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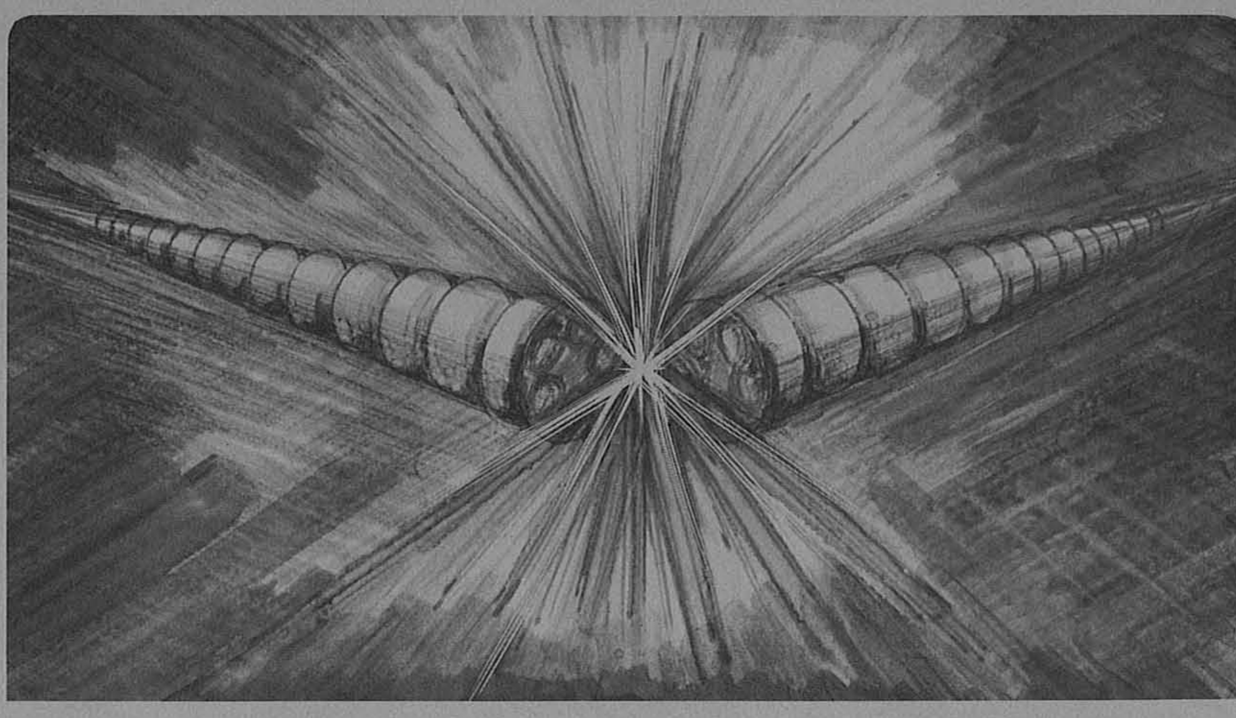

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## Introduction

This report is the third in a series[1,2] which takes into account the boundary condition in electromagnetic problems such as used by the program POISSON. Here we extend the analysis to permit the use of an elliptical boundary both for two dimensional and axisymmetric cylindrical problems. The use of an elliptical boundary instead of a circular one can reduce the mesh size when using the program POISSON and thereby save computing time. Saving cpu time can be significant for problems such as the 2-in-1 dipole proposed for the SSC or other magnets such as solenoids. We therefore expect the use of an elliptical boundary to be more general and the previous spherical boundary solution to be a special case.

## Two Dimensional Problems with Elliptical Cylindrical Coordinates

The analysis is identical to the one reported in Ref. 1,2 except that here we replace the two circular arcs with two confocal ellipses (Fig. 1) and employ elliptic cylindrical coordinates (u, v, ø). This

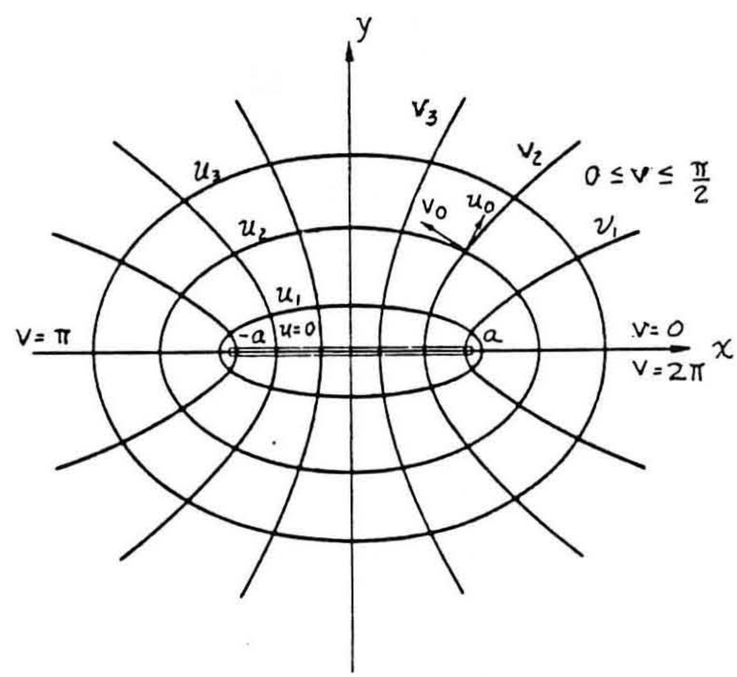


Fig. 1

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can be conveniently done by the use of the conformal transformation $x + iy = c \operatorname{Cosh}(u + iv)$ for which $x = c \operatorname{Cosh} u \cos v$ and $y = c \operatorname{Sinh} u \sin v$ and results in the curves of constant v forming a set of confocal ellipses, concentric with the origin, whose major semi-axes are $\underline{a = c \operatorname{Cosh} u}$ (coincident with the x-axis) and $\underline{b = c \operatorname{Sinh} u}$ (coincident with the y-axis). The variable v is a distorted analogue of the polar coordinate angle $\theta$ and numerically covers the same angle as $\theta$ in the successive quadrants.

For the case $\left(\frac{b}{a}\right)^2 < 1$ we write:

$$c^2 = a^2 - b^2 \tag{1a}$$

$$u = \operatorname{Tanh}^{-1}\left(\frac{b}{a}\right) \tag{1b}$$

$$v = \tan^{-1}[(y/x)/(b/a)] \tag{1c}$$

If the outer elliptical curve has a value of $u_2$ and semi-axes $a_2$, $b_2$ and similarly the inner curve has a value of $u_1$ with $a_1$ and $b_1$, we have:

$$a_1^2 - b_1^2 = a_2^2 - b_2^2 \tag{2a}$$

$$u_2 = \operatorname{Tanh}^{-1} \frac{b_2}{a_2} \tag{2b}$$

$$u_1 = \operatorname{Tanh}^{-1} \frac{b_1}{a_1} = \operatorname{Tanh}^{-1} \left[1 - \left(\frac{a_2}{a_1}\right)^2 + \left(\frac{b_2}{a_1}\right)^2\right]^{1/2} \tag{2c}$$

Note that out of the four semi-axes only three are independent and the fourth one is determined according to (2a) (this is expressed in relation 2c).

