

# Combinatorial Congruences and $\psi$ -Operators

Daqing Wan\*  
 dwan@math.uci.edu  
 Department of Mathematics  
 University of California, Irvine  
 CA 92697-3875

December 23, 2013

## Abstract

The  $\psi$ -operator for  $(\varphi, \Gamma)$ -modules plays an important role in the study of Iwasawa theory via Fontaine's big rings. In this note, we prove several sharp estimates for the  $\psi$ -operator in the cyclotomic case. These estimates immediately imply a number of sharp  $p$ -adic combinatorial congruences, one of which extends the classical congruences of Fleck (1913) and Weisman (1977).

## 1 Combinatorial Congruences

Let  $p$  be a prime,  $n \in \mathbb{Z}_{>0}$ . Throughout this paper, let  $[x]$  denote the integer part of  $x$  if  $x \geq 0$  and  $[x] = 0$  if  $x < 0$ . In the author's course lectures [4] on Fontaine's theory and  $p$ -adic L-functions given at UC Irvine (spring 2005) and at the Morningside Center of Mathematics (summer 2005), the following two congruences were discovered.

**Theorem 1.1.** *For integers  $r \in \mathbb{Z}$ ,  $j \geq 0$ , we have*

$$\sum_{k \equiv r \pmod{p}} (-1)^{n-k} \binom{n}{k} \binom{\frac{k-r}{p}}{j} \equiv 0 \pmod{p^{\lfloor \frac{n-1-jp}{p-1} \rfloor}}.$$

We shall see that the theorem comes from a simple estimate of  $\psi(\pi^n)$  for the cyclotomic  $\varphi$ -module.

---

\*Partially supported by NSF. The author thanks Z.W. Sun for helpful discussions on combinatorial congruences

**Theorem 1.2.** For integer  $j \geq 0$ , we have

$$\sum_{\substack{i_0 + \dots + i_{p-1} = n \\ i_1 + 2i_2 + \dots \equiv r \pmod{p}}} \binom{n}{i_0 i_1 \dots i_{p-1}} \binom{\frac{i_1 + 2i_2 + \dots + r}{p}}{j} \equiv 0 \pmod{p^{\lfloor \frac{n(p-1) - jp - 1}{p-1} \rfloor}}.$$

As we shall see, this theorem comes from a simple estimate of  $\psi(\pi^{-n})$  for the cyclotomic  $\varphi$ -module. Note that when  $p = 2$ , Theorem 1.2 is equivalent to Theorem 1.1.

The above two congruences can be extended from  $p$  to  $q = p^a$ , where  $a$  is a positive integer. To do so, it suffices to estimate the  $a$ -th iterate  $\psi^a(\pi^n)$ . This can be done by induction. The estimate of  $\psi^a(\pi^n)$  for  $n > 0$  leads to

**Theorem 1.3.** For integers  $r \in \mathbb{Z}$ ,  $j \geq 0$  and  $a > 0$ , we have

$$\sum_{k \equiv r \pmod{p^a}} (-1)^{n-k} \binom{n}{k} \binom{\frac{k-r}{p^a}}{j} \equiv 0 \pmod{p^{\lfloor \frac{n-p^{a-1}-jp^a}{p^{a-1}(p-1)} \rfloor}}.$$

The estimate of  $\psi^a(\pi^n)$  for  $n < 0$  leads to

**Theorem 1.4.** Let

$$S_j(n, r, p^a) = \sum_{\substack{i_0 + \dots + i_{p^a-1} = n \\ i_1 + 2i_2 + \dots \equiv r \pmod{p^a}}} \binom{n}{i_0 \dots i_{p^a-1}} \binom{(i_1 + 2i_2 + \dots + r)/p^a}{j}.$$

Then for integer  $j \geq 0$ , we have

$$S_j(n, r, p^a) \equiv 0 \pmod{p^{\lfloor \frac{(an-a+1)(p-1) - j(ap-a+1) - 1}{p-1} \rfloor}}.$$

