_3' \\ P_3 = J_3 J_1' \\ Q_1 = J_1 J_2 J_3' \\ Q_2 = J_1' J_2' J_3 \end{cases}$$

□

Remark 3.2.3. The difficulties to generalize Theorem 3.2.2 to higher dimensions are:

(1) The classification of conjugacy classes of finite subgroups of $\mathrm{GL}(n, \mathbb{Z})$ is difficult and not complete which often only provides algorithms and requires the help of computer even for small n. Also, the number of those finite subgroups grows very fast as n increases. For example, there are total of 73 for $\mathrm{GL}(3, \mathbb{Z})$ and 710 for $\mathrm{GL}(4, \mathbb{Z})$.

(2) The K-group $K_0(X_l)$ in question may not stay a permutation module after blow-ups if X is not a surface.

CHAPTER 4

Construction of Separable Algebras

4.1 Construction of Separable Algebras

Let X be a smooth projective toric T-variety over k that splits over l, and X^* be its associated toric model, cf §2.2. [MP97, Theorem 5.7] states that there is a split monomorphism $u : X^* \to A$ in the motivic category $\mathcal{C}$ from X^* to an étale k-algebra A and u is represented by an element Q in $\operatorname{Pic}(X^* \otimes_k A)$. We can construct $u' : X \to B$ out of u. [MP97, Theorem 7.6] states that u' is also a split monomorphism in $\mathcal{C}$. In this section, we will recall the construction of u' and consider the case when u is an isomorphism.

Write $X_A = X \otimes_k A$ and we have $f : X_l \to X_l^*$, a T_l-isomorphism. Consider the diagram:

$$\begin{array}{ccc} X_{A\otimes_k l} & \xrightarrow{f_A} & X^*_{A\otimes_k l} \\ \Big\downarrow{\scriptstyle \pi_{X_A}} & & \Big\downarrow{\scriptstyle \pi_{X_A^*}} \\ X_A & & X_A^* \end{array} \tag{4.1.1}$$

Let $P' = f^*(\pi^*_{X_A^*}(Q))$, then $B = \operatorname{End}_{X_A}(\pi_{X_A *}(P')) \in \operatorname{Br}(A)$ and $u' : X \to B$ is represented by $\pi_{X_A *}(P')$, namely, $u' = \phi_*(P') \in K_0(X, B)$ where ϕ is the projection $X_{A\otimes_k l} \to X$.

The following criterion, which is [MP97, Proposition 4.5], checks when a toric model is isomorphic to an étale algebra in $\mathcal{C}$:

Proposition 4.1.1. *Let X^* be a smooth projective toric model over k that splits over l and $G = \operatorname{Gal}(l/k)$. Assume that $K_0(X_l^*)$ is a permutation G-module, then $X^* \cong \operatorname{Hom}_G(P, l)$ in the motivic category $\mathcal{C}$ for any permutation G-basis P of $K_0(X_l^*)$.*

Remark 4.1.2. In particular, this implies that for any split smooth projective toric variety Y over k, $Y \cong k^n$ in $\mathcal{C}$ where n equals to the rank of $K_0(Y)$ (also equals to the number of maximal cones of the fan). Note that a smooth projective toric variety Y over k where the fan of Y_l has no symmetry is automatically split.

Lemma 4.1.3. *Let X^*, G be the same as before, then there is an isomorphism $u : X^* \to A$ in $\mathcal{C}$ where A is an étale k-algebra and u is represented by an element $Q \in \mathrm{Pic}(X^*_A)$ if and only if $K_0(X^*_l)$ has a permutation G-basis of line bundles on X^*_l.*

Proof. $\Rightarrow$: decompose A as $\prod_{i=1}^t k_i$ where k_i is a finite separable field extension of k, then $X^*_A = \coprod_{i=1}^t X^*_{k_i}$ is the disjoint union of $X^*_{k_i}$ and $Q = \coprod_{i=1}^t Q_i$ where Q_i is a line bundle on $X^*_{k_i}$. Let $q_i : X^*_{k_i} \to X^*$ be the projections, then $u = \bigoplus_{i=1}^t q_{i*}Q_i$. Let $p_i : X^*_{k^s} \to X^*_{k_i}$ be the projections and $G_i = \mathrm{Gal}(k_i/k)$, then $u_{k_s} = \bigoplus_{i=1}^t p_i^* q_i^* q_{i*}(Q_i) = \bigoplus_{i=1}^t \bigoplus_{g \in G_i} p_i^*(gQ_i)$ and $A_{k^s} \cong (k^s)^n$ where $n = \sum_{i=1}^t |G_i|$. View u as $u^{\mathrm{op}} : A^{\mathrm{op}} = A \to X^*$, then $u^{\mathrm{op}}_{k^s}$ induces an isomorphism $K_0((k^s)^n) \to K_0(X^*_{k^s})$ where the canonical basis of the former sends to $\{p_i^*(gQ_i) \,|\, g \in G_i, 1 \leqslant i \leqslant t\}$ and this set gives a permutation Γ-basis of $K_0(X^*_{k^s})$ of line bundles. As $\mathrm{Gal}(k^s/l)$ acts trivially on $K_0(X^*_{k^s})$, this basis descends to X^*_l.

