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ABSTRACT

The asymptotic feature of diffraction scattering of hadrons is tentatively assumed to mean asymptotically pure-imaginary partial waves. An essential requirement is then inferred to be the existence of an infinity of reaction channels and asymptotically infinite inelasticity (assuming asymptotic vanishing of partial-wave amplitudes). This view is supported by construction of a physically sensible inelastic model which has all the above features. Finally, these assumptions are shown to imply asymptotic vanishing of form factors; this resolves a puzzling feature of conventional dispersion theory solutions for form factors.

Introduction

The plausible conjecture that partial-wave scattering amplitudes become asymptotically pure imaginary (as c.m. energy $\sqrt{s} \rightarrow \infty$) is of current interest; we refer to this as diffraction scattering, henceforth DS. In this note we offer heuristic and illustrative arguments that DS depends on the existence of an infinity of reaction channels and on the associated asymptotically infinite inelasticity. Accepting this view, we then demonstrate that, as a consequence, form factors probably vanish as $s \rightarrow \infty$. Although this result is not surprising, and indeed is expected from our intuitive ideas about composite particles, it has previously not been clear what dynamical feature would be responsible for such behavior. We here clarify this point and resolve a problem encountered in conventional dispersion-theory solutions for a form factor F.

If, for a single elastic strong-interaction channel, we can obtain an expression for the relevant partial-wave scattering amplitude in the usual form N/D, then $F = C\,D^{-1}$ (C an appropriate constant) satisfies the analyticity and unitarity requirements[1] for a form factor. However, conventionally, it is possible to normalize $D(\infty) \rightarrow$ a constant, so that apparently this solution for a form factor does not vanish[2] asymptotically. The same conclusion applies to the generalization of D^{-1} for a finite number of strong-interaction channels. Here, however, the new feature of asymptotically infinite inelasticity is shown to be responsible for the asymptotic vanishing of F.

In Section 1 we present a heuristic argument that an infinity of channels is essential for DS. Then, in Section 2, we exhibit a model for a single channel with inelasticity which possesses the asymptotic feature of DS, namely it becomes pure imaginary. Finally, in Section 3 we explicitly construct a "D^{-1}" type of solution for F which vanishes asymptotically.

Throughout, we shall employ only two-body unitarity for two-body partial-wave amplitudes; inelasticity can, of course, represent many-body scattering channels also.

1. Strong-Interaction Amplitudes

One usually defines diffractive scattering as the process in which forward total scattering amplitudes become pure imaginary, and the differential-scattering cross sections become strongly peaked in the forward direction. We assume that this situation also implies pure imaginary partial-wave amplitudes,[3] as has been shown to be the case if we have Regge asymptotic behavior[4,5] with dominance by Pomeranchuk exchange. In this case, we need consider only a subset of channels for which vacuum-quantum-number exchange is a possible reaction mechanism.[6] For this subset of channels it is a plausible assumption that all the partial-wave amplitudes are asymptotically pure imaginary. This is reasonable especially if for $s \to \infty$ we can think of all the relevant particles as belonging to a representation of some approximate group, for which all the relevant scattering is "elastic".

We next show that this diffractive behavior cannot be achieved with an elastic one-channel model. We denote a partial-wave amplitude by t and permanently suppress all spin indices; from unitarity ($t = e^{i\delta} \sin\delta$ in elastic region) we then obtain

$$\text{Im}\, t = |t|^2 . \tag{1}$$

Therefore, if $t \to i\, \text{Im}\, t \equiv i\, t_I$, then (1) implies that $t_I \to 1$. This implication contradicts our usual ideas about partial-wave amplitudes. Certainly, in a bootstrap theory with no undetermined parameters, one should be able to write an unsubtracted dispersion relation for t, which requires $t_I(s\to\infty) \to 0$. This premise is also true if Regge asymptotic behavior occurs (but is not true if there is a strictly non-shrinking diffraction peak).[2]