As Z.W. Sun informed me, the special case  $j = 0$  of Theorem 1.1.1 was first proved by Fleck [1] in 1913, and the special case of Theorem 1.1.3 for  $j = 0$  was first proved by Weisman [5] in 1977. A different extension of Theorem 1.1.1 and Weisman's congruence has been obtained by Z.W. Sun [2] using different combinatorial arguments. Motivated by applications in algebraic topology, Sun-Davis [3] proved yet another extension:

$$\sum_{k \equiv r \pmod{p^a}} (-1)^{n-k} \binom{n}{k} \binom{\frac{k-r}{p^a}}{j} \equiv 0 \pmod{p^{\left(\text{ord}_p([n/p^{a-1}]!) - j - \text{ord}_p(j!)\right)}}.$$

## 2 The operator $\psi$

Let  $p$  be a fixed prime. Let  $\pi$  be a formal variable. Let

$$A^+ = \mathbb{Z}_p[[\pi]]$$

be the formal power series ring over the ring of  $p$ -adic integers. Let  $A$  be the  $p$ -adic completion of  $A^+[\frac{1}{\pi}]$ , and let  $B = A[\frac{1}{p}]$  be the fraction field of  $A$ . The rings  $A^+$ ,  $A$  and  $B$  correspond to  $A_{\mathbb{Q}_p}^+$ ,  $A_{\mathbb{Q}_p}$  and  $B_{\mathbb{Q}_p}$  in Fontaine's theory.

We shall not discuss the Galois action on  $A$ , which is not needed for our present purpose. The Frobenius map  $\varphi$  acts on the above rings by

$$\varphi(\pi) = (1 + \pi)^p - 1.$$

If we let  $[\varepsilon] = 1 + \pi$ , then  $\varphi([\varepsilon]) = [\varepsilon]^p$ . The map  $\varphi$  is injective of degree  $p$ . This gives

**Proposition 2.1.**  $\{1, \pi, \dots, \pi^{p-1}\}$  (and  $\{1, [\varepsilon], \dots, [\varepsilon]^{p-1}\}$ ) is a basis of  $A$  over the subring  $\varphi(A)$ .

**Definition 2.2.** The operator  $\psi : A \rightarrow A$  is defined by

$$\psi(x) = \psi \left( \sum_{i=0}^{p-1} [\varepsilon]^i \varphi(x_i) \right) = x_0 = \frac{1}{p} \varphi^{-1}(\text{Tr}_{A/\varphi(A)}(x)),$$

where  $x : A \rightarrow A$  denotes the multiplication by  $x$  as  $\varphi(A)$ -linear map.

**Example 2.3.**

$$\psi([\varepsilon]^n) = \begin{cases} [\varepsilon]^{n/p}, & \text{if } p \mid n; \\ 0, & \text{if } p \nmid n. \end{cases}$$

It is clear that  $\psi$  is  $\varphi^{-1}$ -linear:

$$\psi(\varphi(a)x) = a\psi(x) \quad \forall a, x \in A.$$

**Example 2.4.** Let  $a$  be a positive integer relatively prime to  $p$ . Then

$$\psi\left(\frac{1}{(1 + \pi)^a - 1}\right) = \frac{1}{(1 + \pi)^a - 1}.$$

In fact,

$$\begin{aligned} \psi\left(\frac{1}{[\varepsilon]^a - 1}\right) &= \psi\left(\frac{1}{[\varepsilon]^{ap} - 1} \cdot \frac{[\varepsilon]^{ap} - 1}{[\varepsilon] - 1}\right) \\ &= \frac{1}{[\varepsilon]^a - 1} \psi\left(1 + [\varepsilon]^a + \dots + [\varepsilon]^{(p-1)a}\right) \\ &= \frac{1}{[\varepsilon]^a - 1} = \frac{1}{(1 + \pi)^a - 1}. \end{aligned}$$

By  $p$ -adic continuity, the above example holds for any  $p$ -adic unit  $a \in \mathbb{Z}_p^*$ . In the general theory of  $(\varphi, \Gamma)$ -modules, it is important to find the fix points of  $\psi$  for applications to  $p$ -adic L-functions and Iwasawa theory. In the simplest cyclotomic case, we have the following description for the fixed points (see [4]).