$\Leftarrow$: Assume P is a permutation G-basis of $K_0(X^*_l)$ of line bundles on X^*_l and P divides into t G-orbits. Let $\{P_i \,|\, 1 \leqslant i \leqslant t\}$ be the set of representatives of G-orbits, and $\mathrm{Gal}(l/k_i)$ be the stabilizer of P_i. Set $A = \mathrm{Hom}_G(P, l)$, then $A \cong \prod_{i=1}^t k_i$. As X^* has a rational point, by [CKM08, Proposition 5.1], $P_i \in \mathrm{Pic}(X^*_l)^{\mathrm{Gal}(l/k_i)} \cong \mathrm{Pic}(X^*_{k_i})$, namely $P_i \cong p_i^*(Q_i)$ for some $Q_i \in \mathrm{Pic}(X^*_{k_i})$ where $p_i : X^*_l \to X^*_{k_i}$ is the projection. There is a morphism $u : X^* \to A$ which is represented by $\coprod_{i=1}^t Q_i \in \mathrm{Pic}(X^*_A)$, and by construction, u_l induces an isomorphism $K_0(X^*_l) \cong K_0(A_l)$. Using the following lemma, we have u is an isomorphism. $\square$

Lemma 4.1.4. *Let X^* be the same as before and A is an étale k-algebra. If $u : X^* \to A$ is a morphism in $\mathcal{C}$ such that $K_0(u_{k^s}) : K_0(X^*_{k^s}) \to K_0(A_{k^s})$ is an isomorphism, then so is u.*

Proof. There is a commutative diagram:

$$\begin{array}{ccc} K_0(X^*) & \xrightarrow{K_0(u)} & K_0(A) \\ \downarrow & & \downarrow \\ K_0(X^*_{k^s})^\Gamma & \xrightarrow{K_0(u_{k^s})} & K_0(A_{k^s})^\Gamma \end{array}$$

The right vertical map is an isomorphism as A is étale and so is $K_0(u_{k^s})$ by assumption. The left vertical map is an isomorphism by [MP97, Corollary 5.8]. Thus, $K_0(u)$ is also an isomorphism.

Write $w = u^{\mathrm{op}} : A \to X^*$, then by the splitting principle [MP97, Proposition 6.1 and the proof], $K_0^{X^*}(w) : K_0(X^*, A) \to K_0(X^* \times X^*)$ is surjective. Thus, there exists $v \in K_0(X^*, A) : X^* \to A$ such that $w \circ v = K_0^{X^*}(w)(v) = 1_{X^*}$, and then $K_0(w \circ v) = K_0(w)K_0(v) = 1_{K_0(X^*)}$. Since $K_0(w) = \phi$ is an isomorphism, $K_0(v) = \phi^{-1}$ and $K_0(v \circ w) = K_0(v)K_0(w) = 1_{K_0(A)}$. This implies $v \circ w = 1_A$ and thus v is a two sided inverse of w in $\mathcal{C}$. □

The proof of (3) $\Leftrightarrow$ (4) in [MP97, Proposition 7.9] shows that the T_l-isomorphism $f : X_l \to X_l^*$ induces a $G = \mathrm{Gal}(l/k)$-module isomorphism $f^* : K_0(X_l^*) \to K_0(X_l)$. Thus, $K_0(X_l^*)$ has a permutation G-basis of line bundles on X_l^* if and only if $K_0(X_l)$ has such a basis. Note that the proof (1) $\Rightarrow$ (2) (an isomorphism $u : X^* \to A$ gives an isomorphism $u' : X \to B$), which uses the construction (4.1.1) recalled at the beginning of the section, works only when u is represented by an element $Q \in \mathrm{Pic}(X_A^*)$. Thus, we have the following instead:

Theorem 4.1.5. *Let X be a smooth projective toric T-variety over k that splits over l and $G = \mathrm{Gal}(l/k)$. Assume $K_0(X_l)$ has a permutation G-basis P of line bundles on X_l. Let $\{P_i \mid 1 \leqslant i \leqslant t\}$ be G-orbits of P, and let $\pi : X_l \to X$ be the projection. For any $S_i \in P_i$, set $B_i = \mathrm{End}_{\mathcal{O}_Y}(\pi_*(S_i))$ and $B = \prod_{i=1}^t B_i$, then the map $u = \bigoplus_{i=1}^t \pi_*(S_i) : X \to B$ gives an isomorphism in the motivic category $\mathcal{C}$.*

Proof. By Lemma 4.1.3, we have an isomorphism $u : X^* \to A$ represented by $Q \in \mathrm{Pic}(X_A^*)$. Here $A \cong \prod_{i=1}^t k_i$ where $\mathrm{Gal}(l/k_i)$ is the stabilizer of S_i under G-action. $Q = \coprod_{i=1}^t Q_i$ and $Q_i \in \mathrm{Pic}(X_{k_i}^*)$ descends from $(f^*)^{-1}(S_i) \in \mathrm{Pic}(X_l^*)^{\mathrm{Gal}(l/k_i)}$. Now we run the construction (4.1.1) for Q_i:

$$\begin{tikzcd} X_{k_i \otimes_k l} \arrow[r, "f_i"] \arrow[d, "\pi_X"] & X^*_{k_i \otimes_k l} \arrow[d, "\pi_{X^*}"] \\ X_{k_i} & X^*_{k_i} \end{tikzcd}$$