In the free space region the vector potential can be expressed as a sum of harmonic terms, each employing powers of $e^{-u}$.

$$A_i = \sum_{\ell=1} e^{-\alpha_\ell} D_\ell F_\ell(v_i) \tag{3}$$

The vector potential A of mesh point i on the elliptic arc u is expressed in terms of a series of circular functions $F_\ell(v_i)$, their coefficients $D_\ell$ and the problem type symmetry $\alpha_\ell$.

In the spirit of the analysis given in Ref. 2 we express the vector potential on the outer elliptical arc as a linear combination of the vector potential of each mesh point on the inner elliptical arc (Fig. 2).

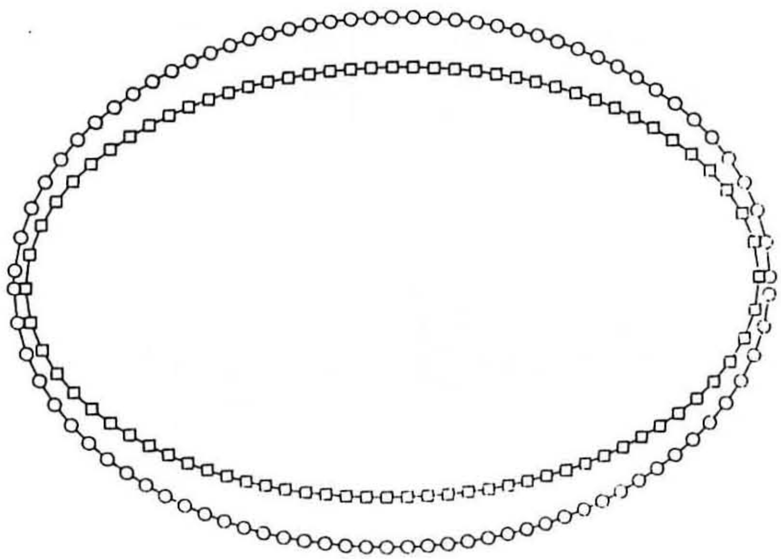


Fig. 2

$$A_k^{outer} = \sum_{n=1}^{N} E_{kn} A_n^{inner} \tag{4}$$

where

$$E_{kn} = \sum_{i=1}^{m} \sum_{j=1}^{m} e^{\alpha_j (u_1 - u_2)} W_n (M^{-1})_{ji} F_j(v_k) F_i(v_n) \tag{5}$$

N = total number of points on inner arc

m = total number of harmonics

and M is as defined on p. 3 of Ref. 2, wherein $M_{ij} = \sum_{n=1}^{N} W_n F_i(v_n) F_j(v_n)$.

Relation 5 can be further reduced:

$$u_1 - u_2 = \text{Tanh}^{-1}\left(\frac{b}{a}\right)_1 - \text{Tanh}^{-1}\left(\frac{b}{a}\right)_2 = \frac{1}{2} \ln \frac{(a + b)_1 (a - b)_2}{(a - b)_1 (a + b)_2}$$

and using relation 2a

$$\frac{(a + b)_1 (a - b)_2}{(a - b)_1 (a + b)_2} = \left[\frac{(a + b)_1}{(a + b)_2}\right]^2$$

we can then write

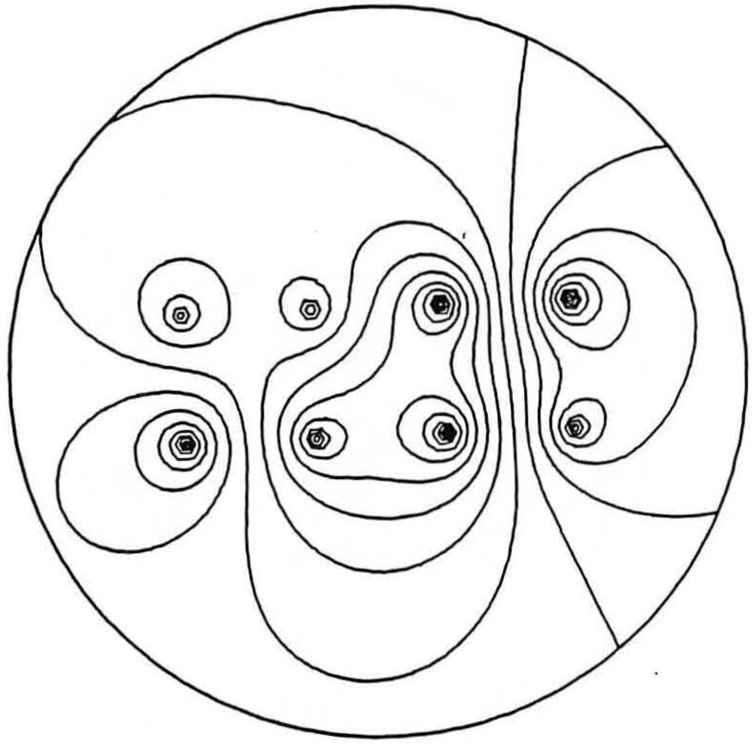
$$e^{\alpha_j \Delta u} = e^{\alpha_j \frac{1}{2} \ln \left[\frac{(a + b)_1}{(a + b)_2}\right]^2} = \left[\frac{(a + b)_1}{(a + b)_2}\right]^{\alpha_j} \tag{6}$$

If we now substitute

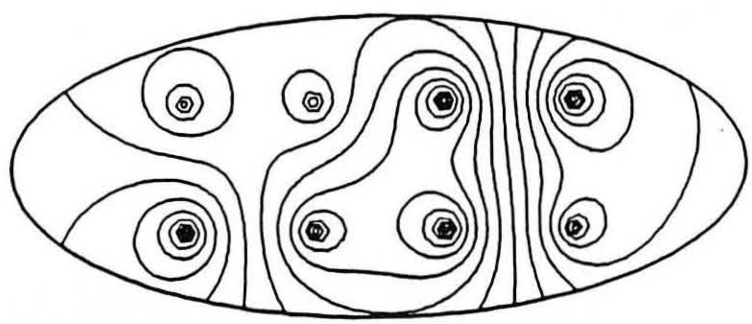
$$R = \frac{(a + b)_2}{2}$$

$$H = \frac{(a_2 - a_1) + (b_2 - b_1)}{2}$$

we have $$\left[\frac{(a + b)_1}{(a + b)_2}\right]^{\alpha_j} = \left(\frac{R - H}{R}\right)^{\alpha_j} \tag{7}$$



(a) $\eta = \infty$



(b) $\eta = 1.097$

Fig. 3 Flux lines around current filaments in a two-dimensional problem, which has no symmetry (CON 126 = 10), with circular boundary (a) and elliptical boundary (b). The problem converged in 350 cycles in case (a) and 160 cycles in case (b).