**Proposition 2.5.**

$$A^{\psi=1} = \frac{1}{\pi} \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \left\{ \sum_{k=0}^{\infty} \varphi^k(x) \mid x \in \bigoplus_{i=1}^{p-1} [\varepsilon]^i \varphi(a_i + \pi \mathbb{Z}_p[[\pi]]), \sum_{i=1}^{p-1} a_i = 0 \right\},$$

where  $a_i \in \mathbb{Z}_p$ .

For example, if  $a$  is a positive integer relatively prime to  $p$ , then the element

$$\frac{a}{(1 + \pi)^a - 1} - \frac{1}{\pi} \in (A^+)^{\psi=1}$$

gives the cyclotomic units and the Euler system. This element is the Amice transform of a  $p$ -adic measure which produces the  $p$ -adic zeta function of  $\mathbb{Q}$ . This type of connections is conjectured to be a general phenomenon for  $(\varphi, \Gamma)$ -modules coming from global  $p$ -adic Galois representations.

### 3 Sharp estimates for $\psi$

The ring  $A$  is a topological ring with respect to the  $(p, \pi)$ -topology. A basis of neighborhoods of 0 is the sets  $p^k A + \pi^n A^+$ , where  $k \in \mathbb{Z}$  and  $n \in \mathbb{N}$ . The operator  $\psi$  is uniformly continuous. This continuity will give rise to combinatorial congruences.

For  $s \in A^+$ , one checks that

$$\begin{aligned} \psi(\pi^p s) &= \psi([\varepsilon] - 1)^p s \\ &= \psi([\varepsilon]^p - 1)s + p s s_1 \\ &= \pi \psi(s) + p \psi(s s_1) \in (p, \pi) \psi(s A^+). \end{aligned}$$

In particular,

$$\psi(\pi^p A^+) \subset (p, \pi) A^+.$$

Thus, by iteration, we get

**Proposition 3.1 (Weak Estimate).** *Let  $n \geq 0$ . Then*

$$\psi(\pi^n A^+) \subset (p, \pi)^{[n/p]} A^+ = \sum_{j=0}^{[n/p]} \pi^j p^{[n/p]-j} A^+.$$

Since the exponent  $[(n - jp)/p]$  is decreasing in  $j$ , this proposition implies that for  $x \in \pi^n A^+$ , we have

$$\psi(x) = \sum_{j=0}^{\infty} a_j \pi^j, \quad a_j \in \mathbb{Z}_p, \quad \text{ord}_p(a_j) \geq [(n - jp)/p].$$

This already gives a non-trivial combinatorial congruence. Let  $r$  be an integer. Let us calculate  $\psi(\pi^n [\varepsilon]^{-r})$  in a different way.

**Lemma 3.2.**

$$\psi(\pi^n [\varepsilon]^{-r}) = \sum_{j \geq 0} \pi^j \sum_{k \equiv r \pmod{p}} (-1)^{n-k} \binom{n}{k} \binom{(k-r)/p}{j}.$$

*Proof.* Since  $\pi = [\varepsilon] - 1$  and  $[\varepsilon] = 1 + \pi$ , we have

$$\begin{aligned} \psi(\pi^n [\varepsilon]^{-r}) &= \psi(([\varepsilon] - 1)^n [\varepsilon]^{-r}) \\ &= \psi \left( \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} [\varepsilon]^{k-r} \right) \\ &= \sum_{k \equiv r \pmod{p}} (-1)^{n-k} \binom{n}{k} [\varepsilon]^{(k-r)/p} \\ &= \sum_{k \equiv r \pmod{p}} (-1)^{n-k} \binom{n}{k} \sum_{j \geq 0} \binom{(k-r)/p}{j} \pi^j \\ &= \sum_{j \geq 0} \pi^j \sum_{k \equiv r \pmod{p}} (-1)^{n-k} \binom{n}{k} \binom{(k-r)/p}{j}. \end{aligned}$$