Let $p : X_l \to X_{k_i}$ and $q : X_{k_i} \to X$ be the projections, then $\pi_{X*} f_i^* \pi_{X^*}^*(Q_i) \cong p_*(S_i) \otimes_k k_i$ where its $\mathcal{O}_{X_{k_i}}$-module structure comes from the one on $p_*(S_i)$. Thus,

$$\mathrm{End}_{\mathcal{O}_{X_{k_i}}}(\pi_{X*} f_i^* \pi_{X^*}^*(Q_i)) \cong \mathrm{End}_{\mathcal{O}_{X_{k_i}}}(p_*(S_i)) \otimes_k \mathrm{End}_k(k_i)$$

is Brauer equivalent to $B_i' = \mathrm{End}_{\mathcal{O}_{X_{k_i}}}(p_* S_i)$. It remains to prove that $B_i \cong B_i'$. There is a G-isomorphism:

$$\begin{aligned} B_i \otimes_k l &\cong \mathrm{End}_{\mathcal{O}_{X_l}}(\pi^* \pi_*(S_i)) \cong \mathrm{End}_{\mathcal{O}_{X_l}}(p^* q^* q_* p_*(S_i)) \\ &\cong \mathrm{End}_{\mathcal{O}_{X_l}}(p^* p_*(S_i) \otimes_k k_i) \cong \mathrm{End}_{\mathcal{O}_{X_l}}(p^* p_*(S_i)) \otimes_k k_i \\ &\cong (B_i' \otimes_{k_i} l) \otimes_k k_i \cong B_i' \otimes_k l. \end{aligned}$$

The fourth isomorphism follows from Lemma 4.1.6. Take G-invariants on both sides, we have $B_i \cong B_i'$. □

Lemma 4.1.6. *Let X be a proper variety over k and assume that there is a finite group G acting on Cartier divisors* $\mathrm{CDiv}(X)$. *Let $D \in \mathrm{CDiv}(X)$ and $g \in G$ such that D and gD are not linearly equivalent, then* $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X(gD)) = 0$.

Proof. Assume that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X(gD)) \neq 0$, which is equivalent to $\mathcal{O}_X(gD - D)$ has a nonzero global section s. Since G is a finite group, $g^n = 1$ for some n. Therefore, $\mathcal{O}_X(D - gD) = (g^{n-1} \otimes \cdots \otimes g \otimes 1)\mathcal{O}_X(gD - D)$ has a nonzero global section $t = g^{n-1}s \otimes \cdots \otimes s$. View s, t as $s : \mathcal{O}_X(D) \to \mathcal{O}_X(gD)$ and $t : \mathcal{O}_X(gD) \to \mathcal{O}_X(D)$. Since $st, ts \in \Gamma(X, O_X) = k$ are nonzero, $\mathcal{O}(gD - D) \cong \mathcal{O}_X$, contradiction. □

Remark 4.1.7. There is a more "economical" description of an algebra isomorphic to X in $\mathcal{C}$:

Write $S_i = \mathcal{O}(-D_i)$ where D_i is torus invariant. Let $\mathrm{Gal}(l/l_i)$ be the stabilizer of D_i under G-action and let $\pi_i : Y_{l_i} \to Y$ be the projections. D_i and thus S_i descend to Y_{l_i}, and we use the same notation. Then $Y \cong \prod_{i=1}^t \mathrm{End}_{\mathcal{O}_X}(\pi_{i*}(S_i))$. In effect, it replaces all $\mathrm{M}_n(k)$ in B constructed in the theorem by k which is an isomorphism in $\mathcal{C}$.

Remark 4.1.8. A question remains: If $K_0(X_l)$ is a permutation G-module, can we always find a permutation G-basis of line bundles?

Recall that for $n \geqslant 0$, K_n defines a functor $K_n : \mathcal{C} \to Ab$, hence we have

Corollary 4.1.9. $K_n(X) \cong \prod_{i=1}^{t} K_n(B_i)$.

4.2 Separable Algebras for Toric Surfaces

4.2.1 Separable algebras for minimal toric surfaces

Recall the families of minimal toric surfaces described in Theorem 3.2.2: Let X be a minimal smooth projective toric T-surface over k that splits over l, and X^* be its associated toric model. Let $\pi : X_l \to X$ be the projection. All isomorphisms below are taken in the motivic category $\mathcal{C}$.

(i) $X_l \cong F_a, a \geqslant 2$. $X^* \cong k^4$ and $X \cong k \times Q \times k \times Q$ where $Q \cong \mathrm{End}_{\mathcal{O}_X}(\pi_* J_1)$ is a quaternion k-algebra.

(ii) More generally, let $X = \mathrm{SB}(A)$ be a Severi-Brauer variety of dimension n and $J = \mathcal{O}_{X_l}(-1)$. $X^* \cong k^{n+1}$ and $X \cong k \times \prod_{i=1}^{n} A^{\otimes i}$ where $A^{\otimes i} \cong \mathrm{End}_{\mathcal{O}_X}(\pi_* J^i)$, cf Example 2.2.5.

(iii) $X_l \cong \mathbb{P}^1 \times \mathbb{P}^1$. $X^* \cong k \times K \times k$ where K is a quadratic étale algebra and the discriminant extension of X. $X \cong k \times B \times A$ where $B \cong \mathrm{End}_{\mathcal{O}_X}(\pi_* J_1)$ is an Azumaya K-algebra of rank 4 and $A \cong \mathrm{End}_{\mathcal{O}_X}(\pi_*(J_1 J_2))$ is a central simple k-algebra of degree 4, cf Example 2.2.7.

(iv) See Example 2.2.6 where $X^* \cong k \times K \times L$ and $P \cong \mathrm{End}_{\mathcal{O}_X}(\pi_* P_1)$ and $Q \cong \mathrm{End}_{\mathcal{O}_X}(\pi_* Q_1)$.