For an n-channel situation,[6] we have

$$t_I^{\,ii} \approx \sum_n (t_R^{\,in})^2 + (t_I^{\,in})^2 . \tag{2}$$

If $t_R(s\to\infty)/t_I(s\to\infty) \to 0$, then

$$t_I(s) \approx n(s)\; c(s) t_I^{\,2} , \tag{3}$$

where $n(s)$ is the number of open channels at energy s connected to channel 1 via vacuum trajectory exchange, and $c(s)$ is $\leqslant O(1)$. We thus obtain the rough estimate

$$t_I \approx 1/[n(s)\; c(s)] . \tag{4}$$

From this and our assumptions that $t \to i\, t_I \to 0$ as $s \to \infty$ we infer that

$$n(s) \to \infty \quad \text{as} \quad s \to \infty \,. \tag{5}$$

This rather sketchy argument leads us to believe that an infinity of reaction channels is essential to DS. A corollary is obtained by considering inelastic unitarity for a single channel. We define

$$R = \sum_n |t^{in}|^2 / |t^{ii}|^2 \,. \tag{6}$$

It then follows from unitarity and the assumption $t \to i\, t_I$ that

$$t_I \to 1/R \,, \tag{6a}$$

so that according to our previous assumptions we must have $R(s) \to \infty$.

2. Sufficient Conditions for DS: a Model

Thus far we have given only heuristic arguments about conditions which are necessarily implied by DS. In this section we shall employ N/D two-body partial-wave equations to examine possible situations which might suffice to give DS.

We shall therefore examine the ratio

$$X = t_I/t_R = -\mathrm{Im}\, D/\mathrm{Re}\, D$$

$$= \rho\, R\, N \Big/ \left[1 - \frac{1}{\pi} \oint_{s_1}^{\infty} \frac{\rho\, R\, N(s-a)}{(s'-s)(s'-a)}\, ds' \right] \,. \tag{7}$$

Here we subtracted D at $s = a$; R as usual is the inelasticity. We choose a pole-model force, with the left-hand cut of t being a δ function at $s = a$; we then find that independent of R

$$N = G/(s - a) \quad \text{where } G \text{ is a constant.} \tag{8}$$

The simplest model we might consider is that of a constant R, leading to

$$X = -R\, N(s)/[1 - R\, I(s)] \; . \tag{9}$$

For large s (assuming that the integral does not tend to zero as $s \to \infty$) we thus find that X does not increase with increasing, large R.

The next complication we can study is contained in a system of n degenerate channels, with

$$N = f(s)\, G \; . \tag{10}$$

Here N and G are matrices, with G independent of s. For simplicity we can look at an "average" ratio X ; we now find that

$$\bar{X} = \sum \text{Im}\, t^{ii}/\text{Re}\, t^{ii}$$

turns out to have the same value as for a single channel. Thus, this model also fails to guarantee that $X_{ii} = \text{Im}\, t_{ii}/\text{Re}\, t_{ii}$ increases with the number of channels, i.e., with the inelasticity.

We feel that these preliminary models lack an essential feature, namely, the existence of an infinite number of channels with thresholds above any given energy. We shall therefore concoct a model having these features; this exercise is amusing in that it incorporates all physically reasonable features and predicts DS asymptotically.

The main assumption of our model is the form of R :

$$R - 1 = \sum_{n=1}^{\infty} A_n \, \theta(s - c_n) . \qquad (11)$$

The step functions admittedly introduce logarithmic singularities at the thresholds, but we tolerate this because the relevant features for our purposes are retained in this approximation, which has the virtue of giving simple expressions. We will disperse the amplitude $t/p^2 = N/D$, and take a one-pole model force so that we have

$$N(s) = G/(s - a) . \qquad (12)$$

We also use the approximation

$$\rho = \text{phase-space factor} = 2p/\sqrt{s} \approx 1 .$$

This is both convenient and accurate except when $p \to 0$.