□

Comparing the coefficients of  $\pi^j$  in this equation and the weak estimate, we get

**Corollary 3.3 (Weak Congruence).** *Let  $n \geq 0$ . We have*

$$\sum_{k \equiv r \pmod{p}} (-1)^{n-k} \binom{n}{k} \binom{(k-r)/p}{j} \equiv 0 \pmod{p^{[(n-jp)/p]}}.$$

The above simple estimate is crude and certainly not optimal since we ignored a factor of  $\pi$ . We now improve on it.

**Theorem 3.4 (Sharp Estimate I).** *For  $n \geq 0$ , we have*

$$\psi(\pi^n A^+) \subset \sum_{j=0}^{\lfloor n/p \rfloor} \pi^j p^{\lfloor \frac{n-1-jp}{p-1} \rfloor} A^+.$$

*Proof.* We prove the theorem by induction. The theorem is trivial if  $n \leq p-1$ . Write

$$\varphi(\pi) = (1 + \pi)^p - 1 = \pi^p - p\pi s_1, \quad s_1 \in A^+.$$

Then,

$$\psi(\pi^p s) = \psi((\varphi(\pi) + p\pi s_1)s) = \pi\psi(s) + p\psi(\pi s_1 s).$$

This proves that the theorem is true for  $n = p$ . Let  $n > p$ . Assume the theorem holds for  $\leq n-1$ . It follows that

$$\psi(\pi^n A^+) = \psi(\pi^p \pi^{n-p} A^+) \subseteq \pi\psi(\pi^{n-p} A^+) + p\psi(\pi^{n+1-p} A^+).$$

By the induction hypothesis, the right side is contained in

$$\begin{aligned} & \pi \sum_{j=0}^{\lfloor (n-p)/p \rfloor} \pi^j p^{\lfloor \frac{n-p-1-jp}{p-1} \rfloor} A^+ + p \sum_{j=0}^{\lfloor (n+1-p)/p \rfloor} \pi^j p^{\lfloor \frac{n-p-jp}{p-1} \rfloor} A^+ \\ &= \sum_{j=1}^{\lfloor n/p \rfloor} \pi^j p^{\lfloor \frac{n-1-jp}{p-1} \rfloor} A^+ + \sum_{j=0}^{\lfloor (n+1-p)/p \rfloor} \pi^j p^{\lfloor \frac{n-1-jp}{p-1} \rfloor} A^+. \end{aligned}$$

□

The function  $\lfloor (n-1-jp)/(p-1) \rfloor$  is decreasing in  $j$  and vanishes for  $j \geq \lfloor n/p \rfloor$ . Comparing the coefficients of  $\pi^j$  in the lemma and the above sharp estimate, we deduce

**Corollary 3.5 (Sharp Congruence I).** *Let  $r \in \mathbb{Z}$ .*

$$\sum_{k \equiv r \pmod{p}} (-1)^{n-k} \binom{n}{k} \binom{(k-r)/p}{j} \equiv 0 \pmod{p^{\lfloor \frac{n-1-jp}{p-1} \rfloor}},$$

where  $j \geq 0$  is a non-negative integer.

**Theorem 3.6 (Sharp Estimate II).** For  $n > 0$ , we have

$$\psi\left(\frac{1}{\pi^n}A^+\right) \subseteq \sum_{j=0}^{\lfloor n(p-1)/p \rfloor} \frac{1}{\pi^{n-j}} p^{\lfloor \frac{n(p-1)-jp-1}{p-1} \rfloor} A^+.$$