Now let X be a smooth projective toric T-variety over k that splits over l and $G = \mathrm{Gal}(l/k)$. Recall that X is uniquely determined by the associated toric model X^*, which corresponds to $\rho : \Gamma \to \mathrm{GL}(n, \mathbb{Z})$ and fan Σ such that $\rho(\Gamma) \subseteq \mathrm{Aut}_\Sigma$, and a principal homogeneous space $U \in H^1(k, T)$. Every variety within a family above has the same fan. Let $\rho' : G \hookrightarrow \mathrm{Aut}_\Sigma(X_l)$ be the inclusion induced by ρ. We want to see how the separable algebras described above relate to ρ' and U.

Let $\dim X = n$ and N be the number of rays in the fan Σ, then the Picard rank of X_l is $m = N - n$. Write M for the group of characters of T_l and CDiv_{T_l} for T_l-invariant Cartier divisors. There is a natural action of $\mathrm{Aut}_\Sigma(X_l)$ on M and $\mathrm{CDiv}_{T_l}(X_l)$ and an induced action on $\mathrm{Pic}(X_l)$ via the canonical morphism $\mathrm{CDiv}_{T_l}(X_l) \to \mathrm{Pic}(X_l)(D \mapsto \mathcal{O}_{X_l}(D))$.

We have a short exact sequence of $\mathrm{Aut}_\Sigma(X_l)$-modules and therefore of G-modules via ρ':

$$0 \to M \to \mathrm{CDiv}_{T_l}(X_l) \to \mathrm{Pic}(X_l) \to 0, \tag{4.2.1}$$

or simply

$$0 \to \mathbb{Z}^n \to \mathbb{Z}^N \to \mathbb{Z}^m \to 0.$$

It corresponds to the short exact sequence of tori over l:

$$1 \to \mathbb{G}_{m,l}^m \to \mathbb{G}_{m,l}^N \to \mathbb{G}_{m,l}^n \to 1$$

and the sequence descends to

$$1 \to S \to V \to T \to 1. \tag{4.2.2}$$

Let $i : \mathrm{Aut}_\Sigma(X_l) \hookrightarrow S_N$ where S_N is the group of permutations of the canonical $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^N$ and it induces $i_* : H^1(G, \mathrm{Aut}_\Sigma) \to H^1(G, S_N)$. Let $[\alpha] = i_*[\rho']$ and E be the corresponding étale k-algebra of degree N, then $V = \mathrm{R}_{E/k}(\mathbb{G}_{m,E})$. Let $j : \mathrm{Aut}_\Sigma(X_l) \to \mathrm{GL}(m, \mathbb{Z})$ be the map induced by the action of $\mathrm{Aut}_\Sigma(X_l)$ on $\mathrm{Pic}(X_l)$ and it induces $j_* : H^1(G, \mathrm{Aut}_\Sigma) \to H^1(G, \mathrm{GL}(m, \mathbb{Z}))$. Let $[\beta] = j_*[\rho']$, then S is the torus corresponding to $[\beta]$.

The short exact sequence of tori over k gives

$$0 \to H^1(G, T) \xrightarrow{\delta} H^2(G, S) \to \mathrm{Br}(E).$$

Here $H^1(G, V) = H^1(G, \mathrm{R}_{E/k}(\mathbb{G}_{m,E})) = \prod H^1(\mathrm{Gal}(E_t/k), \mathbb{G}_{m,E_t}) = 0$ by Hilbert 90 Theorem where $E = \prod E_t$ and the E_t are finite separable field extensions of k.

Let $S^* = \mathrm{Hom}(S_l, G_{m,l})$ be the group of characters over l. Then sequence (4.2.1) can be rewritten as

$$0 \to T^* \to V^* \to S^* \to 0$$

which induces $H^0(G, S^*) \xrightarrow{\partial} H^1(G, T^*)$. Geometrically, ∂ is the map $\mathrm{Pic}(X^*) \to \mathrm{Pic}(T)$ which sends $Q \in \mathrm{Pic}(X^*)$ to its restriction $Q|_T$ on T.

There is a G-equivariant bilinear map $S(l) \otimes S^* \to l^\times$ which sends $x \otimes \chi$ to $\chi(x)$, and it induces a pairing of Galois cohomology groups $\cup : H^2(G, S) \otimes H^0(G, S^*) \to \mathrm{Br}(k)$. Similarly, we have $\cup : H^1(G, T) \otimes H^1(G, T^*) \to \mathrm{Br}(k)$.

Lemma 4.2.1. *The following diagram is commutative:*

$$\begin{array}{ccc} H^1(G,T) \otimes H^0(G,S^*) & \xrightarrow{1\otimes\partial} & H^1(G,T) \otimes H^1(G,T^*) \\ \downarrow{\scriptstyle \delta\otimes 1} & & \downarrow{\scriptstyle \cup} \\ H^2(G,S) \otimes H^0(G,S^*) & \xrightarrow{\quad\cup\quad} & \mathrm{Br}(k) \end{array}$$

Proof. Let $a \in H^1(G, T), \varphi \in H^0(G, S^*)$. For each $a_g \in T(l), g \in G$, pick $b_g \in V(l)$ that maps to a_g, then $(\delta a)_{g,h} = b_{gh}^{-1} b_g {}^g b_h, g, h \in G$. Pick $\phi \in V^*$ that maps to φ, then $(\partial\varphi)_g = \phi^{-1 g}\phi$. Let $\alpha = a \cup (\partial\varphi)$ and $\beta = (\delta a) \cup \varphi$, then

$$\alpha_{g,h} = {}^g(\partial\varphi)_h(a_g) = {}^g(\phi^{-1h}\phi)(b_g) = ({}^g\phi^{-1})(b_g) \cdot ({}^{gh}\phi)(b_g),$$

$$\beta_{g,h} = ({}^{gh}\varphi)((\delta a)_{g,h}) = ({}^{gh}\phi)(b_{gh}^{-1}) \cdot ({}^{gh}\phi)(b_g) \cdot ({}^{gh}\phi)({}^g b_h).$$