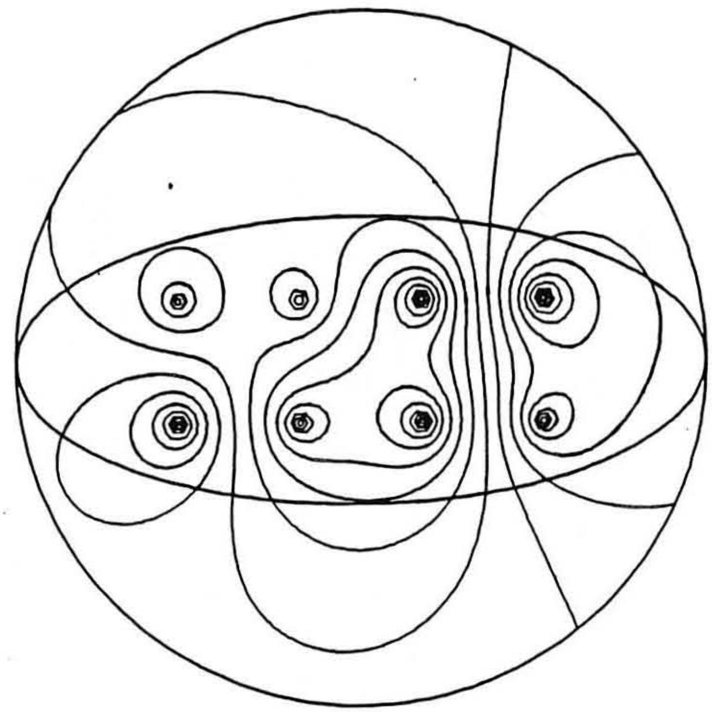
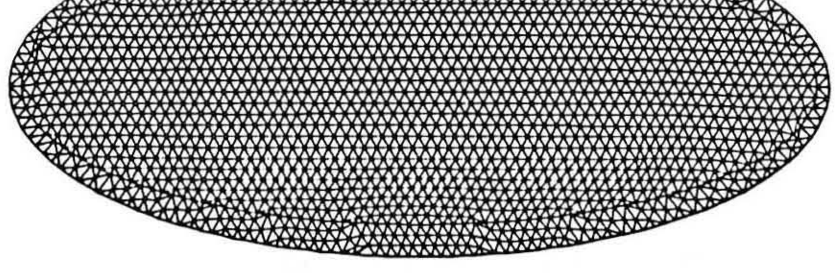


Fig. 4 Overlap of flux lines for the circular and elliptical boundary problem. Below, the triangular mesh, showing the two confocal ellipses used to generate the "universe".



## TABLE 1

Comparison of the vector potential values along y = 0 as resulted from direct computation (theoretical) and from POISSON (iterative)

| y = 0<br>x cm | Theoretical<br>A | circular B.C.<br>Calculated<br>A | <br>Error<br>Δ% | elliptical B.C.<br>Calculated<br>A | <br>Error<br>Δ% |
|---|---|---|---|---|---|
| 1.6 | 154.45 | 154.71 | 0.17 | 155.40 | 0.61 |
| 1.4 | 187.77 | 187.99 | 0.12 | 188.80 | 0.54 |
| 1.2 | 228.34 | 228.50 | 0.07 | 229.31 | 0.42 |
| 1.0 | 247.14 | 247.30 | 0.06 | 248.03 | 0.36 |
| 0.8 | 148.04 | 148.18 | 0.09 | 148.68 | 0.43 |
| 0.6 | -70.41 | -70.330 | -0.11 | -70.09 | -0.45 |
| 0.4 | -288.05 | -288.06 | 0.003 | -288.13 | 0.03 |
| 0.2 | -376.96 | -376.97 | 0.002 | -377.28 | 0.08 |
| 0.0 | -338.0 | -337.970 | -0.009 | -388.42 | 0.12 |
| -0.2 | -273.84 | -273.88 | 0.01 | -274.44 | 0.20 |
| -0.4 | -204.42 | -204.42 | 0.0 | -205.02 | 0.29 |
| -0.6 | -130.25 | -130.27 | 0.015 | -130.84 | 0.45 |
| -0.8 | -65.25 | -65.26 | 0.015 | -65.81 | 0.85 |
| -1.0 | -31.65 | -31.71 | 0.19 | -32.27 | 1.9 |
| -1.2 | -27.68 | -27.75 | 0.25 | -28.33 | 2.3 |
| -1.4 | -30.19 | -30.27 | 0.26 | -30.86 | 2.2 |
| -1.6 | -32.06 | -32.14 | 0.25 | -32.77 | 2.2 |

TABLE 2

Comparison of the vector potential values along x = 0 as resulted from direct computation (theoretical) and from POISSON (iterative)

| x = 0<br>y cm | Theoretical<br>A | circular B.C.<br>Calculated<br>A | <br>Error<br>Δ% | elliptical B.C.<br>Calculated<br>A | <br>Error<br>Δ% |
|---|---|---|---|---|---|
| 1.6 | -45.43 | -45.51 | 0.17 | - | - |
| 1.4 | -56.66 | -56.72 | 0.10 | - | - |
| 1.2 | -72.31 | -72.35 | 0.05 | - | - |
| 1.0 | -94.70 | -94.73 | 0.03 | - | - |
| 0.8 | -127.37 | -127.39 | 0.015 | - | - |
| 0.6 | -174.51 | -174.54 | 0.017 | -175.40 | -0.50 |
| 0.4 | -234.34 | -234.37 | 0.013 | -234.97 | 0.27 |
| 0.2 | -287.14 | -287.14 | 0.0 | -287.62 | 0.17 |
| 0.0 | -338.0 | -337.97 | -0.009 | -338.42 | 0.12 |
| -0.2 | -393.51 | -393.53 | 0.005 | -394.01 | 0.13 |
| -0.4 | -377.01 | -377.06 | -0.01 | -377.75 | 0.19 |
| -0.6 | -286.90 | -286.91 | 0.003 | -287.96 | 0.37 |
| -0.8 | -203.97 | -203.98 | 0.005 | - | - |
| -1.0 | -146.12 | -146.14 | 0.013 | - | - |
| -1.2 | -107.46 | -107.50 | 0.04 | - | - |
| -1.4 | -81.34 | -81.40 | 0.07 | - | - |
| -1.6 | -63.24 | -63.33 | 0.14 | - | - |

Correction

In our previous report (Ref. 2) we have stated that the symmetry condition in problems which have midplane symmetry or no-symmetry is $\alpha_k = k - 1$, with k = 1,2,...,and CON 126 = 11. Choosing $\alpha_\ell = 0$ (for k = 1) implies the introduction of a constant term into the solution. From the convention that the vector potential is equal to zero at $r \to \infty$ the value of this constant should be zero. If this constant is not specified the vector potential will change from one iteration to the other so as to accumulate a small but finite average value.