*Proof.* Note that

$$\varphi(\pi)/\pi = \pi^{p-1} + \binom{p}{1}\pi^{p-2} + \cdots + \binom{p}{p-1} \in (\pi^{p-1}, p),$$

so  $(\varphi(\pi)/\pi)^n \in (\pi^{p-1}, p)^n$ . Then

$$\begin{aligned} \psi\left(\frac{1}{\pi^n}A^+\right) &= \psi\left(\frac{1}{\varphi(\pi)^n} \left(\frac{\varphi(\pi)}{\pi}\right)^n A^+\right) \\ &= \frac{1}{\pi^n} \psi\left(\left(\frac{\varphi(\pi)}{\pi}\right)^n A^+\right) \\ &\subseteq \frac{1}{\pi^n} \sum_{i=0}^n p^{n-i} \psi(\pi^{i(p-1)} A^+). \end{aligned}$$

By Sharp Estimate I, we have

$$\psi(\pi^{i(p-1)} A^+) \subseteq \sum_{j=0}^{\lfloor i(p-1)/p \rfloor} \pi^j p^{\lfloor \frac{i(p-1)-1-jp}{p-1} \rfloor} A^+.$$

Then,

$$\begin{aligned} \psi\left(\frac{1}{\pi^n}A^+\right) &\subseteq \sum_{j=0}^{\lfloor n(p-1)/p \rfloor} \frac{1}{\pi^{n-j}} \sum_{\lfloor jp/(p-1) \rfloor \leq i \leq n} p^{n-i + \lfloor \frac{i(p-1)-jp-1}{p-1} \rfloor} A^+ \\ &\subseteq \sum_{j=0}^{\lfloor n(p-1)/p \rfloor} \frac{1}{\pi^{n-j}} p^{\lfloor \frac{n(p-1)-jp-1}{p-1} \rfloor} A^+. \end{aligned}$$

□

**Corollary 3.7 (Sharp Congruence II).** Let

$$S_j(n, r, p) = \sum_{\substack{i_0 + \cdots + i_{p-1} = n \\ i_1 + 2i_2 + \cdots \equiv r \pmod{p}}} \binom{n}{i_0 \cdots i_{p-1}} \binom{(i_1 + 2i_2 + \cdots - r)/p}{j}.$$

Then integer  $j \geq 0$ , we have

$$S_j(n, r, p) \equiv 0 \pmod{p^{\lfloor \frac{n(p-1)-1-jp}{p-1} \rfloor}}.$$

*Proof.*

$$\begin{aligned}
& \psi\left(\frac{1}{\pi^n}[\varepsilon]^{-r}\right) \\
&= \frac{1}{\pi^n} \psi\left(\left(\frac{[\varepsilon]^p - 1}{[\varepsilon] - 1}\right)^n [\varepsilon]^{-r}\right) \\
&= \frac{1}{\pi^n} \psi\left((1 + [\varepsilon] + \cdots + [\varepsilon]^{p-1})^n \cdot [\varepsilon]^{-r}\right) \\
&= \frac{1}{\pi^n} \sum_{\substack{i_0 + \cdots + i_{p-1} = n \\ i_1 + 2i_2 + \cdots \equiv r \pmod{p}}} [\varepsilon]^{(i_1 + 2i_2 + \cdots - r)/p} \binom{n}{i_0 \cdots i_{p-1}} \\
&= \frac{1}{\pi^n} \sum_{\substack{i_0 + \cdots + i_{p-1} = n \\ i_1 + 2i_2 + \cdots \equiv r \pmod{p}}} \sum_{j \geq 0} \pi^j \binom{n}{i_0 \cdots i_{p-1}} \binom{(i_1 + 2i_2 + \cdots - r)/p}{j} \\
&= \sum_{j=0}^{\infty} \frac{1}{\pi^{n-j}} S_j(n, r, p).
\end{aligned}$$

The function  $[(n(p-1) - jp - 1)/(p-1)]$  is decreasing in  $j$  and vanishes for  $j \geq [n(p-1)/p]$ . Comparing the coefficients of  $\frac{1}{\pi^{n-j}}$ , the congruence follows.  $\square$

## 4 Sharp estimates for $\psi^a$

Let  $a$  be a positive integer. In this section, we extend the sharp estimates for  $\psi$  to  $\psi^a$ .