Set $\theta_g = ({}^g\phi)(b_g)$, then $\beta_{g,h} = \theta_{gh}^{-1}\theta_g {}^g\theta_h \alpha_{g,h}$. Thus, α and β give the same cycle class in $\mathrm{Br}(k)$. $\square$

Let $P \in \mathrm{Pic}(X_l)$ be a line bundle on X_l with stabilizer group $\mathrm{Gal}(l/\kappa)$ under G-action. Since $P \in \mathrm{Pic}(X_l)^{\mathrm{Gal}(l/\kappa)} \cong (S^*)^{\mathrm{Gal}(l/\kappa)}$, P corresponds to a character $\chi : S_\kappa \to \mathbb{G}_{m,\kappa}$ over κ, or equivalently $\chi' : S \to \mathrm{R}_{\kappa/k}(\mathbb{G}_{m,\kappa})$. Let $\pi : X_l \to X$ be the projection.

Proposition 4.2.2. *Consider the composition of maps $\delta_P : H^1(G, T) \xrightarrow{\delta} H^2(G, S) \xrightarrow{\chi'} \mathrm{Br}(\kappa)$, then $\delta_P[U] = [\mathrm{End}_{\mathcal{O}_X}(\pi_* P)] \in \mathrm{Br}(\kappa)$.*

Proof. First we prove the case when $\kappa = k$. In this case, the line bundle $P \in \mathrm{Pic}(X_l)^G \cong \mathrm{Pic}(X^*)$. Thus, there is $Q \in \mathrm{Pic}(X^*)$ such that $P \cong f^*\pi_{X^*}^* Q$ where $\pi_{X^*} : X_l^* \to X^*$ is the projection and $f : X_l \to X_l^*$ is the T_l-isomorphism. [MP97, Lemma 7.3] shows that

$[U] \cup [Q|_T] = [\mathrm{End}_{\mathcal{O}_X}(\pi_* P)] \in \mathrm{Br}(k)$. On the other hand, $\delta_P([U]) = \delta[U] \cup [\chi'] = \delta[U] \cup [Q]$. By Lemma 4.2.1, $\delta_P([U]) = [U] \cup [\partial Q] = [U] \cup [Q|_T]$.

In general, let $H = \mathrm{Gal}(l/\kappa)$ and consider the restriction map $\mathrm{Res} : H^1(G,T) \to H^1(H, T_\kappa)$ which sends $[U]$ to $[U_\kappa]$. There is a commutative diagram:

$$\begin{tikzcd} H^1(G,T) \arrow[r, "\delta"] \arrow[d, "\mathrm{Res}"] & H^2(G,S) \arrow[r, "\chi'"] \arrow[d, "\mathrm{Res}"] & \mathrm{Br}(\kappa) \arrow[d, equal] \\ H^1(H,T_\kappa) \arrow[r, "\delta"] & H^2(H,S_\kappa) \arrow[r, "\chi"] & \mathrm{Br}(\kappa) \end{tikzcd}$$

Thus, $\delta_P[U] = [\mathrm{End}_{\mathcal{O}_{X_\kappa}}(\pi_{\kappa *} P)]$ where $\pi_\kappa : X_l \to X_\kappa$ is the projection. By the proof of Lemma 4.1.3, $\mathrm{End}_{\mathcal{O}_{X_\kappa}}(\pi_{\kappa *} P) \cong \mathrm{End}_{\mathcal{O}_X}(\pi_* P)$. □

Corollary 4.2.3. *Let X be a smooth projective toric variety over k that splits over l and $G = \mathrm{Gal}(l/k)$. Assume $\mathrm{Pic}(X_l)$ is a permutation G-module, i.e, S is quasi-trivial, hence S has the form $\prod_{i=1}^t \mathrm{R}_{k_i/k}\,\mathbb{G}_{m,k_i}$ where k_i is a finite separable field extension of k. Then the principal homogeneous space U is uniquely determined by $(B_i \in \mathrm{Br}(k_i))_{1 \leqslant i \leqslant t}$ where B_i splits over E. Let $\{P_i \,|\, 1 \leqslant i \leqslant t\}$ be the set of representatives for G-orbits of $\mathrm{Pic}(X_l)$, then B_i comes from $\mathrm{End}_{\mathcal{O}_X}(\pi_* P_i)$.*

Proof. The result follows from the exact sequence $0 \to H^1(k,T) \to \prod_{i=1}^t \mathrm{Br}(k_i) \to \mathrm{Br}(E)$ and Proposition 4.2.2. □

Remark 4.2.4. Families (i)(ii)(iii) and their blow-ups have permutation Picard groups.

(ii): $X = \mathrm{SB}(A)$ is a Severi-Brauer variety of dimension n, $\mathrm{Aut}_\Sigma(X_l) = S_{n+1}$. We have

$$1 \to \mathbb{G}_{m,k} \to \mathrm{R}_{E/k}(\mathbb{G}_{m,E}) \to T \to 1$$

which induces

$$0 \to H^1(G,T) \xrightarrow{\delta} \mathrm{Br}(k) \to \mathrm{Br}(E).$$

Then $\delta(U) = [A]$ and A splits over E, cf [MP97, Example 8.5].