So that the problem will be better defined it is advisable to change our series to $\alpha_k = k$ and start from k = 1 or CON 126 = 10. By doing so we have forced the vector potential around the boundary to be averaged out to zero.

Axisymmetrical Problems with Prolate Spheroidal Coordinates

A cross-section of prolate spheroidal coordinate $(u, v, \phi)$ is shown below. We wish a vector-potential expression such that

$$\vec{A} = A_\phi \hat{e}_\phi$$

and $\nabla \times \nabla \times \vec{A} = 0$, with $\vec{A}_\phi$ a function of $\xi$ and $\eta$ or of u and v, to represent in a source free region the vector-potential of an internal source system that has cylindrical symmetry of rotation ($\phi$ independence). We shall first develop and solve the differential equation for the vector-potential $A_\phi$.

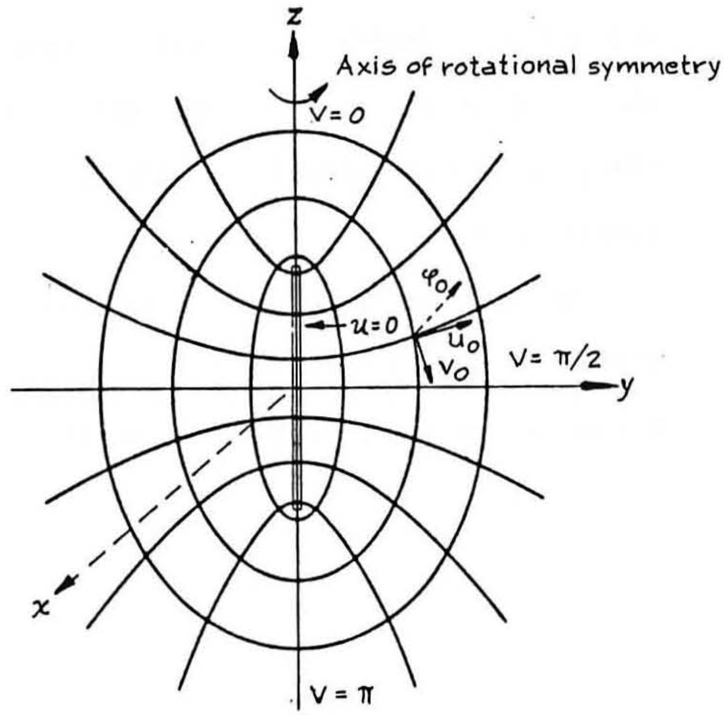


Fig. 5

The explicit form is written

$$\nabla \times [\nabla \times (A_\phi \hat{e}_\phi] = - \hat{e}_\phi \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial x_1} \left[ \frac{h_2}{h_1 h_3} \frac{\partial}{\partial x_1} (h_3 A_\phi) \right] + \frac{\partial}{\partial x_2} \left[ \frac{h_1}{h_2 h_3} \frac{\partial}{\partial x_2} (h_3 A_\phi) \right] \right\} \quad (9)$$

with $h_i$ the metric coefficients. In terms of u and v [e.g. see Arfken, p. 103] the transformation equations are

$$x = c \operatorname{Sinh} u \quad \sin v \cos\phi$$

$$y = c \operatorname{Sinh} u \quad \sin v \sin\phi$$

$$z = c \operatorname{Cosh} u \quad \cos v$$

$$0 \le u \le \infty \; ; \; 0 \le v \le \pi ; \; 0 \le \phi \le 2\pi$$

The scale factors for this system are:

$$h_1 = h_u = c(\text{Sinh}^2 u + \sin^2 v)^{1/2}$$

$$= c(\text{Cosh}^2 u - \cos^2 v)^{1/2}$$

$$h_2 = h_v = c(\text{Sinh}^2 u + \sin^2 v)^{1/2}$$

$$h_3 = h_\phi = c \ \text{Sinh}\ u \sin v$$

If we use a coordinate system ($\xi$, $\eta$, $\phi$) such that

$$\xi = \cos v$$

$$\eta = \text{Cosh}\ u$$

then the transformation equations are:

$$x = \rho \cos\phi; \quad \rho = c[(1 - \xi^2)(\eta^2 - 1)]^{1/2}$$

$$y = \rho \sin\phi$$

$$z = c\xi\eta$$

The scale factors are

$$h_1 = h_\xi = c\left(\frac{\eta^2 - \xi^2}{1 - \xi^2}\right)^{1/2}$$

$$h_2 = h_\eta = c\left(\frac{\eta^2 - \xi^2}{\eta^2 - 1}\right)^{1/2}$$

$$h_3 = h_\phi = \rho = c[(1 - \xi^2)(\eta^2 - 1)]^{1/2}$$

(e.g. W.R. Smyth in Section 5.28)

The constant c is defined as follows:

The curves (or surfaces of rotation) of constant u are given by

$$\left(\frac{z}{c\ \text{Cosh}u}\right)^2 + \left(\frac{\rho}{c\ \text{Sinh}u}\right)^2 = 1$$

so that we have the semi-major axis

$$a = c\ \text{Cosh}\ u = c\eta$$

and the semi-minor axis is

$$b = c\ \text{Sinh}\ u = c(\eta^2 - 1)^{1/2} \quad ;$$

$$\text{thus} \qquad c = (a^2 - b^2)^{1/2}$$

$$\text{and } u = \text{Tanh}^{-1}\left(\frac{b}{a}\right) \text{ and } \eta = \frac{a}{c} = \frac{a}{(a^2 - b^2)^{1/2}}$$

The explicit evaluations of (9) for coordinates $\xi, \eta, \phi$ and for coordinates $u, v, \phi$ are shown below.

<u>In terms of $\xi, \eta, \phi$</u>

$\nabla \times [\nabla \times (A_\phi \hat{e}_\phi)] =$

$$-\frac{\hat{e}_\phi}{h_\xi h_\eta}\left\{\frac{\partial}{\partial\xi}\left[\frac{1}{(\eta^2-1)^{1/2}}\frac{\partial}{\partial\xi}\left[(1-\xi^2)^{1/2}A_\phi\right]\right] + \frac{\partial}{\partial\eta}\left[\frac{1}{(1-\xi^2)^{1/2}}\frac{\partial}{\partial\eta}\left[(\eta^2-1)^{1/2}A_\phi\right]\right]\right\}$$

so that we require (setting the curly bracket equal to zero, and multiplying by $[(1-\xi^2)(\eta^2-1)]^{1/2}$):

$$(1-\xi^2)^{1/2}\frac{\partial^2}{\partial\xi^2}\left[(1-\xi^2)^{1/2}A_\phi\right] + (\eta^2-1)^{1/2}\frac{\partial^2}{\partial\eta^2}\left[(\eta^2-1)^{1/2}A_\phi\right] = 0$$