**Theorem 4.1 (Sharp Estimate I).** *For  $n \geq 0$ , we have*

$$\psi^a(\pi^n A^+) \subseteq \sum_{j=0}^{[n/p^a]} \pi^j p^{\lfloor \frac{n-p^a-1-jp^a}{p^a-1(p-1)} \rfloor} A^+.$$

*Proof.* We prove the theorem by induction on  $a$ . The theorem is true if  $a = 1$ . Assume now  $a \geq 2$  and assume that the theorem holds for  $a - 1$ .



Then,

$$\begin{aligned}
\psi^a(\pi^n A^+) &= \psi(\psi^{a-1} \pi^n A^+) \\
&\subseteq \psi\left(\sum_{i=0}^{\lfloor n/p^{a-1} \rfloor} \pi^i p^{\lfloor \frac{n-p^{a-2}-ip^{a-1}}{p^{a-2}(p-1)} \rfloor} A^+\right) \\
&\subseteq \sum_{i=0}^{\lfloor n/p^{a-1} \rfloor} \sum_{j=0}^{\lfloor i/p \rfloor} \pi^j p^{\lfloor \frac{n-p^{a-2}-ip^{a-1}}{p^{a-2}(p-1)} \rfloor + \lfloor \frac{i-1-jp}{p-1} \rfloor} A^+ \\
&\subseteq \sum_{j=0}^{\lfloor n/p^a \rfloor} \pi^j \sum_{pj \leq i \leq \lfloor n/p^{a-1} \rfloor} p^{\lfloor \frac{n-p^{a-2}-ip^{a-1}}{p^{a-2}(p-1)} \rfloor + \lfloor \frac{i-1-jp}{p-1} \rfloor} A^+.
\end{aligned}$$

The exponent of  $p$  for a fixed  $j$  is decreasing in  $i$  and hence the minimum exponent of  $p$  is attained when  $i = \lfloor n/p^{a-1} \rfloor$ . The minimum exponent is computed to be

$$\left\lfloor \frac{n-p^{a-2}-\lfloor n/p^{a-1} \rfloor p^{a-1}}{p^{a-1}-p^{a-2}} \right\rfloor + \left\lfloor \frac{\lfloor n/p^{a-1} \rfloor - 1 - jp}{p-1} \right\rfloor = \left\lfloor \frac{n-p^{a-1}-jp^a}{p^{a-1}(p-1)} \right\rfloor.$$

□

The proof of the lemma gives more general

**Lemma 4.2.**

$$\psi^a(\pi^n [\varepsilon]^{-r}) = \sum_{j \geq 0} \pi^j \sum_{k \equiv r \pmod{p^a}} (-1)^{n-k} \binom{n}{k} \binom{(k-r)/p^a}{j}.$$

Comparing the coefficients of  $\pi^j$  in the lemma and the sharp estimate for  $\psi^a$ , we get

**Corollary 4.3 (Sharp Congruence I).** *Let  $r \in \mathbb{Z}$ . Then*

$$\sum_{k \equiv r \pmod{p^a}} (-1)^{n-k} \binom{n}{k} \binom{(k-r)/p^a}{j} \equiv 0 \pmod{p^{\lfloor \frac{n-p^{a-1}-jp^a}{p^{a-1}(p-1)} \rfloor}},$$

where  $j \geq 0$  is a non-negative integer.