(i): $X_l = F_a, a \geqslant 2, \mathrm{Aut}_\Sigma = S_2$ and E factors as $k \times F \times k$ where F is the quadratic étale k-algebra corresponding to $[\rho'] \in H^1(G, S_2)$. We have

$$1 \to \mathbb{G}_{m,k} \to \mathbb{G}_{m,k} \times \mathrm{R}_{F/k}(\mathbb{G}_{m,F}) \to T \to 1$$

where $\mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ is the a-th power homomorphism. It induces

$$0 \to H^1(G,T) \xrightarrow{\delta} \mathrm{Br}(k) \to \mathrm{Br}(k) \times \mathrm{Br}(F)$$

where $[U] \mapsto [Q] \mapsto ([Q^{\otimes a}], [Q_F])$. By Lemma 3.1.9, the toric surface X is a $\mathbb{P}^1$-bundle over some conic curve Z. We have the torus of Z is $T' = \mathrm{R}_{F/k}(\mathbb{G}_{m,F})/\mathbb{G}_{m,k}$. There is a commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_{m,k} \arrow[r] \arrow[d, equal] & \mathbb{G}_{m,k} \times \mathrm{R}_{F/k}(\mathbb{G}_{m,F}) \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, "h"] & 1 \\
1 \arrow[r] & \mathbb{G}_{m,k} \arrow[r] & \mathrm{R}_{F/k}(\mathbb{G}_{m,F}) \arrow[r] & T' \arrow[r] & 1
\end{tikzcd}$$

Hence, the image of $[U]$ under $\delta \circ h_* : H^1(G,T) \to H^1(G,T') \to \mathrm{Br}(k)$ is $[Q]$, and then $Z = \mathrm{SB}(Q)$. Since a quaternion algebra has a period at most 2 in the Brauer group, $[Q^{\otimes a}] \in \mathrm{Br}(k)$ is trivial implies that $Q = \mathrm{M}_2(k)$ if a is odd. Thus we have

Proposition 4.2.5. *Let X be a toric surface that is a form of F_{2a+1}, then $X \cong F_{2a+1}$.*

Remark 4.2.6. Iskovskih showed that any form of F_{2a+1} is trivial [Isk79, Theorem 3(2)]. The above proposition reproves this result in the case of toric surfaces.

(iii): $X_l = \mathbb{P}^1 \times \mathbb{P}^1, \mathrm{Aut}_\Sigma = D_8$. In this case, $\beta : G \to \mathrm{GL}(2,\mathbb{Z})$ factors through $\gamma : G \to S_2$ where S_2 permutes $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$. Then the quadratic étale algebra K corresponds to γ. We have

$$1 \to \mathrm{R}_{K/k}(\mathbb{G}_{m,K}) \to \mathrm{R}_{E/k}(\mathbb{G}_{m,E}) \to T \to 1$$

which induces

$$0 \to H^1(G,T) \xrightarrow{\delta} \mathrm{Br}(K) \to \mathrm{Br}(E)$$

Then $\delta(U) = [B]$ and B splits over E. Let $\mathrm{N}_{K/k} : \mathrm{R}_{K/k}(\mathbb{G}_{m,K}) \to \mathbb{G}_{m,k}$ be the norm map which induces $\mathrm{cor}_{K/k} : \mathrm{Br}(K) \to \mathrm{Br}(k)$, then $[A] = \mathrm{cor}_{K/k}[B]$.

4.2.2 Separable algebras for toric surfaces

Let X be a smooth projective toric T-surface over k that splits over l and $G = \mathrm{Gal}(l/k)$. Recall that we have a finite chain of blow-ups of toric T-surfaces

$$X = X_0 \to X_1 \to \cdots \to X_n = X'$$

where X' is minimal. For $1 \leqslant i \leqslant n$, let $f_i : (X_{i-1})_l \to (X_i)_l$ and this is a blow-up of a G-set of T_l-invariant points. Let E_i be the G-set of the exceptional divisors of f_i and $X' \cong B$ in $\mathcal{C}$.

Proposition 4.2.7. *$X \cong B \times \prod_{i=1}^{n} \mathrm{Hom}_G(E_i, l)$ in $\mathcal{C}$.*

Proof. We only need to consider the following simple case:

$f : Y \to Z$ is a blow-up of toric T-surfaces and $E = \{P_j\}$ is the G-set of the exceptional divisors of $g = f_l$. We assume further that the G-action on E is transitive.

Let $p : Y_l \to Y$ and $q : Z_l \to Z$ be the projections, then we have a commutative diagram:

$$\begin{tikzcd} Y_l \arrow[r, "g"] \arrow[d, "p"] & Z_l \arrow[d, "q"] \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

Recall that if $K_0(Z_l)$ has a G-basis γ, then $g^*(\gamma) \cup E$ is a G-basis of $K_0(Y_l)$. As Z is a toric surface, we can assume γ consists of line bundles over Z_l. Let $P \in \gamma$, then

$$\mathrm{End}_{\mathcal{O}_Y}(p_* g^* P) \cong \mathrm{End}_{\mathcal{O}_Y}(f^* q_* P) \cong \mathrm{Hom}_{\mathcal{O}_Z}(q_* P, f_* f^*(q_* P)) \cong \mathrm{End}_{\mathcal{O}_Z}(q_* P)$$

where $f_* f^*$ is identity as f is flat proper and $f_* \mathcal{O}_Y = \mathcal{O}_Z$.