From these specifications we now obtain, for $s \in (c_N, c_{N+1})$,

$$X(s) = \left(G \sum_{1}^{N} A_n\right)/Y \qquad (13a)$$

$$Y(s) = 1 - \frac{G}{\pi}\left[1 + \frac{s - s_1}{s - a}\left\{\ln\left(\frac{s_1 - a}{s - s_1}\right) - \sum_{n=1}^{\infty} A_n \ln\left|\frac{c_n - s}{c_n - a}\right|\right\} + \sum_{1}^{\infty} \frac{s_1 - a}{c_n - a} A_n\right] , \qquad (13b)$$

where s_1 is the first channel threshold and Y is just Re D, with the above approximations used to simplify the integral. Notice that for any finite number of channels N_{total}, we find

$$X(s \to \infty) \approx N_{total}/N_{total} \times \ln(s) \to 0 .$$

However, a radically new feature can emerge if we let $N_{total} \to \infty$. To illustrate this, we choose the following parameters for simplicity:

$$a = 0; \quad c_n = n^2 s_0; \quad A_n = A \quad \text{for all } n . \tag{14}$$

Consequently,

$$\sum_1^N A_n = A\, N_{\text{total open channels}}(s) \tag{15a}$$

$$\sum_1^\infty A_n \ln\left|\frac{c_n - s}{c_n}\right| = A \ln\left|\prod_{n=1}^{\infty}\left(1 - \left(\frac{\pi\sqrt{s/s_0}}{n\pi}\right)^2\right)\right| \tag{15b}$$

and

$$s_1 \sum_1^\infty \frac{A_n}{c_n} = A\,\frac{s_1}{s_0} \sum_1^\infty \frac{1}{n^2} . \tag{15c}$$

With these parameters, each new channel provides an equal increment to the inelasticity (which is consonant with classical ideas of energy equipartition). The channel spacing would be typical of multi-pion channels, for instance, with $s_0 = m_\pi^2$. For $s \to \infty$, we note that $R \simeq \sqrt{s}$, and also $R\, p^2 \times$ numerator "N function" $\approx \sqrt{s}$. According to Olesen and Squires,[3] this behavior can result in DS. Furthermore, in a once-subtracted dispersion relation for D, the highenergy integrand in our model is proportional to $s^{-3/2}$. Thus, in spite of $R(s\to\infty)\to\infty$, the low-energy phenomena in this partial wave can still be dominated by long-range forces (i.e., the lower energy range of the integral for D).

The expressions of Eqs. (15) can be easily evaluated[7] with the result

$$Y = 1 - \frac{G}{\pi}\left[1 - \frac{s - s_1}{s}\left\{\ell n\,\frac{s - s_1}{s_1} + A\,\ell n\left|\frac{\sin \pi\sqrt{s/s_0}}{\pi\sqrt{s/s_0}}\right|\right\} + A\,\frac{s_1}{s_0}\,\frac{\pi^2}{6}\right] . \tag{16}$$

For $s \to \infty$ it follows that

$$X \approx \frac{\sqrt{s}}{\ell n\ s} \longrightarrow \infty \tag{17}$$

as promised.

Again, we wish to emphasize the "cancellation" arising within the infinite series of channel contributions to Y. (Incidentally, if we took $\sqrt{c_n}$ = nth root of a Bessel function, then in Eq. (16) the asymptotic behavior would turn out to be unchanged.)

Apparently the existence of channels opening up above any arbitrary energy gives essential features which cannot be obtained with any finite number of channels. This need not be disconcerting, since it is impossible to have one production channel without an infinity of many-body inelastic channels.

For the interested reader, we sketch in the Appendix the consequences of a choice of R with more appropriate threshold properties.