In terms of u, v, ø

$$\nabla \times [\nabla \times (A_\phi \hat{e}_\phi)] = -\frac{\hat{e}_\phi}{h_u h_v}\left\{\frac{\partial}{\partial u}\left[\frac{1}{\text{Sinh } u}\frac{\partial}{\partial u}(\text{Sinh } u\, A_\phi)\right]\right.$$

$$\left. + \frac{\partial}{\partial v}\left[\frac{1}{\sin v}\ \frac{\partial}{\partial v}(\sin v\, A_\phi)\right]\right\}$$

so that we may require

$$\frac{\partial}{\partial u}\left[\frac{1}{\text{Sinh } u}\frac{\partial}{\partial u}(\text{Sinh } u\, A_\phi)\right] + \frac{\partial}{\partial v}\left[\frac{1}{\sin v}\ \frac{\partial}{\partial v}(\sin v\, A_\phi)\right] = 0$$

With (as suggested earlier) $\eta$ = Cosh u and $\xi$ = cos v

$$\frac{\partial}{\partial u}(\text{Sinh } u\, A_\phi) = \text{Sinh } u\,\frac{\partial}{\partial \eta}\left[(\eta^2-1)^{1/2}A_\phi\right]$$

$$\frac{\partial}{\partial u}\left[\frac{1}{\text{Sinh } u}\frac{\partial}{\partial u}(\text{Sinh } u\, A_\phi)\right] = \frac{\partial}{\partial u}\frac{\partial}{\partial \eta}\left[(\eta^2-1)^{1/2}A_\phi\right]$$

$$= \text{Sinh } u\ \frac{\partial^2}{\partial \eta^2}\left[(\eta^2-1)^{1/2}\ A_\phi\right] = (\eta^2-1)^{1/2}\ \frac{\partial^2}{\partial \eta^2}\left[(\eta^2-1)^{1/2}A_\phi\right]$$

and similarly

$$\frac{\partial}{\partial v}\left[\frac{1}{\sin v}\ \frac{\partial}{\partial v}(\sin v\, A_\phi)\right] = (1-\xi^2)^{1/2}\ \frac{\partial^2}{\partial \xi^2}\left[(1-\xi^2)^{1/2}A_\phi\right]$$

so that the differential equation in terms of u, v, becomes

$$(1-\xi^2)^{1/2}\ \frac{\partial^2}{\partial \xi^2}\left[(1-\xi^2)^{1/2}A_\phi\right] + (\eta^2-1)^{1/2}\ \frac{\partial^2}{\partial \eta^2}\left[(\eta^2-1)^{1/2}A_\phi\right]$$

as was obtained directly above.

With the partial differential equation for $A_\phi$, as written in terms of $\xi$, $\eta$, we may separate variables by writing $A_\phi = F(\eta)G(\xi)$--to obtain

$$(\eta^2 - 1)^{1/2} \frac{d^2}{d\eta^2} [(\eta^2 - 1)^{1/2} F(\eta)] = n(n+1)\ F(\eta)$$

$$(1 - \xi^2)^{1/2} \frac{d^2}{d\xi^2} [(1 - \xi^2)^{1/2} G(\xi)] = -\ n(n+1)\ G(\xi)$$

where we have written the separation constant as n(n+1).

It then becomes permissible to introduce terms in a development of $A_\phi$ that involve

$$P_n^1\ (\xi)\ Q_n^1\ (\eta)$$

or

$$Q_n^1\ (\text{Cosh } u)\ \ .\ \ P_n^1\ (\cos v),$$

since such forms provide solutions to the ordinary differential equations (for F and G) that result from the procedure of separation of variables.

Expressing the vector-potential as a series of the associated Legendre functions

$$P_n^1\ (\xi) Q_n^1\ (\eta)$$

has one disadvantage when the elliptical boundary becomes increasingly circular. In such a case the values of the $Q_n^1$, as expressed in our earlier formulation, approach zero and therefore would preclude the use of ellipses which approach circularity.

So that we can avoid such numerical problems we make use of the relation

$$Q_n^1(\eta) = -(\eta^2 - 1)^{1/2} Q_n'(\eta) \qquad (10)$$

and choose a series constructed from terms

$$(\eta^2 - 1)^{1/2} \text{ times } [P_n^{\,1}(\xi)][-Q_n'(\eta)]$$

(where a prime denotes differentiation with respect to the argument).

NOTE: The functions $Q_n^1$ or $Q_n'$ satisfy the same second-order recursion relation vs. n, since they differ only by the n-independent factor $-(\eta^2 - 1)^{1/2}$. (Appendix A).

For a nested prolate ellipsoid we have

$$\eta = \text{Cosh } u = \frac{a}{c}$$

$$c = (a^2 - b^2)^{1/2}$$

$$(\eta^2 - 1)^{1/2} = \text{Sinh } u = \frac{b}{c}$$

We shall need functions $Q_n'(\eta)$ for values of n commencing with n = 1 and could include availability of values of such functions for n = 0.

We now define the function

$$G_n(\eta) = \eta^{n+2}[-Q'_n(\eta)] \qquad (11)$$

for the argument $\eta$.

Note that:

$$G_o = \eta^2[-Q'_o(\eta)] = \frac{\eta^2}{\eta^2 - 1} = \left(\frac{a}{b}\right)^2$$

We are thus in a position to visualize that

$$A_\phi = \sum C_n P_n^1(\xi) c^{n+1} \frac{(\eta^2-1)^{1/2}}{\eta^{n+2}} G_n(\eta)$$

POISSON uses $A_\phi^* = \rho A_\phi$ , $\rho = c[(1-\xi^2)(\eta^2-1)]^{1/2}$

$$A_\phi^* = \sum C_n c^{n+2} (1-\xi^2)^{1/2} P_n^1(\xi) \frac{\eta^2-1}{\eta^{n+2}} G_n(\eta)$$

So that we can comply with the same problem in polar coordinates (circular boundary) we normalize with respect to n and substitute $B_n$ for $C_n c^{n+2} n$

and make use of the relations $G_o = \frac{\eta^2}{\eta^2-1} = \left(\frac{a}{b}\right)^2$ ; $(1-\xi^2)^{1/2} = \sin v$

to arrive at:

$$A_\phi^* = \sum_{n=1} \eta^{-n} B_n \left[\frac{\sin v \; P_n^1(\cos v)}{n}\right] \frac{G_{n(\eta)}}{G_o(\eta)}$$

We wish to reduce $A_\phi^*$ further into a form such that in the limiting case when $a = b$ $(\eta \to \infty)$ it will be identical to the one for a circular boundary condition (Ref. 2). In order to do so we introduce the function $G_n^\infty$ which is defined as

$$G_n^\infty = \lim_{\eta \to \infty} G_n(\eta).$$

We are now in a position to define:

$$H_n(\eta) = \frac{G_n(\eta)}{G_o(\eta)\ G_n^{\infty}} \tag{12}$$

Introducing $H_n(\eta)$ into the $A_\phi^*$ and substituting $D_n$ for $B_n \cdot G_n^{\infty}$, we write the vector potential as follows:

$$A_\phi^* = \sum_{n=1} D_n \eta^{-n} \left[H_n(\eta)\right]\left[\frac{\sin v\ P_n^1(\cos v)}{n}\right] \tag{13}$$