**Theorem 4.4 (Sharp Estimate II).** *For  $n > 0$  and  $a > 0$ , we have*

$$\psi^a\left(\frac{1}{\pi^n} A^+\right) \subseteq \sum_{j=0}^{\lfloor \frac{(an-a+1)(p-1)}{ap-a+1} \rfloor} \frac{1}{\pi^{n-j}} p^{\lfloor \frac{(an-a+1)(p-1)-j(ap-a+1)-1}{p-1} \rfloor} A^+.$$

*Proof.* The theorem is true for  $a = 1$ . Assume now that  $a > 1$  and assume that the theorem is true for  $a - 1$ . Then

$$\begin{aligned}
\psi^a \left( \frac{1}{\pi^n} A^+ \right) &= \psi \left( \psi^{a-1} \left( \frac{1}{\pi^n} A^+ \right) \right) \\
&\subseteq \psi \left( \sum_{j=0}^{\lfloor \frac{(a-1)n-a+2)(p-1)}{(a-1)p-a+2} \rfloor} \frac{1}{\pi^{n-j}} p^{\lfloor \frac{(a-1)n-a+2)(p-1)-j((a-1)p-a+2)-1}{p-1} \rfloor} A^+ \right) \\
&\subseteq \sum_j \sum_i \frac{1}{\pi^{n-j-i}} p^{\lfloor \frac{(a-1)n-a+2)(p-1)-j((a-1)p-a+2)-1}{p-1} \rfloor + \lfloor \frac{(n-j)(p-1)-ip-1}{p-1} \rfloor} A^+,
\end{aligned}$$

where the indices  $i$  and  $j$  satisfy

$$0 \leq j \leq \lfloor \frac{(a-1)n-a+2)(p-1)}{(a-1)p-a+2} \rfloor, \quad 0 \leq i \leq \lfloor \frac{(n-j)(p-1)}{p} \rfloor.$$

For fixed  $i + j = k$ , the exponent of  $p$  is decreasing in  $j$  and the minimum value is attained when  $j = k$  and  $i = 0$ . It follows that

$$\begin{aligned}
\psi^a \left( \frac{1}{\pi^n} A^+ \right) &\subseteq \sum_{k \geq 0} \frac{1}{\pi^{n-k}} p^{\lfloor \frac{(a-1)n-a+2)(p-1)-k((a-1)p-a+2)-1}{p-1} \rfloor + \lfloor n-k-1 \rfloor} A^+ \\
&\subseteq \sum_{k=0}^{\lfloor \frac{(an-a+1)(p-1)}{ap-a+1} \rfloor} \frac{1}{\pi^{n-k}} p^{\lfloor \frac{(an-a+1)(p-1)-k(ap-a+1)-1}{p-1} \rfloor} A^+,
\end{aligned}$$

where we stop at  $k = \lfloor \frac{(an-a+1)(p-1)}{ap-a+1} \rfloor$  in the summation as the exponent of  $p$  is zero if  $k \geq \lfloor \frac{(an-a+1)(p-1)}{ap-a+1} \rfloor$ .  $\square$

**Corollary 4.5 (Sharp Congruence II).** *Let*

$$S_j(n, r, p^a) = \sum_{\substack{i_0 + \dots + i_{p^a-1} = n \\ i_1 + 2i_2 + \dots \equiv r \pmod{p^a}}} \binom{n}{i_0 \dots i_{p^a-1}} \binom{(i_1 + 2i_2 + \dots - r)/p^a}{j}.$$

*Then for integer  $j \geq 0$ , we have*

$$S_j(n, r, p^a) \equiv 0 \pmod{p^{\lfloor \frac{(an-a+1)(p-1)-j(ap-a+1)-1}{p-1} \rfloor}}.$$

## References

- [1] L.E. Dickson, History of the Theory of Numbers, Vol. I, AMS Chelsea Publ., 1999, Chapter XI, pp. 270-275.
- [2] Z.W. Sun, Polynomial extension of Fleck's congruence, preprint, 2005, arXiv:math.NT/0507008.
- [3] Z.W. Sun and D.M. Davis, A combinatorial congruence for polynomials, arXiv:math.NT/0508087.
- [4] D. Wan, Fontaine's Rings and  $p$ -Adic L-Functions, Lecture Notes at the Morningside Center of Mathematics, 2005.
- [5] C.S. Weisman, Some congruences for binomial coefficients, Michigan Math. J., 24(1977), 141-151.