As for G-orbit E, we have $\bigoplus_j P_j = p^* Q$ for some locally free sheaf Q on Y. By Lemma 4.1.6 and the assumption that G acts transitively on E, $\mathrm{End}_{\mathcal{O}_Y}(Q) \cong \mathrm{Hom}_G(E, l)$. It is Brauer equivalent to $\mathrm{End}_{\mathcal{O}_Y}(p_* P_j)$ for any $P_j \in E$. Thus the result follows from Theorem 4.1.5. $\square$

CHAPTER 5

Derived Categories of Toric Varieties

5.1 Derived categories of toric varieties

Let X be a smooth projective variety over k and $D^b(X)$ be the bounded derived category of coherent sheaves on X. We will define exceptional objects and collections in a generalized way.

Definition 5.1.1. Let A be a finite simple k-algebra (i,e a central simple algebra where the center is a finite separable field extension of k). An object V in $D = D^b(X)$ is called *A-exceptional* if $\mathrm{Hom}_D(V, V) = A$ and $\mathrm{Ext}^i_D(V, V) = 0$ for $i \neq 0$.

Definition 5.1.2. A set of objects $\{V_1, \ldots, V_n\}$ in $D = D^b(X)$ is called an *exceptional collection* if for each $1 \leqslant i \leqslant n$, V_i is A_i-exceptional for some finite simple k-algebra A_i, and $\mathrm{Ext}^r_D(V_i, V_j) = 0$ for any integer r and $i > j$. The collection is *full* if the thick triangulated subcategory $\langle V_1, \ldots, V_n \rangle$ generated by the V_i is equivalent to $D^b(X)$.

Definition 5.1.3. A set of objects $\{V_1, \ldots, V_n\}$ in $D \in D^b(X)$ is called an *exceptional block* if it is an exceptional collection and $\mathrm{Ext}^r_D(V_i, V_j) = 0$ for any integer r and $i \neq j$. Note that the ordering of the V_i in this case does not matter.

Assume $\{V_1, \ldots, V_n\}$ is a full exceptional collection as above. Since $\langle V_i \rangle$ is equivalent to $D^b(A_i)$, the bounded derived category of right A_i-modules, we have semiorthogonal decompositions $D^b(X) = \langle V_1, \ldots, V_n \rangle = \langle D^b(A_1), \ldots, D^b(A_n) \rangle$.

The semiorthogonal decomposition of $D^b(X)$ can be lifted to the world of dg categories. For details about dg categories, see [Kel06]. There is a dg enhancement of $D^b(X)$, denoted as $D^b_{dg}(X)$ where $D^b_{dg}(X)$ is the dg category with same objects as $D^b(X)$ and whose morphism has a dg k-module structure such that $H^0(\mathrm{Hom}_{D^b_{dg}(X)}(x,y)) = \mathrm{Hom}_{D^b(X)}(x,y)$. $\mathrm{perf}_{dg}(X)$ is the dg subcategory of perfect complexes. Since X is smooth projective, $\mathrm{perf}_{dg}(X)$ is quasi-equivalent to $D^b_{dg}(X)$. For an A-exceptional object V, the pretriangulated dg subcategory $\langle V\rangle_{dg}$ generated by V is quasi-equivalent to $D^b_{dg}(A)$. Therefore, there is a dg enhancement of the semiorthogonal decomposition $D^b_{dg}(X) = \langle V_1, \ldots, V_n\rangle_{dg}$, which is quasi-equivalent to $\langle D^b_{dg}(A_1), \ldots, D^b_{dg}(A_n)\rangle_{dg}$.

Let *dgcat* be the category of all small dg categories, there is a universal additive functor $U : dgcat \to Hmo_0$ where Hmo_0 is the category of noncommutative motives, see [Tab15, §2.1-2.4]. We have $U(\mathrm{perf}_{dg}(X)) \simeq \bigoplus_{i=1}^n U(D^b_{dg}(A_i)) \simeq \bigoplus_{i=1}^n U(A_i)$. On the other hand, the motivic category $\mathcal{C}$ is a full subcategory of Hmo_0 by sending a pair (X, A) to $\mathrm{perf}_{dg}(X, A)$, the dg category of complexes of right $\mathcal{O}_X \otimes_k A$-modules which are also perfect complexes of $\mathcal{O}_X$-modules [Tab14, Theorem 6.10] or [Tab15, Theorem 4.17]. The above discussion gives the following well-known fact:

Theorem 5.1.4. *Let X be a smooth projective variety over k. Assume $D^b(X)$ has a full exceptional collection of objects $\{V_1, \ldots, V_n\}$ where V_i is A_i-exceptional, then $X \cong \prod_{i=1}^n A_i$ in the motivic category $\mathcal{C}$.*

Now we explore the existence of a full exceptional collection for a smooth projective toric variety of higher dimension.