If we define:

$$F_\ell(v) = \frac{\sin v\ P_{\alpha_\ell}^1(\cos v)}{\alpha_\ell} \tag{14}$$

$$\left(\frac{R-H}{R}\right)^{\alpha_j} = \left(\frac{a_{inner}}{a_{outer}}\right)^{\alpha_j} \frac{H_{\alpha_j}(\eta_{outer})}{H_{\alpha_j}(\eta_{inner})} \quad ; \quad \eta = \frac{a}{c}$$

and if M is defined in terms of the $F_\ell(v)$ in the manner employed previously (see p. 3, Ref. 2), we finally arrive at our canonical form for the solution of the vector potential

$$A_k^* = \sum_{n=1}^{N} E_{kn} A_n^{inner}$$

$$E_{kn} = \sum_{i=1}^{m} \sum_{j=1}^{m} \left(\frac{R-H}{R}\right)^{\alpha_j} W_n (M^{-1})_{ji} F_j(v_k) F_i(v_n)$$

$$R = a_2 \ ; \quad \eta = \frac{a}{c} = \frac{a}{(a^2 - b^2)^{1/2}} \ ; \ v = \tan^{-1}\left[\frac{(y/b)}{(x/a)}\right]$$

Concerning the Weight Factor $W_n$

The inverted matrix $(M^{-1})_{ji}$ is evaluated numerically by an LINPACK inversion routine. The "goodness" of the inversion is judged by a condition number which reflects the invertibility of the matrix. A "good" matrix for inversion has a condition of 1. Conditions of ~ 0.9 are typical in two-dimensional problems with circular and elliptical boundaries.

Initially the condition number in axisymmetrical problems with polar coordinates or prolate speriodal coordinates was found to be of the order of 0.01. We then assumed that this could be improved by readjusting the weights. We further assumed that expressions which conform to those used to calculate the norm in integral form are favorable for numerical inversion.

In accordance therewith we should use terms such as

$$\sin v \; P_j^1(\cos v) \; P_i^1(\cos v)$$

The element in our M matrix differs from the above by a factor sin v. We accordingly have proceeded to adopt a weight function proportional to 1/sin v in combination with the relative angular spacing $\Delta v$ between nodal points. The weight function can be expressed as

$$W_n = \frac{\Delta v_n}{\Delta v_1} \frac{1}{\sin v_n}$$

Such a weight function was tested to give an improvement in the inversion condition (~0.5). A choice of weights proportional to $1/\sin^2 v$ or $1/\sin^{1/2} v$ was also tested and shown to give a poorer inversion.

Note that the possible singularities in $W_n$ when $v_n = 0$ or $\pi$ is removed by excluding from the analysis the two points on the axis of rotation. These two points do not contribute anyway, since their vector potential is always equal to zero.

## Concerning the Function $H_n(\eta)$

The recursion relation for $H_n(\eta)$ as derived in Appendix B is as follows.

$$H_n(\eta) = H_{n+1}(\eta) - \frac{(n + 1)(n + 3)}{(2n + 1)(2n + 5)} \frac{1}{\eta^2} H_{n+2}(\eta) \; ; \qquad (15)$$

we note that $$H_n(\eta) \leq H_{n+1}(\eta) \leq H_{n+2}(\eta) \ldots$$

[With the equality signs holding for

$$\lim_{\eta \to \infty} H_n(\eta) = 1$$

relations (14) reduce to the identical form derived for circular boundaries.]

In practice, to derive $H_n(\eta)$, we may commence the recursion relation by adopting some Nmax (rather greater than the largest value of n for which good results are required, e.g. Nmax = 2n) and assigning $H_{Nmax}$ and $H_{Nmax-1}$ values that are in about the correct ratio into the recursion relation (15). We then proceed to calculate $H_n(\eta)$ for decreasing n down to n = 1 (this will result in terms which are correctly related through the recursion relation). With the value for $H_n$ at n = 0 known, namely $H_o(\eta) = 1$, we can write

$$H_o(\eta) = H_1(\eta) - 0.2 \frac{1}{\eta^2} H_2(\eta) = 1$$

and define a scale factor

$$SF = 1./[H_1(\eta) - 0.2 \frac{1}{\eta^2} H_2(\eta)]$$

by which all $H_n(\eta)$ whould now be normalized (multiplied) in order to arrive at their correct absolute values.

The initial values for $H(\eta)$ are somewhat arbitrary for large $\eta$; however to avoid difficulties (especially when $\eta \simeq 1$) we have adopted the following (Appendix C).

$$H_{Nmax}(\eta) = 2^{Nmax + \frac{1}{2}} \frac{(b/a)^{1/2}}{(1 + b/a)^{Nmax + 1/2}}$$

and

$$H_{Nmax-1}(\eta) = \frac{(1 + b/a)}{2} H_{Nmax}(\eta)$$

This has tried out and verified to give good results even for $\eta \simeq 1$, $(\eta > 1)$.

Example

To illustrate the use of an elliptical boundary condition in axisymmetrical problems, we picked eight current loops with strengths such that no symmetry exists across the equatorial plane.

| Location | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| z(cm) | -2.5 | -2.0 | -1.5 | -1.0 | 1.0 | 1.5 | 2.0 | 2.5 |
| r(cm) | 0.4 | 0.4 | 0.4 | 0.4 | -0.4 | -0.4 | -0.4 | -0.4 |
| I(A) | 1100.0 | 1200.0 | 1300.0 | 1400.0 | 1500.0 | 1600.0 | 1700.0 | 1800.0 |

Using POISSON, the vector potential was calculated for circular and elliptical boundaries and compared with theoretical prediction, see Tables 3,4. The flux lines are plotted respectively in Fig. 6 a,b,c for a circular boundary $\eta = \infty$, and elliptical $\eta = 1.333$, $\eta = 1.075$. The reduction in cpu time is reflected by the reduction of the number of iteration cycles from 650 in case (a) to 320 and 160 in cases (b) and (c) respectively.