3. Form Factors

In this section we shall adopt a "truncated" inelastic unitarity relation to examine the asymptotic behavior of a form factor F . This relation is

$$\mathrm{Im}\, F \equiv F_I = r\, t^* F\,, \tag{18}$$

where

$$r = \Big(\sum_{\text{all } n} t^*_{1n} F_{n1}\Big) \Big/ t^*_{11} F_{11}\ ;$$

here r can generally be a complex number. If we define

$$r = \rho\, e^{i\theta}$$

$$F = \mathcal{F}\, e^{i\chi}$$

$$t = \tau\, e^{i\omega},\ \theta,\ \chi,\ \omega \ \text{functions of } s\,, \tag{19}$$

then Eq. (18) can be written

$$\mathcal{F}\,\sin\chi = \tau\,\rho\,\mathcal{F}\, e^{i(\theta+\chi-\omega)}\,. \tag{18a}$$

We emphasize the important consequence

$$r\, t^* = e^{-i\chi} \sin\chi\,. \tag{18b}$$

As long as the "inelasticity" r is finite, our usual assumptions about $t(s \to \infty)$ lead to the conclusion that $\chi \to 0$ as $s \to \infty$. However, if (and only if) r is asymptotically infinite, the possibility arises that $r\, t^*$ does not vanish asymptotically, and therefore $\chi \to$ a nonzero constant as $s \to \infty$. To see the relevance of this, we additionally assume that F' satisfies a dispersion relation (possibly once subtracted). The solution to Eq. (18) is then given, e.g., in Goldberger and Watson,[8] by:

$$F(s) = F(0) \exp\left[\frac{s}{\pi}\int_{s_1}^{\infty} \frac{ds'\,\Phi(s')}{(s'-s)s'}\right] \qquad (20)$$

$$\tan\Phi = \frac{\mathrm{Re}(r^*t)}{1-\mathrm{Im}(r^*t)} = \tan X\ , \qquad (20a)$$

that is

$$\Phi = X\ .$$

Equation (20) has the asymptotic behavior

$$F(|s|\to\infty) = F(0)\ |s|^{-\Phi(\infty)/\pi}\ e^{i\Phi(\infty)}\ (1 + O(\ell n\ s)\cdots)\ . \qquad (21)$$

If r remains finite, then our above discussion implies that $F(\infty) \neq 0$. However, if $r(s\to\infty)\to\infty$, then it is possible to have $X(\infty)\neq 0$, which implies an asymptotically vanishing form factor [barring the possibility of $X(\infty) < 0$, not reasonable physically].

An example, which is not necessarily realistic, is furnished by assuming that asymptotically $r \approx R$. In this case, DS implies that $X \to \pi/2$, so that asymptotically $F \approx 1/\sqrt{s}$. Asymptotic Regge behavior with Pomeranchuk trajectory exchange provides an example for such DS, giving rise to partial waves[5] of the form

$$\frac{1}{\ell n\ s}\,[i - \pi/2\ \ell n\ s]\ .$$

Note that once we concede the possibility of $rt^* \not\to 0$, we also encounter the possibility of $rt^* = \sin X\ e^{iX}$ oscillating, with $X(s)$ increasing steadily as $s \to \infty$. Again, this is a feature not expected with a finite number of channels, but unfortunately we are now unlikely to be able to infer such behavior solely from our sparse knowledge of "t_{ni}".

In particular, for $X(s) \approx \sqrt{s}$ asymptotically, the following behavior of F is possible:

(a) F(s) does not vanish asymptotically but oscillates for $s \to +\infty$;

(b) F(s) vanishes as $\exp\left[-\sqrt{|s|}\right]$ for $s \to -\infty$.

[The reader may convince himself that this is possible by noting the following identity,[9] relevant for evaluating Eq. (20):

$$\mathcal{P}\int_0^\infty \frac{\sqrt{s'}\ ds'}{s'(s'-s)} \approx |s|^{-1/2}, \quad s < 0$$

$$= 0, \quad s > 0$$

and $F(s) \approx \exp(i\sqrt{s})$ everywhere.]

Currently we do not have any good model for the form-factor inelasticity r (which probably will not be asymptotically the same as R). Nevertheless the presence of asymptotically infinite "inelasticity" enables us to see how a form factor vanishes asymptotically, in principle, when calculated via present dispersion techniques.[10]

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Appendix: An Inelastic N/D Model.