Lemma 5.1.5. *Let Y be a split smooth projective toric variety over k of dimension $n \geqslant 3$ and let D be a Cartier divisor on Y, then $H^i(Y, \mathcal{O}(D)) = 0$ for $0 < i < n$.*

Proof. Let Σ be the fan structure of Y, then the support $|\Sigma| = \mathbb{R}^n$. Let φ be the support function corresponding to D. [Dan78, Theorem 7.2] states that

$$H^i(Y, \mathcal{O}(D)) = \bigoplus_{u \in M} H^i_{Z(u)}(Y, \mathcal{O}(D))$$

where M is the dual lattice and

$$H^i_{Z(u)}(Y, \mathcal{O}(D)) = H^i(|\Sigma|, |\Sigma| \setminus Z(u))$$

where $Z(u) = \{v \in |\Sigma| \mid \langle u, v\rangle \geqslant \varphi(v)\}$. Therefore, we have short exact sequences:

$$0 \to H^0_{Z(u)}(Y, \mathcal{O}(D)) \to H^0(|\Sigma|) \to H^0(|\Sigma| \setminus Z(u)) \to H^1_{Z(u)}(Y, \mathcal{O}(D)) \to 0$$

and $H^i_{Z(u)}(Y, \mathcal{O}(D)) \cong H^{i-1}(|\Sigma| \setminus Z(u))$ for $i \geqslant 2$. Since $Z(u)$ is a union of cones, $|\Sigma| \setminus Z(u) = \emptyset, \mathbb{R}^n \setminus \{0\}$ or is homotopy equivalent to $\mathbb{R}^{n-1} \setminus \{p_1, \ldots, p_m\}$ for $m \geqslant 0$. From topology, we have

$$H^i(\mathbb{R}^{n-1} \setminus \{p_1, \ldots, p_m\}) = \begin{cases} k, & i = 0 \\ k^m, & i = n-2 \\ 0, & \textit{otherwise} \end{cases}$$

Therefore, $H^i(Y, \mathcal{O}(D)) = 0$ for any D and $1 \leqslant i < n-1$. By Serre duality, $H^{n-1}(Y, \mathcal{O}(D)) = 0$. □

Lemma 5.1.6. *Let Y, D be the same as before. Let G be a finite group that acts on Cartier divisors* $\mathrm{CDiv}(Y)$ *and fixes K_Y. Assume that D is not linear equivalent to gD for any $g \in G$, then $\{\mathcal{O}(gD) \mid g \in G\}$ is an exceptional block.*

Proof. It suffices to show that $H^i(Y, \mathcal{O}(gD - D)) = 0$ for any $g \in G$ and $i = 0, n$. The case $i = 0$ is proved in Lemma 4.1.6. By Serre duality, $H^n(Y, \mathcal{O}(gD - D))$ is the dual of $H^0(Y, \mathcal{O}(K_Y + D - gD))$. Write $E = K_Y + D - gD$ and there exists $m \geqslant 1$ such that $g^m = 1$. Assume there is a nonzero global section $s \in \Gamma(Y, \mathcal{O}(E))$, then there is a nonzero global section $\otimes_{i=0}^{m-1} g^i s$ of $\mathcal{O}(E + gE + \cdots + g^{m-1}E) = \mathcal{O}(mK_Y)$. But since Y is rational, $H^0(Y, \mathcal{O}(mK_Y)) = 0$, contradiction. □

Assume X satisfies the conditions of Theorem 4.1.5, i.e, X is a smooth projective toric variety over k that splits over l where $K_0(X_l)$ has a permutation $G = \mathrm{Gal}(l/k)$-basis P of line bundles over X_l. Assume $\dim X \geqslant 3$, then by Lemma 5.1.6, each G-orbit of P is an exceptional block. Let $\pi : X_l \to X$ be the projection.

Theorem 5.1.7. *If there is an ordering for G-orbits $\{P_i\}_{i=1}^t$ of P such that $\{P_1, \ldots, P_t\}$ forms a full exceptional collection of $D^b(X_l)$, then for any $S_i \in P_i$, $\{\pi_* S_1, \ldots, \pi_* S_t\}$ is a full exceptional collection of $D^b(X)$.*

Proof. First we show that $\{\pi_*S_1, \dots, \pi_*S_t\}$ is an exceptional collection. Since π is flat and finite, both $\pi^* : D^b(X) \to D^b(X_l)$ and $\pi_* : D^b(X_l) \to D^b(X)$ are exact functors. The result follows from

$$\mathrm{Ext}^r_{D^b(X)}(\pi_*S_i, \pi_*S_j) \otimes_k l \cong \mathrm{Ext}^r_{D^b(X_l)}(\pi^*\pi_*S_i, \pi^*\pi_*S_j) \cong \bigoplus_{g,g' \in G} \mathrm{Ext}^r_{D^b(X_l)}(gS_i, g'S_j).$$

In particular, π_*S_i is an exceptional object, thus $\langle \pi_*S_i \rangle$ is an admissible subcategory of $D^b(X)$. Since $\langle \pi_*S_i \otimes_k l \rangle = \langle P_i \rangle$ and $D^b(X_l) = \langle P_1, \dots, P_t \rangle$, by [AB15, Lemma 2.3], $D^b(X) = \langle \pi_*S_1, \dots, \pi_*S_t \rangle$. $\square$

Remark 5.1.8. In the cases of smooth projective toric surfaces X, using the classification of minimal toric surfaces and the known results of semiorthogonal decompositions for projective spaces, projective bundles, del Pezzo surfaces and blow-ups, cf [Kuz14][BSS11][AB15], there exists such an ordering for the permutation G-basis of line bundles of $K_0(X_l)$ constructed in Theorem 3.2.2. Therefore, the decomposition of X in the motivic category $\mathcal{C}$ constructed in §4.2 can be lifted to the semiorthogonal decomposition of $D^b(X)$.

Question 2. *Let X be a smooth projective toric variety satisfying the conditions of Theorem 4.1.5 and assume $\dim X \geqslant 3$. Is there always a choice of ordering for G-orbits $\{P_i\}_{i=1}^t$ of P such that $\{P_1, \dots, P_t\}$ forms a full exceptional collection of $D^b(X_l)$?*

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