## TABLE 3

Comparison Between the Theoretical Vector Potential r A and Three Boundary Cases (Axialsymmetry)

| z = 0.0 | | Case 1 $\eta \to \infty$ | | Case 2 $\eta = 1.333$ | | Case 3 $\eta = 1.075$ | |
|---|---|---|---|---|---|---|---|
| r(cm) | Theoretical (r A) | Circular Boundary r A | Error Δ% | Elliptical Boundary r A | Error Δ% | Elliptical Boundary r A | Error Δ% |
| 0.2 | 7.03 | 7.04 | 0.14 | 7.04 | 0.14 | 7.04 | 0.14 |
| 0.4 | 25.513 | 25.54 | 0.10 | 25.54 | 0.10 | 25.54 | 0.10 |
| 0.6 | 49.427 | 49.46 | 0.06 | 49.45 | 0.04 | 49.468 | 0.08 |
| 0.8 | 72.966 | 73.01 | 0.06 | 73.00 | 0.04 | 73.018 | 0.07 |
| 1.0 | 92.78 | 92.82 | 0.04 | 92.79 | 0.01 | 92.827 | 0.05 |
| 1.2 | 107.89 | 107.928 | 0.03 | 107.89 | 0.0 | 107.933 | 0.04 |
| 1.4 | 118.622 | 118.65 | 0.02 | 118.61 | -0.01 | - | - |
| 1.8 | 130.034 | 130.065 | 0.02 | 130.00 | -0.02 | - | - |
| 2.3 | 132.59 | 132.62 | 0.01 | 132.52 | -0.05 | - | - |

## TABLE 4

Comparison Between the Theoretical Vector Potential rA and Values Calculated From Three Boundary Cases (Axialsymmetry)

| r = 0.6 | | Case 1 $\eta \to \infty$ | | Case 2 $\eta = 1.333$ | | Case 3 $\eta = 1.075$ | |
|---|---|---|---|---|---|---|---|
| z(cm) | Theoretical (r A) | Circular Boundary r A | Error Δ% | Elliptical Boundary r A | Error Δ% | Elliptical Boundary r A | Error Δ% |
| -3.4 | 24.83 | 24.80 | -0.12 | 24.60 | -0.9 | | - |
| -3.0 | 57.90 | 57.81 | -0.15 | 57.97 | 0.12 | 57.99 | 0.15 |
| -2.50 | 174.408 | 175.18 | 0.44 | 175.11 | 0.4 | 174.297 | -0.06 |
| -2.0 | 221.08 | 219.7 | -0.6 | 221.04 | -0.02 | 220.12 | -0.4 |
| -1.5 | 236.41 | 236.2 | -0.08 | 236.39 | 0.0 | 236.74 | 0.14 |
| -1.0 | 211.26 | 211.0 | -0.12 | 210.48 | -0.37 | 211.056 | -0.09 |
| -0.5 | 78.70 | 78.74 | 0.0 | 78.73 | 0.04 | 78.75 | 0.06 |
| -0.0 | 49.427 | 49.46 | 0.06 | 49.45 | 0.04 | 49.47 | 0.08 |
| 0.5 | 86.49 | 86.57 | 0.09 | 86.55 | 0.07 | 86.57 | 0.09 |
| 1.0 | 238.42 | 238.3 | -0.05 | 238.18 | -0.1 | 238.29 | -0.05 |
| 1.5 | 293.96 | 293.12 | -0.28 | 293.84 | -0.04 | 293.03 | -0.3 |
| 2.0 | 306.86 | 306.86 | 0.0 | 306.95 | 0.03 | 306.19 | -0.2 |
| 2.5 | 266.75 | 266.00 | -0.28 | 266.53 | -0.08 | 266.46 | -0.1 |
| 3.0. | 87.53 | 87.78 | 0.28 | 87.61 | 0.09 | 87.12 | -0.4 |
| 3.4 | 36.58 | 36.58 | 0.0 | 36.18 | -1.1 | - | - |

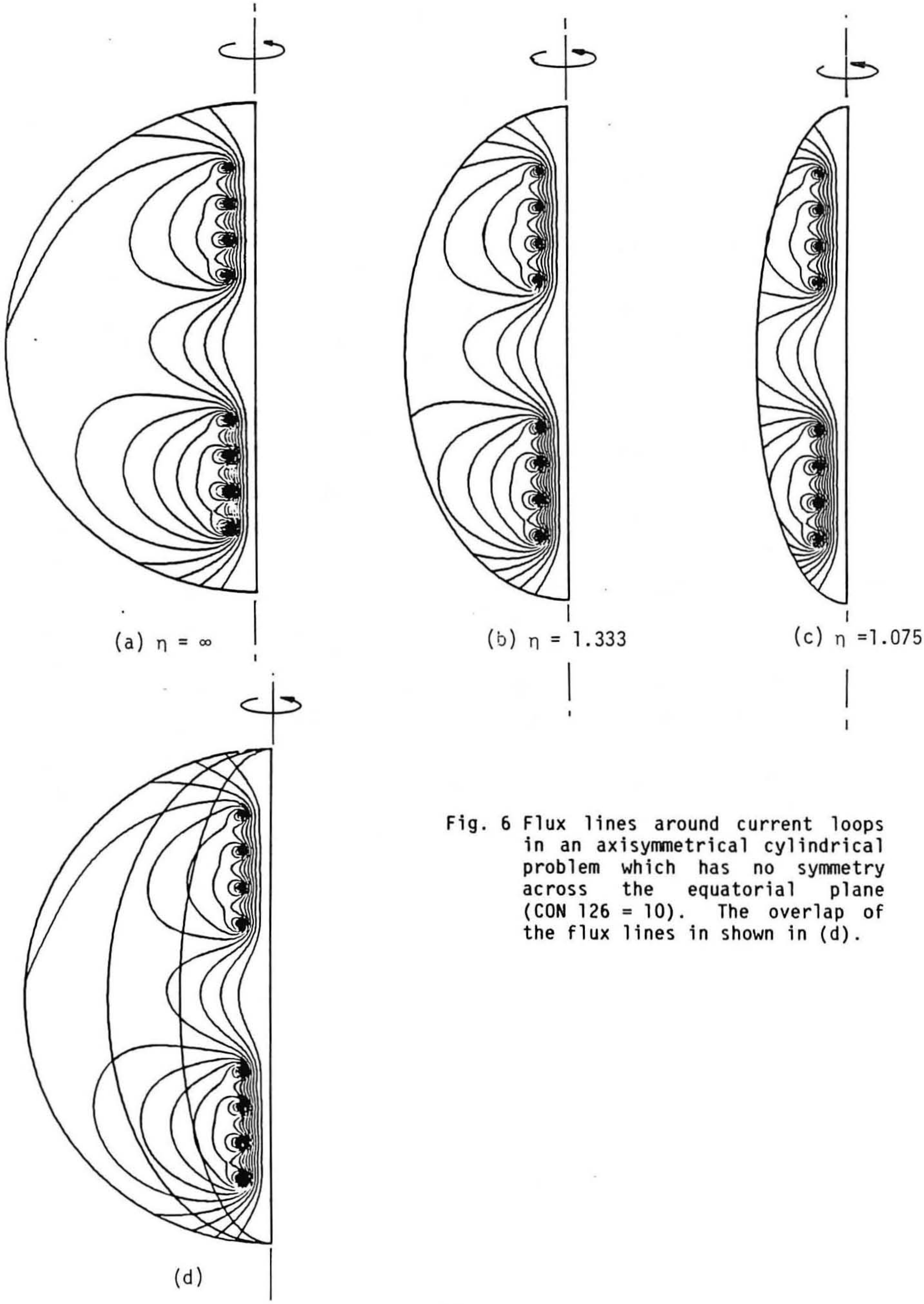



Fig. 6 Flux lines around current loops in an axisymmetrical cylindrical problem which has no symmetry across the equatorial plane (CON 126 = 10). The overlap of the flux lines in shown in (d).