We briefly sketch the consequences of choosing an R which more appropriately preserves inelastic-threshold analyticity. For all P-wave channels an appropriate choice would be

$$R = \sum_n A_n \left(\frac{p_n}{p_1} \right)^3 \theta(s - c_n) \,. \tag{A1}$$

To provide easily integrable expressions we could approximate

$$\left(\frac{2p_1}{W} \right) \frac{p_1^{\,2}}{s' - a} \left(\frac{p_n}{p_1} \right)^3 \equiv \left(\frac{2p_n}{W} \right) \frac{p_n^{\,2}}{s' - a} \quad \text{by} \quad \frac{p_n^{\,2}}{s' - a} \,. \tag{A2}$$

This at least prevents the occurrence of singularities at inelastic thresholds as in Section 2. As a result we now have

$$\text{Re } D - 1 = \frac{-1}{\pi} \text{Re} \left[\sum_n A_n \left\{ 1 - \left(1 - \frac{c_n}{s}\right) \ln\left(1 - \frac{s}{c_n}\right) \right\} \right] . \tag{A3}$$

For $c_n \to \infty$ the individual terms at fixed s are $O(s/c_n)$ so that our previous choice of parameters still gives a convergent result. Choosing $A_n = A$, we sum the series using the identity

$$\int^s ds \, \ln\left(1 - \frac{s}{c_n}\right) = (s - c_n) \ln\left(1 - \frac{s}{c_n}\right) - s \tag{A4}$$

to obtain

$$\text{Re } D - 1 = -\frac{A}{\pi} \left[\frac{1}{s} \int^s ds \, \ln \left| \frac{\sin \pi \sqrt{s/s_0}}{\pi \sqrt{s/s_0}} \right| \right] . \tag{A5}$$

For $s \to \infty$, $\ln(\;) \sim \ln s$ and therefore (A5) is of order $\ln s$ also. Our conclusions of Section 2 are therefore unaffected.

FOOTNOTES AND REFERENCES

* This work was done under the auspices of the U. S. Atomic Energy Commission.

1. S. D. Drell and F. Zachariasen, Electromagnetic Structure of Nucleons (Oxford University Press, London, 1961).
2. G. F. Chew (Lawrence Radiation Laboratory), private communication.
3. P. Olesen and E. J. Squires, Nuovo Cimento 39, 956 (1956).
4. E. J. Squires, Complex Angular Momentum and Particle Physics (W. A. Benjamin, Inc., New York, 1963).
5. E. J. Squires, Nuovo Cimento 34, 1277 (1964).

6. Here we expect all Pomeranchuk exchange amplitudes to have the same s behavior, and ignore coupling to all other amplitudes which are expected to vanish faster as a power of s than Pomeranchuk exchange amplitudes. As long as the number of channels of the latter type increases more slowly than this power of s, our assumption is valid. For our assumption to be invalid, we would require an asymptotically infinite number of "non-Pomeranchuk" channels; with our assumption, we still will need an infinity of channels (Eq. 5). Thus, either way, we infer the need for $n(s) \to \infty$.

7. P. M. Morse and H. Feshback, Methods of Theoretical Physics (McGraw-Hill Book Company, Inc., New York, 1953). See page 385.
8. M. L. Goldberger and K. Watson, Collision Theory (J. Wiley and Sons, Inc., New York, 1964).
9. Bateman Manuscript Project, Tables of Integral Transforms, Vol. II (McGraw-Hill Book Company, Inc., New York, 1954), p. 249, Entry 28.

10. After this work was done, the author received a preprint by D. H. Lythe, University of Birmingham (November 1965), which covers much of the relevant material from a more mathematical approach. (He does not discuss form factors, however.) He notes that the features of DS as presently known experimentally are also consistent with the following assumptions

(a) $t \to i\, a, \quad a \neq 0,$ but = 1/2 for "complete absorption"

(b) R constant, = 1/a for one-channel case.

Allowing $t(\infty) \neq 0$ deviates from conventional assumptions in the literature of dynamical calculations; this results in the possibility

$$r\, t^{*}\Big)_{s \to \infty} \neq 0 \,.$$

Thus, it is again possible for $X(\infty) \neq 0$, with the consequences

$$F(s \to \infty) \to 0 \,.$$