## APPENDIX A

### Concerning Associated Legendre Functions $P_n^1(x)$ and $Q_n^1(x)$

The functions $P_n^1(x)$ and $Q_n^1(x)$ satisfy the same recursion relation

$$nP_{n+1}^1(x) = (2n+1)x \quad P_n^1(x) - (n+1)P_{n-1}^1(x)$$

and

$$nQ_{n+1}^1(x) = (2n+1)x \quad Q_n^1(x) - (n+1)Q_{n-1}^1(x)$$

or

$$(n-1)P_n^1(x) = (2n-1)x \quad P_{n-1}^1(x) - nP_{n-2}^1(x) \tag{1A}$$

$$(n-1)Q_n^1(x) = (2n-1)x \quad Q_{n-1}^1(x) - nQ_{n-2}^1(x)$$

For the functions $Q_n^1(x)$ with $x > 1$ use of the recursion relation may be stable only for n <u>decreasing</u> --- see Introduction of "Handbook of Mathematical Functions", Abramowitz and Stegun, Section 7, p. 13 (Dover, New York).

We expect that the functions $P_n^1(x)$ satisfy the differential equation

$$(1 - x^2)^{1/2} \frac{d^2}{dx^2} [(1 - x^2)^{1/2} P_n^1(x)] + n(n + 1)P_n^1(x) = 0$$

and similarly the functions $Q_n^1$ satisfy the equation that we write conveniently as

$$(x^2-1)^{1/2} \frac{d^2}{dx^2} \left[(x^2- 1)^{1/2}Q_n^1(x)\right] + n(n + 1)Q_n^1(x) = 0$$

If we make use of relation (10) and the recursion relation (1A) we get:

$$-Q_n'(x) = \frac{(2n+3) \ x \ [-Q_{n+1}'(x)] - (n+1)[-Q_{n+2}'(x)]}{n + 2} \tag{2A}$$

## APPENDIX B

### Concerning the Recursion Relation for $H_n(\eta)$

We shall need the functions $G_n(\eta)$, $G_o(\eta)$ and $G_n^\infty$ to arrive at the function $H_n(\eta)$ as defined in (12).

From (11) $$G(\eta) = \eta^{n+2}[-Q'(\eta)]$$

If we now substitute relation (2A), we get

$$G_n(\eta) = \frac{2n + 3}{n + 2} G_{n+1}(\eta) - \frac{n + 1}{n + 2} \frac{1}{\eta^2} G_{n+2}(\eta) \tag{1B}$$

$G_n(\eta)$ satisfies a linear recursion relation vs. n for any possible $\eta > 1$.

From (10) we get:

$$G_o(\eta) = \frac{\eta^2}{\eta^2 - 1} = \left(\frac{a}{b}\right)^2 \tag{2B}$$

To arrive at the $G_n^\infty$ we perform the following

$$G_n^\infty = \lim_{\eta\to\infty} G_n(\eta) = \frac{2n + 3}{n + 2} G_{n+1}^\infty$$

We further write

$$G_n^{\infty} = \frac{n + 1}{2n + 1} \; G_{n-1}^{\infty}$$

$$G_n^{\infty} = \frac{n + 1}{2n + 1} \; \frac{n}{2n - 1} \; \frac{n - 1}{2n - 3} \; \ldots\ldots G_o^{\infty}$$

and from (2B) $G_o^{\infty} = 1$

We thus can write $G_n^{\infty}$ as the series

$$G_n^{\infty} = \frac{2^n n! \; (n + 1)!}{(2n + 1)!} \tag{3B}$$

We are now in the position to arrive at the recursion relation for $H_n(\eta)$

$$H_n(\eta) = H_{n+1}(\eta) - \frac{(n + 1)(n + 3)}{(2n + 3)(2n + 5)} \frac{1}{\eta^2} H_{n+2}(\eta) \tag{4B}$$

## APPENDIX C

### Concerning the Asymptotic Value of $H_n(\eta)$ for Large n

We would like to start with a "good" guess for $H_{Nmax}(\eta)$ in the recursion relation (15). We make use of the definition of $H_n(\eta)$ and the asymptotic formula for $Q_n^1(\eta)$ as suggested by...

$$Q_n^1(\eta) \simeq -(n\pi)^{1/2} \frac{e^{-(n+1/2)\eta}}{(2\, \text{Sinh}\, \eta)^{1/2}} \; ; \quad n \text{ large}$$

Introducing the definition of $G_n(\eta)$ we get:

$$G_n(\eta) = (n\pi)^{1/2} \eta^{n+2} \frac{e^{-(n+1/2)\eta}}{(2\, \text{Sinh}\, \eta)^{1/2}} \frac{1}{(\eta^2-1)^{1/2}} \qquad (1C)$$

We note that:

$$e^{-\eta} = \eta - (\eta^2-1)^{1/2} = \frac{(a-b)}{(a^2-b^2)^{1/2}}$$

$$e^{\eta} = \eta + (\eta^2-1)^{1/2}$$

$$2\, \text{Sinh}\, \eta = 2(\eta^2-1)^{1/2} = \frac{2b}{(a^2-b^2)^{1/2}}$$

We substitute into (1C) and get

$$G_n(\eta) = \left(\frac{\pi n}{2}\right)^{1/2} \frac{1}{(b/a)^{3/2}(1+b/a)^{n+1/2}} \qquad (2C)$$

Introducing $G_n(\eta)$, $G_o(\eta)$ and the definition $G_n^\infty$ from (3B) into the expression of $H_n(\eta)$, we get:

$$H_n(\eta) = \frac{G_n\ (b/a)^2\ (2n+1)!}{2^n n!(n+1)!}$$

or

$$H_n(\eta) = \frac{(\pi n)^{1/2}(2n+1)!}{2^{n+1/2} n!\ (n+1)!}\ \frac{(b/a)^{1/2}}{(1+b/a)^{n+1/2}}$$

Using Stirling's formula $n! = n^n e^{-n}(2\pi n)^{1/2}$, we can approximate

$$\frac{(\pi n)^{1/2}(2n+1)!}{2^{n+1/2}\ n!(n+1)!} \simeq 2^{n+1/2}$$

and so finally we have

$$H_n(\eta) \simeq 2^{n+1/2}\ \frac{(b/a)^{1/2}}{(1+b/a)^{n+1/2}}\ ;\ \text{large } n$$

References

1. S. Caspi, M. Helm, and L.J. Laslett, "Incorporation of a Circular Boundary Condition into the Program POISSON", LBID-887, Lawrence Berkeley Laboratory. February 1984.

2. S. Caspi, M. Helm, and L.J. Laslett, "The Generalization of a Circular Boundary Condition into the Program POISSON to Include the Symmetry and Axis-Symmetry of Revolution", LBL-18063, Lawrence Berkeley Laboratory. July 1984.